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Analysis of an efficient integrator for a size-structured population model with a dynamical resource

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ABSTRACT

In this paper, an efficient numerical method for the approximation of a nonlinear size-structured population model is presented. The nonlinearity of the model is given by dependency on the environment through the consumption of a dynamical resource. We analyse the properties of the numerical scheme and optimal second-order convergence is derived. We report experiments with academical tests to demonstrate numerically the predicted accuracy of the scheme. The model is applied to solve a biological problem: the dynamics of an ectothermic population (the water flea, *Daphnia magna*). We analyse its long time evolution and describe the asymptotically stable steady states, both equilibria and limit cycles.

1. Introduction

In nature we observe that some physiological characteristics, like age, level of satiation (of a predator), energy reserves or the body size of the individual, play an important role in its behaviour. Physiologically structured population models express the dynamics of the population in terms of the processes taking place at the basic unit (individual level) considering physiological differences. They describe the changes in the number of individuals of a population due to growth, death and reproduction and reflect the effect of the physiological state of the individuals on the population dynamics.

Unlimited population growth does not exist either. A population influences its environment and therefore its own behaviour. In addition, many biological feedback loops can only be described properly in terms of the interaction of the physiological processes within the individuals (e.g. the availability of food). Consequently, the use of nonlinear structured population models is an ideal tool to give a realistic mathematical formulation of density dependence. They allow us to take into account the effect of competition for natural resources in the structured-specific growth, mortality and fertility rates.

In this paper, we study a situation in which the vital rates are influenced by the availability of food in the environment, for which the individuals in the population compete. The dynamics of food density is determined by two processes: the regeneration of the food within the environment (which models the changes in the food density in the absence of any consumers) and the feeding by the individuals of the physiological structured population. It could be explained as a predator–prey model in which the predator is considered physiologically structured and the prey is unstructured. A similar theoretical framework could be found when we try to model the dynamics of a cellular population for a continuously stirred batch culture in a tank reactor [1]. In this case, the model includes a different integral term which takes into account how cells divide. With respect to individual growth rate, a fraction of the ingested food is channelled to maintenance and growth. The necessity of maintenance could make this quantity negative. Therefore, the animal shrinks in fact (sea-anemones, flatworms, water fleas, etc.) [2–4].

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Theoretical analysis of these kinds of problems is highly difficult. With respect to models with a nonlinear dependence on weighted populations, the analysis of existence and uniqueness was proven in [5]. Calsina et al. [6] made an in depth theoretical study in two settings: a finite and infinite size range, although it was made only in the case of the nonlinear autonomous case. It was pointed out in [4], that for an analysis of models that involves a dynamical resource we had to perform as carried out in [6]. The study of this theoretical framework appeared first in a Thieme's presentation http://math.la.asu.edu/~dieter/workshop/schedule.html. Nevertheless, a problem with only positive growth was analysed in [7]. They showed stability properties and bifurcation phenomena within a study of the renewal equation for the consumer population birth rate coupled to a delay differential equation for the resource concentration. We can find an extensive study of physiologically structured population models, with analytical studies of aspects such as existence and uniqueness, smoothness and the asymptotic behaviour of solutions in [8,2,9,10,4].

Often, models such as those discussed above cannot be solved analytically and require numerical integration to obtain an approximation to the solution. Nevertheless, standard numerical methods cannot be indiscriminately applied, because they could lead to inaccurate results and, therefore, wrong conclusions. We could introduce the following examples. We showed that the application of a non A-stable numerical method for the simulation to this problem in [11] was not suitable for the approximation of its asymptotically stable steady state. Some spurious oscillations occur if we use a wrong choice of the numerical integrator [12]. The choice of the selection procedure makes the approximation to singular asymptotically stable steady states better [13]. Also, the lack of regularity of the solution plays an essential role in using an *ad hoc* method [14].

The numerical solution to the model, due to its obvious mathematical complexity, entails a serious challenge. De Roos [3] introduced a semidiscrete method, the "escalator boxcar train", in terms of momenta of the original density function. However, it did not provide a direct approximation to the density function and its convergence has not yet been considered yet. In [15], we consider a direct integration of the system by means of a version of a numerical method, successfully employed in [11], to obtain this missing information. Nevertheless, it was shown that some numerical methods were not appropriate for a long-time integration. In that work, we presented a new suitable numerical method based on the modified Euler method and the mid-point quadrature rule. On the other hand, a modification of such a numerical procedure was introduced in [13] in order to approximate singular asymptotic states for these kinds of models. Details about the numerical integration of physiologically-structured population models can be found in [16] and the references therein.

On the other hand, numerical methods for the solution to these kinds of models have been successfully applied to structured models to replicate the available field and/or laboratory data for a variety of different systems: rotifers [17,18], where we showed the existence of an asymptotic stable equilibrium state and the existence of a stable periodic solution with *ad hoc* schemes due to the lack of regularity of the problem; slugs [19], in which we studied equilibrium and oscillatory solutions of a general mass structured system with a boundary delay: the numerical calculations revealed oscillations, pulse solutions and irregular dynamics; marine invertebrates [20,21], where we approximated accurately the steady states and analysed the asymptotic behaviour of the solutions to the linear model and provided original knowledge about the mechanisms that govern the stability of a nonlinear system with a dynamical larvae behaviour; forest dynamics [22,23], in which we described coexistence mechanisms in a size-structured model in terms of competitive differences at the regeneration state, etc.

Finally, when we have to face a numerical simulation in a problem, we must carry out the analysis of the following numerical properties: consistency, stability and convergence. These properties guarantee the goodness of the method to approach the solution.

In this paper, we have developed a new, more efficient second-order numerical method for the problem and performed its complete convergence analysis. We have developed numerical simulations for an academic problem to confirm numerically the convergence order. Also, we have applied it to a significant biological example: the dynamics of a *Daphnia magna* population. It has been studied numerically, but the convergence analysis for the numerical integration has not yet finished [3]. Nevertheless, an analysis of a different scheme that utilized an intermediate value to perform the numerical integration was initiated in [24]. Here, we place emphasis on the approximation to the asymptotically stable states of the model. In Section 2 we proceed to present the general model. Section 3 is devoted to introducing the numerical method employed to approximate the solution to the model. The analysis of the convergence properties is shown in Sections 4–7 and the numerical results performed complete the last section.

2. The model

We consider the following nonlinear size-structured population model where the population feedback on the individuals life history is given by an integro-ordinary differential equation,

$$u_t + (g(x, S(t), t)\, u)_x = -\mu(x, S(t), t)\, u, \quad 0 < x < x_M(t),\ t > 0, \tag{2.1}$$

$$g(0, S(t), t)\, u(0, t) = \int_0^{x_M(t)} \alpha(x, S(t), t)\, u(x, t)\, dx, \quad t > 0, \tag{2.2}$$

$$u(x, 0) = u_0(x), \quad 0 \le x \le x_M(0), \tag{2.3}$$

$$S'(t) = f(S(t), I(t), t), \quad t > 0, \tag{2.4}$$

$$S(0) = S^0. \tag{2.5}$$

The model involves a nonlinear hyperbolic partial differential equation which represents a balance law, a nonlinear and ~~a~~ nonlocal boundary condition which reflects the reproduction process and an initial size-distribution for the population. This size-structured problem is coupled with an initial value one.

The independent variables x and t represent size and time, respectively. The dependent variable $u(x,t)$ is the size-specific density of individuals with size x at time t. The influence of the environment is given by a function $S(t)$ which represents a physiological resource. Its dynamics is driven with the initial value problem (2.4)–(2.5), which is coupled with (2.1)–(2.3). The evolution of such a resource also depends on the population, which is performed by means of the nonlocal term $I(t)$ defined by

$$I(t) = \int_0^{x_M(t)} \gamma(x, S(t), t)\, u(x,t)\, dx, \quad t \geq 0. \tag{2.6}$$

Functions g, α and μ represent the growth, fertility and mortality rates, respectively. These are usually called the vital functions and define the life history of an individual. Functions α and μ are nonnegative. Note that all the vital functions (g, μ and α) depend on the size x (the structuring internal variable), on time t and on the value of the resource at time t, which can reflect the influence of the environmental changes on the vital functions. The function f on the right-hand side of (2.4) depends on the value of the resource at time t, on the total amount of individuals in the population by means of the weighted functional $I(t)$ (which represents the way of weighting the size distribution density in order to model the different influence of the individuals of different sizes on such dynamics) and on time t. In general, the size of any individual varies according to the following ordinary differential equation

$$\frac{dx}{dt} = g(x, S(t), t). \tag{2.7}$$

Therefore, as the growth rate does not have a determined sign, the maximum size $x_M(t)$ that an individual could have at time t, changes with time and its dynamics is described by

$$\begin{aligned} &\frac{d}{dt} x_M(t) = g(x_M(t), S(t), t), \quad t > 0, \\ &x_M(0) = x_M. \end{aligned} \tag{2.8}$$

Model (2.1)–(2.5) represents a way of describing the dynamics of a *Daphnia magna* population into an algal food source and was introduced first in [3].

With regard to the theoretical and numerical analyses of the model, they were mentioned in the Introduction. We note that the mathematical analysis of the solutions to the problem (2.1)–(2.5) in its general setting is beyond the scope of this paper and we will concentrate on the numerical aspects of the problem. Thus, we assume the existence and uniqueness of sufficiently smooth solutions under suitable hypotheses.

In the present paper, we introduce a new, more efficient second-order numerical method for this nonlinear model. We also validate it with its convergence analysis. Throughout the paper we assume the following regularity conditions on the data functions and the solution to the problem (2.1)–(2.6):

(H1) $u \in \mathcal{C}^2([0, x_M(t)] \times [0,T])$, $u(x,t) \geq 0$, $x \in [0, x_M(t)]$, $t \geq 0$.

(H2) $S \in \mathcal{C}^2([0,T])$, $S(t) \geq 0$, $t \geq 0$.

(H3) $\gamma \in \mathcal{C}^2([0, x_M(t)] \times D \times [0,T])$, where D is a compact neighbourhood of

$$\{S(t),\, 0 \leq t \leq T\}.$$

(H4) $\mu \in \mathcal{C}^2([0, x_M(t)] \times D \times [0,T])$, is nonnegative and D is a compact neighbourhood of

$$\{S(t),\ 0 \leq t \leq T\}.$$

(H5) $\alpha \in \mathcal{C}^2([0, x_M(t)] \times D \times [0,T])$, is nonnegative and D is a compact neighbourhood of

$$\{S(t),\ 0 \leq t \leq T\}.$$

(H6) $f \in \mathcal{C}^2(D \times D_I \times [0,T])$, is nonnegative, D is a compact neighbourhood of

$$\{S(t),\ 0 \leq t \leq T\},$$

and D_I is a compact neighbourhood of

$$\left\{ \int_0^{x_M(t)} \gamma(x, S(t), t)\, u(x,t)\, dx,\ 0 \leq t \leq T \right\}.$$

(H7) $g \in \mathcal{C}^3([0, x_M(t)] \times D \times [0,T])$, where D is a compact neighbourhood of

$$\{S(t),\ 0 \leq t \leq T\}.$$

and $g(0, S, t) \geq C > 0$, $t \geq 0$, $S \in \mathbb{R}$. In addition, the characteristic curves $x(t; t^*, x^*)$ defined in (2.7) are continuous and differentiable with respect to the initial values $(t^*, x^*) \in [0,T] \times [0, x_M(t)]$.

The above hypotheses may be based on three possible reasons. First, biological assumptions such as the nonnegativity of some of the vital functions or, in (H7), to reflect that any individual in the studied population could shrink [2]. Second, the

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mathematical requirements to obtain the existence and uniqueness of solutions for the problem (2.1)–(2.6) [2]. Finally, the regularity properties needed in the numerical analysis to derive optimal rates of convergence [15].

3. Numerical integration

The numerical method we employ to approximate the solution to (2.1)–(2.6) is based on the discretization of a representation of the solution along the characteristic curves [11]. First of all we rewrite the partial differential equation (2.1) in a more suitable form for its numerical treatment. So we define

$$\mu^*(x,z,t) = \mu(x,z,t) + g_x(x,z,t).$$

Thus, Eq. (2.1) has the form

$$u_t + g(x,S(t),t)\,u_x = -\mu^*(x,S(t),t)\,u, \quad 0 < x < x_M(t),\ t > 0. \tag{3.9}$$

We denote by $x(t;t^*,x^*)$ the characteristic curve of Eq. (3.9) that takes the value x^* at time t^*. Such a characteristic curve is the solution to the initial value problem

$$\begin{aligned} &\frac{d}{dt}x(t;t^*,x^*) = g(x(t;t^*,x^*),S(t),t), \quad t \ge t^*, \\ &x(t^*;t^*,x^*) = x^*. \end{aligned} \tag{3.10}$$

Now, we consider the function that represents the solution to (3.9) along the characteristic curves

$$w(t;t^*,x^*) = u(x(t;t^*,x^*),t), \quad t \ge t^*,$$

which satisfies the initial value problem

$$\begin{aligned} &\frac{d}{dt}w(t;t^*,x^*) = -\mu^*\left(x\left(t;t^*,x^*\right),S(t),t\right)w(t;t^*,x^*), \quad t \ge t^*, \\ &w(t^*;t^*,x^*) = u(x^*,t^*), \end{aligned}$$

and, therefore, it can be represented in the following integral form

$$w(t;t^*,x^*) = u(x^*,t^*)\,\exp\left\{-\int_{t^*}^{t}\mu^*\left(x\left(\tau;t^*,x^*\right),S(\tau),\tau\right)d\tau\right\}, \quad t \ge t^*. \tag{3.11}$$

Given a constant step $k > 0$, we introduce the discrete time levels $t^n = n\,k$, $n = 0, 1, 2, \ldots$. We also take J a positive integer, as a parameter related to the size variable which describes the number of points in the uniform initial grid. The diameter of such a mesh grid is $h = x_M/J$, and the initial grid nodes are $X_j^0 = j\,h$, $0 \le j \le J$. In order to start the integration, we consider as an approximation to the density at the initial time (t^0), the grid restriction of the initial condition in (2.3), $U_j^0 = u_0(X_j^0)$, $0 \le j \le J$. Also, we use S^0 in (2.5) as the initial value of the resource. Then, the numerical method provides, at each discrete time level, a mesh grid on the size interval in $\mathbf{X}^n$, the approximation to the density on such a mesh grid in $\mathbf{U}^n$ and the approximation of the value of the resource S^n, from the approximations we computed at the previous time level, by using discretizations of Eqs. (3.10), (3.11), (2.2), (2.5), (2.6). Thus, for $n = 0, 1, 2, \ldots$, the numerical solution at time $t^{n+1} = t^n + k$, is obtained from the known values of the numerical solution at time t^n as follows,

$$X_0^{n+1} = 0, \tag{3.12}$$

$$X_{j+1}^{n+1} = X_j^n + \frac{k}{2}\left(g(X_j^n,S^n,t^n) + g(X_{j+1}^{n+1,*},S^{n+1,*},t^{n+1})\right), \quad 0 \le j \le J, \tag{3.13}$$

$$S^{n+1} = S^n + \frac{k}{2}\left(f\left(S^n,\mathcal{Q}(\mathbf{X}^n,\boldsymbol{\gamma}^n\cdot\mathbf{U}^n),t^n\right) + f\left(S^{n+1,*},\mathcal{Q}(\mathbf{X}^{n+1,*},\boldsymbol{\gamma}^{n+1,*}\cdot\mathbf{U}^{n+1,*},t^{n+1})\right)\right), \tag{3.14}$$

$$U_{j+1}^{n+1} = U_j^n\exp\left(-\frac{k}{2}\left(\mu^*\left(X_j^n,S^n,t^n\right) + \mu^*\left(X_{j+1}^{n+1,*},S^{n+1,*},t^{n+1}\right)\right)\right), \tag{3.15}$$

$0 \le j \le J$,

$$U_0^{n+1} = \frac{\mathcal{Q}(\mathbf{X}^{n+1},\boldsymbol{\alpha}^{n+1}\cdot\mathbf{U}^{n+1})}{g(0,S^{n+1},t^{n+1})}, \tag{3.16}$$

where we have to compute approximations at time level t^{n+1},

$$X_0^{n+1,*} = 0, \quad X_{j+1}^{n+1,*} = X_j^n + k\,g(X_j^n,S^n,t^n), \quad 0 \le j \le J, \tag{3.17}$$

$$S^{n+1,*} = S^n + kf\left(S^n,Q^n(\mathbf{X}^n,\boldsymbol{\gamma}^n\cdot\mathbf{U}^n),t^n\right), \tag{3.18}$$

$$U_{j+1}^{n+1,*} = U_j^n\exp\left(-k\,\mu^*\left(X_j^n,S^n,t^n\right)\right), \quad 0 \le j \le J, \tag{3.19}$$

$$U_0^{n+1,*} = \frac{\mathcal{Q}(\mathbf{X}^{n+1,*},\boldsymbol{\alpha}^{n+1,*}\cdot\mathbf{U}^{n+1,*})}{g(0,S^{n+1,*},t^{n+1})}. \tag{3.20}$$

Note that the general step of the method increases the number of grid points and also the dimension of the vector with the numerical densities: at time t^n, we have $J+1$ grid nodes in X^n and the $(J+1)$-dimensional vector $\mathbf{U}^n$, and at time t^{n+1} we obtain $J+2$ grid nodes in X^{n+1} and the $(J+2)$-dimensional vector $\mathbf{U}^{n+1}$. In order to maintain the number of grid points suitable to perform the next step, we eliminate at time t^{n+1} the first grid node X_l^{n+1} which satisfies

$$|X_{l+1}^{n+1} - X_{l-1}^{n+1}| = \min_{1\le j\le J+1} |X_{j+1}^{n+1} - X_{j-1}^{n+1}|. \tag{3.21}$$

We reproduce the same reduction in the corresponding vector $\mathbf{U}^{n+1}$. However, we do not recompute the approximations to the nonlocal terms at such a time level.

In the description of the method, we use the following notation; vectors $\boldsymbol{\alpha}^p$ and $\boldsymbol{\gamma}^p$ contain the evaluations of the functions α and γ in (2.2) and (2.6), respectively, at the grid points in $\mathbf{X}^p$, at the resource value S^p and at time t^p. Products $\boldsymbol{\gamma}^p.\mathbf{U}^p$ and $\boldsymbol{\alpha}^p.\mathbf{U}^p$ must be considered componentwise. In order to approximate integrals over the interval $[0, x_M(t^p)]$, we use the composite trapezoidal quadrature rule based on the grid points $\mathbf{X}^p = [X_0^p, X_1^p, \dots, X_J^p]$, that is

$$\mathcal{Q}(\mathbf{X}^p, \mathbf{V}^p) = \sum_{j=1}^{J} \frac{X_j^p - X_{j-1}^p}{2} \left(V_{j-1}^p + V_j^p\right). \tag{3.22}$$

Note that the method is implicit: all the expressions provide explicit equations for the numerical values at the highest time level, except those which involve the numerical density U_0^p at the first grid point, but it is easy to implement the method in an explicit form.

4. Convergence analysis: preliminaries

Below, we will analyse numerical methods based on integration along characteristics that use a general quadrature rule with suitable properties to approximate the integral terms. Convergence will be obtained by means of consistency and nonlinear stability. We use the discretization framework developed by López-Marcos et al. [25].

We assume that the spatial discretization parameter, h, takes values in the set $H = \{h > 0 : h = x_M/J,\ J \in \mathbb{N}\}$. Now, we suppose that the time step, k, satisfies $k = r\,h$, where r is an arbitrary and positive constant fixed throughout the analysis. In addition, we set $N = [T/k]$. For each $h \in H$, we define the space

$$\mathcal{A}_h = \prod_{n=0}^{N} \left(\mathbb{R}^{J+n} \times \mathbb{R}^{J+n+1}\right) \times \mathbb{R}^{N+1},$$

where $\mathbb{R}^{J+n}$ is used for the approximations to the interior grid nodes and $\mathbb{R}^{J+n+1}$ for the approaches to the theoretical solution on them and on the left boundary node, at time level t^n, $0 \le n \le N$; and $\mathbb{R}^{N+1}$ is employed for the approximations to the theoretical solution to the initial value problem. We also consider the space

$$\mathcal{B}_h = \left(\mathbb{R}^J \times \mathbb{R}^{J+1} \times \mathbb{R}\right) \times \mathbb{R}^N \times \prod_{n=1}^{N} \left(\mathbb{R}^{J+n} \times \mathbb{R}^{J+n}\right) \times \mathbb{R}^N,$$

where $\left(\mathbb{R}^J \times \mathbb{R}^{J+1} \times \mathbb{R}\right)$ is employed to compare with the initial approximations; $\mathbb{R}^N$ considers the residuals that take place on the boundary node for every time step; and $\prod_{n=1}^{N} \left(\mathbb{R}^{J+n} \times \mathbb{R}^{J+n}\right)$, is used for the residuals which arise in the formulae that define the grid nodes and the solution values; and $\mathbb{R}^N$ considers the residuals that are computed for the resource. Both spaces have the same dimension.

In order to measure the size of the errors, we define

$$\|\boldsymbol{\eta}\|_\infty = \max_{1\le j\le p} |\eta_j|, \quad \boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_p) \in \mathbb{R}^p,$$

$$\|\mathbf{V}^n\|_1 = \sum_{j=0}^{J+n} h\,|V_j^n|, \quad \mathbf{V}^n \in \mathbb{R}^{J+n+1}.$$

Thus, we endow the spaces $\mathcal{A}_h$ and $\mathcal{B}_h$ with the following norms.

If $\left(\mathbf{y}^0, \mathbf{V}^0, \dots, \mathbf{y}^N, \mathbf{V}^N, \mathbf{a}\right) \in \mathcal{A}_h$, then

$$\|\left(\mathbf{y}^0, \mathbf{V}^0, \dots, \mathbf{y}^N, \mathbf{V}^N, \mathbf{a}\right)\|_{\mathcal{A}_h} = \max\left(\max_{0\le n\le N} \|\mathbf{y}^n\|_\infty, \max_{0\le n\le N} \|\mathbf{V}^n\|_\infty, \|\mathbf{a}\|_\infty\right).$$

On the other hand, if $\left(\mathbf{Y}^0, \mathbf{Z}^0, A_0, \mathbf{Z}_0, \mathbf{Y}^1, \mathbf{Z}^1, \dots, \mathbf{Y}^N, \mathbf{Z}^N, \mathbf{A}\right) \in \mathcal{B}_h$, thus

$$\begin{aligned}\|\left(\mathbf{Y}^0, \mathbf{Z}^0, A_0, \mathbf{Z}_0, \mathbf{Y}^1, \mathbf{Z}^1, \dots, \mathbf{Y}^N, \mathbf{Z}^N, \mathbf{A}\right)\|_{\mathcal{B}_h} &= \|\mathbf{Y}^0\|_\infty + \|\mathbf{Z}^0\|_\infty + |A_0| + \|\mathbf{Z}_0\|_\infty \\ &\quad + \sum_{n=1}^{N} k\,\|\mathbf{Z}^n\|_\infty + \sum_{n=1}^{N} k\,\|\mathbf{Y}^n\|_\infty + \sum_{n=1}^{N} k\,|\mathbf{A}_n|.\end{aligned}$$

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Now, for each $h \in H$, we define

$$\mathbf{x}_h = (\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N), \qquad \mathbf{x}^n = (x_1^n, \dots, x_{J+n}^n) \in \mathbb{R}^{J+n},$$
$$x_j^0 = jh, \quad 1 \le j \le J,$$
$$x_j^n = x(t_n; t_{n-1}, x_{j-1}^{n-1}), \quad 1 \le j \le J+n,\ 1 \le n \le N; \tag{4.23}$$

and we denote $x_0^n = 0$, $n \ge 0$. Recall that $x(t; t^*, x^*)$ represents the theoretical solution to problem (2.7), $t^* \in [0, T]$, $x^* \in [0, x_M(t)]$. In addition, if u represents the theoretical solution to (2.1)–(2.6) we define

$$\mathbf{u}_h = (\mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^N), \qquad \mathbf{u}^n = (u_0^n, u_1^n, \dots, u_{J+n}^n) \in \mathbb{R}^{J+n+1},$$
$$u_j^n = u(x_j^n, t_n), \quad 0 \le j \le J+n,\ 0 \le n \le N. \tag{4.24}$$

Finally, if S is the theoretical solution to (2.5) then we define

$$\mathbf{s}_h = (s_0, s_1, s_2, \dots, s_N), \qquad s_n = S(t_n), \quad 0 \le n \le N. \tag{4.25}$$

Therefore, $\tilde{\mathbf{u}}_h = (\mathbf{x}^0, \mathbf{u}^0, \mathbf{x}^1, \mathbf{u}^1, \dots, \mathbf{x}^N, \mathbf{u}^N, \mathbf{s}_h) \in \mathcal{A}_h$.

Next, we introduce the discretization operator. Let R be a positive constant and we denote by $B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h, R\,h^p) \subset \mathcal{A}_h$ the open ball with centre $\tilde{\mathbf{u}}_h$ and radius $R\,h^p$, $1 < p < 2$,

$$\boldsymbol{\Phi}_h : B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h, R\,h^p) \to \mathcal{B}_h,$$
$$\boldsymbol{\Phi}_h\left(\mathbf{y}^0, \mathbf{V}^0, \dots, \mathbf{y}^N, \mathbf{V}^N, \mathbf{a}\right) = \left(\mathbf{Y}^0, \mathbf{P}^0, A_0, \mathbf{P}_0, \mathbf{Y}^1, \mathbf{P}^1, \dots, \mathbf{Y}^N, \mathbf{P}^N, \mathbf{A}\right), \tag{4.26}$$

defined by the following equations:

$$\mathbf{Y}^0 = \mathbf{y}^0 - \mathbf{X}^0 \in \mathbb{R}^J, \tag{4.27}$$
$$\mathbf{P}^0 = \mathbf{V}^0 - \mathbf{U}^0 \in \mathbb{R}^{J+1}, \tag{4.28}$$
$$A^0 = a^0 - S^0 \in \mathbb{R}. \tag{4.29}$$

Vectors $\mathbf{X}^0$, $\mathbf{U}^0$ and value S^0 represent approximations at $t = 0$, respectively, to the initial grid nodes, to the theoretical solution at these nodes and to the initial resource. Also,

$$P_0^{n+1} = V_0^{n+1} - \frac{\mathcal{Q}\left(\mathbf{y}^{n+1}, \boldsymbol{\alpha}^{n+1} \cdot \mathbf{V}^{n+1}\right)}{g\left(0, a^{n+1}, t^{n+1}\right)}, \tag{4.30}$$
$$Y_{j+1}^{n+1} = \frac{1}{k}\left\{y_{j+1}^{n+1} - y_j^n - \frac{k}{2}\left(g(y_j^n, a^n, t^n) + g(y_{j+1}^{n+1,*}, a^{n+1,*}, t^{n+1})\right)\right\}, \tag{4.31}$$
$$P_{j+1}^{n+1} = \frac{1}{k}\left\{V_{j+1}^{n+1} - V_j^n \exp\left(-\frac{k}{2}\left(\mu^*\left(y_j^n, a^n, t^n\right) + \mu^*\left(y_{j+1}^{n+1,*}, a^{n+1,*}, t^{n+1}\right)\right)\right)\right\}, \tag{4.32}$$

$0 \le j \le J+n-1$,

$$A_{n+1} = \frac{1}{k}\left\{a^{n+1} - a^n - \frac{k}{2}\left(f\left(a^n, \mathcal{Q}(\mathbf{y}^n, \boldsymbol{\gamma}^n \cdot \mathbf{V}^n), t^n\right) + f\left(a^{n+1,*}, \mathcal{Q}(\mathbf{y}^{n+1,*}, \boldsymbol{\gamma}^{n+1,*} \cdot \mathbf{V}^{n+1,*}), t^{n+1}\right)\right)\right\}, \tag{4.33}$$

$0 \le n \le N-1$, where, with the notation introduced in Section 3,

$$y_{j+1}^{n+1,*} = y_j^n + k\,g(y_j^n, a^n, t^n), \tag{4.34}$$
$$V_{j+1}^{n+1,*} = V_j^n \exp\left(-k\,\mu^*\left(y_j^n, a^n, t^n\right)\right), \tag{4.35}$$

$0 \le j \le J+n-1$,

$$V_0^{n+1,*} = \frac{\mathcal{Q}(\mathbf{y}^{n+1,*}, \boldsymbol{\alpha}^{n+1,*} \cdot \mathbf{V}^{n+1,*})}{g(0, a^{n+1,*}, t_{n+1})}, \tag{4.36}$$
$$a^{n+1,*} = a^n + k f\left(a^n, \mathcal{Q}(\mathbf{y}^n, \boldsymbol{\gamma}^n \cdot \mathbf{V}^n), t^n\right), \tag{4.37}$$

$0 \le n \le N-1$. We denote by $\mathcal{Q}(\mathbf{X}, \mathbf{V}) = \sum_{l=0}^{M} q_l(\mathbf{X})\, V_l$, the general quadrature rule employed in (4.30)–(4.37). We have to highlight that the number of nodes considered at each time level is $J+n+1$, which does not coincide with the number of nodes of $\mathbf{X}^n$, $0 \le n \le N$. This is due to the fact that the quadrature rules use the fixed values of the nodes $X_0^n = x_0^n = 0$, $0 \le n \le N$. This notation is also valid if we consider quadrature rules whose nodes are, at each time level, a subgrid of $\mathbf{X}^n$, $0 \le n \le N$.

Note that, $\boldsymbol{\Phi}_h$ takes into account all the possible nodes and their corresponding solution values at each time level, and it employs quadrature rules possibly based on a subgrid. If $\tilde{\mathbf{U}}_h = (\mathbf{X}^0, \mathbf{U}^0, \mathbf{X}^1, \mathbf{U}^1, \ldots, \mathbf{X}^N, \mathbf{U}^N, \mathbf{s}_h) \in B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h, R\,h^p)$, satisfies

$$\boldsymbol{\Phi}_h(\tilde{\mathbf{U}}_h) = \mathbf{0} \in \mathcal{B}_h, \tag{4.38}$$

the nodes and the corresponding values of the solution at such nodes of $\tilde{\mathbf{U}}_h$ are a numerical solution to the scheme defined by (3.13)–(3.16) when the composite trapezoidal quadrature rule is given. On the other hand, the numerical solution to the scheme defined by (3.13)–(3.16) satisfies (4.38).

Henceforth, C will denote a positive constant, independent of h, k $(k = r\,h)$, j $(0 \le j \le J+n)$ and n $(0 \le n \le N)$; C may have different values in different places.

We assume that the quadrature rule satisfies the following properties:

(P1) $|I(t^n) - \mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}^n \cdot \mathbf{u}^n)| \le C\,h^2$, when $h \to 0$, $0 \le n \le N$.

(P2) $\left|\int_0^{x_M(t)} \alpha(x, S(t^n), t^n)\, u(x, t^n)\, dx - \mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\alpha}^n \cdot \mathbf{u}^n)\right| \le C\,h^2$, when $h \to 0$, $0 \le n \le N$.

(P3) $|q_j(\mathbf{x}^n)| \le q\,h$, where q is a positive constant independent of h, k, j $(0 \le j \le J+n)$ and n $(0 \le n \le N)$, for $0 \le j \le J+n$, $0 \le n \le N$.

(P4) Let R and p be positive constants with $1 < p < 2$. The quadrature weights q_j are Lipschitz continuous functions on $B_\infty(\mathbf{x}^n, R\,h^p)$, $0 \le j \le J+n$, $0 \le n \le N$.

(P5) Let R and p be positive constants with $1 < p < 2$. If $\mathbf{y}^n, \mathbf{z}^n \in B_\infty(\mathbf{x}^n, R\,h^p)$, $\mathbf{V}^n \in B_\infty(\mathbf{u}^n, R\,h^p)$ and $a^n \in B_\infty(S^n, R\,h^p)$, then

$$\left|\sum_{i=0}^{J+n} \left(q_i(\mathbf{y}^n) - q_i(\mathbf{z}^n)\right)\, \gamma\,(z_i^n, a^n, t^n)\, V_i^n\right| \le C\,\|\mathbf{y}^n - \mathbf{z}^n\|_\infty,$$

when $h \to 0$, $0 \le n \le N$.

(P6) Let R and p be positive constants with $1 < p < 2$. If $\mathbf{y}^n, \mathbf{z}^n \in B_\infty(\mathbf{x}^n, R\,h^p)$, $\mathbf{V}^n \in B_\infty(\mathbf{u}^n, R\,h^p)$ and $a^n \in B_\infty(S^n, R\,h^p)$, then

$$\left|\sum_{i=0}^{J+n} \left(q_i(\mathbf{y}^n) - q_i(\mathbf{z}^n)\right)\, \alpha\left(z_i^n, a^n, t^n\right)\, V_i^n\right| \le C\,\|\mathbf{y}^n - \mathbf{z}^n\|_\infty,$$

when $h \to 0$, $0 \le n \le N$.

The following result establishes that the composite trapezoidal rule used in our experiments satisfies these properties [15].

Theorem 1. *Assume that the hypotheses (H1)–(H7) hold. If the quadrature rules are the composite trapezoidal quadrature on subgrids* $\left\{x^n_{j^n_l}\right\}_{l=0}^{M(n)}$, $0 \le n \le N$ *with the property*

(SR) *There exists a positive constant C such that, for h sufficiently small,* $x^n_{j^n_{l+1}} - x^n_{j^n_l} \le C\,h$, $0 \le l \le M(n)-1$, $x^n_{j^n_0} = 0$, $x^n_{j^n_{M(n)}} = x_{J+n}$, *with* $\left\{x^n_{j^n_l}\right\}_{l=1}^{M(n)-1}$ *contained in* $\mathbf{x}^n$, $0 \le n \le N$.

Then, properties (P1)–(P6) hold.

The following result shows that operator (4.26) is well defined.

Proposition 1. *Assume that hypotheses (H1)–(H7) hold and that the quadrature rules used in (4.30)–(4.37) satisfy the properties (P1)–(P6). If*

$$\left(\mathbf{X}^0, \mathbf{V}^0, \ldots, \mathbf{X}^N, \mathbf{V}^N, \mathbf{S}\right) \in B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h, R\,h^p),$$

where R is a fixed positive constant and $1 < p < 2$, *then, for h sufficiently small,*

$$\mathcal{Q}\,(\mathbf{X}^n, \boldsymbol{\gamma}^n \cdot \mathbf{V}^n) \in D_I, \tag{4.39}$$

$0 \le n \le N$. *Furthermore, as* $h \to 0$, $\mathbf{X}^{n,*} \in B_\infty(\mathbf{x}^n, R'\,h^p)$, $S^{n,*} \in B_\infty(s^n, R'\,h^p)$ *and* $\mathbf{V}^{n,*} \in B_\infty(\mathbf{u}^n, R'\,h^p)$, *and*

$$\mathcal{Q}\,(\mathbf{X}^{n,*}, \boldsymbol{\gamma}^{n,*} \cdot \mathbf{V}^{n,*}) \in D_I, \tag{4.40}$$

$1 \le n \le N$.

Proof. The definition of $\mathcal{Q}$, the hypotheses (H1)–(H7), the properties (P1)–(P6) and that $\mathbf{V}^n$ is bounded, allow us to obtain

$$\begin{aligned}
\left|\mathcal{Q}\,(\mathbf{X}^n, \boldsymbol{\gamma}^n \cdot \mathbf{V}^n) - I(t^n)\right| \le{}& \left|\mathcal{Q}\,(\mathbf{X}^n, \boldsymbol{\gamma}(\mathbf{X}^n, S^n) \cdot \mathbf{V}^n) - \mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}(\mathbf{X}^n, S^n) \cdot \mathbf{V}^n)\right| \\
&+ \left|\mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}(\mathbf{X}^n, S^n) \cdot \mathbf{V}^n) - \mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}(\mathbf{X}^n, S(t^n)) \cdot \mathbf{V}^n)\right| \\
&+ \left|\mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}(\mathbf{X}^n, S(t^n)) \cdot \mathbf{V}^n) - \mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}^n \cdot \mathbf{u}^n)\right| + \left|\mathcal{Q}\,(\mathbf{x}^n, \boldsymbol{\gamma}^n \cdot \mathbf{u}^n) - I(t^n)\right|
\end{aligned}$$

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$$\leq \left|\sum_{j=0}^{J+n} \left(q_j(\mathbf{X}^n) - q_j(\mathbf{x}^n)\right) \gamma(X_j^n, S^n)\, V_j^n\right| + \left|\sum_{j=0}^{J+n} q_j(\mathbf{x}^n) \left(\gamma(X_j^n, S^n) - \gamma(x_j^n, s^n)\right) V_j^n\right|$$
$$+ \left|\sum_{j=0}^{J+n} q_j(\mathbf{x}^n)\, \gamma(x_j^n, s^n) \left(V_j^n - u_j^n\right)\right| + o(1)$$
$$\leq C\, R\, h^p\, (J+n+1) \|\gamma\|_\infty \|\mathbf{V}^n\|_\infty + C\, q\, R\, h^{p+1} (J+n+1) \|\mathbf{V}^n\|_\infty$$
$$+ R\, q\, h^{p+1} (J+n+1) \|\gamma\|_\infty + o(1), \tag{4.41}$$

$0 \leq n \leq N$, $h \to 0$. Therefore (4.39) holds, for h sufficiently small. On the other hand, (4.40) is derived following the same arguments. □

5. Consistency

We define *the local discretization error* as

$$\mathbf{l}_h = \boldsymbol{\Phi}_h(\tilde{\mathbf{u}}_h) \in \mathcal{B}_h,$$

and we say that the discretization (4.26) is *consistent* if, as $h \to 0$,

$$\lim \|\boldsymbol{\Phi}_h(\tilde{\mathbf{u}}_h)\|_{\mathcal{B}_h} = \lim \|\mathbf{l}_h\|_{\mathcal{B}_h} = 0.$$

The following theorem establishes the consistency of the numerical scheme defined by Eqs. (4.27)–(4.32).

Theorem 2. *Assume that hypotheses (H1)–(H7) hold and that the considered quadrature rules satisfy properties (P1)–(P6). Then, as $h \to 0$, the local discretization error satisfies,*

$$\|\boldsymbol{\Phi}_h(\tilde{\mathbf{u}}_h)\|_{\mathcal{B}_h} = \|\mathbf{u}^0 - \mathbf{U}^0\|_\infty + \|\mathbf{x}^0 - \mathbf{X}^0\|_\infty + |s^0 - S^0| + O(h^2 + k^2). \tag{5.1}$$

Proof. We denote $\boldsymbol{\Phi}_h(\tilde{\mathbf{u}}_h) = (\mathbf{Z}^0, \mathbf{L}^0, \sigma^0, \mathbf{L}_0, \mathbf{Z}^1, \mathbf{L}^1, \ldots, \mathbf{Z}^N, \mathbf{L}^N, \sigma)$.

First, we set the next bounds for the auxiliary values. Then, by means of regularity hypotheses (H1)–(H7), properties (P1), (P2) of the quadrature rule and error bounds for the explicit Euler method and the rectangular quadrature rule, we obtain

$$|x_j^n - x_j^{n,*}| = \left|x\left(t^n; t^{n-1}, x_{j-1}^{n-1}\right) - x_{j-1}^{n-1} - k\, g(x_{j-1}^{n-1}, s^{n-1}, t^{n-1})\right|$$
$$\leq C\, k^2, \tag{5.2}$$

$1 \leq j \leq J+n$,

$$|s^n - s^{n,*}| \leq \left|S(t^n) - s^{n-1} - k f\left(s^{n-1}, I(t^{n-1}), t^{n-1}\right)\right|$$
$$+ C\, k \left|f\left(s^{n-1}, I(t^{n-1}), t^{n-1}\right) - f\left(t^{n-1}, s^{n-1}, \mathcal{Q}(\mathbf{x}^{n-1}, \boldsymbol{\gamma}^{n-1} \cdot \mathbf{u}^{n-1})\right)\right|$$
$$\leq C \left(k^2 + k \left|I(t^{n-1}) - \mathcal{Q}(\mathbf{x}^{n-1}, \boldsymbol{\gamma}^{n-1} \cdot \mathbf{u}^{n-1})\right|\right)$$
$$\leq C\, (k^2 + h^2), \tag{5.3}$$

and

$$|u_j^n - u_j^{n,*}| = |u_{j-1}^{n-1}| \left|\exp\left\{-\int_{t^{n-1}}^{t^n} \mu^*\left(x(\tau; t^{n-1}, x_{j-1}^{n-1}), S(\tau), \tau\right) d\tau\right\}\right.$$
$$\left.- \exp\left\{-k\, \mu^*\left(x_{j-1}^{n-1}, s^{n-1}, t^{n-1}\right)\right\}\right| \leq C\, k^2, \tag{5.4}$$

$1 \leq j \leq J+n$, $1 \leq n \leq N$. By means of inequality (5.3) we arrive at

$$|u_0^n - u_0^{n,*}| \leq C \left|g\left(0, s^{n,*}, t^n\right) u_0^n - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\, \mathbf{u}^{n,*}\right)\right|$$
$$\leq C \left|g\left(0, s^{n,*}, t^n\right) - g\left(0, s^n, t^n\right)\right| |u_0^n|$$
$$+ \left|\int_0^{x_M(t^n)} \alpha(x, s^n, t^n)\, u(x, t^n)\, dx - \mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\, \mathbf{u}^n\right)\right|$$
$$+ \left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\, \mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\, \mathbf{u}^{n,*}\right)\right|$$
$$\leq C\, (h^2 + k^2) + \left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\, \mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\, \mathbf{u}^{n,*}\right)\right|. \tag{5.5}$$

Now, by means of inequalities (5.2)–(5.5), hypotheses (H1)–(H7) and the properties of the quadrature rule (P3) and (P6), we obtain

$$\begin{aligned}\left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\,\mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\,\mathbf{u}^{n,*}\right)\right| &\le \left|\sum_{j=0}^{J+n}\left(q_j(\mathbf{x}^n) - q_j(\mathbf{x}^{n,*})\right)\,\alpha\left(x_j^{n,*}, s^{n,*}, t^n\right)\,u_j^{n,*}\right| \\ &\quad + \sum_{j=0}^{J+n}\left|q_j(\mathbf{x}^n)\,\left(\alpha\left(x_j^{n,*}, s^{n,*}, t^n\right) - \alpha\left(x_j^n, s^n, t^n\right)\right)\,u_j^{n,*}\right| \\ &\quad + \sum_{j=0}^{J+n}\left|q_j(\mathbf{x}^n)\,\alpha\left(x_j^n, s^n, t^n\right)\,\left(u_j^n - u_j^{n,*}\right)\right| \\ &\le C\,(h^2+k^2) + C\,h\,\left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\,\mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\,\mathbf{u}^{n,*}\right)\right|,\end{aligned}$$

thus, for h sufficiently small, we arrive at

$$\left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\alpha}^n\,\mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\alpha}^{n,*}\,\mathbf{u}^{n,*}\right)\right| \le C\,(h^2+k^2). \tag{5.6}$$

Therefore, by means of (5.5) and (5.6), it follows that

$$|u_0^n - u_0^{n,*}| \le C\,(h^2+k^2). \tag{5.7}$$

Finally, using the inequalities (5.2)–(5.4) and (5.6)–(5.7), the hypotheses (H1)–(H7), the properties of the quadrature rule (P3) and (P5) in (5.6), we obtain

$$\left|\mathcal{Q}\left(\mathbf{x}^n, \boldsymbol{\gamma}^n\,\mathbf{u}^n\right) - \mathcal{Q}\left(\mathbf{x}^{n,*}, \boldsymbol{\gamma}^{n,*}\,\mathbf{u}^{n,*}\right)\right| \le C\,(h^2+k^2). \tag{5.8}$$

Now, we set the bounds for $\mathbf{Z}^n$, $1 \le n \le N$. By means of (3.10) and (4.31), the regularity hypotheses (H1)–(H7), inequalities (5.2)–(5.3) and the error bound of R–K schemes employed, we have

$$\begin{aligned}|Z_j^n| &\le \frac{1}{k}\left\{\left|x_j^n - x_{j-1}^{n-1} - \frac{k}{2}\,\left(g\left(x_{j-1}^{n-1}, s^{n-1}, t^{n-1}\right) + g\left(x_j^n, s^n, t^n\right)\right)\right|\right. \\ &\quad + \frac{k}{2}\,\left|g\left(x_j^n, s^n, t^n\right) - g\left(x_j^{n,*}, s^n, t^n\right)\right| \\ &\quad \left. + \,\frac{k}{2}\,\left|g\left(x_j^{n,*}, s^n, t^n\right) - g\left(x_j^{n,*}, s^{n,*}, t^n\right)\right|\right\} \\ &\le C\,\left\{k^2 + \left|x_j^n - x_j^{n,*}\right| + |s^n - s^{n,*}|\right\} \\ &\le C\,(k^2+h^2),\end{aligned} \tag{5.9}$$

$1 \le j \le J+n$, $1 \le n \le N$.

Next, arguments analogous to those used to derive (5.9) lead us to establish the bound for the truncation errors produced by the resource solution. By means of (4.33), the regularity hypotheses (H1)–(H7), the property of the quadrature rule (P1), inequalities (5.3) and (5.8) and the error bound of R–K schemes employed, we arrive at

$$\begin{aligned}|\sigma^n| &\le \frac{1}{k}\left\{\left|s^n - s^{n-1} - \frac{k}{2}\,\left(f\left(s^{n-1}, I(t^{n-1}), t^{n-1}\right) + f\left(s^n, I(t^n), t^n\right)\right)\right|\right. \\ &\quad + \frac{k}{2}\,\left|f\left(s^{n-1}, I(t^{n-1}), t^{n-1}\right) - f\left(s^{n-1}, \mathcal{Q}(\mathbf{x}^{n-1}, \boldsymbol{\gamma}^{n-1}\cdot\mathbf{u}^{n-1}), t^{n-1}\right)\right| \\ &\quad + \frac{k}{2}\,\left|f\left(s^n, I(t^n), t^n\right) - f\left(s^{n,*}, I(t_n), t^n\right)\right| \\ &\quad \left. + \frac{k}{2}\,\left|f\left(s^{n,*}, I(t_n), t^n\right) - f\left(s^{n,*}, \mathcal{Q}(\mathbf{x}^{n,*}, \boldsymbol{\gamma}^{n,*}\cdot\mathbf{u}^{n,*}), t^n,\right)\right|\right\} \\ &\le C\,\left\{k^2 + \left|s^n - s^{n,*}\right| + \left|I(t^{n-1}) - \mathcal{Q}(\mathbf{x}^{n-1}, \boldsymbol{\gamma}^{n-1}\cdot\mathbf{u}^{n-1})\right|\right. \\ &\quad \left. + \left|I(t^n) - \mathcal{Q}(\mathbf{x}^n, \boldsymbol{\gamma}^n\cdot\mathbf{u}^n)\right| + \left|\mathcal{Q}(\mathbf{x}^n, \boldsymbol{\gamma}^n\cdot\mathbf{u}^n) - \mathcal{Q}(\mathbf{x}^{n,*}, \boldsymbol{\gamma}^{n,*}\cdot\mathbf{u}^{n,*})\right|\right\} \\ &\le C\,\left(k^2+h^2\right),\end{aligned} \tag{5.10}$$

$1 \le n \le N$. Finally, we establish the bounds for the truncation errors produced by the solution to the PDE with a similar reasoning. By means of (3.11) and (4.32), the regularity hypotheses (H1)–(H7), the property of the quadrature rule (P1),

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inequalities (5.2)–(5.3) and the error bound of the trapezoidal quadrature rule, we have

$$
\begin{aligned}
|L_j^n| &\le \frac{C}{k}\left\{\left|\int_{t^{n-1}}^{t^n} \mu^*\left(x(\tau;t^{n-1},x_{j-1}^{n-1}),s(\tau),\tau\right)\,d\tau-\frac{k}{2}\left(\mu^*\left(x_j^{n-1},s^{n-1},t^{n-1}\right)\right.\right.\right.\\
&\qquad\left.\left.\left.+\mu^*\left(x_j^n,s^n,t^n\right)\right)\right|+\frac{k}{2}\left|\mu^*\left(x_j^n,s^n,t^n\right)-\mu^*\left(x_j^{n,*},s^{n,*},t^n\right)\right|\right\}\\
&\le C\left\{k^2+\left|x_j^n-x_j^{n,*}\right|+\left|s^n-s^{n,*}\right|\right\}\\
&\le C\,(k^2+h^2), \qquad (5.11)
\end{aligned}
$$

$1\le j\le J+n$, $1\le n\le N$. And, to find an estimation for the boundary terms, Eq. (4.30), hypothesis (H7) and property (P2), allow us to obtain

$$
\begin{aligned}
|L_0^n| &\le C\left|g\left(0,s^n,t^n\right)\,u_0^n-\mathcal{Q}\left(\mathbf{x}^n,\boldsymbol{\alpha}^n\cdot\mathbf{u}^n\right)\right|\\
&\le C\left|\int_0^{x_M(t^n)}\alpha(x,s^n,t^n)\,u(x,t^n)\,dx-\mathcal{Q}\left(\mathbf{x}^n,\boldsymbol{\alpha}^n\cdot\mathbf{u}^n\right)\right|\\
&\le C\,h^2, \qquad (5.12)
\end{aligned}
$$

$1\le n\le N$. Therefore, (5.1) follows from (5.9)–(5.11) and (5.12). □

6. Stability

Another notion that plays an important role in the analysis of the numerical method is the *stability with h-dependent thresholds*. For $h\in H$, let R_h be a real number (*the stability threshold*) with $0<R_h<\infty$, we say that the discretization (4.26) is *stable* for $\tilde{\mathbf{u}}_h$ restricted to the thresholds R_h, if there exist two positive constants h_0 and S (*the stability constant*) such that, for any $h\in H$ with $h\le h_0$, the open ball $B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h,R_h)$ is contained in the domain of $\boldsymbol{\Phi}_h$, and, for all $\tilde{\mathbf{V}}_h$, $\tilde{\mathbf{W}}_h$ in that ball,

$$\|\tilde{\mathbf{V}}_h-\tilde{\mathbf{W}}_h\|\le S\,\|\boldsymbol{\Phi}_h(\tilde{\mathbf{V}}_h)-\boldsymbol{\Phi}_h(\tilde{\mathbf{W}}_h)\|.$$

We begin with the following auxiliary result whose demonstration was made in [24] where the same quadrature rule is given.

Proposition 2. *Assume that hypotheses (H1)–(H7) hold and that the considered quadrature rules satisfy properties (P1)–(P6). Let* $\mathbf{y}^n,\mathbf{z}^n\in B_\infty(\mathbf{x}^n,R\,h^p)$, $\mathbf{V}^n,\mathbf{W}^n\in B_\infty(\mathbf{u}^n,R\,h^p)$ *and* $a^n,b^n\in B_\infty(s^n,R\,h^p)$. *Then, as* $h\to 0$,

$$|\mathcal{Q}(\mathbf{y}^n,\boldsymbol{\gamma}^n\cdot\mathbf{V}^n)-\mathcal{Q}(\mathbf{z}^n,\boldsymbol{\gamma}^n\cdot\mathbf{W}^n)|\le C\left(\|\mathbf{V}^n-\mathbf{W}^n\|_1+\|\mathbf{y}^n-\mathbf{z}^n\|_\infty+|b^n-a^n|\right), \qquad (6.1)$$

$$|\mathcal{Q}(\mathbf{y}^{n,*},\boldsymbol{\gamma}^{n,*}\cdot\mathbf{V}^{n,*})-\mathcal{Q}(\mathbf{z}^{n,*},\boldsymbol{\gamma}^{n,*}\cdot\mathbf{W}^{n,*})|\le C\left(\|\mathbf{V}^{n,*}-\mathbf{W}^{n,*}\|_1+\|\mathbf{y}^{n,*}-\mathbf{z}^{n,*}\|_\infty+|b^{n,*}-a^{n,*}|\right), \qquad (6.2)$$

$1\le n\le N$.

Now, we introduce the theorem that establishes the *stability* of the discretization defined by Eqs. (4.27)–(4.32).

Theorem 3. *Assume that hypotheses (H1)–(H7) hold and that the considered quadrature rules satisfy properties (P1)–(P6). Then, the discretization is stable for* $\tilde{\mathbf{u}}_h$ *with* $R_h=R\,h^p$, $1<p<2$.

Proof. We denote

$$
\begin{aligned}
&\boldsymbol{\Phi}_h\left(\mathbf{y}^0,\mathbf{V}^0,\mathbf{y}^1,\mathbf{V}^1,\ldots,\mathbf{y}^N,\mathbf{V}^N,\mathbf{a}\right)=\left(\mathbf{Y}^0,\mathbf{P}^0,A_0,\mathbf{P}_0,\mathbf{Y}^1,\mathbf{P}^1,\ldots,\mathbf{Y}^N,\mathbf{P}^N,\mathbf{A}\right),\\
&\boldsymbol{\Phi}_h\left(\mathbf{z}^0,\mathbf{W}^0,\mathbf{z}^1,\mathbf{W}^1,\ldots,\mathbf{z}^N,\mathbf{W}^N,\mathbf{b}\right)=\left(\mathbf{Z}^0,\mathbf{R}^0,B_0,\mathbf{R}_0,\mathbf{Z}^1,\mathbf{R}^1,\ldots,\mathbf{Z}^N,\mathbf{R}^N,\mathbf{B}\right),
\end{aligned}
$$

$\left(\mathbf{y}^0,\mathbf{V}^0,\mathbf{y}^1,\mathbf{V}^1,\ldots,\mathbf{y}^N,\mathbf{V}^N,\mathbf{a}\right),\left(\mathbf{z}^0,\mathbf{W}^0,\mathbf{z}^1,\mathbf{W}^1,\ldots,\mathbf{z}^N,\mathbf{W}^N,\mathbf{b}\right)\in B_{\mathcal{A}_h}(\tilde{\mathbf{u}}_h,R_h)$.

Now, we set

$$\mathbf{E}^n=\mathbf{V}^n-\mathbf{W}^n\in\mathbb{R}^{J+n+1},\ \boldsymbol{\Delta}^n=\mathbf{y}^n-\mathbf{z}^n\in\mathbb{R}^{J+n},\ \sigma^n=b^n-a^n\in\mathbb{R},\quad 0\le n\le N.$$

$$\mathbf{E}^{n,*}=\mathbf{V}^{n,*}-\mathbf{W}^{n,*}\in\mathbb{R}^{J+n+2},\ \boldsymbol{\Delta}^{n,*}=\mathbf{y}^{n,*}-\mathbf{z}^{n,*}\in\mathbb{R}^{J+n+1},\ \sigma^{n,*}=b^{n,*}-a^{n,*}\in\mathbb{R},$$

$1\le n\le N$. From (4.34) and by means of hypotheses (H7), we obtain

$$
\begin{aligned}
|\Delta_j^{n,*}| &\le |\Delta_{j-1}^{n-1}|+k\left|g\left(y_{j-1}^{n-1},a^{n-1},t^{n-1}\right)-g\left(z_{j-1}^{n-1},b_{n-1},t^{n-1}\right)\right|\\
&\le (1+C\,k)\,|\Delta_{j-1}^{n-1}|+C\,k\left|\sigma^{n-1}\right|, \qquad (6.3)
\end{aligned}
$$

$1 \le j \le J+n$. Thus

$$\left\|\boldsymbol{\Delta}^{n,*}\right\|_\infty \le (1+C\,k)\,\left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + C\,k\,\left|\sigma^{n-1}\right|, \tag{6.4}$$

$1 \le n \le N$. Next, from (4.37), by means of hypothesis (H6) and inequality (6.1), we arrive at

$$\begin{aligned} |\sigma^{n,*}| &\le |\sigma^{n-1}| + k\,\left|f\left(a^{n-1}, Q\left(\mathbf{y}^{n-1}, \boldsymbol{\gamma}^{n-1}\cdot\mathbf{V}^{n-1}\right), t^{n-1}\right) - f\left(b^{n-1}, Q\left(\mathbf{z}^{n-1}, \boldsymbol{\gamma}^{n-1}\cdot\mathbf{W}^{n-1}\right), t^{n-1}\right)\right| \\ &\le (1+C\,k)\,|\sigma^{n-1}| + C\,k\,\left\{\left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1\right\}, \end{aligned} \tag{6.5}$$

$1 \le n \le N$. Now, from (4.35), by means of hypotheses (H4), (H7), $\|\mathbf{W}^{n-1}\|_\infty \le C$, we have

$$\begin{aligned} |E_j^{n,*}| &\le |E_{j-1}^{n-1}|\,\exp\left(-k\,\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right)\right) \\ &\quad + |W_{j-1}^{n-1}|\,\left|\exp\left(-k\,\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right)\right) - \exp\left(-k\,\mu^*\left(z_{j-1}^{n-1}, b^{n-1}, t^{n-1}\right)\right)\right| \\ &\le |E_{j-1}^{n-1}| + C\,k\,\left|\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right) - \mu^*\left(z_{j-1}^{n-1}, b^{n-1}, t^{n-1}\right)\right| \\ &\le |E_{j-1}^{n-1}| + C\,k\,\left\{\left|\sigma^{n-1}\right| + \left|\Delta_{j-1}^{n-1}\right|\right\}, \end{aligned}$$

$1 \le j \le J+n$. Thus,

$$|E_j^{n,*}| \le \left\|\mathbf{E}^{n-1}\right\|_1 + C\,k\,\left\{\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty\right\}, \tag{6.6}$$

$1 \le j \le J+n$. And, from (4.36), (H7) and inequality (6.2), enable us to write

$$\begin{aligned} |E_0^{n,*}| &\le C\,\left\{\left|g\left(0, a^{n,*}, t^n\right) - g\left(0, b^{n,*}, t^n\right)\right|\,\left|\mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}\,\mathbf{V}^{n,*})\right|\right. \\ &\quad \left. + \left|g\left(0, a^{n,*}, t^n\right)\right|\,\left|\mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}\,\mathbf{V}^{n,*}) - \mathcal{Q}(\mathbf{z}^{n,*}, \boldsymbol{\gamma}^{n,*}\,\mathbf{W}^{n,*})\right|\right\} \\ &\le C\,\left\{\left|\sigma^{n,*}\right| + \left\|\boldsymbol{\Delta}^{n,*}\right\|_\infty + \left\|\mathbf{E}^{n,*}\right\|_1\right\}, \end{aligned} \tag{6.7}$$

$1 \le n \le N$. Next, we use (6.4) and (6.5) in (6.7) to obtain, for h sufficiently small,

$$|E_0^{n,*}| \le C\,\left\{\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 + \left\|\mathbf{E}^{n,*}\right\|_1\right\}, \tag{6.8}$$

$1 \le n \le N$. Now, multiplying $|E_j^{n,*}|$ by h and summing in j, $0 \le j \le J+n+1$, from (6.6), (6.8) and that $k = r\,h$, we obtain

$$\begin{aligned} \|E^{n,*}\|_1 &\le C\,h\left\{\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 + \left\|\mathbf{E}^{n,*}\right\|_1\right\} + h\sum_{j=1}^{J+n+1}\left\{\left\|\mathbf{E}^{n-1}\right\|_1 + C\,k\left(\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty\right)\right\} \\ &\le C\,h\,\left\|\mathbf{E}^{n,*}\right\|_1 + C\,\left\{\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1\right\}, \end{aligned} \tag{6.9}$$

$1 \le n \le N$. Therefore, for h sufficiently small,

$$\|E^{n,*}\|_1 \le C\,\left\{\left|\sigma^{n-1}\right| + \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1\right\}, \tag{6.10}$$

$1 \le n \le N$.

Now, by means of (4.31), hypotheses (H6)–(H7), (6.3) and (6.5) enable us to write

$$\begin{aligned} |\Delta_j^n| &\le |\Delta_{j-1}^{n-1}| + \frac{k}{2}\,\left|g(y_j^{n-1}, a^{n-1}, t^{n-1}) - g(z_j^{n-1}, b^{n-1}, t^{n-1})\right| \\ &\quad + \frac{k}{2}\,\left|g(y_j^{n,*}, a^{n,*}, t^n) - g(z_j^{n,*}, b^{n,*}, t^n)\right| + k\,\left|Y_j^n - Z_j^n\right| \\ &\le |\Delta_{j-1}^{n-1}| + C\,k\left\{\left|g(y_j^{n-1}, a^{n-1}, t^{n-1}) - g(z_j^{n-1}, a^{n-1}, t^{n-1})\right|\right. \\ &\quad + \left|g(z_j^{n-1}, a^{n-1}, t^{n-1}) - g(z_j^{n-1}, b^{n-1}, t^{n-1})\right| + \left|g(y_j^{n,*}, a^{n,*}, t^n) - g(z_j^{n,*}, a^{n,*}, t^n)\right| \\ &\quad \left. + \left|g(z_j^{n,*}, a^{n,*}, t^n) - g(z_j^{n,*}, b^{n,*}, t^n)\right| + k\,\left|Y_j^n - Z_j^n\right|\right\} \\ &\le (1+C\,k)\,|\Delta_{j-1}^{n-1}| + C\,k\,\left\{|\sigma^{n-1}| + \left|\Delta_j^{n,*}\right| + \left|\sigma^{n,*}\right|\right\} + k\,\left|Y_j^n - Z_j^n\right| \\ &\le |\Delta_{j-1}^{n-1}| + C\,k\,\left\{\left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 + \left|\sigma^{n-1}\right|\right\} + k\,\left|Y_j^n - Z_j^n\right|, \end{aligned} \tag{6.11}$$

$1 \le j \le J+n$, $1 \le n \le N$. Thus, when $N \ge n > j \ge 1$, from (6.11), we have

$$|\Delta_j^n| \le C\,k\sum_{l=0}^{j-1}\left\{\|\mathbf{E}^{n-1-l}\|_1 + \|\boldsymbol{\Delta}^{n-1-l}\|_\infty + \left|\sigma^{n-1-l}\right|\right\} + k\sum_{l=0}^{j-1}|Y_{j-l}^{n-l} - Z_{j-l}^{n-l}|. \tag{6.12}$$

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Therefore, when $N \geq n > j \geq 1$, by means of (6.12), we establish

$$|\Delta_j^n| \leq C \left\{ \sum_{m=n-j}^{n-1} k \, \|\mathbf{E}^m\|_1 + \sum_{m=n-j}^{n-1} k \, \|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=n-j}^{n-1} k \, \left|\sigma^m\right| \right\} + \sum_{m=n-j+1}^{n} k \, \|\mathbf{Y}^m - \mathbf{Z}^m\|_\infty. \tag{6.13}$$

On the other hand, when $J + n \geq j \geq n \geq 1$, due to (6.11) it follows that

$$|\Delta_j^n| \leq |\Delta_{j-n}^0| + C\,k \sum_{l=0}^{n-1} \left\{ \|\mathbf{E}^{n-1-l}\|_1 + \|\boldsymbol{\Delta}^{n-1-l}\|_\infty + \left|\sigma^{n-1-l}\right| \right\} + k \sum_{l=0}^{n-1} |Y_{j-l}^{n-l} - Z_{j-l}^{n-l}|. \tag{6.14}$$

Thus, when $J + n \geq j \geq n \geq 1$, (6.14) yields

$$|\Delta_j^n| \leq \|\boldsymbol{\Delta}^0\|_\infty + C \left\{ \sum_{m=0}^{n-1} k \, \|\mathbf{E}^m\|_1 + \sum_{m=0}^{n-1} k \, \|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n-1} k \, \left|\sigma^m\right| \right\} + \sum_{m=1}^{n} k \, \|\mathbf{Y}^m - \mathbf{Z}^m\|_\infty. \tag{6.15}$$

Then, by means of (6.13) and (6.15), we can conclude that

$$\|\boldsymbol{\Delta}^n\|_\infty \leq \|\boldsymbol{\Delta}^0\|_\infty + C \left\{ \sum_{m=0}^{n-1} k \, \|\mathbf{E}^m\|_1 + \sum_{m=0}^{n-1} k \, \|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n-1} k \, \left|\sigma^m\right| \right\} + \sum_{m=1}^{n} k \, \|\mathbf{Y}^m - \mathbf{Z}^m\|_\infty. \tag{6.16}$$

$1 \leq n \leq N$. On the other hand, from (4.33), hypothesis (H6) and inequalities (6.1)–(6.2), (6.4)–(6.5) and (6.10), we arrive at

$$\begin{aligned}
|\sigma^n| \leq\ & |\sigma^{n-1}| + k\,|A^n - B^n| + C\,k\, \Big\{ \Big| f(a^{n-1}, \mathcal{Q}(\mathbf{y}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{y}, a) \cdot \mathbf{V}^{n-1}), t^n) \\
& - f(b^{n-1}, \mathcal{Q}(\mathbf{y}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{y}, a) \cdot \mathbf{V}^{n-1}), t^n) \Big| \\
& + \Big| f(b^{n-1}, \mathcal{Q}(\mathbf{y}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{y}, a) \cdot \mathbf{V}^{n-1}), t^n) \\
& - f(b^{n-1}, \mathcal{Q}(\mathbf{z}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{z}, b) \cdot \mathbf{V}^{n-1}), t^n) \Big| \\
& + \Big| f(a^{n,*}, \mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{y}, a) \cdot \mathbf{V}^{n,*}), t^n) \\
& - f(b^{n,*}, \mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{y}, a) \cdot \mathbf{V}^{n,*}), t^n) \Big| \\
& + \Big| f(b^{n,*}, \mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{y}, a) \cdot \mathbf{V}^{n,*}), t^n) \\
& - f(b^{n,*}, \mathcal{Q}(\mathbf{z}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{z}, b) \cdot \mathbf{V}^{n,*}), t^n) \Big| \Big\} \\
\leq\ & (1 + C\,k)\,|\sigma^{n-1}| + k\,|A^n - B^n| + C\,k\, \Big\{ \left|\sigma^{n,*}\right| \\
& + \left| \mathcal{Q}(\mathbf{y}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{y}, a) \cdot \mathbf{V}^{n-1}) - \mathcal{Q}(\mathbf{z}^{n-1}, \boldsymbol{\gamma}^{n-1}(\mathbf{z}, b) \cdot \mathbf{V}^{n-1}) \right| \\
& + \left| \mathcal{Q}(\mathbf{y}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{y}, a) \cdot \mathbf{V}^{n,*}) - \mathcal{Q}(\mathbf{z}^{n,*}, \boldsymbol{\gamma}^{n,*}(\mathbf{z}, b) \cdot \mathbf{V}^{n,*}) \right| \Big\} \\
\leq\ & (1 + C\,k)\,|\sigma^{n-1}| + k\,|A^n - B^n| + C\,k\, \left\{ \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 \right\} \\
& + C\,k\, \left\{ \left\|\boldsymbol{\Delta}^{n,*}\right\|_\infty + \left\|\mathbf{E}^{n,*}\right\|_1 + |\sigma^{n,*}| \right\} \\
\leq\ & (1 + C\,k)\,|\sigma^{n-1}| + k\,|A^n - B^n| + C\,k\, \left\{ \left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 \right\},
\end{aligned} \tag{6.17}$$

$1 \leq n \leq N$. Thus,

$$|\sigma^n| \leq (1 + C\,k)^n\,|\sigma^0| + \sum_{l=0}^{n-1} k\,(1 + C\,k)^l\,|A^{n-l} - B^{n-l}| + \sum_{l=0}^{n-1} C\,k\,(1 + C\,k)^l\, \left\{ \left\|\boldsymbol{\Delta}^{n-l-1}\right\|_\infty + \left\|\mathbf{E}^{n-l-1}\right\|_1 \right\}, \tag{6.18}$$

$1 \leq n \leq N$. Therefore,

$$|\sigma^n| \leq C \left\{ |\sigma^0| + \sum_{m=1}^{n} k\,|A^m - B^m| + \sum_{m=0}^{n-1} k\, \left\|\boldsymbol{\Delta}^m\right\|_\infty + \sum_{m=0}^{n-1} k\, \left\|\mathbf{E}^m\right\|_1 \right\}, \tag{6.19}$$

$1 \leq n \leq N$.

On the other hand, from (4.32) we arrive at

$$\begin{aligned}
|E_j^n| \leq\ & |E_{j-1}^{n-1}| \exp\left\{ -\frac{k}{2} \left(\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right) + \mu^*\left(y_j^{n,*}, a^{n,*}, t^n\right) \right) \right\} \\
& + |W_{j-1}^{n-1}| \left| \exp\left\{ -\frac{k}{2} \left(\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right) + \mu^*\left(y_j^{n,*}, a^{n,*}, t^n\right) \right) \right\} \right. \\
& \left. - \exp\left\{ -\frac{k}{2} \left(\mu^*\left(z_{j-1}^{n-1}, b^{n-1}, t^{n-1}\right) + \mu^*\left(z_j^{n,*}, b^{n,*}, t^n\right) \right) \right\} \right) + k\,|P_j^n - R_j^n|,
\end{aligned} \tag{6.20}$$

$1 \le j \le J+n$, $1 \le n \le N$. Now, by means of hypotheses (H4) and (H7), we have

$$\exp\left\{-\frac{k}{2}\left(\mu^*\left(y_{j-1}^{n-1}, a^{n-1}, t^{n-1}\right) + \mu^*\left(y_j^{n,*}, a^{n,*}, t^n\right)\right)\right\} \le 1 + C\,k, \tag{6.21}$$

$1 \le j \le J+n$, $1 \le n \le N$. Thus, (6.1), (6.4)–(6.5), (6.20)–(6.21), hypotheses (H4) and (H6)–(H7), and that $\|\mathbf{W}^{n-1}\|_\infty \le C$, enable us to write

$$\begin{aligned}
|E_j^n| \le\ & k\,|P_j^n - R_j^n| + (1 + C\,k)\,|E_{j-1}^{n-1}| + C\,k\,\Big\{\left|\mu^*\left(y_j^{n-1}, a^{n-1}, t^{n-1}\right) - \mu^*\left(z_j^{n-1}, a^{n-1}, t^{n-1}\right)\right| \\
& + \left|\mu^*\left(y_j^{n,*}, a^{n,*}, t^n\right) - \mu^*\left(z_j^{n,*}, a^{n,*}, t^n\right)\right| \\
& + \left|\mu^*\left(z_j^{n-1}, a^{n-1}, t^{n-1}\right) - \mu^*\left(z_j^{n-1}, b^{n-1}, t^{n-1}\right)\right| \\
& + \left|\mu^*\left(z_j^{n,*}, a^{n,*}, t^n\right) - \mu^*\left(z_j^{n,*}, b^{n,*}, t^n\right)\right|\Big\} \\
\le\ & k\,|P_j^n - R_j^n| + (1 + C\,k)\,|E_{j-1}^{n-1}| + C\,k\,\left\{\left|\Delta_{j-1}^{n-1}\right| + \left|\sigma^{n-1}\right| + \left|\Delta_j^{n,*}\right| + \left|\sigma^{n,*}\right|\right\} \\
\le\ & k\,|P_j^n - R_j^n| + (1 + C\,k)\,|E_{j-1}^{n-1}| + C\,k\,\left\{\left\|\boldsymbol{\Delta}^{n-1}\right\|_\infty + \left\|\mathbf{E}^{n-1}\right\|_1 + \left|\sigma^{n-1}\right|\right\},
\end{aligned} \tag{6.22}$$

$1 \le j \le J+n$, $1 \le n \le N$. Now, from (4.30) and hypotheses (H7) it follows that

$$\begin{aligned}
|E_0^n| \le\ & |P_0^n - R_0^n| + \left|\frac{\mathcal{Q}\left(\mathbf{y}^n, \boldsymbol{\alpha}^n(\mathbf{y}, a)\,\mathbf{V}^n\right)}{g\,(0, a^n, t^n)} - \frac{\mathcal{Q}\left(\mathbf{z}^n, \boldsymbol{\alpha}^n(\mathbf{z}, b)\,\mathbf{W}^n\right)}{g\,(0, b^n, t^n)}\right| \\
\le\ & |P_0^n - R_0^n| + C\,\Big\{\left|g\left(0, b^n, t^n\right) - g\left(0, a^n, t^n\right)\right|\,\left|\mathcal{Q}\left(\mathbf{z}^n, \boldsymbol{\alpha}^n(\mathbf{z}, b)\,\mathbf{W}^n\right)\right| \\
& + \left|g\left(0, t^n, b^n\right)\right|\,\left|\mathcal{Q}\left(\mathbf{y}^n, \boldsymbol{\alpha}^n(\mathbf{y}, a)\,\mathbf{V}^n\right) - \mathcal{Q}\left(\mathbf{z}^n, \boldsymbol{\alpha}^n(\mathbf{z}, b)\,\mathbf{W}^n\right)\right|\Big\},
\end{aligned} \tag{6.23}$$

$1 \le n \le N$. Next, (H5), property (P3), and that $\|\mathbf{W}^n\|_\infty \le C$, enable us to obtain

$$\left|\mathcal{Q}\left(\mathbf{z}^n, \boldsymbol{\alpha}^n(\mathbf{z}, b)\,\mathbf{W}^n\right)\right| \le C, \tag{6.24}$$

$1 \le n \le N$. Furthermore, the definition of α_i and hypotheses (H5) yield

$$\left|\alpha_i\left(y_i^n, a^n, t^n\right) - \alpha_i\left(z_i^n, b^n, t^n\right)\right| \le C\,\left\{|\Delta_i^n| + |\sigma^n|\right\}, \tag{6.25}$$

$1 \le n \le N$. Next, by means of (6.25), hypotheses (H5), properties (P3) and (P6), and that $\|\mathbf{W}^n\|_\infty \le C$, we arrive at

$$\begin{aligned}
\left|\mathcal{Q}\left(\mathbf{y}^n, \boldsymbol{\alpha}^n(\mathbf{y}, a)\,\mathbf{V}^n\right) - \mathcal{Q}\left(\mathbf{z}^n, \boldsymbol{\alpha}^n(\mathbf{z}, b)\,\mathbf{W}^n\right)\right| \le\ & \left|\sum_{i=0}^{J+n} q_i(\mathbf{y}^n)\,\alpha(y_i^n, t^n, a^n)\,(V_i^n - W_i^n)\right| \\
& + \left|\sum_{i=0}^{J+n}\left(q_i(\mathbf{y}^n)\,\alpha(y_i^n, a^n, t^n) - q_i(\mathbf{z}^n)\,\alpha(z_i^n, b^n, t^n)\right)\,W_i^n\right| \\
\le\ & C\,\|\mathbf{E}^n\|_1 + \left|\sum_{i=0}^{J+n}\left(q_i(\mathbf{y}^n) - q_i(\mathbf{z}^n)\right)\,\alpha(y_i^n, t^n, a^n)\,W_i^n\right| \\
& + \left|\sum_{i=0}^{J+n} q_i(\mathbf{z}^n)\,\left(\alpha(y_i^n, t^n, a^n) - \alpha(z_i^n, t^n, b^n)\right)\,W_i^n\right| \\
\le\ & C\,\left\{\|\mathbf{E}^n\|_1 + \|\boldsymbol{\Delta}^n\|_\infty + \left|\sigma^n\right|\right\},
\end{aligned} \tag{6.26}$$

$1 \le n \le N$. Therefore, we complete the derivation of the stability estimate for the boundary node taking into account (6.23)–(6.24) and (6.26), and hypotheses (H7),

$$|E_0^n| \le |P_0^n - R_0^n| + C\,\left\{\left|\sigma^n\right| + \left\|\boldsymbol{\Delta}^n\right\|_\infty + \left\|\mathbf{E}^n\right\|_1\right\}, \tag{6.27}$$

$1 \le n \le N$. Thus, when $N \ge n > j \ge 1$, from (6.22), we obtain

$$\begin{aligned}
|E_j^n| \le\ & (1 + C\,k)^j\,|E_0^{n-j}| + k\,\sum_{l=0}^{j-1}(1 + C\,k)^l\,|P_{j-l}^{n-l} - R_{j-l}^{n-l}| \\
& + C\,k\,\sum_{l=0}^{j-1}(1 + C\,k)^l\,\left\{\|\mathbf{E}^{n-1-l}\|_1 + \|\boldsymbol{\Delta}^{n-1-l}\|_\infty + |\sigma^{n-1-l}|\right\}.
\end{aligned} \tag{6.28}$$

Therefore, we establish

$$|E_j^n| \le C\left\{|E_0^{n-j}| + \sum_{m=n-j}^{n-1} k\,\|\mathbf{E}^m\|_1 + \sum_{m=n-j}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=n-j+1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + \sum_{m=n-j}^{n-1} k\,|\sigma^m|\right\}. \tag{6.29}$$

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On the other hand, when $J+n \geq j \geq n \geq 1$, due to (6.22) it follows that

$$|E_j^n| \leq (1+C\,k)^n\,|E_{j-n}^0| + k\sum_{l=0}^{n-1}(1+C\,k)^l\,|P_{j-l}^{n-l} - R_{j-l}^{n-l}|$$

$$+ C\,k\sum_{l=0}^{n-1}(1+C\,k)^l\left\{\|\mathbf{E}^{n-1-l}\|_1 + \|\boldsymbol{\Delta}^{n-1-l}\|_\infty + |\sigma^{n-1-l}|\right\}. \qquad (6.30)$$

Thus, we can conclude that

$$|E_j^n| \leq C\left\{\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n-1} k\,\|\mathbf{E}^m\|_1 + \sum_{m=0}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n-1} k\,|\sigma^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty\right\}. \qquad (6.31)$$

Now, multiplying $|E_j^n|$ by h and summing in j, $0 \leq j \leq J+n$, $1 \leq n \leq N$, from (6.27), (6.29) and (6.31) and that $k = r\,h$, we have

$$\|\mathbf{E}^n\|_1 = h\,|E_0^n| + \sum_{j=1}^{n-1} h\,|E_j^n| + \sum_{j=n}^{J+n} h\,|E_j^n|$$

$$\leq h\,|P_0^n - R_0^n| + C\,h\left(|\sigma^n| + \|\mathbf{E}^n\|_1 + \|\boldsymbol{\Delta}^n\|_\infty\right)$$

$$+ C\sum_{j=1}^{n-1} h\left\{|E_0^{n-j}| + \sum_{m=n-j}^{n-1} k\,\|\mathbf{E}^m\|_1 + \sum_{m=n-j}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=n-j}^{n-1} k\,|\sigma^m| + \sum_{m=n-j+1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty\right\}$$

$$+ C\sum_{j=n}^{J+n-1} h\left\{\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n-1} k\,\|\mathbf{E}^m\|_1 + \sum_{m=0}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n-1} k\,|\sigma^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty\right\}$$

$$\leq C\left\{\|\mathbf{E}^0\|_1 + h\,\|\mathbf{E}^n\|_1 + \sum_{m=0}^{n-1} k\,\|\mathbf{E}^m\|_1 + \sum_{j=1}^{n-1} h\,|E_0^{n-j}| + h\,|\sigma^n| + \sum_{m=0}^{n-1} k\,|\sigma^m|\right.$$

$$\left. + h\,\|\boldsymbol{\Delta}^n\|_\infty + \sum_{m=0}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + h\,|P_0^n - R_0^n|\right\}$$

$$\leq C\left\{\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n} k\,\|\mathbf{E}^m\|_1 + \sum_{m=0}^{n} k\,|\sigma^m| + \sum_{j=1}^{n-1} h\left(|P_0^{n-j} - R_0^{n-j}| + \|\mathbf{E}^{n-j}\|_1 + \|\boldsymbol{\Delta}^{n-j}\|_\infty + |\sigma^{n-j}|\right)\right.$$

$$\left. + \sum_{m=0}^{n} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + h\,|P_0^n - R_0^n|\right\}$$

$$\leq C\left\{\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n} k\,\|\mathbf{E}^m\|_1 + \sum_{m=0}^{n} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n} k\,|\sigma^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + \sum_{m=1}^{n} h\,|P_0^m - R_0^m|\right\},$$

$1 \leq n \leq N$. Then,

$$\|\mathbf{E}^n\|_1 \leq C\left\{\|\mathbf{E}^0\|_1 + \sum_{m=1}^{n} k\,\|\mathbf{E}^m\|_1 + \sum_{m=0}^{n} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n} k\,|\sigma^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + \|\mathbf{P}_0 - \mathbf{R}_0\|_\infty\right\}, \qquad (6.32)$$

$1 \leq n \leq N$. Thus, by means of the discrete Gronwall lemma,

$$\|\mathbf{E}^n\|_1 \leq C\left\{\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n} k\,\|\boldsymbol{\Delta}^m\|_\infty + \sum_{m=0}^{n} k\,|\sigma^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + \|\mathbf{P}_0 - \mathbf{R}_0\|_\infty\right\}, \qquad (6.33)$$

$1 \leq n \leq N$. Next, we substitute (6.33) into (6.16) to have

$$\|\boldsymbol{\Delta}^n\|_\infty \leq \|\boldsymbol{\Delta}^0\|_\infty + C\left\{k\,\|\mathbf{E}^0\|_1 + \sum_{m=0}^{n-1} k\,\|\boldsymbol{\Delta}^m\|_\infty\right.$$

$$+ \sum_{m=1}^{n-1} k\left(\|\mathbf{E}^0\|_1 + \sum_{l=0}^{m} k\,\|\boldsymbol{\Delta}^l\|_\infty + \sum_{l=0}^{m} k\,|\sigma^l| + \sum_{l=1}^{m} k\,\|\mathbf{P}^l - \mathbf{R}^l\|_\infty\right.$$

$$+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty\Bigg)+\sum_{m=0}^{n-1}k\,|\sigma^m|+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\Bigg\}$$

$$\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\sum_{m=1}^{n-1}k\,\|\boldsymbol{\Delta}^m\|_\infty+\sum_{m=0}^{n-1}k\,|\sigma^m|\right.$$

$$\left.+\sum_{m=1}^{n-1}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}, \tag{6.34}$$

$1\le n\le N$. Again, by means of the discrete Gronwall lemma, it follows that

$$\|\boldsymbol{\Delta}^n\|_\infty\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\sum_{m=0}^{n-1}k\,|\sigma^m|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n-1}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}, \tag{6.35}$$

$1\le n\le N$. Next, we substitute (6.35) in (6.33) to obtain

$$\|\mathbf{E}^n\|_1\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\sum_{m=0}^{n}k\,|\sigma^m|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}, \tag{6.36}$$

$1\le n\le N$. Next, we substitute (6.35) and (6.36) in (6.19)

$$|\sigma^n|\le|\sigma^0|+\sum_{m=1}^{n}k\,|A^m-B^m|+C\sum_{m=0}^{n-1}k\,\left|\sigma^m\right|$$

$$+C\sum_{m=1}^{n-1}k\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\sum_{l=0}^{m-1}k\,|\sigma^l|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{l=1}^{m}k\,\|\mathbf{P}^l-\mathbf{R}^l\|_\infty+\sum_{l=1}^{m}k\,\|\mathbf{Y}^l-\mathbf{Z}^l\|_\infty\right\}$$

$$+C\sum_{m=1}^{n-1}k\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\sum_{l=0}^{m}k\,|\sigma^l|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{l=1}^{m}k\,\|\mathbf{P}^l-\mathbf{R}^l\|_\infty+\sum_{l=1}^{m}k\,\|\mathbf{Y}^l-\mathbf{Z}^l\|_\infty\right\},$$

$$\le C\left\{|\sigma^0|+\sum_{m=1}^{n}k\,|A^m-B^m|+C\sum_{m=1}^{n-1}k\,\left|\sigma^m\right|\right.$$

$$\left.+\ \|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n-1}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n-1}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\} \tag{6.37}$$

$1\le n\le N$. Again by means of the discrete Gronwall lemma, it follows that

$$|\sigma^n|\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+|\sigma^0|+\sum_{m=1}^{n}k\,|A^m-B^m|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty\right.$$

$$\left.+\sum_{m=1}^{n-1}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n-1}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}. \tag{6.38}$$

$1\le n\le N$. Then, we substitute (6.38) in (6.35) and (6.36) to obtain

$$\|\boldsymbol{\Delta}^n\|_\infty\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+|\sigma^0|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n-1}k\,|A^m-B^m|\right.$$

$$\left.+\sum_{m=1}^{n-1}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}, \tag{6.39}$$

and

$$\|\mathbf{E}^n\|_1\le C\left\{\|\boldsymbol{\Delta}^0\|_\infty+\|\mathbf{E}^0\|_1+|\sigma^0|+\|\mathbf{P}_0-\mathbf{R}_0\|_\infty+\sum_{m=1}^{n-1}k\,|A^m-B^m|\right.$$

$$\left.+\sum_{m=1}^{n}k\,\|\mathbf{P}^m-\mathbf{R}^m\|_\infty+\sum_{m=1}^{n}k\,\|\mathbf{Y}^m-\mathbf{Z}^m\|_\infty\right\}, \tag{6.40}$$

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$1 \le n \le N$. And, finally, we substitute (6.38)–(6.40) in (6.27), (6.29) and (6.31) to arrive at

$$\|\mathbf{E}^n\|_\infty \le C \left\{ \|\boldsymbol{\Delta}^0\|_\infty + \|\mathbf{E}^0\|_1 + |\sigma^0| + \|\mathbf{P}_0 - \mathbf{R}_0\|_\infty + \sum_{m=1}^{n-1} k\,|A^m - B^m| + \sum_{m=1}^{n} k\,\|\mathbf{P}^m - \mathbf{R}^m\|_\infty + \sum_{m=1}^{n} k\,\|\mathbf{Y}^m - \mathbf{Z}^m\|_\infty \right\}, \tag{6.41}$$

$1 \le n \le N$. So, due to (6.38), (6.39) and (6.41) we have

$$\|\left(\boldsymbol{\Delta}^0, \mathbf{E}^0, \dots, \boldsymbol{\Delta}^N, \mathbf{E}^N, \boldsymbol{\sigma}\right)\|_{\mathcal{A}_h} \le C\,\|(\boldsymbol{\Delta}^0, \mathbf{E}^0, \sigma^0, \mathbf{P}_0 - \mathbf{R}_0, \mathbf{Y}^1 - \mathbf{Z}^1, \mathbf{P}^1 - \mathbf{R}^1, \dots, \mathbf{Y}^N - \mathbf{Z}^N, \mathbf{P}^N - \mathbf{R}^N, \mathbf{A} - \mathbf{B})\|_{\mathcal{B}_h}. \quad \square$$

7. Convergence

The *global discretization error* is defined as

$$\tilde{\mathbf{e}}_h = \tilde{\mathbf{u}}_h - \tilde{\mathbf{U}}_h \in \mathcal{A}_h.$$

We say that the discretization (4.26) is *convergent* if there exists $h_0 > 0$ such that, for each $h \in H$ with $h \le h_0$, (4.38) has a solution $\tilde{\mathbf{U}}_h$ for which, as $h \to 0$,

$$\lim \|\tilde{\mathbf{u}}_h - \tilde{\mathbf{U}}_h\|_{\mathcal{A}_h} = \lim \|\tilde{\mathbf{e}}_h\|_{\mathcal{A}_h} = 0.$$

In our analysis, we shall use the following result of the general discretization framework introduced by López-Marcos et al. [25].

Theorem 4. *Assume that (4.26) is consistent and stable with thresholds R_h. If Φ_h is continuous in $B(\tilde{\mathbf{u}}_h, R_h)$ and $\|\mathbf{l}_h\|_{\mathcal{B}_h} = o(R_h)$ as $h \to 0$, then:*

(i) *For h sufficiently small, the discrete equations (4.38) possess a unique solution in $B(\tilde{\mathbf{u}}_h, R_h)$.*
(ii) *As $h \to 0$, the solutions converge and $\|\tilde{\mathbf{e}}_h\|_{\mathcal{A}_h} = O(\|\mathbf{l}_h\|_{\mathcal{B}_h})$.*

Finally, we propose the next theorem which establishes the *convergence* of the numerical method defined by Eqs. (4.27)–(4.32).

Theorem 5. *Assume that hypotheses (H1)–(H7) hold and that the considered quadrature rules satisfy properties (P1)–(P6). Then, for h sufficiently small, the numerical method defined by Eqs. (4.27)–(4.32) has a unique solution $\tilde{\mathbf{U}}_h \in B(\tilde{\mathbf{u}}_h, R_h)$ and*

$$\|\tilde{\mathbf{U}}_h - \tilde{\mathbf{u}}_h\|_{\mathcal{A}_h} \le C\left(\|\mathbf{x}^0 - \mathbf{X}^0\|_\infty + \|\mathbf{u}^0 - \mathbf{U}^0\|_\infty + |s^0 - S^0| + O(h^2 + k^2)\right). \tag{7.1}$$

The proof of Theorem 5 is immediately derived by means of consistency (Theorem 2), stability (Theorem 3) and Theorem 4. Specifically, if $\mathbf{X}^0 = \mathbf{x}^0$, $\mathbf{U}^0 = \mathbf{u}^0$ and $S^0 = s^0$, the proposed numerical scheme is second-order accurate.

At this moment, we have obtained convergence of the numerical method (4.27)–(4.33) which does not employ selection at each time level. Also, we have proven the convergence of numerical methods which employ a selection criterion, whenever the positions, which are determined by the criterion we have chosen, lead us to subgrids which satisfy property (SR). For the criterion presented in this paper, this property may be shown in two stages. First, as proven in [15], it leads us to subgrids with such a property when we applied it over nodes which are in a neighbourhood of the theoretical ones with radius $R\,h^p$. In a second stage, it is proven that the nodes, which in fact the numerical method computes, are in such neighbourhoods. In order to do this, it is enough to realize that such nodes could be seen, up to each time level, as the solutions obtained by a discrete operator which has the form of that defined in (4.26).

Remark. It is well known that regularity hypotheses are necessary, in numerical analysis, to derive optimal rates of convergence for numerical quadrature rules and numerical methods for differential equations. This is the meaning of our smoothness assumptions. However, an interesting question is to consider how the numerical scheme analysed in this paper should be used to carry out the numerical integration of problems with non-smooth biological data functions. In such case, sometimes it is possible to locate these singular points, thus we would obtain the convergence result taking into account that the method is based on the approximation along the characteristics curves by means of quadrature rules. For example, with respect to the discontinuities caused by the lack of compatibility among the initial and boundary conditions, the discussion introduced in [15], for the model without dynamical resource, is also valid in this case. In other physiologically structured models, we can observe finite jump discontinuities caused by the problem itself (as in [14]), or by non-smooth coefficients (as in [26,27]) which propagate along the characteristic curves. While these points are located, a proper choice of the mesh grid and a suitable simple adaptation of the quadrature rules, keep the order of convergence of the method. The same studies could be made for our model if they would be necessary. Also we have found situations, as in [13], providing unbounded solutions close to the maximum size. Again, we can perform a different selection procedure and a modification of the quadrature rule to describe the approximation to the solution. Finally, we refer to [28,29], and the references therein, for numerical approximation to weak solutions for a similar kind of models.

Table 1
Error and experimental order of convergence.

$k \setminus h$	2.5e−2	1.25e−2	6.25e−2	3.13e−2	1.56e−2
2.5e−2	1.565536e−4	2.674563e−4	2.852092e−4	2.888894e−4	2.897345e−4
1.25e−2	6.434510e−4	3.741278e−5 2.07	6.469671e−5	6.920614e−5 2.04	7.013337e−5 2.04
6.25e−2	7.647515e−4	1.656411e−4 1.96	2.05 9.157412e−6	1.590974e−5 2.02	1.704878e−5 2.02
3.13e−2	8.238862e−4	1.916386e−4 2.00	2.03 4.209129e−5	2.265792e−6 2.01	3.944845e−6 2.01
1.56e−2	8.438404e−4	2.051189e−4 2.01	1.98 4.790993e−5 2.00	1.060032e−5 1.99	5.635547e−7 2.01

8. Numerical results

We have carried out different numerical experiments with the scheme defined in Section 3.

First, we have considered an academical problem. It consists of a theoretical test problem with meaningful nonlinearities (both from a mathematical and a biological ~~point~~ of view). In this case, the numerical integration for the experiment was carried out on the time interval [0, 10]. The size interval was taken as [0, 1]. Below, we describe the functions involved in the experiment. The size-specific growth, fertility and mortality moduli are chosen as

$$g(x,z,t) = \frac{\lambda}{2}\,\frac{1+z}{z}\left(\left(\frac{z}{1+z}\right)^2 - x^2\right) + \frac{xr}{1+z}\left(\frac{29}{30} - \frac{z}{k}\right),$$

$$\alpha(x,z,t) = \frac{3}{2}\lambda\,\frac{1+\left(\frac{z}{C\left(\frac{29}{30}-\frac{z}{k}\right)}\right)^{\frac{-29\lambda}{30r}}}{1+2\left(\frac{z}{C\left(\frac{29}{30}-\frac{z}{k}\right)}\right)^{\frac{-29\lambda}{30r}}},$$

$$\mu(x,z,t) = \lambda\,\frac{1+z}{z}\left(\frac{z}{1+z} + 2x\right) - \frac{3r}{1+z}\left(\frac{29}{30} - \frac{z}{k}\right).$$

The weight function is taken as $\gamma(x,z,t) = x^2$ and, finally,

$$f(z,i,t) = rz\left(1 - \frac{z}{k}\right) - rzi\frac{(1+z)^5}{z^5(1+4e^{-\lambda t})}.$$

With this choice of data functions, the problem (2.1)–(2.6) has the following solution

$$u(x,t) = \left(\frac{S(t)}{1+S(t)} - x\right)^2 - e^{-\lambda t}\left(\left(\frac{S(t)}{1+S(t)}\right)^2 - x^2\right),$$

$$S(t) = \frac{29}{30}\,\frac{Ce^{29rt/30}}{1+Ce^{29rt/30}/k},$$

$r = 0.1$, $C = 24$, $k = 5$, $\lambda = 0.3$. Since we know the exact solution to the problem, we can show numerically that our method is second-order accurate by means of an error table. In Table 1, each entry in columns two to seven of the upper value represents the global error

$$e_{h,k} = \max\left\{\max_{0\le j\le J}|u(X_j^0,t^0) - U_j^0|,\ |S(t^0) - S^0|,\ \max_{1\le n\le N}\left\{\max_{0\le j\le J}|u(X_j^n,t^n) - U_j^n|\right\},\ |S(t^n) - S^n|\right\}$$

and the lower number is the experimental order s of the method as computed from

$$s = \frac{\log(e_{2h,2k}/e_{h,k})}{\log 2}.$$

Each column and each row of the table correspond to different values of the spatial and time discretization parameter, respectively. The results in the table clearly confirm the expected second-order convergence. The property of convergence

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in finite time interval is important to carry out experiments in which the long-time behaviour of the population is investigated. Therefore, the numerical integration of the model with an efficient method allows us to consider a more realistic test problem.

Second, the numerical method has been employed to describe the dynamics of a population of ectothermic invertebrates: the water flea, *Daphnia magna*. In this particular case, the data functions employed in [3] are given by

$$g(x,z,t) = g\left(\frac{z}{1+z} - x\right),$$

$$\mu(x,z,t) = \mu,$$

$$\alpha(x,z,t) = \alpha\frac{z}{1+z}x^2,$$

$$f(z,i,t) = r\,z\left(1-\frac{z}{K}\right) - i\frac{z}{1+z},$$

$$\gamma(x,z,t) = x^2.$$

Q2 Simple calculations provide the following steady state of the model,

$$u^*(x) = \frac{\alpha r}{g}(1+S^*)\left(1-\frac{S^*}{K}\right)\left(1-\frac{x}{x_M^*}\right)^{\frac{\mu}{g}-1}, \quad 0 \le x \le x_M^*, \tag{8.1}$$

with the following maximum size and resource

$$x_M^* = \sqrt[3]{\frac{\mu(\mu+g)(\mu+2g)}{2\alpha g^2}}, \qquad S^* = \frac{x_M^*}{1-x_M^*}. \tag{8.2}$$

This model, with the following set of parameter values $\mu = 0.1$, $\alpha = 0.75$, $r = 3$, $K = 8.3$ and $x_M = 1$, was analysed with different numerical techniques in [3,13]. This set is beyond the theoretical analysis settings because the steady state is unbounded (although integrable in whatever case $g > \mu$).

We have performed simulations with $g = 0.0075$, $\mu = 0.1$, $\alpha = 0.75$, $r = 3$ and $x_M = 1$. This set of parameters makes the solution be bounded and therefore, within the theoretical analysis settings. In a first experiment, we take the value $K = 8.3$. We employ initial conditions (2.3) and (2.5) which ensure compatibility between u_0, S^0 and the problem,

$$u_0(0) = \frac{\alpha}{g}\int_0^{x_M} x^2\, u_0(x)\, dx, \qquad \frac{S^0}{1+S^0} = \frac{3\,g\,u_0(0)}{2\,\alpha\,\int_0^{x_M} x\,u_0(x)\,dx + g\,u_0'(0)},$$

and integrate numerically until $T = 1000$. We employ small perturbations of (8.1) and (8.2) as initial conditions: $\widetilde{x}_M = 0.875$, $S^0 = 7$, and

$$u_0(x) = \begin{cases} \frac{\alpha r}{g}(1+S^0)\left(1-\frac{S^0}{K}\right)\left(1-\frac{x}{\widetilde{x}_M}\right)^{\beta} & \text{if } 0 \le x \le \widetilde{x}_M, \\ 0 & \text{if } \widetilde{x}_M < x \le x_M. \end{cases} \tag{8.3}$$

The value of β is taken in order to ensure the compatibility conditions

$$(\beta+3)(\beta+2)(\beta+1) = \frac{2\,\alpha\,\widetilde{x}_M^3}{g},$$

which results in $\beta \approx 0.5153057420$.

The computations performed with different values of the parameters of the discretization, k and J, show that the numerical solution is attracted to a stationary state. Therefore, we can state that equilibrium is asymptotically stable. For a fixed number of grid points on the size interval, J, the numerical steady state converges to the theoretical one as the time step k decreases. However, we observe that a critical value of the time step k, depending on J, appears. Below this critical value, the numerical method does not produce better approximations to the theoretical values of the equilibrium state. Our experience allows us to estimate that an optimal ratio of the discretization parameters is about $kJ \approx 250$. In Table 2, we present the errors produced by the numerical approximations to the theoretical equilibrium state with different values of the discretization parameters (J and k). Note that the convergence is second-order. For this choice of the final time integration, the approach to the theoretical steady state is good enough as shown in Table 2.

In Fig. 1, we show the long time evolution of the numerical solution obtained with the values of the discretization parameters $k = 0.0625$, $J = 4000$. In the figure, we draw the evolution of the density function, the total population computed from the population density by the composite trapezoidal quadrature rule, the maximum individual size and the dynamical resource. As we can observe, all of them evolve towards the equilibrium state.

Taking into account the good behaviour of the numerical solution in a predictable situation, we now consider a more complicated situation. As the value of the parameter K increases, the equilibrium state becomes unstable. In a second

Table 2
Errors in the approximations with respect to the theoretical values: $N_0^* = \int_0^{x_M^*} u^*(x)\ dx \approx 46.6763867098$ (total population), $S^* \approx 4.0859721214$ (resource).

k	J	Total population	Resource
1	250	4.946335e−3	4.112717e−2
5e−1	500	1.004377e−3	9.992129e−3
2.5e−1	1 000	2.270716e−4	2.461613e−4
1.25e−1	2 000	5.403338e−5	6.108879e−4
6.25e−2	4 000	1.318184e−5	1.521058e−4
3.125e−2	8 000	3.255560e−6	3.788916e−5
1.5625e−2	16 000	8.089576e−7	9.394259e−6
7.8125e−3	32 000	2.016265e−7	2.277898e−6

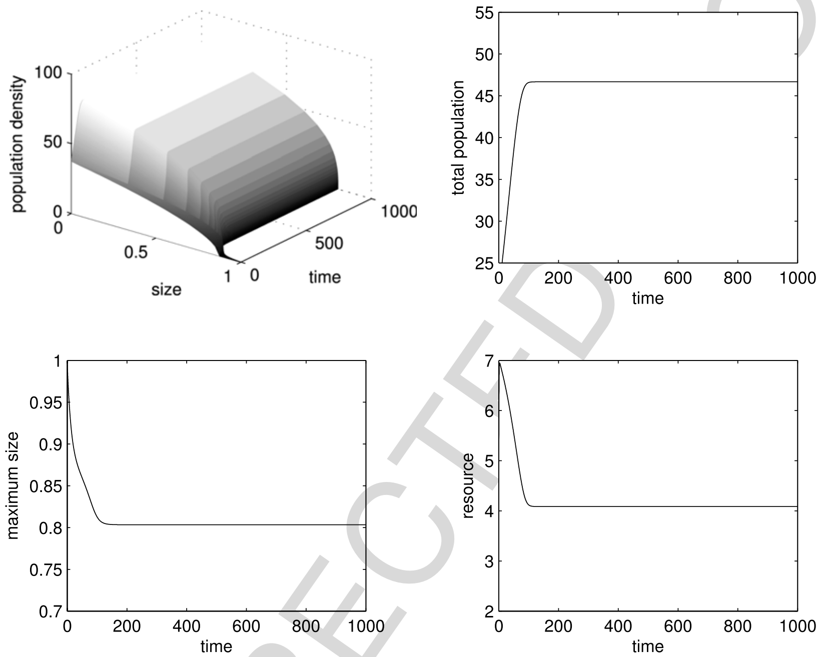

Fig. 1. Evolution of the numerical solution. Case of a stable steady state.

experiment, we present an example of this situation by taking $K = 9.64$. As an initial condition, we take the unstable equilibrium state (8.1)–(8.2) (we extend this function to the whole interval $[0, x_M]$ by taking $u_0(x) = 0$ if $x_M^* < x \leq x_M$). As in the previous experiment, the values of the discretization parameters are $k = 0.0625, J = 4000$. We can observe the instability of the equilibrium and the solution evolving towards a cycled situation (Fig. 2). Taking into account that the numerical solution is attracted to a limit cycle, considering a sufficiently large time, the numerical solution obtained after this long time integration lies practically on such a cycle. In this way, the numerical method provides an approximation to the limit cycle. In Fig. 3, the representation of such a cycle in the tridimensional space defined by the total population, the maximum individual size and the dynamical resource, is drawn. From the numerical results obtained for the total population, the maximum size and the dynamical resource, we can obtain an in depth analysis of these quantities throughout a period of the limit cycle. For example, in Table 3, we present the behaviour of some numerical quantities of the solution throughout a period of the limit cycle. Also, we can estimate such a period by interpolation: it is about 64.6824.

9. Conclusions

We have analysed a second-order numerical method for a problem that describes a population with a possible shrinking size and with a dependency on the environment managed by the evolution of a vital resource. The method involves only one level of time each step. The second-order convergence has been theoretically proven by means of an argument of consistency and stability of the scheme. The academical test problem allows us to report numerical experiments which demonstrate the predicted accuracy of the scheme.

This knowledge leads us to make a long time integration of the model. The biological problem we consider is well known. It describes the dynamics of a *Daphnia magna* population. With its long time integration, we observe the good stability

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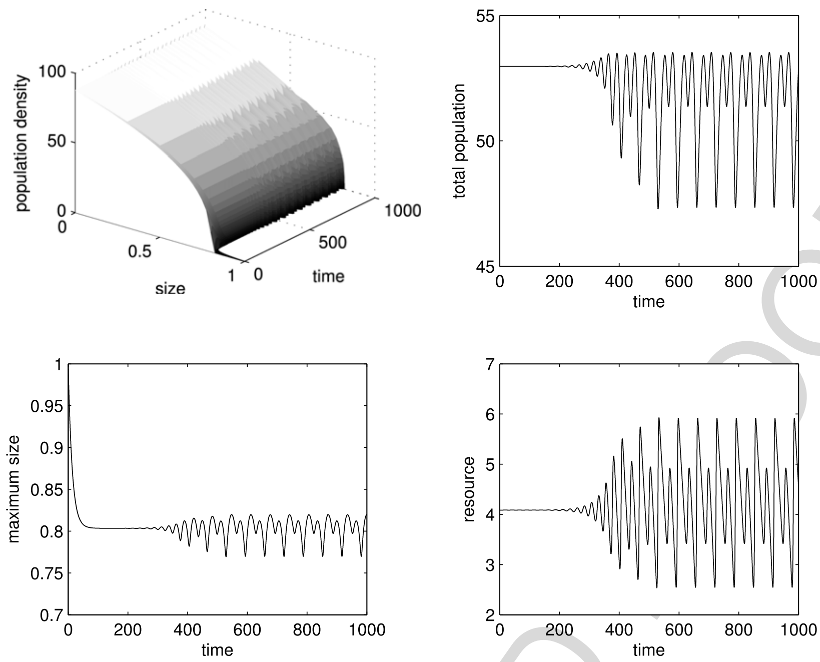

Fig. 2. Evolution of the numerical solution. Case of an unstable steady state.

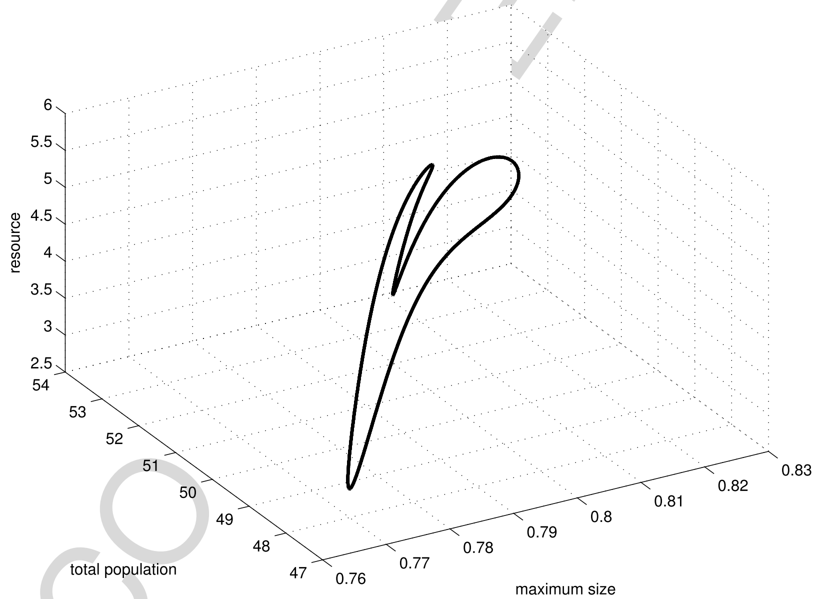

Fig. 3. Limit cycle appearing in the case of an unstable steady state.

properties that our numerical scheme possesses. In the settings of the theoretical analysis, our numerical scheme makes it possible to determine the rich dynamics of the model. It presents an equilibrium that is stable until the parameter K reaches the bifurcation value. After that, it becomes unstable. In the stable case, we show how the numerical steady state approaches the theoretical steady state. When the equilibrium is unstable, a stable limit cycle appears, the characteristics of which are also described.

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Table 3
Behaviour of some numerical quantities of the solution throughout a period of the limit cycle.

Time	Resource	Total population	Maximum size
0	Abs. maximum (5.91)	↗	↗
14.22	↘	↗	Abs. maximum(0.82)
19.57	↘	Rel. maximum(53.41)	↘
27.01	Rel. minimum(3.42)	↘	↘
30.90	↗	↘	Rel. minimum(0.80)
32.15	↗	Rel. minimum(51.37)	↗
36.21	Rel. maximum(4.92)	↗	↗
43.09	↘	↗	Rel. maximum(0.81)
45.92	↘	Abs. maximum(53.52)	↘
57.50	Abs. minimum(2.55)	↘	↘
60.20	↗	↘	Abs. minimum(0.77)
61.79	↗	Abs. minimum(47.35)	↗
64.68	Abs. maximum(5.92)	↗	↗

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