# Optimal Pollution Standards and Non-Compliance in a Dynamic Framework *

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**Abstract**

In this paper we present a Stackelberg differential game to study the dynamic interaction between a polluting firm and a regulator who sets pollution limits overtime. At each time, the firm settles emissions taking into account the fine for non-compliance with the pollution limit, and balances current costs of investments in a capital stock which allows for future emission reductions. We derive two main results. First, we show that the optimal pollution limit decreases as the capital stock increases, while both emissions and the level of non-compliance decrease. Second, we find that offering fine discounts in exchange for firm's capital investment is socially desirable. We numerically obtain the optimal value of such discount, which crucially depends on the severity of the fine. In the limiting scenario with a very large severity of the fine, the optimal discount implies that no penalties are levied, since the firm shows adequate adaptation progress through capital investment.

**Key words:** dynamic regulation; Stackelberg differential games; non-compliance; fines; pollution standards.

**JEL Codes:** C61, C73, K32, K42, L51, Q28.

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# 1 Introduction

Pollution limits or standards are frequently used by regulators to deal with pollution problems. A pollution standard defines the maximum allowed amount of a certain pollutant that polluting sources, typically firms, can release. When deciding on the level of the standard, regulators must take into consideration that firms may decide to exceed the limit. This may occur if firms' expected costs of violating the limit are smaller than firms' compliance costs. Then, the choice of the appropriate level of the standard must account for monitoring and enforcement elements (basically inspection frequencies and fines for non-compliance), which crucially affect firms' induced pollution levels and the resulting environmental damages.

The analysis of pollution standard setting and compliance issues has been traditionally studied in static contexts. Several authors have explored the relationship between regulatory stringency and monitoring and enforcement strategies under alternative modeling structures in one period models, see, for example, Downing and Watson (1974), Harford (1978), Veljanovski (1984), Kambhu (1989), Jones (1989), Jones and Scotchmer (1990), Keeler (1995), or Arguedas (2008, 2013).

However, the static approach is of limited applicability when one seeks to explain the progressive implementation of more stringent limits overtime, as well as firms' progressive adaptation to harsher regulatory environments. A dynamic framework that addresses this adaptation process while at the same time allows firms to exceed the emission limits (while paying the corresponding fines) is crucial in addressing these issues. But, to our knowledge, this problem has not been explored before, as we show in the literature discussion later on. This is, therefore, the purpose of the present paper.

In particular, we aim to answer two research questions. The first question is whether pollution limits should be soft in the beginning and progressively tighter while the firm adapts its production technology. Our second question is whether it would be socially beneficial that regulators offer penalty discounts in exchange for environmentally-friendly actions to help facilities to better adequate to more stringent regulations.

Our analysis is motivated by some common characteristics of many regulatory programs dealing with pollution. For example, the US National Pollutant Discharge Elimination System (NPDES) permit program under the Clean Water Act contains a number of programs and initiatives which require polluting facilities to obtain permits to release specific amounts of pollution into waters of the US.[1] Among others, the Multi-Sector General Permit for stormwater discharges associated with industrial activity and the Vessel General Permit for Vessel Discharges have been progressively tightened since their respective initial issuances.[2]

An example about fine discounts can be found in the EPA's Audit Policy, which lists some conditions for enforcement relief with the NPDES programs. In particular, fines for noncompliance can be reduced up to 100% of the nongravity-based part and up to 75% of the gravity-based component if firms promptly disclose and correct any discovered violations as a result of a self-audit procedure.[3] These practices generally involve large investment efforts in compliance-promoting activities, including information costs such as training personnel about regulatory requirements. Overtime, the likelihood of violations decreases because facilities are

[1] Consult the Environmental Protection Agency (EPA)'s webpage for additional information (http://cfpub.epa.gov/npdes/).

[2] The first one has been progressively tightened in 2000, 2008 and 2013, since the initial issuance on September 29, 1995 (http://cfpub.epa.gov/npdes/home.cfm?program_id=6 for details). The second one was issued initially in 2008, and then reissued in 2013 (http://cfpub.epa.gov/npdes/home.cfm?program_id=350).

[3] Incentives for Self-Policing: Discovery, Disclosure, Correction and Prevention of Violations – Final Policy Statement, 60 Fed. Reg. 66,706 (December 22, 1995), revised on April 11, 2000.

able to identify and correct problems before they become violations, see Stafford (2005).

We model the dynamic interaction between a representative polluting firm and a regulator by means of a differential game played à la Stackelberg. Emissions are considered as an input to produce a consumption good. The regulator sets the pollution standard overtime, while the firm settles emissions and invests to enlarge a capital stock (which is a substitute for emissions in the production of output). At each point in time, the firm can either comply with the standard or pollute above the standard and then be subject to a fine for non-compliance. The firm faces a double dilemma. On the one hand, the firm balances the productive gains from higher emissions with the associated increase in the fine for non-compliance. On the other hand, the firm balances the current cost of investment in the capital stock with the future fine reductions from lower emissions. At each point in time the level of this capital stock determines the regulator's decision on the pollution standard. We assume that the firm seeks to maximize discounted profits throughout an infinite time horizon. By contrast, the regulator is concerned about social welfare, that is, firm's profits net of environmental damages and the administrative costs of imposing sanctions.

To answer our first research question, we consider a base model where the fine is quadratic in the degree of non-compliance and penalty discounts are not allowed. Assuming that the initial capital stock is small, we analytically show that the pollution standard decreases while the capital stock increases overtime. Interestingly, the induced emissions and the induced degree of non-compliance decrease as well. Note that the feature that a tighter standard induces a lower rate of non-compliance cannot be explained in a static setting (in there, a tighter standard always predicts a larger gap between emissions and the standard). However, in a dynamic context the firm can progressively adapt to a stricter standard by means of capital investment. Within this base model, we also perform a comparative statics exercise with respect to the severity of the fine (that is, for a given degree of non-compliance, we analyze variations in the resulting penalty). We find that the severity of the fine must be kept bounded to reach maximum social welfare.

Next, we extend our base model to explore the consequences of offering discounts in the fine for non-compliance in exchange for the firm's investments in capital, which is the purpose of our second research question. Technically, we do so by introducing the concept of *effective standard*, which is the pollution level above which the fine for non-compliance is truly imposed. This effective standard is laxer than the announced standard if the firm invests to increase the capital stock. The level of non-compliance is now a broader concept, which depends on the stock of capital, the level of emissions, the capital investment and the pollution limit. In this extended scenario, we confirm similar patterns for the evolution of the relevant variables as those found in the base model, that is, increasing capital stock and decreasing standards, emissions and degree of non-compliance. But most importantly, this extended model allows us to find that offering fine discounts in exchange for firm's capital investment is socially desirable. We indeed obtain the value of the discount that maximizes social welfare numerically. In contrast with the base model, the severity of the fine does not need to be kept bounded to reach maximum social welfare when penalty discounts are allowed. In fact, for large fine discounts, the severity of the fine that maximizes social welfare is infinitely high. This limiting situation induces full compliance with the effective pollution limit, since the firm is threatened by a very large punishment if it pollutes above that limit. Emissions are still above the pollution limit imposed by the regulator, but due to fine discounts no penalties are levied. Indeed, the corresponding social welfare level obtained in this limiting situation is very close to the first-best level.[4]

In the dynamic context, the endogenous determination of the regulatory standards and the monitoring and enforcement issues have been considered separately in the literature.[5] On the one hand, all the papers that deal with standard setting assume perfect compliance and, therefore, abstract from modeling inspection frequencies and fines for non-compliance. On the other hand, all the papers that deal with dynamic enforcement do so assuming exogenous (or constant overtime) regulatory regimes. We now elaborate on these two related strands of the literature.

Regarding the literature that considers dynamic problems of regulatory standard setting assuming perfect compliance, Beavis and Dobbs (1986) is probably the first study of this kind. There, the environmental authority selects the pollution limit and the time at which that limit comes into force. Fines for non-compliance are not specifically modelled, since it is assumed that the firm complies with the standard once the regulatory period starts. Our paper is similar to Beavis and Dobbs (1986) in two respects. First, we share a similar spirit, which calls for the social convenience of polluters' gradual adaptation to new environmental regulations. Second, we do not consider cumulative pollution effects, as theirs. Cases that we have in mind are those of degradable pollution, assimilative waste or flow pollutants, in which per-period environmental

[4]In a static framework, Arguedas (2013) finds three critical conditions for the social convenience of penalty reductions: (i) administrative costs of sanctioning, (ii) imperfect compliance, and (iii) fines progressive in the degree of non-compliance. We confirm that such penalty discounts are socially convenient in a dynamic setting as well. Arguedas (2013) does not calculate the value of the optimal discount. In contrast, we numerically obtain that value, and we find that it crucially depends on the severity of the fine.

[5]In the case of tradable permits, only Innes (2003), Stranlund *et al.* (2005) and Lappi (2013) jointly consider enforcement issues and intertemporal permit trading, although neither of them jointly considers the endogeneity of the permits and the possibility of non-compliance with these permits overtime. Both Innes (2003) and Stranlund *et al.* (2005) constrain the analysis to enforcement policies that induce full compliance, such that the former assumes an enforcement strategy which consists only of a costly penalty for permit violations, while the second considers the reporting and monitoring functions of enforcement altogether. Lappi (2013) allows for the possibility of non-compliance, assuming that the auditing probability is subjective and, therefore, compliance decisions are made according to the firms' beliefs about that probability. However, emission permits are also exogenous in that setting.

damages are mainly affected by per-period pollution.

However, the main difference with Beavis and Dobbs (1986) is that we model a more realistic progressive tightening of the standard, which induces a decrease in the degree of non-compliance overtime. Beavis and Dobbs (1986) only consider a first period of no regulation where compliance is not an issue, and a second period of full compliance with the regulation. This difference allows us to explain the progressive decrease in non-compliance levels, even under more stringent pollution limits, a result that cannot be explained in the context of Beavis and Dobbs (1986).

Some extensions of pollution standards setting in dynamic contexts include uncertainty about the exact pollution limit and the time at which the regulatory period starts (Hartl, 1992), persistent pollution (Conrad, 1992, Falk and Mendelsohn, 1993) or incentive-compatibility issues (Benford, 1998) among others. Two distinctive features of all these studies versus the present paper are that (i) polluters are assumed to comply with pollution limits, and the enforcement aspects of the policy are absent; and (ii) they all consider cumulative pollution effects.

Regarding the literature on dynamic enforcement, relevant references are Harrington (1988), Harford and Harrington (1991), Raymond (1999) and Friesen (2003). A common feature of all these studies is the specific modelling of monitoring decisions. There, budget constrained regulators distribute inspections in response to compliance information, thus prompting increased compliance. However, in all these studies firms are assumed to undertake discrete decisions whether to comply or to violate a given regulation, and then be moved to the bad group in the case of non-compliance. Therefore, the progressive tightening of the regulations cannot be explained in these models either.

An extensive number of works in the literature suggests that fines for non-compliance should be bounded. Our paper also fits with this literature showing the conditions under which the fine which maximizes social welfare is bounded. Besides the above mentioned papers that include the possibility of targeting enforcement in dynamic settings, the literature gives alternative justifications for such a conclusion. Examples include self-reporting of emission levels (Livernois and Mckenna 1999), penalty evasion (Kambhu 1989), possible inverse relationships between fines and conviction probabilities (Andreoni 1991), marginal deterrence (Shavell 1992), specific versus general enforcement (Shavell 1991), hierarchical governments (Saha and Poole 2000; Decker 2007), regulatory dealing (Heyes and Rickman 1999), or others.

Finally, our paper is also related to the literature that analyzes the incentives set by different policy measures for investing in greener technologies, see Jaffe *et al.* (2003) or Requate (2005) for detailed overviews. The first works within this literature are those of Downing and White (1986) and Milliman and Prince (1989), who show that taxes and auctioned permits generally provide the largest innovation and adoption incentives. Later studies analyze the consequences of partial diffusion of new technologies (Requate and Unold, 2003), imperfect competition in the supply of abatement technology (Maia and Sinclair-Desgagné, 2005), uncertainty with regards to abatement costs or permit prices (Zhao, 2003 and Insley, 2003), uncertainty with regards to regime changes (Nishide and Nomi, 2009), international competition (Feenstra *et al.*, 2001), induced technological progress (Krysiak, 2011), etc.

The remainder of the paper is organized as follows. In section 2 we present the base model, where the fine depends only on the degree of non-compliance. In section 3, we present the characteristics of the optimal policy in the base model. In section 4, we extend the model to allow for fine discounts. In section 5, we discuss on the possiblity of over-compliance in our setting. We conclude in Section 6. All the proofs are in the Appendix.

## 2 The base model

A representative firm produces a consumption good with capital and emissions as inputs.[6] Let $Y(t)$, $K(t)$, $E(t)$ respectively denote the levels of production, capital and emissions of the facility at time $t$. For mathematical convenience we assume that the production function is linear quadratic, as follows:

$$Y(K(t), E(t)) = K(t) + \sigma E(t) - \frac{[K(t) + \sigma E(t)]^2}{2}. \tag{1}$$

Note that this is a concave function with a constant marginal rate of technical substitution between emissions and capital given by $\sigma > 0$.

The flow of emissions is a decision variable of the firm, while capital is accumulated over time according to the investment decision, $I(t)$, such that

$$\dot{K}(t) = I(t) - \delta K(t), \quad K(0) = k_0, \tag{2}$$

where $\delta > 0$ is the rate of depreciation of capital and $k_0 \geq 0$ is the level of capital at $t = 0$. At each time $t$, investment costs are a quadratic function of the investment level, given by $C(I(t)) = cI^2(t)/2$, with $c > 0$.

From the point of view of the firm, capital accumulation is costly while emissions are free in the absence of any regulation. However, instantaneous emissions cause environmental damages, given by the quadratic expression $D(E(t)) = d(E(t))^2/2$, with $d > 0$. [7]

We assume that an environmentally concerned regulator wants to make the firm (partially) responsible for the environmental damages caused. The regulator imposes an emission tar-

[6] Copeland and Taylor (1994) and some other papers cited therein also consider emissions as an input in the production process.

[7] As mentioned in the Introduction, in this paper we do not consider cumulative pollution effects.

get or pollution limit which can vary across time, $L(t)$. At each time $t$, the firm faces a fine which depends on the level of non-compliance with the regulation, given by $N(E(t), L(t)) = E(t) - L(t)$, as long as $E(t) > L(t)$. We assume the fine to be quadratic in the degree of non-compliance, that is, $F(E(t), L(t)) = f[E(t) - L(t)]^2/2$, or zero under compliance, where $f \geq d$ measures the severity of the fine.[8]

One possible interpretation for the fine is to consider that the firm is risk neutral and the fine is actually an expected fine, which embeds the sanction itself times the probability of being discovered and penalized. The convex specification of the fine well reflects the fact that firms are more likely caught and punished when the degree of non-compliance is large than when it is small. In particular, it captures the feature that the marginal (expected) fine for an infinitesimal violation is negligible. We depart from this latter assumption in Section 5.[9]

The specific level of the standard upon which the firm can be penalized is variable through time, as well as the firm's decisions on emissions. This implies that the amount of the fine also varies, since it depends on the degree of non-compliance. However, we assume that the specific shape for the fine is time-invariant. In particular, this implies that the severity of the fine, $f$, is exogenous in the model. However, later on in Sections 3 and 4 we perform several comparative statics exercises to obtain the desirable level for this parameter from the regulator and the firm viewpoints.

Apart from tractability considerations, the reason for this modeling is that we believe that in practice the general structure of the fines is less flexible than the level of the standard itself. As mentioned in the Introduction, the different permit programs under the Clean Water Act contain

[8]The assumption $f \geq d$ allows us to concentrate on the subset of most intuitive results. In particular, this assumption ensures a non-negative optimal standard.

[9]For tractability reasons, we abstract from specifically modeling monitoring issues.

specific timings for the progressive tightening of the standards. However, the general structure of the sanctions is common to all the programs.[10] Thus, from the perspective of the design of a particular permit program, there is flexibility to change standards through time, although the nature of the sanctions cannot be varied within that program itself.

Following the standard literature on pollution regulation, we consider a differential game played *à la Stackelberg* in feedback strategies, where the regulator is the leader and the firm is the follower. We assume that the regulator has a stagewise first-mover advantage, i.e. an instantaneous advantage at each time. This type of solution, denoted as *stagewise feedback Stackelberg solution* following the terminology in Başar and Olsder (1982), is commonly used in the literature (see, for example, Dockner *et al.*, 2000, Long, 2010, and Haurie *et al.*, 2012). As it is usual for this type of differential games with an infinite time horizon, we assume that the agents (the firm and the regulator) employ stationary strategies, i.e, their strategies and value functions do not explicitly depend on time, but exclusively on the capital stock (see, for example, Dockner *et al.*, 2000).

An alternative approach is to find the *global Stackelberg solution*, under which the leader selects and announces his strategy at time zero which gives him a global first-mover advantage (see Dockner *et al.* (2000)). Long (2010) provides some examples to illustrate that the leader attains a higher payoff under the global Stackelberg solution (when the leader is restricted to linear stationary strategies) than under the stagewise feedback Stackelberg solution. However, we consider that the stagewise feedback Stackelberg solution is more appropriate in our context, for two main reasons. The first reason is that the global Stackelberg solution requires

[10] Consult the Criminal Provisions of the Clean Water Act in the EPA webpage (http://www2.epa.gov/enforcement/criminal-provisions-clean-water-act) for more information on this issue.

commitment for the leader to follow the announced strategy throughout the time horizon. If this commitment is credible, then the global Stackelberg solution gives the leader a mechanism to influence the follower's valuation of the state variable. However, as pointed out in the literature (see, for example, Dockner *et al.* (2000), Haurie *et al.* (2012)) this commitment could not be always credible.

The second reason is that we consider a model of a representative firm, that is, a firm that represents a large number of firms that are subject to the same regulation.[11] In this context, a single firm is too small to recognize that its investment in "its own" capital stock would affect the "aggregate" capital stock,[12] and, hence, future values of the announced pollution limit. Therefore, the mechanism through which the leader influences the firm's valuation of the capital stock does not operate.

To characterize the stagewise feedback Stackelberg solution, the firm, who acts as the Stackelberg follower, chooses emissions and investment in order to maximize the present value of profits over an infinite time horizon, taking into account the time evolution of the capital stock. Instantaneous profits are given by the income from production, minus the investment costs and the fine for non-compliance. Considering the consumption good as the *numéraire*, the dynamic

[11] An example of the use of the stagewise feedback Stackelberg solution for the case of a continuum of followers can be found in Long (2010, page 217).

[12] Considering a continuum of firms defined over the unit interval, the firm's capital stock and the aggregate capital stock coincide.

maximization problem for the firm is the following:[13]

$$\max_{I,E} \int_0^\infty \left[Y(K,E) - C(I) - F(E,L)\right] e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = I - \delta K, \quad K(0) = k_0, \tag{3}$$

where $\rho > 0$ is the discount factor.

The regulator acts as a Stackelberg leader and decides upon the optimal pollution limit rule, $L(K)$, taking into account the firm's optimal response to the policy, $\hat{I}(K;L)$ and $\hat{E}(K;L)$ (i.e., the investment and emissions reaction functions that solve problem $(3)$ as functions of the state variable, $K$, and the pollution limit, $L$, settled by the regulator). The regulator is concerned about the firm's profits, the environmental damages and the social cost of sanctioning. We define the latter as a proportion $h \in (0,1)$ of the fine for non-compliance, which reflects the existence of positive social costs associated with imposing penalties, such as administrative costs of court processes, citizens' discomfort with law transgressions, etc.[14]

The dynamic maximization problem for the regulator is then the following:

$$\max_{L} \int_0^\infty \Big[Y(K,\hat{E}(K;L)) - C(\hat{I}(K;L)) - D(\hat{E}(K;L)) - hF(\hat{E}(K;L),L)\Big] e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = \hat{I}(K;L) - \delta K, \quad K(0) = k_0. \tag{4}$$

Once the optimal pollution limit policy is determined, the corresponding optimal policies for capital investment and emissions, as well as the optimal time-path of the capital stock can be obtained. All the characteristics of the solution are analyzed in the following section.

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[13]The time argument is omitted here and henceforth when no confusion can arise. As a general principle, upper-case letters denote time-dependent (either state or control) variables, while lower-case letters denote time-independent parameters.

[14]We assume that the administrative costs of imposing sanctions are increasing in the level of non-compliance. Sanctioning costs may increase with this level (and eventually, with the level of the fine) since individuals can strongly resist to the imposition of larger fines, engage in avoidance activities,etc., see Polinsky and Shavell (1992). Stranlund (2007) or Arguedas (2008, 2013) also consider sanctioning costs dependent on the level of the fines.

# 3 The optimal solution in the base model

In this section, we characterize the optimal policy that solves problem $(4)$ and the corresponding optimal policies for emissions and investments. From these, the optimal time-path of the capital stock can be computed. The regulator fixes the pollution limit, $L(K)$, knowing the best-reaction functions of the firm. The firm's best-reaction function for emissions is positively related with the emission limit. Interestingly, the firm's best-reaction function for investment in this base model does not directly depend on the pollution limit set by the regulator (see $(28)$ in the Appendix). Therefore, the accumulation of capital is unaffected by the decision taken by the regulator, who in consequence behaves as a static player.

The following proposition presents the characteristics of the optimal policy in the base model (all the proofs are in the Appendix). In view of the linear-quadratic structure of the problem we conjecture quadratic value functions leading to optimal linear strategies.[15]

**Proposition 1** *In the base model, the optimal pollution limit strategy is given by:*[16]

$$L^{*b}(K) = L_0^{*b}(1-K), \qquad L_0^{*b} = \frac{\sigma(f + h\sigma^2 - d)}{\Psi} > 0, \tag{5}$$

*where* $\Psi = f(d+\sigma^2) + h\sigma^4$; *the induced optimal emissions and capital investment strategies*

[15] For tractability, in this model and also in the extended model presented in the following section we substitute the piecewise fine by the quadratic expression $F(E,L) = f[E-L]^2/2$, both under non-compliance and over-compliance. This assumption artificially rules out all the solutions that imply over-compliance, since their opposed solutions with non-compliance lead to the same penalty but higher production. However this is not problematic if (as we will prove) we find strict non-compliance along the optimal trajectory at any time. That being true, perfect compliance is never optimal under this artificial specification. Under the real piecewise fine, over-compliance would imply the same (null) penalty but lower production, and hence it could not be optimal either. In consequence the optimal solution obtained under the artificial specification of the fine coincides with the optimal solution for the real piecewise fine. In Section 5, we discuss on the possibility of over-compliance with alternative penalty-subsidy schemes.

[16] Superscript $b$ stands for base model.

*are:*

$$E^{*b}(K) = E_0^{*b}(1-K), \qquad E_0^{*b} = \frac{\sigma(f+h\sigma^2)}{\Psi} > 0, \tag{6}$$

$$I^{*b}(K) = I_0^{*b} + I_1^{*b}K, \qquad I_0^{*b} = \frac{b_{\scriptscriptstyle F}^b}{c}, \quad I_1^{*b} = \frac{a_{\scriptscriptstyle F}^b}{c}, \tag{7}$$

*where*

$$a_{\scriptscriptstyle F}^b = \frac{c(\rho+2\delta)-\sqrt{\Delta^b}}{2} < 0, \quad b_{\scriptscriptstyle F}^b = \frac{2cd^2f(f+\sigma^2)}{(c\rho+\sqrt{\Delta^b})\Psi^2} > 0, \tag{8}$$

$$\Delta^b = c^2(\rho+2\delta)^2 + \frac{4cd^2f(f+\sigma^2)}{\Psi^2}; \tag{9}$$

*and the optimal time-path of the capital stock is given by:*

$$K^b(t) = (k_0 - \bar{K}^b)e^{\theta^b t} + \bar{K}^b, \tag{10}$$

*where the steady-state value of the capital stock and the speed of convergence towards this value are respectively given by*

$$\bar{K}^b = \frac{d^2f(f+\sigma^2)}{d^2f(f+\sigma^2)+c\delta(\delta+\rho)\Psi^2} \in (0,1), \quad \theta^b = \frac{c\rho-\sqrt{\Delta^b}}{2c} < 0. \tag{11}$$

The optimal policy for the pollution limit, given by expression $(5)$, is inversely related to the stock of capital, which is characterized in $(10)$. If initially the stock of capital is small ($k_0 < \bar{K}^b$), then the stock of capital increases towards its long-run value from below.[17] Therefore, the regulator fixes a lax emission limit at the beginning, and this limit becomes more stringent over time as the capital grows. The pollution standard converges towards its steady-state value, $\bar{L}^b$,

[17]This is the interesting case, in which the firm faces the dilemma between adaptation through capital investment or paying the fine for non-compliance. The alternative case where $k_0 > \bar{K}^b$ would lead the firm to destroy capital along the optimal trajectory.

which is given by the following expression:

$$\bar{L}^b = \frac{\sigma(f + h\sigma^2 - d)\left(1 - \bar{K}^b\right)}{\Psi} = \frac{c\delta(\delta+\rho)\sigma(f + h\sigma^2 - d)\Psi}{d^2 f(f+\sigma^2) + c\delta(\delta+\rho)\Psi^2}. \tag{12}$$

From expression (5), and since $K^b(t) \leq \bar{K}^b < 1$, we can infer that the standard is always positive (recall that we have assumed that $f \geq d$).

Correspondingly, both the optimal emissions and capital investment policies characterized in (6) and (7) are also inversely related to the capital stock. Therefore, provided $k_0 < \bar{K}^b$, both the optimal emissions and capital investment decrease to their steady-state values, $\bar{E}^b$ and $\bar{I}^b$, which are the following:

$$\bar{E}^b = \frac{\sigma(f + h\sigma^2)\left(1 - \bar{K}^b\right)}{\Psi} = \frac{c\delta(\delta+\rho)\sigma(f + h\sigma^2)\Psi}{d^2 f(f+\sigma^2) + c\delta(\delta+\rho)\Psi^2}, \tag{13}$$

$$\bar{I}^b = \delta\bar{K}^b = \frac{\delta d^2 f(f+\sigma^2)}{d^2 f(f+\sigma^2) + c\delta(\delta+\rho)\Psi^2}. \tag{14}$$

The optimal pollution limit decreases over time, as well as the associated degree of non-compliance. The reason is that the firm progressively adapts to more stringent standards through capital investment and, therefore, the induced emissions are closer to the required limits, even if those limits have become more stringent. This can be easily seen substracting $(5)$ from $(6)$, which results in:

$$E^{*b}(K) - L^{*b}(K) = (1-K)\frac{\sigma d}{f(d+\sigma^2) + h\sigma^4}. \tag{15}$$

The optimal degree of non-compliance in the long run converges to a positive value, given by the combination of expressions $(12)$ and $(13)$, as follows:

$$\bar{E}^b - \bar{L}^b = \frac{c\delta(\delta+\rho)\sigma d\Psi}{d^2 f(f+\sigma^2) + c\delta(\delta+\rho)\Psi^2}. \tag{16}$$

Finally, it is easy to prove that $D(E^{*b}(K)) - f(E^{*b}(K) - L^{*b}(K))^2/2 > 0$ if and only if

$df - (f + h\sigma^2)^2 < 0$. Therefore, whenever $f \geq d$, the fine for non-compliance does not exceed environmental damages, no matter how severe the fine is (i.e., how large $f$ is).

We now compare optimal emissions, investment and capital time-paths under the optimal policy characterized in Proposition 1 with the corresponding paths under no regulation and the first-best scenario. In the scenario of no regulation, the firm does not have to comply with any pollution limit. Therefore, the maximization problem of the firm in this case simply reads:

$$\max_{I,E} \int_0^\infty [Y(K,E) - C(I)]\, e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = I - \delta K, \quad K(0) = k_0. \tag{17}$$

The solution is denoted as $(E^{\text{NR}}, I^{\text{NR}}, K^{\text{NR}})$, where the superscript stands for the no-regulation scenario, and is given in (30) in the Appendix. Under no regulation, the firm does not have any incentives to invest in capital. Then, the optimal capital stock depreciates until zero while the emissions increase towards the long-run value $1/\sigma$ (the inverse of the marginal rate of substitution between emissions and capital).

On the other hand, the first-best solution arises when the firm fully accounts for the environmental damages in its optimization problem. This solution is defined by the optimal investment and emission policies that maximize the difference between the value of production and the sum of investment costs and environmental damages, as follows:

$$\max_{I,E} \int_0^\infty [Y(K,E) - C(I) - D(E)]\, e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = I - \delta K, \quad K(0) = k_0. \tag{18}$$

The solution is denoted as $(E^{\text{FB}}, I^{\text{FB}}, K^{\text{FB}})$, where the superscript stands for the first-best scenario, and is given in (31) and (32) in the Appendix. In the first-best solution, if the initial capital stock is small the investment effort is initially strong and, as the optimal capital stock increases towards its long-run value, investments are reduced to the amount strictly necessary to replace the depleted capital. Correspondingly, emissions decrease to the steady-state value.

The following proposition shows that the optimal emissions and capital accumulation time-paths associated with the optimal policy characterized in Proposition 1 evolve between the paths in the two extreme scenarios presented in $(17)$ and $(18)$, for all $t$, while it is not necessarily true that capital investment levels under the optimal policy are lower than those under the first-best scenario for all $t$.[18]

**Proposition 2** *Assuming $f \geq d$, the steady-state values and optimal paths in the base model can be related to the no regulation and the first-best outcomes as follows:*[19]

*(i) Capital stock:*

$\bar{K}^{FB} > \bar{K}^{b} > 0$, *and* $K^{FB}(t) > K^{b}(t) > K^{NR}(t)$ *for all* $t > 0$.

*(ii) Investment:*

$\bar{I}^{FB} > \bar{I}^{b} > 0$, *although it is not necessarily true that* $I^{FB}(t) > I^{b}(t)$ *for all* $t \geq 0$. *In the particular case where* $k_0 = 0$, $I^{FB}(0) > I^{b}(0) > 0$.

*(iii) Emissions:*

$\bar{E}^{FB} < \bar{E}^{b} < 1/\sigma$ *and* $E^{FB}(t) < E^{b}(t) < E^{NR}(t)$ *for all* $t > 0$.

[18] In the particular case with $h = 0$, the regulator could induce the firm to act as in the first-best scenario, if the severity of the fine were $f = d$, which then implies $L^{*b}(K) = 0$ at any time. Note that under this strict liability specification, the firm's maximization problem is identical to the first-best maximization problem. This particular case is out of the scope of this paper.

[19] $K^i(t), E^i(t), I^i(t)$ refer to the optimal capital stock, emissions and investment time paths for the different scenarios, $i \in \{NR, FB, b\}$.

We are now interested in performing a comparative statics exercise about the effect of the severity of the fine, $f$, on the optimal policy characterized in Proposition 1. In particular, we want to analyze the consequences of a rise in $f$ on the long-run values of emissions, the capital stock, the pollution limit and the degree of non-compliance. Remember that the severity of the fine is not a regulator's decision variable, but a given parameter which determines the final fine the firm has to face in case of non-compliance, as well as the social costs born by the regulator. In consequence, the outcome of the optimal policy characterized in Proposition 1 is clearly influenced by this parameter.

Starting at a low level of $f$, an increase in $f$ results in a greater capital stock, lower emissions and a better compliance, while the effect on the pollution limit is ambiguous. However, if the social cost per unit of the fine is not too large, these effects are reversed at certain values of $f$, except for the degree of non-compliance, which decreases monotonically with $f$. In particular, increases in $f$ above a certain value $\hat{f}_L^b$, result in a loosening of the pollution limit. Thus, from this value on, the severity of the fine and the pollution limit are substitute instruments in the regulatory problem. The following proposition summarizes this interesting finding.

**Proposition 3** *The effect of a more severe penalty, characterized by a higher value of parameter $f$, over steady-state values of the more relevant variables of the model can be summarized as:*[20]

*i) If $0 < h < h_{\max} \equiv \frac{d+\sigma^2}{2\sigma^2}$, then there exist a finite $\hat{f}_K^b$ and $\hat{f}_E^b$ such that:*

$$(\bar{K}^b)_f, (\bar{I}^b)_f, |\theta^b|_f \begin{cases} > 0 & \textit{if} \quad f \in [0, \hat{f}_K^b), \\ < 0 & \textit{if} \quad f > \hat{f}_K^b, \end{cases} \qquad (\bar{E}^b)_f \begin{cases} < 0 & \textit{if} \quad f \in [0, \hat{f}_E^b), \\ > 0 & \textit{if} \quad f > \hat{f}_E^b, \end{cases}$$

$$0 < \hat{f}_K^b \equiv \frac{h\sigma^4}{d + \sigma^2(1-2h)} < \hat{f}_E^b. \tag{19}$$

[20]The subscript $f$ denotes partial derivative with respect to $f$.

*ii) If* $\frac{2d}{2d+\sigma^2}\frac{d+\sigma^2}{2\sigma^2} \equiv h_{\min} < h < h_{\max} \equiv \frac{d+\sigma^2}{2\sigma^2}$, *then:*

*There exists a* $\hat{f}_L^b \geq 0$ *such that* $(\bar{L}^b)_f > 0$ *for any* $f > \hat{f}_L^b$.

*Furthermore,* $\max\left\{d, \hat{f}_L^b\right\} < \hat{f}_K^b < \hat{f}_E^b$.

*iii)* $(\bar{E}^b - \bar{L}^b)_f < 0,$ *for any* $h > 0,\ f \geq 0.$

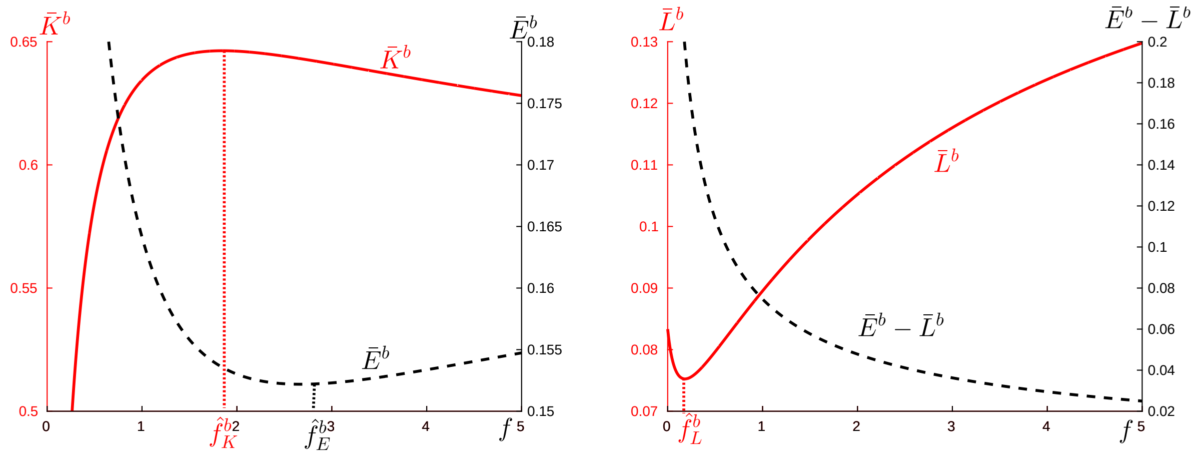


Figure 1: $\bar{K}^b,\ \bar{E}^b$ w.r.t. $f$ (left); $\quad \bar{L}^b,\ \bar{E}^b - \bar{L}^b$ w.r.t. $f$ (right)

The analytical results in Proposition 3 can be explained with the help of Figure 1. For the numerical illustration, here and hencefoth, the parameters values are:

$$c = d = 1,\ \rho = 0.05,\ \delta = 0.15,\ \sigma = 2,\ k_0 = 0,\ h = 0.3. \tag{20}$$

Figure 1 illustrates[21] the relationship between the severity of the fine (i.e., the value of $f$, in the horizontal axes) and the long-run values of the capital stock, emissions, the pollution limit and the degree of non-compliance (in the vertical axes), for $h \in (h_{\min}, h_{\max})$.[22] For low values of $f$, an increase in this parameter is associated with higher capital and lower emissions in the long

[21] Capital and emissions are measured respectively on the left and right vertical axes of Figure 1 (left), while the pollution limit and the degree of noncompliance are measured, respectively, on the left and right vertical axes of Figure 1 (right).

[22] For this specification $h \in (h_{\min}, h_{\max}) = (0.2083, 0.625)$.

run. Moreover, the speed of convergence of these variables towards their steady-state values is also increased. However, the effect of a more severe penalty on these variables is reversed for a certain value of $f$. Thus, further increases in $f$ above $\hat{f}_K^b$ will decrease the capital stock, and above $\hat{f}_E^b$ will increase the emissions in the long run, see Figure 1 (left). Correspondingly, as shown in Figure 1 (right), for this example[23] an increase of $f$ below $\hat{f}_L^b$ tightens the emission limit, although in any case an increase in the severity of the fine above $\hat{f}_L^b$ will lead to a laxer emission limit in the long run.[24] Thus, for values of the fine above $\hat{f}_L^b$, the pollution limit and the severity of the fine are substitutes, while in this example they are complementary instruments for values of $f$ below this bound. Finally, the degree of non-compliance decreases monotonically with the severity of the fine.

Note that the effect of a higher $f$ on the evolution of the variables over time is the same as that at the steady state, with some exceptions: the effect of a higher $f$ on emissions at a particular time $t$ is ambiguous in the interval $\left(\hat{f}_K^b, \hat{f}_E^b\right)$; the corresponding effect on the optimal pollution limit at time $t$ is ambiguous in the interval $\left(\hat{f}_L^b, \hat{f}_K^b\right)$ ; and, finally, the effect of a higher $f$ on the degree of non-compliance at time $t$ is ambiguous for $f > \hat{f}_K^b$.

The next question is to analyze the effect of a more severe fine on the accumulated profits of the firm and, more importantly, on the accumulated social welfare. Due to its analytical complexity, we rely on a numerical simulation, which is robust to changes in the parameters values. Here and in the next section, a numerical result is robust when it is valid for the parameters values considered in (20), but also after a variation in any one of the parameters keeping all

[23]For $f \in [0, \hat{f}_L^b)$, and for the vast majority of the values for $h$, $\bar{L}^b$ decreases monotonously with $f$. However, for $h$ close to $h_{\min}$, $\bar{L}^b$ either always increases or it increases within a first period and decreases henceforth.

[24]In the example, $\hat{f}_L^b < d$, although for greater values of $h$ (for example $h = 0.5$), $\hat{f}_L^b > d$, and $\bar{L}^b$ would be U-shaped within the interval we are interested in, that is, $f \in [d, \infty)$.

others unchanged. Parameters changes analyzed are:

$$c, d \in \{0.8, 1.2\}, \rho \in \{0.04, 0.07\}, \delta \in \{0.1, 0.2\}, \sigma \in \{1.5, 2.5\}, h \in \{0.21, 0.4\}, k_0 = 0.5. \quad (21)$$

Figure 2 shows the relationship between the value function of the firm, $V_{\text{F}}^b(k_0)$, and the social welfare, $V_{\text{R}}^b(k_0)$, with respect to the severity of the fine, $f$.

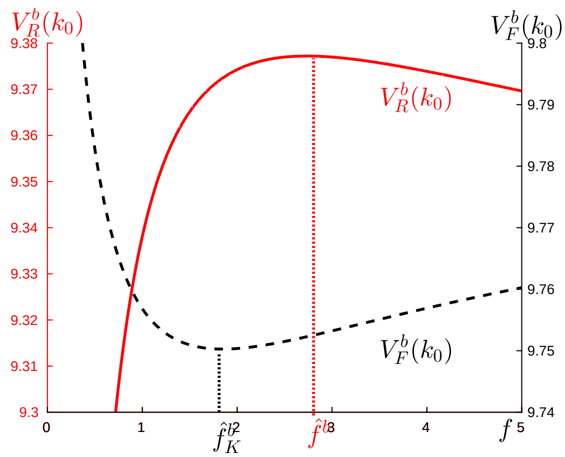


Figure 2: $V_{\text{R}}^b(k_0),\ V_{\text{F}}^b(k_0)$ w.r.t. $f$

We first observe that social welfare is improved and firm's profits lessens when moving from no regulation ($f = 0$) to a regulatory scenario in which the firm is marginally penalized when polluting above the pollution standard specified by the regulator. But the effect of successive increments in the severity of the fine is not monotonous. Firm's profits reach a minimum at $\hat{f}_K^b$, which coincides with the value of $f$ for which the firm's capital investment is maximum. Focusing on social welfare, we also observe that it improves with the severity of the fine up to a maximum, $\hat{f}^b$, which lays not only to the right of $\hat{f}_K^b$, but it is also greater than $\hat{f}_E^b$.[25]

Our interpretation of this non-monotonic effect of the severity of the fine $f$ on firm's profits and social welfare is centered on the different valuation of non-compliance by the firm and

[25]The parameter values presented in (20) result in $\hat{f}_L^b = 0.18223 < \hat{f}_K^b = 1.84615 < \hat{f}_E^b = 2.6723 < \hat{f}^b = 2.73391$. The numerical result $\hat{f}^b > \hat{f}_K^b$ is robust to parameter changes, as long as $h \in (h_{\min}, h_{\max})$.

the regulator. Since $h \in (0, 1)$ , a more severe penalty affects the firm more strongly than the regulator. In consequence, when the severity of the fine is low, a more severe penalty will push the firm to better comply with the environmental standard, increasing capital investments and reducing emissions. This will reduce firm's profits, but will increase society's welfare, which bears a lower damage from pollution. Nevertheless, from the viewpoint of the regulator (less concerned on compliance than the firm), an excessively severe penalty might eventually lead the firm to focus too much on compliance and too little on production. Hence, the regulator would decide a laxer standard, making it easier for the firm to comply with higher emissions and production. Higher production will increase firm's welfare, but the damage linked with higher emissions will lead social welfare down.

To conclude this section, Table 1 summarizes the effects of increases in $f$ on the long-run values of the relevant variables of the model (the capital stock, emissions, the pollution limit, and the degree of non-compliance) as well as on the optimal firm's profits and social welfare, for the values of the per-unit social cost of imposing fines such that $h_{\min} < h < h_{\max}$.

If the social cost satisfies $h \in (0, h_{\min})$, the results presented in Table 1 are still valid. However, since the relevant range of values for $f$ is $[d, \infty)$, some of the non-monotonic effects disappear. Since now $\hat{f}_L^b < \hat{f}_K^b < d$, increments in $f$ always reduce the capital stock and increase firm's profits and the pollution limit. As $h$ becomes smaller, $\hat{f}_E^b$ might turn lower than $d$, which implies that emissions increase monotonously with $f$. Likewise $\hat{f}^b$ decreases as $h$ tends to zero.[26] Then for very small values of $h$, $\hat{f}^b$ might also be lower than $d$ and a more severe fine would unequivocally reduce social welfare.

[26]Numerical simulations show that as $h$ decreases, $V_{\mathrm{R}}^b(k_0)$ reaches a higher maximum at a lower $\hat{f}^b$, and hence social welfare increases as the social costs associated with imposing penalties decrease.

| $f < \hat{f}^b_L$ | $\hat{f}^b_L < f < \hat{f}^b_K$ | $\hat{f}^b_K < f < \hat{f}^b_E$ | $\hat{f}^b_E < f < \hat{f}^b$ | $f > \hat{f}^b$ |
|---|---|---|---|---|
| $\bar{K}^b \uparrow$ | $\bar{K}^b \uparrow$ | $\bar{K}^b \downarrow$ | $\bar{K}^b \downarrow$ | $\bar{K}^b \downarrow$ |
| $\bar{E}^b \downarrow$ | $\bar{E}^b \downarrow$ | $\bar{E}^b \downarrow$ | $\bar{E}^b \uparrow$ | $\bar{E}^b \uparrow$ |
| $\bar{L}^b \downarrow$ | $\bar{L}^b \uparrow$ | $\bar{L}^b \uparrow$ | $\bar{L}^b \uparrow$ | $\bar{L}^b \uparrow$ |
| $\bar{E}^b - \bar{L}^b \downarrow$ | $\bar{E}^b - \bar{L}^b \downarrow$ | $\bar{E}^b - \bar{L}^b \downarrow$ | $\bar{E}^b - \bar{L}^b \downarrow$ | $\bar{E}^b - \bar{L}^b \downarrow$ |
| $V^b_{\mathrm{F}}(k_0) \downarrow$ | $V^b_{\mathrm{F}}(k_0) \downarrow$ | $V^b_{\mathrm{F}}(k_0) \uparrow$ | $V^b_{\mathrm{F}}(k_0) \uparrow$ | $V^b_{\mathrm{F}}(k_0) \uparrow$ |
| $V^b_{\mathrm{R}}(k_0) \uparrow$ | $V^b_{\mathrm{R}}(k_0) \uparrow$ | $V^b_{\mathrm{R}}(k_0) \uparrow$ | $V^b_{\mathrm{R}}(k_0) \uparrow$ | $V^b_{\mathrm{R}}(k_0) \downarrow$ |

Table 1: Effect of a more severe penalty for different ranges of the penalty $f$

# 4 The extended model: allowing for fine discounts

In this section, we analyze the optimal policy if the firm is not or is less penalized for polluting above the emission limit set by the regulator as long as it invests to increase the stock of capital. We broaden the concept of non-compliance, as follows:

$$N(K(t), E(t), I(t), L(t)) = E(t) - [L(t) + \beta \dot{K}(t)] = E(t) - [L(t) + \beta(I(t) - \delta K(t))], \quad (22)$$

where $\beta \geq 0$, and the equation (2) has been considered.

Note that $\beta = 0$ corresponds to the base case analyzed in the previous section. Now, $\beta > 0$ allows the regulator to be more flexible with respect to the imposition of the fine and take into account the investment efforts of the firm. In this case, the *effective* standard is $L(t) + \beta \dot{K}(t)$. Therefore, a laxer effective limit is imposed if the firm invests to increase the capital stock. The level of non-compliance now depends on the stock of capital, the level of emissions, the capital investment and the pollution limit, i.e., $N(K, E, I, L)$. Then again, the fine is quadratic in the degree of non-compliance, that is, $F(K, E, I, L) = f[E - (L + \beta(I - \delta K))]^2/2$, or zero under compliance.[27] The fine is now a function of the gap between emissions and the effective

[27] Applying the same reasoning as in footnote 15, the equilibrium is characterized substituting the piecewise fine by the quadratic expression. For this extended model we will show a positive degree of non-compliance at any moment in time through numerical simulations.

emission limit, $L+\beta(I-\delta K)$. Considering this broader concept of the fine in the dynamic maximization problem for the firm described in (3), modifies the best-response functions of the firm.[28] Now, the pollution standard chosen by the regulator does not only affect the emissions of the firm, but it also has a direct effect on its investment decisions. A tighter pollution limit (a lower $L$) induces a rise in the investment effort of the firm, and correspondingly, a higher pollution limit discourages investment affecting the accumulation of capital. In consequence, the regulator who fixes the optimal emission limit knowing the best-response functions chosen by the firm in (35), is no longer a static player as in the case with $\beta=0$.

The following proposition presents the characteristics of the optimal policy in the extended model.

**Proposition 4** *In the extended model where fine discounts are allowed (i.e. $\beta>0$), the optimal strategies for the pollution limit, emissions and investment in capital are given by:*

$$L^*(K)=L_0^*+L_1^*K, \qquad E^*(K)=E_0^*+E_1^*K, \qquad I^*(K)=I_0^*+I_1^*K, \tag{23}$$

*with*

$$V_{F}(K)=a_{F}\frac{K^2}{2}+b_{F}K+c_{F}, \quad V_{R}(K)=a_{R}\frac{K^2}{2}+b_{R}K+c_{R}, \tag{24}$$

*being the value function of the firm and the regulator, respectively, and*

$$\begin{aligned}
&L_0^*=(cd+(b_{R}-b_{F})\beta\sigma^3)\frac{\sigma\Phi-2h\sigma^3\Omega}{\Phi^2}, \quad L_1^*=-(cd+(a_{F}-a_{R})\beta\sigma^3)\frac{\sigma\Phi-2h\sigma^3\Omega}{\Phi^2},\\
&E_0^*=\frac{f^2\beta\sigma^3(b_{F}-b_{R})+\Phi-cdf^2}{\sigma\Phi}, \quad E_1^*=-\frac{f^2\beta\sigma^3(a_{R}-a_{F})+\Phi-cdf^2}{\sigma\Phi}.\\
&I_0^*=\frac{b_{F}}{c}+\frac{f^2\beta\sigma[cd+(b_{R}-b_{F})\beta\sigma^3]}{c\Phi}, \quad I_1^*=\frac{a_{F}}{c}-\frac{f^2\beta\sigma[cd+(a_{F}-a_{R})\beta\sigma^3]}{c\Phi}.\\
&\Omega=cf+(c+f\beta^2)\sigma^2>0, \qquad \Phi=c\Psi+f^2\beta^2\sigma^4>0.
\end{aligned}$$

[28] See the proof of Proposition 4 in the Appendix for the detailed expressions.

*The optimal time-path of the capital stock reads:*

$$K(t) = (k_0 - \bar{K})e^{\theta t} + \bar{K}, \tag{25}$$

*where*

$$\bar{K} = \frac{f^2\beta^2\sigma^4 b_R + f^2\beta\sigma cd + b_F c\Psi}{f^2\beta^2\sigma^4(a_F - a_R) + f^2\beta\sigma cd + \Phi(c\delta - a_F)},$$

$$\theta = \frac{f^2\beta^2\sigma^4(a_R - c\delta) - f^2\beta\sigma cd + (a_F - c\delta)c\Psi}{c\Phi}.$$

In contrast with the results presented in the previous section, no closed form solutions can be obtained from the Riccati equations for the coefficients of the value functions. In consequence in the subsequent analysis we will rely on numerical simulations. Now we cannot provide general conclusions about the evolution of the different variables through time. However, for the parameters values presented in (20), the capital stock converges towards a positive long-run value from below, with a speed of convergence given by $|\theta|$. The temporal evolution of the optimal time-paths of emissions and the pollution limit are equivalent to those without fine discounts. The effective pollution standard is initially laxer, and it becomes tighter over time as the capital stock grows. Nevertheless, although the effective pollution limit becomes more stringent, capital growth allows the firm to reduce the degree of non-compliance overtime. These results are robust to the changes in parameters values collected in (21). Moreover, focusing on the steady state, the environmental damage always surpasses the fine from non-compliance. Again this result is robust to the same sensitivity analysis.

In what follows, we are interested in performing two exercises. The first is similar to the one performed in the previous section: When fine discounts are allowed, is there a bounded value of the severity of the fine for which social welfare is maximum? The second one is idiosyncratic of this section: Does the existence of fine discounts associated with investment in capital give room for social welfare improvements?

We then perform a comparative statics analysis on parameter $f$, which measures the severity of the fine, and parameter $\beta$, which defines to what extent the emission limit can be raised with the capital investment of the firm. This analysis is carried out numerically, and results are robust to the changes in the parameters values presented in (21) unless said otherwise.

We first start with the comparative statics on the severity of the fine (that is, parameter $f$). Let us recall that in the base model, ($\beta = 0$), where the pollution limit was independent of the investment decisions, the effect of a more severe penalty was, in general, non-monotone (see Proposition 2). In particular, when the fine was low, a rise in $f$ increased social welfare but the reverse occurred when the fine was large (see Figure 2). In consequence, we could compute the magnitude of the fine, $\hat{f}^b$, which resulted in the highest social welfare level.

The following claim[29] characterizes the effect of a change in the severity of the fine on social welfare when fine discounts are allowed, i.e. assuming $\beta > 0$.

**Claim 1** *For a given fine discount $\beta > 0$,*

*i) If $\beta$ is small, then there exists $\hat{f} \in (\hat{f}^b, \infty)$ such that:*

$$(V_R(k_0))_f \begin{cases} > 0 & \text{if} \quad f \in [0, \hat{f}), \\ < 0 & \text{if} \quad f > \hat{f}. \end{cases}$$

*Furthermore, $\hat{f}$ increases with $\beta$, $\lim_{\beta \to 0} \hat{f} = \hat{f}^b$ and $\lim_{\beta \to 0} V_R(k_0) = V_R^b(k_0)$.*

*ii) If $\beta$ is large enough, then $(V_R(k_0))_f > 0$ for all $f > 0$.*

[29]From now on since our results have been generated numerically, but cannot be proven analytically, we state them as claims rather than propositions.

When the allowed fine discount is not too large (part i of the claim), initial rises in the severity of the fine increase social welfare, up to a certain value, from which further increases result in social losses. Then, there exists a finite value $\hat{f}$ at which the society attains the highest social welfare level. Also, the level of the fine which maximizes social welfare is more severe the larger the discount, i.e., $\hat{f}$ increases with $\beta$. Moreover, when the fine discount is very small ($\beta$ approaches zero), $\hat{f}$ approaches $\hat{f}^b$ and social welfare matches its value when fine discounting were not allowed.

When the firm is allowed to discount a large part of its investments (part ii of the claim), social welfare increases monotonously with the severity of the fine. In this case, from a social perspective the fine should be set as large as possible.[30]

Next, we perform a comparative statics exercise on the fine discount $\beta$, taking the severity of the fine, $f$, as given. We ask whether fine discounts are socially desirable. The result is presented next.

**Claim 2** *For a given severity of the fine $f > 0$, there always exists a bounded fine discount $\hat{\beta} \in (0, \infty)$ such that*

$$(V_R(k_0))_\beta \begin{cases} > 0 & \textit{if} \quad \beta \in [0, \hat{\beta}), \\ < 0 & \textit{if} \quad \beta > \hat{\beta}. \end{cases}$$

[30] We cannot characterize analytically the value of $\beta$ above which an infinite punishment for non-compliance would be socially desirable. For the values in (20) it holds that this threshold (0.217) is smaller than the marginal rate of technical substitution $1/\sigma = 0.5$ between capital and emissions.

*Furthermore, there exist* $\hat{\beta}_F, \hat{\beta}_E$ *and* $\hat{\beta}_L$ *with* $0 < \hat{\beta}_F < \hat{\beta}_L < \hat{\beta} < \hat{\beta}_E$ *such that*

$$(V_F(k_0))_\beta \begin{cases} < 0 & \textit{if} \quad \beta \in [0, \hat{\beta}_F), \\ > 0 & \textit{if} \quad \beta > \hat{\beta}_F, \end{cases}$$

$$\left(\bar{E}\right)_\beta \begin{cases} < 0 & \textit{if} \quad \beta \in [0, \hat{\beta}_E), \\ > 0 & \textit{if} \quad \beta > \hat{\beta}_E, \end{cases} \qquad \left(\bar{L}\right)_\beta \begin{cases} < 0 & \textit{if} \quad \beta \in [0, \hat{\beta}_L), \\ > 0 & \textit{if} \quad \beta > \hat{\beta}_L, \end{cases}$$

$$\left(\bar{K}\right)_\beta > 0, \quad \left(N(\bar{K}, \bar{E}, \bar{I}, \bar{L})\right)_\beta < 0 \quad \forall \beta > 0.$$

The previous claim suggests that switching from the base scenario where fine discounts are not allowed to the extended model in which fine discounts are allowed results in a social welfare improvement as long as $\beta$ is small enough (within $0$ and $\hat{\beta}$). However, when the allowed fine discount is sufficiently large (larger than $\hat{\beta}$), further discounts reduce social welfare. Thus, for a given $f$, society would attain a maximum welfare level at $\hat{\beta}$.

The explanation for this result is the following. Starting from the base scenario without fine discounts, we analyze successive increases in the fine discount, $\beta$. Due to the greater incentive to invest in capital, the stock of capital at the steady state increases. A greater capital stock allows the firm to produce with lower emissions and the regulator to implement a more stringent pollution standard at the steady state. Still the degree of non-compliance decreases. The firm is pushed to invest but since the regulator has tightened the standard, the associated investment costs exceed the gains from a better compliance, hence firm's profits are reduced. Correspondingly, the social welfare is increased due to a lower environmental damage together with a lower social cost from non-compliance. Because the social costs associated with non-compliance are softer than the penalty paid by the firm ($h < 1$), the incentive to rise investment linked to a rise in the fine discount is differently perceived by the firm and the regulator. Thus, when the fine discount is already too high, the firm reaction to further increases is still a rise in investments in order to reduce the fine for non-compliance. However, for the regulator, the reduction in the social costs associated with non-compliance, the increase in production and the lower environmental damage might not be enough to counterbalance the rise in the investment costs. Thus, instead of tightening the standard, the regulator would relax the standard making it easier for the firm to comply and therefore, alleviating the incentive to invest in capital. A softer emission standard increases firm's profits, but the slight increase in emissions worsens social welfare.

By comparing the corresponding long-run values under $\beta = 0$ and $\beta = \hat{\beta}$ (where the social optimum is maximum), we observe that long-run emissions, the effective pollution limit and the degree of non-compliance are lower under $\beta = \hat{\beta}$. In fact, these results (and also the fact that the capital stock is larger under $\beta = \hat{\beta}$) also hold for any time period, as shown in Figures 3 and 4.[31]

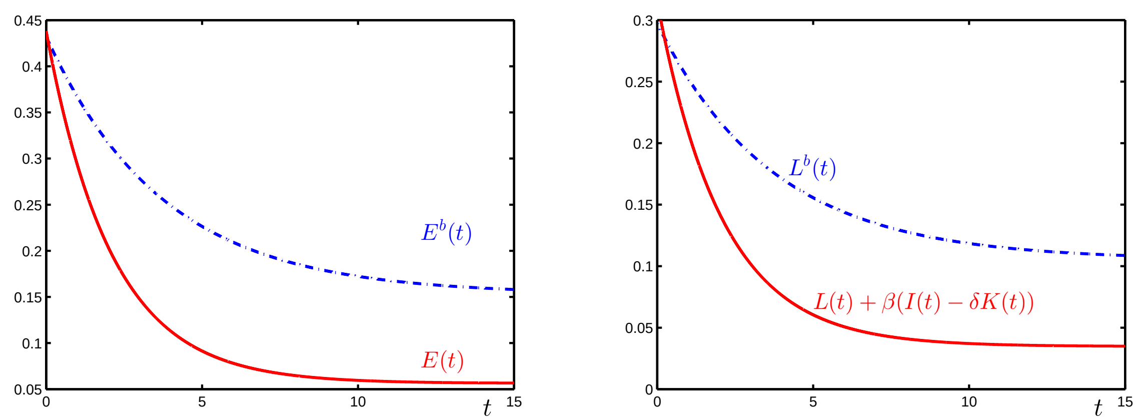


Figure 3: Emissions (left); Effective emission limit (right)

In Claim 1 we have analyzed the effect of the magnitude of the fine assuming that the allowed discount is fixed (which may or may not be monotone). Analogously, Claim 2 describes

[31] We have considered the same parameter values as in (20), and also $f = 2 > d = 1, \beta = \hat{\beta} = 0.807$.

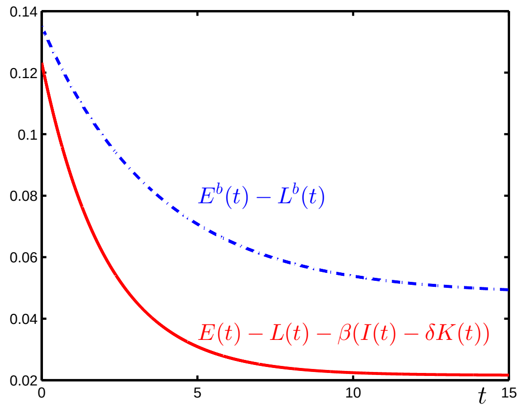


Figure 4: Degree of non-compliance

the effect of the fine discount given the magnitude of the fine. Our next question is whether a combination of parameters values $(f, \beta)$ exists, for which the solution of the Stackelberg game reaches its maximum.

Figure 5 depicts the social welfare level when the initial capital stock is zero as a function of the fine discount, $\beta$, for different values of the magnitude of the fine, $f$. For $f = 1$, the dashed line reaches its maximum at $\hat{\beta} \backsimeq 1.1232$, with $V_{\text{R}}(k_0) = 9.5482$. Increasing the magnitude of the fine to $f = 2$ leads to a new and higher maximum of social welfare at $\hat{\beta} \backsimeq .807$, with $V_{\text{R}}(k_0) = 9.55416$. Further increases in the magnitude of the fine are associated with new $\hat{\beta}$ at which the social welfare is increased. This process continues indefinitely.[32]

This new finding is then summarized in the following:

**Claim 3** *There is no pair of parameters $(f, \beta) \in (0, \infty) \times (0, \infty)$ for which $V_{\text{R}}(k_0)$ reaches a maximum.*

In consequence, the best option for the society would be to penalize as much as possible,

[32]We have numerically computed a rise in the maximum value of $V_{\text{R}}(k_0)$ when switching from $f = 10^{12}$ to $f = 10^{13}$. Interestingly, the value of $\beta$ that maximizes social welfare when the severity of the fine tends to infinity converges to a strictly positive finite value, $\hat{\beta}$.

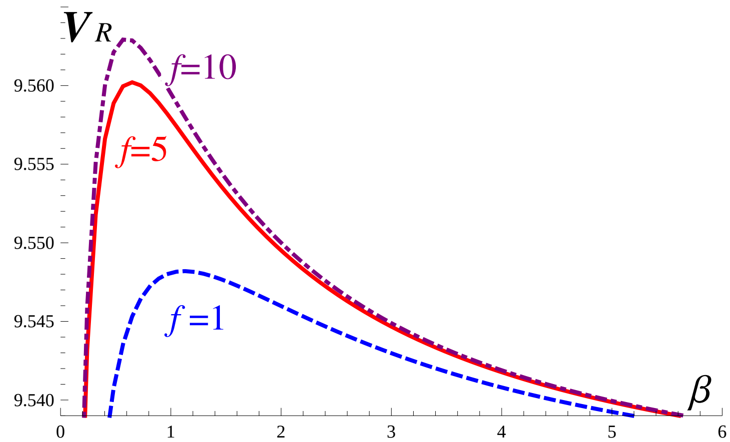


Figure 5: $V_{\mathrm{R}}(k_0)$ w.r.t. $\beta$ for different values of $f$.

and for this magnitude of the fine select the fine discount $\hat{\beta}$ which maximizes social welfare. This limiting case when $f$ tends to infinity is equivalent to a new scenario where *full compliance* with the effective pollution limit is assumed (and therefore, no penalty is levied). In this context, the firm is allowed to exceed the pollution standard set by the regulator in its net investment times the fine discount $\hat{\beta}$. Once the pollution standard is chosen, investment must guarantee full compliance with the effective pollution limit, $\tilde{I}(E, L, K) = \delta K + (E - L)/\hat{\beta}$. The firm maximization problem then reads:

$$\max_{E} \int_0^\infty \left[ Y(K, E) - C(\tilde{I}(E, K, L)) \right] e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = \tilde{I}(E, L, K) - \delta K, \quad K(0) = k_0.$$

Knowing the best-reaction function of the firm, $\hat{E}(L; K)$, the regulator solves the following dynamic optimization problem:

$$\max_{L} \int_0^\infty \left[ Y(K, \hat{E}(K; L)) - C(\tilde{I}(\hat{E}(K; L), K, L)) - D(\hat{E}(K; L)) \right] e^{-\rho t} dt$$
$$\text{s.t.: } \dot{K} = \tilde{I}(\hat{E}(K; L), K, L) - \delta K, \ K(0) = k_0.$$

Denoting the social welfare in this full compliance scenario as $V_{\mathrm{R}}^{\mathrm{FC}}(K)$, for the parameters values considered we obtain the following ranking of social welfare levels:

$$V^{\text{NR}}(k_0) = 7.5 < V_{\text{R}}^{b}(k_0) = 9.3772\,(\text{for } \hat{f}^b) < V_{\text{R}}^{\text{FC}}(k_0) = 9.56632\,(\text{for } \hat{\beta}) < V^{\text{FB}}(k_0) = 9.56702.$$

If we compute the distance to the social welfare in the first-best scenario, we observe that the scenario with no fine discounts ($V_{\text{R}}^{b}(k_0)$) distances $1.98\%$, while the scenario with discounts and full compliance ($V_{\text{R}}^{\text{FC}}(k_0)$) distances $0.0073\%$. Therefore, this limiting scenario results in a social welfare level roughly equal to the first-best level.

# 5 A discussion on the possibility of over-compliance

All the previous analysis in both the base and the extended models is based on the assumption that the fine is quadratic under non-compliance and zero under compliance. Particularly, we have assumed that the marginal fine for an infinitesimal violation is negligible, and this is the reason that results in strict non-compliance along the optimal trajectory. This means that this particular specification for the fine rules out the possibility of full compliance or over-compliance, and this is the reason why considering a quadratic fine under non-compliance and zero fine under compliance gives the same result as considering a quadratic fine for both non-compliance and over-compliance, see footnotes 15 and 27.

In this section, we analyze the possibility of over-compliance under an alternative fine specification that punishes non-compliance but subsidizes over-compliance, as follows[33]:

$$F^{\text{S}}(E, L) = f\left[\frac{(E-L)^2}{2} + \eta(E-L)\right].$$

The introduction of the linear term $\eta(E-L)$ increases the fine in the case of non-compliance ($E > L$), and induces a fine discount (that indeed turns it into a subsidy) for over-compliance

[33]From now on, superscript S refers to the scenario with the subsidy.

($E < L$). Parameter $f$ again measures the severity of the fine in case of non-compliance, and in this new scenario, it also measures the generosity of the subsidy in case of over-compliance, when the full term $(E - L)^2/2 + \eta(E - L)$ is negative. Parameter $\eta$ is also a measure of the magnitude of the subsidy.[34]

The firm maximizes the same problem as in (3), replacing $F(E, L)$ by $F^{\mathrm{s}}(E, L)$. Taking into account the investment and emission reaction functions of this problem, the regulator maximizes a problem identical to (4), where the social costs associated with either non-compliance or over-compliance are given by:

$$H(\hat{E}^{\mathrm{s}}(K; L), L) = hg(f, \eta)\frac{(E - L)^2}{2}.$$

In the event of non-compliance, this function represents the administrative costs associated with collecting fines previously described in the model section, which are increasing in the level of non-compliance. However, in the event of over-compliance, this function also accounts for the social opportunity costs associated with giving subsidies, which are increasing in the level of over-compliance.[35]

Given the lack of evidence on the comparison between the social costs of imposing fines and the social costs of giving subsidies, we treat them in a symmetric way, and we assume that these costs depend on the distance of the pollution level to the emissions standard (since the larger the gap, the larger the fine or subsidy, everything else equal). Also, we assume these costs to be increasing in the intensity of the fine/subsidy, $f$, and also non-decreasing in the subsidizing

[34]It represents the level of over-compliance at which the subsidy reaches its maximum (further emissions reductions will decrease the total subsidy).

[35]In many instances, raising funds for subsidies requires exacerbating tax distortions elsewhere in the economy, imposing a loss on society in the range of \$0.20 – \$0.30 for every \$1 raised, see, for example, Laffont and Tirole (1996a, 1996b) and Ruggeri (1999).

parameter, $\eta$. Hence, function $g$ is assumed to satisfy $g_f > 0$, $g_\eta \geq 0$. Finally, for comparison purposes, we assume $g(f, 0) = f$. Thus, the case $\eta = 0$ collapses into the base model with no subsidy analyzed in Section 3.

The optimal level of non-compliance for this problem is given by:

$$N^{\mathrm{s}}(K, \eta) = E^{\mathrm{s}}(K) - L^{\mathrm{s}}(K) = \frac{d\sigma(1-K) - \eta f(d+\sigma^2)}{\hat{\Psi}}, \tag{26}$$

where $\hat{\Psi} = f(d + \sigma^2) + h\sigma^4 g/f$. In consequence, for a given value of $K$ at a given time $t$, over-compliance results if and only if:

$$N^{\mathrm{s}}(K, \eta) < 0 \Leftrightarrow \eta > \frac{d}{f}\frac{\sigma(1-K)}{d+\sigma^2} \equiv \frac{d}{f}E^{\mathrm{FB}}(K).$$

According to this condition, over-compliance is linked to parameter $\eta$ being larger than a specific bound. This bound however, decreases as the capital stock grows towards its long run value, giving rise to several possibilities. If $\eta$ is greater than this lower bound for the initial capital stock $k_0$, the firm over-complies along the whole planning period. If $\eta$ is lower than the lower bound at the limit (when the capital stock reaches its maximum), non-compliance is the recurrent behaviour of the firm. Finally, a situation of non-compliance in the beginning and over-compliance henceforth is also possible.

To characterize the different scenarios, we specify a functional form for $g$. For the ease of presentation we consider $g(f, \eta) = f + \xi\eta$, with $\xi \geq 0$, and we study two cases. First, we analytically study the case $\xi = 0$ in which the social costs are not dependent on parameter $\eta$. Second, we numerically analyze the case $\xi = 1$. As we will see later on, results are qualitatively similar in both cases.

The new fine, which penalizes further non-compliance and subsidizes over-compliance, gives the firm an incentive to reduce emissions. To partially counterbalance this emission reduction, the regulator correspondingly fixes a laxer standard. The following proposition characterizes the conditions under which this double effect results in over-compliance for the particular case $\xi = 0$. The proof is in the Appendix.

**Proposition 5** *When $g(f,\eta) = f$, the compliance behaviour of the firm crucially depends on parameter $\eta$ as follows:*

1. *Non-compliance at all times, if $\eta \leq \tilde{\eta}$;*
2. *Non-compliance followed by over-compliance, if $\eta \in (\tilde{\eta}, \eta^0)$;*
3. *Full-compliance exclusively at the starting date followed by over-compliance, if $\eta = \eta^0$;*
4. *Always over-compliance, if $\eta \in (\eta^0, \bar{\eta}]$,*

*with $\tilde{\eta} < \min\{\eta^0, \bar{\eta}\}$, and $\tilde{\eta}$, $\eta^0$ and $\bar{\eta}$ given in the Appendix. For parameters values satisfying $\eta^0 > \bar{\eta}$, cases 3 and 4 would not be feasible, and case 2 would hold for $\eta \in (\tilde{\eta}, \bar{\eta})$.*

As a result, over-compliance arises as long as the subsidizing parameter $\eta$ is large enough. Furthermore, in this case with $g(f,\eta) = f$, it is easy to prove that over-compliance is associated with emissions below the first best, $E^{\text{S}}(K) < E^{\text{FB}}(K) < L^{\text{S}}(K)$, while non-compliance leads to emissions above the first-best, $L^{\text{S}}(K) < E^{\text{FB}}(K) < E^{\text{S}}(K)$. Obviously full-compliance would imply $L^{\text{S}}(K) = E^{\text{FB}}(K) = E^{\text{S}}(K)$.

We now turn to the case where $g(f,\eta) = f + \eta$. Here, we carry out a numerical analysis considering parameters values already used in previous numerical simulations, see (20), and we also assume $f = 5$. For these values, we obtain $\tilde{\eta} = 0.026 < \eta^0 = 0.08 < \bar{\eta} = 0.188$. For $\eta \in (0, \tilde{\eta})$ the firm does never comply. For a higher $\eta$ in the interval $(\tilde{\eta}, \eta^0)$ the firm emits above the limit within a first period, but over-complies henceforth. By contrast, for $\eta \in (\eta^0, \bar{\eta}]$ the firm always over-complies. Figure 6 shows that for the cases $\eta = 0$, the base model with no subsidy, and $\eta = 0.02 < \tilde{\eta}$, non-compliance is observed at all time. For $\eta = 0.06 \in (\tilde{\eta}, \eta^0)$ the firm does not comply within a first period and over-complies from a given time on. Finally, for $\eta = 0.1 > \eta^0$ over-compliance holds at any time.

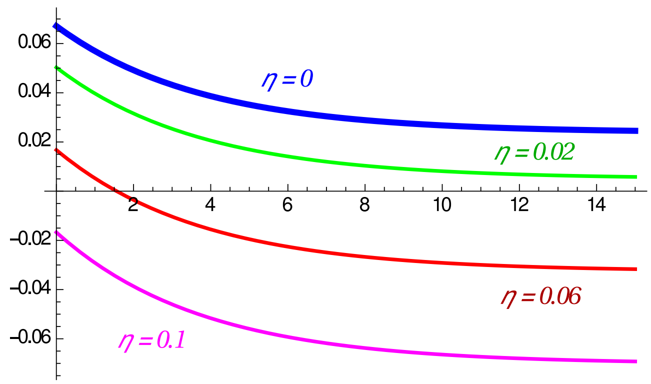


Figure 6: Non-compliance for different $\eta$.

From the definition of non-compliance in (26) and regardless of whether $\xi = 0$ or $\xi = 1$, the degree of non-compliance decreases with $\eta$. In fact, Proposition 5 and also our numerical simulation show that over-compliance can emerge for a sufficiently large $\eta$.

Regardless of the value of $\eta$ and the consideration of $\xi$ being zero or positive, so far we have seen that the possibility to subsidize over-compliance induces emissions reductions and laxer emission standards. An interesting question now is to see if this necessarily leads to emissions closer to the standard, which may result in lower social costs of administering sanctions/ subsidies, as compared to the models without subsidies. Or put differently, is this new enforcement mechanism that considers the possibility of subsidizing over-compliance more effective in inducing emission levels closer to the standard? We will prove (again analytically for $\xi = 0$ and numerically for $\xi = 1$) that although this is generally true, this effectiveness worsens if parameter $\eta$ is very large.

We define the gap between emissions and the emission standard as $|N^{\mathrm{s}}(K,\eta)| = |E^{\mathrm{s}}(K) - L^{\mathrm{s}}(K)|$. Therefore, the effectiveness of the emission limit when a subsidy for over-compliance exists ($\eta > 0$) is greater than in the case with no subsidy ($\eta = 0$) if and only if[36] $m(K^b(t), K^{\mathrm{s}}(t), \eta) = N^{\mathrm{s}}(K^b(t), 0) - |N^{\mathrm{s}}(K^{\mathrm{s}}(t), \eta)|$ is positive.

**Proposition 6** *When $g(f,\eta) = f$, a subsidy for over-compliance might increase or decrease the effectiveness of the enforcement mechanism, depending on $\eta$, as follows:*

1. *Higher effectiveness at every time if $\eta \leq 2\tilde{\eta}$.*
2. *Higher effectiveness followed by lower effectiveness if $\eta \in (2\tilde{\eta}, 2\eta^0)$.*
3. *Lower effectiveness at every time if $\eta \in [2\eta^0, \bar{\eta})$.*

*For parameters values satisfying $\bar{\eta} < 2\eta^0$, case 3 never applies and case 2 would hold for $\eta \in (2\tilde{\eta}, \bar{\eta})$. If further $\bar{\eta} < 2\tilde{\eta}$ the subsidy always increases effectiveness for any $\eta \leq \bar{\eta}$.*

According to this proposition, incorporating a subsidy for over-compliance would increase effectiveness if parameter $\eta$ is small. As this parameter increases above certain value, the gains in effectiveness only last for a first period, while effectiveness worsens from a certain time on. In fact, for sufficiently large $\eta$, it follows that the subsidy implies a loss of effectiveness through the whole period. We have also observed this same behaviour numerically, for the more general case where $g(f,\eta) = f + \eta$. This is shown in Figure 7, which displays the absolute value of non-compliance for the base case (thick solid line) and different values for

[36]The solution without subsidy is characterized by non-compliance and hence $|N^{\mathrm{s}}(K,0)| = N^{\mathrm{s}}(K,0)$ for any $K \leq \bar{K}^b$.

$\eta$. For small values of $\eta < 2\tilde{\eta}$ considering a subsidy is more effective at any time, either with non-compliance at any time $\eta = 0.025$ (dotted line), or with non-compliance followed by over-compliance $\eta = 0.04$ (dashed line). For a higher $\eta = 0.07 \in (2\tilde{\eta}, 2\eta^0)$ (solid line) the subsidy increases effectiveness within a first period and reduces effectiveness henceforth. Finally, for a sufficiently large $\eta = 0.17 > 2\eta^0$ incorporating the subsidy always implies a lower effectiveness.

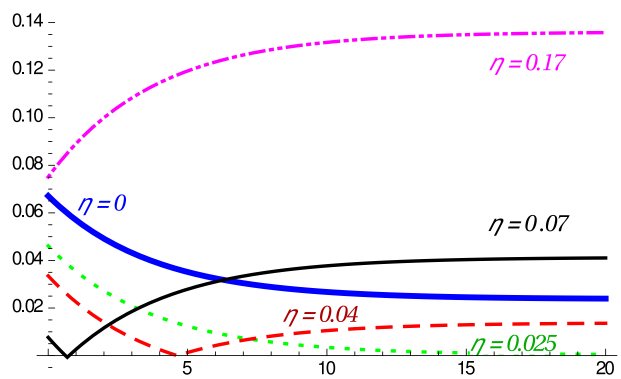


Figure 7: Absolute value of non-compliance for different $\eta$.

# 6 Conclusions

Our paper constitutes a first attempt in the literature to address the determination of optimal pollution standards overtime in a context of imperfect compliance. Considering a representative firm which uses emissions as an input to produce a consumption good and invests to increase a capital stock, we have assumed that the firm can decide to exceed the pollution limit set by the regulator at any time, and then be subject to a fine if it finds it profitable to do so. In the base model, we have considered a fine quadratic in the degree of non-compliance, defined as the difference between firm's emissions and the pollution limit set by the regulator. In the extended model, we have introduced the concept of effective pollution limit, and we have redefined the fine as being quadratic in the difference between emissions and the effective pollution limit, which is laxer than the pollution limit set by the regulator if the firm invests to increase the capital stock.

We have shown that the optimal time-path for the effective pollution limit gradually decreases while the capital stock increases, resulting in a decrease in emissions. Interestingly, while the firm faces a more stringent pollution limit, however the degree of non-compliance decreases. This result is valid in both the base and the extended models. While this result cannot be found in static models of regulatory enforcement typically considered in the literature (in those settings, a tighter standard always results in a larger gap between emissions and the standard), however in a dynamic context the firm can progressively adapt to more stringent regulations by means of capital investment. In fact, a fine dependent on emissions in excess of the effective pollution limit helps the firm to decrease the level of non-compliance even more, since the firm is induced to invest in capital accumulation even more than without discounts.

We have analyzed the question of how severe the penalty for non-compliance should be. To that aim, we have performed a comparative statics analysis regarding the severity of the fine. In the base model where the fine depends on emissions in excess of the pollution limit set by the regulator, we have generally found non-monotone relationships between the main variables and the severity of the fine, *ceteris paribus*. In general, the optimal pollution limit decreases with the severity of the fine for rather low values of the fine, but it always increases for large values of the fine. This means that the pollution limit and the fine act as substitute instruments provided the level of the fine exceeds a certain threshold, whereas they might be substitute or complementary instruments when the level of the fine is below this threshold. The induced emissions and capital stock are, respectively, U-shaped and inverse U-shaped in the severity of the fine and, finally, the degree of non-compliance decreases with the severity of the fine. All this results in a first decreasing and then increasing relationship between firm's profits and the severity of the fine, and also in a first increasing and then decreasing relationship between social welfare and the severity of the fine. The latter implies that there always exists a bounded value of the severity of the fine that maximizes social welfare.

In the extended model where fine discounts are allowed, we are able to affirmatively answer our second research question: *Should the fine be reduced in exchange for firm's capital investments?* Moreover, for a given severity of the fine, we can always find the finite value of the fine discount at which social welfare attains its maximum. The first question on the severity of the fine can also be studied in the extended model. We have numerically found the same sort of non-monotonicity only when the fine discount is small. If, to the contrary, the fine discount is large, social welfare increases monotonically with the severity of the fine. Finally, if we compute the fine discount which maximizes social welfare for successive values of the severity of the fine, we observe that the value of social welfare increases with the severity of the fine at the social maximizing fine discount. A limiting scenario with a very large severity of the fine would be equivalent to an alternative scenario of full compliance with the effective pollution limit, since the firm is threatened by a very large punishment if it pollutes above that limit. Emissions are still above the pollution limit imposed by the regulator, but no penalties are levied. This scenario roughly mimics the first-best scenario in terms of social welfare levels.

All these results have been derived assuming that the marginal fine for an infinitesimal violation is negligible. This has resulted in strict non-compliance along the optimal trajectory and, therefore, the possibility of over-compliance has been ruled out. We have then modified the penalty to incorporate a discount in the fine if the firm decides to pollute below the standard, which finally results in a net subsidy for over-compliance. Taking into account that both administering fines and giving subsidies may be costly for society, we have found that over-compliance can arise as long as the discount for over-compliance is large enough, although this does not necessarily result in lower social administration costs.

Several extensions of our model are possible, such as including monitoring issues, cumulative pollution effects, or imperfect information on the part of the regulator and/or the polluting firm. All these issues deserve further research.

# 7 Appendix

## Proof of Proposition 1.

This proof characterizes the stagewise feedback Stackelberg equilibrium strategies (see, for example Haurie *et al.* (2012), page 278). Under the assumption of stationary strategies, it involves the resolution of two Hamilton-Jacobi-Bellman equations:

$$\begin{aligned}
\rho V_{\mathrm{F}}^{b}(K) &= Y(K,\phi^{\mathrm{E}}(K)) - C(\phi^{\mathrm{I}}(K)) - f\frac{(\phi^{\mathrm{E}}(K)-\phi^{\mathrm{L}}(K))^2}{2} + (V_{\mathrm{F}}^{b})'(K)(\phi^{\mathrm{I}}(K)-\delta K),\\
\rho V_{\mathrm{R}}^{b}(K) &= Y(K,\phi^{\mathrm{E}}(K)) - C(\phi^{\mathrm{I}}(K)) - D(\phi^{\mathrm{E}}(K)) - hf\frac{(\phi^{\mathrm{E}}(K)-\phi^{\mathrm{L}}(K))^2}{2}\\
&+ (V_{\mathrm{R}}^{b})'(K)(\phi^{\mathrm{I}}(K)-\delta K),
\end{aligned}$$

where $(\boldsymbol{\phi}_{\mathrm{F}}(K), \phi_{\mathrm{R}}(K))$ is a feedback strategy pair, with $\boldsymbol{\phi}_{\mathrm{F}}(K) = (\phi^{\mathrm{E}}(K), \phi^{\mathrm{I}}(K))$ and $\phi_{\mathrm{R}}(K) = \phi^{\mathrm{L}}(K)$. To obtain this pair, we first compute the best reaction functions of the firm as the solution to:

$$\max_{I,E}\left\{Y(K,E) - C(I) - f\frac{(E-\phi^{\mathrm{L}}(K))^2}{2} + (V_{\mathrm{F}}^{b})'(K)(I-\delta K)\right\}.$$

In view of the linear-quadratic structure of the problem we conjecture a quadratic value function for the firm in $K$:

$$V_{\mathrm{F}}^{b}(K) = \frac{a_{\mathrm{F}}^{b}}{2}K^{2} + b_{\mathrm{F}}^{b}K + c_{\mathrm{F}}^{b}. \tag{27}$$

Therefore, the best-response functions of the firm are then given by:

$$\hat{E}^{b}(K;\phi^{\mathrm{L}}(K)) = \underline{E}^{b} + E^{\mathrm{L}b}\phi^{\mathrm{L}}(K) + E^{\mathrm{K}b}K, \qquad \hat{I}^{b}(K;\phi^{\mathrm{L}}(K)) = \hat{I}^{b}(K) = \underline{I}^{b} + I^{\mathrm{L}b}\phi^{\mathrm{L}}(K) + I^{\mathrm{K}b}K, \tag{28}$$

$$\underline{E}^{b} = \frac{\sigma}{\sigma^{2}+f}, \quad E^{\mathrm{L}b} = \frac{f}{\sigma}\underline{E}^{b}, \quad E^{\mathrm{K}b} = -\underline{E}^{b}, \qquad \underline{I}^{b} = \frac{b_{\mathrm{F}}^{b}}{c}, \quad I^{\mathrm{L}b} = 0, \quad I^{\mathrm{K}b} = \frac{a_{\mathrm{F}}^{b}}{c}.$$

Knowing the best-reaction functions of the firm presented in $(28)$, now the regulator fixes the pollution limit, to maximize:

$$\max_{L}\left\{Y(K,\hat{E}^{b}(K;L)) - C(\hat{I}^{b}(K;L)) - D(\hat{E}^{b}(K;L)) - hf\frac{[\hat{E}^{b}(K;L)-L]^{2}}{2}\right.$$
$$\left. +(V_{\mathrm{R}}^{b})'(K)(\hat{I}^{b}(K;L)-\delta K)\right\}.$$

Notice, however that $I^{\mathrm{L}b} = 0$ and therefore, $\hat{I}^{b}(K;L)$ does not actually depend on $L$. The regulator cannot directly influence investment, and hence behaves as a static player. His optimal strategy is then to set an emission limit which satisfies:

$$Y_{E}(K,\hat{E}^{b}(K;L)) = D'(\hat{E}^{b}(K;L)) + hf[\hat{E}^{b}(K;L)-L]\frac{E^{\mathrm{L}b}-1}{E^{\mathrm{L}b}}.$$

From this condition, the optimal emission limit is the affine function of the stock of capital, $L^{*b}(K)$, given in (5). As long as $h<1$, it follows that $\left(L_{0}^{*b}\right)_{f} > 0$.

From the optimal emission limit in (5) and the best-response functions of the firm, the optimal emissions and investment are given by the expressions in (6) and (7). And it is easy to see that $\left(E_{0}^{*b}\right)_{f} < 0$ for all $f$. Further, from the Riccati equations the coefficients which define the firm's value function, $V_{\mathrm{F}}^{b}(K)$, defined in (27) can now be computed and are given by (8) and:

$$c_{\mathrm{F}}^{b}=\frac{(b_{\mathrm{F}}^{b})^2+c}{2c\rho}-\frac{d^2 f(f+\sigma^2)}{2\rho\Psi^2}>0. \tag{29}$$

Once the optimal investment strategy is known, integrating the differential equation which defines the capital stock dynamics, the optimal time path of the capital stock is given by (10).

## Proof of Proposition 2

We first characterize the optimal strategies for the case of no regulation and the first-best scenario, and then we compare these two cases with the stagewise feedback Stackelberg equilibrium characterized in Proposition 1.

First, with no regulator the Hamilton-Jacobi-Bellman equation for problem $(17)$ is:

$$\rho V_{\mathrm{F}}^{\mathrm{NR}}(K)=\max_{I,E}\left\{K+\sigma E-\frac{(K+\sigma E)^2}{2}-c\frac{I^2}{2}+(V_{\mathrm{F}}^{\mathrm{NR}})'(K)(I-\delta K)\right\}.$$

We again conjecture a quadratic value function in $K$, denoted by $V_{\mathrm{F}}^{\mathrm{NR}}(K)$:

$$V_{\mathrm{F}}^{\mathrm{NR}}(K)=\frac{a_{\mathrm{F}}^{\mathrm{NR}}}{2}K^2+b_{\mathrm{F}}^{\mathrm{NR}}K+c_{\mathrm{F}}^{\mathrm{NR}},$$

and $a_{\mathrm{F}}^{\mathrm{NR}}, b_{\mathrm{F}}^{\mathrm{NR}}, c_{\mathrm{F}}^{\mathrm{NR}}$ are unknowns to be determined. The only feasible solution for these unknowns associated with a convergent time path of the capital stock is: $a_{\mathrm{F}}^{\mathrm{NR}}=b_{\mathrm{F}}^{\mathrm{NR}}=0, c_{\mathrm{F}}^{\mathrm{NR}}=1/(2\rho)$. Therefore, the solution under no regulation is the following:

$$I^{*\mathrm{NR}}=0, \qquad E^{*\mathrm{NR}}(K)=\frac{1}{\sigma}(1-K), \quad \text{and} \quad K^{\mathrm{NR}}(t)=k_0e^{-\delta t}, \tag{30}$$

where the optimal time path of capital is obtained by solving (2) once the optimal investment strategy has been replaced.

In the particular case $k_0=0$, then $K(t)=0$, $E(t)=1/\sigma$ for any $t\geq 0$ and the social welfare is $V^{\mathrm{NR}}(0)=1/(2\rho)-d/(2\rho\sigma^2)$.

The first-best solution to problem $(18)$ can be found by solving the following Hamilton-Jacobi-Bellman equation:

$$\rho V^{\text{FB}}(K) = \max_{I,E} \left\{ (K+\sigma E) - \frac{(K+\sigma E)^2}{2} - c\frac{I^2}{2} - d\frac{E^2}{2} + (V^{\text{FB}})'(K)(I-\delta K) \right\},$$

where $V^{\text{FB}}(K)$ denotes the value function of the problem. Following the same procedure as above, we then easily obtain the following optimal strategies:

$$I^{*\text{FB}}(K) = \frac{b^{\text{FB}}}{c} + \frac{a^{\text{FB}}}{c}K, \qquad E^{*\text{FB}}(K) = \frac{\sigma}{d+\sigma^2}(1-K), \tag{31}$$

where $V^{\text{FB}}(K) = a^{\text{FB}}K^2/2 + b^{\text{FB}}K + c^{\text{FB}}$, with

$$a^{\text{FB}} = \frac{c(\rho+2\delta)-\sqrt{\Delta^{\text{FB}}}}{2} < 0, \qquad b^{\text{FB}} = \frac{2cd}{(d+\sigma^2)\left[c\rho + \sqrt{\Delta^{\text{FB}}}\right]} > 0,$$
$$c^{\text{FB}} = \frac{(b^{\text{FB}})^2}{2c\rho} + \frac{\sigma^2}{2\rho(d+\sigma^2)} > 0, \quad \Delta^{\text{FB}} = c^2(\rho+2\delta)^2 + \frac{4cd}{d+\sigma^2},$$

and the optimal time-path of the capital stock is:

$$K^{\text{FB}}(t) = (k_0 - \bar{K}^{\text{FB}})e^{\theta^{\text{FB}}t} + \bar{K}^{\text{FB}}, \tag{32}$$

$$\bar{K}^{\text{FB}} = \frac{d}{d+c(d+\sigma^2)\delta(\delta+\rho)} > 0, \quad \theta^{\text{FB}} = \frac{c\rho-\sqrt{\Delta^{\text{FB}}}}{2c} < 0,$$

where $\bar{K}^{\text{FB}}$ is the long-run value of the capital stock, and $|\theta^{\text{FB}}|$ is the speed of convergence towards this value.

The corresponding long-run values of emissions and investment read:

$$\bar{E}^{\text{FB}} = \frac{\sigma}{d+\sigma^2}\left(1-\bar{K}^{\text{FB}}\right) = \frac{c\delta(\delta+\rho)\sigma}{d+c(d+\sigma^2)\delta(\delta+\rho)}, \qquad \bar{I}^{\text{FB}} = \delta\bar{K}^{\text{FB}} = \frac{d\delta}{d+c(d+\sigma^2)\delta(\delta+\rho)}.$$

Finally, we compare the optimal time paths under no regulation, the first best and the base model.

(i) Since $f \geq d$, it follows that $\bar{K}^{\text{FB}} > \bar{K}^b$ and both are positive. Moreover, under this condition it is also true that $\Delta^{\text{FB}} > \Delta^b > 0$, hence inequality $\theta^{\text{FB}} < \theta^b$ follows. It is immediate to verify that $\theta^b < -\delta$. Then $K^{\text{FB}}(t) > K^b(t) > K^{\text{NR}}(t)$ for all $t > 0$ follows.

(ii) $\bar{I}^{\text{FB}} > \bar{I}^b > 0$ follows straightforwardly from part (i). Since $f \geq d$, then $a^{\text{FB}} < a^b_{\text{F}} < 0$ immediately follows from $\Delta^{\text{FB}} > \Delta^b$. Moreover, proving $b^{\text{FB}} > b^b_{\text{F}}$ is equivalent to proving:

$$\frac{\Delta^{\text{FB}} - c^2(\rho + 2\delta)^2}{2cd(c\rho + \sqrt{\Delta^{\text{FB}}})} > \frac{\Delta^b - c^2(\rho + 2\delta)^2}{2cd(c\rho + \sqrt{\Delta^b})}.$$

Provided that $f(x) = (x - c^2(\rho+2\delta)^2)/(c\rho + \sqrt{x})$ is an increasing function and $\Delta^{\text{FB}} > \Delta^b$, $b^{\text{FB}} > b^b_{\text{F}}$ follows. Thus, $I^{*\text{FB}}(0) = b^{\text{FB}}/c > b^b_{\text{F}}/c = I^{*b}(0) > 0$. However, nothing can be said of the comparison between $I^b(t)$ and $I^{\text{FB}}(t)$ for $t > 0$.

(iii) The results immediately follow from part (i), the expressions of $E^{*\text{NR}}(K)$, $E^{*\text{FB}}(K)$, $E^{*b}(K)$, and inequalities:

$$\frac{1}{\sigma} > \frac{\sigma(f + h\sigma^2)}{\Psi} > \frac{\sigma}{d + \sigma^2}.$$

## Proof of Proposition 3

i) From (8), (29) and (11) it follows that sign$|\theta^b|_f$ =sign$(\bar{K}^b)_f$ =sign$(\bar{I}^b)_f$ is given by:

$$\text{sign}(h\sigma^2(2f + \sigma^2) - (d + \sigma^2)f) = \text{sign}(2h\sigma^2(f + \sigma^2) - \Psi). \quad (33)$$

Here and henceforth we will be assuming $d + \sigma^2(1 - 2h) > 0$, i.e. $h < (d + \sigma^2)/(2\sigma^2) = h_{\max}$. Under this condition, the sign in (33) is positive for $f \in [d, \hat{f}^b_K)$, and negative for $f > \hat{f}^b_K$, with $\hat{f}^b_K$ given in (19).[37]

[37]If $h > (d + \sigma^2)/(2\sigma^2) = h_{\max}$ the sign in (33) would be positive for all $f \geq 0$.

The effect on emissions is obtained from (13),

$$(\bar{E}^b)_f = -\frac{dh\sigma^3}{\Psi^2}\left(1-\bar{K}^b\right) - \frac{\sigma(f+h\sigma^2)}{\Psi}(\bar{K}^b)_f.$$

In consequence, if $(\bar{K}^b)_f > 0$ then $(\bar{E}^b)_f < 0$, implying $\hat{f}_E^b > \hat{f}_K^b > 0$ for all $h > 0$. To prove that $\hat{f}_E^b$ is finite it is sufficient to see than $(\bar{E}^b)_f$ is positive for sufficiently large values of $f$: $\lim_{f\to\infty}(\bar{E}^b)_f = \infty$.

ii) The effect of $f$ on the emission limits can be computed from (12):

$$(\bar{L}^b)_f = \frac{d\sigma(d+(1-h)\sigma^2)}{\Psi^2}\left(1-\bar{K}^b\right) - \frac{\sigma(-d+f+h\sigma^2)}{\Psi}(\bar{K}^b)_f, \tag{34}$$

where $1-\bar{K}^b$ can be written as a function of $(\bar{K}^b)_f$, and then:

$$(\bar{L}^b)_f = \left\{\frac{[d+(1-h)\sigma^2][d^2 f(f+\sigma^2)+c\delta(\delta+\rho)\Psi^2]}{\Psi d\sigma(2h\sigma^2(f+\sigma^2)-\Psi)} - \frac{\sigma(f+h\sigma^2-d)}{\Psi}\right\}(\bar{K}^b)_f.$$

We know that $(\bar{K}^b)_f < 0$ if and only if $f > \hat{f}_K^b$, i.e. $2h\sigma^2(f+\sigma^2)-\Psi < 0$. Then, a sufficient condition for the expression in brackets to be negative is $f + h\sigma^2 - d > 0$. In consequence, if $\hat{f}_K^b > d$, then $(\bar{K}^b)_f < 0$ implies $(\bar{L}^b)_f > 0$ and therefore $\hat{f}_L^b < \hat{f}_K^b$. It is easy to prove that

$$\hat{f}_K^b > d \Leftrightarrow h > \frac{2d}{2d+\sigma^2}\frac{d+\sigma^2}{2\sigma^2} \equiv h_{min}.$$

Thus $(\bar{L}^b)_f > 0$ for all $f > \hat{f}_L^b$. To study the sign of $(\bar{L}^b)_f$ for $f \in [0, \hat{f}_L^b)$, expression (34) can be re-written as an expression which sign is given by a second-order convex polynomial in $f$. Depending on whether this polynomial has no real roots or it has two roots with different or equal sign, the different behaviours in footnote 23 appear.

iii) The derivative $(\bar{E}^b - \bar{L}^b)_f$ can be computed and proved negative for any positive $h$.

# Proof of Proposition 4

For the dynamic maximization problem for the firm described in (3), assuming a quadratic value function for the firm, $V_{\text{F}}(K)$ in (24), and following the same reasoning as in the proof of Proposition 1, the best-response functions of the firm are given by:

$$\hat{E}(K;\zeta^{\text{L}}(K)) = \underline{E} + E^{\text{L}}\zeta^{\text{L}}(K) + E^{\text{K}}K, \qquad \hat{I}(K;\zeta^{\text{L}}(K)) = \underline{I} + I^{\text{L}}\zeta^{\text{L}}(K) + I^{\text{K}}K, \tag{35}$$

where $(\boldsymbol{\zeta}_{\text{F}}(K), \zeta_{\text{R}}(K))$ is a feedback strategy pair, with $\boldsymbol{\zeta}_{\text{F}}(K) = (\zeta^{\text{E}}(K), \zeta^{\text{I}}(K))$ and $\zeta_{\text{R}}(K) = \zeta^{\text{L}}(K)$ and:

$$\begin{aligned}
&E^{\text{L}} = \frac{cf}{\Omega}, \quad \underline{E} = \frac{\sigma}{f}E^{\text{L}} + \frac{f\beta(b_{\text{F}} + \beta\sigma)}{\Omega}, \quad E^{\text{K}} = \frac{E^{\text{L}} - 1}{\sigma} + \frac{f\beta(a_{\text{F}} - c\delta)}{\Omega},\\
&I^{\text{L}} = -\frac{f\beta\sigma^2}{\Omega}, \quad \underline{I} = -\frac{I^{\text{L}}}{\sigma} + \frac{b_{\text{F}}(f + \sigma^2)}{\Omega}, \quad I^{\text{K}} = \frac{I^{\text{L}}(1 - \beta\delta\sigma)}{\sigma} + \frac{a_{\text{F}}(f + \sigma^2)}{\Omega},
\end{aligned}$$

with $\Omega = cf + (c + f\beta^2)\sigma^2$, $E^{\text{L}} \in (0,1)$, and $I^{\text{L}} < 0$. This last inequality is the main difference with the base model with $\beta = 0$ where $I^{\text{L}b} = 0$ (see (28)). Therefore, the regulator, taking into account the best-response functions in (35), now behaves as a dynamic player. It fixes the pollution limit in order to maximize:

$$\max_L \Bigg\{ Y(K, \hat{E}(K;L)) - C(\hat{I}(K;L)) - D(\hat{E}(K;L)) - hf\frac{[\hat{E}(K;L) - (L + \beta(\hat{I}(K;L) - \delta K))]^2}{2} + (V_{\text{R}}^b)'(K)(\hat{I}(K;L) - \delta K) \Bigg\}.$$

Assuming a quadratic value function for the regulator, $V_{\text{R}}(K)$ in (24), the optimal emission standard strategy, $L^*(K)$ in (23) follows. The optimal emissions and investment strategies, $E^*(K)$ and $I^*(K)$ in (23) immediately follow from the best-response functions in (35) once $L^*(K)$ is known.

From the optimal investment strategy in (23) and the capital stock dynamics in (2), the optimal time path of the capital stock can be written as in (25).

## Proof of Proposition 5

For $g(f,\eta) = f$, the optimal strategies of the problem in Section 5 are:

$$E^{\mathrm{s}}(K) = \sigma\frac{(f+h\sigma^2)(1-K) - \eta f h\sigma}{\Psi}, \quad I^{\mathrm{s}}(K) = \frac{(V_{\mathrm{F}}^{\mathrm{s}})'(K)}{c}, \tag{36}$$

$$L^{\mathrm{s}}(K) = \frac{(f-d+h\sigma^2)(1-K) + \eta f(d+(1-h)\sigma^2)}{\Psi}, \tag{37}$$

with $V_{\mathrm{F}}^{\mathrm{s}}(K)$ being the value function of the firm when a subsidy for over-compliance exists. The optimal time path of the capital stock reads:

$$K^{\mathrm{s}}(t) = (k_0 - \bar{K}^{\mathrm{s}})e^{\theta^b t} + \bar{K}^{\mathrm{s}}, \quad \bar{K}^{\mathrm{s}} = \frac{df(f+\sigma^2)(d+h\eta\sigma^3)}{d^2 f(f+\sigma^2) + 4c\delta(\rho+\delta)\Psi^2} > \bar{K}^b. \tag{38}$$

The capital stock grows higher than in the case without a subsidy, and the speed of convergence is identical, $\theta^b$. Assuming the same initial value, $k_0$, it follows that $K^b(t) < K^{\mathrm{s}}(t) < \bar{K}^{\mathrm{s}}$ for all $t > 0$.

From (36), an upper bound for $\eta$ ensures non-negative emissions. This bound is minimum for the long-run capital stock. Hence, a necessary and sufficient condition which guarantees positive emissions along the whole planning horizon is:

$$\eta \leq \bar{\eta} \equiv \frac{c\delta(\rho+\delta)(f+h\sigma^2)\Psi}{fh\sigma[d(f+\sigma^2) + c\delta(\rho+\delta)\Psi]},$$

with $\bar{\eta}$ being the value of $\eta$ for which emissions are null at the steady state. This condition also ensures that $\bar{K}^{\mathrm{s}} < 1$.

Focusing on the starting date, we define $\eta^0$ as the value of $\eta$ for which full compliance initially holds, that is, $N(k_0, \eta^0) = 0$:

$$\eta^0 = \frac{d\sigma(1-k_0)}{f(d+\sigma^2)}.$$

Since $N_\eta^{\mathrm{s}} < 0$, then the firm initially over-complies if $\eta > \eta^0$ and vice versa. Furthermore, since $N_K^{\mathrm{s}} < 0$ and $K$ grows with time, then if the firm initially over-complies, $\eta > \eta^0$, it will over-comply henceforth.

Focusing on the steady state, $\bar{K}^{\mathrm{s}}(\eta)$, we define $\tilde{\eta}$ as the value of $\eta$ for which full compliance holds, that is, $N(\bar{K}^{\mathrm{s}}(\tilde{\eta}), \tilde{\eta}) = 0$:

$$\tilde{\eta} = \frac{dc\delta(\rho+\delta)\sigma\Psi}{f[d^2(f+\sigma^2) + c\delta(\rho+\delta)(\sigma^2+d)\Psi]} < \eta^0.$$

Since $dN^{\mathrm{s}}/d\eta < 0$, then at the steady state the firm over-complies if $\eta > \tilde{\eta}$ and vice versa. Further, since $N^{\mathrm{s}}_K < 0$ and $K$ grows with time, then if the firm ends-up with non-compliance, $\eta \leq \tilde{\eta}$, the firm does not comply along the whole period.

From the behaviour at the initial time and at the steady state, and the fact that $\tilde{\eta} < \min\{\eta^0, \bar{\eta}\}$, the results in Proposition 5 follow.

## Proof of Proposition 6

This proof distinguishes two cases depending on the value of $\eta$ :

a) $\eta \leq \tilde{\eta}$ (Non-compliance for all $t \geq 0$)

$$m(K^b(t), K^{\mathrm{s}}(t), \eta) = N^{\mathrm{s}}(K^b(t), 0) - N^{\mathrm{s}}(K^{\mathrm{s}}(t), \eta) = \frac{d\sigma(K^{\mathrm{s}}(t) - K^b(t))}{\Psi} + \eta\frac{f(d+\sigma^2)}{\Psi}.$$

Since $K^{\mathrm{s}}(t) > K^b(t)$ for all $t > 0$, then $m(K^b(t), K^{\mathrm{s}}(t), \eta) > 0$ for all $t \geq 0$ and the consideration of the subsidy increases the effectiveness of the enforcement mechanism along the whole period.

b) $\tilde{\eta} < \eta$

b1) $\tilde{\eta} < \eta < \eta^0$ (Non-compliance for $t \in [0, \hat{t})$ and over-compliance for $t \geq \hat{t}$)

Within the interval $[0, \hat{t})$ effectiveness increases following the argument in case a).

Within the interval $[\hat{t}, \infty)$ function $m$ reads:

$$\begin{aligned} m(K^b(t), K^{\mathrm{s}}(t), \eta) &= N^{\mathrm{s}}(K^b(t), 0) + N^{\mathrm{s}}(K^{\mathrm{s}}(t), \eta) \\ &= \frac{d\sigma(2 - K^{\mathrm{s}}(t) - K^b(t))}{\Psi} - \eta\frac{f(d+\sigma^2)}{\Psi}. \end{aligned} \tag{39}$$

At time $\hat{t}$, $N^{\mathrm{s}}(K^{\mathrm{s}}(\hat{t}), \eta) = 0$ and therefore, $m(K^b(\hat{t}), K^{\mathrm{s}}(\hat{t}), \eta) > 0$. Taking into account (10) and (38), it immediately follows that $m(K^b(t), K^{\mathrm{s}}(t), \eta)$ is monotonously decreasing with time. Finally, at the steady state $m(\bar{K}^b, \bar{K}^{\mathrm{s}}(\eta), \eta)$ is a decreasing function of $\eta$ which vanishes for $\eta = 2\tilde{\eta}$. Therefore:

i. If $\eta \leq 2\tilde{\eta}$, then $m(\bar{K}^b, \bar{K}^{\mathrm{s}}(\eta), \eta) \geq 0$ and since $m$ is decreasing with time, $m(\bar{K}^b, \bar{K}^{\mathrm{s}}(\eta), \eta) > 0$ for all $t \in [\hat{t}, \infty)$. In consequence, the effectiveness of the enforcement mechanism increases for the whole period.

ii. If $\eta > 2\tilde{\eta}$, then $m(\bar{K}^b, \bar{K}^{\mathrm{s}}(\eta), \eta) < 0$ but, at time $\hat{t}$, $m(K^b(\hat{t}), K^{\mathrm{s}}(\hat{t}), \eta) > 0$. In consequence, the effectiveness is improved within a first interval longer than $[0, \hat{t})$, but worsens from a given time on.

b2) $\tilde{\eta} < \eta^0 \leq \eta$ (Over-compliance for all $t > 0$)

Function $m$ is given in equation (39). At time $0$, $m(k_0, k_0, \eta)$ is positive if and only if $\eta < 2\eta^0$. Under this condition, the same reasoning as in the case b1) for the interval $[\hat{t}, \infty)$ now applies for $[0, \infty)$. Therefore either the effectiveness of the enforcement mechanism increases for the whole period if $\eta \leq 2\tilde{\eta}$, or it increases within a first period but worsens henceforth, if $\eta > 2\tilde{\eta}$.

Finally, $\eta > 2\eta^0$ implies $m(k_0, k_0, \eta) < 0$ and since this function decreases with time it will remain negative. Therefore, the consideration of the subsidy would imply a reduction in the effectiveness of the enforcement mechanism along the whole period.

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