

Factorization Approach to Superintegrable Systems: Formalism and Applications*

Á. Ballesteros¹⁾** , F. J. Herranz¹⁾*** , Ş. Kuru²⁾**** , and J. Negro³⁾*****

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Abstract—The factorization technique for superintegrable Hamiltonian systems is revisited and applied in order to obtain additional (higher-order) constants of the motion. In particular, the factorization approach to the classical anisotropic oscillator on the Euclidean plane is reviewed, and new classical (super)integrable anisotropic oscillators on the sphere are constructed. The Tremblay–Turbiner–Winternitz system on the Euclidean plane is also studied from this viewpoint.

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1. INTRODUCTION

In some previous works [1, 2], a new method to deal with the symmetries of some superintegrable systems was introduced. This method consists essentially in extending the factorization method for quantum-mechanical Hamiltonians [3] to some classical separable systems depending on several variables. We recall that if an integrable classical Hamiltonian H can be separated in a certain coordinate system, it is well known that each coordinate leads to an integral of motion. Then two sets of “ladder” B^\pm and “shift” functions A^\pm can be found and, if certain conditions are fulfilled, additional constants of motion can be explicitly constructed in a straightforward manner by combining these B^\pm and A^\pm functions. It is worth stressing that such integrals are, in the general case, of higher-order on the momenta.

This “extended” factorization method has many advantages that we briefly enumerate: (i) The method is valid for quantum as well as for classical systems [4], and the classical-quantum correspondence becomes manifest at each stage of the factorization procedure. (ii) The approach can be applied either for second-order or for higher-order symmetries. (iii) The symmetries so obtained close a quite simple

symmetry algebra [2] from which it is straightforward to write the associated polynomial symmetries. (iv) For classical systems the results allow to find in a simple way the associated phase space trajectories, and in the case of quantum systems the discrete spectrum can be explicitly computed.

The aim of this work is to provide an introduction to this method by means of some (known and new) examples of two-dimensional superintegrable systems, where we have restricted ourselves to the classical framework in order to make the presentation simpler. However, we stress that the very same procedure can also be applied to quantum Hamiltonians, thus leading to the corresponding ladder and shift operators.

The structure of the paper is as follows. In the next section we revisit the anisotropic oscillator on the Euclidean plane. Section 3 is devoted to propose an anisotropic oscillator Hamiltonian on the sphere, which is a completely new model. In Section 4 we study the classical Tremblay–Turbiner–Winternitz (TTW) system on the Euclidean plane from the factorization point of view. Finally, some remarks and conclusions close the paper.

2. ANISOTROPIC OSCILLATORS ON THE EUCLIDEAN PLANE

Let us consider the anisotropic oscillator Hamiltonian with unit mass and frequencies ω_x and ω_y on the Euclidean plane in Cartesian coordinates:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2). \quad (1)$$

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¹⁾Departamento de Física, Universidad de Burgos, Spain.

²⁾Department of Physics, Faculty of Science, Ankara University, Turkey.

³⁾Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, Spain.

** E-mail: angelb@ubu.es

*** E-mail: fjherranz@ubu.es

**** E-mail: kuru@science.ankara.edu.tr

***** E-mail: jnegro@fta.uva.es

Obviously, this Hamiltonian is integrable since it Poisson-commutes with the (quadratic in the momenta) integrals of motion

$$I_x = \frac{1}{2}p_x^2 + \frac{1}{2}\omega_x^2 x^2, \quad I_y = \frac{1}{2}p_y^2 + \frac{1}{2}\omega_y^2 y^2, \\ H = I_x + I_y.$$

It is well known that for commensurate frequencies $\omega_x : \omega_y$ the Hamiltonian (1) defines a superintegrable system [5–7], endowed with an “additional” integral of motion. In the sequel we study the Hamiltonian (1) by following a factorization approach [1, 2, 4] (see also [8, 9] and references therein) which is different from the one applied in [5–7].

Firstly, let us introduce a positive real parameter γ

$$\omega_x = \gamma\omega_y, \quad \omega_y = \omega, \quad (2)$$

which encodes the anisotropy. This leads us to define a new coordinate ξ as

$$\xi = \gamma x, \quad p_\xi = p_x/\gamma. \quad (3)$$

Hence, the Hamiltonian (1) is expressed as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\omega^2}{2}((\gamma x)^2 + y^2) \\ = \frac{1}{2}p_\xi^2 + \frac{\omega^2}{2}y^2 + \gamma^2 \left(\frac{1}{2}p_\xi^2 + \frac{\omega^2}{2\gamma^2}\xi^2 \right). \quad (4)$$

In this latter form, the two one-dimensional Hamiltonians H^ξ and H^y

$$H^\xi = \frac{1}{2}p_\xi^2 + \frac{\omega^2}{2\gamma^2}\xi^2, \quad H^y = \frac{1}{2}p_y^2 + \frac{\omega^2}{2}y^2, \quad (5)$$

are two integrals of motion quadratic in the momenta, since $\{H, H^\xi\} = \{H, H^y\} = \{H^\xi, H^y\} = 0$. Therefore, since the function H^ξ is a constant of motion, the complete Hamiltonian (4) is just

$$H = H^y + \gamma^2 H^\xi,$$

and H is reduced to a one-dimensional system on the phase space submanifold defined by H^ξ being a certain constant.

As we will see in the sequel, the factorization approach requires, firstly, to factorize the integral H^ξ in terms of ladder functions B^\pm and, secondly, to factorize H through H^y in terms of shift functions A^\pm (we remark that in this specific example the names “ladder” and “shift” can be interchanged).

2.1. Factorization

Ladder (lowering and raising) functions B^\pm for the constant of motion H^ξ (5) are obtained by imposing that

$$H^\xi = B^+ B^- + \lambda_B, \quad (6)$$

yielding

$$B^\pm = \mp \frac{i}{\sqrt{2}} p_\xi + \frac{1}{\sqrt{2}} \frac{\omega}{\gamma} \xi, \quad \lambda_B = 0. \quad (7)$$

The three functions H^ξ and B^\pm obey the Poisson brackets given by

$$\{H^\xi, B^\pm\} = \mp i \frac{\omega}{\gamma} B^\pm, \quad \{B^-, B^+\} = -i \frac{\omega}{\gamma}, \quad (8)$$

so that, together with the constant function 1, they close the harmonic oscillator Poisson–Lie algebra \mathfrak{h}_4 .

As far as the shift functions A^\pm is concerned, we factorize H^y (5) by imposing that

$$H^y = A^+ A^- + \lambda_A, \quad (9)$$

giving rise to

$$A^\pm = \mp \frac{i}{\sqrt{2}} p_y - \frac{\omega}{\sqrt{2}} y, \quad \lambda_A = 0. \quad (10)$$

Again, the four functions $(H^y, A^\pm, 1)$ span the Poisson–Lie algebra \mathfrak{h}_4 since

$$\{H^y, A^\pm\} = \pm i\omega A^\pm, \quad \{A^-, A^+\} = i\omega.$$

Thus, the two-dimensional Hamiltonian (4) can be expressed in terms of these ladder and shift functions as

$$H = A^+ A^- + \gamma^2 B^+ B^-,$$

and the following Poisson algebra is generated

$$\{H, B^\pm\} = \mp i\gamma\omega B^\pm, \quad \{H, A^\pm\} = \pm i\omega A^\pm.$$

2.2. Higher-Order Integrals of Motion

The remarkable result arises if we consider a rational value for the parameter γ , namely,

$$\gamma = \frac{\omega_x}{\omega_y} = \frac{m}{n}, \quad m, n \in \mathbb{N}^*. \quad (11)$$

In such a case “additional” integrals of motion X^\pm can be constructed for the Hamiltonian H (4) by combining the ladder (7) and shift (10) functions in the form

$$X^\pm = (B^\pm)^n (A^\pm)^m, \quad \{H, X^\pm\} = 0. \quad (12)$$

Notice that the integrals of motion (12) are of $(m + n)$ th-order in the momenta. However, since X^\pm are, in fact, complex functions we can obtain two real constants of motion by considering their real and imaginary parts, namely,

$$X = \frac{1}{2}(X^+ + X^-), \quad Y = \frac{1}{2i}(X^+ - X^-), \quad (13)$$

whose maximal order in the momenta is at the most equal to $(m + n)$. Alternatively, the modulus and the

phase functions of (12) could be considered (as, e.g., in [6]).

In this way we recover the known results on the (super)integrability of anisotropic oscillators on the Euclidean plane [5–7]:

Theorem 1. (i) *The Hamiltonian $H(4)$ is integrable for any value of γ , since it is endowed with the quadratic constant of motion given by $H^\xi(5)$.*

(ii) *When $\gamma = m/n$ is a rational parameter (11) the Hamiltonian (4) defines a superintegrable anisotropic oscillator with commensurate frequencies $\omega_x : \omega_y$, and the additional real constant of motion is given by (13), which is at most of $(m + n)$ th order in the momenta. The set (H, H^ξ, X) (or (H, H^ξ, Y)) is formed by three functionally independent integrals.*

Some comments concerning the specific anisotropic Euclidean oscillators comprised by the Hamiltonian $H(4)$ seem to be pertinent.

The isotropic or 1 : 1 oscillator corresponds to $\gamma = m = n = 1$, $\omega_x = \omega_y = \omega$, $\xi = x$ and $p_\xi = p_x$. In this case, the function $X(13)$ is a quadratic integral, corresponding to one of the components of the Demkov–Fradkin tensor [10, 11], meanwhile $Y(13)$ is a linear integral in the momenta which is proportional to the angular momentum.

The 2 : 1 oscillator comes out by setting $\gamma = 2$, $m = 2$, $n = 1$, $\omega_x = 2\omega_y = 2\omega$, $\xi = 2x$ and $p_\xi = p_x/2$. In this case, X is a quadratic integral, meanwhile Y is a cubic one. Thus, the 2 : 1 oscillator is considered as a superintegrable system with quadratic constants of motion. Obviously, the 1 : 2 ($\gamma = 1/2$) oscillator is a completely equivalent system to the 2 : 1 ($\gamma = 2$) oscillator.

We remark that the 1 : 1 and 2 : 1 (or 1 : 2) oscillators are the only anisotropic oscillators endowed with quadratic integrals according to the classification of superintegrable systems on the two-dimensional Euclidean space [12] (see [13] for three dimensions). In other words, all the remaining $m : n$ oscillators have higher-order integrals.

It is also worth stressing that for the study of anisotropic Euclidean oscillators it is not necessary to introduce neither the parameter $\gamma(2)$ nor the new variable $\xi(3)$ (see the procedure developed in [5–7]). Nevertheless, such parameter and variable turn out to be essential for defining anisotropic oscillators on spaces of constant curvature which, to the best of our knowledge, were so far unknown. In the next section we solve this problem, for the first time, on the two-dimensional sphere.

3. ANISOTROPIC OSCILLATORS ON THE SPHERE

Let us consider the two-dimensional sphere \mathbf{S}^2 with unit radius embedded in the three-dimensional space \mathbb{R}^3 with ambient coordinates (x_0, x_1, x_2) such that

$$x_0^2 + x_1^2 + x_2^2 = 1.$$

We set the origin O in \mathbf{S}^2 as the point given by $O = (1, 0, 0) \in \mathbb{R}^3$ and we parametrize the ambient coordinates in terms of two intrinsic quantities (r, ϕ) and (x, y) in the form

$$\begin{aligned} x_0 &= \cos r = \cos x \cos y, \\ x_1 &= \sin r \cos \phi = \sin x \cos y, \\ x_2 &= \sin r \sin \phi = \sin y, \end{aligned} \tag{14}$$

whose geometrical meaning is as follows [12, 14, 15].

Let l_1, l_2 be two base geodesics orthogonal at O and let l be the geodesic that joins a point P (the particle) and the origin O . The so called *geodesic polar coordinates* (r, ϕ) are defined by the distance r between O and P measured along l and the angle ϕ of l relative to l_1 . Thus these generalize the polar coordinates to the sphere. Now let P_1 be the intersection point of l_1 with its orthogonal geodesic l' through P . Then the so called *geodesic parallel coordinates* (x, y) are determined by the distance x between O and the point P_1 measured along l_1 and the distance y between P_1 and P measured along l' (see Fig. 1). Hence, these generalize the Cartesian coordinates to \mathbf{S}^2 . The domain of these variables reads

$$\begin{aligned} 0 < r < \pi, & \quad 0 \leq \phi < 2\pi, \\ -\pi < x \leq \pi, & \quad -\frac{\pi}{2} < y < \frac{\pi}{2}. \end{aligned}$$

The metric on \mathbf{S}^2 in the above coordinates is so given by:

$$\begin{aligned} ds^2 &= (dx_0^2 + dx_1^2 + dx_2^2)|_{\mathbf{S}^2} \\ &= dr^2 + \sin^2 r d\phi^2 = \cos^2 y dx^2 + dy^2. \end{aligned}$$

By denoting (p_r, p_ϕ) and (p_x, p_y) the conjugate momenta of (r, ϕ) and (x, y) , respectively, we obtain the free Hamiltonian T that determines the free motion on \mathbf{S}^2 :

$$T = \frac{1}{2} \left(p_r^2 + \frac{p_\phi^2}{\sin^2 r} \right) = \frac{1}{2} \left(\frac{p_x^2}{\cos^2 y} + p_y^2 \right).$$

Now, by having in mind the Euclidean Hamiltonian (4), we are able to propose an “appropriate” Hamiltonian that determines anisotropic oscillators

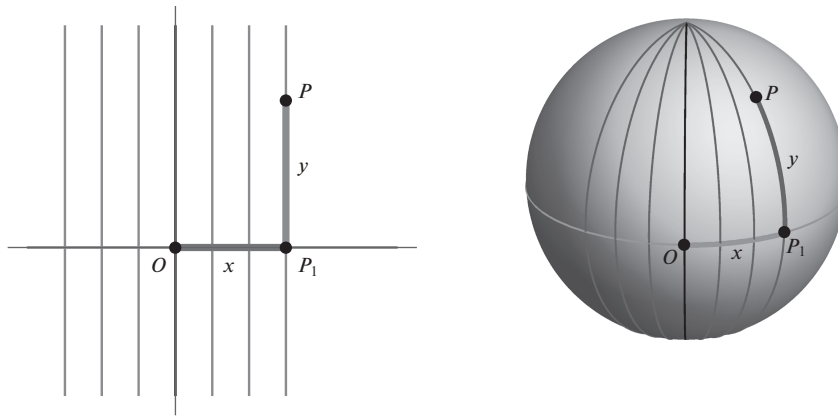


Fig. 1. Cartesian coordinates in the plane versus geodesic parallel coordinates on the sphere.

on the sphere. Explicitly, we shall consider the following Hamiltonian, expressed in terms of the geodesic parallel variables, and given by

$$H = T + U^\gamma = \frac{1}{2} \left(\frac{p_x^2}{\cos^2 y} + p_y^2 \right) + \frac{\omega^2}{2} \left(\frac{\tan^2(\gamma x)}{\cos^2 y} + \tan^2 y \right). \quad (15)$$

We remark that due to the term $\tan(\gamma x)$ in the potential, the domain of the variable x and the value of the real parameter γ are restricted in the form

$$-\frac{\pi}{2} < \gamma x < \frac{\pi}{2}, \quad \gamma \geq \frac{1}{2},$$

so avoiding a multivalued Hamiltonian.

Next we introduce the variable ξ (3) and we write the Hamiltonian (15) as

$$H = \frac{p_y^2}{2} + \frac{1}{\cos^2 y} \left(\frac{p_x^2}{2} + \frac{\omega^2}{2 \cos^2(\gamma x)} \right) - \frac{\omega^2}{2} = \frac{p_y^2}{2} + \frac{\gamma^2}{\cos^2 y} \left(\frac{p_\xi^2}{2} + \frac{\omega^2}{2\gamma^2 \cos^2 \xi} \right) - \frac{\omega^2}{2}, \quad (16)$$

which leads to a quadratic integral of motion H^ξ such that

$$H = \frac{p_y^2}{2} + \frac{\gamma^2 H^\xi}{\cos^2 y} - \frac{\omega^2}{2}, \quad H^\xi = \frac{p_\xi^2}{2} + \frac{\omega^2}{2\gamma^2 \cos^2 \xi}, \quad \{H, H^\xi\} = 0. \quad (17)$$

Hence, when H^ξ is taken as a constant, H is reduced to a one-dimensional Hamiltonian which always determines an integrable anisotropic oscillator on \mathbf{S}^2 for any value of γ .

In the sequel we will factorize the one-dimensional Hamiltonians H^ξ and H (17). The resulting (ladder and shift) factor functions will provide additional

integrals of motion of the two-dimensional Hamiltonian (16) whenever γ is a rational number, similarly to what happens for the previous anisotropic Euclidean oscillators.

3.1. Factorization

Let us consider the one-dimensional integral of motion H^ξ (17) and look for some ladder functions B^\pm fulfilling the Poisson brackets

$$\{H^\xi, B^\pm\} = f_\pm(H^\xi)B^\pm, \quad (18)$$

for certain functions f_\pm . We define the following function

$$h^\xi = \cos^2 \xi \left(\frac{p_\xi^2}{2} - H^\xi \right).$$

As in (6), we require that

$$h^\xi = B^+ B^- + \lambda_B,$$

which yields, as particular solutions, to

$$B^\pm = \mp \frac{i}{\sqrt{2}} \cos \xi p_\xi + \sqrt{H^\xi} \sin \xi, \quad \lambda_B = -H^\xi. \quad (19)$$

Although is clear that $h^\xi \equiv -\omega^2/(2\gamma^2)$, the remarkable point is that B^\pm are ladder functions for H^ξ (17) since the three functions (H^ξ, B^\pm) verify the Poisson brackets

$$\{H^\xi, B^\pm\} = \mp i\sqrt{2H^\xi}B^\pm, \quad \{B^-, B^+\} = -i\sqrt{2H^\xi}, \quad (20)$$

which are of the type (18). Notice that we have thus obtained a “deformation” of the Poisson–Lie algebra \mathfrak{h}_4 (8).

The expressions (20) suggest to write the integral H^ξ (17) as

$$\mathcal{E} = \sqrt{2H^\xi}. \quad (21)$$

The Poisson brackets among the Hamiltonian H (17) and the ladder functions B^\pm (19) read

$$\{H, B^\pm\} = \mp i \frac{\gamma^2 \mathcal{E}}{\cos^2 y} B^\pm, \quad \{B^-, B^+\} = -i\mathcal{E}.$$

The second step in the factorization approach is to search for shift functions A^\pm for the Hamiltonian H (17), written in terms of (21) as

$$H = \frac{p_y^2}{2} + \frac{\gamma^2 \mathcal{E}^2}{2 \cos^2 y} - \frac{\omega^2}{2},$$

which must verify the Poisson brackets

$$\{H, A^\pm\} = g_\pm(H^\xi, U^\gamma) A^\pm = g_\pm(\mathcal{E}, y) A^\pm,$$

for certain functions g_\pm and where U^γ is the potential given in (15). We impose the relation (9), finding now that

$$A^\pm = \mp \frac{i}{\sqrt{2}} p_y - \frac{\gamma \mathcal{E}}{\sqrt{2}} \tan y, \quad \lambda_A = \frac{1}{2} (\gamma^2 \mathcal{E}^2 - \omega^2). \quad (22)$$

From this result we find the following Poisson brackets

$$\{H, A^\pm\} = \pm i \frac{\gamma \mathcal{E}}{\cos^2 y} A^\pm, \quad \{A^-, A^+\} = i\gamma \frac{\mathcal{E}}{\cos^2 y}.$$

3.2. Higher-Order Integrals of Motion

Similarly to the anisotropic oscillators on the Euclidean plane, it is a matter of straightforward computations to show that for rational values $\gamma = m/n$ (11), we obtain (higher-order) additional integrals of motion for H (15), namely

$$X^\pm = (B^\pm)^n (A^\pm)^m, \quad \{H, X^\pm\} = 0, \quad X = \frac{1}{2}(X^+ + X^-), \quad Y = \frac{1}{2i}(X^+ - X^-), \quad (23)$$

where B^\pm and A^\pm are given in (19) and (22), respectively.

Consequently, the generalization of Theorem 1 on the sphere is obtained, and a new infinite family of (super)integrable curved systems are found:

Theorem 2. (i) *The Hamiltonian H (15) defines an integrable anisotropic oscillator on \mathbf{S}^2 for any value of the positive real parameter γ . The (quadratic in the momenta) constant of motion for H is given by H^ξ (17).*

(ii) *When γ is a rational parameter (11), the Hamiltonian H (15) provides a superintegrable anisotropic oscillator on \mathbf{S}^2 with additional constants of motion given by (23), which are at most of $(m+n)$ th order in the momenta. The set*

(H, H^ξ, X) (also (H, H^ξ, Y)) is formed by three functionally independent integrals.

We stress that this statement covers two well-known particular cases which correspond to $\gamma = 1$ and $\gamma = 2$, respectively. Indeed, these are the two cases appearing in the classification of quadratic superintegrable systems on the sphere [12, 16, 17], meanwhile all the remaining ones are new superintegrable systems on \mathbf{S}^2 .

In particular, the curved oscillator system coming from $\gamma = m = n = 1$, $\xi = x$ and $p_\xi = p_x$ is just the so called Higgs oscillator [18, 19] whose potential, expressed in geodesic polar coordinates (14), is simply $\tan^2 r$. This curved oscillator has been widely studied (see, e.g., [15, 20–30] and references therein). In both sets of coordinates (14), the Higgs potential reads

$$U^{\gamma=1} = \frac{\omega^2}{2} \left(\frac{\tan^2 x}{\cos^2 y} + \tan^2 y \right) = \frac{\omega^2}{2} \tan^2 r.$$

The case with $\gamma = 2$ was firstly introduced in the classification presented in [12] (see also [30, 31]). In geodesic parallel and polar coordinates (14) this potential is given by

$$U^{\gamma=2} = \frac{\omega^2}{2} \left(\frac{\tan^2(2x)}{\cos^2 y} + \tan^2 y \right) = \frac{\omega^2}{2} \left(\frac{4 \tan^2 r \cos^2 \phi}{(1 - \sin^2 r \sin^2 \phi)(1 - \tan^2 r \cos^2 \phi)^2} + \frac{\sin^2 r \sin^2 \phi}{1 - \sin^2 r \sin^2 \phi} \right).$$

The latter expression clearly justifies the use of geodesic parallel variables instead of the geodesic polar ones when looking for generic anisotropic oscillators on \mathbf{S}^2 . We also remark that, in contradistinction with the anisotropic oscillators on the Euclidean plane, the curved potentials U^γ and $U^{1/\gamma}$ are no longer equivalent.

4. THE TTW SYSTEM ON THE EUCLIDEAN PLANE

In this section we will consider another example of superintegrable system on the Euclidean plane, but using polar coordinates instead of Cartesian ones. This is the case of the well-known TTW system [32]. In what follows we will apply the factorization method to find the symmetries according to [1], although we will introduce some inessential changes in order to simplify the discussion.

The TTW Hamiltonian, in polar coordinates (r, ϕ) , has the following expression (the factor 1/2 has been suppressed to accommodate with the notation of [1]):

$$H = p_r^2 + \omega^2 r^2$$

$$+ \frac{1}{r^2} \left(p_\phi^2 + \frac{\gamma^2 \alpha^2}{\cos^2(\gamma\phi)} + \frac{\gamma^2 \beta^2}{\sin^2(\gamma\phi)} \right). \quad (24)$$

Here it is assumed that $\gamma \geq 1/4$, in such a manner that the potential is well defined for $0 < \gamma\phi < \pi/2$. If we perform the usual canonical transformation $\theta = \gamma\phi$, $p_\theta = p_\phi/\gamma$, then the TTW Hamiltonian becomes

$$\begin{aligned} H &= p_r^2 + \omega^2 r^2 + \frac{\gamma^2 H_\theta}{r^2}, \\ H_\theta &= p_\theta^2 + \frac{\alpha^2}{\cos^2 \theta} + \frac{\beta^2}{\sin^2 \theta}, \end{aligned} \quad (25)$$

yielding two one-dimensional systems, that is, the angular Hamiltonian H_θ and the radial one $H \equiv H_r$, provided that H_θ is assumed to take a fixed constant value.

In this case we shall also deal with “ladder” and “shift” functions separately for the angular and radial parts of the TTW Hamiltonian function (25), and the symmetries will be constructed in the same way as in the previous examples.

4.1. Factorization

We recall that the Hamiltonian H_θ (25) is known as the two-parameter Pöschl–Teller Hamiltonian [33]. The ladder functions have a similar expression as in the quantum case [33], namely

$$B^\pm = \pm i \sin 2\theta p_\theta + \sqrt{H_\theta} \cos 2\theta + \frac{\beta^2 - \alpha^2}{\sqrt{H_\theta}}. \quad (26)$$

Along with the Hamiltonian H_θ , they satisfy the following Poisson brackets

$$\begin{aligned} \{H_\theta, B^\pm\} &= \mp 4i \sqrt{H_\theta} B^\pm, \\ \{B^-, B^+\} &= -4i \sqrt{H_\theta} \left(1 - \frac{(\beta^2 - \alpha^2)^2}{H_\theta^2} \right). \end{aligned} \quad (27)$$

The product of these two functions gives another function which depends only on H_θ :

$$B^+ B^- = H_\theta + \frac{(\beta^2 - \alpha^2)^2}{H_\theta} - 2(\beta^2 + \alpha^2).$$

The second Hamiltonian involved in (25) is the radial oscillator Hamiltonian:

$$H = H_r = p_r^2 + \omega^2 r^2 + \frac{\gamma^2 \mathcal{E}^2}{r^2}, \quad \mathcal{E} = \sqrt{H_\theta}.$$

This is factorized in a similar way to the quantum case [8] by requiring the relation (9) which yields two types of shift functions:

$$\begin{aligned} A_1^\pm &= \mp i p_r + \omega r - \frac{\gamma \mathcal{E}}{r}, & \lambda_{1A} &= 2\omega \gamma \mathcal{E}, \\ A_2^\pm &= \mp i p_r + \omega r + \frac{\gamma \mathcal{E}}{r}, & \lambda_{2A} &= -2\omega \gamma \mathcal{E}. \end{aligned}$$

These functions together with H_r satisfy the following Poisson brackets

$$\begin{aligned} \{H_r, A_1^\pm\} &= \mp 2i \left(\omega + \frac{\gamma \mathcal{E}}{r^2} \right) A_1^\pm, \\ \{A_1^-, A_1^+\} &= -2i \left(\omega + \frac{\gamma \mathcal{E}}{r^2} \right), \\ \{H_r, A_2^\pm\} &= \mp 2i \left(\omega - \frac{\gamma \mathcal{E}}{r^2} \right) A_2^\pm, \\ \{A_2^-, A_2^+\} &= -2i \left(\omega - \frac{\gamma \mathcal{E}}{r^2} \right). \end{aligned}$$

They are “mixed” ladder-shift functions [1, 33], but we can construct “pure” shift functions by taking the following products:

$$A^+ = A_1^+ A_2^-, \quad A^- = A_1^- A_2^+, \quad (28)$$

satisfying

$$\begin{aligned} \{H_r, A^\pm\} &= \mp 4i \frac{\gamma \mathcal{E}}{r^2} A^\pm, \\ \{A^-, A^+\} &= -8i \frac{\gamma \mathcal{E}}{r^2} H_r. \end{aligned} \quad (29)$$

The above shift functions are different from those given in [33], but they lead to similar results.

4.2. Higher-Order Integrals of Motion and Trajectories

Whenever $\gamma = m/n$, additional symmetries X^\pm for the Hamiltonian (25) can be easily obtained in terms of the functions B^\pm (26) and A^\pm (28) in the same form given in (23). This is proved, directly, with the help of (27) and (29). Consequently, for a rational value of γ , the TTW Hamiltonian (24) determines a superintegrable system.

These symmetries are helpful in order to find quite easily the phase trajectories for the Hamiltonian (25). When we fix the value of any of these real symmetry functions (i.e. X or Y (23)) together with H and H_θ , then we get a trajectory. Some examples for different values of γ have been plotted in Figs. 2 and 3 in the (r, θ) plane. If one prefers to deal with the (r, ϕ) plane, it is enough to change the angle, i.e. $\theta = \gamma\phi$, but the shape will remain quite similar.

For the discussion concerning the algebra generated by the integrals of motion and the corresponding polynomial symmetries we refer the reader to [2].

5. CONCLUDING REMARKS

The factorization approach to integrable systems has been revisited, and we have illustrated it by considering known integrable Hamiltonians, such as the

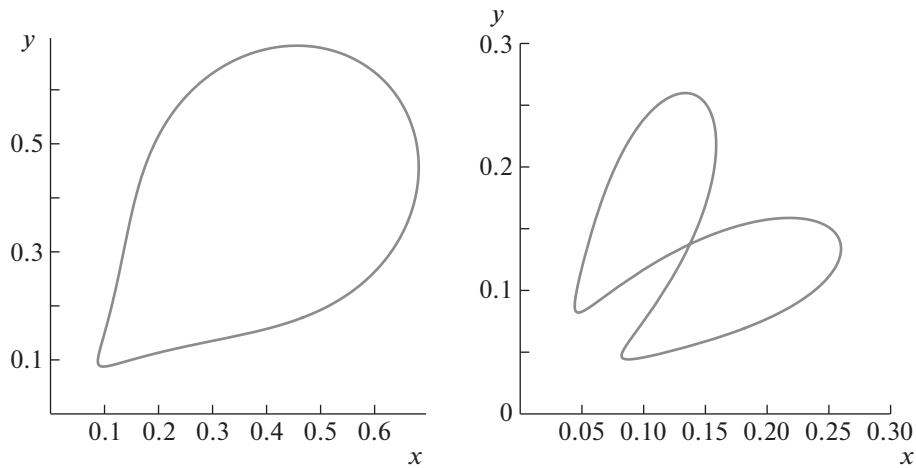


Fig. 2. Trajectories in the (r, θ) plane corresponding to: $\gamma = 1$ (left) and $\gamma = 2$ (right).

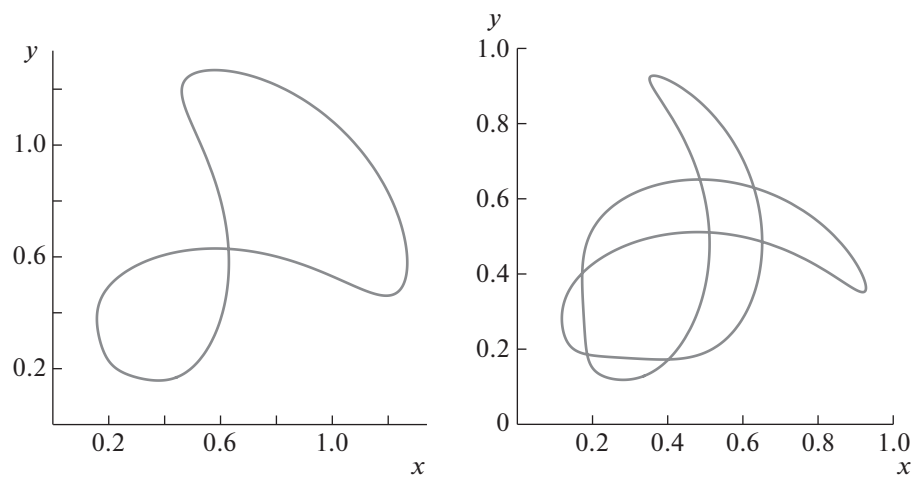


Fig. 3. Trajectories in the (r, θ) plane corresponding to: $\gamma = 1/2$ (left) and $\gamma = 2/3$ (right).

anisotropic oscillator and the TTW system on the Euclidean plane. Also, new systems like the anisotropic oscillator on the sphere have been introduced with the aid of this technique. It would be indeed interesting to apply this approach to other relevant problems, for instance:

To construct curved anisotropic oscillators on other spaces of constant curvature such as the hyperbolic space as well as onto the relativistic (anti-)de Sitter and Minkowski spacetimes (see [26] for the 1 : 1 oscillator on these spacetimes and, more recently, [34, 35] for the oscillator problem on the $SO(2, 2)$ hyperboloid).

The addition of two “centrifugal potentials”, that on the sphere can be considered as noncentral oscillators, by keeping superintegrability [15, 30]. We recall that in the Euclidean space the superposition of the anisotropic oscillator with centrifugal terms leads

to the so called caged anisotropic oscillator, studied in [36].

To study superintegrable Hamiltonian systems on spaces of nonconstant curvature. Recently, examples of this type of systems have been presented in [37] (a curved Kepler–Coulomb problem on the Taub-NUT space [38]) and in [39] (a curved oscillator on the Darboux III space [40]).

And, finally and more importantly, to define and solve the corresponding quantum systems.

Work on all these lines is currently in progress and will be presented elsewhere.

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