Metrizable ordinal proximity measures and their aggregation

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Abstract

Ordered qualitative scales formed by linguistic terms are frequently used for evaluating sets of alternatives in different decision-making problems. These scales are usually implicitly considered as uniform in the sense that the psychological proximity between consecutive terms is perceived as identical. However, sometimes agents can perceive different proximities between the linguistic terms of the scale, and an appropriate method is required for aggregating these perceptions. In this paper we introduce the notion of metrizable ordinal proximity measure, discuss some aggregation procedures and propose a method based on metrics for aggregating experts' opinions on proximities between linguistic terms on ordered qualitative scales.

Keywords: group decision-making; qualitative scales; ordinal proximity measures; judgment aggregation.

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1. Introduction

In many real problems in various disciplines (Economics, Management, Marketing, Psychology, Sociology, and Tourism, among others) the use of Ordered Qualitative Scales (**OQS**s) formed by linguistic terms is quite widespread as a way of collecting the opinions given by a group of agents concerning a number of alternatives. It is usually implicitly assumed that these OQSs are uniform, in the sense that the psychological proximity between all the pairs of consecutive terms of the OQS is perceived as identical.

Some OQSs with an odd number of linguistic terms are devised as uniform by fixing a neutral central linguistic term and arranging the other terms around it symmetrically (e.g. the 5-term OQS {'very bad', 'bad', 'regular', 'good', 'very good'}). However, not all OQSs with an odd number of linguistic terms are necessarily uniform[1].

OQSs with an even number of linguistic terms do not have a neutral central linguistic term. Consequently, it is easier to find non-uniform OQSs in even cases than in odd ones. For instance, the 4-term OQS {'reject', 'major revision', 'minor revision', 'accept'} used by some scientific journals in evaluating papers can be understood as non-uniform (see García-Lapresta and Pérez-Román [11, 2.4] for empirical evidence). Another example of an OQS with an even number of linguistic terms is provided by Balinski and Laraki [1] when they introduce the *Majority Judgment* voting system. These authors consider the 6-term OQS {'to reject', 'poor', 'acceptable', 'good', 'very good', 'excellent'} for evaluating candidates in political elections. Again, it is not clear that this scale is uniform.

There are numerous contributions in the literature that deal with non-uniform OQSs that follow fuzzy techniques involving cardinal approaches (see Herrera-Viedma and López-Herrera [17], Herrera *et al.* [16], among others). Other authors tackle non-uniform OQSs by means of ordinal ranges avoiding either cardinal or ordinal measurements of the proximities between linguistic terms (see Franceschini *et al.* [9]).

To deal with non-uniform OQSs in a purely ordinal way via psychological proximities between linguistic terms of OQSs, García-Lapresta and Pérez-Román [11] introduce the notion of ordinal proximity measure. This new ap-

[1] Question HS140 in the European Union Statistics on Income and Living Conditions (EU-SILC) survey conducted by Eurostat uses the 3-term OQS {'a heavy burden', 'somewhat a burden', 'not burden at all'} for asking individuals about the financial burden of total housing cost. In this OQS the central term is not neutral, so it is not clear that the scale is uniform.

proach has some similarities with *difference measurement* within classical measurement theory (see Krantz *et al.* [18, chapter 4] and Roberts [26, section 3.3]), and with *non-metric multidimensional scaling*, where only the ranks of the psychological distances or proximities are known (see Bennett and Hays [2], Shepard [27], Coombs [4], Kruskal and Wish [19], Cox and Cox [5] and Borg and Groenen [3, chapter 9], among others).

Given an OQS, the behavior of some ordinal proximity measures is better than others. This is why in this paper we introduce metrizable ordinal proximity measures. They behave as if the ordinal comparisons between the terms of an OQS were managed through a linear metric on the OQS[2].

For instance, taking into account the above mentioned OQS {'reject', 'major revision', 'minor revision', 'accept'}, saying that *'minor revision' is closer to 'accept' than to 'major revision'* is equivalent to saying that *the numerical distance between 'minor revision' and 'accept' is shorter than the numerical distance between 'minor revision' and 'major revision'*, whatever the corresponding linear metric that generates the ordinal proximity measure[3].

Deciding whether a given OQS is uniform or not, and in the latter case determining what the ordinal proximities are between the terms of the scale is an important issue for constructing metrizable ordinal proximity measures. This problem can be solved by a well-informed expert. However, sometimes the corresponding metrizable ordinal proximity measure is not easy to construct. In this paper we present an algorithm which generates a metrizable ordinal proximity measure by means of appropriate sequences of questions about the proximities between the terms of the scale.

If a group of experts is asked about these proximities, they may have different opinions and different metrizable ordinal proximity measures may therefore emerge. In these situations, it is advisable to find a collective metrizable ordinal proximity measure that represents individual opinions as faithfully as possible. Therefore, an appropriate aggregation procedure is needed. This aggregation problem is not trivial, and different procedures can generate different outcomes

[2]A linear metric on an OQS is a metric satisfying the requirement that the distance between two terms must be the sum of the distance between the first term and any intermediate term between the two given terms and the distance between that intermediate term and the second given term.

[3]Notice that this approach is similar to classical preference modeling of weak orders, where one alternative is preferred to another if and only if the utility of the first alternative is greater than the utility of the second, whatever the corresponding numerical utility function that represents the weak order may be.

and even inconsistencies.

To avoid these problems, we propose some solutions in this paper in the setting of *judgment aggregation theory* (see Dietrich and List [6], List [22], Mongin [24] and Grossi and Pigozzi [15], among others). In particular, we have devised a weighted-metric-based procedure that provides the metrizable ordinal proximity measure that minimizes the sum of distances (square distances) between itself and the metrizable ordinal proximity measures of the experts. This procedure solves the aforementioned problems. It is illustrated with two real case studies.

The rest of the paper is organized as follows. Section 2 introduces, analyzes and generates metrizable ordinal proximity measures. Section 3 studies the problem of how to aggregate experts' opinions through certain voting systems in order to generate metrizable ordinal proximity measures, and the inconsistencies that can result. Section 4 presents some proposals for avoiding inconsistencies. Section 5 contains some concluding remarks.

2. Ordinal proximity measures

Consider an OQS $\mathcal{L} = \{l_1, \dots, l_g\}$ whose terms are arranged from worst to best, with granularity at least 3, i.e., $g \geq 3$. In order to recall the notion of ordinal proximity measure on $\mathcal{L}$, introduced by García-Lapresta and Pérez-Román [11], we shall use a linear order $\Delta = \{\delta_1, \dots, \delta_h\}$, with $\delta_1 \succ \cdots \succ \delta_h$, for representing different degrees of proximity (with no meaning) among the terms of $\mathcal{L}$, being δ_1 and δ_h the maximum and minimum degrees, respectively.

As usual in the setting of linear orders, $\delta_r \succeq \delta_s$ means $\delta_r \succ \delta_s$ or $\delta_r = \delta_s$; $\delta_r \prec \delta_s$ means $\delta_s \succ \delta_r$; and $\delta_r \preceq \delta_s$ means $\delta_r \prec \delta_s$ or $\delta_r = \delta_s$.

Definition 1. *([11])* An *ordinal proximity measure (**OPM**) on $\mathcal{L}$ with values in* Δ is a mapping $\pi : \mathcal{L}^2 \longrightarrow \Delta$, where $\pi(l_r, l_s) = \pi_{rs}$ means the degree of proximity between l_r and l_s, satisfying the following conditions:

1. *Exhaustiveness*: For every $\delta \in \Delta$, there exist $l_r, l_s \in \mathcal{L}$ such that $\delta = \pi_{rs}$.
2. *Symmetry*: $\pi_{sr} = \pi_{rs}$, for all $r, s \in \{1, \dots, g\}$.
3. *Maximum proximity*: $\pi_{rs} = \delta_1 \Leftrightarrow r = s$, for all $r, s \in \{1, \dots, g\}$.
4. *Monotonicity*: $\pi_{rs} \succ \pi_{rt}$ and $\pi_{st} \succ \pi_{rt}$, for all $r, s, t \in \{1, \dots, g\}$ such that $r < s < t$.

We note that the previous conditions are independent (see García-Lapresta and Pérez-Román [11, Prop. 1]).

Every OPM $\pi : \mathcal{L}^2 \longrightarrow \Delta$ can be represented by a $g \times g$ symmetric matrix with coefficients in Δ, where the elements in the main diagonal are $\pi_{rr} = \delta_1$,

$r = 1, \ldots, g$:

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1s} & \cdots & \pi_{1g} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{r1} & \cdots & \pi_{rs} & \cdots & \pi_{rg} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{g1} & \cdots & \pi_{gs} & \cdots & \pi_{gg} \end{pmatrix}.$$

This matrix is called the *proximity matrix associated with* π.

Taking into account the conditions appearing in Definition 1, it is only necessary to show the upper half proximity matrix

$$\begin{pmatrix} \delta_1 & \pi_{12} & \pi_{13} & \cdots & \pi_{1(g-1)} & \pi_{1g} \\ & \delta_1 & \pi_{23} & \cdots & \pi_{2(g-1)} & \pi_{2g} \\ & & & \cdots & \cdots & \cdots \\ & & & & \delta_1 & \pi_{(g-1)g} \\ & & & & & \delta_1 \end{pmatrix}.$$

We note that the minimum proximity between linguistic terms is only reached when comparing the extreme linguistic terms: $\pi_{rs} = \delta_h \Leftrightarrow (r, s) \in \{(1, g), (g, 1)\}$ (see García-Lapresta and Pérez-Román [11, Prop. 2]).

We also note that the cardinality of Δ is located between the granularity of $\mathcal{L}$ and a polynomial of degree 2 of that granularity (see García-Lapresta and Pérez-Román [11, Prop. 4]):

$$g \leq h \leq \frac{g \cdot (g-1)}{2} + 1.$$

2.1. Linear metrics

The notion of metrizable OPM is based on linear metrics on OQSs. So, we first introduce the notion of linear metric in the setting of OQSs.

Definition 2. A *linear metric* on an OQS $\mathcal{L}$ is a mapping $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ satisfying the following conditions for all $r, s, t \in \{1, \ldots, g\}$:

1. *Positiveness*: $d(l_r, l_s) \geq 0$.
2. *Identity of indiscernibles*: $d(l_r, l_s) = 0 \Leftrightarrow r = s$.
3. *Symmetry*: $d(l_s, l_r) = d(l_r, l_s)$.
4. *Linearity*: $r < s < t \Rightarrow d(l_r, l_t) = d(l_r, l_s) + d(l_s, l_t)$.

As shown in the following remark, it is possible to generate a linear metric from the distances between consecutive terms of the OQS.

Remark 1. Given $d(l_r, l_{r+1}) = \rho_r > 0$ for $r = 1, \ldots, g-1$, there exists a unique linear metric on $\mathcal{L}$, $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$, satisfying the given conditions: $d(l_r, l_{r+2}) = d(l_r, l_{r+1}) + d(l_{r+1}, l_{r+2}) = \rho_r + \rho_{r+1}$ for every $r \in \{1, \ldots, g-2\}$. Iterating this process, we have $d(l_r, l_{r+t}) = \rho_r + \rho_{r+1} + \cdots + \rho_{r+t-1}$ for all $r, t \in \{1, \ldots, g-1\}$ such that $r + t \leq g$. It suffices to define $d(l_s, l_r) = d(l_r, l_s)$ and $d(l_r, l_r) = 0$ for all $r, s \in \{1, \ldots, g\}$.

We now justify that the family of linear metrics is a proper subset of the family of metrics.

Proposition 1. *Every linear metric $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ satisfies the triangle inequality, i.e., $d(l_r, l_t) \leq d(l_r, l_s) + d(l_s, l_t)$ for all $r, s, t \in \{1, \ldots, g\}$.*

Proof. There exist 6 cases.

1. $r \leq s \leq t$: $d(l_r, l_t) = d(l_r, l_s) + d(l_s, l_t)$.
2. $r \leq t \leq s$: $d(l_r, l_t) \leq d(l_r, l_t) + d(l_t, l_s) = d(l_r, l_s) \leq d(l_r, l_s) + d(l_s, l_t)$.
3. $s \leq r \leq t$: $d(l_r, l_t) \leq d(l_r, l_t) + d(l_s, l_r) = d(l_s, l_t) \leq d(l_r, l_s) + d(l_s, l_t)$.
4. $s \leq t \leq r$: $d(l_r, l_t) \leq d(l_r, l_t) + d(l_s, l_t) = d(l_s, l_r) \leq d(l_r, l_s) + d(l_s, l_t)$.
5. $t \leq r \leq s$: $d(l_r, l_t) \leq d(l_r, l_t) + d(l_r, l_s) = d(l_t, l_s) \leq d(l_r, l_s) + d(l_s, l_t)$.
6. $t \leq s \leq r$: $d(l_r, l_t) = d(l_t, l_r) = d(l_t, l_s) + d(l_s, l_r) = d(l_r, l_s) + d(l_s, l_t)$. $\square$

Consequently, every linear metric on $\mathcal{L}$ is a metric. The reciprocal is not true, as shown in the following remark.

Remark 2. There exist metrics on $\mathcal{L}$ that are not linear metrics. For instance, consider $\mathcal{L} = \{l_1, l_2, l_3\}$ and $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ the mapping defined as $d(l_1, l_1) = d(l_2, l_2) = d(l_3, l_3) = 0$, $d(l_1, l_2) = d(l_2, l_1) = 2$, $d(l_2, l_3) = d(l_3, l_2) = 3$ and $d(l_1, l_3) = d(l_3, l_1) = 4$. It is easy to check that d is a metric on $\mathcal{L}$. However, it is not linear: $4 = d(l_1, l_3) < d(l_1, l_2) + d(l_2, l_3) = 5$.

2.2. Metrizable OPMs

Before introducing the notion of metrizable OPM, we justify that every linear metric on an OQS defines in a natural way an OPM.

Proposition 2. *Let $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ be a linear metric. If $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is an exhaustive mapping defined as $\pi_{rs} \succ \pi_{tu} \Leftrightarrow d(l_r, l_s) < d(l_t, l_u)$, then π is an OPM on $\mathcal{L}$.*

Proof.

1. Exhaustiveness: By hypothesis.
2. Symmetry: By symmetry of d.

3. Maximum proximity: Since π is exhaustive, $\delta_1 = \pi_{rs}$ for some $r, s \in \{1, \dots, g\}$. If $r \neq s$, then $d(l_r, l_r) = 0 < d(l_r, l_s)$. Then, $\pi_{rr} \succ \pi_{rs} = \delta_1$, that is a contradiction. Consequently, $r = s$, i.e., $\delta_1 = \pi_{rr}$. In order to prove that $\pi_{tt} = \delta_1$ for every $t \in \{1, \dots, g\}$, suppose that $\pi_{tt} \neq \pi_{rr}$ for some $t \in \{1, \dots, g\}$. If $\pi_{tt} \succ \pi_{rr}$, then $0 = d(l_t, l_t) < d(l_r, l_r)$, that is a contradiction. Analogously, from $\pi_{rr} \succ \pi_{tt}$ we obtain a contradiction.
4. Monotonicity: Consider $r, s, t \in \{1, \dots, g\}$ such that $r < s < t$. Since $d(l_r, l_t) = d(l_r, l_s) + d(l_s, l_t)$, $d(l_r, l_s) > 0$ and $d(l_s, l_t) > 0$, we have $d(l_r, l_t) > d(l_r, l_s)$, i.e., $\pi_{rs} \succ \pi_{rt}$, and $d(l_r, l_t) > d(l_s, l_t)$, i.e., $\pi_{st} \succ \pi_{rt}$. □

Definition 3. An OPM $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is *metrizable* if there exists a linear metric $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ such that $\pi_{rs} \succ \pi_{tu} \Leftrightarrow d(l_r, l_s) < d(l_t, l_u)$, for all $r, s, t, u \in \{1, \dots, g\}$. We say that π is generated by d.

Remark 3. If $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is a metrizable OPM generated by a linear metric $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ and $d' : \mathcal{L}^2 \longrightarrow \mathbb{R}$ is defined as $d'(l_r, l_s) = \lambda \cdot d(l_r, l_s)$ for some $\lambda > 0$, then d' is a linear metric and π is also generated by d'.

However, a metrizable OPM can be generated by non proportional linear metrics. Consider $\mathcal{L} = \{l_1, l_2, l_3\}$ and $\pi : \mathcal{L}^2 \longrightarrow \Delta$ the OPM associated with the matrix A_{32} (see Subsection 2.3). Let $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ be the mapping defined as $d(l_1, l_1) = d(l_2, l_2) = d(l_3, l_3) = 0$, $d(l_1, l_2) = d(l_2, l_1) = 2$, $d(l_2, l_3) = d(l_3, l_2) = 1$ and $d(l_1, l_3) = d(l_3, l_1) = 3$. Let $d' : \mathcal{L}^2 \longrightarrow \mathbb{R}$ be the mapping defined as $d'(l_1, l_1) = d'(l_2, l_2) = d'(l_3, l_3) = 0$, $d'(l_1, l_2) = d'(l_2, l_1) = 3$, $d'(l_2, l_3) = d'(l_3, l_2) = 1$ and $d'(l_1, l_3) = d'(l_3, l_1) = 4$. It is easy to see that d and d' are linear metrics and that π is simultaneously generated by d and d'. Nevertheless, $d' \neq \lambda \cdot d$ for every $\lambda > 0$.

2.3. Constructing metrizable OPMs

We now show the proximity matrices associated with all the metrizable OPMs for $g = 3, 4$.

The subindices of the matrices A's correspond to the subindices of the δ's appearing in the coefficients just over the main diagonal, $\pi_{12}, \pi_{23}, \dots, \pi_{(g-1)\,g}$. These ordinal proximities correspond to the comparisons between all the pairs of consecutive linguistic terms.

For $g = 3$ there are three OPMs and all of them are metrizable. If an expert declares $\pi_{12} = \pi_{23}$, $\pi_{12} \succ \pi_{23}$ or $\pi_{12} \prec \pi_{23}$, then the matrix associated with the corresponding metrizable OPM will be A_{22} (see Figure 1), A_{23} (see Figure

2) or A_{32} (see Figure 3), respectively:

$$A_{22} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 \\ & \delta_1 & \delta_2 \\ & & \delta_1 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 \\ & \delta_1 & \delta_3 \\ & & \delta_1 \end{pmatrix}, \quad A_{32} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 \\ & \delta_1 & \delta_2 \\ & & \delta_1 \end{pmatrix}.$$

Figure 1: Ordinal proximity measure with associated matrix A_{22}.

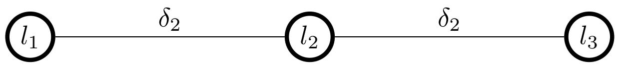


Figure 2: Ordinal proximity measure with associated matrix A_{23}.

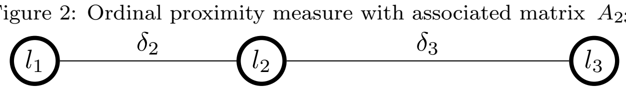
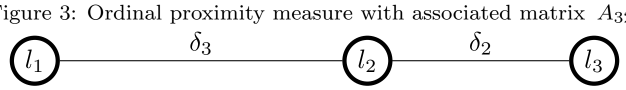


Figure 3: Ordinal proximity measure with associated matrix A_{32}.



For $g > 3$, experts may find some difficulties in directly constructing the OPM (or the associated proximity matrix) that reflects their own opinions about the proximities between the terms of the OQS. This can be done through appropriate sequences of questions, the answers to which lead to a metrizable OPM.

For $g = 4$ there are 51 OPMs, but only 25 of them are metrizable. Figure 4 contains an algorithm for $g = 4$ that guides the sequence of questions, depending on the answers provided by an expert, in order to obtain one of the 25 metrizable OPMs. This algorithm starts by asking the expert about ordinal proximities π_{12} and π_{23}. The next question differs depending on whether one of these ordinal proximities is greater than the other or they are the same. The procedure continues with similar questions comparing the ordinal proximities between the remaining pairs of terms of the OQS until the OPM is obtained. It is interesting to note that 4, 12 and 9 matrices are achieved after answering 2, 3 and 4 questions, respectively (3.2 questions on average). Unfortunately, for $g > 4$ the complexity of the algorithm dramatically increases.

The associated proximity matrices of these 25 metrizable OPMs are:

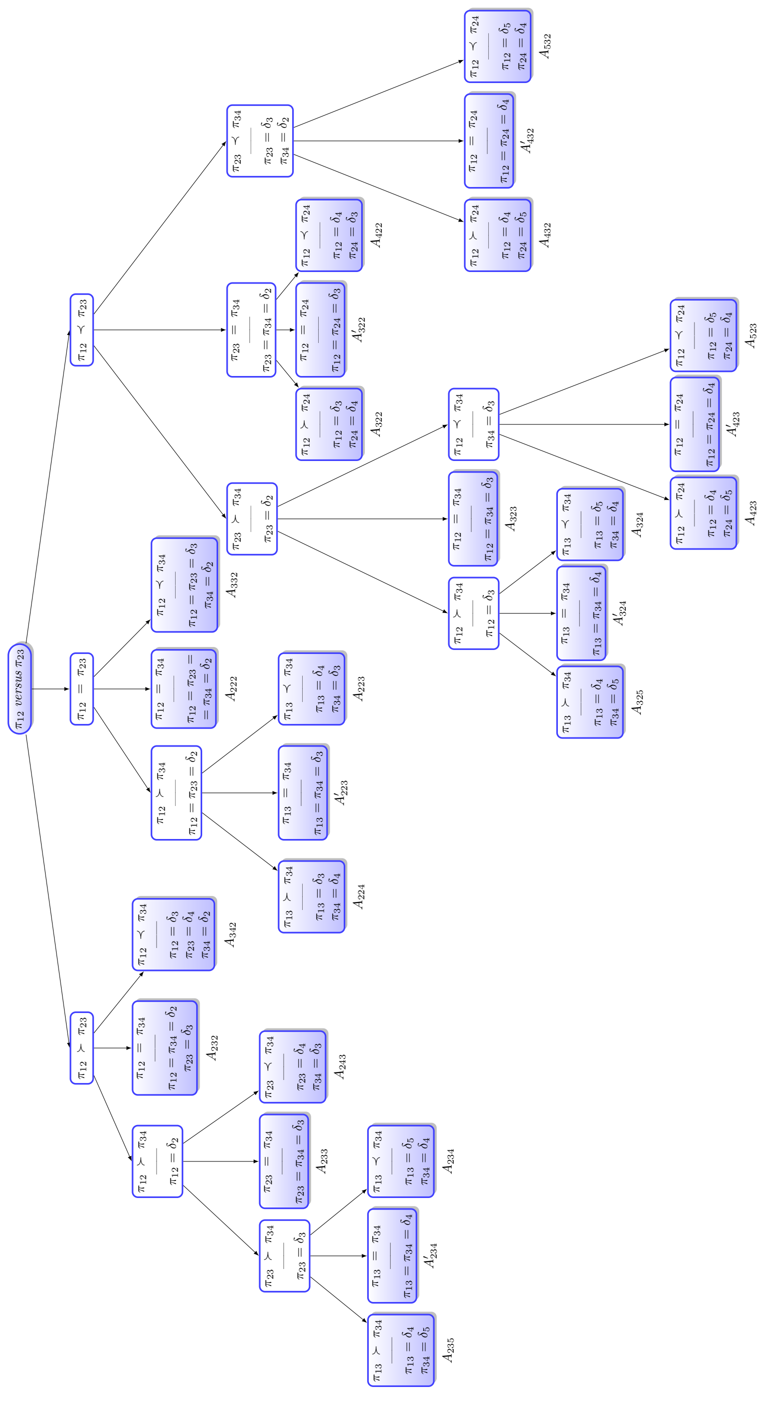


Figure 4: Algorithm for $g = 4$.

$$A_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A'_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A_{224} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{232} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A_{233} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{234} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_6 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}, \quad A'_{234} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{235} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_6 \\ & & \delta_1 & \delta_5 \\ & & & \delta_1 \end{pmatrix}, \quad A_{243} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_4 & \delta_6 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{322} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A'_{322} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{323} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A_{324} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_6 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

$$A'_{324} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}, \quad A_{325} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_6 \\ & & \delta_1 & \delta_5 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{332} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A_{342} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_4 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{422} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A_{423} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A'_{423} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A'_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A_{523} = \begin{pmatrix} \delta_1 & \delta_5 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A_{532} = \begin{pmatrix} \delta_1 & \delta_5 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}.$$

We now explain why the above matrices correspond to metrizable OPMs, by showing the basic information of some linear metrics that generate these OPMs (take into account Remarks 1 and 3):

A_{22}: $d(l_1, l_2) = d(l_2, l_3) = 1$.

A_{23}: $d(l_1, l_2) = 1$, $d(l_2, l_3) = 2$.

A_{32}: $d(l_1, l_2) = 2$, $d(l_2, l_3) = 1$.

A_{222}: $d(l_1, l_2) = d(l_2, l_3) = d(l_3, l_4) = 1$.

A_{223}: $d(l_1, l_2) = d(l_2, l_3) = 1,\ d(l_3, l_4) = 1.5$.

A'_{223}: $d(l_1, l_2) = d(l_2, l_3) = 1,\ d(l_3, l_4) = 2$.

A_{224}: $d(l_1, l_2) = d(l_2, l_3) = 1,\ d(l_3, l_4) = 3$.

A_{232}: $d(l_1, l_2) = d(l_3, l_4) = 1,\ d(l_2, l_3) = 2$.

A_{233}: $d(l_1, l_2) = 1,\ d(l_2, l_3) = d(l_3, l_4) = 2$.

A_{234}: $d(l_1, l_2) = 1,\ d(l_2, l_3) = 1.5,\ d(l_3, l_4) = 2$.

A'_{234}: $d(l_1, l_2) = 1,\ d(l_2, l_3) = 2,\ d(l_3, l_4) = 3$.

A_{235}: $d(l_1, l_2) = 1,\ d(l_2, l_3) = 1.5,\ d(l_3, l_4) = 3$.

A_{243}: $d(l_1, l_2) = 1,\ d(l_2, l_3) = 3,\ d(l_3, l_4) = 2$.

A_{322}: $d(l_1, l_2) = 2,\ d(l_2, l_3) = d(l_3, l_4) = 1.5$.

A'_{322}: $d(l_1, l_2) = 2,\ d(l_2, l_3) = d(l_3, l_4) = 1$.

A_{323}: $d(l_1, l_2) = d(l_3, l_4) = 1,\ d(l_2, l_3) = 2$.

A_{324}: $d(l_1, l_2) = 2.5,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 3$.

A'_{324}: $d(l_1, l_2) = 2,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 3$.

A_{325}: $d(l_1, l_2) = 2,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 4$.

A_{332}: $d(l_1, l_2) = d(l_2, l_3) = 2,\ d(l_3, l_4) = 1$.

A_{342}: $d(l_1, l_2) = 2,\ d(l_2, l_3) = 3,\ d(l_3, l_4) = 1$.

A_{422}: $d(l_1, l_2) = 3,\ d(l_2, l_3) = d(l_3, l_4) = 1$.

A_{423}: $d(l_1, l_2) = 2.5,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 2$.

A'_{423}: $d(l_1, l_2) = 3,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 2$.

A_{432}: $d(l_1, l_2) = 2.5,\ d(l_2, l_3) = 2,\ d(l_3, l_4) = 1$.

A'_{432}: $d(l_1, l_2) = 3,\ d(l_2, l_3) = 2,\ d(l_3, l_4) = 1$.

A_{523}: $d(l_1, l_2) = 3,\ d(l_2, l_3) = 1,\ d(l_3, l_4) = 1.5$.

A_{532}: $d(l_1, l_2) = 3,\ d(l_2, l_3) = 1.5,\ d(l_3, l_4) = 1$.

Remark 4. The Appendix includes the associated proximity matrices of the 26 non metrizable OPMs for $g = 4$.

As an example, we show that the OPM with associated proximity matrix

$$A^1_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}$$

is not metrizable.

Since $\pi_{13} = \delta_3 \succ \delta_4 = \pi_{24}$, we have $d(l_1, l_3) < d(l_2, l_4)$. If π was metrizable, we had $d(l_1, l_2) + d(l_2, l_3) < d(l_2, l_3) + d(l_3, l_4)$. Then, $d(l_1, l_2) < d(l_3, l_4)$, i.e., $\pi_{12} \succ \pi_{34}$, that is a contradiction, because $\pi_{12} = \pi_{34} = \delta_2$.

2.4. Uniform OPMs

We now consider uniform and totally uniform OPMs, and analyze their relationships with metrizable OPMs.

Definition 4. An OPM $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is *uniform* if $\pi_{r\,(r+1)} = \pi_{s\,(s+1)}$ for all $r, s \in \{1, \ldots, g-1\}$, and *totally uniform* if $\pi_{r\,(r+t)} = \pi_{s\,(s+t)}$ for all $r, s, t \in \{1, \ldots, g-1\}$ such that $r + t \leq g$ and $s + t \leq g$.

Remark 5. Let $\pi : \mathcal{L}^2 \longrightarrow \Delta$ be a totally uniform OPM on $\mathcal{L}$. Taking into account Definitions 1 and 4, $\succ$ is determined by $\pi_{rr} = \pi_{ss} \succ \pi_{r\,(r+1)} = \pi_{(r+1)\,r} = \pi_{s\,(s+1)} = \pi_{(s+1)\,s} \succ \pi_{r\,(r+2)} = \pi_{(r+2)\,r} = \pi_{s\,(s+2)} = \pi_{(s+2)\,s} \succ \cdots \succ \pi_{r\,(r+g-2)} = \pi_{(r+g-2)\,r} = \pi_{s\,(s+g-2)} = \pi_{(s+g-2)\,s} \succ \pi_{1g} = \pi_{g1}$, for all admissible $r, s \in \{1, \ldots, g\}$. Consequently, $\pi_{rs} \succ \pi_{tu} \Leftrightarrow |s - r| < |u - t|$, for all $r, s, t, u \in \{1, \ldots, g\}$.

Obviously, every totally uniform OPM is uniform. We now justify that the converse is also true in the case of metrizable OPMs. We also prove that totally uniform OPMs are always metrizable.

Proposition 3. *If $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is a uniform metrizable OPM on $\mathcal{L}$, then it is also totally uniform.*

Proof. Let $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ be a linear metric such that $\pi_{rs} \succ \pi_{tu} \Leftrightarrow d(l_r, l_s) < d(l_t, l_u)$. Since π is uniform, there exists $\rho > 0$ such that $d(l_r, l_{r+1}) = \rho$ for every $r \in \{1, \ldots, g-1\}$. Then, $d(l_r, l_{r+2}) = d(l_r, l_{r+1}) + d(l_{r+1}, l_{r+2}) = 2\rho$ for every $r \in \{1, \ldots, g-2\}$, i.e., $\pi_{r(r+2)} = \pi_{s(s+2)}$ for all $r, s \in \{1, \ldots, g-2\}$. Iterating this process, we have $d(l_r, l_{r+t}) = t \cdot \rho$ for all $r, t \in \{1, \ldots, g-1\}$ such that $r + t \leq g$, hence $\pi_{r(r+t)} = \pi_{s(s+t)}$ for all $r, s, t \in \{1, \ldots, g-1\}$ such that $r + t, s + t \leq g$. $\square$

Proposition 4. *If $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is a totally uniform OPM on $\mathcal{L}$, then π is metrizable and it is only generated by the family of linear metrics $d_\rho : \mathcal{L}^2 \longrightarrow \mathbb{R}$ defined as $d_\rho(l_r, l_s) = \rho \cdot |s - r|$ with $\rho > 0$.*

Proof. It is easy to see that the mapping $d_1 : \mathcal{L}^2 \longrightarrow \mathbb{R}$ defined as $d_1(l_r, l_s) = |s - r|$ for all $r, s \in \{1, \ldots, g\}$ is a linear metric. By Remark 5, $\pi_{rs} \succ \pi_{tu} \Leftrightarrow |s - r| < |u - t|$, for all $r, s, t, u \in \{1, \ldots, g\}$. Then, π is metrizable and it is generated by d_1. Let $d : \mathcal{L}^2 \longrightarrow \mathbb{R}$ a linear metric that generates π. Since π is uniform, there exists $\rho > 0$ such that $d(l_r, l_{r+1}) = \rho$ for every $r \in \{1, \dots, g-1\}$, and $d(l_r, l_{r+t}) = \rho \cdot t$ for all $r, t \in \{1, \dots, g-1\}$ such that $r + t \leq g$. Let $r, s \in \{1, \dots, g\}$. If $r \leq s$, then $d(l_r, l_s) = d(l_r, l_{r+(s-r)}) = \rho \cdot (s-r) = \rho \cdot |s-r|$. If $s < r$, then $d(l_r, l_s) = d(l_s, l_r) = d(l_s, l_{s+(r-s)}) = \rho \cdot (r-s) = \rho \cdot |s-r|$. □

Remark 6. For each granularity $g \geq 3$, there is one and only one totally uniform OPM. For $g = 3$ and $g = 4$ these are the ones with associated proximity matrices A_{22} and A_{222}, respectively. For $g = 4$, two of the 51 OPMs are uniform but not totally uniform (hence, non metrizable), the ones with associated proximity matrices A^1_{222} and A^2_{222} (see the Appendix). For $g \geq 5$, finding all the OPMs is a tedious task. However, the unique totally uniform OPM is directly obtained from Remark 5: $\delta_1 = \pi_{rr} = \pi_{ss} \succ \delta_2 = \pi_{r\,(r+1)} = \pi_{(r+1)\,r} = \pi_{s\,(s+1)} = \pi_{(s+1)\,s} \succ \delta_3 = \pi_{r\,(r+2)} = \pi_{(r+2)\,r} = \pi_{s\,(s+2)} = \pi_{(s+2)\,s} \succ \delta_{g-1} = \cdots \succ \pi_{r\,(r+g-2)} = \pi_{(r+g-2)\,r} = \pi_{s\,(s+g-2)} = \pi_{(s+g-2)\,s} \succ \delta_g = \pi_{1g} = \pi_{g1}$, for all admissible $r, s \in \{1, \dots, g\}$.

3. Aggregating experts' opinions through voting systems

Consider that a set of experts $E = \{1, \dots, m\}$ compares the ordinal proximities between pairs of linguistic terms of an OQS $\mathcal{L}$ in a purely ordinal way.

Given $(l_r, l_s), (l_t, l_u) \in \mathcal{L}^2$, with $\pi_{rs} \succ_e \pi_{tu}$ we denote that expert $e \in E$ declares that l_r is closer to l_s than l_t is to l_u. Analogously, $\pi_{rs} =_e \pi_{tu}$ denotes that expert $e \in E$ declares that the proximity between l_s and l_t is the same as that between l_t and l_u.

We now introduce some methods for generating social proximity outcomes through different voting systems: simple and qualified majorities, and scoring rules. Given $(l_r, l_s), (l_t, l_u) \in \mathcal{L}^2$, our aim is to determine, if possible, whether the social outcome is $\pi_{rs} \succ \pi_{tu}$, $\pi_{tu} \succ \pi_{rs}$ or $\pi_{rs} = \pi_{tu}$, after applying the corresponding voting system to the comparisons drawn up by the experts π_{rs} *versus* π_{tu}, for all $e \in E$.

3.1. Majorities

We consider that the *status quo* is $\pi_{rs} = \pi_{tu}$: this should be the social outcome whenever there is no previously fixed majority, declaring $\pi_{rs} \succ \pi_{tu}$ or $\pi_{tu} \succ \pi_{rs}$.

Simple majority is the most decisive majority (when indifferences are allowed, the winner can have very poor support). Qualified majorities require more support before a winner can be declared; they range from absolute to unanimous majorities.

1. If simple majority is applied:
   (a) $\pi_{rs} \succ \pi_{tu} \Leftrightarrow \big(\#\{e \in E \mid \pi_{rs} \succ_e \pi_{tu}\} > \#\{e \in E \mid \pi_{tu} \succ_e \pi_{rs}\}$ and $\#\{e \in E \mid \pi_{rs} \succ_e \pi_{tu}\} > \#\{e \in E \mid \pi_{rs} =_e \pi_{tu}\}\big)$,
   (b) $\pi_{tu} \succ \pi_{rs} \Leftrightarrow \big(\#\{e \in E \mid \pi_{tu} \succ_e \pi_{rs}\} > \#\{e \in E \mid \pi_{rs} \succ_e \pi_{tu}\}$ and $\#\{e \in E \mid \pi_{tu} \succ_e \pi_{rs}\} > \#\{e \in E \mid \pi_{rs} =_e \pi_{tu}\}\big)$,
   (c) $\pi_{rs} = \pi_{tu}$, otherwise.
2. If the qualified majority of threshold $q \in [0.5, 1)$ is applied[4]:
   (a) $\pi_{rs} \succ \pi_{tu} \Leftrightarrow \#\{e \in E \mid \pi_{rs} \succ_e \pi_{tu}\} > q \cdot m$,
   (b) $\pi_{tu} \succ \pi_{rs} \Leftrightarrow \#\{e \in E \mid \pi_{tu} \succ_e \pi_{rs}\} > q \cdot m$,
   (c) $\pi_{rs} = \pi_{tu}$, otherwise.

3.2. Scoring rules

We assume that when an expert $e \in E$ declares $\pi_{rs} \succ_e \pi_{tu}$, then that expert prefers this assessment to $\pi_{rs} =_e \pi_{tu}$, and the latter to $\pi_{tu} \succ_e \pi_{rs}$. Analogously, when an expert $e \in E$ declares $\pi_{tu} \succ_e \pi_{rs}$, then that expert prefers this assessment to $\pi_{rs} =_e \pi_{tu}$, and the later one to $\pi_{rs} \succ_e \pi_{tu}$. We also assume that when an expert $e \in E$ declares $\pi_{rs} =_e \pi_{tu}$, then that expert prefers this assessment to both $\pi_{rs} \succ_e \pi_{tu}$ and $\pi_{tu} \succ_e \pi_{rs}$, and is indifferent between these two latter assessments[5].

These three situations can be visualized in the usual way of showing linear and weak orders

$\pi_{rs} \succ_e \pi_{tu}$	$\pi_{tu} \succ_e \pi_{rs}$	$\pi_{rs} =_e \pi_{tu}$	
$\pi_{rs} =_e \pi_{tu}$	$\pi_{rs} =_e \pi_{tu}$	$\pi_{rs} \succ_e \pi_{tu}$	$\pi_{tu} \succ_e \pi_{rs}$
$\pi_{rs} \prec_e \pi_{tu}$	$\pi_{tu} \prec_e \pi_{rs}$		

If a scoring rule with normalized scoring vector $(1, s, 0)$, with $0 \le s \le 1$, is applied, then 1 point is assigned to the first ranked alternative, s points to the second ranked alternative and 0 points to the third ranked alternative. As usual, when indifferences appear, the average of the corresponding scores is assigned. Thus, in the third situation a score of $s/2$ is assigned to the two alternatives that are in a tie.

[4] Absolute majority corresponds to the case $q = 0.5$.

[5] Clearly, an expert $e \in E$ could prefer $\pi_{rs} \succ_e \pi_{tu}$ to $\pi_{tu} \succ_e \pi_{rs}$ or vice-versa. With this simplification we avoid asking the experts more questions.

3.3. Consistency

For all $r, s, t, u, v, w \in \{1, \dots, g\}$ the following conditions must be satisfied:

$$\big(\pi_{rs} \succ \pi_{tu} \;\wedge\; \pi_{tu} \succ \pi_{vw}\big) \;\Rightarrow\; \pi_{rs} \succ \pi_{vw} \tag{1}$$

$$\big(\pi_{rs} \succ \pi_{tu} \;\wedge\; \pi_{tu} = \pi_{vw}\big) \;\Rightarrow\; \pi_{rs} \succ \pi_{vw} \tag{2}$$

$$\big(\pi_{rs} = \pi_{tu} \;\wedge\; \pi_{tu} \succ \pi_{vw}\big) \;\Rightarrow\; \pi_{rs} \succ \pi_{vw} \tag{3}$$

$$\big(\pi_{rs} = \pi_{tu} \;\wedge\; \pi_{tu} = \pi_{vw}\big) \;\Rightarrow\; \pi_{rs} = \pi_{vw} \tag{4}$$

$$r < s < t \;\Rightarrow\; \pi_{rs} \succ \pi_{rt} \tag{5}$$

$$r < s < t \;\Rightarrow\; \pi_{st} \succ \pi_{rt} \tag{6}$$

Conditions (1), (2) and (3) refer to the linear order $\succ$ on Δ; (4) to the transitivity of $=$; and (5) and (6) to the monotonicity of π.

3.4. A case study

To show how some voting systems can be applied for generating an OPM from the opinions of a group of experts, we consider the data from the case study reported in García-Lapresta and Pérez-Román [11, 2.4].

We now present the results of a survey conducted on 76 members of the *Spanish Society for Fuzzy Logic and Technology* (*ESTYLF*) about the degrees of proximity between the usual decisions of some journal editors (see Table 1).

l_1	l_2	l_3	l_4
Reject	Major revision	Minor revision	Accept

Table 1: Meaning of the linguistic terms.

Table 2 contains the data obtained in the survey.

We now present the social outcomes generated by simple majority and some qualified majorities when aggregating experts' opinions about the ordinal proximities between pairs of linguistic terms.

- Simple majority: $\pi_{34} \succ \pi_{23} \succ \pi_{12} \succ \pi_{24} \succ \pi_{13}$. This information allows us to assign the following degrees of proximity

$$\pi_{rr} = \delta_1 \succ \pi_{34} = \delta_2 \succ \pi_{23} = \delta_3 \succ \pi_{12} = \delta_4 \succ \pi_{24} = \delta_5 \succ \pi_{13} = \delta_6 \succ \pi_{14} = \delta_7$$

π_{12} *versus* π_{23}	Number	%
$\pi_{12} \succ \pi_{23}$	27	35.5
$\pi_{12} \prec \pi_{23}$	32	42.1
$\pi_{12} = \pi_{23}$	17	22.4

π_{23} *versus* π_{34}	Number	%
$\pi_{23} \succ \pi_{34}$	6	7.9
$\pi_{23} \prec \pi_{34}$	69	90.8
$\pi_{23} = \pi_{34}$	1	1.3

π_{12} *versus* π_{34}	Number	%
$\pi_{12} \succ \pi_{34}$	10	13.2
$\pi_{12} \prec \pi_{34}$	54	71.0
$\pi_{12} = \pi_{34}$	12	15.8

π_{12} *versus* π_{24}	Number	%
$\pi_{12} \succ \pi_{24}$	42	55.3
$\pi_{12} \prec \pi_{24}$	18	23.7
$\pi_{12} = \pi_{24}$	16	21.0

π_{13} *versus* π_{34}	Number	%
$\pi_{13} \succ \pi_{34}$	0	0.0
$\pi_{13} \prec \pi_{34}$	75	98.7
$\pi_{13} = \pi_{34}$	1	1.3

π_{13} *versus* π_{24}	Number	%
$\pi_{13} \succ \pi_{24}$	1	1.3
$\pi_{13} \prec \pi_{24}$	53	69.7
$\pi_{13} = \pi_{24}$	22	29.0

Table 2: Data of the survey.

and the metrizable OPM with associated proximity matrix

$$A_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}.$$

- Qualified majorities of thresholds $q \in [0.5, 0.552)$: $\pi_{34} \succ \pi_{12} = \pi_{23} \succ \pi_{24} \succ \pi_{13}$. This information allows us to assign the following degrees of proximity

$$\pi_{rr} = \delta_1 \succ \pi_{34} = \delta_2 \succ \pi_{12} = \pi_{23} = \delta_3 \succ \pi_{24} = \delta_4 \succ \pi_{13} = \delta_5 \succ \pi_{14} = \delta_6$$

and the metrizable OPM with associated proximity matrix

$$A_{332} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}.$$

- Qualified majorities of thresholds $q \in [0.697, 0.710)$: $\pi_{34} \succ \pi_{12} = \pi_{23} =$

$\pi_{13} = \pi_{24}$. Since $\pi_{13} = \pi_{12}$ and $\pi_{24} = \pi_{23}$, π violates the monotonicity conditions (5) and (6) and, consequently, π is not an OPM. Nevertheless, the obtained ordinal proximities can be arranged in the following matrix

$$\begin{pmatrix} \delta_1 & \delta_3 & \delta_3 & \delta_4 \\ & \delta_1 & \delta_3 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}.$$

- Qualified majorities of thresholds $q \in [0.710, 0.907)$: $\pi_{12} = \pi_{23}$, $\pi_{12} = \pi_{34}$, $\pi_{34} \succ \pi_{13}$, $\pi_{34} \succ \pi_{23}$, $\pi_{12} = \pi_{24}$ and $\pi_{13} = \pi_{24}$. Since $\pi_{34} = \pi_{12}$, $\pi_{12} = \pi_{23}$ and $\pi_{34} \succ \pi_{23}$, π violates condition (4), the transitivity of $=$, and, consequently, it is not an OPM. In this case it is not possible to arrange the ordinal proximities obtained in any matrix.

We now present the outcomes generated by some scoring rules when aggregating experts' opinions on the proximities between pairs of linguistic terms. Consider the normalized scoring vector $(1, s, 0)$, with $0 \le s \le 1$.

- $s \in [0, 0.2]$: we obtain the metrizable OPM with associated proximity matrix A_{432}.

- $s \in [0.3, 0.5)$: we obtain the metrizable OPM with associated proximity matrix A_{332}.

- $s \in (0.5, 0.7]$: $\pi_{34} \succ \pi_{23} = \pi_{12} = \pi_{24}$. Since $\pi_{24} = \pi_{23}$, π violates the monotonicity condition (5) and, consequently, π is not an OPM.

- $s \in [0.8, 0.9]$: $\pi_{12} = \pi_{23}$, $\pi_{12} = \pi_{34}$, $\pi_{34} \succ \pi_{13}$, $\pi_{34} \succ \pi_{23}$, $\pi_{12} = \pi_{24}$ and $\pi_{13} = \pi_{24}$. Since $\pi_{34} = \pi_{12}$, $\pi_{12} = \pi_{23}$ and $\pi_{34} \succ \pi_{23}$, π violates condition (4), the transitivity of $=$, and, consequently, it is not an OPM. Again, it is not possible to arrange the obtained ordinal proximities in any matrix.

All these inconsistencies may be considered as specific problems of judgment aggregation within social choice theory, in the sense that the aggregation of judgments over multiple interconnected issues may produce inconsistent outcomes (see, for instance, Grossi and Pigozzi [15]). Some proposals aimed at avoiding these inconsistencies are made in the next section.

4. Avoiding inconsistencies

In order to avoid the inconsistencies shown in Section 3 when experts' opinions are aggregated through voting systems, we now introduce some distance-based procedures. Two of them are related to the characterizations of the median and the mean, in the sense that the outcomes are the metrizable OPMs that minimize the sum of distances and squared distances, respectively, to the metrizable OPMs of the agents.

First we introduce metrics on OPMs.

4.1. Distances between OPMs

With $P(\mathcal{L})$ we denote the set of OPMs on $\mathcal{L}$. With $M(\mathcal{L})$ we denote the set of metrizable OPMs on $\mathcal{L}$.

Definition 5. Let $\pi^1 : \mathcal{L}^2 \longrightarrow \Delta_1$ and $\pi^2 : \mathcal{L}^2 \longrightarrow \Delta_2$ be two OPMs, $w : \mathbb{N} \longrightarrow \mathbb{R}$ a weighting function such that $w(1) = 1 \geq w(2) \geq \cdots \geq w(g-1) > 0$, $\beta : (\Delta_1 \cup \Delta_2)^2 \longrightarrow \mathbb{N} \cup \{0\}$ the mapping defined as $\beta(\delta_i, \delta_j) = |i - j|$ and $S = \{(r, s) \in \{1, \ldots, g-1\}^2 \mid r + s \leq g\}$. Then, $D_w\left(\pi^1, \pi^2\right)$ is defined as

$$D_w\left(\pi^1, \pi^2\right) = \sum_{(r,s)\in S} w(s) \cdot \beta\big(\pi^1_{r\,(r+s)}, \pi^2_{r\,(r+s)}\big). \tag{7}$$

Remark 7. An equivalent formulation of Eq. (7) is

$$\begin{aligned} D_w\left(\pi^1, \pi^2\right) \;\; = \;\; & \sum_{r=1}^{g-1} \beta\big(\pi^1_{r\,(r+1)}, \pi^2_{r\,(r+1)}\big) + w(2) \cdot \sum_{r=1}^{g-2} \beta\big(\pi^1_{r\,(r+2)}, \pi^2_{r\,(r+2)}\big) + \\ & + w(3) \cdot \sum_{r=1}^{g-3} \beta\big(\pi^1_{r\,(r+3)}, \pi^2_{r\,(r+3)}\big) + \cdots \end{aligned}$$

Remark 8. Some simple examples of weighting functions are:

1. Power: $w(s) = \dfrac{1}{s^\alpha}$, with $\alpha \geq 1$.
2. Exponential: $w(s) = \dfrac{1}{\alpha^{s-1}}$, with $\alpha > 1$.
3. Linear: $w(s) = 1 - \dfrac{1-\alpha}{g-2} \cdot (s-1)$, with $0 < \alpha < 1$ (note that $\alpha = w(g-1)$).

Proposition 5. *If $w : \mathbb{N} \longrightarrow \mathbb{R}$ is a weighting function such that $w(1) = 1 \geq w(2) \geq \cdots \geq w(g-1) > 0$, then the mapping $D_w : P(\mathcal{L})^2 \longrightarrow \mathbb{R}$ defined from Eq. (7) is a metric.*

Proof. Taking into account that $w(s) > 0$ for every $s \in \{1, \ldots, g-1\}$ and the mapping $\beta : \{\delta_i \mid i \in \mathbb{N}\} \longrightarrow \mathbb{R}$ defined as $\beta(\delta_i, \delta_j) = |i - j|$ is a metric, we have that $D_w\left(\pi^1, \pi^2\right)$ is a positive linear combination of distances. Consequently, D_w is a metric. □

Remark 9. For $g = 4$, let $w : \mathbb{N} \longrightarrow \mathbb{R}$ be the weighting functions with power weights for $\alpha = 1$, i.e., $w(1) = 1$, $w(2) = \frac{1}{2}$ and $w(3) = \frac{1}{3}$, and $\alpha = 2$, i.e., $w(1) = 1$, $w(2) = \frac{1}{4}$ and $w(3) = \frac{1}{9}$, and exponential weights for $\alpha = 2$, i.e., $w(1) = 1$, $w(2) = \frac{1}{2}$ and $w(3) = \frac{1}{4}$, and $\alpha = 3$, i.e., $w(1) = 1$, $w(2) = \frac{1}{3}$ and $w(3) = \frac{1}{9}$. The closest metrizable OPM, with respect to D_w, to the non metrizable OPMs with associated proximity matrices A^1_{ijk} and A^2_{ijk} (see the Appendix) are just the ones associated with A_{ijk} or A'_{ijk} (see Subsection 2.3). However, linear weighting functions do not follow this pattern. For instance, if $\alpha = 0.3$, i.e., $w(1) = 1$, $w(2) = 0.85$ and $w(3) = 0.7$, then the closest metrizable OPM, with respect to D_w, to the non metrizable OPMs with associated proximity matrices A^1_{222} and A^2_{222} (see the Appendix) is not the one associated with A_{222}, but the ones associated with A'_{223} and A'_{322}, respectively. For this reason, on the sequel we do not consider linear weighting functions.

In what follows we focus on metrizable OPMs. Consider a profile of metrizable OPMs $(\pi^1, \ldots, \pi^m) \in M(\mathcal{L})^m$ associated with m experts[6]. We are interested in finding a metrizable OPM that represents, as faithfully as possible, the opinions of the experts. As shown in Subsection 3.4, applying a voting system for generating the ordinal proximities between pairs of linguistic terms from the comparisons of the experts may generate inconsistencies. In the following subsections we propose different approaches for solving these problems.

4.2. Voting systems

In spite of the inconsistency problems mentioned in Section 3, in some cases it is possible to use a voting system to obtain a matrix $(\overline{\pi}_{rs})$ that represents the ordinal proximities between pairs of linguistic terms[7]. If $(\overline{\pi}_{rs})$ corresponds to a metrizable OPM $\overline{\pi} \in M(\mathcal{L})$, then $\overline{\pi}$ is the outcome. Otherwise, once a

[6]Notice that experts are not required directly to provide metrizable OPMs. They only need to answer some simple questions. In particular, for $g = 4$, following the algorithm included in Figure 4, the corresponding metrizable OPMs are easily generated once between two and four questions have been answered.

[7]As shown in the examples provided in Subsection 3.4, such a matrix cannot exist if condition (4) is not satisfied.

metric D_w has been fixed, we find[8]

$$\{\pi \in M(\mathcal{L}) \mid \forall \pi' \in M(\mathcal{L}) \quad D_w(\pi, \overline{\pi}) \leq D_w(\pi', \overline{\pi})\},$$

i.e., the solution of

$$\arg \min_{\pi \in M(\mathcal{L})} D_w(\pi, \overline{\pi}).$$

If this set contains more than one metrizable OPM, a tie-breaking procedure needs to be applied. One possibility is to set a sequence of weighting functions and apply them lexicographically.

4.3. Minimizing aggregated distances

We now introduce a proposal for obtaining collective metrizable OPMs from the individual ones[9]. It is based on the characterizations of the statistical notions of median and mean.

4.3.1. The median

It is well known that the medians of a list of numbers is the set of real numbers that minimize the sum of distances to the numbers of the list. In this way, the metrizable OPMs that minimize the sum of distances (for a fixed metric D_w) to the metrizable OPMs of the agents, $\left(\pi^1, \ldots, \pi^m\right)$, can be said to be their medians. Thus, the medians of $\left(\pi^1, \ldots, \pi^m\right)$ are the elements of the following set

$$\operatorname{med}_w\left(\pi^1, \ldots, \pi^m\right) =$$
$$\left\{\pi \in M(\mathcal{L}) \mid \forall \pi' \in M(\mathcal{L}) \quad \sum_{i=1}^{m} D_w\left(\pi, \pi^i\right) \leq \sum_{i=1}^{m} D_w\left(\pi', \pi^i\right)\right\},$$

i.e., the solution of

$$\operatorname{med}_w\left(\pi^1, \ldots, \pi^m\right) = \arg \min_{\pi \in M(\mathcal{L})} \sum_{i=1}^{m} D_w\left(\pi, \pi^i\right). \tag{8}$$

---

[8] We extend Eq. (7) to matrices that do not necessarily correspond to OPMs.

[9] This proposal is related to the one provided by Grossi and Pigozzi [15, 4.3.3] in a specific problem of judgment aggregation. See also Eckert and Klamler [7] and Lang *et al.* [20].

4.3.2. The mean

It is also well known that the mean of a list of numbers is the real number that minimizes the sum of the squared distances to the numbers of the list. The metrizable OPM that minimizes the sum of squared distances (for a fixed metric D_w) to the metrizable OPMs of the agents, $\left(\pi^1, \ldots, \pi^m\right)$, can be said to be its mean. Thus, the mean of $\left(\pi^1, \ldots, \pi^m\right)$ is the following set

$$\mathrm{mean}_w \left(\pi^1, \ldots, \pi^m\right) =$$

$$\left\{\pi \in M(\mathcal{L}) \mid \forall \pi' \in M(\mathcal{L}) \quad \sum_{i=1}^{m} \left(D_w\left(\pi, \pi^i\right)\right)^2 \leq \sum_{i=1}^{m} \left(D_w\left(\pi', \pi^i\right)\right)^2\right\},$$

i.e., the solution of

$$\mathrm{mean}_w \left(\pi^1, \ldots, \pi^m\right) = \underset{\pi \in M(\mathcal{L})}{\arg} \min \sum_{i=1}^{m} \left(D_w\left(\pi, \pi^i\right)\right)^2. \tag{9}$$

Notice that the sets of medians and means may have more than one metrizable OPM. In such cases, a tie-breaking procedure needs to be applied. As in Subsection 4.2, one possibility is to set a sequence of weighting functions and apply them lexicographically.

Example 1. Table 3 shows the matrices associated with the OPMs of the 76 participants in the survey shown in Subsection 3.4 after the algorithm included in Figure 4 for $g = 4$ has been applied.

If we consider D_w for the weighting functions appearing in Remark 9, power weights for $\alpha = 1$, i.e., $w(1) = 1$, $w(2) = \frac{1}{2}$ and $w(3) = \frac{1}{3}$, and $\alpha = 2$, i.e., $w(1) = 1$, $w(2) = \frac{1}{4}$ and $w(3) = \frac{1}{9}$, and exponential weights for $\alpha = 2$, i.e., $w(1) = 1$, $w(2) = \frac{1}{2}$ and $w(3) = \frac{1}{4}$, and $\alpha = 3$, i.e., $w(1) = 1$, $w(2) = \frac{1}{3}$ and $w(3) = \frac{1}{9}$, then the median and the mean of $\left(\pi^1, \ldots, \pi^{76}\right)$ is the metrizable OPM with associated matrix A_{332}, just the same as the one obtained when absolute majority and closed qualified majorities are applied in Subsection 3.4.

Remark 10. The sums appearing in 4.3.1 and 4.3.2 can be changed for an aggregation function F (for instance an OWA operator, such as the median or a trimmed mean, a quasiarithmetic mean, etc.). Thus, (8) and (9) become

$$\underset{\pi \in M(\mathcal{L})}{\arg} \min F\Big(D_w\left(\pi, \pi^1\right), \ldots, D_w\left(\pi, \pi^m\right)\Big)$$

and

$$\underset{\pi \in M(\mathcal{L})}{\arg} \min F\Big(\left(D_w\left(\pi, \pi^1\right)\right)^2, \ldots, \left(D_w\left(\pi, \pi^m\right)\right)^2\Big),$$

Matrix	Frequency	%
A_{222}	4	5.3
A_{223}	1	1.3
A_{232}	5	6.6
A_{243}	6	7.9
A_{323}	1	1.3
A_{332}	12	15.8
A_{342}	16	21.1
A_{432}	11	14.5
A'_{432}	4	5.3
A_{532}	16	21.1
	76	100

Table 3: Matrices of the survey in Example 1.

respectively.

For instance, in Example 1 if the power weighting function for $\alpha = 1$, i.e., $w(1) = 1$, $w(2) = \frac{1}{2}$ and $w(3) = \frac{1}{3}$ is considered, and the distances are aggregated through the trimmed mean that removes the three highest and the three lowest values, then the metrizable OPM with associated matrix A_{432} is obtained, which is just the same as the one obtained when simple majority is applied in Subsection 3.4.

Notice that the metrizable OPMs with the highest frequency (16), those with associated proximity matrices A_{342} and A_{532}, are not selected as the social outcome for any of the procedures considered. This is not surprising at all, since plurality rule[10] could not faithfully represent individual opinions when there are more than two alternatives (see, for instance, Morales [25] –English translation in McLean and Urken [23]– and Laslier [21]).

Example 2. In this example we apply the proposed procedure to a 4-term OQS used in the Trends in International Mathematics and Science Study (TIMSS). This study evaluates the home, community, school, and student factors associated with student achievement in mathematics and science at fourth and eighth grade levels in more than 60 countries. To that end, data is collected through questionnaires completed by students, parents, teachers, and school principals. The TIMSS has been conducted every four years since 1995. The latest data

[10]In this voting system each agent votes for only one alternative and the winners are the alternatives with most number of votes.

available is from TIMSS 2015 (see [28]).

In this example, a 4-term OQS used in TIMSS 2015 to assess some school problems (arriving late at school, vandalism, cheating, classroom disturbances, etc.) is considered. The linguistic terms of the scale appear in Table 4.

l_1	l_2	l_3	l_4
Serious problem	Moderate problem	Minor problem	Not a problem

Table 4: Meaning of the linguistic terms.

As in Example 1, we carried out an on-line survey as to the proximities between the terms of the scale. A total of 19 Spanish school principals who participated in TIMSS 2015 contributed to this survey.

The algorithm included in Figure 4 for $g = 4$ is applied and the matrices associated with the OPMs of the 19 participants are then shown in Table 5.

Matrix	Frequency	%
A_{222}	6	31.59
A_{232}	1	5.26
A_{234}	1	5.26
A_{322}	2	10.53
A_{332}	3	15.79
A_{342}	1	5.26
A_{432}	1	5.26
A_{532}	4	21.05
	19	100

Table 5: Matrices of the survey in Example 2.

Considering D_w for the same weighting functions as in Example 1, the metrizable OPM obtained using the median and the mean is the matrix A_{332}.

5. Concluding remarks

Given an OQS, determining whether the scale is uniform or not (and if not what the ordinal proximities between the linguistic terms of the scale are) is an important issue. Our proposal is to ask some experts their opinions about these ordinal proximities and aggregate these opinions to obtain a metrizable OPM on the OQS. This is not a trivial problem, since inconsistencies could appear (see Section 3).

We propose various procedures for generating a metrizable OPM from the opinions of the experts, paying special attention to OQSs with three or four terms. Once this problem is solved, the OPM obtained can be used in different decision-making and classification problems in which agents show their opinions about a set of alternatives through an OQS equipped with a metrizable OPM: Measuring consensus in a group of agents on a subset of alternatives, and consensus-based clustering procedures, as in García-Lapresta and Pérez-Román [11]; consensus-reaching processes, as in García-Lapresta and Pérez-Román [13]; implementing an appropriate voting system, such as the one introduced and analyzed in García-Lapresta and Pérez-Román [14]; etc.

As shown in Subsection 2.3, for $g = 3$, answering a single question is sufficient to assign the corresponding metrizable OPM; for $g = 4$, the number of questions should be between 2 and 4 (see Figure 4). Following this pattern, it is possible to determine all the metrizable OPMs of OQSs with granularity $g \in \{5, 6, 7\}$ (OQSs with more than seven linguistic terms are neither usual nor appropriate). This tedious task needs to be carried out if the proposal included in Subsections 4.2 and 4.3 is applied.

Taking into account Proposition 4, totally uniform OPMs are metrizable only through $\pi_{rs} \succ \pi_{tu} \;\Leftrightarrow\; \rho \cdot |s - r| < \rho \cdot |u - t|$, with $\rho > 0$. Thus, it is appropriate to associate a real number with each term of the OQS (for instance, l_r can be identified with r). However, this identification is not possible for non-uniform OQSs (equipped with the corresponding OPMs), even when they are metrizable, since different non proportional linear metrics may generate the same OPM (see Remark 3). On the meaningless of assigning numerical values to the linguistic terms of non-uniform OQSs, see Roberts [26], Franceschini *et al.* [9] and Fattore *et al.* [8], among others.

As further research, the present analyses could be extended to the framework of intervals of linguistic terms, when agents are allowed to assign several consecutive terms of the OQS, if they hesitate (see García-Lapresta and Pérez-Román [12] and García-Lapresta and González del Pozo [10]).

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Annex: The non metrizable OPMs for $g = 4$

$$A^1_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{232} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{232} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{233} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{233} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{234} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{234} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{243} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_4 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{243} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_4 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{322} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{322} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{323} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{323} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_4 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{324} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{324} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{332} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{332} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_5 \\ & \delta_1 & \delta_3 & \delta_4 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{342} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_4 & \delta_6 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{342} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_4 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{423} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_6 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{423} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix},$$

$$A^1_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_6 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}, \quad A^2_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_5 & \delta_6 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}.$$