Basic results on the equations of magnetohydrodynamics of partially ionized inviscid plasmas

Manuel Núñez[a)]
Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain

(Received 7 April 2009; accepted 17 September 2009; published online 15 October 2009)

The equations of evolution of partially ionized plasmas have been far more studied in one of their many simplifications than in its original form. They present a relation between the velocity of each species, plus the magnetic and electric fields, which yield as an analog of Ohm's law a certain elliptic equation. Therefore, the equations represent a functional evolution system, not a classical one. Nonetheless, *a priori* estimates and theorems of existence may be obtained in appropriate Sobolev spaces. © *2009 American Institute of Physics.* [doi:10.1063/1.3246611]

I. INTRODUCTION

One of the most instructive aspects of fluid mechanics is to detail how the Boltzmann equations describing statistically how a fluid transports heat, electric charge, and magnetic fields are drastically simplified to obtain manageable magnetohydrodynamics (MHD) systems.[1] The culmination of this process is the single fluid description of classical MHD. While the range of phenomena described satisfactorily by the equations of MHD is extremely wide, covering from dynamo theory in astrophysics[2] to industrial liquid metals,[3] there are still many phenomena where the multicomponent character of the plasma is a key feature of its behavior. This includes fast magnetic reconnection,[4,5] the consequences of the Hall effect,[6] ambipolar drift,[7] and the propagation of beams in a plasma.[8] It is therefore worthwhile to analyze a more general frame by considering a three-species plasma formed by positive ions, electrons, and neutral particles. This covers many instances, although not certainly all; moreover, in order to bring the equations to manageable length, we will assume each of the species incompressible and of constant density. This excludes, in particular, ionization and recombination effects that effectively change one species to another.[9,10] Moreover, for the case of a bounded domain, we will take boundary conditions for the magnetic field intended to avoid any interchange of electromagnetic energy with the outside of the domain. The logic of the equations is clear: each of the species satisfies a momentum equation with an electromagnetic forcing and collisional effects. We will omit the possible existence of viscosity, thus degrading these equations from a Navier–Stokes type to an Euler one. The viscous case was studied in Ref. 11, but in most applications,[12] the plasma is considered inviscid because the rarified astrophysical plasmas have indeed practically no viscosity. The collision frequencies between species a and b, ν_{ab}, satisfy $\rho_a \nu_{ab} = \rho_b \nu_{ba}$, where ρ_a is the material density of species a. This density is related to the number density n_a by $\rho_a = m_a n_a$, where m_a is the mass of a single particle of this species. The system has been known for quite a long time.[13,14] The collision coefficients ν_{ab} may be expressed in terms of fractional ionizations and plasma temperature,[15] but we will take them as constant for further simplification. Also we will abbreviate the notation by taking units so that the speed of light is $c=1$ and the magnetic permeability is $\mu=1$. If we denote by $\mathbf{v}_a$, $a=e,i,n$, the velocities of ions, electrons, and neutrals, respectively, by P_a the respective kinetic pressures, by $\mathbf{E}$ the electric field, and by $\mathbf{B}$ the magnetic one, the momentum equations are

a)Electronic mail: mnjmhd@am.uva.es.

$$\frac{\partial \mathbf{v}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -\frac{e}{m_e}(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \frac{\nabla P_e}{\rho_e} - \sum_{j=i,n} \nu_{ej}(\mathbf{v}_e - \mathbf{v}_j), \tag{1}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = \frac{e}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \frac{\nabla P_i}{\rho_i} - \sum_{j=e,n} \nu_{ij}(\mathbf{v}_i - \mathbf{v}_j), \tag{2}$$

$$\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\frac{\nabla P_n}{\rho_n} - \sum_{j=e,i} \nu_{nj}(\mathbf{v}_n - \mathbf{v}_j). \tag{3}$$

The magnetic field satisfies Faraday's equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{4}$$

which implies that $\nabla \cdot \mathbf{B} = 0$ for all time if the initial conditions satisfy this. The system still lacks one condition because we have five unknowns and four equations. This is given by the fact that the current density $\mathbf{J}$ is the flow of positive charges,

$$\mathbf{J} = en_e(\mathbf{v}_i - \mathbf{v}_e). \tag{5}$$

On the other hand, as usual in these relatively low-frequency plasmas, we omit the displacement current from Ampère's law, and we get

$$\nabla \times \mathbf{B} = \mathbf{J}. \tag{6}$$

Later we will join these laws together in a single self-contained system.

II. SIMPLIFIED VERSIONS OF THE GENERAL SYSTEM

To see how the character of the equations changes with every one of the several simplifications used in the study of particular phenomena, we will consider some of the most important. For fast time scales, electrons will respond much more quickly than ions or neutrals, so we may take $\mathbf{v}_i = \mathbf{v}_n = \mathbf{0}$ in (1)–(3) and (5). Writing

$$\mathbf{v}_e \cdot \nabla \mathbf{v}_e = -\mathbf{v}_e \times (\nabla \times \mathbf{v}_e) + \tfrac{1}{2} \nabla (|\mathbf{v}_e|^2), \tag{7}$$

(1) may be written as

$$-\frac{1}{en_e}\frac{\partial \mathbf{J}}{\partial t} - \mathbf{v}_e \times (\nabla \times \mathbf{v}_e) + \frac{1}{2} \nabla (|\mathbf{v}_e|^2) = -\frac{1}{m_e n_e} \nabla P_e - \frac{e}{m_e}\mathbf{E} - \frac{e}{m_e}(\mathbf{v}_e \times \mathbf{B}) - \frac{\nu_{ei} + \nu_{en}}{en_e}\mathbf{J}. \tag{8}$$

Taking rotationals and using the fact that $\nabla \times \mathbf{J} = -\Delta \mathbf{B}$,

$$\frac{1}{en_e}\frac{\partial \Delta \mathbf{B}}{\partial t} - \nabla \times \left[\mathbf{v}_e \times \frac{1}{en_e}\Delta \mathbf{B} \right] = \frac{e}{m_e}\frac{\partial \mathbf{B}}{\partial t} - \frac{e}{m_e} \nabla \times (\mathbf{v}_e \times \mathbf{B}) - \frac{\nu_{ei} + \nu_{en}}{en_e}\Delta \mathbf{B}, \tag{9}$$

i.e.,

$$\frac{\partial}{\partial t}\left(\mathbf{B} - \frac{m_e}{e^2 n_e}\Delta \mathbf{B} \right) = \nabla \times \left[\mathbf{v}_e \times \left(\mathbf{B} - \frac{m_e}{e^2 n_e}\Delta \mathbf{B} \right) \right] + \frac{(\nu_{ei} + \nu_{en})m_e}{e^2 n_e}\Delta \mathbf{B}. \tag{10}$$

The values

$$d_e^2 = \frac{m_e}{e^2 n_e}, \quad \eta = \frac{(\nu_{ei} + \nu_{en}) m_e}{e^2 n_e}$$

are called electron skin depth and resistivity, respectively. The equation

$$\frac{\partial}{\partial t}(\mathbf{B} - d_e^2 \Delta \mathbf{B}) = -\frac{1}{e n_e} \nabla \times (\mathbf{J} \times (\mathbf{B} - d_e^2 \Delta \mathbf{B})) + \eta \Delta \mathbf{B} \tag{11}$$

forms the basis of electron MHD (EMHD).[16–18]

Another simplification concerns the case when the plasma is totally ionized and the velocities are small enough for the quadratic terms $\mathbf{v}_j \cdot \nabla \mathbf{v}_j$ to be safely ignored. Substituting $\mathbf{v}_e = \mathbf{v}_i - \mathbf{J}/e n_e$ into the electron momentum equation (1), we find

$$-\frac{1}{e n_e}\frac{\partial \mathbf{J}}{\partial t} + \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -\frac{1}{n_e m_e} \nabla P_e - \frac{e}{m_e}\mathbf{E} - \frac{e}{m_e}\mathbf{v}_i \times \mathbf{B} + \frac{1}{m_e n_e}\mathbf{J} \times \mathbf{B} + \frac{\nu_{ei}}{e n_e}\mathbf{J}. \tag{12}$$

Combining this with the ion momentum equation (2), one finds

$$n_e\left(1 + \frac{m_e}{m_i}\right)\mathbf{E} + n_e\left(1 - \frac{m_e}{m_i}\right)(\mathbf{v}_i \times \mathbf{B}) - \frac{1}{e}\mathbf{J} \times \mathbf{B} - \frac{m_e}{e^2}\frac{\partial \mathbf{J}}{\partial t} + \frac{1}{e}\nabla\left(P_e - \frac{m_e}{m_i}P_i\right) - \frac{m_e \nu_{ei}}{e^2}\left(1 - \frac{m_e}{m_i}\right)\mathbf{J} = 0. \tag{13}$$

Since the ions are at least protons and, therefore, $m_e \ll m_i$, we may take the quotient m_e/m_i as zero and obtain

$$\mathbf{E} + \mathbf{v}_i \times \mathbf{B} = -\frac{\nabla P_e}{e n_e} + \frac{m_e \nu_{ie}}{e^2 n_e}\mathbf{J} + \frac{1}{e n_e}\mathbf{J} \times \mathbf{B} + \frac{m_e}{e^2 n_e}\frac{\partial \mathbf{J}}{\partial t}, \tag{14}$$

which is the two-fluid Ohm equation. The value $(e n_e)^{-1}$ is called the Hall coefficient. The pressure term is the cause of the so-called Biermann battery, which in the case that n_e is not constant may provide a thermal source for the magnetic field.[19] By using Faraday's law, this yields the two-fluid induction equation,

$$\frac{m_e}{e^2 n_e}\frac{\partial(\nabla \times \mathbf{J})}{\partial t} + \frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} + \nabla \times (\mathbf{v}_i \times \mathbf{B}) - \frac{1}{e n_e}\nabla \times (\mathbf{J} \times \mathbf{B}). \tag{15}$$

The electron inertia term

$$\frac{m_e}{e^2 n_e}\frac{\partial \mathbf{J}}{\partial t} \tag{16}$$

is extremely small and may be omitted in low-frequency, large scale phenomena. This procedure yields the Hall induction equation; and when the term in $\nabla \times (\mathbf{J} \times \mathbf{B})$ is also ignored, the classical MHD system.

Another different approximation involves canceling the electron velocity altogether, based on the small contribution of the electrons' mass to the whole momentum. This makes (1) an equilibrium equation,

$$\mathbf{E} + \mathbf{v}_i \times \mathbf{B} = -\frac{\nabla P_e}{e n_e} - \frac{1}{e n_e}\mathbf{J} \times \mathbf{B} + \frac{m_e(\nu_{ie} + \nu_{in})}{e^2 n_e}\mathbf{J} + \frac{m_e \nu_{en}}{e n_e}(\mathbf{v}_i - \mathbf{v}_n), \tag{17}$$

which is similar to (14), except by the absence of electron inertia and the presence of the difference between the ion and neutral velocities. This is estimated by assuming that the accelerations of ions and neutrals are similar, $\partial \mathbf{v}_i/\partial t = \partial \mathbf{v}_n/\partial t$. With these assumptions, by adding (2) and (3) (without the quadratic terms), we obtain the momentum equation for the *mean velocity*

$$\mathbf{v} = \frac{\rho_i \mathbf{v}_i + \rho_n \mathbf{v}_n}{\rho_i + \rho_n}, \tag{18}$$

which jointly with (17), yields a new induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\mathbf{v} \times \mathbf{B} - \frac{m_e(\nu_{ie} + \nu_{in})}{e^2 n_e} \mathbf{J} - \frac{r}{e n_e} \mathbf{J} \times \mathbf{B} + \frac{r^2}{\rho_i \nu_{in}} (\mathbf{J} \times \mathbf{B}) \times \mathbf{B} + \frac{r^2}{\rho_i \nu_{in}} \left(\frac{\rho_i}{\rho_n} \nabla P_n - \nabla P_i - \nabla P_e \right) \times \mathbf{B} \right], \tag{19}$$

where $r = \rho_i / (\rho_i + \rho_n)$. This approximation is useful when studying the drift of charged with respect to neutral fluid, i.e., the so-called phenomenon of ambipolar diffusion. Sometimes a further drastic (strong coupling) simplification is taken,

$$\mathbf{v}_i - \mathbf{v}_n = \frac{1}{\nu_{in} \rho_i \rho_n} \mathbf{J} \times \mathbf{B}. \tag{20}$$

III. OHM'S LAW AND CLOSED FORM OF THE SYSTEM

The mathematical problem with these simplified equations is that one often suppresses the highest order term so that the character of the system changes radically and it seems impossible to find a general theorem of existence covering all cases. We will consider the original systems (1)–(4). The set where it is defined is either the whole space $\mathbb{R}^3$, a periodic box Q, or a bounded smooth domain Ω. In the last case, boundary conditions must be set. Since we do not want the fluid to leave Ω, all the velocities must be tangential to the boundary: $\mathbf{v}_j \cdot \mathbf{n}|_{\partial\Omega} = 0$. Given the definition of the current (5), also $\mathbf{J} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Among all the solenoidal fields $\mathbf{B}$ satisfying $\nabla \times \mathbf{B} = \mathbf{J}$, there is precisely one satisfying as well $\mathbf{B} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}$.[20] Other possible fields are of the form $\mathbf{B} + \nabla\Phi$, where Φ is a harmonic function in Ω. The jump $[\mathbf{B} \times \mathbf{n}]$ at the boundary is the surface current $\mathbf{K}$. As for the electric field, in the absence of free charges, it satisfies $\nabla \cdot \mathbf{E} = 0$. Faraday's equation shows that $\mathbf{E}$ is a vector potential for $-\partial \mathbf{B} / \partial t$. This may be chosen in a unique way if we set $\mathbf{E} \cdot \mathbf{n}|_{\partial\Omega} = 0$; any sum $\mathbf{E} + \nabla\Psi$, Ψ, a harmonic function, also is a vector potential, and the value $[\mathbf{E} \cdot \mathbf{n}] = \partial\Psi / \partial n = \rho_s$ is the surface charge density. Although we could work with fixed $\mathbf{K}$ and ρ_s, it is highly convenient for simplicity to assume them to be zero, i.e., the magnetic field is normal to the boundary and the electric field parallel to it. In these conditions, differentiating (5),

$$\frac{\partial \mathbf{J}}{\partial t} = e n_e \left(\frac{\partial \mathbf{v}_i}{\partial t} - \frac{\partial \mathbf{v}_e}{\partial t} \right). \tag{21}$$

When we take this value to the difference between (1) and (2), we obtain

$$\frac{1}{e n_e} \frac{\partial \mathbf{J}}{\partial t} = -\mathbf{v}_i \cdot \nabla \mathbf{v}_i + \mathbf{v}_e \cdot \nabla \mathbf{v}_e - \frac{\nabla P_i}{\rho_i} + \frac{\nabla P_e}{\rho_e} + e \left(\frac{1}{m_i} + \frac{1}{m_e} \right) \mathbf{E} + e \left(\frac{\mathbf{v}_i}{m_i} + \frac{\mathbf{v}_e}{m_e} \right) \times \mathbf{B} - (\nu_{ei} + \nu_{ie} + \nu_{in}) \mathbf{v}_i + (\nu_{ei} + \nu_{ie} + \nu_{en}) \mathbf{v}_e + (\nu_{in} - \nu_{en}) \mathbf{v}_n. \tag{22}$$

On the other hand, since $\nabla \cdot \mathbf{E} = 0$,

$$\frac{\partial \mathbf{J}}{\partial t} = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\nabla \times \mathbf{E}) = \Delta \mathbf{E}, \tag{23}$$

which implies that

$$\frac{1}{en_e}\Delta\mathbf{E} - e\left(\frac{1}{m_i}+\frac{1}{m_e}\right)\mathbf{E}$$
$$= -\mathbf{v}_i\cdot\nabla\mathbf{v}_i + \mathbf{v}_e\cdot\nabla\mathbf{v}_e - \frac{\nabla P_i}{\rho_i} + \frac{\nabla P_e}{\rho_e} + e\left(\frac{\mathbf{v}_i}{m_i}+\frac{\mathbf{v}_e}{m_e}\right)\times\mathbf{B}$$
$$-(\nu_{ei}+\nu_{ie}+\nu_{in})\mathbf{v}_i + (\nu_{ei}+\nu_{ie}+\nu_{en})\mathbf{v}_e + (\nu_{in}-\nu_{en})\mathbf{v}_n. \tag{24}$$

This is the complete law of Ohm. Unlike its simplified versions, it is not a point relation between **E** and the remaining variables, but an elliptic equation. The operator acting on **E** is

$$T{:}\mathbf{E} \to \frac{1}{en_e}\Delta\mathbf{E} - e\left(\frac{1}{m_i}+\frac{1}{m_e}\right)\mathbf{E}. \tag{25}$$

T is elliptic when defined in the spaces

$$H_s = \{\mathbf{E} \in H^1(\mathbb{R}^3)^3 {:} \nabla\cdot\mathbf{E} = 0\}, \tag{26}$$

$$H_p = \left\{\mathbf{E} \in H^1(Q)^3 {:} \nabla\cdot\mathbf{E} = 0,\ \ \mathbf{E}\ \text{ periodic}, \int_Q \mathbf{E}dV = \mathbf{0}\right\}, \tag{27}$$

$$H_d = \{\mathbf{E} \in H^1(\Omega)^3 {:} \nabla\cdot\mathbf{E} = 0,\ \ (\nabla\times\mathbf{E})\times\mathbf{n}|_{\partial\Omega} = \mathbf{0},\ \ \mathbf{E}\cdot\mathbf{n}|_{\partial\Omega} = 0\}. \tag{28}$$

This follows from the identity

$$(\Delta\mathbf{E},\mathbf{E}) = -\int_{\partial\Omega}(\nabla\times\mathbf{E},\mathbf{E},\mathbf{n})d\sigma - \|\nabla\times\mathbf{E}\|_2^2. \tag{29}$$

The boundary integral disappears for $\mathbb{R}^3$; it vanishes for Q because of the antiperiodicity of the normal vector and in a bounded domain because $(\nabla\times\mathbf{E})\times\mathbf{n}=\mathbf{0}$. It is well known that in these spaces the norm of H^1 is equivalent to the L^2 norm of $\nabla\times\mathbf{E}$ (for this reason, we impose in the periodic case the condition of zero mean).

The way to unify all the equations and constitutive relations into a single self-contained system is as follows: first, we set

$$\mathbf{E} = \mathbf{E}(\mathbf{v}_e,\mathbf{v}_i,\mathbf{v}_n,P_e,P_i,\mathbf{B}) \tag{30}$$

as the solution of Eq. (24) within the spaces (26)–(28). Then, we substitute in the original system **E** as a (nonlocal) function of the remaining variables. The resulting system is not a partial differential one; since T^{-1} may be written as an integral on the right hand side of (24), it is an integrodifferential one. Anyway it is perfectly defined. Let us see that the solutions to it really satisfy all the requirements.

For Ω bounded, the fact that **E** lies within H_d means that $(\nabla\times\mathbf{E})\times\mathbf{n}=\mathbf{0}$. By (4), also $(\partial\mathbf{B}/\partial t)\times\mathbf{n}=\mathbf{0}$. It is therefore enough for $\mathbf{B}\times\mathbf{n}=\mathbf{0}$ to hold for all time that it holds for $t=0$. Since (4) also implies $\Delta\mathbf{E}=\partial\mathbf{J}/\partial t$, we find that $\partial\mathbf{J}/\partial t$ satisfies precisely the same equation as the one obtained for $en_e(\mathbf{v}_i-\mathbf{v}_e)$ by substracting (1) and (2); hence, it is enough that $\mathbf{J}=en_e(\mathbf{v}_i-\mathbf{v}_e)$ at time zero for this to hold for all time. The same thing (minus the boundary conditions) holds for the whole space case and the periodic one. As for the velocities, obviously we will choose them to lie in the space

$$H = \{\mathbf{v} \in (H^1)^3 {:} \nabla\cdot\mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n} = 0\}. \tag{31}$$

Moreover, the classic theorems on elliptic equations (see, e.g., Ref. 21) guarantee that T is an isomorphism between the spaces $H^{m+2}(\mathbb{R}^3)^3 \cap H_s$ and the subspace of $H^m(\mathbb{R}^3)^3$ formed by solenoidal fields and the same for the subspaces H_p and H_d. The bound $\|\mathbf{E}\|_{m+2} \le C\|T(\mathbf{E})\|_m$ will be fundamental in Sec. IV.

IV. *A PRIORI* ESTIMATES AND EXISTENCE THEOREMS

It is known that the Sobolev space H^m is an algebra provided $m>3/2$. Since we will work with spaces such that H^{m-1} is an algebra, we will need $m>5/2$; it is always simpler to take m as an integer, so $m=3$ will be the smallest integer such that we have an existence theorem within H^m. We will denote by $(,)_m$ the scalar product in H^m,

$$(\mathbf{u},\mathbf{v})_m = \sum_{|\boldsymbol{\alpha}|\leq m} \int D^{\boldsymbol{\alpha}}\mathbf{u}\cdot D^{\boldsymbol{\alpha}}\mathbf{v}dV, \tag{32}$$

where the integral is extended to the domain under consideration. It will be more convenient, as well as physically meaningful, to define the following H^m norm in the space of 12-dimensional variables:

$$\|(\mathbf{v}_e,\mathbf{v}_i,\mathbf{v}_n,\mathbf{B})\|_m^2 = \rho_e\|\mathbf{v}_e\|_m^2 + \rho_i\|\mathbf{v}_i\|_m^2 + \rho_n\|\mathbf{v}_n\|_m^2 + \|\mathbf{B}\|_m^2. \tag{33}$$

Then we have the following *a priori* estimates:

Theorem 4.1: *There exists a constant $C>0$ such that any solution $\mathbf{w}=(\mathbf{v}_e,\mathbf{v}_i,\mathbf{v}_n,\mathbf{B})$ of (1)–(5) satisfies*

$$\frac{d}{dt}\|\mathbf{w}\|_m \leq C(\|\mathbf{w}\|_m + 1)^2. \tag{34}$$

Proof: As usual, the main difficulty lies in the pressure terms. By taking the divergence of (1)–(3), we find that that each P_j satisfies the following equation:

$$\frac{1}{\rho_e}\Delta P_e = -\partial_a v_{ea}\partial_b v_{ea} - \frac{e}{m_e}\nabla\cdot(\mathbf{v}_e\times\mathbf{B}), \tag{35}$$

$$\frac{1}{\rho_i}\Delta P_i = -\partial_a v_{ia}\partial_b v_{ia} + \frac{e}{m_i}\nabla\cdot(\mathbf{v}_i\times\mathbf{B}), \tag{36}$$

$$\frac{1}{\rho_n}\Delta P_n = -\partial_a v_{na}\partial_b v_{na}. \tag{37}$$

When $\mathbf{w}=(\mathbf{v}_e,\mathbf{v}_i,\mathbf{v}_n,\mathbf{B})$ belongs to H^m, the right hand side of (35)–(37) belongs to H^{m-1} since this as well as H^m are algebras. Thus, Eqs. (35)–(37) have a unique solution in $H^{m+1}(\mathbb{R}^3)$ in the whole space case. In the periodic case, we may impose that each P_j has a zero mean to obtain again a unique solution within $H^{m+1}(Q)$; since all solutions differ by a constant and we will deal with ∇P_j, the choice is indifferent. The hardest case is the bounded domain one, on which we need boundary conditions. These are found by multiplying (1)–(3) by the normal vector. Since $\mathbf{E}\cdot\mathbf{n}=0$, $\mathbf{B}\times\mathbf{n}=\mathbf{0}$, this means that

$$\frac{1}{\rho_j}\frac{\partial P_j}{\partial n} = -\mathbf{v}_j\cdot\nabla\mathbf{v}_j\cdot\mathbf{n}. \tag{38}$$

We follow the technique of Temam.[22] Since the boundary $\partial\Omega$ is locally the level set of a smooth function $\phi=0$,

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}, \tag{39}$$

$$-\mathbf{v}_j\cdot\nabla\mathbf{v}_j\cdot\mathbf{n} = -\frac{1}{|\nabla\phi|}\mathbf{v}_j\cdot\nabla\mathbf{v}_j\cdot\nabla\phi \tag{40}$$

$$= -\frac{1}{|\nabla\phi|}(\mathbf{v}_j \cdot \nabla(\phi\mathbf{v}_j) - \mathbf{v}_j \cdot \nabla^2\phi \cdot \mathbf{v}_j). \tag{41}$$

The last term means

$$\mathbf{v}_j \cdot \nabla^2\phi \cdot \mathbf{v}_j = v_{ja}v_{jb}\partial_a\partial_b\phi.$$

Since $\mathbf{v}_j \cdot \nabla$ involves only tangential derivatives of the function $\phi\mathbf{v}_j$, whose value is $\mathbf{0}$ in $\partial\Omega$, we are left with

$$-\mathbf{v}_j \cdot \nabla\mathbf{v}_j \cdot \mathbf{n} = \frac{1}{|\nabla\phi|}\mathbf{v}_j \cdot \nabla^2\phi \cdot \mathbf{v}_j. \tag{42}$$

When $\mathbf{v}_j \in H^m(\Omega)^3$, $\mathbf{v}_j|_{\partial\Omega} \in H^{m-1/2}(\partial\Omega)^3$ by the trace theorems. Since $H^{m-1/2}(\partial\Omega)$ is again an algebra for $m > 3/2$ and ϕ is smooth, we find

$$\left\| \frac{\partial P_j}{\partial n} \right\|_{m-1/2,\partial\Omega} \leq C\|\mathbf{v}_j\|^2_{m-1/2,\partial\Omega} \leq C\|\mathbf{v}_j\|^2_m. \tag{43}$$

From now on we will denote all constants depending only on the domain by C to avoid a pointless collection of subindices. Considering the right hand side of (35)–(37) plus the boundary estimate (43) in the case of a bounded domain, we find that for all cases

$$\begin{aligned}\|\nabla P_j\|_m \leq \|P_j\|_{m+1} \leq C(&\|\nabla\mathbf{v}_j\|^2_{m-1} + \|\nabla \cdot (\mathbf{v}_j \times \mathbf{B})\|_{m-1} + \|\mathbf{v}_j\|^2_m \\ &+ \|\mathbf{v}_j\|_m\|\mathbf{B}\|_m) \leq C(\|\mathbf{v}_j\|^2_m + \|\mathbf{v}_j\|_m\|\mathbf{B}\|_m) \leq C\|\mathbf{w}\|^2_m.\end{aligned} \tag{44}$$

We will prove a similar bound for all the terms in the right hand side of (1)–(4). The first difficulty lies in $\mathbf{v}_j \cdot \nabla\mathbf{v}_j$ since it apparently involves a term of the order of $m+1$. However, since Ref. 23 we know how to deal with this: in the product $(\mathbf{v}_j \cdot \nabla\mathbf{v}_j, \mathbf{v}_j)_m$ the troublesome terms

$$\int (\mathbf{v}_j \cdot \nabla D^{\boldsymbol{\alpha}}\mathbf{v}_j) \cdot D^{\boldsymbol{\alpha}}\mathbf{v}_j dV,$$

with $|\boldsymbol{\alpha}| = m$ integrate to zero. Thus,

$$|(\mathbf{v}_j \cdot \nabla\mathbf{v}_j, \mathbf{v}_j)_m| \leq C\|\mathbf{v}_j\|^3_m \leq C\|\mathbf{w}\|^3_m. \tag{45}$$

The products $(\nabla P_j, \mathbf{v}_j)_m$ integrate to zero in the whole space and periodic cases; nonetheless, the bound (44) will be needed to estimate $\mathbf{E}$ even in these cases. For a bounded domain Ω, (44) implies

$$\frac{1}{\rho_j}|(\nabla P_j, \mathbf{v}_j)_m| \leq C\|\mathbf{w}\|^3_m. \tag{46}$$

As for the terms in $\mathbf{v}_j \times \mathbf{B}$, we have

$$\frac{e}{m_j}|(\mathbf{v}_j \times \mathbf{B}, \mathbf{v}_j)_m| \leq C\|\mathbf{v}_j\|^2_m\|\mathbf{B}\|_m \leq C\|\mathbf{w}\|^3_m. \tag{47}$$

It remains to study the nonlocal term $\mathbf{E}$. Notice that in the right hand side of (24) all the terms belong to H^m, except for $\mathbf{v}_j \cdot \nabla\mathbf{v}_j$, which lies within H^{m-1}. Considering our previous bounds on the pressure, we get

$$\|T(\mathbf{E})\|_{m-1} \leq C(\|\mathbf{w}\|^2_m + \|\mathbf{w}\|_m). \tag{48}$$

which implies that

$$\|\mathbf{E}\|_{m+1} \leq C\|T(\mathbf{E})\|_{m-1} \leq C(\|\mathbf{w}\|_m^2 + \|\mathbf{w}\|_m). \tag{49}$$

This bound holds *a fortiori* for $\|\mathbf{E}\|_m$ so that

$$|(\mathbf{E}, \mathbf{v}_j)_m| \leq C(\|\mathbf{w}\|_m^3 + \|\mathbf{w}\|_m^2). \tag{50}$$

As for the collisional terms, we could bound them in a simpler way, but it is worth noticing that with our choosing of the product (33) we have that their contribution is

$$-\rho_e \nu_{ei}\|\mathbf{v}_e - \mathbf{v}_i\|_m^2 - \rho_i \nu_{in}\|\mathbf{v}_i - \mathbf{v}_n\|_m^2 - \rho_n \nu_{ne}\|\mathbf{v}_n - \mathbf{v}_e\|_m^2 \leq 0 \tag{51}$$

so that the total effect is negative and may be ignored. This makes good physical sense; collisions tend to decrease the size of the velocities, and this continues for as long as not all the species have the same motion, i.e., for as long as some difference $\|\mathbf{v}_a - \mathbf{v}_b\|_m$ is positive. It only remains as the magnetic field in (4). (49) implies

$$|(-\nabla \times \mathbf{E}, \mathbf{B})_m| \leq C\|\mathbf{E}\|_{m+1}\|\mathbf{B}\|_m \leq C(\|\mathbf{w}\|_m^3 + \|\mathbf{w}\|_m^2). \tag{52}$$

Finally, the products in the left hand side of (1)–(4) are identical to

$$\sum_j \rho_j \left(\frac{\partial \mathbf{v}_j}{\partial t}, \mathbf{v}_j\right)_m + \left(\frac{\partial \mathbf{B}}{\partial t}, \mathbf{B}\right)_m = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_m^2. \tag{53}$$

Therefore,

$$\frac{d}{dt}\|\mathbf{w}\|_m^2 \leq C(\|\mathbf{w}\|_m^3 + \|\mathbf{w}\|_m^2), \tag{54}$$

so that

$$\frac{d}{dt}\|\mathbf{w}\|_m \leq C(\|\mathbf{w}\|_m + 1)^2 \tag{55}$$

as stated.

We can now state the main theorem.

Theorem 4.2: *Let the initial condition* $\mathbf{w}(0) \in H^m$. *Then, there exists* $T_* > 0$ *such that a solution of (1)–(5) exists for all* $t \in [0, T_*]$.

Proof: Elementary calculations on the *a priori* estimates (34) yield formally

$$\|\mathbf{w}(t)\|_m \leq \frac{\|\mathbf{w}(0)\|_m + 1}{1 - Ct(1 + \|\mathbf{w}(0)\|_m)} - 1. \tag{56}$$

Thus, if, e.g., we take $T_* = [2C(\|\mathbf{w}_0\|_m + 1)]^{-1}$ for all $t \in [0, T_*]$, we have $\|\mathbf{w}(t)\|_m \leq 2\|\mathbf{w}(0)\|_m + 1$. Going back to (1)–(4), we see that except for the terms $\mathbf{v}_j \cdot \nabla \mathbf{v}_j$, which belong to H^{m-1}, all the remaining term in the right hand side belong to H^m for as long as $\mathbf{w}$ belongs to this space. In particular, for all $t \in [0, T]$,

$$\left\|\frac{\partial \mathbf{w}}{\partial t}\right\|_{m-1} \leq C(\|\mathbf{w}(0)\|_m + 1)^2. \tag{57}$$

This implies that for $\mathbf{w}(0) \in H^m$ (plus the conditions described before), there exists a time interval $[0, T]$ such that there exists a unique solution,

$$\mathbf{w} \in \mathcal{C}([0,T], H^m(\cdot)^{12}) \cap \mathcal{AC}([0,T], H^{m-1}(\cdot)^{12}), \tag{58}$$

where $(\cdot)$ means $\mathbb{R}^3$, Q, or Ω, $\mathcal{AC}$ represents the absolutely continuous functions, and, in fact, $\mathbf{w}$ takes values in a certain subspace, described as follows: $\nabla \cdot \mathbf{v}_j = 0$ for all j, $\mathbf{v}_j \cdot \mathbf{n} = 0$ when appro-

priate; $\mathbf{B}\times\mathbf{n}=\mathbf{0}$. When the initial conditions satisfy $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{B}(0)=en_e(\mathbf{v}_i(0)-\mathbf{v}_e(0))$, this holds for all time. $\mathbf{E}$ is found from these variables through (24).

The method of proof is standard (see, e.g., Ref. 24). One approximates the system by projection in a finite-dimensional space X_k: these approximate systems may be solved finding solutions $\mathbf{w}_k$ in a finite time interval $[0,T_k]$, and the *a priori* estimates ensure that T_k does not shrink to zero as we approach the initial condition $\mathbf{w}(0)$ by $\mathbf{w}_k(0)$. Moreover, the estimates provide uniform bounds of the approximate solutions in

$$L^\infty([0,T],H^m)\cap \mathrm{Lip}([0,T],H^{m-1}).$$

The Lions–Aubin compactness theorem proves that there exists a limit $\mathbf{w}$ of the $\mathbf{w}_k$, which is a solution of the system, and, in fact, belongs to the spaces described in (58).

V. CONCLUSIONS

The equations describing the evolution of partially ionized, inviscid, and incompressible plasmas are formed by three momentum equations for each of the species: electrons, ions, and neutral particles plus the Faraday equation for the magnetic field. A number of different simplifications exist, dealing with particular phenomena, such as EMHD and ambipolar diffusion. Since these simplifications often work by suppressing the higher order terms, they have no common mathematical characteristics and a separate study is needed for each of them. The original unsimplified system possesses an unusual feature: the law of Ohm relating electric and magnetic fields to the velocities of each species is not a functional identity as in the classic cases, but an elliptic equation. This makes the whole system a functional (integrodifferential) evolution equation. The main problem consists of tying down several equations and state laws into a single self-contained system; once this is achieved, *a priori* estimates on certain Sobolev spaces are not excessively hard to obtain. As it often happens, the case where the domain is a smooth bounded one proves more difficult that the whole space and periodic box cases. Once these *a priori* estimates are found, a local theorem of existence and uniqueness may be proved by routine methods.

[1] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum, New York, 1983).
[2] A. Brandenburg and K. Subramanian, Phys. Rep. **417**, 1 (2005).
[3] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2001).
[4] D. Biskamp, E. Schwarz, and J. F. Drake, Phys. Rev. Lett. **75**, 3850 (1995).
[5] D. Biskamp, *Magnetic Reconnection in Plasmas* (Cambridge University Press, Cambridge, 2000).
[6] M. A. Shay, J. F. Drake, B. N. Rogers, and R. E. Denton, J. Geophys. Res. **106**, 3759 (2001).
[7] A. Brandenburg and E. G. Zweibel, Astrophys. J. **427**, L91 (1994).
[8] T. M. Abdalla, B. N. Kushinov, T. J. Schep, and E. Westerhof, Phys. Plasmas **8**, 3957 (2001).
[9] A. Brandenburg and E. G. Zweibel, Astrophys. J. **448**, 734 (1995).
[10] M. Núñez, Eur. J. Mech. B/Fluids **28**, 465 (2009).
[11] M. Núñez, Proc. R. Soc. London, Ser. A **464**, 1571 (2008).
[12] B. P. Pandey and M. Wardle, Mon. Not. R. Astron. Soc. **385**, 2269 (2008).
[13] L. Spitzer, Jr., *Physics of Fully Ionized Plasmas* (Interscience, New York, 1956).
[14] S. I. Braginsky, Rev. Plasma Phys. **1**, 205 (1965).
[15] B. T. Draine, W. G. Roberge, and A. Dalgarno, Astrophys. J. **264**, 485 (1983).
[16] A. V. Gordeev, A. S. Kingsep, and L. I. Rudakov, Phys. Rep. **243**, 215 (1994).
[17] A. S. Kingsep, K. V. Chukbar, and V. V. Yan'kov, in *The Electron Magnetohydrodynamic*, Review of Plasma Physics Vol. 16, edited by B. B. Kadontsev (Consultants Bureau, New York, 1990), pp. 243–291.
[18] S. Dastgeer and G. P. Zank, Astrophys. J. **599**, 715 (2003).
[19] L. Mestel and I. W. Roxburgh, Astrophys. J. **136**, 615 (1962).
[20] R. Dautray and J. L. Lions, *Spectre des Opérateurs*, Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques Vol. 5 (Masson, Paris, 1988).
[21] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer, New York, 2001).
[22] R. Temam, J. Funct. Anal. **20**, 32 (1975).
[23] T. Kato, J. Funct. Anal. **9**, 296 (1972).
[24] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Variables*, Applied Mathematical Science Vol. 53 (Springer, New York, 1984).