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TESIS DOCTORAL:

ROBUST STABILIZATION AND OBSERVATION FOR POSITIVE TAKAGI-SUGENO SYSTEMS

Presentada por **Ines Zaidi** para optar al grado de
doctora por la Universidad de Valladolid

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To my great parents Mohamed and Zouhour.
To my grandfather Hbib and my aunt Kalthoum.
To my sister Asma and brother Anis.
To all those who offered me their help and support,
in my ongoing journey,
to reach what I have being dreaming of,
and to become who I want to be.

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I strongly believe that everything happens for a reason. Looking back at my life, I can see clearly the major events and circumstances which I have encountered along my way and which have had an influence on my life progress, even it was bad or good. Obviously, I have learnt to exploit the consequences of these events in my favor and neglect the effects of those which were against my destination. While struggling to achieve what I am meant to do, nothing has been more valuable to me than the support and assistance of those ones who have not hesitated to provide me with love, compassion, kindness and even a big sense of responsibility. I am really grateful for their help and for making the level of my self-confidence as high as possible during my PhD work. That is why, at the beginning of my dissertation, I have the honor to acknowledge these precious persons who made my PhD time one of the greatest steps in my whole life.

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Keep your trust in me, I will make you proud..

"Measure what is measurable and make measurable what is not so".

Galileo Galilei.

"The sacred formula of positivists: Love as principle, Order as basis and Progress as end".

August Comte.

"They did not know it was impossible so they did it"

Marc Twain.

Summary

List of Figures

Notations

- **Matrices and Vectors**

I	identity matrix of appropriate dimensions
$I_{n \times n}$	$n \times n$ Identity matrix
$0_{p \times q}$	$p \times q$ matrix with all entries equal to 0.
0	Matrix of appropriate dimensions with all entries equal to 0.
$P > 0$ $(P \geq 0)$	Square and positive definite matrix (resp. positive semi-definite)
$P < 0$ $(P \leq 0)$	Square and negative definite matrix (resp. semi-negative definite)
$X > Y$ $(X \geq Y)$	The matrix $X - Y$ is positive definite (resp. semi-positive definite)
$diag(d)$	Diagonal matrix with diagonal components the elements of d
P^T	Transpose of P
P^{-1}	Inverse of P
P^{-T}	Shorthand for $(P^{-1})^T$
$\lambda(P)$	Eigenvalue of P
$\lambda_{min}(P)$	Minimum eigenvalue of P
$\lambda_{max}(P)$	Maximum eigenvalue of P
$\begin{pmatrix} P_{11} & P_{12} \\ * & P_{22} \end{pmatrix}$	Symmetric matrix with $(*)$ representing P_{12}^T
$Re(P)$	Real part of the eigenvalues of the matrix P
$Im(P)$	Imaginary part of the eigenvalues of the matrix P
$|a|$	Absolute value of a

$\|X\|$	Norm 2 of the vector X: $\sqrt{|x_1|^2 + \cdots + |x_n|^2}$
H_2	Norm 2 of a system
H_∞	Norm ∞ of a system
ΔA	Additive uncertainty on the matrix A
$[A]_{ij}$	Element located at the ith row and jth column of matrix A
$A \succcurlyeq 0$	Nonnegative matrix: $\forall(i,j), [A]_{ij} \geq 0$
$A \succ 0$	Positive matrix: $\forall(i,j), [A]_{ij} > 0$
$A \succcurlyeq B$	The matrix $A - B \succcurlyeq 0$
$A \in [\underline{A}, \overline{A}]$	$\underline{A} \preccurlyeq A \preccurlyeq \overline{A}$
$\forall$	For all
$\in$	Belongs to
$\mu(A)$	The spectral abscissa of the matrix A (max of the real parts of the eigenvalues of A)
$\rightarrow$	Tends toward
$\Rightarrow$	Implies
■	End of the proof
$\displaystyle\sum_{i,j,k=1}^{r} h_i h_j h_k$	$\displaystyle\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i h_j h_k$
$\dot{X}(t)$	The derivative of the vector X with respect of time
$\hat{X}(t)$	The estimation of the vector X
Metzler A	All its off-diagonal elements are nonnegative, i.e., $\forall(i,j), i \neq j, [A]_{ij} \geq 0$

- **Sets and Areas**

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^{n \times n}$	Space of matrices with real entries
$\mathbb{R}_+$	Set of positive or null real numbers
$\mathbb{R}_+^*$	Set of positive real numbers
$\mathbb{R}_+^{n \times n}$	Positive orthant of $\mathbb{R}^{n \times n}$
$\mathcal{C}^1$	Set of the continuously differentiable functions

- **Acronyms**

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
LP	Linear Programming
LTI	Linear Time Invariant
LTV	Linear Time Variant
PDC	Parallel Distributed Control
PPDC	Proportional Parallel Distributed Control
OPDC	Output Parallel Distributed Control
CDF	Compensation and Division by Fuzzy model
GEVP	General Eigenvalue Problem
T-S	Takagi-Sugeno
MDV	Measurable Decision Variables
UDV	Unmeasurable Decision Variables

- **Notations of Multimodels**

$x(t)$ The state of the system: $x(t) \in \mathbb{R}^{n_x}$

$u(t)$ The input of the system: $u(t) \in \mathbb{R}^{n_u}$

$y(t)$ The output of the system: $y(t) \in \mathbb{R}^{n_y}$

A_i The state matrix of the ith LTI submodel: $A_i \in \mathbb{R}^{n_x \times n_x}$

B_i The input matrix of the ith LTI submodel: $B_i \in \mathbb{R}^{n_x \times n_u}$

C_i The output matrix of the ith LTI submodel: $C_i \in \mathbb{R}^{n_y \times n_x}$

Introduction

In this chapter, we discuss the motivation behind this thesis and provide an overview of the material presented in the manuscript.

0.1. Introduction

The assumption of linearity in practical systems makes possible to develop simple models that approximate its behavior. These linear models have been extensively studied in different contexts: identification, state estimation, control, diagnosis, etc. However, such models allow the representation of the behavior of a system only around a given operating point, as the linearity assumption is verified only in a restricted area of the operating space. Given that real systems are not linear in nature, the performance of control and diagnosis systems based on linear models degrade when moving away from the operating point. In order to improve the system performance, it is imperative to take into account the nonlinearities in the modeling phase. This allows describing the behavior of a real system over a wide operating range with better accuracy than with linear models. Control and diagnostic systems developed using nonlinear models are then more efficient than those developed from linear models. The main drawback of nonlinear models is the complexity of their mathematical structures, which makes them difficult to use. For this reason, studies on nonlinear systems do not have a general framework, but relate to specific classes of nonlinear models, such as Lipchitz systems, bilinear systems, Takagi-Sugeno (T-S) systems, LPV systems, etc. In this dissertation, we concentrate on T-S systems, as the tools used resemble those of linear systems.

In many investigations on dynamical systems, the state vector is assumed to be available for measurement. However, such an assumption is not always true in practice, as for technical and/or economic reasons, it is not possible to measure all state variables. However, the need to know fully the state variables of the system is often crucial, which requires the use of tools to estimate variables which are not accessible to measurement. This makes the problem of observer design a fundamental issue in control systems.

The first works on the problem of state reconstruction were dedicated to linear systems (Luenberger, 1971). Many theoretical results were then proposed and are widely used in control and estimation. For example, the diagnostic of operating systems is based on linear models (Gertler, 1998), (Patton et al, 1989), (Isermann, 2007), (Ding, 2008). However, the linearity of the model is a strong assumption

which limits the validity of the results obtained. Furthermore, a direct extension of the methods developed for linear models to nonlinear models is tricky. Many techniques have been then dedicated to state estimation of particular classes of nonlinear systems (based on changes in a canonical form of observability, Kalman filter, observer, Luenberger extended observers...) (Kalman, 1960), (Chen & Patton, 1999). However, these techniques are often difficult to apply because of the imposed constraints. In addition, the wealth of results for linear systems is very little exploitable in the context of nonlinear systems.

The strategy of state reconstruction proposed in this thesis uses a technique to obtain a model, taking into account the nonlinearities of the system and providing a simple and exploitable mathematical structure. This is generally called a *multimodel approach*. Several types of multimodels have been introduced in recent years, such as multimodels with coupled states and multimodels with single states, called Takagi-Sugeno (T-S) models (Tanaka & Wang, 2001), (Orjuela, 2008).

T-S models are the most studied in the literature: they are described by a set of submodels sharing a single state vector (Takagi & Sugeno, 1985). Two categories can be considered depending on the nature of the variables involved in the weighting functions. Indeed, these variables, called decision variables or premise variables, can be known (input, output of the system, etc.) or unknown (system state, etc.). The category of T-S models with measurable decision variables (VDM) has been the subject of many developments in control, stabilization, state estimation (Tanaka & Wang, 2001) and diagnosis (Nagy et al, 2009).

A new constraint is currently been added in the synthesis of control and estimation systems, on the sign of the variables. Positive systems are those whose states remain nonnegative for all future times, once started from nonnegative initial conditions. Positivity is not an inherent property of a system; we might be able to turn a nonpositive system into a positive system with a simple change of variable (Zaidi et al, 2012). Such systems can be found in practice in different areas of science and technology, such as: biology and physiology where biochemical models have a common important characteristic which is that most variables take only nonnegative values, since they usually represent chemical concentrations (Sontag, 2005), (Vahid, 2012), (Haddad & Chellaboina, 2005), in Communications, mainly in congestion control in TCP networks (Shorten et al, 2006), (Jacquez & Simon, 1993), Economics (Leontief, 1936), (Neumann, 1945),

Compartmental systems (Benvenuti & Farina, 2002), Ecology and population dynamics (Lotka, 1925), (Volterra, 1926), etc.

Even though stability properties of positive linear time-invariant systems are now well investigated (Bolajraf, 2012), (Benzaouia et al, 2011), (Benzaouia & El Hajjaji, 2011), (Benzaouia et al, 2014), (Rami & Tadeo, 2008), (Rami & Tadeo, 2007a), (Rami & Tadeo, 2007b) there are still a lot of unanswered questions in other classes of positive systems, such as uncertain nonlinear systems, or time-delay systems. Although there is a rich literature on properties of uncertain systems, they have been rarely studied in the context of nonlinear positive systems. To deal with this deficiency, in this thesis, we deal with stabilization and α-stabilization of nonlinear systems, especially positive Takagi-Sugeno and time-delay systems. We also present conditions for stability and stabilization of positive time-delay systems, when the size of delay is fixed or variable. We also applied static memoryless state-feedback control laws, with and without memory, in order to guarantee a performing stabilization for such systems.

0.2. Overview

We begin by setting the context and providing the state of the art for much of the later work in Chapter 1. We define various concepts and results that will be used in the following sections. Firstly, we will have an overview on the Takagi-Sugeno (T-S) modeling approach and the stabilization and estimation of this type of systems. Secondly, a background is deserved to the classification and modeling of time-delay T-S systems. Then, we focus on the stability and stabilization of time-delay T-S systems. Later on, the design of observers and observer-based controllers will be introduced, concentrating on positive systems and their properties. Then, we focus on positivity of linear systems and their properties. Classes of time-delay T-S systems, which will be repeatedly used in the following chapters, are discussed in the context of guaranteeing their stability and positivity. α-stability will be discussed thoroughly in the different chapters of the thesis; therefore, we discuss in this chapter basic stability properties.

In Chapter 2, we are interested in the analysis of stability and stabilization of positive nonlinear systems described by T-S models that only involve nonnegative states. Firstly, we introduce the concept of asymptotic stability and α-stability of positive T-S systems, where the main stability and stabilization approaches depend on the type of the premise variables: measurable or unmeasurable.

Moreover, we are interested in robust stabilization and robust α-stabilization of positive T-S systems with interval uncertainties. Memory state-feedback controllers for positive interval systems have been then established.

In Chapter 3, firstly, LMI conditions are established in order to synthesize interval observers for positive linear systems, that can provide lower and upper estimates on the unmeasurable states. This can be done by minimizing an adequate bound on the interval errors and can be solved via an LMI optimization problem. Secondly, we establish approaches for the design of positive observer-based controllers for positive T-S systems with measurable and unmeasurable premise variables, with and without interval uncertainties. Numerical and practical examples are presented to show the effectiveness of the proposed methods.

Chapter 4 is dedicated to the study of positive time-delay systems, with and without interval uncertainties. Firstly, we introduce stability results with constant and multiple delays. Secondly, we will deal with the asymptotic stabilization and the robust α-stabilization. Finally, necessary and sufficient conditions are provided for the asymptotic stabilization and robust α-stabilization of positive interval T-S time-delay systems by means of state-feedback laws with or without memory. We also consider the decomposition of the state-feedback controller gains in order to reduce the conservatism.

Chapter 5 is firstly devoted to the design of positive observers for positive linear time-delay systems, with and without interval uncertainties: necessary and sufficient conditions have been established and expressed in terms of LMIs, taking into account the positivity constraints. Secondly, observer-based controllers have been synthesized for positive linear time-delay systems to guarantee the stability and positivity of the closed-loop system. Necessary conditions are formulated in order to check the existence of any solutions to the problem of continuous-time observer-based control. Once satisfied, we study the sufficient conditions and the corresponding synthesis for this problem. Moreover, extensions of these approaches are applied for positive interval Takagi-Sugeno systems with variable time-delay. In this issue, we consider when the decision variables are measurable and when they are not.

Finally, illustrative results of numerical and practical examples have been given to show the effectiveness of these approaches.

In the Conclusion, we summarize the results and outlining possible directions for extending those results.

0.3. Publications

Journal papers

1. Zaidi, I., Chaabane, M., Tadeo, F., & Benzaouia, A. (2015). Static state-feedback controller and observer design for interval positive systems with time-delay. IEEE Transactions on Circuits and Systems II, Accepted in December 2014.

2. Zaidi, I., Chaabane, M., Tadeo, F., & Benzaouia, A. (2015). Robust α-stabilization of uncertain positive T-S systems via memory state-feedback. Circuits, Systems and Signal Processing (CSSP), Submitted in October 2014.

3. Zaidi, I., Chaabane, M., Tadeo, F., & Benzaouia, A. (2015). Stabilization of positive Takagi-Sugeno systems by a decomposed memory state-feedback controller. Nonlinear analysis: Hybrid Systems, Submitted in October 2014.

Conference papers

1. Zaidi, I., Tadeo, F., Chaabane, M., & Benzaouia, A. (2012a). Positive observation of Takagi-Sugeno Systems. In the 17th International Conference on Methods and Models in Automation and Robotics (MMAR), Miedzyzdrojie, Poland.

2. Zaidi, I., & Tadeo, F. (2012b). Control of Continuous-time Takagi-Sugeno Systems for Stability and Positivity. In the 17th International Conference on Methods and Models in Automation and Robotics (MMAR), Miedzyzdrojie, Poland.

3. Zaidi, I., Tadeo, F., & Chaabane, M. (2012c). Stability and Positive State Estimation for Takagi-Sugeno Systems". In the 14th Sciences and Techniques of Automatic Conference (STA), Sousse, Tunisia.

4. Zaidi, I., Tadeo, F., & Chaabane, M. (2013a). Positive Observer Based Tracking Controller Design for Takagi-Sugeno Systems". In the 10th

Conference on Systems, Signals and Devices Conference (SSD), Hammamet, Tunisia.

5. Zaidi, I., Tadeo, F., & Chaabane, M. (2013b). Positive State Estimation with H_∞ performance for Positive Continuous-time Takagi Sugeno Systems. In European Student paper Competition (ESPC-ISA) Conference, Lisboa, Portugal.

6. Zaidi, I., Tadeo, F., & Chaabane, M. (2013c). Positive observer design for continuous-time Takagi-Sugeno systems. In Proc. of the 52nd Decision and Control Conference (CDC), 5030-5035, Florence, Italy.

7. Zaidi, I., Tadeo, F., & Chaabane, M. (2014a). Positive Observers for Continuous time Positive Interval Systems with time-delay. In the 15th Sciences and Techniques of Automatic Conference (STA), Sousse, Tunisia.

8. Zaidi, I., Tadeo, F., & Chaabane, M. (2014b). Design of Robust Observers for Positive Delayed Interval Systems. In the 11th Conference on Systems, Signals and Devices Conference (SSD), Barcelona, Spain.

9. Zaidi, I., Tadeo, F., & Chaabane, M. (2015). Stabilization of positive Takagi-Sugeno systems with unmeasurable premise variables. In the 4th International Conference on Systems and Control (ICSC), Accepted.

Communications

1. Zaidi, I. (2014). Robust stabilization and observation of positive Takagi-Sugeno systems. XII Simposio CEA de Ingeniería de Control "Ingeniería de Control: de la Teoría a la Práctica", Universidad de Valladolid, Spain: 6-7 de Febrero.

2. Workshop on "Intelligent Planning and Control: Bringing together adaptive control and reinforcement learning for guaranteeing Optimal Performance and Robustness". In Proc. of the 52nd Decision and Control Conference (CDC), 5030-5035, Florence, Italy.

Chapter I

State of the Art

Background

This chapter provides the mathematical basis required for the results presented in the following chapters. More precisely, we recall some basic definitions and results related to dynamical systems.

1.1. State of the Art of Takagi-Sugeno (T-S) systems

Firstly, we provide an overview on the Takagi-Sugeno (T-S) modeling approach, concentrating on the stabilization and estimation of this type of systems. Secondly, the modeling of time-delay T-S systems is discussed. Then, we focus on the stability, stabilization of time-delay T-S systems, followed by the design of observers and observer-based controllers. At the end of this chapter, we focus on the positivity of linear and T-S systems and their properties, including time-delay positive systems, which will be repeatedly used in the following chapters.

1.1.1. TAKAGI-SUGENO MODELS

Recently, many investigations have revealed the importance of Takagi-Sugeno systems in modern system control, as it can be considered as a nonlinear combination of a set of linear systems interconnected by nonlinear weighting functions (Tanaka et al, 2001), (Ichalal , 2009). This type of modeling is very useful thanks to its ability of approximating different complex systems, and then applying simple control and estimation approaches to get remarkable results.

The representation of nonlinear systems introduced as T-S models (Takagi & Sugeno, 1985) is an interesting alternative in the field of control, observation and diagnosis. Specifically, a Takagi-Sugeno system is described by fuzzy IF-THEN rules, which locally represent linear input-output relations called subsystems. Each of these rules is of the following form:

Rule i: IF $z_1(t)$ is F_i^1 and …and $z_p(t)$ is F_i^p THEN:

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{1.1}$$

$$y(t) = C_i x(t) + D_i u(t) \tag{1.2}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector , $u(t) \in \mathbb{R}^{n_u}$ is the control input, r is the number of IF-THEN rules, with p is the number of the premise variables,

$z_1(t), \dots, z_p(t)$ and $F_i^1, \dots, F_i^p$ are respectively the premise variables and the grade of membership of $z_i(t)$ in F_i^j, $i, j = 1, \dots, r$.

A T-S model can also be expressed by a finite set of interconnected linear models through nonlinear functions, satisfying a convex sum property that we will introduce in (1.4).

Thus, the general mathematical formulation of T-S models is given by the following equations:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(z(t))(C_i x(t) + D_i u(t)) \end{cases} \tag{1.3}$$

The r subsystems are defined by known matrices: $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n_x}$ and $D_i \in \mathbb{R}^{n_y \times n_u}$. The activation functions $h_i(z(t))$ are nonlinear functions which depend on the vector of premise variables $z(t)$ (which can be measurable, for example the input $u(t)$ or the output $y(t)$, or unmeasurable such as the state $x(t)$). These activation functions satisfy the following properties, $\forall t \geq 0$:

$$\begin{cases} 0 \leq h_i(z(t)) \leq 1, \quad i = 1, \dots, r \\ \sum_{i=1}^{r} h_i(z(t)) = 1 \end{cases} \tag{1.4}$$

1.1.2. STABILITY AND STABILIZATION OF TAKAGI-SUGENO SYSTEMS

The stability of nonlinear systems represented by T-S models has been the target of many developments. The particular structure of this type of model has enabled the extension of the study of the stability of linear systems to the case of nonlinear systems.

Consider an autonomous Takagi-Sugeno system represented by (1.3).

1.1.2.1. The direct Lyapunov approach

To realize the knowledge of the trajectories, we use the direct method of Lyapunov. The idea is to study the variation of a positive definite scalar function to conclude about the stability of the system. This method is related to the concept of energy: "If the energy of a system, being linear or nonlinear, is

continuously dissipated, then this system, called dissipative in this case, may tend towards an equilibrium point". We can refer to (Borne, 1993), (Khalil, 1996).

Thus, the following theorems state the conditions under which an equilibrium point of a system is stable, according to the direct method of Lyapunov.

Theorem 1.1. (Local stability)

If there exists, in the box $B(\beta) = \{x, \|x\| \leq \beta\}$, a scalar function $V(x)$ whose first partial derivatives are continuous, such that:

- *$V(x(t))$ is positive definite.*
- *$\dot{V}(x(t))$ is negative semi-definite.*

Then, the equilibrium point is stable. If $\dot{V}(x)$ is negative definite in $B(\beta)$, then, the equilibrium point is asymptotically stable.

Theorem 1.2. (Global stability)

If there exists a scalar function $V(x)$ whose first partial derivatives are continuous, such that:

- *$V(x(t))$ is positive definite.*
- *$lim_{\|x\| \to \infty} V(x(t)) \to \infty$*
- *$\dot{V}(x(t))$ is negative definite.*

Then, the equilibrium point is globally asymptotically stable.

The general definition of the Lyapunov function does not allow finding all the forms they can take. The choice of this type of function and structure of the system under study plays an important role in the development of stability conditions. Several functions that meet the definition of Lyapunov functions have been used to study the stability of systems. These functions depend on the structure of the studied system and the problem of the results conservatism is often due to the choice of these functions.

In general case, there does not exist a method to find all Lyapunov candidate functions. Therefore, Lyapunov theory leads to sufficient stability conditions where pessimism depends on the particular form of the given function $V(x(t))$ and the system structure. However, we often use well-known Lyapunov functions,

according to the nature of the studied system: linear systems, piecewise continuous systems, nonlinear systems, uncertain systems, systems with time-delay, etc.

- Quadratic function

When we focus on Lyapunov functions, the first type that firstly comes to mind is the quadratic function; it is the most classic form given by:

$$V\big(x(t)\big) = x(t)^T P x(t), \; P > 0 \tag{1.5}$$

The study of the stability using this type of functions has been the basic theory of several works. We can cite for example (Garcia, 1997), (Boyd et al, 1994).

In order to ameliorate the pessimism of the quadratic stability, it is necessary to use other Lyapunov candidate functions such that polyquadratic functions.

- Polyquadratic function

This function is of the form:

$$V\big(x(t), z(t)\big) = x(t)^T \sum_{i=1}^{r} h_i(z(t)) P_i x(t) \tag{1.6}$$

with $P_i > 0$, $h_i(z(t)) > 0$, $\sum_{i=1}^{r} h_i(z(t))$. It allows to relax the constraints that are imposed by the quadratic method, in the case of the multimodel approach. This type is a general case of quadratic functions when $P_i = P$, $i = 1, \dots, r$. It is also noted that, in contrast with the quadratic functions, this type of function has the advantage of taking the variation speed of the decision variables of the continuous multimodel into account. This may lead to less conservative stability conditions (Jadbabaie, 1999), (Chadli et al, 2000), (Morère & Guerra, 2000), (Blanco et al, 2001), (Tanaka & Wang, 2001).

- Parametric affine function

This function is of the following form:

$$V\big(x(t)\big) = x(t)^T P(\theta) x(t) \tag{1.7}$$

where $P(\theta) = P_0 + \theta_1 P_1 + \cdots + \theta_k P_k > 0$. This type of functions is usually used for linear systems with uncertain time-varying parameters: $\dot{x}(t) = A(\theta)x(t)$ with $A(\theta) = A_0 + \theta_1 A_1 + \cdots + \theta_k \theta_k$ where the parameters θ_i and their variations are bounded.

The expression (1.7) generalizes the quadratic Lyapunov functions corresponding to $P_1 = \cdots = P_k = 0$. They are less conservative than quadratic functions because they take the parameters variations into account (Gahinet at al, 1996), (Bara, 2001).

1.1.2.2. Stability of Takagi-Sugeno systems

Regarding the T-S system (1.3), the quadratic stability lies on the quadratic Lyapunov function mentioned there before.

The following theorem shows the stability conditions of the system:

Theorem 1.3. (Tanaka et al, 1992)

The equilibrium of the system described by (1.3) is globally asymptotically stable if there exists a common positive definite matrix P such that:

$$A_i^T P + PA_i < 0, i = 1, \dots, r \tag{1.8}$$

The proof of this theorem is obtained by using the theory of the stability in the sense of Lyapunov (Theorem 1.2) by considering the Lyapunov function (1.5) along the trajectory of the system (1.3).

1.1.2.3. Stabilization of Takagi-Sugeno systems

A dynamic system requires a control law which makes it stable, robust and performant. In practice, the objectives of the control are complicated. Indeed, it is often desirable to impose additional constraints on the characteristics of the closed-loop response of the system such as overshoot, rise time, response time, etc. In addition, certain robustness with respect to parametric variations and external disturbances is requested. Studies in this area focus on the synthesis of a control law obeying these conditions (Borne et al, 1990), (Bernussou, 1996). The goal from diversifying the control laws is to minimize the conservatism and the number of the unknown parameters of the stabilization conditions.

Among the used laws of the multimodel control, we can cite:

- **PDC control law (Parallel Distributed Compensation)**

The T-S control law of the global system $u(t)$ is obtained by the fusion of linear control laws with state-feedback of the submodels (Wang et al, 1996). It has the form:

$$u(t) = -\sum_{i=1}^{r} h_i\big(z(t)\big)K_i x(t) \tag{1.9}$$

where r is the number of submodels and K_i, for $i = 1, ..., r$ are the linear state-feedback gains.

The advantage of this law is that it takes into account the recovery rate through the activation functions $h_i(z(t))$, but its drawback is that it involves the cross-terms (A_i, B_j) for $i, j = 1, ..., r$, $i \neq j$ of the submodels and the gains K_j, $j = 1, ..., r$. A typical control law of the form $u(t) = -Kx(t)$ can be considered to overcome the problem of cross-terms, but the problem is that the law requires a common gain K has to stabilize all r submodels.

- **DPDC control law (Dynamic PDC)**

The control law DPDC is based on the output-feedback of the system. It is considered as described by a multi-system (Li & Wang, 2000):

$$\begin{cases} \dot{x}_c(t) = \sum_{i,j=1}^{r} h_i\big(z(t)\big)h_j\big(z(t)\big)\big[A_{cij}x_c(t) + B_{ci}y(t)\big] \\ u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)(C_{ci}x_c(t) + D_c y(t)) \end{cases} \tag{1.10}$$

The determination of this law is the identification of its parameters $(A_{cij}, B_{ci}, C_{ci}, D_c)$. In the works of (Chilali & Gahinet, 1996), linearization techniques are provided to resolve the BMIs found in the quadratic stabilization conditions.

- **OPDC control law (Output PDC)**

The OPDC control law is inspired by the PDC control law based on output-feedback, usually nonlinear, and expressed by (Chadli et al, 2002a), (Chadli et al, 2002b):

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)F_i y(t) \tag{1.11}$$

where F_i, $i = 1, \dots, r$ are the output-feedback gains.

Another law is inspired by the CDF control law when the input matrices B_i, $i = 1, \dots, r$ are linearly dependent, it is expressed as follows (Chadli et al, 2002a):

$$u(t) = -\frac{\sum_{i=1}^{r} h_i\big(z(t)\big)\alpha_i F_i}{\sum_{i=1}^{r} h_i\big(z(t)\big)\alpha_i} y(t) \tag{1.12}$$

The T-S system described in (1.3) and the PDC control law (1.9) are considered. We denote K_i, $i = 1, \dots, r$ the state-feedback control gains of the system.

Replacing (1.9) in (1.3), we get the following closed-loop system:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i\big(z(t)\big)h_i\big(z(t)\big)\big(A_i - B_i K_j\big) x(t) \tag{1.13}$$

We denote $G_{ij} = A_i - B_i K_j$

If the pairs (A_i, B_i) are controllable (stabilizable) $(rank(B_i \; A_i B_i \dots A_i^{n-1} B_i) = r)$, then the multimodel (1.3) is controllable (stabilizable).

Theorem.1.4. (Tanaka & Sugeno, 1992)

The equilibrium of the system described by (1.13) is generally asymptotically stable if there exists a common positive definite matrix P such that:

$$X A_i^T - M_j^T B_i^T + A_i X - B_i M_j < 0, i, j = 1, \dots, r \tag{1.14}$$

where:

$X = P^{-1}$

$M_i = K_i X \tag{1.15}$

After solving the LMI (1.14), the state-feedback gains are given by:

$K_i = M_i X^{-1}, i = 1, \dots, r \tag{1.16}$

1.1.3. OBSERVER DESIGN FOR TAKAGI-SUGENO SYSTEMS

Control techniques of dynamical systems lead to control laws using a state-feedback (Gauthier et al, 1992), (Miekzarski, 1988), (O'Reilly, 1983). Indeed, some state variables may have no physical meaning; and therefore, they are not measurable. Similarly, when some variables are measurable, their measurements require the installation of new transmitters, which increases the cost of the

control. Moreover, the directly measured variables are not usually able to describe the behavior of the process. We can then deal with the problem of the information reconstruction not directly measurable; it is the role of the observer or state estimator (Zhou et al, 1995), (Farza, 2000), (Tlili & Belhadj, 2002).

1.1.3.1. Measurable decision variables (MDV)

We will recall the main results concerning the design of observers for T-S systems. For this, consider the T-S system (1.3), supposing that $D_i = 0$, $i = 1, \dots, r$.

The T-S observer can then be given as follows (Tanaka et al, 1998):

$$
\begin{cases}
\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\Big(A_i \hat{x}(t) + B_i u(t) + L_i\big(y(t) - \hat{y}(t)\big)\Big) \\
\hat{y}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big) C_i \hat{x}(t)
\end{cases}
\tag{1.17}
$$

where L_i, $i = 1, \dots, r$ are the observer gains of the submodels.

If the pairs (A_i, C_i) are observable, then, the multimodel (1.3) is called observable (Ichalal et al, 2008a).

The problem is to determine these gains while ensuring the convergence of the observed state (1.17) to the state of the real system (1.3).

The majority of the works about the design of state observers for T-S systems is based on the assumption that decision variables are available. Therefore, the observer uses the same decision variables as the system model ones, which allows a factorization by the activation functions when we make the evaluation of the dynamics of the state estimation error. More specifically, it is written as follows:

$$
e(t) = x(t) - \hat{x}(t)
\tag{1.18}
$$

$$
\dot{e}(t) = \sum_{i,j=1}^{r} h_i\big(z(t)\big) h_j\big(z(t)\big)\big(A_i - L_i C_j\big) e(t)
\tag{1.19}
$$

We denote: $S_{ij} = A_i - L_i C_j$ (1.20)

The stability conditions of the system (1.19) are given in the following theorem:

Theorem 1.4. (Tanaka & Sugeno, 1992):

The equilibrium of the system described by (1.19) is globally asymptotically stable, if there exists a common positive definite matrix Q such that:

$$A_i^T Q - C_j^T N_j^T + Q A_i - N_i C_j < 0, \qquad i, j = 1, \dots, r \tag{1.21}$$

The proof of this theorem is also obtained by using the theory of the stability in the sense of Lyapunov along the path (1.19). The condition (1.21) is in the form of a BMI. In order to linearize it, we simply assume the following change of variables:

$$N_i = Q L_i, \qquad i = 1, \dots, r \tag{1.22}$$

After solving the LMI (1.21), the observer gains are given by:

$$L_i = Q^{-1} N_i, \qquad i = 1, \dots, j \tag{1.23}$$

More recently, in (Akhenak, 2004) and (Rodrigues, 2005), the authors generalized the unknown input observers proposed in (Darouach et al, 1994) for linear systems. Stability was studied by the Lyapunov theory and the obtained conditions are formulated using LMIs. In (Akhenak, 2004), observers with variable structures (sliding mode) have also been developed for T-S uncertain systems.

However, the representation (1.3) assumes that the decision variables $z(t)$ are measurable; which is not always the case where the decision variables are not measurable.

1.1.3.2. Unmeasurable decision variables (UDV)

In case where the decision variables are not known, their factorization is no longer possible and the observer for the system (1.3) ($D_i = 0$, $i = 1, \dots, r$) can be written as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\hat{z}(t)) \left(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t)) \right) \\ \hat{y}(t) = \sum_{i=1}^{r} h_i(\hat{z}(t)) C_i \hat{x}(t) \end{cases} \tag{1.24}$$

where $h_i(\hat{z}(t)) = \hat{h}_i(z(t))$ is the weighting function which is estimated from $h_i(z(t))$ and $\hat{z}(t)$ is the vector of the decision variables constructed from $z(t)$.

Then, the dynamics of the state estimation error can be written as:

$$\dot{e}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\big(A_i x(t) + B_i u(t)\big)$$

$$- \sum_{i,j=1}^{r} h_i\big(\hat{z}(t)\big) h_j\big(z(t)\big) \Big(A_i \hat{x}(t) + B_i u(t) + L_i C_j e(t)\Big) \qquad (1.25)$$

By analyzing the state equation (1.24), we conclude that the results obtained in the case of T-S systems with measurable decision variables are not applicable for the determination of the observer gains L_i. Few works have been done to solve this problem. Nevertheless, one can cite (Bergsten & Palm, 2000) and (Bergsten et al, 2001), where the authors propose conditions for convergence of the state estimation error to zero, based on the observer of Thau-Luenberger (Thau, 1973). The activation functions are assumed to be Lipschitzian.

Theorem 1.5. (Bergsten & Palm, 2000)

The state estimation error between the T-S model and the observer converges asymptotically to zero if there exist symmetric and positive definite matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $Q \in \mathbb{R}^{n_x \times n_x}$, matrices $K_i \in \mathbb{R}^{n_x \times n_y}$ and a positive scalar γ such that:

$$A_i^T P + P A_i - C_j^T K_i^T - K_i C_j + Q < 0 \qquad (1.26)$$

$$\begin{pmatrix} -Q + \gamma^2 & P \\ P & -I \end{pmatrix} < 0 \qquad (1.27)$$

The demonstration is available in (Bergsten & Palm, 2000).

The conditions on the error estimation will be treated through the next chapters.

1.1.4. OBSERVER-BASED CONTROL FOR TAKAGI-SUGENO SYSTEMS

Depending on the nature of the decision variables, there are two cases: the case when decision variables are measurable (Patton et al, 1998) and when they are unmeasurable (Bergsten & Palm, 2002).

1.1.4.1. Measurable decision variables (MDV)

We consider system (1.3) and the following control law:

$$u(t) = - \sum_{i=1}^{r} h_i\big(z(t)\big) K_i \hat{x}(t) \qquad (1.28)$$

Substituting (1.28) into (1.3) and considering the estimation error given by (1.18), the following system is obtained:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)h_j\big(z(t)\big)\Big((A_i - B_iK_j)x(t) + B_iK_je(t)\Big) \\ \hat{y}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\,C_i\hat{x}(t) \end{cases} \tag{1.29}$$

Thus, a system can be built up as follows:

$$\dot{\tilde{x}}(t) = \sum_{i,j=1}^{r} h_i\big(z(t)\big)h_j\big(z(t)\big)\,M_{ij}\,\tilde{x}(t) \tag{1.30}$$

where $\tilde{x}(t) = \begin{pmatrix} x(t) \\ e(t) \end{pmatrix}$, $M_{ij} = \begin{pmatrix} G_{ij} & B_iK_j \\ 0 & S_{ij} \end{pmatrix}$ \hfill (1.31)

$$G_{ij} = A_i - B_iK_j$$

$$S_{ij} = A_i - L_iC_j$$

The following theorem can be stated for the study of the stability of the system (1.30), as follows:

Theorem 1.6. (Tanaka et al, 1998)

The balance of the system described by (1.30) is globally asymptotically stable if there exists a common positive definite matrix P such that:

$$M_{ij}^T P + P M_{ij} < 0, \qquad i,j = 1, \dots, r \tag{1.32}$$

Similarly, the proof of this theorem is obtained by using the theory of Lyapunov stability considering the Lyapunov function (1.5) along the path (1.3).

The procedures of LMI (1.32) feasibility may be conducted by applying the separation method between the control part and observation one (Chadli et al, 2002b), (Chadli et al, 2002c), (Ma et al, 1998). Indeed, we assume that the matrix P is of the form:

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tag{1.33}$$

By developing conditions (1.32) and supposing the following variables changes:

$$X = P^{-1}, M_i = K_iX \text{ and } N_i = P_2L_i \tag{1.34}$$

we get:

$$\begin{cases} XA_i^T - M_j^T B_i^T + A_i X - B_i M_j < 0 \\ A_i^T P_2 - C_j^T N_i^T + P_2 A_i - N_i C_j < 0 \end{cases} \tag{1.35}$$

If the LMIs (1.35) are feasible, the state-feedback and observation gains are given by:

$$K_i = M_i X^{-1}, \; L_i = P_2^{-1} N_i \tag{1.36}$$

1.1.4.2. Unmeasurable decision variables (UDV)

In this case, the state observer can be presented as given in (1.17).

The PDC control law has the following form:

$$u(t) = -\sum_{i=1}^{r} h_i\big(\hat{z}(t)\big) K_i \hat{x}(t) \tag{1.37}$$

The works in this case are important ; we can cite among them (Kruszewski, 2006), (Guerra et al, 2006), (Ichalal et al, 2007), (Yoneyama et al, 2000) and the techniques used to develop stabilization conditions will be addressed in the next chapters.

1.2. State of the Art of time-delay Takagi-Sugeno systems

1.2.1. TIME-DELAY SYSTEMS

The evolution of time-delay systems depends not only on the present information but also on a part of its past. They appear naturally in the modeling of physical processes frequently encountered in physics, economics, mechanic, chemistry, biology, population dynamics, ecology, physiology, etc. (Gopalsamy, 1992), (Kosko, 1992), (Kolmanovskii et al, 1999).

In practice, time-delay often occurs in the transmission of information or material between different parts of a system. Transportation systems, communication systems, chemical processing systems, environmental systems and power systems are examples of time-delay systems. Also, it has been shown that the existence of time-delay usually becomes the source of instability and deteriorates the performance of systems. Therefore, the T-S model has been extended to deal with nonlinear time-delay systems.

1.2.2. CLASSIFICATION OF DELAY MODELS

Several types of delays can be considered for the class of the studied systems. In the sequel, we present the different models of punctual delays, namely the constant delay and the different forms of variable delays.

1.2.2.1. Constant delay

Two decades ago, several criteria of the robust stability analysis of systems with constant delay have been developed (Li & Souza, 1997), (Li & Souza, 1999), (Kolmanovskii et al, 1999) and (Niculescu, 2001). They have been proposed for known or unknown constant delays which can be bounded or not.

The notion of constant delay leads to define it by a positive number τ $(\tau(t) = \tau)$.

1.2.2.2. Plus variable delay

The assumption of constant delay is rarely verified in reality (Lopez et al, 2006). However, if the variable delay (known or unknown) has been the subject of much research. In this case, the study of this class of systems requires an increase in the delay. Then, there is a known real scalar $\overline{\tau} > 0$ such that (Hale, 1977):

$$0 \leq \tau(t) \leq \overline{\tau} \tag{1.38}$$

This type of delay with a zero lower bound is usually called in literature: "small delay".

1.2.2.3. Bounded variable delay

We define the bounded delay $\tau(t)$ for which there are two real numbers $\underline{\tau}$ and $\overline{\tau}$ such that :

$$0 < \underline{\tau} \leq \tau(t) \leq \overline{\tau} \tag{1.39}$$

This type of delay with a nonzero inferior bound is usually called in litterature "non small delay". A typical example of dynamical systems with variable and bounded delays in time is the Networked Control Systems, for which the delay is induced by the communication networks used to convey information to control systems from sensors fitted to these systems. The study of stability and stabilization of these systems has been the subject of several studies (Yue & Won, 2002), (Yue et al, 2005), (Tian et al, 2007), (Ariba, 2009), (Dilanech, 2009).

1.2.2.4. Variable delay with constraint on the derivative

Many results require a condition on the derivative of the delay function, such as:

$$\dot{\tau}(t) \leq \beta < 1 \tag{1.40}$$

Therefore, if we consider a function $g(t)$ such that $g(t) = t - h(t)$, then the condition (1.40) implies that $g(t)$ is a strictly increasing function. This means that delayed informations arrive in chronological order.

1.2.2.5. Variable delay with bounded derivative

We present the following theorems and propositions according to time-delay T-S systems.

We assume that the first derivative of the delay has an upper bound $\overline{d}$ and the lower bound $\underline{d}$: it is more general than that studied previously given by (1.40) conditions. This case was investigated primarily by (Saadni & Mehdi, 2004).

$$\underline{d} \leq \dot{\tau}(t) \leq \overline{d} < 1 \tag{1.41}$$

1.2.3. CLASSIFICATION OF TIME-DELAY TAKAGI-SUGENO MODELS

Recently, the T-S model has been extended to study the nonlinear delay systems. In the sequel, we present the classes of time-delay T-S models:

1.2.3.1. Takagi-Sugeno model with delayed state

✓ *Nominal system*

Rule i: If z_1 is F_i^1 and … and z_p is F_i^p Then

$$\dot{x}(t) = A_i x(t) + A_{\tau i} x(t - \tau(t)) + B_i u(t) \tag{1.42}$$

where $x(t - \tau(t)) \in \mathbb{R}^{n_x}$ is the vector of the delayed state and $A_{\tau i} \in \mathbb{R}^{n_x \times n_x}$ is the delayed state matrix.

By adopting a barycentric defuzzification, the overall dynamics is defined by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_{\tau i} x(t - \tau(t)) + B_i u(t)] \tag{1.43}$$

If the model (1.43) is subject to external disturbances, it is written in the general form:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_{\tau i} x(t - \tau(t)) + B_i u(t) + B_{wi} w(t)] \\ y(t) = \sum_{i=1}^{r} h_i(z(t))[C_i x(t) + C_{\tau i} x(t - \tau(t)) + D_i u(t)] \end{cases}$$
(1.44)

where: $w(t)$ presents the bounded external disturbances and it is given by L_2 norm, that is, $\|w(t)\|_2^2 = \int_0^{\infty} w(t)^T w(t) dt < \infty$, $y(t)$ is the vector of the controlled outputs and $C_{\tau i}$ is the delayed observation matrix.

✓ *Uncertain system*

Mathematical models from modeling do not fully represent the physical systems. In fact, these models are often obtained through many simplifications. In order to illustrate this, we consider the following uncertain time-delay T-S model:

Rule i: If z_1 is F_i^1 and … and z_p is F_i^p Then

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + (A_{\tau i} + \Delta A_{\tau i})x(t - \tau(t)) + (B_i + \Delta B_i)u(t)$$
(1.45)

Then, the overall uncertain T-S system with time-delay and subject to external disturbances can be written as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[(A_i + \Delta A_i)x(t) + (A_{\tau i} + \Delta A_{\tau i})x(t - \tau(t)) + (B_i + \Delta B_i)u(t) + B_{wi} w(t)] \\ y(t) = \sum_{i=1}^{r} h_i(z(t))[(C_i + \Delta C_i)x(t) + (C_{\tau i} + \Delta C_{\tau i})x(t - \tau(t)) + (D_i + \Delta D_i)u(t)] \end{cases}$$

(1.46)

1.2.3.2. Takagi-Sugeno model with delayed control

This type of systems has the following form: (Lee et al, 2005)

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + B_i u(t) + B_{\tau i} u(t - \tau(t))]$$
(1.47)

1.2.3.3. Takagi-Sugeno model with delayed state and control

This type of system has the following form: (Chen et al, 2009)

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_{\tau i} x(t - \tau_a(t)) + B_i u(t) + B_{\tau i} x(t - \tau_b(t))]$$
(1.48)

We have presented in this section a brief state of the art of delay models to place better our work in the general context.

1.2.4. STABILIZATION BY PDC CONTROL OF T-S SYSTEMS WITH TIME-DELAY

In this section, we address the problem of stabilizing a time-delay T-S model under the assumption:

Assumption 1.1. All the model states are measurable.

To stabilize this type of T-S models, the control law of PDC type (Wang et al, 1995) is often used. This corresponds to using a linear control law for each submodel. The PDC regulator is defined in (1.9).

Applying this control law to different classes of time-delay T-S models:

  ✓ Nominal system (1.43), the closed-loop system is written:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\big[A_{cij}x(t) + A_{\tau i}x(t - \tau(t))\big] \tag{1.49}$$

where : $A_{cij} = A_i - B_i K_j$ $\tag{1.50}$

which can be rewritten :

$$\dot{x}(t) = A_c(t)x(t) + A_\tau(t)x(t - \tau(t)) \tag{1.51}$$

where:

$$A_c(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))A_{cij} \tag{1.52}$$

The first results of stabilizing time-delay T-S models by PDC control are proposed by (Cao & Frank, 2001), (Cao & Frank, 2000).

Theorem 1.8. (Cao & Frank, 2000)

The system (1.49) with a plus delay $(0 \le \tau(t) \le \overline{\tau})$ and $\dot{\tau}(t) \le d < 1$ is asymptotically stable if there exist matrices $X > 0$, $Q > 0$ and Y such that the following LMIs are satisfied:

$$\begin{pmatrix} XA_i^T + A_i X - B_i Y_i - Y_i^T B_i^T + \dfrac{1}{1-d}Q & A_{\tau i}X \\ * & -Q \end{pmatrix} < 0 \tag{1.53}$$

$$\begin{pmatrix} \Delta_{ij} + \dfrac{2}{1-d}Q & (A_{\tau i} + A_{\tau j})X \\ * & -2Q \end{pmatrix} \le 0, \qquad i < j = 1, \dots, r \tag{1.54}$$

with:

$$\Delta_{ij} = XA_i^T + A_i X + XA_j^T + A_j X - B_i Y_j - Y_j^T B_i^T - B_j Y_i - Y_i^T B_j^T \tag{1.55}$$

Then, the controller gains K_i are given by:

$$K_i = Y_i X^{-1}, \qquad i = 1, \dots, r \tag{1.56}$$

The main limitation of this approach is that the stabilization conditions are independent of the delay size, they are conservative. In order to reduce this conservatism, (Chen & Liu, 2005a) have proposed stabilization conditions which are dependent of the size of delay using a quadratic FLK with a double integral.

This section presents a state of art on the PDC stabilization of a class of T-S models with delayed states. Relaxations of stabilization conditions are given by introducing additional variables. The improvement of these conditions makes a significant increase in the complexity of the problem to solve. We find ourselves faced to a compromise between conservatism and complexity. The complexity depends on the number of variables, the number of control parameters and the size of the used T-S models. There have been other recent results (Gassara et al, 2009a), (Gassara et al, 2009b), (Gassara et al 2009c), (Gassara et al, 2010b) and (Gassara et al. 2010c) for the stabilization of uncertain and disturbed time-delay T-S models. These results make it possible to find an acceptable solution to compromise between the reduction of conservatism and computational complexity.

1.2.5. Relaxation techniques

In this section, the main techniques of stability conditions relaxation of Takagi-Sugeno models with time-delay are presented:

- **Quadratic FLK with a simple integral** (Cao & Franck, 2000)

$$V\big(x(t)\big) = x(t)^T P x(t) + \int_{t-\tau(t)}^{t} x(\alpha)^T S x(\alpha) d\alpha \tag{1.57}$$

with $P > 0$ and $S > 0$ are symmetric and positive definite matrices (thus ensuring that the functional $V\big(x(t)\big)$ is positive definite).

- **Quadratic FLK with a double integral** (Li et al, 2004)

$$V\big(x(t)\big) = x(t)^T P x(t) + \int_{t-\tau(t)}^t x(\alpha)^T S x(\alpha) d\alpha + \int_{-\bar\tau}^0 \int_{t+\sigma}^t \dot{x}(\alpha)^T Z \dot{x}(\alpha) d\alpha d\sigma \quad (1.58)$$

where P, S and Z are symmetric and positive definite matrices.

- **Polyquadratic FLK with a double integral** (Lin et al, 2007)

$$V\big(x(t)\big) = x(t)^T P(t) x(t) + \int_{t-\tau(t)}^t x(\alpha)^T S x(\alpha) d\alpha + \int_{-\bar\tau}^0 \int_{t+\sigma}^t \dot{x}(\alpha)^T Z \dot{x}(\alpha) d\alpha d\sigma$$

$$(1.59)$$

where:

$$P(t) = \sum_{j=1}^r h_j(z(t)) P_j \qquad (1.60)$$

where S, Z and P_j, $j = 1, \dots, r$, are symmetric and positive definite matrices.

This functional is more general because fixing only $P_j = P$ gives the quadratic FLK. However, it requires that the activation functions $h_j(z(t))$ are continuously differentiable and the derivative is bounded:

$$\big\|\dot{h}_k(t)\big\| \le \mathcal{X}_k, k = 1, \dots, r \qquad (1.61)$$

with $\mathcal{X}_k \ge 0$.

- **Polyquadratic FLK with a double integral** (Wu & Li, 2007)

$$V\big(x(t)\big) = x(t)^T P(t) x(t) + \int_{t-\tau(t)}^t x(\alpha)^T S(t) x(\alpha) d\alpha + \int_{-\bar\tau}^0 \int_{t+\sigma}^t \dot{x}(\alpha)^T Z(t) \dot{x}(\alpha) d\alpha d\sigma$$

$$(1.62)$$

where:

$$S(t) = \sum_{j=1}^r h_j(z(t)) S_j \qquad (1.63)$$

$$Z(t) = \sum_{j=1}^r h_j(z(t)) Z_j \qquad (1.64)$$

Recently, other functionals have been proposed for Takagi-Sugeno models with a bounded delay, for example, in (Jiang & Han, 2007), the following FLK has been constructed:

$$V\big(x(t)\big) = V_1\big(x(t)\big) + V_2\big(x(t)\big) + V_3\big(x(t)\big) \tag{1.65}$$

with:

$$V_1\big(x(t)\big) = x(t)^T P(t) x(t) \tag{1.66}$$

$$V_2\big(x(t)\big) = \int_{t-\tau_1}^{t} x(s)^T Q x(s) ds \tag{1.67}$$

$$V_3\big(x(t)\big) = \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R_1 \dot{x}(s) ds d\theta + \int_{-\underline{\tau}}^{\overline{\tau}} \int_{t+\theta}^{t} \dot{x}(s)^T R_2 \dot{x}(s) ds d\theta \tag{1.68}$$

where P, Q, R_1 and R_2 are symmetric and positive definite matrices.

1.3. State of the Art of Positive Systems

1.3.1. INTRODUCTION

In this chapter, we define various concepts and results that will be used in the following chapters. A part from the results concerning positive systems and their structural properties. Such systems must satisfy a sign inherent constraint: the components of the state of the system have to remain nonnegative when they are initialized by nonnegative initial values. The rest of the results presented in this chapter are well-known results in the literature (Benzaouia et al, 2011), (Benzaouia et al, 2014), (Benzaouia & Oubah, 2014), (Benzaouia & El Hajjaji, 2014). We focus here on positivity of linear systems and their properties that have been successfully used in the last works. α-stability and stabilization techniques of positive linear systems, with delay and without delay, will be also discussed thoroughly in this section. Finally, we state some problems encountered when designing observers for this class of systems.

1.3.2. DEFINITION AND PROPERTIES OF POSITIVE SYSTEMS

1.3.2.1. Notations and Definitions

We define here several notations and operations affected to vectors and matrices that maintain positivity. These notations and operations are in conformity with

those classically introduced in literature, see (Luenberger, 1979; Farina & Rinaldi, 2000). We now introduce the following relations of partial order in $\mathbb{R}^n$, then:

$$\begin{cases} x > y \ \leftrightarrow\ x_i > y_i\,, \ \ \forall\, i = 1, \dots, n. \\ x \geqslant y \ \leftrightarrow\ x_i \geq y_i\,, \ \ \forall\, i = 1, \dots, n. \end{cases}$$

This above partial order relations suggest the following notations

- x is a positive vector $(x > 0)$ if all its elements are strictly greater than zero.

- x is a nonnegative vector $(x \geqslant 0)$ if all elements are nonnegative.

In the same way, we define the partial order relations for matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right] \in \mathbb{R}^{n_x \times n_u}$.

$$\begin{cases} A > B \ \Leftrightarrow\ a_{ij} > b_{ij}\,, \ \ \forall\,(i,j) \\ A \geqslant B \ \Leftrightarrow\ a_{ij} \geq b_{ij}\,, \ \ \forall\,(i,j) \end{cases}$$

Also, we use the following notations for matrices

- A is a positive matrix if all its elements are strictly greater than zero $(A > 0)$.

- A is a nonnegative matrix if all its elements are nonnegative $(A \geqslant 0)$.

It is also worth to notice the following rules that hold for matrices A, B and vectors x, y with appropriate dimensions

$$\begin{cases} A > 0\,, B > 0 \ \Leftrightarrow\ AB > 0 \\ A \geqslant 0\,, B \geqslant 0 \ \Leftrightarrow\ AB \geqslant 0 \\ A > 0\,, x \geqslant 0 \ \Leftrightarrow\ Ax \geqslant 0 \\ A \geqslant 0\,, x \geqslant 0 \ \Leftrightarrow\ Ax \geqslant 0 \\ x \geqslant 0\,, y \geqslant 0 \ \Rightarrow\ x^T y = y^T x \ \geqslant 0 \end{cases}$$

1.3.2.2. Properties of Metzler Matrices

First, we introduce the notion of Metzler matrix, proposed by (Metzler, 1945). This matrix plays an important role in the mathematical developments that demonstrate how this matrix can be connected to positive systems. For this reason, we are devoted to present the structural properties of such matrices:

Definition 1.9.

A real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, that is, $m_{ij} \geq 0$, $\forall\, i \neq j$.

Remark 1.1.

A matrix M is a Metzler matrix if and only if there exists $\beta \in \mathbb{R}_+$ such that $(M + \beta I) \succcurlyeq 0$.

An important property of Metzler matrices is given by the following result (Farina & Rinaldi, 2000)

Lemma 1.4.

Given a Metzler matrix M, then, e^{tM} is nonnegative for all $t \geq 0$.

We will use the following definition in the sequel:

Definition 1.10.

A real matrix M is called a Hurwitz matrix if all its eigenvalues have a strictly negative real part.

There are many results that concentrate on positive matrices. They were mainly developed for positive systems in discrete-time (Luenberger, 1979), (Senata, 1981), (Bapat & Raghavan, 1997), (Farina & Rinaldi, 2000). We state here Frobenius-Perron theorem (Perron, 1907), (Frobenius, 1908) as it seems one of the most important results on positive matrices:

Theorem 1.12.

If $M > 0$, then there exists a scalar $\lambda_0 > 0$ and a vector $x_0 > 0$ such that

- $M x_0 = \lambda_0 x_0$ *(λ_0 is an eigenvalue of M and x_0 is a right eigenvector).*
- *For any other eigenvalue $\lambda \neq \lambda_0$, we have $|\lambda| \leq \lambda_0$.*
- *λ_0 is an eigenvalue of geometric and algebraic multiplicity one.*

Luenberger extended many works resulting from Theorem 1.12. for positive systems to Metzler matrices (Luenberger, 1979). He provided the following results, which will be extremely useful for the study of positive systems.

Theorem 1.13. (Luenberger 1979)

Given a Metzler matrix M and a vector $b > 0$, the matrix M has all of its eigenvalues strictly within the left half of the complex plane if and only if there exists a vector $\lambda > 0$ satisfying: $M\lambda + b = 0$

Theorem 1.14. (Luenberger, 1979)

Let M be a Metzler matrix. Then, $-M^{-1}$ exists and is a positive matrix if and only if M has all of its eigenvalues strictly within the left half of the complex plane.

Also, there are some revealing results for Metzler matrices. This theorem is also a consequence of Theorem 1.12.

Theorem 1.15. (Smith, 1986)

Let M, N be two Metzler matrices. Then,

- *The dominant eigenvalue of M is real and its associate eigenvector is positive.*
- *If $N \geq M$, then, the dominant eigenvalue of N is greater than or equal to the dominant eigenvalue of M.*

1.3.2.3. Properties of Positive Systems

Now, we are meant to present and analyze an important class of dynamical systems that consists of positive systems.

Beginning with the general case of a nonlinear autonomous dynamical system, we consider the following expression:

$$\dot{x} = f(x(t)), \, x(0) = x_0, \, t \geq 0 \tag{1.68}$$

with: $x(t) \in \mathbb{R}^{n_x}$ is the system state vector, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$.

The following definition is about the positivity of system (1.130).

Definition 1.11.

System (1.68) is positive if all the trajectories $x(t, x_0)$ generated by (1.68), with nonnegative initial conditions $(x_0 \succcurlyeq 0)$ remain nonnegative $(x(t) \succcurlyeq 0, \forall t \geq 0)$.

The above definition 1.11. means that the positive orthant $\mathbb{R}_+^{n_x}$ is invariant for system (1.68). The invariance of the positive orthant $\mathbb{R}_+^{n_x}$ is equivalent to the fact

that the borders of $\mathbb{R}_+^{n_x}$ represent barriers for the trajectories of the system, when initialized in $\mathbb{R}_+^{n_x}$. Thus, system (1.68) is positive, if and only if, for any $i = 1, \dots, n_x$, such that $x_i(t) = 0$ at a time t (with $x(t) \geqslant 0$), then, its derivative $\dot{x}_i(t)$ is positive. This fact can be used to derive a useful property to check out the positivity of the system. Hence, we can note that the previous definition may be equivalent to the property below on the function f.

Property 1.1.

System (1.68) remains positive if and only if $\forall\, i = 1, \dots, n_x$, we have that for any $x \geqslant 0$ with $x_i = 0$

$$f_i(x_1, \dots, x_i(=0), \dots, x_n) \geq 0,$$

with $f(x) = [f_1(x) \dots f_n(x)]^T$

Otherwise, the above property means that each face of the positive orthant $\mathbb{R}_+^{n_x}$ is repulsive for system (1.68) when it starts within the positive orthant. The necessity of this property is to demonstrate that a linear system expressed by $f(x) = Ax$ is positive if and only if the matrix A is Metzler.

1.3.2.4. Positive Linear Systems

This section is devoted to present some necessary and sufficient conditions for stability of positive linear systems. Note that the stability of linear positive systems was studied by (Luenberger, 1979). Particularly, some works (Farina & Rinaldi, 2000), (Kaczorek, 2002) deal with other properties of positive systems such that controllability and observability. In this section, we are not interested in these notions but we should talk about monotonicity and stability properties. We are going to state some well-known stability results and, in particular, we will see that there exists a structural relation between a positive equilibrium point of a linear positive system and its global stability. In order to obtain further structural properties about linear positive systems, one can see (Haddad et al, 2010), (De Leenheer & Aeyels, 2001), (Hof, 1997).

We consider the linear dynamic system in $\mathbb{R}^{n_x}$ described by:

$$\dot{x}(t) = Ax(t) + b(t), \ x(0) = x_0 \tag{1.69}$$

Definition 1.12.

Let $T > 0$. A real function $u: [0, T] \to \mathbb{R}^{n_u}$ is a nonnegative (respectively positive) function if $u(t) \geqslant 0$ (respectively $u(t) > 0$) on the interval $[0, T]$.

Since now, we will look for the stability of system (1.69).

Theorem 1.16. (Luenberger, 1979)

For any nonnegative function $b(t)$, system (1.69) is positive if and only if A is a Metzler matrix.

Theorem 1.17. (Bolajraf, 2012)

Consider two dynamic systems:

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + b_1(t), & x_1(t) = x_{10} \\ \dot{x}_2(t) = A_2 x_2(t) + b_2(t), & x_2(t) = x_{20} \\ \text{with } x_{10} \leqslant x_{20} \end{cases}$$

Assume that A_2 is a Metzler matrix and $A_1 \leqslant A_2$, $b_1(t) \leqslant b_2(t)$, $\forall t \geq 0$. Then, we have that $x_1(t) \leqslant x_2(t)$, $\forall t \geq 0$.

Theorem 1.18. (Bolajraf, 2012)

Consider System (1.69) with A Metzler and Hurwitz. If there exists a nonnegative constant vector $g \in \mathbb{R}_+^{n_x}$ such that $b(t) \leqslant g$; then, the trajectory $x(t)$ of system (1.69) converges towards the box

$$\mathcal{B}(0, x^*) := \{v \in \mathbb{R}^{n_x} \mid 0 \leqslant v \leqslant -A^{-1} g\}.$$

The above result is considered to be a very important solution for convergence analysis in the part of estimation of positive linear systems.

1.3.2.5. Stability of Positive Linear Systems

In this part, we will provide some useful stability conditions for linear positive systems. We can express these conditions in terms of Linear Programming (LP) or in terms of Linear Matrix Inequalities (LMI), so that they can be numerically checked easily and in an efficient way.

Consider the following autonomous linear system:

$$\dot{x}(t) = Ax(t), \, x(0) = x_0 \in \mathbb{R}_+^{n_x} \tag{1.70}$$

The analysis of the asymptotic stability of the system (1.70) is presented in the following result. We can also find more general results for the case of definite positive matrix linear systems (El Ghaoui & Rami, 1996).

Theorem 1.19. (Rami & Tadeo, 2007a)

Assume that system (1.70) is positive (or equivalently the state matrix is Metzler); then, the following statements are equivalent:

  - *(i)   A is a Hurwitz matrix.*
  - *(ii)   System (1.70) is asymptotically stable for every initial condition.*
  - *(iii)   System (1.70) is asymptotically stable for an arbitrary initial condition x_0 in the interior of $\mathbb{R}_+^{n_x}$.*
  - *(iv)   There exists $\lambda > 0$ such that $A\lambda < 0$.*

Next, we formulate the theorem of positive linear systems stability through linear matrix inequalities (LMIs).

Theorem 1.20. (Bolajraf, 2012)

Assume that system (1.70) is positive (or equivalently that the state matrix is Metzler); then the following statements are equivalent:

  - *(i)   A is a Hurwitz matrix.*
  - *(ii)   System (1.70) is asymptotically stable.*
  - *(iii)   There exist $d > 0$ such that $A\,diag(d) + diag(d)A^T < 0$.*

1.3.3. POSITIVE LINEAR SYSTEMS WITH TIME-DELAY

Motivated by the fact that stability is the prime objective in control system design, we present in this section a review on approaches that analyze the stability behavior of time-delay linear systems. Specifically, we are interested in analyzing time-delay systems that maintain positivity and stability against delay factors and/or interval uncertainties (Bolajraf, 2012). One of the most revealing facts is that stability of time-delay linear systems is independent of the delays that affect them. Moreover, guaranteeing the stability of a positive system with delays can be easily done through the analysis of a specific Metzler matrix, connected to its dynamic that is independent of the system delay.

We will focus on the main stability results (Bolajraf et al, 2009), (Bolajraf et al, 2010) on linear time-delay positive systems. We will also define the robust α-stability that guarantees a specified decay rate in front of possible uncertainties on the system. In addition, necessary and sufficient conditions will be provided for the stabilization of this type of systems by means of state-feedback laws that can be with or without memory.

1.3.3.1. Stability Analysis

- **Asymptotic Stability**

In this section, we revise necessary and sufficient stability conditions which are presented in terms of LP or LMI conditions.

Consider the class of autonomous linear system with delays, represented as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i \, x(t - \tau_i) \tag{1.71}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $A \in \mathbb{R}^{n_x \times n_x}$, $A_i \in \mathbb{R}^{n_x \times n_x}$, $i = 1, \dots, m$ are constant matrices and τ_i, $i = 1, \dots, m$ are constant delays.

Initial conditions are given by:

$$x(t) = \phi(t) \in \mathbb{R}_+^{n_x}, \, t \in [-\tau, 0], \, \tau = \max_{1 \le i \le m} \tau_i \tag{1.72}$$

According to Definition 1.11., the following result provides checkable conditions for the positivity of system (1.71). We can cite, for example, (Haddad & Chellaboina, 2004) and (Rami, 2009).

Proposition 1.1.

System (1.71) is positive if and only if A is a Metzler matrix and A_i, $i = 1, \dots, m$ are nonnegative matrices.

Next, necessary and sufficient conditions regarding the stability of system (1.71) are given. We can refer to some works: (Hmamed et al, 2008a), (Rami, 2009) and (Liu, 2009).

Theorem 1.21. (Bolajraf, 2012)

System (1.71) is asymptotically stable for every initial condition (1.72) if and only if there exists $\lambda \in \mathbb{R}^{n_x}$ such that:

$$\left(A + \sum_{i=1}^{m} A_i\right)\lambda < 0, \lambda > 0 \tag{1.73}$$

Remark 1.2.

We can note that checking the asymptotic stability of system (1.72) is equivalent to checking the asymptotic stability of the following positive system without delays $\dot{x}(t) = A_0 x(t)$, with:

$$A_0 = A + \sum_{i=1}^{m} A_i.$$

Corollary 1.1.

The system (1.71) is asymptotically stable for every nonnegative intial condition (1.72) if and only if one of the following conditions holds :

i) The inverse of $A + \sum_{1=1}^{m} A_i$ exists and all its components are nonnegative; that is

$$(A + \sum_{i=1}^{m} A_i)^{-1} \le 0 \tag{1.74}$$

ii) There exists $\lambda \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + \sum_{i=1}^{m} A_i\right)\lambda < 0 \\ \lambda > 0 \end{cases} \tag{1.75}$$

iii) There exists $d \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + \sum_{i=1}^{m} A_i\right)diag(d) + diag(d)\left(A^T + \sum_{i=1}^{m} A_i^T\right) < 0 \\ d > 0 \end{cases} \tag{1.76}$$

iv) The matrix $A + \sum_{i=1}^{p} A_i$ is Hurwitz.

- **α-Stability**

If system (1.71) is asymptotically stable then it will decrease exponentially to the origin with a decay rate which is inherent from its dynamic. In this part, we show how one can check that the decay rate of convergence of the system is higher than a given decay rate α which will generate a lower bound on the real decay rate of the system.

Definition 1.13.

For a real scalar $\alpha > 0$, the system (1.71) is α-stable, if all its possible trajectories satisfy $\lim_{t \to \infty} e^{\alpha t} x(t) = 0$. In this case, we say that system (1.71) has at least a decay rate of order α.

Theorem 1.22.

For a real scalar $\alpha > 0$, the system (1.71) is α-stable if and only if one of the following conditions holds:

i) The positive system with delays (1.71) is asymptotically stable.

ii) There exists $\lambda \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + \alpha I + \sum_{i=1}^{m} e^{\alpha \tau_i} A_i \right) \lambda \prec 0 \\ \lambda > 0 \end{cases} \tag{1.77}$$

iii) There exists $d \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + \sum_{i=1}^{m} e^{\alpha \tau_i} A_i \right) diag(d) + diag(d) \left(A^T + \sum_{i=1}^{m} e^{\alpha \tau_i} A_i^T \right) + 2\alpha diag(d) < 0 \\ d > 0 \end{cases} \tag{1.78}$$

Regarding unknown delays, we can extend the previous result.

Corollary 1.2.

Assume that the time delays are unknown but bounded $0 \le \tau_i \le \overline{\tau}, \ i = 1, ..., m$. Then, for a given real scalar $\alpha > 0$, the following statements are equivalent.

i) The positive system (1.71) is α-stable for any delays bounded by $\overline{\tau}$.

ii) There exists $\lambda \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + \alpha I + e^{\alpha \bar{\tau}} \sum_{i=1}^{m} A_i \right) \lambda < 0 \\ \lambda > 0 \end{cases} \tag{1.79}$$

iii) There exists $d \in \mathbb{R}^n$ such that

$$\begin{cases} \left(A + e^{\alpha \bar{\tau}} \sum_{i=1}^{m} A_i \right) diag(d) + diag(d) \left(A^T + e^{\alpha \bar{\tau}} \sum_{i=1}^{m} A_i^T \right) + 2\alpha diag(d) < 0 \\ d > 0 \end{cases} \tag{1.80}$$

1.3.3.2. PDC Controller Design

Consider the following forced time-delay system:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i \, x(t - \tau_i) + Bu(t) \tag{1.81}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input and τ_i, $i = 1, \dots, m$ are the delays. Matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$ represent the nominal system without delays. A_i, $i = 1, \dots, m$ are known matrices and the initial condition is given by (1.72).

We are meant to design for system (1.71) a state-feedback controller of the form:

$$u(t) = Kx(t), K \in \mathbb{R}^{n_u \times n_x} \tag{1.82}$$

For which the closed-loop system is positive and asymptotically stable.

Applying the control law (1.82) to system (1.71), we get the closed-loop system given by:

$$\dot{x}(t) = (A + BK)x(t) + \sum_{i=1}^{m} A_i \, x(t - \tau_i) \tag{1.83}$$

Hence, we need to find necessary and sufficient conditions on matrices A, $A_1, \dots, A_m$ and B, such that there exists a matrix K satisfying:

* Positivity of the closed-loop system: the matrix $A + BK$ is Metzler $(A_1, \dots, A_m)$ are necessarily nonnegative)

* Stability of the closed-loop system: the matrix $A + BK$ is Hurwitz.

- **State-feedback α-Stabilization**

In the folowing, we develop necessary and sufficient conditions for positivity and α-stability of the closed-loop system (1.83).

Theorem 1.23. (Bolajraf, 2012)

Assume that the matrices A_i, $i = 1, \dots, m$ are nonnegative. Then, for a prescribed decay rate $\alpha > 0$, the closed-loop system (1.83) is positive and α-stable if and only if one of the following equivalent conditions is satisfied :

i) The following LP problem in the variables $\beta \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+^{n_x}$ and $Z \in \mathbb{R}^{n_x \times n_u}$ is feasible

$$\begin{cases} A_0\lambda + BZ1_{n_x} < 0 \\ Adiag(\lambda) + BZ + \beta 1_{n_x} \geqslant 0 \\ \lambda > 0 \end{cases} \tag{1.84}$$

where:

$$A_0 = A + \alpha I + \sum_{i=1}^{m} e^{\alpha \tau_i} A_i$$

Moreover, the gain matrix K satisfying (1.84) can be computed as follows

$$K = Zdiag(\lambda)^{-1} \tag{1.85}$$

where λ and Z are any feasible solution to the LP problem (1.84).

ii) The following LMI problem in the variables $\beta \in \mathbb{R}_+$, $d \in \mathbb{R}^{n_x}$ and $Y \in \mathbb{R}^{n_x \times n_u}$ is feasible

$$\begin{cases} A_0 diag(d) + diag(d)A_0^T + BY + Y^T B^T < 0 \\ Adiag(d) + BY + \beta I_{n_x} \geqslant 0 \\ d > 0 \end{cases} \tag{1.86}$$

Moreover, the gain matrix K satisfying (1.86) can be computed as follows

$$K = Ydiag(d)^{-1} \tag{1.87}$$

where d and Y are any feasible solution to the LMI problem (1.86).

Results on robust stabilization and estimation of postive interval systems are also studied in (Bolajraf, 2012).

1.3.4. POSITIVE TAKAGI-SUGENO SYSTEMS

We have explained how to model a Takagi-Sugeno system. However, we are meant to study a particular class of continuous-time Takagi-Sugeno systems: the positive ones. Our thesis work considers an additional problem that we frequently encounter in several dynamical systems: the nonnegativity of the states. (Benzaouia et al, 2006), (Benzaouia & Tadeo, 2010), (Benzaouia & Tadeo, 2008), (Benzaouia et al, 2007), (Boukas & El Hajjaji, 2006), (El Hajjaji et al, 2006), (El Hajjaji & Chadli, 2008). The study of systems with nonnegative states is important in practice because many chemical, physical and biological processes involve quantities that have intrinsically constant and nonnegative signs: the concentration of substances, the levels of liquids, etc, are always nonnegative. In the literature, systems whose states are nonnegative whenever the initial conditions are nonnegative are referred to as positive (Farina & Rinaldi, 2000). The design of controllers for these positive systems has been studied by (Rami & Tadeo, 2006), (Rami & Tadeo, 2007a), where the authors provide a new treatment for the stabilization of positive linear systems. All the proposed conditions are necessary, sufficient and expressed in terms of Linear Programming (LP). These results were then extended to systems with delay by (Hmamed et al, 2007), (Hmamed et al, 2008). One could think that LMI techniques can easily handle this new constraint of nonnegativity of the states. Nevertheless, this is not usually possible without taking care of the use of the adequate Lyapunov function. The model of a real plant is used to show the need for such controllers in practice, especially for fuzzy systems where the model is global involving the whole state and not a state of variation around a set point. This idea, which was earlier used for positive switching systems in (Benzaouia & Tadeo, 2010), (Benzaouia & Tadeo, 2008), has a different impact on positive fuzzy systems due to the form of the obtained global matrix in closed-loop. Sufficient conditions of asymptotic stability for positive discrete-time fuzzy systems, represented by Takagi-Sugeno models, were obtained for the first time in (Benzaouia et al, 2001). Positive Takagi-Sugeno are defined as follows:

Definition 1.14.

The T-S system (1.3) is said to be controlled positive if, given any nonnegative initial state and any input function $u(t) \geqslant 0$, the corresponding trajectory remains in the positive orthant for all t: $x(t) \in \mathbb{R}_+^n$.

Lemma 1.2.

The autonomous system (1.3) is positive if and only if A_i is a Metzler matrix and B_i, C_i and D_i are nonnegative matrices for $i = 1, \dots, r$.

Next, in the following chapters, we will deal with the stabilization and estimation of positive T-S systems.

1.3.5. REVIEW ON POSITIVE TIME-DELAY TAKAGI-SUGENO SYSTEMS

However, a new constraint is added to this type of systems: the positivity. The aim becomes then to maintain the stability and positivity of the states at each time.

Consider a nonlinear system with time-delay which could be represented by a T-S time-delay model described by (1.43).

Then, the overall T-S system can be inferred as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\Big(A_i x(t) + A_{1i} x\big(t - \tau(t)\big) + B_i u(t)\Big) \tag{1.88}$$

such that (1.4) is satisfied.

Definition 1.15.

The T-S system (1.88) is said to be controlled positive if, given any nonnegative initial state and any input function $u(t) \geqslant 0$, the corresponding trajectory remains in the positive orthant for all t, $x \in \mathbb{R}_+^n$.

Lemma 1.3. (Benzaouia et al, 2011)

The autonomous delayed system (1.88) is positive if and only if A_i is a Metzler matrix and A_{1i} and B_i are nonnegative matrices for $i = 1, \dots, r$.

Conditions of stability and stabilization of a time-delay T-S system (1.88), using LMI method as presented in (Benzaouia et al, 2011), are recalled.

Theorem 1.24.

For positive matrices A_{1i} and Metzler matrices A_i, the autonomous system (1.88) is asymptotically stable, if there exist a diagonal matrix $P = P^T > 0$ and a matrix $Q = Q^T > 0$ satisfying the following LMIs:

$$\begin{pmatrix} A_i^T P + PA_i + Q & PA_{1i} \\ * & -Q \end{pmatrix} < 0, i = 1, \dots, r \tag{1.89}$$

We consider the stabilization techniques of positive time-delay T-S systems. Consider that the control used in this work is the so called PDC control:

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big) K_i x\,(t) \tag{1.90}$$

By using (1.90) in (1.88), the closed-loop system can be written as:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i\big(z(t)\big) h_j\big(z(t)\big)\big[(A_i + B_i K_j)x(t) + A_{1i}x(t - \tau(t))\big] \tag{1.91}$$

Theorem 1.25.

For positive matrices A_{1i}, if there exist a diagonal matrix $X = X^T > 0$, matrices $Y_j; j = 1, \dots, r$ and Z satisfying the following LMIs:

$$\begin{cases} M_{ij} + M_{ji} < 0 \\ A_i X + B_i Y_j \text{ is Metzler} \end{cases}, i,j = 1, \dots, r, i \le j \tag{1.92}$$

where:

$$M_{ij} = \begin{pmatrix} XA_i^T + Y_j^T B_i^T + A_i X + B_i Y_j + Z & A_{1i}X \\ * & -Z \end{pmatrix} \tag{1.93}$$

with $P = X^{-1}$, $K_j = Y_j X^{-1}$, $Q = X^{-1}ZX^{-1}$

Then, system (1.91) is asymptotically stable and controlled positive.

To establish these conditions, the Lyapunov-Krakovskii functional (1.57) was used.

Note that these results are a particular case of the ones given by (Benzaouia & El Hajjaji, 2011).

1.4. Conclusion

This chapter was devoted to review some definitions related to the modeling of T-S systems as well as the stabilization and observers design for this type of

systems, taking into account if the decision variables are measurable or not. Recent approaches have been presented. A discussion of nonlinear systems was then used to introduce T-S models with time-delay, presenting briefly the classification of delay models and time-delay T-S models. We have also introduced methods for the stabilization and estimation of this class of systems. Finally, we have presented useful properties and results about positive dynamic systems. In particular, we have analyzed positivity and stability of linear systems and we have provided necessary and sufficient conditions for their stabilization. Equivalent conditions for checking α-stability have been given in terms of LP and LMIs. These results were applied for the control design of positive systems with time-delays. Then, we made a short background on positive T-S time-delay systems. The previous results will be useful for the several approaches of the next chapters.

Chapter II

Stability and Stabilization of Positive Takagi-Sugeno systems

2.1. Introduction

In this chapter, we are interested in the analysis of stability and stabilization of positive dynamical systems. Regarding linear systems, we may recall some basic properties of positive systems, developed in (Luenberger, 1979). Other structural and stability properties can be found in (Farina & Rinaldi, 2000). More investigation works were developed in this issue (Kaczorek, 2002), (Haddad et al, 2010), (Benzaouia et al, 2010), (Benzaouia et al, 2011), etc. For this, we will provide our results for positive T-S systems. We note that positive linear systems are a particular case of positive Takagi-Sugeno systems.

2.2. Stability and Stabilization of Positive Takagi-Sugeno systems with measurable premise variables

Control synthesis for the systems that are represented by Takagi-Sugeno (T-S) models may be considered as a convex problem soluble by LMI optimization techniques (Kim & Park, 2003). Despite the benefits of LMIs, the existence of a solution which satisfies the sufficient stabilization conditions is not always guaranteed, specially, when the number of submodels increases, or when several constraints are imposed on the control performance such as the convergence rate, the problem becomes unsolvable (Luoh, 2002).

To overcome this problem, several studies have proposed less stringent stabilization conditions to minimize the conservatism of these techniques (Tanaka et al, 1998), (Lim & Lee, 2000), (Xiaodong & Qingling, 2003), (Chadli et al, 2004). However, maximizing the speed of convergence is often not considered.

An extension of this work was proposed to define new conditions of asymptotic and exponential stability with maximization of the convergence rate for a Takagi-Sugeno system with measurable variable decisions. In addition to the LMIs, other optimization techniques have been used, such as the resolution of the problem of generalized eigenvalues (GEVP) and the maximization of a function under constraints (Salem, 2013).

After a presentation of the results established in the literature regarding the stability of T-S systems, new and improved conditions for both continuous and discrete systems are proposed. These conditions are then used for the synthesis of

control laws based on PDC (Parallel Distributed Compensation) (Wang et al, 1995a), (Wang et al, 1995b), (Wang et al, 1996).

In this section, the decision variables are assumed to be measurable.

2.2.1. POSITIVE STABILIZATION BY MEMORYLESS STATE-FEEDBACK

In recent years, there have been many works on synthesis of control laws for multimodels. We can cite, for example, (Tanaka et al, 1998), (Guerra & Vermeiren, 2001), (Blanco et al, 2001), (Kim & Lee, 2000), (Chadli et al, 2001). The synthesis of control laws by the quadratic method requires the existence of a symmetric positive definite matrix P satisfying the Lyapunov equation. Some relaxations seem to be crucial to the PDC control law (Kim & Lee, 2000). In addition, the use of CDF control law (Compensation and Division for Fuzzy models), when the input matrices are positively collinear, leads to less pessimistic conditions.

2.2.2.1. Positive Asymptotic Stabilization

Let us consider again the following continuous-time T-S system:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) x(t) \tag{2.1}$$

where the normalized activation functions $h_i(z(t))$ corresponding to the ith submodel verify the proprieties (1.4).

Definition 2.3. (Chadli, 2002)

If the pairs (A_i, B_i) are controllable (stabilizable), then, the multimodel (2.1) is locally controllable (stabilizable).

To stabilize a positive Takagi-Sugeno model (2.1), a Takagi-Sugeno controller can be designed using the PDC technique. In this case, the global control law is obtained interpolating local state-feedback linear laws associated to the various submodels. For this, the system (2.1) is assumed to be stabilizable, i.e., all pairs (A_i, B_i), $\forall i = 1, \dots, r$ are stabilizable.

The global control law is given by:

$$u(t) = -\sum_{i=1}^{r} h_i\big(z(t)\big)K_i x(t) \tag{2.2}$$

where $h_i(z(t))$ are the activation functions verifying the constraints (2.2).

Substituting (2.2) in (2.1), we obtain the continuous-time closed loop system:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))G_{ij}x(t) \tag{2.3}$$

$$= \sum_{i=1}^{r} h_i^2\big(z(t)\big)G_{ii}x(t) + 2\sum_{1\leq i<j}^{r} h_i\big(z(t)\big)h_j\big(z(t)\big)\left(\frac{G_{ij}+G_{ji}}{2}\right)x(t) \tag{2.4}$$

where: $G_{ij} = A_i - B_i K_j$ $\tag{2.5}$

The equation (2.4) highlights the submodels called *dominant* characterized by the matrices G_{ii} and the submodels called *couples* caracterized by $\left(\frac{G_{ij}+G_{ji}}{2}\right)$.

We then use (2.4) the previous results to analyse the stability of positive Takagi-Sugeno systems, as an extension of (Chadli, 2002) for positive T-S systems.

Theorem 2.1.

The system described by (2.4) is globally asymptotically stable and positive, if there exist a symmetric positive definite matrix P and a positive scalar β such that:

$$\begin{cases} G_{ii}^{T}P + PG_{ii} < 0; 1 \leq i \leq r \\ \left(\frac{G_{ij}+G_{ji}}{2}\right)^{T}P + P\left(\frac{G_{ij}+G_{ji}}{2}\right) \leq 0; 1 \leq i < j \leq r \\ G_{ij}P + \beta P \geqslant 0 \end{cases} \tag{2.6}$$

Proof:

The first two conditions in Theorem 2.1. are given from (Chadli, 2002). Moreover, to guarantee the positivity of the closed-loop system (2.4), we have to prove that $G_{ij} = A_i - B_i K_j$ is Metzler, $\forall\, i,j = 1, \dots, r$: From Remark 1.1, (2.4) is positive if and only if there exists $\beta \in \mathbb{R}_+$ such that $\big(G_{ij} + \beta I\big) \geqslant 0$ $\tag{2.7}$
Multiplying this inequality on the right by P, we get: $G_{ij}P + \beta P \geqslant 0$. This ends the proof. ∎

Note that conditions (2.6) are conservative because they require the stability of all submodels (dominant and coupled). This result also shows that the positive stabilization of the controlled system can be reduced to a problem of existence of a common symmetric positive definite matrix P. However, if r is high, it becomes difficult to find the common matrix P satisfying the conditions (2.6). To minimize this conservatism, Tanaka et al. (Tanaka et al, 1998) proposed better conditions that require only the stability of the dominant submodels. Based on these works, we propose less conservative stability conditions for positive T-S systems.

Theorem 2.1.

Let s be the number of the simultaneously active submodels with $2 \leq s \leq r$. The equilibrium of the controlled system described by (2.12) is asymptotically stable, if there is a symmetric positive definite matrix P and a positive semi-definite symmetric matrix Q, and a positive scalar β such that:

$$\begin{cases} G_{ii}^T P + P G_{ii} + (s-1)Q < 0; 1 \leq i \leq r \\ \left(\dfrac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\dfrac{G_{ij} + G_{ji}}{2}\right) - Q \leq 0; 1 \leq i < j \leq r \\ G_{ij}P + \beta P \geqslant 0 \end{cases} \qquad (2.8)$$

The conditions of this theorem are also conservative since they require a common matrix Q for the stability of all submodels (dominant and coupled). Based on the work of Lim and Lee (Lim & Lee, 2000), we extend Theorem 2.1 in order to reduce the conservatism.

Theorem 2.2.

The equilibrium of the controlled system described by (2.4) is asymptotically stable and positive, if there is a symmetric positive definite matrix P and symmetric matrices $Q_{ij} = Q_{ji}$, $i,j = 1, \ldots, r$, and a positive scalar β such that:

$$\begin{cases} G_{ii}^T P + P G_{ii} + Q_{ii} < 0; 1 \leq i \leq r \\ \left(G_{ij} + G_{ji}\right)^T P + P\left(G_{ij} + G_{ji}\right) - 2Q_{ij} \leq 0; 1 \leq i < j \leq r \\ G_{ij}P + \beta P \geqslant 0 \end{cases} \qquad (2.9)$$

with

$$Q_{ij} = \begin{pmatrix} Q_{11} & \cdots & Q_{1r} \\ \vdots & \ddots & \vdots \\ Q_{1r} & \cdots & Q_{rr} \end{pmatrix}.$$

The conditions (2.9) of Theorem 2.2. impose $Q_{ij} = Q_{ji}, i = 1, \dots, r$ and that these matrices are all symmetric; so we have extended the works of Xiaodong and Qingling (Xiaodong & Qingling, 2003) to improve them:

Theorem 2.3.

The equilibrium of the controlled system described by (2.4) is asymptotically stable and positive, if there is a symmetric positive definite matrix P, symmetric matrices Q_{ii}, $\forall\, i = 1, \dots, r$ and matrices $Q_{ji} = Q_{ij}^T$, $1 \leq i < j \leq r$, and a positive scalar β such that:

$$
\begin{cases}
G_{ii}^T P + P G_{ii} + Q_{ii} < 0; 1 \leq i \leq r \\
(G_{ij} + G_{ji})^T P + P(G_{ij} + G_{ji}) + (Q_{ij} + Q_{ij}{}^T) \leq 0; 1 \leq i < j \leq r \\
G_{ij} P + \beta P \geqslant 0
\end{cases}
\tag{2.10}
$$

with:

$$
Q_{ij} = \begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1r} \\
Q_{12}^T & Q_{22} & \cdots & Q_{2r} \\
\vdots & . & \ddots & \vdots \\
Q_{1r}^T & Q_{2r}^T & \cdots & Q_{rr}
\end{pmatrix}.
$$

2.2.2.2. Positive Exponential Stabilization

In addition to the stabilization, there exist other important control performances such as the decay rate, also called stabilization degree, defined as a scalar $\alpha > 0$ such that:

$$
\lim_{t \to \infty} e^{\alpha t} \|x(t)\| = 0
\tag{2.11}
$$

for all the trajectories $x(t)$ of the system (2.4).

Condition (2.11) is guaranteed if:

$$
\dot{V}(x(t)) \leq -2\alpha V(x(t))
\tag{2.12}
$$

where: $V(x(t))$ is a quadratic Lyapunov function with $P > 0$.

Chadli et al. (Chadli et al, 2002d) have developed the global exponential stability conditions of the system (2.4) where only the dominant submodels are meant to be stable but with a common matrix Q, and have characterized the minimum

decay rate of the system. We have then extended the results to the positive Takagi-Sugeno systems in the following.

Theorem 2.4.

Let s be the number of the simultaneously active submodels with $2 \leq s \leq r$. The equilibrium of the controlled system (2.4) is globally exponentially stable and positive if there exist two symmetric positive definite matrices P and Q, a scalar $0 < \varepsilon < 1$ and a scalar $\beta \in \mathbb{R}_+$ such that:

$$\begin{cases} G_{ii}^T P + P G_{ii} + (s - \varepsilon)Q < 0; 1 \leq i \leq r \\ \left(\dfrac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\dfrac{G_{ij} + G_{ji}}{2}\right) - \varepsilon Q \leq 0; 1 \leq i < j \leq r \\ G_{ij}P + \beta P \geqslant 0 \end{cases} \qquad (2.13)$$

The minimum decay rate is given by:

$$\alpha = (1 - \varepsilon)\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} \qquad (2.14)$$

Remark 2.2.

- A system that verifies conditions (2.13) necessarily verifies that: $\dot{V}(x) + (1 - \varepsilon)x^T Q x < 0$.

- A system that verifies conditions (2.13) necessarily verifies that: $\dot{V}(x) + \frac{1}{2}x^T Q x < 0$.

- The generalized conditions (2.13) of Theorem 2.4. are:

- less conservative if $\frac{1}{2} < \varepsilon < 1$ because $\dot{V}(x) + (1 - \varepsilon)x^T Q x < \dot{V}(x) + \frac{1}{2}x^T Q x$.

- more conservative if $\varepsilon < \frac{1}{2}$ because $\dot{V}(x) + \frac{1}{2}x^T Q x < \dot{V}(x) + (1 - \varepsilon)x^T Q x$.

However, the minimum decay rate may reach a value greater than that obtained for $\varepsilon = \frac{1}{2}$. (Salem, 2013)

2.2.2.3. Relaxed Positive Exponential Stabilization Conditions

Conditions of exponential stability can be deduced from Theorem 2.3, if matrices Q_{ij} are all positive. Thus, regarding the conditions of exponential stability given in (Salem, 2013), a corollary is given as follows:

Corollary 2.1.

The equilibrium of the controlled system (2.4) is globally exponentially stable and positive if there exist a symmetric positive definite matrix P, symmetric positive definite matrices $Q_{ij} = Q_{ji}$, $i, j = 1, ..., r$ such that the conditions (2.13) are verified.

Then, the minimum decay rate is:

$$\alpha = \frac{\min_{1 \leq i < j \leq r} \left(\lambda_{min}(Q_{ij}) \right)}{2\lambda_{max}(P)} \tag{2.15}$$

Remark 2.3.

The conditions (2.13) have the benefit that they do not require a common matrix Q for the stability of submodels and the inconvenient that they require the stability of all submodels (dominant and coupled).

To overcome the drawback mentioned in Remark 2.2. and control the decay rate, we propose the following Theorem 2.5 which is an extension of the stabilization conditions developed by Lian et Liou (Lian & Liou, 2006) and Chadli (Chadli, 2004).

Theorem 2.5.

Let s be the number of simultaneously active submodels with $2 \leq s \leq r$. The controlled system described by (2.4) is globally exponentially stable if there exists a symmetric positive definite matrix P, symmetric matrices $Q_{ii}, 1 \leq i \leq r$, matrices $Q_{ij} = Q_{ji}^T, 1 \leq i < j \leq r$, a positive real number γ and a positive scalar $\beta \geq 0$ such that:

$$\begin{cases} G_{ii}^T P + PG_{ii} + Q_{ii} + 2s\gamma P < 0; 1 \leq i \leq r \\ (G_{ij} + G_{ji})^T P + P(G_{ij} + G_{ji}) + (Q_{ij} + Q_{ij}^T) \leq 0; 1 \leq i < j \leq r \\ G_{ij}P + \beta P \succcurlyeq 0 \\ Q_{ij} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1r} \\ Q_{12}^T & Q_{22} & \cdots & Q_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{1r}^T & Q_{2r}^T & \cdots & Q_{rr} \end{pmatrix} \end{cases} \tag{2.16}$$

The decay rate is equal to α.

Remark 2.3.

- We can notice that the Corollary 2.1 is a special case of Theorem 2.5 for $\alpha = 0$ and positive definite $Q_{ij} = Q_{ij}^T = Q_{ji}$.

- The advantage of Theorem 2.5 is the ability to impose a decay rate and overcome the requirement of the positive definite matrices Q_{ij} in Corollary 2.1.

2.2.3. Positive Stabilization of Nonpositive Takagi-Sugeno systems

A positive system is one whose states remain nonnegative along its trajectories, starting from any nonnegative initial conditions (Kaczorek, 2002), (Kaczorek, 2010). These positive systems arise in many practical applications, such as economics, biology, chemistry, etc. The stability of positive systems was investigated in (Kaczorek, 2010), (Kaczorek, 2008), (Kaczorek, 2002), (d'Alessandro & de Santis, 1994). Controllers have been proposed for positive systems (see for example (Liu, 2009) and (Benzaouia et al, 2010)). For further information on positive systems, we refer to Kaczorek book (Kaczorek, 2002) or Farina and Rinaldi book (Farina & Rinaldi, 2000).

This section concentrates on positive Takagi-Sugeno (T-S) systems, as they provide a representation of positive nonlinear systems. In the literature, positive Takagi-Sugeno systems have been discussed in (Liu, 2009), (Kaczorek, 1998) and (Fornasini & Valcher, 2010), where stability and stabilization results were presented, based on the use of a piecewise quadratic Lyapunov function. Unfortunately, the Lyapunov function used by the authors assumes equilibrium at the origin, which constitutes a trivial solution for most applications of positive systems. In addition, it is not possible to simply translate the coordinates of the

T-S system to obtain an equivalent system with a nonzero equilibrium because of the complex switching associated with the T-S system. This limits the applicability of the results of (Gao et al, 2005) and (Shim & Jo, 2008).

2.2.3.1. Positive Asymptotic Stabilization of T-S systems

Let us consider the following Takagi-Sugeno continuous-time model given in (1.3).

Assume that the state variables $x(t)$ can be directly measured. We propose, in this section, the design of a state-feedback control law of the following form:

$$u(t) = \sum_{i=1}^{r} h_i(z(t))(K_i x(t) + v(t)) \ , \qquad i = 1, \dots, r \tag{2.17}$$

where $v(t)$ is a new input that can eventually represent the set point. The closed-loop system is then given by:

$$\begin{cases} \dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))(\bar{A}_{ij}x(t) + B_i v(t)) \\ y(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))(\bar{C}_{ij}x(t) + D_i v(t)) \end{cases} \tag{2.18}$$

where: $\bar{A}_{ij} = [\bar{a}_{kl}] = A_i + B_i K_j$, $\bar{C}_{ij} = [\bar{c}_{ok}] = C_i + D_i K_j$ and $K_i = [k_{jk}] \in \mathbb{R}^{n_u \times n_x}$ are the controller gains to be determined.

The following lemma presents the conditions that guarantee the positivity of the closed loop T-S system (2.18).

Lemma 2.1.

The closed-loop system (2.18) is positive if and only if $\bar{A}_{ij}$ is a Metzler matrix and $B_i \in \mathbb{R}_+^{n_x \times n_u}$, $\bar{C}_{ij} \in \mathbb{R}_+^{n_y \times n_x}$, $D_i \in \mathbb{R}_+^{n_y \times n_u}$ for $i = 1, \dots, r$.

Proof:

The proof is based on the work of positivity of T-S systems of Kaczoreck (Kaczoreck, 2010). ∎

Thus, we state the following theorem that deals with the problem of positive stabilization of the closed-loop system (2.18).

Theorem 2.6. (Zaidi et al, 2012b)

The closed-loop system (2.18) is asymptotically stable and positive if and only if there exist a positive definite diagonal matrix $Q_i = diag\{q_1, \dots, q_n\}$ and a matrix $\overline{K}_i = [\overline{k}_{jk}] \in \mathbb{R}^{n_u \times n_x}$ satisfying:

$$
\begin{cases}
\quad QA_i^T + A_i Q + Q\overline{K}_i^T B_i^T + B_i \overline{K}_i Q < 0; \ 1 \le i \le r & (2.20.1) \\[2mm]
Q\left(\dfrac{A_i + A_j}{2}\right)^T + \left(\dfrac{A_i + A_j}{2}\right)^T Q + \dfrac{B_i\overline{K}_j + \overline{K}_j^T B_i}{2} + \dfrac{B_j\overline{K}_i + \overline{K}_i^T B_j}{2} \le 0; \ 1 \le i < j \le r & (2.20.2) \\[2mm]
\quad a_{ij}q_{ij} + \displaystyle\sum_{z=1}^{l} b_{jz}\,\overline{k}_{zk} \ge 0 \qquad , 1 \le j \ne k \le n & (2.20.3) \\[2mm]
\quad c_{ij}q_{ij} + \displaystyle\sum_{z=1}^{l} d_{jz}\,\overline{k}_{zk} \ge 0 \qquad , 1 \le j, k \le n & (2.20.4) \\[2mm]
\quad \forall(i,j), h_i\big(z(t)\big)h_j\big(z(t)\big) \ne 0, \forall t &
\end{cases}
$$

Under the above conditions, the matrix gains of a desired controller (2.17) are:

$$K_i = \overline{K}_i\, Q^{-1}, \qquad i = 1, \dots, r \tag{2.21}$$

Proof:

For the sufficiency of the condition,

First, from (2.21), we have $k_{zk} = \overline{k}_{zk}q_k^{-1}, z = 1, \dots, n_y, \ k = 1, \dots, n_x$. By noticing $q_k > 0$, (2.20.3) and (2.20.4) trivially ensure that $\overline{A}_i \in \mathbb{R}_+^{n_x \times n_x}$ is a Metzler matrix and $\bar{C}_i \in \mathbb{R}_+^{n_y \times n_x}$. Then, by the positivity of $B_i \in \mathbb{R}_+^{n_x \times n_u}$ and $D_i \in \mathbb{R}_+^{n_y \times n_u}$, from Lemma 2.1, we conclude that the closed-loop system (2.18) is positive.

Second, from (2.21) , we have $\overline{K}_i = K_i\, Q_i$, $i = 1, 2, \dots, r$ $\tag{2.22}$

By substituting (2.22) into (2.20.1) and (2.20.2), we obtain

$$QA_i^{\ T} + Q\, K_i^{\ T} B_i^{\ T} + A_i Q + B_i K_i\, Q \ < 0 \tag{2.23}$$

$$Q\left(\frac{A_i + A_j}{2}\right)^T + \left(\frac{A_i + A_j}{2}\right)^T Q + Q\left(\frac{B_i K_j + B_j K_i}{2}\right)^T + \left(\frac{B_i K_j + B_j K_i}{2}\right)Q \le 0 \tag{2.24}$$

By applying the congruence transformation defined by $Q_i^{\ -1}$ to (2.22) and (2.23), one gets:

$$A_i^T Q^{-1} + K_i^T B_i^T Q^{-1} + Q^{-1}A_i + Q^{-1}QB_iK_i \ < 0, \qquad \forall\ 1 \le i \le r \qquad (2.25)$$

$$\left(\frac{A_i + A_j}{2}\right)^T Q^{-1} + Q^{-1}\left(\frac{A_i + A_j}{2}\right)^T + \left(\frac{B_iK_j + B_jK_i}{2}\right)^T Q^{-1} + Q^{-1}\left(\frac{B_iK_j + B_jK_i}{2}\right) \le 0$$

$$\forall\ 1 \le i < j \le r \qquad (2.26)$$

By defining $P_i = Q_i^{-1}$, we obtain the stability condition Then, from (2.13), we deduce that the closed-loop system (2.18) is asymptotically stable.

To prove the necessity of the condition, we suppose that there exists a controller (1.17) such that the closed-loop system (1.18) is asymptotically stable and positive.

Then, from Lemma 2.1 and condition (2.13), we impose that $\bar{A}_i$ is a Metzler matrix, $\bar{C}_i \in \mathbb{R}_+^{n_y * n_x}$, and that there exists a positive definite diagonal matrix $P = diag\{p_1, p_2, \ldots, p_n\} \in \mathbb{R}^{n_x * n_x}$ satisfying (2.25) and (2.26).

First, by applying the congruence transformation defined by P^{-1} to (2.25) and (2.26) and keeping in mind (2.18), one obtains

$$P^{-1}A_i^T + P^{-1}K_i^T B_i^T + A_iP^{-1} + B_iK_i\ P^{-1} < 0, \forall\ 1 \le i \le r \qquad (2.27)$$

$$P^{-1}\left(\frac{A_i + A_j}{2}\right)^T + \left(\frac{A_i + A_j}{2}\right)^T P^{-1} + P^{-1}\left(\frac{B_iK_j + B_jK_i}{2}\right)^T + \left(\frac{B_iK_j + B_jK_i}{2}\right)P^{-1} \le 0$$

$$\forall\ 1 \le i < j \le r \qquad (2.28)$$

By defining: $Q = P^{-1}$ and $\bar{K}_i = K_iQ$, we get (2.20.1) and (2.20.2).

Secondly, if $\bar{A}_i$ is a Metzler matrix and $\bar{C}_i \in \mathbb{R}_+^{n_y \times n_x}$. This implies that:

$$a_{jk} + \sum_{z=1}^{l} b_{jz}\bar{k}_{zk} \ge 0\ , 1 \le j \ne k \le n \qquad (2.29)$$

$$c_{jk} + \sum_{z=1}^{l} b_{jz}\bar{k}_{zk} \ge 0\ , 1 \le j, k \le n \qquad (2.30)$$

These sets of conditions are trivially equivalent to **(2.20.3)** and **(2.20.4)**, respectively, which ends the proof. ∎

Remark 2.5.

Theorem 2.6 presents a necessary and sufficient condition for the existence of the desired controllers. Conditions (2.20) are all LMIs, that is, they are convex in the matrix variables Q_i and $\overline{K}_i$; therefore, these conditions can be readily checked by using standard numerical software.

In the next section, we provide some examples to illustrate the developed theories.

2.2.3.2. Illustrative Examples

Example 2.1.

Consider the following continuous-time Takagi-Sugeno system (1.3) composed of two subsystems ($r = 2$), where:

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 1 & 2 & -4 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The initial condition is given by: $x_0 = [7 \ 10 \ 9.5]^T$.

We could easily show that the above system is stable but not positive. Our purpose is to design a state-feedback controller of the form (2.17) such that the closed-loop system is asymptotically stable and positive. By applying Theorem 2.6, we obtain the following matrix variables:

$$Q_1 = \begin{bmatrix} 113.8534 & 0 & 0 \\ 0 & 76.9663 & 0 \\ 0 & 0 & 245.0669 \end{bmatrix}, Q_2 = \begin{bmatrix} 9.2870 & 0 & 0 \\ 0 & 20.9911 & 0 \\ 0 & 0 & 38.3648 \end{bmatrix}$$

$$\overline{K}_1 = [-9.0733 \ 0.2518 \ 77.0801 \,], \overline{K}_2 = [-9.0733 \ 0.2518 \ 77.0801 \,]$$

Then, according to (2.41), the feedback gain matrices K_1 and K_2 of the controller (2.17) are given by:

$$K_1 = [-0.0797 \ 0.0033 \ 0.3145], K_2 = [-0.9770 \ 0.0120 \ 2.0091]$$

From the previous results, the matrices for the closed-loop system (2.18) are given by:

$$\bar{A}_1 = \begin{bmatrix} -2..0797 & 1.0033 & 1.3145 \\ 0.9602 & -2.9984 & 0.1573 \\ 1.9801 & 1.0008 & -1.9214 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} -3.9770 & 2.0120 & 0.0091 \\ 4.5115 & -2.9940 & 1.0046 \\ 0.7558 & 2.0030 & -3.4977 \end{bmatrix}$$

$\bar{B}_i = B_i$, $\forall i = 1,2$.

$$\bar{C}_1 = \begin{bmatrix} 0.9203 & 1.0033 & 1.3145 \\ 0.9203 & 0.0033 & 1.3145 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} 0.0230 & 1.0120 & 3.0091 \\ 0.0230 & 0.0120 & 3.0091 \end{bmatrix}$$

$\bar{D}_i = D_i$, $\forall i = 1, 2$.

It can be seen that for $i = 1,2$, $\bar{A}_i$ is a Metzler matrix and $\bar{B}_i \in \mathbb{R}_+^{n_x \times n_u}$, $\bar{C}_i \in \mathbb{R}_+^{n_y \times n_x}$ and $\bar{D}_i \in \mathbb{R}_+^{n_y \times n_u}$. In addition, $\bar{A}_i$ is a stable matrix. So, we can conclude that the closed-loop system is effectively stable and positive.

From Figure 2.1, which plots the evolution of the system states, we can easily see that the T-S system is stable and its states remain nonnegative.
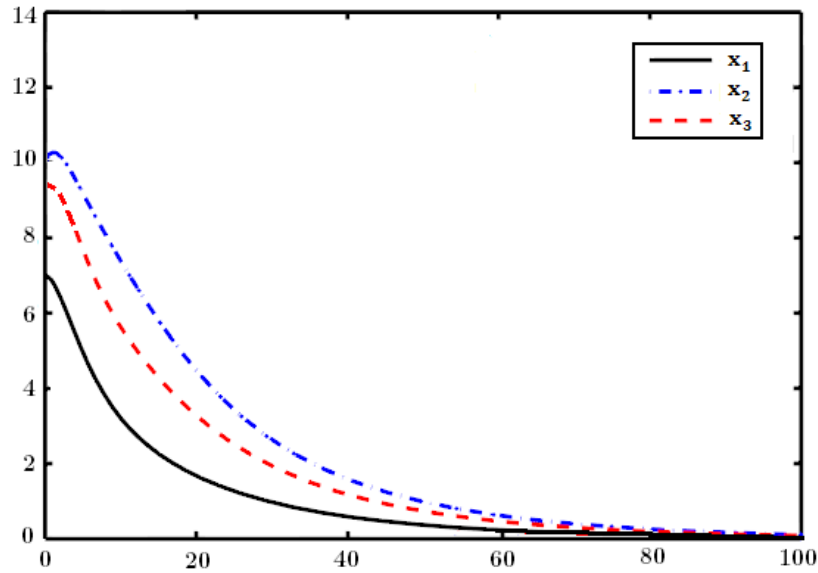
Figure 2. 1. The evolution of the states x_1, x_2 and x_3 for the closed-loop system in Example 2.1

Example 2.2.

Then, we consider a numerical example a T-S system (1.3) which has two nonlinearities, where:

$$A_1 = \begin{bmatrix} 0.5 & -0.3 \\ 0.5 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & 0.7 \\ -1 & -0.3 \end{bmatrix}, A_3 = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & -1 \end{bmatrix}, A_4 = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.9 \end{bmatrix},$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$C_1 = C_2 = [1 \quad 0.5], D_1 = D_2 = [1 \quad 1].$

The initial condition is given by: $x_0 = [2.3 \quad 3]^T$.

We could easily show that the above system is neither stable nor positive. Our purpose is to design a state-feedback controller (2.17) such that the closed-loop system is asymptotically stable and positive. By applying Theorem 2.6, we obtain the following matrix variables:

$$Q_1 = \begin{bmatrix} 6733.5 & 0 \\ 0 & 9647.0 \end{bmatrix}, Q_2 = \begin{bmatrix} 4865.7 & 0 \\ 0 & 3045.4 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 5928.7 & 0 \\ 0 & 2899.3 \end{bmatrix}, Q_4 = \begin{bmatrix} 2875.9 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{K}_1 = \begin{bmatrix} -10385 & 8947 \\ 7509 & -8947 \end{bmatrix}, \bar{K}_2 = \begin{bmatrix} -9433.3 & 7995.4 \\ 6557.4 & -7995.4 \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} -9341.6 & 7903.6 \\ 6465.6 & -7903.6 \end{bmatrix}, \bar{K}_4 = \begin{bmatrix} -9698.1 & 8260.2 \\ 6822.2 & -8260.2 \end{bmatrix}$$

Then, according to (2.21), the feedback gain matrices K_i, $\forall i = 1, \dots, 4$ of the controller (2.17) are:

$$K_1 = \begin{bmatrix} -1.5423 & 0.9274 \\ 1.1152 & -0.9274 \end{bmatrix}, K_2 = \begin{bmatrix} -1.9388 & 2.6254 \\ 1.3477 & -2.6254 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -1.5757 & 2.7261 \\ 1.0906 & -2.7261 \end{bmatrix}, K_4 = \begin{bmatrix} 0 & 4.8442 \\ 0.1031 & -4.8442 \end{bmatrix}$$

From the previous results, the matrices for the closed-loop (2.18) are given by:

$$\bar{A}_1 = \begin{bmatrix} -1.0423 & 0.6274 \\ 1.6152 & -1.8274 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} -1.0388 & 3.3254 \\ 0.3477 & -2.9254 \end{bmatrix}$$

$$\bar{A}_3 = \begin{bmatrix} -0.9757 & 3.9261 \\ 0.2906 & -3.7261 \end{bmatrix}, \bar{A}_4 = \begin{bmatrix} 0 & 4.8442 \\ 0 & -4.8442 \end{bmatrix}$$

$\bar{B}_i = B_i$, $\forall i = 1, \dots, 4$.

$\bar{C}_1 = [0.5729 \quad 0.5000], \bar{C}_2 = [0.4089 \quad 0.5000]$

$\bar{C}_3 = [0.5149 \quad 0.5000], \bar{C}_4 = [0 \quad 0.5000]$

$\bar{D}_i = D_i$, $\forall i = 1, \dots, 4$.

It can be seen that $\forall i = 1, \dots, 4$, $\bar{A}_i$ is a Metzler matrix, $\bar{B}_i \in \mathbb{R}_+^{n_x \times n_u}$ and $\bar{C}_i \in \mathbb{R}_+^{n_y \times n_x}$. In addition, $\bar{A}_i$ is a stable matrix.

We have then guaranteed both stability and positivity of the system, that we can see from Figure 2.2.
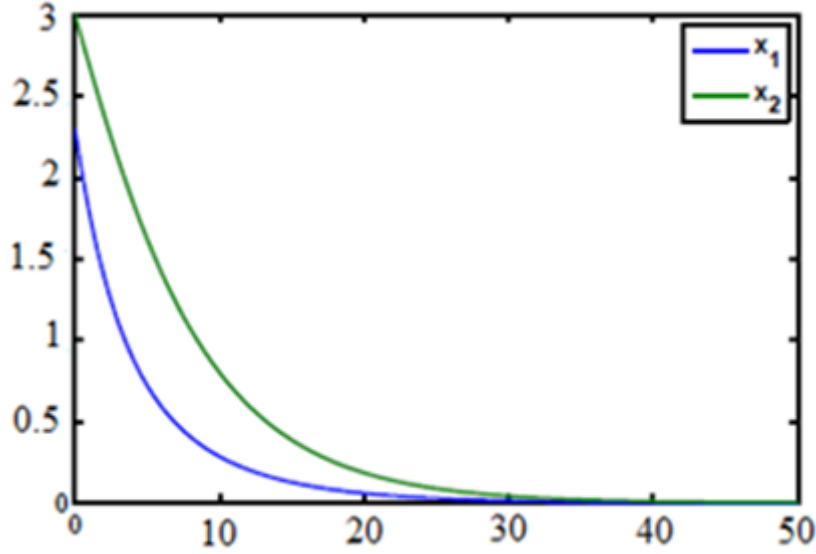
Figure 2. 2. The evolution of the states x_1 and x_2 for the closed-loop system in Example 2.2

2.2.4. STABILIZATION BY MEMORY STATE-FEEDBACK

In this section, we are interested in the design of design of a memory T-S state-feedback controller to guarantee the stability and positivity of T-S systems.

The control law applied to system (1.3) is given by:

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)K_i x(t) + \sum_{j=1}^{r} h_j\big(z(t)\big)K_j x\big(t - \tau_j\big) \tag{2.31}$$

where τ_j, $j = 1, \dots, r$ are supposed constant.

The closed-loop T-S system is then given as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i,j,k=1}^{r} h_i(z(t))h_j(z(t))\left(\big(A_i + B_i K_j\big)x(t) + B_i K_k x(t - \tau_k)\right) \\ y(t) = \sum_{i,j,k=1}^{r} h_i(z(t))h_j(z(t))\left(\big(C_i + D_i K_j\big)x(t) + D_i K_k x(t - \tau_k)\right) \end{cases} \tag{2.32}$$

In order to reduce the conservatism of the stability conditions of T-S systems, we partition the controller gains: For any matrices K_j and K_k, $j, k = 1, \dots, r$, it is

obvious that there exist nonnegative matrices K_j^-, K_j^+, K_k^- and K_k^+, $i, j = 1, \dots, r$ such that:

$$K_j = K_j^+ - K_j^-, j = 1, \dots, r \tag{2.33}$$

$$K_k = K_k^+ - K_k^-, k = 1, \dots, r \tag{2.34}$$

Using this fact, the corresponding closed-loop T-S system becomes:

$$
\begin{cases}
\dot{x}(t) = \displaystyle\sum_{i,j=1}^{r} h_i\big(z(t)\big) h_j\big(z(t)\big)\big(A_i + B_i K_j^+ - B_i K_j^-\big)x(t) \\
\quad + \displaystyle\sum_{i,k=1}^{r} h_i\big(z(t)\big) h_k\big(z(t)\big)(B_i K_k^+ - B_i K_k^-)x(t - \tau_k) \\
y(t) = \displaystyle\sum_{i,j=1}^{r} h_i\big(z(t)\big) h_j\big(z(t)\big)\big(C_i + B_i K_j^+ - D_i K_j^-\big)x(t) \\
\quad + \displaystyle\sum_{i,k=1}^{r} h_i(z(t)) h_k(z(t))(D_i K_k^+ - D_i K_k^-)x(t - \tau_k)
\end{cases}
$$

$$\tag{2.35}$$

This expression will be used to develop the conditions of stabilization: Design of a state-feedback controller (2.31) such that the closed-loop system (2.35) is positive and globally asymptotically stable.

Next, we will provide necessary and sufficient conditions on the gain matrices K_i^+, K_i^-, K_j^+ and $K_j^- \in \mathbb{R}_+^{n_x \times n_u}$, $i, j = 1, \dots, r$ which satisfy:

- Positivity of the closed-loop system:

$$
\begin{cases}
A_i + B_i K_j^+ - B_i K_j^- \text{ are Metzler}, \forall i, j = 1, \dots, r \\
B_i K_k^+ - B_i K_k^- \succcurlyeq 0, \forall i, k = 1, \dots, r \\
C_i + B_i K_j^+ - D_i K_j^- \succcurlyeq 0, \forall i, j = 1, \dots, r \\
D_i K_k^+ - D_i K_k^- \succcurlyeq 0, \forall i, k = 1, \dots, r
\end{cases}
\tag{2.36}
$$

- Stability of the closed-loop system:

$$\big(A_i + B_i K_j^+ - B_i K_j^-\big)x(t) + (B_i K_k^+ - B_i K_k^-)x(t - \tau_k) \text{ is stable} \tag{2.37}$$

In the following, we develop sufficient LMI conditions for positivity and asymptotic stability of the closed-loop T-S system.

Theorem 2.7. (Zaidi et al.)

The closed-loop system (2.35) is positive and asymptotically stable with the control law (2.31) if the following LMI problem in the variables $\beta_i \in \mathbb{R}_+$, $d \in \mathbb{R}^n$, Y_i^+, Y_i^-, Y_j^+, $Y_j^- \in \mathbb{R}_+^{n_x \times n_u}$ is feasible:

$$\begin{cases} A_i diag(d) + diag(d) A_i^T + B_i Y_{ij}^+ - B_i Y_{ij}^{-1} + \left(Y_{ij}^+\right)^T B_i^T - \left(Y_{ij}^-\right)^T B_i^T < 0 \\ A_i diag(d) + B_i Y_i^+ - B_i Y_i^- + \beta_i I \geqslant 0 \\ B_i Y_j^+ - B_i Y_j^- \geqslant 0 \\ d_i > 0 \end{cases} \tag{2.38}$$

where $Y_{ij}^+ = Y_i^+ + Y_j^+$ and $Y_{ij}^- = Y_i^- + Y_j^-$.

Moreover, the gain matrices K_i^+, K_i^-, K_j^+ and K_j^-, $i, j = 1, \dots, r$ can be computed as follows:

$$\begin{cases} K_i^+ = Y_i^+ diag(d)^{-1}, K_i^- = Y_i^- diag(d)^{-1} \\ K_j^+ = Y_i^+ diag(d)^{-1}, K_j^- = Y_i^- diag(d)^{-1} \end{cases} \tag{2.39}$$

2.2.5. ROBUST INTERVAL α-STABILIZATION OF POSITIVE T-S SYSTEMS WITH MEMORY STATE-FEEDBACK

In this section, we treat the problem of α-stabilization of interval T-S systems with memory state-feedback.

We consider the interval system (1.3), where the matrices A_i, B_i, C_i and D_i are uncertain with known bounds $\underline{A}_i$, $\overline{A}_i$, $\underline{B}_i$, $\overline{B}_i$, $\underline{C}_i$, $\overline{C}_i$, $\underline{D}_i$ and $\overline{D}_i$, $i = 1, \dots, r$, such that: $\underline{A}_i \leqslant A_i \leqslant \overline{A}_i$, $\underline{B}_i \leqslant B_i \leqslant \overline{B}_i$, $\underline{C}_i \leqslant C_i \leqslant \overline{C}_i$ and $\underline{D}_i \leqslant D_i \leqslant \overline{D}_i$. $\tau_1, \dots, \tau_r$ are supposed to be constant.

We consider the decomposed memory state-feedback controller (2.31), where: $\tau_1, \dots, \tau_r$ are constant.

We may use some results regarding the positivity and α-stability of system (1.3), which are derived from the monotonicity property, with respect to the boundedness of the dynamical matrices of system (2.35).

Lemma 2.3. (Bolajraf, 2012)

Consider the two dynamics system:

$$\begin{cases} \dot{x}_1(t) = \sum_{i=1}^{r} h_i(z(t)) A_i^1 x_1(t) \\ \dot{x}_2(t) = \sum_{i=1}^{r} h_i(z(t)) A_i^2 x_2(t) \end{cases} \qquad (2.40)$$

The initial conditions satisfy: $x_1(0) \leqslant x_2(0)$.

Assume that A_i^1 are Metzler, A_i^2 are nonnegative matrices and $A_i^1 \leqslant A_i^2$, $i = 1, \ldots, r$. Then, we have: $0 \leqslant x_1(t) \leqslant x_2(t)$, $\forall t > 0$.

Proposition 2.1. (Bolajraf, 2012)

The interval system (1.3) with $u(t) \equiv 0$ is positive if and only if $\underline{A}_i$ are Metzler matrices and $\underline{C}_i$ are nonnegative matrices, $\forall\, i = 1, \ldots, r$.

In the following theorem, we will develop our conditions for the following stabilization problem: design of a robust controller (2.31) such that the closed-loop T-S system (2.35) is positive and α-stable.

Theorem 2.8. (Zaidi et al.)

For a specific decay rate $\alpha > 0$, the interval closed-loop T-S system (2.35) is positive and α-stable with the memory control law design (2.31) if and only if the following LMI problem in the variables $\beta \in \mathbb{R}_+$, $d \in \mathbb{R}^n$, Y_i^+, Y_i^-, Y_j^+, $Y_j^- \in \mathbb{R}_+^{n_u \times n_x}$ is feasible:

$$\begin{cases} \overline{A}_{0i} diag(d) + diag(d)\overline{A}_{0i}^T + \overline{B}_i Y_{ij}^+ - \underline{B}_i Y_{ij}^- + \left(Y_{ij}^+\right)^T \overline{B}_i^T - \left(Y_{ij}^-\right)^T \underline{B}_i^T < 0 \\ \underline{A}_i diag(d) + \underline{B}_i Y_i^+ - \overline{B}_i Y_i^- + \beta_i I \geqslant 0 \\ \underline{B}_i Y_j^+ - \overline{B}_i Y_j^- \geqslant 0 \\ d_i > 0 \end{cases} \qquad (2.41)$$

where: $\overline{A}_{0i} = \overline{A}_i + \alpha I$, $Y_{ij}^+ = Y_i^+ + Y_j^+$ *and* $Y_{ij}^- = Y_i^- + Y_j^-$.

The gain matrices K_i^+, K_i^-, K_j^+ and K_j^-, $i, j = 1, \ldots, r$ can be deduced as follows:

$$\begin{cases} K_i^+ = Y_i^+ diag(d)^{-1}, K_i^- = Y_i^- diag(d)^{-1} \\ K_j^+ = Y_j^+ diag(d)^{-1}, K_j^- = Y_j^- diag(d)^{-1} \end{cases} \qquad (2.42)$$

2.2.6. POSITIVE STABILIZATION BY OUTPUT-FEEDBACK CONTROLLER

The state-feedback control requires a complete availability of the state variables. Unfortunately, this is not always possible. We then consider problems where the control depends on the measured variables of the system. In Tanaka et al. (Li et al, 2000), a nonlinear control law, called DPDC is proposed. In (Han et al, 2000), some techniques of uncertain systems and piecewise quadratic functions have been proposed to establish linear correctors. We can also cite (Garcia et al, 2001), (Geromel et al, 1996), (Benton & Smith, 1997), (Park et al, 2001), (Chadli et al, 2002a), (Chadli et al, 2002b), (Chadli et al, 2002c) and (Klug et al, 2015). In this part, we deal with the problem of output feedback positive stabilization. We firstly present how to synthesize the OPDC control law using Lyapunov quadratic functions, taking into account the positivity of the system. This is done by the transformation of the initial BMI problem into an LMI one, and by the formulation of the complementarity on the cone.

2.**2.6.1. Output-feedback Stabilization for Positive Takagi-Sugeno systems**

We consider the OPDC control law defined by:

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i y(t) \tag{2.43}$$

Applying the output-feedback control law (2.43) in the continuous-time T-S system (2.1), the state equation of the closed-loop system becomes:

$$\dot{x}(t) = \sum_{i,j,k}^{r} h_i(z(t)) h_j(z(t)) h_k(z(t)) \tilde{A}_{ijk} x(t) \tag{2.44}$$

with

$$\tilde{A}_{ijk} = A_i + B_i F_j C_k \tag{2.45}$$

For reasons of simplicity, we suppose that $C_i = C$, $i = 1, \dots, r$. We then get:

$$\dot{x}(t) = \sum_{i,j}^{r} h_i(z(t)) h_j(z(t)) \bar{\bar{G}}_{ij} x(t) \tag{2.46}$$

with

$$\bar{\bar{G}}_{ij} = A_i + B_i F_j C \tag{2.47}$$

Stability conditions are now deduced:

Theorem 2.9. (Zaidi et al, 2015)

For a given $\beta \in \mathbb{R}_+$, the system described by (2.46) is asymptotically stable and positive, if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices $Q_{ii} \in \mathbb{R}^{n_x \times n_x}$, $\forall\, i = 1, ..., r$ and matrices $Q_{ji} = Q_{ij}^T \in \mathbb{R}^{n_x \times n_x}$, $1 \le i < j \le r$ such that:

$$\begin{cases} \bar{\bar{G}}_{ii}^T P + P\bar{\bar{G}}_{ii} + Q_{ii} < 0; 1 \le i \le r \\ \left(\bar{\bar{G}}_{ij} + \bar{\bar{G}}_{ji}\right)^T P + P\left(\bar{\bar{G}}_{ij} + \bar{\bar{G}}_{ji}\right) + \left(Q_{ij} + Q_{ij}^T\right) \le 0; 1 \le i < j \le r \\ \bar{\bar{G}}_{ij} + \beta I \succcurlyeq 0 \end{cases} \quad (2.48)$$

where $\bar{\bar{G}}_{ii}$ is defined in (2.47).

Unfortunately, the two first matrix inequalities are not convex in P and F_i, $\forall\, i = 1, ..., r$. For this reason, it seems impossible to obtain equivalent convex functions through variables changes: this is the major difficulty in positive output-feedback stabilization. We then propose a new LMI formulation.

Assumption 2.1. The matrix C is of full rank.

Once this hypothesis is verified, we propose the convex formulation in Theorem 2.10. We note that this method may bring some additional algebraic constraints: it is not such difficult to resolve them.

Theorem 2.10. (Zaidi et al.)

For a given $\beta \in \mathbb{R}_+$, if there exist symmetric positive definite matrix X, symmetric matrices S_{ii}, $\forall\, i = 1, ..., r$ and matrices $S_{ji} = S_{ij}^T$, $1 \le i < j \le r$ such that:

$$\begin{cases} A_i X + X A_i^T + B_i N_i C + C^T N_i^T B_i^T + S_{ii} < 0; 1 \le i \le r \\ (A_i + A_j)X + X(A_i + A_j)^T + (B_i N_j + B_j N_i)C + C^T(B_i N_j + B_j N_i)^T + (S_{ij} + S_{ji}^T) \le 0; \\ 1 \le i < j \le r \\ A_i X + B_i N_j C + \beta X \succcurlyeq 0 \end{cases}$$

$$(2.49)$$

with

$$S_{ij} = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1r} \\ S_{12}^T & S_{22} & \cdots & S_{2r} \\ \vdots & \cdot & \ddots & \vdots \\ S_{1r}^T & S_{2r}^T & \cdots & S_{rr} \end{pmatrix}$$

Then, the Takagi-Sugeno system (2.44) is positive and asymptotically stable. The output-feedback gains are defined by:

$$F_i = N_i M^{-1}, \qquad i = 1, \dots, r \tag{2.50}$$

such as $CX = MC$.

Proof:

Taking into account the definition of $\bar{\bar{G}}_{ij}$, the first condition of (2.48) is equivalent to:

$X(A_i + B_i F_i)^T + (A_i + B_i F_i)X + S_{ii} < 0$ with $P^{-1} = X > 0$ and $S_{ii} = P^{-1} Q_{ii} P^{-1}$.

Taking into account that $CX = MC$ and the variable change $F_i M = N_i$, we get the first inequality of (2.49).

The second inequality is obtained from the second one in (2.48), using the same change of variables.

As the matrix C is supposed to be of full rank, we can deduce from $CX = MC$ and $F_i M = N_i$ that $M = CXC^T(CC^T)^{-1}$ and $F_i = N_i M^{-1}$.

To ensure the positivity of system (2.46), we have to prove that $\bar{\bar{G}}_{ij}$ is a Metzler matrix. Making the necessary changes of variables on the inequality $\bar{\bar{G}}_{ij} + \beta I \geqslant 0$ leads to the last inequality of (2.49). ∎

Remark 2.6.

The stability conditions of this theorem do not require a common matrix S. We guarantee then the reduction of conservatism with respect to conditions (2.49).

Remark 2.1.

If we suppose linear output-feedback $F_i = F$, $\forall i = 1, \dots, r$, then, the stability conditions reduce to the existence of matrices X, N and M such that:
$X > 0, A_i X + X A_i^T + B_i N C + C^T N^T B_i^T < 0 \tag{2.51}$

with $CX = MC$ and the stabilizing feedback gain is: $F = NM^{-1}$.

2.2.6.2. Illustrative Example

Consider the following continuous-time T-S system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} h_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{2.52}$$

where

$$A_1 = \begin{bmatrix} 3 & 6 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & 10 \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$C_1 = C_2 = [1 \quad 0],$$

$$x_0 = [1 \ 1.5]^T$$

For $\beta = 2$, the resolution of the LMIs of Theorem 2.10 generates the following output-feedback control gains:

$$\begin{aligned} F_1 &= -4.1520, \\ F_2 &= -2.6313 \end{aligned} \tag{2.53}$$

The OPDC control law given by (2.43) has then the following expression:

$$u(t) = \big((h_1(y(t)))F_1 + (h_2(y(t)))F_2\big) y(t) \tag{2.54}$$

For simulation, the activation functions are:

$$h_1(y(t)) = \frac{1 - \tanh(y(t))}{2} \tag{2.55}$$

$$h_2(y(t)) = \frac{1 + \tanh(y(t))}{2} \tag{2.56}$$

Figure 2.3 presents some simulations from the initial condition $x(0) = x_0 = [1 \ 1.5]^T$.
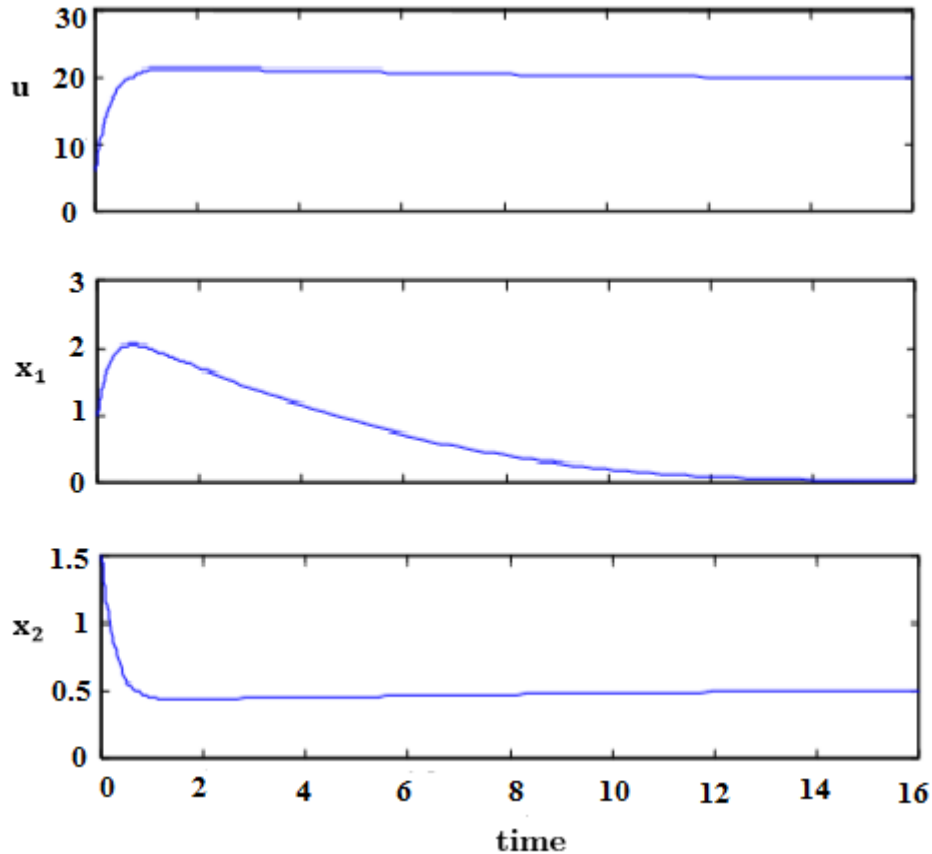
Figure 2. 3. Stabilization of the system (2.52) via OPDC control (2.43)

We can see that the system (2.52) is asymptotically stable and remains positive, which proves that this theorem has succeeded to synthesize an OPDC control law that guarantees the positive stabilization of the system.

2.3. Stability and Stabilization of Positive Takagi-Sugeno Systems with unmeasurable premise variables

Many theoretical results consider that the Takagi-Sugeno submodels depend on measurable decision variables, while in several applications, the membership functions depend on the state variables of the system, which are often unmeasurable (Yoneyama, 2007), (Yoneyama, 2008), (Kruszewski, 2006), (Ichalal et al, 2008a), (Ichalal et al, 2008b), (Guerra et al, 2006), (Yoneyama, 2000), (Ichalal et al, 2008c). Thus, we now consider the case where these functions depend on unmeasurable variables.

To design a state observer for systems with unmeasurable decision variables, many approaches have been developed, including those which consider analytic

development of an estimation error (Ichalal et al, 2007), and those using a description of the error by a Takagi-Sugeno model with uncertainty or unstructured disturbance (Akhenak, 2004).

In this section, we will establish new stability and stabilization conditions for positive T-S systems with unmeasurable premise variables in the continuous-time case. First, we start with the representation of T-S systems with unmeasurable decision variables and their observers. Then, we develop two approaches for the stabilization of these systems; the first one considers the description of the state estimation error with disturbance. In the second approach, we assume that the inputs and states of the system are bounded. Some application examples illustrate the results.

2.3.1. STABILITY AND STABILIZATION BY MEMORYLESS STATE-FEEDBACK CONTROLLER

We consider the following class of positive T-S systems (1.3), where $D_i = 0$, $i = 1, \dots, r$.

Assuming that the decision variables are not measurable, the state observer for this system is given by (Tanaka et al, 1998):

$$
\begin{cases}
\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\Big(A_i\hat{x}(t) + B_i u(t) + L_i\big(y(t) - \hat{y}(t)\big)\Big) \\
\hat{y}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big) C_i\hat{x}(t)
\end{cases}
\tag{2.57}
$$

where $\hat{x}(t)$ and $\hat{y}(t)$ denote the estimated state and output vectors and $\hat{z}(t)$ is the set of the estimated decision variables.

The normalized activation function $h_i\big(z(t)\big)$ corresponding to the ith observer of the ith submodel and verifies (1.4).

We define the state estimation error as follows:

$$
e(t) = x(t) - \hat{x}(t)
\tag{2.58}
$$

The principal role of the observer is to satisfy a rapid convergence of the state estimation error to 0.
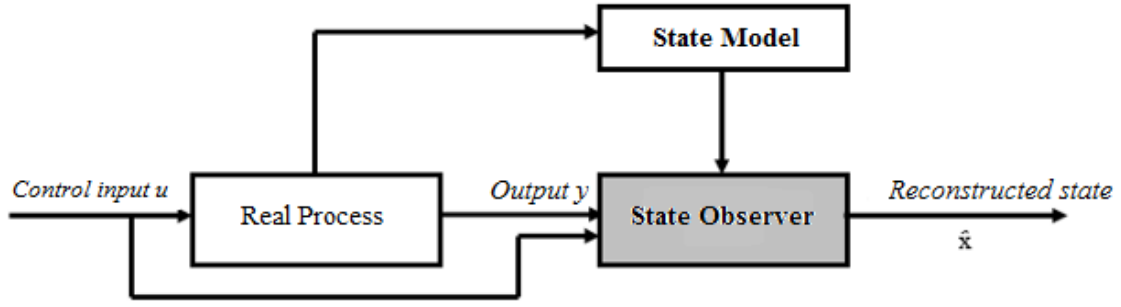
Figure 2. 4. Principle of the state observer

In the remainder of this study, we assume, without loss of generality, that the vector of decision variables is identical to the state vector: $z(t) = x(t)$. For simplicity, t is omitted.

2.3.1.1. Approach using the description of the state estimation error with disturbance

This approach is based on the results in (Ichalal et al, 2008a), (Ichalal et al, 2008b), (Akhenak, 2004), (Bergsten & Palm, 2000) where the estimation error (2.58) is considered with bounded uncertainty. Firstly, we present some results that will be used later to simplify the formulation of the T-S model in order to obtain better results: (Zaidi et al, 2013c)

Lemma 2.4. (Zaidi et al, 2013c)

Let $\{X_i\}$ be a set of matrices and $h_i(x(t))$ fuzzy weighting functions that satisfy (1.4).Then,

$$\sum_{i=1}^{r}\big(h_i(x) - h_i(\hat{x})\big)X_i = \sum_{i,j=1}^{r} h_i(x)h_j(\hat{x})(X_i - X_j) \qquad (2.59)$$

Proof:

$$\sum_{i,j=1}^{r} h_i(x)h_j(\hat{x})\big(X_i - X_j\big) = \sum_{i,j=1}^{r} h_i(x)h_j(\hat{x})X_i - \sum_{i,j=1}^{r} h_j(x)h_i(\hat{x})X_j$$

$$= \sum_{i,j=1}^{r} h_i(x)(1 - h_j(\hat{x}))\, X_i - \sum_{i,j=1}^{r} h_j(\hat{x})\,(1 - h_i(x))\, X_j$$

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$$= \sum_{i=1}^{r} h_i(x) \, X_i - \sum_{j=1}^{r} h_j(\hat{x}) \, X_j - \sum_{i=1}^{r} h_i(x) h_i(\hat{x}) X_i + \sum_{j=1}^{r} h_j(\hat{x}) h_j(x) X_j$$

$$= \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \big(X_i - X_j \big) . \blacksquare$$

Lemma 2.5.

Let $\{X_i\}$ be a set of matrices and $h_i(x(t))$ fuzzy weighting functions that satisfy (1.4). Then,

$$\sum_{i,j=1}^{r} h_i(x) h_j(x) = \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \tag{2.60}$$

Proof:

$$\sum_{i,j=1}^{r} h_i(x) h_j(x) = h_1(x) h_1(x) + h_1(x) h_2(x) + \cdots + h_r(x) h_{r-1}(x) + h_r(x) h_r(x)$$

$$= \big(h_1(x) + \cdots + h_r(x) \big) \big(h_1(x) + \cdots + h_r(x) \big)$$

$$= \big(h_1(x) + \cdots + h_r(x) \big) \big(h_1(\hat{x}) + \cdots + h_r(\hat{x}) \big)$$

$$= \sum_{i=1}^{r} h_i(x) \sum_{j=1}^{r} h_j(\hat{x}) . \blacksquare$$

In fact, by developing the derivative of the estimation error with respect to time and using the previous lemmas, we get:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) - \sum_{i=1}^{r} h_i(\hat{x}) \big(A_i \hat{x} + B_i u + L_i(y - \hat{y}) \big)$$

$$= \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) - \sum_{i=1}^{r} h_i(\hat{x})(A_i x - A_i e + B_i u)$$

$$- \sum_{i,j=1}^{r} h_i(\hat{x}) L_i C_j \big(h_j(x) x - h_j(\hat{x}) \hat{x} \big)$$

$$= \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) - \sum_{i=1}^{r} h_i(\hat{x})(A_i x - A_i e + B_i u)$$

$$- \sum_{i,j=1}^{r} h_i(\hat{x}) L_i C_j \left(\left(h_j(x) - h_j(\hat{x}) \right) x - (x - \hat{x}) h_j(\hat{x}) \right)$$

$$= \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) - \sum_{i=1}^{r} h_i(\hat{x})(A_i x - A_i e + B_i u)$$

$$- \sum_{i,j=1}^{r} h_i(\hat{x}) L_i C_j \left(h_j(x) - h_j(\hat{x}) \right) - \sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x}) L_i C_j e$$

$$= \sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x}) \left(A_i - L_i C_j \right) e + \sum_{i,j=1}^{r} h_i(\hat{x}) \left(h_i(x) - h_i(\hat{x}) \right) \left(\left(A_i - L_j C_i \right) x + B_i u \right)$$

Thus, the estimation error can be written as follows:

$$\dot{e}(t) = \sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x}) \left(A_i - L_i C_j \right) e + \delta_{ij} \tag{2.61}$$

where:

$$\delta_{ij} = \sum_{i,j=1}^{r} h_i(\hat{x}) \left(h_i(x) - h_i(\hat{x}) \right) \left(\left(A_i - L_j C_i \right) x + B_i u \right) \tag{2.62}$$

If we suppose that $C_i = C$, $i = 1, \dots, r$, (2.61) becomes :

$$\dot{e}(t) = \sum_{i=1}^{r} h_i(\hat{x})(A_i - L_i C) e + \delta_i \tag{2.63}$$

where:

$$\delta_i = \sum_{i=1}^{r} h_i(\hat{x}) \left(h_i(x) - h_i(\hat{x}) \right) \left((A_i - L_i C) x + B_i u \right) \tag{2.64}$$

An additional constraint is added to the stabilization of system (1.3): the positivity of the states as well as that of the estimation error. From the equation of the dynamic error, we can guarantee both the positivity of the estimation error and the original system, once the element δ_{ij} contains the original state equation.

This yields: $\forall \, i, j = 1, \dots, r$,

$$\begin{cases} \sum_{i,j=1}^{r} h_i(\hat{x})h_j(\hat{x})\big(A_i - L_iC_j\big) \text{ is Metzler} \\ \delta_{ij} \geqslant 0 \end{cases} \tag{2.65}$$

We can remark that if $e \to 0$, then $\delta_{ij} \to 0$.

The term δ_{ij} is considered an unstructured bounded disturbance:

$$\|\delta_{ij}\| \leq \beta \|e\|, \ \beta > 0 \tag{2.66}$$

The stabilization and positivity conditions of the estimation error (1.3) are given in the following theorem:

Theorem 2.12. (Zaidi et al, 2015)

The estimation error system (2.61) is positive and globally converges to 0, if there exist a symmetric positive definite P, symmetric matrices Q_{ii}, $\forall \, 1 \leq i < j \leq r$, matrices $Q_{ij} = Q_{ij}^T < 0$, $\forall 1 \leq i < j \leq r$, matrices W_i, $\forall \, i = 1, \dots, r$, scalars $\alpha > 0$ and $\xi \geq 0$ such that:

$$\begin{cases} \begin{pmatrix} \mathbb{A}_i & P \\ * & -\alpha I \end{pmatrix} < 0, & \forall \, i = 1, \dots, r \\ \begin{pmatrix} \mathbb{B}_{ij} & P \\ * & -\alpha I \end{pmatrix} < 0, & \forall \, 1 \leq i < j \leq r \\ PA_i - W_iC_i + \xi P \geqslant 0 \end{cases} \tag{2.67}$$

where:

$$\mathbb{A}_i = A_i^T P + PA_i - C_i^T W_i^T - W_iC_i - Q_{ii} + \sigma I$$

$$\mathbb{B}_{ij} = \left(\frac{A_i + A_j}{2}\right)^T P + P\left(\frac{A_i + A_j}{2}\right) - \frac{C_i^T W_j^T + C_j^T W_i^T}{2} - \frac{W_jC_i + W_iC_j}{2} - \frac{Q_{ij} + Q_{ji}}{2} + \alpha\beta^2 I$$

Moreover, the observer gains are given by:

$$L_i = P^{-1}W_i, \ \forall \, i = 1, \dots, r \tag{2.68}$$

Proof:

Consider the following Lyapunov quadratic function:

$$V\big(e(t)\big) = e(t)^T Pe(t) \tag{2.69}$$

with $P > 0$. The derivative of the error equation in (2.61) is given by the following equality:

$$\dot{V}(e(t)) = \sum_{i,j=1}^{r} h_i(\hat{z})h_j(\hat{z}) \left[e^T (A_i - L_i C_j)^T P + P(A_i - L_i C_j)e \right] + \delta^T P e + e^T P \delta$$

$$= \sum_{i,j=1}^{r} h_i(\hat{z})h_j(\hat{z}) \left[e^T U_{ij}^T P + P U_{ij} e \right] + \delta_{ij}^T P e + e^T P \delta_{ij} \qquad (2.70)$$

with:
$$U_{ij} = A_i - L_i C_j \qquad (2.71)$$

By applying Lemma B.2 (Annex), for a scalar $\alpha > 0$, we may write:

$$e^T P \delta + \delta^T P e \leq \alpha^{-1} e^T P^2 e + \alpha \delta^T \delta \qquad (2.72)$$

From (2.72) and (2.70), we may write:

$$\dot{V}(e) \leq \sum_{i,j=1}^{r} h_i(\hat{x})h_j(\hat{x}) \left[e^T (U_{ij}^T P + P U_{ij} + \alpha^{-1} P^2)e \right] + \alpha \delta^T \delta \qquad (2.73)$$

Using the inequality (2.66), we get:

$$\dot{V}(e) \leq \sum_{i,j=1}^{r} h_i(\hat{x})h_j(\hat{x}) \, e^T (U_{ij}^T P + P U_{ij} + \alpha^{-1} P^2 + \alpha \beta^2 I)e \qquad (2.74)$$

The derivative of that Lyapunov function is negative if:

$$U_{ij}^T P + P U_{ij} + \alpha^{-1} P^2 + \alpha \beta^2 I \leq 0 \qquad (2.75)$$

We can then write:

$$\dot{V}(e) \leq -\lambda_{min}\left(-U_{ij}^T P - P U_{ij} - \alpha^{-1} P^2\right)\|e\|^2 + \alpha \beta^2 \|e\|^2 \leq 0 \qquad (2.76)$$

such that:

$$\lambda_{min}\left(-U_{ij}^T P - P U_{ij} - \alpha^{-1} P^2\right) \geq \alpha \beta^2 \qquad (2.77)$$

Using (Liu, 2009), the stability conditions (2.67) are satisfied if there exists a symmetric matrix $P > 0$, symmetric matrices Q_{ii}, $i = 1, \ldots, r$, matrices $Q_{ji} = Q_{ji}^T, 1 \leq i < j \leq r$, observation gains L_i and scalars $\alpha > 0$ and $\beta > 0$ such that:

$$\begin{cases} U_{ii}^T P + PU_{ii} + \alpha^{-1}P^2 + \alpha\beta^2 I < Q_{ii}\,, \quad i = 1, \ldots, r \\ \left(\dfrac{U_{ij} + U_{ji}}{2}\right)^T P + P\left(\dfrac{U_{ij} + U_{ji}}{2}\right) + \alpha^{-1}P^2 + \alpha\beta^2 I \leq \dfrac{Q_{ji} + Q_{ij}^T}{2}\,, 1 \leq i < j \leq r \end{cases} \qquad (2.78)$$

Using the Schur Complement (Boyd et al, 1994), we get:

$$\begin{cases} \begin{pmatrix} U_{ij}^T P + PU_{ij} + \alpha\beta^2 I - Q_{ii} & P \\ * & -\alpha I \end{pmatrix} < 0, \qquad \forall\, i = 1, \ldots, r \\ \begin{pmatrix} \left(\dfrac{U_{ij} + U_{ji}}{2}\right)^T P + P\left(\dfrac{U_{ij} + U_{ji}}{2}\right) + \alpha\beta^2 I - \dfrac{Q_{ji} + Q_{ij}^T}{2} & P \\ * & -\alpha I \end{pmatrix} < 0, \quad 1 \leq i < j \leq r \end{cases}$$

$$(2.79)$$

Inequalities (2.79) are nonlinear in the variables P, L_i and α.

In order to have a problem with LMI, we use the changes of variables $W_i = PL_i$ in (2.79) to give two first inequalities of (2.67).

Regarding the positivity of the estimation error, guaranteeing that $\left(A_i - L_i C_j\right)$ is Metzler and $\delta_{ij} \geqslant 0$ is equivalent to:

$$\begin{cases} A_i - L_j C_i + \xi I \geqslant 0, \ \forall i,j = 1, \ldots, r \\ \left(A_i - L_j C_i\right)x + B_i u \text{ is positive}, \quad \forall i = 1, \ldots, r \end{cases} \qquad (2.81)$$

Multiplying the first inequality on the left side by P implies that:

$$\begin{cases} PA_i - W_i C_i + \xi P \geqslant 0 \\ B_i \geqslant 0 \end{cases} \qquad (2.82)$$

where ξ is a positive scalar.

This ends the proof. ∎

As mentioned in (Salem, 2013), the disturbance term considered in the estimation error depends on the input $u(t)$ and the state $x(t)$. Thus, a high value of the input can increase the bound, which may reduce the feasibility domain of the LMIs in (2.67). To overcome this difficulty, another form of state estimation error is proposed in the following section.

2.3.1.2. Approach assuming bounded inputs and states

This approach is based on (Ichalal et al, 2007) where a development of Taylor series of the activation functions $h_i(x)$ to order 0 adjacent to $\hat{x}$ is applied to calculate the Lipchitz constant applied to $h_i(x)$.

$$h_i(x) = h_i(\hat{x}) + \int_{\hat{x}}^{x} \dot{h}_i(\alpha)d\alpha.$$

$$|h_i(x) - h_i(\hat{x})| \leq \left| \int_{\hat{x}}^{x} \dot{h}_i(\lambda)d\lambda \right| \leq M_i|x - \hat{x}| \tag{2.83}$$

The activation functions are continuous and differentiable, so, it is sufficient to study the extremes of the function $\dot{h}_i(x)$ to find the Lipchitz constant

$$M_i = max_x|\dot{h}_i(x)|.$$

We consider the system (1.3) and the observer (2.57) and we develop the derivative of the estimation error (2.61) to get:

$$\dot{e} = \sum_{i=1}^{r} \left[A_i(h_i(x)x - h_i(\hat{x})\hat{x}) + B_iu(h_i(x) - h_i(\hat{x})) \right.$$

$$\left. - h_i(\hat{x})L_i \sum_{j=1}^{r} C_j\big(h_j(x) - h_j(\hat{x})\big)\hat{x} \right]$$

$$= \sum_{i=1}^{r} h_j(\hat{x})\big[(A_i - L_jC_i)(h_i(x)x - h_i(\hat{x})\hat{x}) + B_iu(h_i(x) - h_i(\hat{x}))\big]$$

$$= \sum_{i=1}^{r} h_j(\hat{x})\big[(A_i - L_jC_i)\big((h_i(x) - h_i(\hat{x}))x + (x - \hat{x})h_i(\hat{x})\big) + B_iu(h_i(x) - h_i(\hat{x}))\big]$$

$$\dot{e} = \sum_{i,j=1}^{r} h_i(\hat{x})h_j(x)U_{ij}e + \sum_{i,j=1}^{r} h_j(\hat{x})\big(U_{ij}\varrho_{ij} + B_i\Delta_i\big) \tag{2.84}$$

where :

$\varrho_i = (h_i(x) - h_i(\hat{x}))x$, $\Delta_i = (h_i(x) - h_i(\hat{x}))u$ and $U_{ij} = A_i - L_jC_i$.

If we suppose that $C_i = C$, $i = 1, \dots, r$, (2.84) becomes :

$$\dot{e} = \sum_{i=1}^{r} h_i(\hat{x})(U_{ii}e + A_i\varrho_i + B_i\Delta_i) \tag{2.85}$$

We consider the following hypothesis for the inequality (2.83):

$$
\begin{cases}
\|x\| \le \beta_1 \\
\|u\| \le \beta_2 \\
\|h_i(x) - h_i(\hat{x})\| \le M_i \|x - \hat{x}\| \\
\|\varrho_{ij}\| \le M_i \beta_1 \|e\| \\
\|\Delta_i\| \le M_i \beta_2 \|e\|
\end{cases}
\tag{2.86}
$$

where M_i is the constant of Lipchitz and β_1 and β_2 are positive scalars.

With these assumptions, the conditions for the convergence of the error described by (2.84) are given in the following theorem:

Theorem 2.13. (Zaidi et al.)

The estimation error system in (2.84) is positive and globally asymptotically converges to zero if there is a symmetric matrix $P > 0$, a symmetric matrix $Q \ge 0$, matrices F_i, $i = 1, \dots, r$ and positive scalars ξ, λ_1 and λ_2 such that:

$$
\begin{cases}
P > 0, Q \ge 0, \lambda_1 > 0, \lambda_2 > 0, \gamma_1 > 0, \gamma_2 > 0 \\
A_i^T P + P A_i - C_i^T F_i - F_i C_i < -Q, i = 1, \dots, r \\
\left(\dfrac{A_i + A_j}{2}\right)^T P + P\left(\dfrac{A_i + A_j}{2}\right) - \dfrac{C_j^T F_i + F_i C_j}{2} - \dfrac{C_i^T F_j + F_j C_i}{2} \le -Q, 1 \le i < j \le r \\
\begin{pmatrix}
-\dfrac{Q}{r} + M_i^2(\gamma_1 + \gamma_2)I & * & * \\
A_i^T P - C_i^T F_j^T & -\lambda_1 I & 0 \\
B_i^T P & 0 & -\lambda_2 I
\end{pmatrix}, i, j = 1, \dots, r \\
P A_i - F_i C_i + \xi P \geqslant 0
\end{cases}
\tag{2.87}
$$

where:

$\gamma_1 = \lambda_1 \beta_1^2$, $\gamma_2 = \lambda_2 \beta_2^2$ *and* $F_i = P L_i$, $i = 1, \dots, r$ *are the observer gains.*

Remark 2.7.

when $C_i = C$, $i = 1, \dots, r$, we get the following result:

$$\begin{cases} P > 0, Q \geq 0, \lambda_1 > 0, \lambda_2 > 0, \gamma_1 > 0, \gamma_2 > 0 \\ A_i^T P + P A_i - C F_i - F_i C < -Q, \qquad i = 1, \dots, r \\ \begin{pmatrix} -\dfrac{Q}{r} + M_i^2(\gamma_1 + \gamma_2)I & * & * \\ A_i^T P - C_i^T F_j^T & -\lambda_1 I & 0 \\ B_i^T P & 0 & -\lambda_2 I \end{pmatrix}, i, j = 1, \dots, r \\ P A_i - F_i C_i + \xi P \succcurlyeq 0 \\ B_i \succcurlyeq 0 \end{cases} \qquad (2.88)$$

Proof:

We consider the Lyapunov function (2.69) applied to the system (2.84) and we develop $\dot{V}(e)$ to obtain:

$$\dot{V}(e) = \sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x}) e^T \left(U_{ij}^T P + P U_{ij} \right) e$$

$$+ \sum_{i,j=1}^{r} h_j(\hat{x}) \left(\varrho_i^T U_{ij}^T P e + e^T P U_{ij} \varrho_i + \Delta_i^T B_i^T P e + e^T P B_i \Delta_i \right) \qquad (2.89)$$

Using Lemma B.2 (Annex) and the hypothesis (2.86), we can write:

$$\varrho_i^T U_{ij}^T P e + e^T P U_{ij} \varrho_i \leq \lambda_1 \varrho_i^T \varrho_i + \lambda_1^{-1} e^T P U_{ij} U_{ij}^T P e \leq \lambda_1 M_i^2 \beta_1^2 e^T e + \lambda_1^{-1} e^T P U_{ij} U_{ij}^T P e$$

$$(2.90)$$

$$\Delta_i^T B_i^T P e + e^T P B_i \Delta_i \leq \lambda_2 \Delta_i^T \Delta_i + \lambda_2^{-1} e^T P B_i B_i^T P e \leq \lambda_2 M_i^2 \beta_2^2 e^T e + \lambda_2^{-1} e^T P B_i B_i^T P e$$

$$(2.91)$$

with λ_1 and λ_2 positive reals.

Thus, the conditions (2.89) can be written as follows:

$$\dot{V}(e) \leq \sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x}) e^T \left(U_{ij}^T P + P U_{ij} \right) e$$

$$+ \sum_{i,j=1}^{r} h_j(\hat{x}) e^T \left[M_i^2 (\lambda_1 \beta_1^2 + \lambda_2 \beta_2^2) I + \lambda_1^{-1} P U_{ij} U_{ij}^T P + \lambda_2^{-1} P B_i B_i^T P \right] e \qquad (2.92)$$

We suppose that there exists a symmetric matrix $P > 0$ and a symmetric matrix $Q \geq 0$ such that:

$$\sum_{i,j=1}^{r} h_i(\hat{x}) h_j(\hat{x})\left(U_{ij}^T P + P U_{ij}\right) < -Q \tag{2.93}$$

Using the results of (Tanaka et al, 1998), condition (2.93) is fulfilled, if there exist matrices P and Q such that:

$$\begin{cases} U_{ii}^T P + P U_{ii} < -Q, & i = 1, \dots, r \\ \left(U_{ij} + U_{ji}\right)^T P + P\left(U_{ij} + U_{ji}\right) \le -2Q, & 1 \le i < j \le r \end{cases} \tag{2.94}$$

Substituting (2.94) in (2.92), we get:

$$\dot{V}(e) \le \sum_{i,j=1}^{r} h_j(\hat{x})\, e^T \left[-\frac{Q}{r} + M_i^2(\lambda_1\beta_1^2 + \lambda_2\beta_2^2)I + \lambda_1^{-1} P U_{ij} U_{ij}^T P + \lambda_2^{-1} P B_i B_i^T P \right] e$$

$$\tag{2.95}$$

To guarantee $\dot{V}(e) < 0$, it suffices that (2.94) and the following condition hold:

$$-\frac{Q}{r} + M_i^2(\lambda_1\beta_1^2 + \lambda_2\beta_2^2)I + \lambda_1^{-1} P U_{ij} U_{ij}^T P + \lambda_2^{-1} P B_i B_i^T P < 0 \tag{2.96}$$

is verified.

Using (2.94) and the Schur complement (2.96), we get the following conditions:

$$\begin{cases} P > 0, Q \ge 0, \lambda_1 > 0, \lambda_2 > 0, \gamma_1 > 0, \gamma_2 > 0 \\ U_{ii}^T P + P U_{ii} < -Q;\ i = 1, \dots, r \\ \left(U_{ij} + U_{ji}\right)^T P + P\left(U_{ij} + U_{ji}\right) \le -2Q; 1 \le i < j \le r \\ \begin{pmatrix} -\dfrac{Q}{r} + M_i^2(\lambda_1\beta_1^2 + \lambda_2\beta_2^2)I & * & * \\ U_{ii}^T P & -\lambda_1 P & 0 \\ B_i^T P & 0 & -\lambda_2 P \end{pmatrix} < 0;\ i,j = 1, \dots, r \end{cases} \tag{2.97}$$

Replacing U_{ij} by $A_i - L_j C_i$, conditions (2.97) become:

$$\begin{cases} P > 0, Q \ge 0, \lambda_1 > 0, \lambda_2 > 0, \gamma_1 > 0, \gamma_2 > 0 \\ A_i^T P + P A_i - C_i^T L_i^T P - P L_i C_i < -Q;\ i = 1, \dots, r \\ \left(A_i + A_j\right)^T P + P\left(A_i + A_j\right) - C_j^T L_i^T P - P L_i C_j - C_i^T L_j^T P - P L_j C_i \le -2Q; \\ \quad 1 \le i < j \le r \\ \begin{pmatrix} -\dfrac{Q}{r} + M_i^2(\lambda_1\beta_1^2 + \lambda_2\beta_2^2)I & * & * \\ U_{ii}^T P & -\lambda_1 P & 0 \\ B_i^T P & 0 & -\lambda_2 P \end{pmatrix} < 0;\ i,j = 1, \dots, r \end{cases} \tag{2.98}$$

Considering the variables change $F_i = PL_i$, we get (2.87).

As for the positivity of the estimation error in (2.84), we impose that:

$$\begin{cases} U_{ij} = A_i - L_j C_i \text{ is Metzler} \\ U_{ij}\varrho_{ij} + B_i\Delta_i \succcurlyeq 0 \end{cases} \tag{2.99}$$

Noting that ϱ_{ij} and Δ_i are necessarily positive, then conditions (2.99) can be rewritten in the following way:

$$\begin{cases} A_i - L_j C_i + \xi I \succcurlyeq 0, \qquad \forall i, j = 1, \dots, r \\ B_i \succcurlyeq 0 \end{cases} \tag{2.100}$$

where ξ is a positive scalar.

This ends the proof. ∎

2.3.1.3. Illustrative Example

To illustrate both approaches, we consider a positive T-S system (1.3) in which the decision variables are unmeasurable, where $r = 2$, $n_x = 3$ and the system matrices are:

$$A_1 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -6 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 2 & 2 \\ 5 & -3 & 1 \\ 0.5 & 0.5 & -4 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix}, B_2 = \begin{pmatrix} 0.5 \\ 1.5 \\ 0.25 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$h_1\big(x(t)\big) = \frac{1 - \tanh(x_1(t))}{2}, h_2\big(x(t)\big) = \frac{1 + \tanh(x_1(t))}{2}$$

For $\xi = 2$, the resolution of Theorem 2.12 gives the following observer gains and disturbance bound:

$$P = \begin{pmatrix} 0.0321 & -0.1250 & 0.0189 \\ -0.1250 & 0.2612 & 0.0243 \\ 0.0189 & 0.0243 & 0.4921 \end{pmatrix}, Q_{11} = \begin{pmatrix} -2.0368 & -1.0111 & -0.2731 \\ -1.0111 & -1.0852 & 0.0012 \\ -0.2731 & 0.0012 & -1.0626 \end{pmatrix}$$

$$Q_{12} = \begin{pmatrix} -0.0131 & -0.1002 & 0.0361 \\ -0.1002 & -0.0002 & 0.1033 \\ 0.0361 & 0.1033 & -0.2351 \end{pmatrix}, Q_{22} = \begin{pmatrix} -0.6521 & 0.0568 & -0.3501 \\ 0.0568 & -1.0212 & -0.0137 \\ -0.3501 & -0.0137 & -1.0128 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 18.4501 & 5.6656 \\ 17.3218 & -3.7137 \\ 4.0970 & -2.0460 \end{pmatrix}, L_2 = \begin{pmatrix} 31.0368 & 16.0254 \\ 21.0221 & 12.0825 \\ 7.0250 & 0.5017 \end{pmatrix}$$

$\alpha = 1.2051, \beta = 0.6641$

When checking the feasibility of Theorem 2.12, we note that for $\beta > 2.7301$, the conditions (2.67) do not have solution.

The resolution of Theorem 2.13, with $\xi = 2$ and the Lipschitz constants given by: $M_1 = M_2 = 0.8$ yields:

$$P = \begin{pmatrix} 1.7292 & -3.6620 & -0.7908 \\ -3.6620 & 5.7213 & 0.1021 \\ -0.7908 & 0.1021 & 10.7057 \end{pmatrix}, Q = \begin{pmatrix} 54.0250 & -7.0012 & 17.9120 \\ -7.0012 & 11.2372 & 25.3218 \\ 17.9120 & 25.3218 & 81.5017 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 21.0368 & 32.0021 \\ 5.0111 & 17.6928 \\ 1.7137 & 8.6665 \end{pmatrix}, L_2 = \begin{pmatrix} 12.2450 & 15.9161 \\ 4.9415 & 12.6620 \\ 2.5621 & 5.2372 \end{pmatrix}$$

$\lambda_1 = 126.0421, \lambda_2 = 25.9043$

$\beta_1 = 0.6182, \beta_2 = 0.3521$

Figures 2.6-2.8 illustrate the state estimation errors of the system for both approaches with the input $u(t)$ in Figure 2.5:
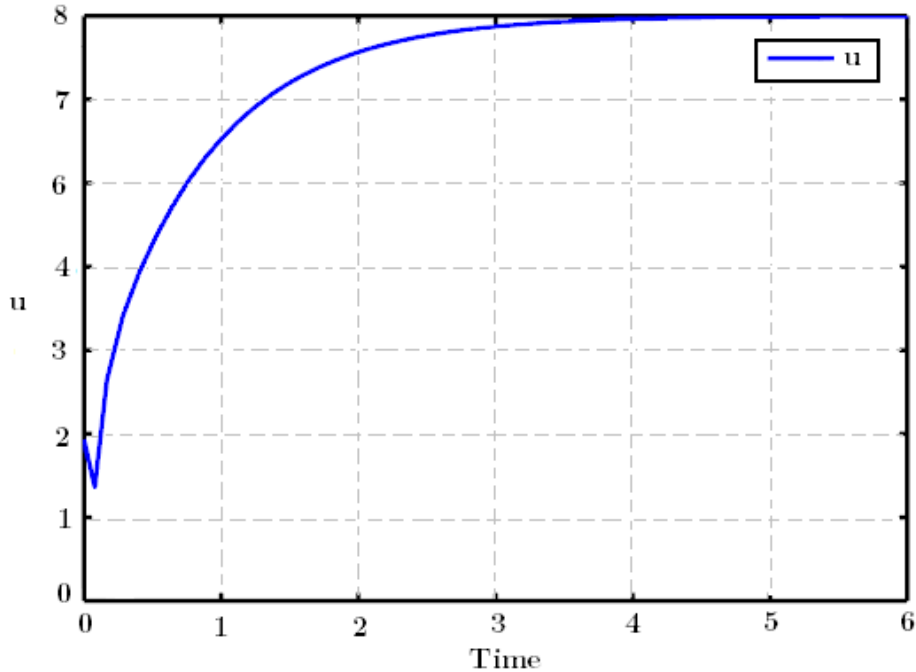
Figure 2. 5. The evolution of the system input $u(t)$
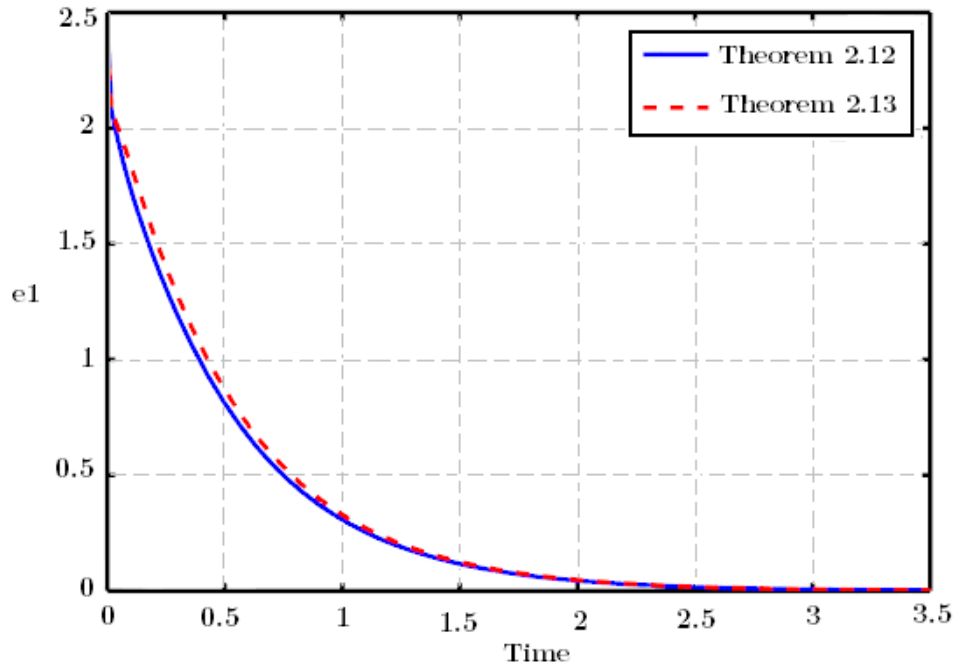
Figure 2. 6. The evolution of the state estimation error $e_1(t)$
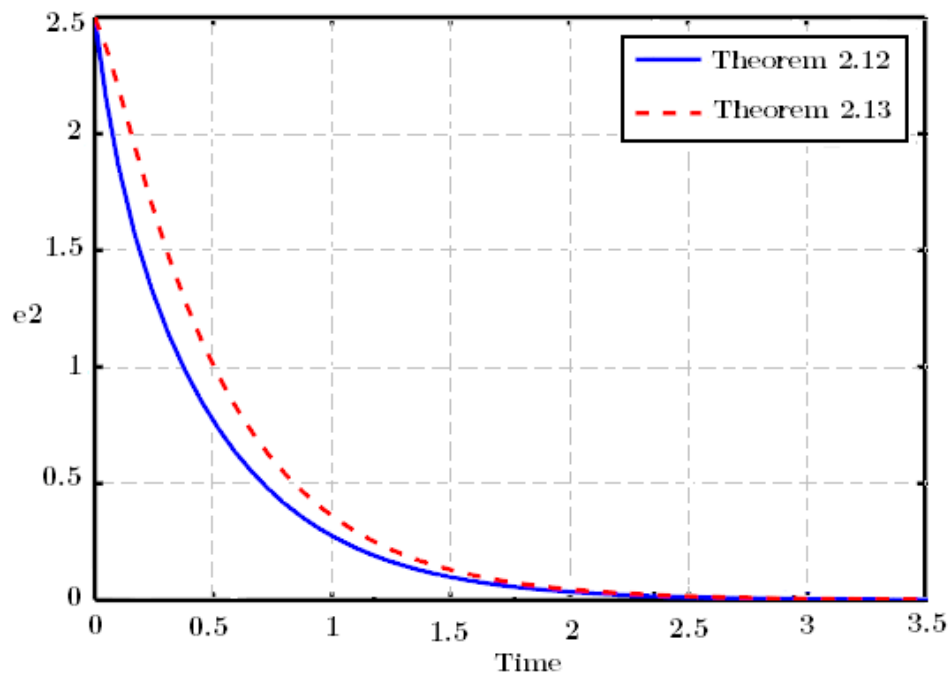
Figure 2. 7. The evolution of the state estimation error $e_2(t)$
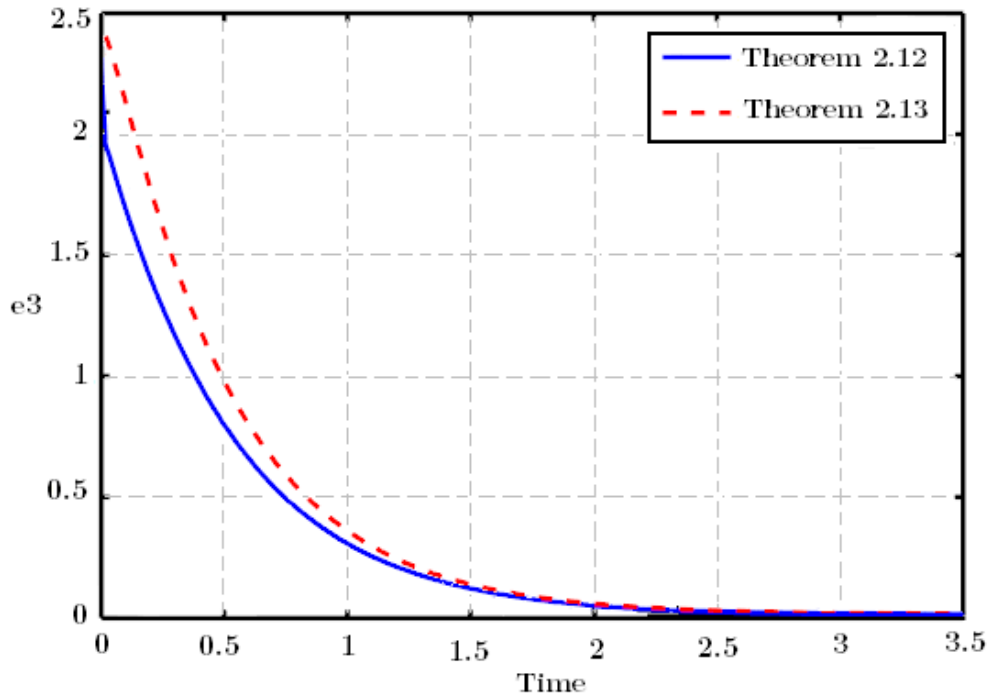
Figure 2. 8. The evolution of the state estimation error $e_3(t)$

We can note the rapid convergence of the state estimation errors for both approaches. In addition, the errors always remain nonnegative. However, Theorem 2.13 presents a slight better performance in guaranteeing the estimation error convergence.

2.4. Conclusion

In this section, we first proposed quadratic stabilization approaches to design state-feedback controllers for positive T-S systems, using the concept of Parallel Distributed Compensation. We developed generalized and relaxed stabilization conditions for the stability of positive T-S systems. We have also proposed the decomposition of controller gains to stabilize and α-stabilize positive interval T-S systems. The results are used to improve the stability conditions and maximize the decay rate of the exponential stabilization of this class of systems.

Then, we developed results for the stabilization of positive T-S systems with unmeasurable decision variables. The first approach is based on a representation of the estimation error in the case when the decision variables are measurable but affected with disturbance. We have shown the limitation of this approach due to the large bounds that this disturbance can take. In the second approach, the

expression of the estimation error involves the bounds of the state variables vector and the bounds of the input vector. This approach has the drawback that it requires a bound on the state variables vector.

Chapter III

Observer-Based Control Design for Positive Systems

3.1. Introduction

In this chapter, we study the simultaneous state estimation and stabilization of positive linear and T-S systems that maybe have interval uncertainties. For this, we present different approaches for positive observer-based control design, giving LMI conditions.

The issue of control and state constraints is also dealt with: the proposed approach is extended to handle component-wise lower and upper bounds on the controls.

3.2. Observation of Linear Positive Systems

The problem of state estimation is important in identification and control design. We can find in literature the stochastic approach (Kalman filter), the H_∞ filtering theory and the set-membership approach. We can make an overview on some classical observers, such as the Luenberger one and the well-known Kalman filter, have been firstly presented in (Luenberger, 1966), (Kalman, 1960) and (Julien & Uhlmann, 2004).

3.2.1. INTERVAL OBSERVERS APPROACH

3.2.1.1. Interval observer design using upper and lower errors

In the following, we suppose that the only information available about the uncertainties is that they are bounded by known upper and lower values. We will present positive interval observers with bounded errors for positive linear systems. Bounding observers can guarantee bounds on the estimated states (Gouzé et al, 2000), (Alamo et al, 2005). Other interval estimation methods for uncertain systems can be found in (Chen et al, 1997), (Jaulin et al, 2001), (Jaulin et al, 2002). The interval observers developed here can be expressed by pairs of estimators.

We consider the following dynamical system:

$$\dot{x}(t) = f\big(x(t), \mu, u(t)\big), \qquad x(0) = x_0 \tag{3.1}$$

where f is a Lipschitz function that depends on unknown parameters μ.

An interval observer of system (3.1) can be expressed using two dynamical systems as follows:

$$\begin{cases} \dot{\overline{\omega}}(t) = g\left(\overline{\omega}(t), \underline{\omega}(t), x(t), \mu, u(t)\right), & \overline{\omega}(0) = \overline{x}_0 \\ \dot{\underline{\omega}}(t) = g\left(\underline{\omega}(t), \overline{\omega}(t), x(t), \mu, u(t)\right), & \underline{\omega}(0) = \underline{x}_0 \end{cases} \tag{3.2}$$

where their initial conditions are such that $\underline{x}_0 \leqslant x_0 \leqslant \overline{x}_0$.

Definition 3.1. (Bolajraf et al, 2011)

An interval observer (3.2) for system (3.1) is a pair of upper and lower estimator functions $\left(\overline{\omega}(t), \underline{\omega}(t)\right)$ of the real state function $x(t)$, that is,

$$\underline{\omega}(t) \leqslant x(t) \leqslant \overline{\omega}(t), t \geq 0 \tag{3.3}$$

We will need some other definitions for the rest of the investigation part.

Definition 3.2. (Bolajraf et al, 2011)

An interval observer for system (3.1) is said to be convergent towards to a box if $\lim_{t \to \infty} \left(\overline{\omega}(t) - \underline{\omega}(t)\right)$ exists or is bounded.

Definition 3.3. (Bolajraf et al, 2011)

An interval observer of system (3.1) is said to be asymptotically convergent if $\lim_{t \to \infty} \left(\overline{\omega}(t) - \underline{\omega}(t)\right) = 0$.

The main goal is now to design interval observers for systems of the form (3.1), where the initial conditions of the system (3.1) are unknown but bounded $\underline{x}_0 \leqslant x_0 \leqslant \overline{x}_0$. In addition, we suppose that some parameters of the model are unknown, so the observer has to be robust. The basic idea investigated is to reconstruct the error dynamics $\dot{e}(t) = \dot{\overline{\omega}}(t) - \dot{\underline{\omega}}(t)$, which is enforced to be positive by the proposed approach. First of all, we have to ensure that if the initial error fulfills $e(0) \geqslant 0$, the error remains nonnegative $e(t) \geqslant 0, \forall t$. So, we can guarantee nonnegative lower and upper bounds on the estimated states.

We first consider the following system subject to interval uncertainties:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.4}$$

where $x(t) \in \mathbb{R}_+^{n_x}$ is the state vector and $y(t) \in \mathbb{R}^{n_y}$ is the output formed by the measurements. The initial conditions $x(0) = x_0 \in \mathbb{R}_+^{n_x}$ are assumed to be unknown and bounded. We suppose $u(t) = 0$.

$$\underline{x}_0 \leqslant x(0) \leqslant \overline{x}_0 \tag{3.5}$$

The matrix $A \in \mathbb{R}^{n_x \times n_x}$, which is Metzler, and the matrix $C \in \mathbb{R}^{n_y \times n_x}$ are assumed to be unknown, but bounded by known constant matrices $\overline{A}$, $\underline{A}$, $\overline{C}$ and $\underline{C}$:

$$\underline{A} \leqslant A \leqslant \overline{A}, \underline{C} \leqslant C \leqslant \overline{C} \tag{3.6}$$

Our goal is to obtain an interval observer of system (3.4) that guarantees bounds on the estimated states. A positive interval observer is proposed in (Bolajraf, 2012) with gain matrix decomposed into matrices:

$$\begin{cases} \dot{\overline{\omega}} = (\overline{A} - L\underline{C} + G\overline{C})\overline{\omega} + (L - G)y, & \overline{\omega}(0) = \overline{x}_0 \\ \dot{\underline{\omega}} = (\underline{A} - L\overline{C} + G\underline{C})\underline{\omega} + (L - G)y, & \underline{\omega}(0) = \underline{x}_0 \\ 0 \leqslant \underline{\omega}(t) \leqslant x(t) \leqslant \overline{\omega}(t) \end{cases} \tag{3.7}$$

where the gains L and G have to fulfill the additional constraints

$$\begin{cases} L \geqslant 0 \\ G \geqslant 0 \\ L\underline{C} - G\overline{C} \geqslant 0 \\ \underline{A} - L\overline{C} + G\underline{C} \text{ is Metzler} \end{cases} \tag{3.8}$$

By fulfilling the conditions in (3.8) on the gains L and G of the pair of dynamical systems in (3.7), we can demonstrate that $\underline{\omega}(t) \leqslant x(t) \leqslant \overline{\omega}(t)$.

For this reason, we can establish the lower and upper errors that follow:

$$\dot{\overline{e}} = (\overline{A} - L\underline{C} + G\overline{C})\overline{e} + \left(\overline{A} - A + L(C - \underline{C}) + G(\overline{C} - C)\right)x \tag{3.9}$$

$$\dot{\underline{e}} = (\underline{A} - L\overline{C} + G\underline{C})\underline{e} + \left(A - \underline{A} + L(\overline{C} - C) + G(C - \underline{C})\right)x \tag{3.10}$$

These errors must be nonnegative for all initial conditions $\overline{e}_0 = \overline{\omega}_0 - x_0 \geqslant 0$ and $\underline{e}_0 = x_0 - \underline{\omega}_0$. In addition, note that the nonnegativity of the lower estimate $\underline{\omega}(t) \geqslant 0$ is guaranteed from the condition $L\underline{C} - G\overline{C} \geqslant 0$. To prove this claim, we can use Theorem 1.17. and the fact that $\overline{A} - L\underline{C} + G\overline{C}$ and $\underline{A} - L\overline{C} + G\underline{C}$ are Metzler, combined with $\left(\overline{A} - A + L(C - \underline{C}) + G(\overline{C} - C)\right)x \geqslant 0$ and $\left(A - \underline{A} + L(\overline{C} - C) + G(C - \underline{C})\right)x \leqslant 0$. Also, we can show that the lower estimate of (3.7)

is nonnegative since $\underline{A} - L\overline{C} + G\underline{C}$ is Metzler and $(L - G)C$ is a nonnegative matrix for all C such as $\underline{C} \leqslant C \leqslant \overline{C}$, by noticing that if $L\underline{C} - G\overline{C}$ is nonnegative then $(L - G)C$ is nonnegative $\left(L\underline{C} - G\overline{C} \leq (L - G)C\right)$.

We propose an approach of interval observers design with bounded errors, based on LMI formulation.

Theorem 3.1. (Zaidi et al, 2015)

Assume that the trajectory of system (3.4) is bounded, that is, $x(t) \leqslant \overline{x}, \ \forall t \geq 0$, then, there exists a positive interval observer of system (3.4) of the form (3.7) with bounded error, if the following LMI problem, in the variables $\beta \in \mathbb{R}_+$, $\lambda \in \mathbb{R}^{n_x}$, X, $Y \in \mathbb{R}_+^{n_y \times n_x}$, is feasible:

$$
\begin{cases}
\overline{A}^T diag(\lambda) + diag(\lambda)\overline{A} - \underline{C}^T X - X\underline{C} + \overline{C}^T Y + Y\overline{C} < 0 & (3.12a) \\
\underline{C}^T X - \overline{C}^T Y \geqslant 0 & (3.12b) \\
& (3.12c) \\
\underline{A}^T diag(\lambda) - \overline{C}^T X + \underline{C}^T Y + \beta I \geqslant 0 & (3.12d) \\
\lambda > 0 & (3.12e) \\
X \geqslant 0 & (3.12f) \\
Y \geqslant 0 &
\end{cases}
$$

Moreover, the gain matrices L and G of the interval observer (3.7) are given by:

$$L = diag(\lambda)^{-1} X^T, \ G = diag(\lambda)^{-1} Y^T \tag{3.13}$$

where β, λ, X and Y are any feasible solution to the LMI problem (3.12).

Proof:

Assume that the conditions (3.12) are verified and define $L = diag(\lambda)^{-1} X^T$ and $G = diag(\lambda)^{-1} Y^T$. By simple manipulations, (3.12) is equivalent to

$$
\begin{cases}
\left(\overline{A}^T - \underline{C}^T L^T + \overline{C}^T G^T\right)\lambda < 0 & (3.14a) \\
\lambda > 0 & (3.14b) \\
L^T diag(\lambda) \geqslant 0 & (3.14c) \\
G^T diag(\lambda) \geqslant 0 & (3.14d) \\
\left(\underline{C}^T L^T - \overline{C}^T G^T\right) diag(\lambda) \geqslant 0 & (3.14e) \\
\left(\underline{A}^T - \overline{C}^T L^T + \underline{C}^T G^T\right) diag(\lambda) + \beta I \geqslant 0 & (3.14f)
\end{cases}
$$

The condition $\left(\underline{A}^T - \overline{C}^T L^T + \underline{C}^T G^T\right) diag(\lambda) + \beta I \succcurlyeq 0$ equivalently means that $\underline{A}^T - \overline{C}^T L^T + \underline{C}^T G^T$ is Metzler, or equivalently, $\underline{A} - L\overline{C} + G\underline{C}$ is Metzler.

The nonnegativity of L, G and $L\underline{C} - G\overline{C}$ are equivalent to the conditions $L^T diag(\lambda) \succcurlyeq 0$, $G^T diag(\lambda) \succcurlyeq 0$ and $(\underline{C}^T L^T - \overline{C}^T G^T)diag(\lambda) \succcurlyeq 0$. Hence, we know that the conditions $L \succcurlyeq 0$, $G \succcurlyeq 0$, $L\underline{C} - G\overline{C} \succcurlyeq 0$ and $\underline{A} - L\overline{C} + G\underline{C}$ is Metzler define a positive interval observer of the form (3.7).

In order to complete the proof, we use the fact that $\overline{A} - L\overline{C} + G\overline{C}$ is Hurwitz. To see this, use the fact that $\underline{A} - L\overline{C} + G\underline{C}$ is Metzler together with the conditions $\left(\overline{A}^T - \underline{C}^T L^T + \overline{C}^T G^T\right)\lambda < 0$, $\lambda > 0$ and Theorem 1.19.

3.2.1.2. Design of Positive Observers for Interval Systems

We now consider the observer design for the following autonomous positive interval system (3.4), where $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$ are unknown constant matrices with known bounds, that fulfill $\underline{A} \in \mathbb{R}_+^{n_x \times n_x}$ is Metzler, $\underline{C} \in \mathbb{R}_+^{n_y \times n_x}$ and $\overline{A}$ is Hurwitz (so A is Hurwitz).

Lemma 3.1. (Minc, 1988), (Berman & Plemmons, 1994)

For matrices $A, B \in \mathbb{R}^{n_x \times n_x}$, if B is Metzler and $A \succcurlyeq B$, then $\mu(A) \geq \mu(B)$.

The more general structure that we will adopt is:

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ly(t) \tag{3.15}$$

where $G \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_x \times n_y}$ are the observer matrices to be identified. Define the error as $e(t) = x(t) - \hat{x}(t)$, then, the augmented system is given by:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ A - LC - G & G \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \tag{3.16}$$

For positive linear systems, we want also to guarantee the nonnegativity of the estimated state $\hat{x}(t)$. Then, referring to Lemma 2.1, it is natural to require that G is Metzler and $L \succcurlyeq 0$. In addition, as seen from (Hof, 1998), the induced error dynamic system is nonnegative for the Luenberger observer. In fact, the nonnegativity of the error $e(t)$ is compulsory for the design of positive interval observers. Therefore, our problem is to design a Metzler G and $L \succcurlyeq 0$ of the

positive observer (3.15) such that the augmented system (3.16) is positive and asymptotically stable for any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$.

Remark 3.2.

The positivity specification on the error $e(t)$ facilitates the synthesis of the desired positive observer, but may cause some conservatism.

Necessary and sufficient conditions for the existence of a robust interval observer for positive linear systems are summarized in the following theorem.

Theorem 3.2. (Shu et al, 2008)

There exists a positive observer of the form (3.15) for system (3.16) if and only if there exist matrices $P = diag(p_{11}, \dots, p_{nn}) > 0$, $Q = diag(q_{11}, \dots, q_{nn}) > 0$, $V \geqslant 0$ *and Metzler* W *such that the following LMIs hold:*

$$\begin{pmatrix} P\overline{A} + \overline{A}^T P & \overline{A}^T Q - \underline{C}^T V^T - W^T \\ Q\overline{A} - V\underline{C} - W & W + W^T \end{pmatrix} < 0 \tag{3.17}$$

$$Q\underline{A} - V\overline{C} - W \geqslant 0 \tag{3.18}$$

Then, the desired observer matrices can be obtained as follows:

$$G = Q^{-1}W, \ L = Q^{-1}V \tag{3.19}$$

Based on these results, we propose an approach to synthesize an observer-based state-feedback controller for positive linear systems.

3.2.1.3. Design of Positive Interval Observer-based state-feedback controller

We consider the observer design for the positive interval linear system (3.4), where: $u(t) \in \mathbb{R}^{n_u}$ is the control vector, $A \in [\underline{A}, \overline{A}], B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$ are unknown constant matrices with known bounds, that fulfill: $\underline{A} \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{B} \geqslant 0 \in \mathbb{R}^{n_x \times n_u}$ and $\underline{C} \geqslant 0 \in \mathbb{R}^{n_y \times n_x}$.

Let us suppose the state observer (3.15), which allows the states of the model (3.4) to be estimated.

Consider the state-feedback control given by the following expression:

$$u(t) = K\hat{x}(t) \tag{3.20}$$

where $K \in \mathbb{R}^{n_y \times n_x}$ is the controller matrix to be determined.

Then, the closed-loop system is written as follows:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,K\hat{x}(t) \\ y(t) = C\,x(t) \end{cases} \tag{3.21}$$

Considering the observer (3.15) and the closed loop system (3.21), we can construct the augmented system as follows:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} = \begin{pmatrix} A & BK \\ LC & G \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} \tag{3.22}$$

Firstly, the aim is to guarantee the asymptotic stability and the positivity of the state $x(t)$. In this context, it is natural to require $\hat{x}(t)$ being nonnegative as well. Therefore, the problem is to design G, L and K such that the augmented system (3.22) is positive and asymptotically stable. Unfortunately, no feasible solution exists for the LMIs associated to this problem.

To solve this issue, the key lies in the nonnegativity of the error $e(t) = x(t) - \hat{x}(t)$. Then, we consider the new augmented system defined by:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \tag{3.23}$$

Therefore, the problem is reduced to the determination of the gains G, L and K of the augmented system such that the augmented system (3.23) is positive and asymptotically stable. In the following, we provide a necessary condition for the existence of solutions to this problem.

Theorem 3.3. (Zaidi et al, 2014c)

If there exists a static state-feedback controller (3.20) that stabilizes system (3.21), using the observer (3.15), then the following inequalities with respect to Metzler G, $L \geqslant 0$ and $K \leqslant 0$ have a solution:

$$trace\big(\underline{A} + G + (\overline{B} - \underline{B})K\big) < 0 \tag{3.24}$$

$$[\overline{A} + \underline{B}K]_{ij} \geq 0\,, 1 \leq i \neq j \leq n \tag{3.25}$$

$$[G - \overline{B}K]_{ij} \geq 0\,, 1 \leq i \neq j \leq n \tag{3.26}$$

$$\overline{A} - L\underline{C} + \underline{B}K - G \geqslant 0 \tag{3.27}$$

Proof:

If the augmented system (2.23) is stable and positive we have that, we have that

$$\mu\left(\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix}\right) < 0 \tag{3.28}$$

$$\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix} \text{ is Metzler} \tag{3.29}$$

As we have that:

$$\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix} \leqslant \begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix}$$

$$\leqslant \begin{bmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{bmatrix} \tag{3.30}$$

Then, we deduce, using (3.28), that

$$\mu\left(\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix}\right) < 0 \tag{3.31}$$

And using

$$\begin{bmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{bmatrix} \text{ is Metzler} \tag{3.32}$$

It follows from (3.31) that

$$trace\left(\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix}\right) < 0 \tag{3.33}$$

which is equivalent to (3.24). Finally, it is obvious that (3.32) is equivalent to (3.25), (3.26) and (3.27), which completes the proof. ∎

Next, we further study sufficient conditions and the corresponding synthesis approach.

Theorem 3.4. (Zaidi et al, 2014c)

There exists a static state-feedback controller (3.20) that stabilizes system (3.21) using the observer (3.15), if for a positive scalar ε, there exist matrices $P = diag(P_1, P_2) > 0$, a Metzler matrix G, $L \geqslant 0$ and $K \leqslant 0$ such that:

$$\begin{pmatrix} \mathcal{A}^T P + P\mathcal{A} - \varepsilon \mathcal{B}\mathcal{B}^T P - \varepsilon P\mathcal{B}^T\mathcal{B} & P\mathcal{B} + \mathcal{C}^T\mathcal{K}^T \\ * & -I \end{pmatrix} < 0 \qquad (3.34)$$

$$[\underline{A} + \overline{B}K]_{ij} \geq 0 \,, 1 \leq i \neq j \leq n \qquad (3.35)$$

$$[G - \underline{B}K]_{ij} \geq 0 \,, 1 \leq i \neq j \leq n \qquad (3.36)$$

$$\underline{A} - L\overline{C} + \overline{B}K - G \succcurlyeq 0, \qquad (3.37)$$

where:

$$\mathcal{A} = \begin{pmatrix} \overline{A} & 0 \\ \underline{A} & 0 \end{pmatrix}, \; \mathcal{B} = \begin{pmatrix} 0 & \underline{B} - \overline{B} & 0 & \underline{B} \\ 0 & \underline{B} - \overline{B} & -I & \underline{B} \end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix} G & L & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & G & L \\ 0 & 0 & K & 0 \end{pmatrix}, \; \mathcal{C} = \begin{pmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C} & 0 \end{pmatrix} \qquad (3.38)$$

Proof:

It follows from (3.35) that $\underline{A} + \overline{B}K$ is Metzler. Combining this with $K \preccurlyeq 0$ and $L \succcurlyeq 0$ yields that, for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$:

$$A + BK \succcurlyeq \underline{A} + \overline{B}K \text{ is Metzler} \qquad (3.39)$$

and

$$A - LC + BK - G \succcurlyeq \underline{A} - L\overline{C} + \overline{B}K - G \succcurlyeq 0 \qquad (3.40)$$

In addition, from G being Metzler and $K \preccurlyeq 0$, we obtain that, for any $B \in [\underline{B}, \overline{B}]$:

$$-BK \succcurlyeq 0 \qquad (3.41)$$

and

$$G - BK \text{ is Metzler} \qquad (3.42)$$

Therefore, from (3.39), (3.40), (3.41) and (3.42), we have that, for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$, the augmented system (3.23) is positive.

It follows from (3.34), by the Schur complement, that:

$$\mathcal{A}^T P + P\mathcal{A} - \varepsilon \mathcal{B}\mathcal{B}^T P - \varepsilon P\mathcal{B}^T\mathcal{B} + \varepsilon^2 \mathcal{B}\mathcal{B}^T + (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C})^T(\mathcal{B}^T P + \mathcal{K}\,\mathcal{C}) < 0 \quad (3.43)$$

Taking into account the following relationship:

$$PBB^T P - \varepsilon BB^T P - \varepsilon PB^T B + \varepsilon^2 BB^T = (PB - \varepsilon\, B)(B^T P - \varepsilon B^T) \geq 0 \qquad (3.44)$$

we obtain the following inequality:

$$\mathcal{A}^T P + P\mathcal{A} - P\mathcal{B}\mathcal{B}^T P + (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C})^T (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C}) < 0 \qquad (3.45)$$

Rewriting (3.56) yields that:

$$(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) + \mathcal{C}^T \mathcal{K}^T \mathcal{K}\,\mathcal{C} < 0 \qquad (3.46)$$

which implies that:

$$(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) < 0 \qquad (3.47)$$

Therefore, we get:

$$\mu(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) < 0 \qquad (3.48)$$

Some algebraic manipulations lead to:

$$\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C} = \begin{pmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{pmatrix} \qquad (3.49)$$

In addition, it is easy to show that:

$$\begin{pmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{pmatrix} \succcurlyeq \begin{pmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{pmatrix} \qquad (3.50)$$

Therefore, by combining (3.48)-(3.50) and using Lemma 3.1, we obtain that:

$$\mu\left(\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix} \right) < 0 \qquad (3.51)$$

which means that (3.23) is asymptotically stable for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$, which completes the proof.∎

Remark 3.3.

Using the designed observer-based controller, the state vectors $x(t), \hat{x}(t)$ and $e(t)$ will be nonnegative if the initial conditions satisfy $x(0) \succcurlyeq 0$ and $\hat{x}(0) \succcurlyeq 0$. A question which may be asked is why (3.22) is not a positive and asymptotically stable system, even when $x(t)$ and $\hat{x}(t)$ are nonnegative and converge to the origin. The reason is that the invariant set associated with (3.22) is not the positive orthant but the cone defined by:

$$ß = \left\{ \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \succcurlyeq 0; \ [I \quad -I] \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \succcurlyeq 0 \right\}$$

Otherwise, any trajectory of (3.22) starting from $[x^T(0), \hat{x}^T(0)]^T \in ß$ will remain in ß for $t > 0$. If a positive system can be called positive orthant invariant, then the system (3.22) with Metzler G, $L \geqslant 0$ and $K \leqslant 0$ obtained through Theorem 3.4 can be viewed as ß invariant. This interpretation may be useful to seek less conservative conditions for such a problem and even to establish solvable necessary and sufficient conditions for the positive stabilization problem with sign-indefinite K.

3.2.1.4. Illustrative example

Let us consider the positive interval system (3.4) with:

$$\underline{A} = \begin{bmatrix} -0.4 & 0.2 \\ 0.1 & -0.5 \end{bmatrix}, \overline{A} = \begin{bmatrix} -0.2 & 0.5 \\ 0.5 & -0.1 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \underline{C} = [0.4 \quad 0.1], \overline{C} = [0.6 \quad 0.4].$$

Then, by using the Scilab 5.3.3 LMI toolbox, it can be seen that the LMIs in Theorems 3.3 and 3.4 are feasible, for $\varepsilon = 15$, with the following solution:

$$P = \begin{bmatrix} 24.6213 & 0 \\ 0 & 12.4880 \end{bmatrix}, G = \begin{bmatrix} -5.2922 & 0.1028 \\ 0.0006 & -0.9139 \end{bmatrix},$$

$$K = [-1.6628 \quad -1.2478], L = \begin{bmatrix} 9.7926 \\ 4.1322 \end{bmatrix}.$$
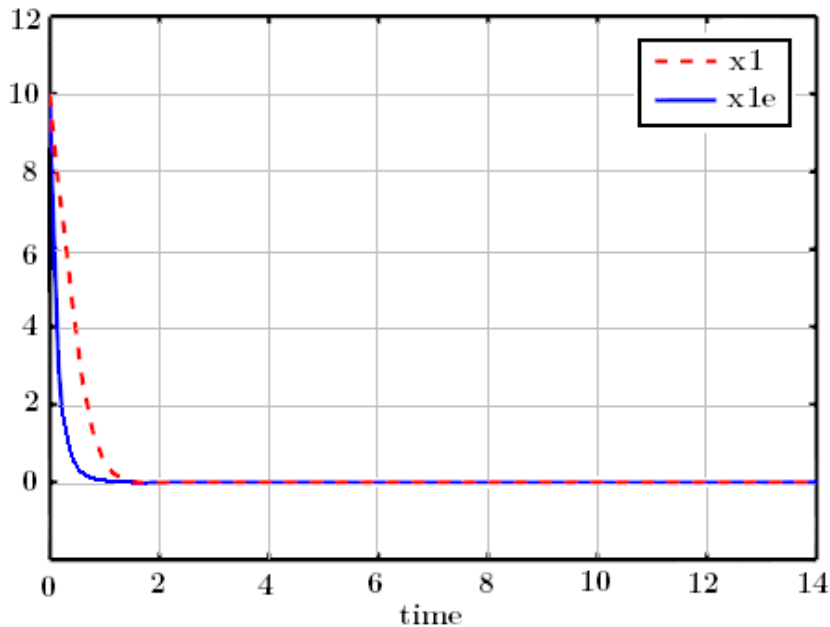
Figure 3. 1. Evolution of the state $x_1(t)$ and its estimation $\hat{x}_1(t)$
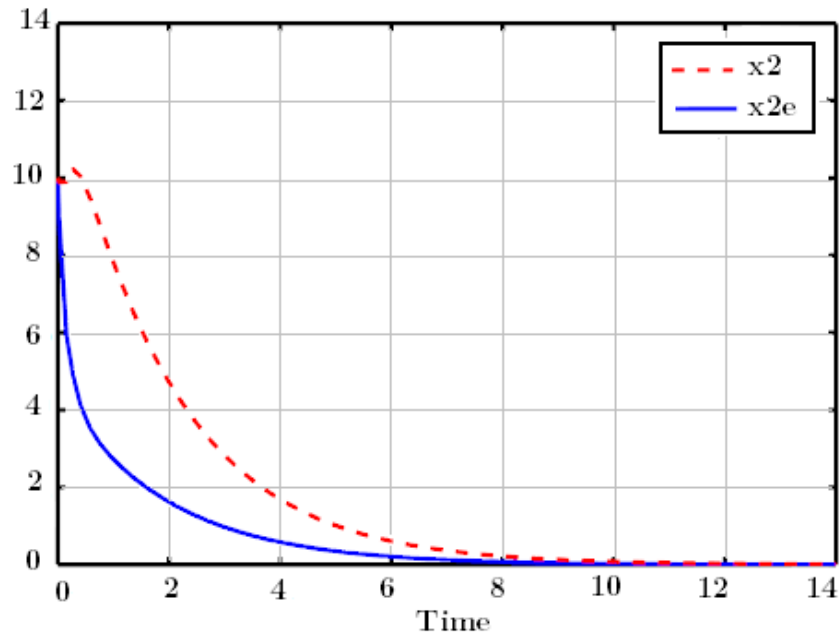
Figure 3. 2. Evolution of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$
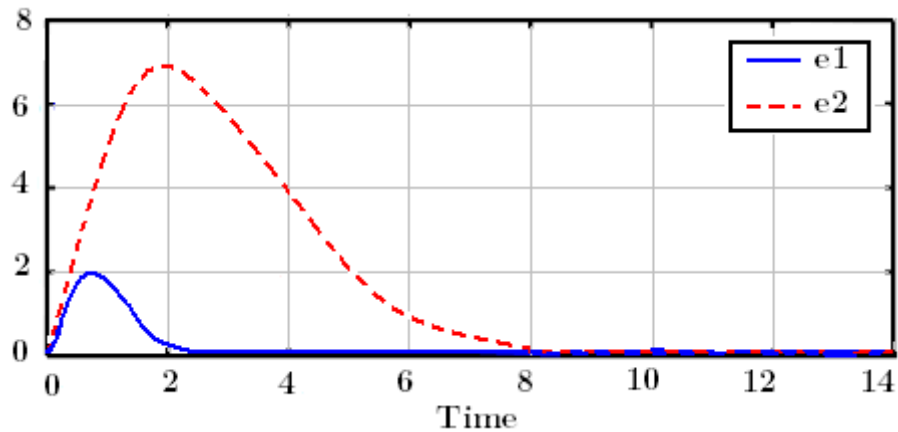
Figure 3. 3. Evolution of the estimation errors $e_1(t)$ and $e_2(t)$

Figures 3.1 to 3.3 show that the evolution of the state vector $x(t)$, as well as its estimated state vector $\hat{x}(t)$, remain always nonnegative and converge (the unknown system was simulated for $A = \underline{A}$, $B = \underline{B}$, $C = \underline{C}$). Figures 3.1 and 3.2 illustrate the good estimation and stabilization. Figure 3.3 shows also the nonnegativity of the estimation errors. These facts show the effectiveness of the proposed approach.

3.3. Observer-based Control Design for Positive T-S Systems

In many practical nonlinear control systems, state variables are unavailable, so, observer-based control is necessary and has attracted some interest for T-S systems (Tanaka et al, 1998), (Ma & Sun, 1998), (Yoneyama et al, 2001), (Marray-Smith & Johansen, 1997), (Xiaodong & Qingling, 2003), (Benhadj Braiek & Rotella, 1995).

Consider the following T-S system for which the output is a linear function of the state model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) = C x(t) \end{cases} \tag{3.52}$$

The developed observer has the following form (Chadli et al, 2002d):

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))) \\ \hat{y}(t) = C \hat{x}(t) \end{cases} \tag{3.53}$$

In the next section, we propose approaches to design observer-based controllers by defining two cases, according to the decision variables: measurable and nonmeasurable.

3.3.1. OBSERVER-BASED CONTROL OF POSITIVE T-S SYSTEMS WITH MEASURABLE PREMISE VARIABLES

The control based on the state observation for nonlinear systems has been actively considered during the last decades in several studies (Marray-Smith & Johansen, 1997), (Tanaka et al, 1998), (Xiaodong & Qingling, 2003), (Benhadj Braiek & Rotella, 1995). It aims to develop more systematic algorithms that guarantee the stability and the specific performance for these systems (Tanaka et al, 1998), (Chadli et al, 2004), (Chadli et al, 2002d), (Zhang & Fei, 2006). However, the problem of the synthesis of positive observer-based controllers has not been frequently dealt with in the literature.

3.3.1.1. Positive T-S observer-based controller (First approach)

To estimate the unmeasured state variables of the T-S model (1.3) (with $D_i = 0$, $i = 1, \dots, r$), an observer can be designed using the PDC technique. In this case,

the overall observer is obtained by linear interpolation of local Luenberger monitors (Luenberger, 1963) associated with different submodels.

The decision variables are supposed measurable, with the T-S system (1.3), locally detectable (i.e, all pairs (A_i, C_i), $i = 1, ..., r$ are observable).

We consider the state estimation error $e(t) = x(t) - \hat{x}(t)$. Its dynamical model is then:

$$\dot{e}(t) = \sum_{i,j=1}^{r} h_i(z)h_j(z)S_{ij}e(t) \tag{3.55}$$

with $S_{ij} = A_i - L_i C_j$.

Let us consider a PDC control law:

$$u(t) = -\sum_{i=1}^{r} h_i(z)K_i\hat{x}(t) \tag{3.56}$$

From (3.54) and (3.56), we get the following observer:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i,j=1}^{r} h_i(z)h_j(z)\big(A_i - B_i K_j\big)\hat{x}(t) + \sum_{i,j=1}^{r} h_i(z)h_j(z)L_i C_j e(t) \\ \hat{y}(t) = \sum_{i=1}^{r} h_i(z)C_i\hat{x}(t) \end{cases} \tag{3.57}$$

By considering the augmented state given by $\tilde{x}(t) = [\hat{x}^T(t) \ e^T(t)]^T$, we can construct then the following augmented system:

$$\dot{\tilde{x}}(t) = \sum_{i,j=1}^{r} h_i(z)h_j(z)M_{ij}\tilde{x}(t)$$

$$= \sum_{i=1}^{r} h_i^2(z)M_{ii}\tilde{x}(t) + 2\sum_{1 \leq i < j}^{r} h_i(z)h_j(z)\left(\frac{M_{ij} + M_{ji}}{2}\right)\tilde{x}(t) \tag{3.58}$$

where

$$M_{ij} = \begin{pmatrix} A_i - B_i K_j & L_i C_j \\ 0 & A_i - L_i C_j \end{pmatrix} = \begin{pmatrix} G_{ij} & L_i C_j \\ 0 & S_{ij} \end{pmatrix}, \ i,j = 1, ..., r \tag{3.59}$$

We can now apply to (3.58) the results of Section 2.2.1 to guarantee the convergence of the error $e(t)$ and the state estimation $\hat{x}(t)$.

The following result presents the condition guaranteeing the positive stabilization of the augmented system (3.58).

Theorem 3.5. (Zaidi et al, 2015)

If there exist symmetric matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$ and $Q_2 > 0$, a scalar ε such that $\frac{1}{2} < \varepsilon < 1$ and scalars $\beta_1 \geq 0$, $\beta_2 \geq 0$ and $s > 1$ such that:

$$\begin{cases} G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1 < 0; i = 1, \dots, r \\ \left(\dfrac{G_{ij} + G_{ji}}{2}\right)^T P_1 + P_1\left(\dfrac{G_{ij} + G_{ji}}{2}\right) - \varepsilon Q_1 \leq 0; \ i < j = 1, \dots, r \\ S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2 < 0; \ i = 1, \dots, r \\ \left(\dfrac{S_{ij} + S_{ji}}{2}\right)^T P_2 + P_2\left(\dfrac{S_{ij} + S_{ji}}{2}\right) - \varepsilon Q_2 \leq 0; i < j = 1, \dots, r \\ P_1(G_{ij} + \beta_1 I) \succcurlyeq 0 \\ P_2(S_{ij} + \beta_2 I) \succcurlyeq 0 \\ L_i C_j \succcurlyeq 0 \\ h_i(z(t))h_j(z(t)) \neq 0, \forall (i,j), \end{cases}$$ (3.60)

where: $G_{ij} = A_i - B_i K_j$ and $S_{ij} = A_i - L_i C_j$ (3.61)

Then, the augmented system (3.58) is positive and globally exponentially stable.

Proof :

Consider the following Lyapunov function: $V(x) \doteq \tilde{x}^T P(\delta)\tilde{x}$; $P(\delta) = \begin{pmatrix} P_1 & 0 \\ 0 & \delta P_2 \end{pmatrix}$.

Applying Theorem 2.4 for the system (3.58), in order to have $\dot{V} < 0$, the following inequalities must hold :

$$M_{ii}^T P + P M_{ii} + (s - \varepsilon)Q < 0, \qquad i = 1, \dots, r$$ (3.62a)

$$\left(\frac{M_{ij} + M_{ji}}{2}\right)^T P + P\left(\frac{M_{ij} + M_{ji}}{2}\right) - \varepsilon Q \leq 0, \qquad i < j = 1, \dots, r$$ (3.62b)

Replacing M_{ij}, P and Q by their expressions, we get :

$$\begin{pmatrix} G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1 & P_1 L_i C_i \\ (L_i C_i)^T P_1 & \delta(S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2) \end{pmatrix} < 0$$ (3.63)

To get **(3.63)** < 0, using the Schur complement (Annex B.1), we have to verify that:

- $S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2 < 0$, which is verified by the conditions (3.60).

- $G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1 < 0$, which is verified by the conditions (3.60).
- $G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1 < \frac{1}{\delta} P_1 L_i C_i [S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2]^{-1} (L_i C_i)^T P_1$

From the last condition, we can get also:

$$\lambda_{max}(G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1)$$

$$< \frac{1}{\delta} \lambda_{min}(P_1 L_i C_i [S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2]^{-1} (L_i C_i)^T P_1)$$

$$\Rightarrow G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1 < 0,$$

which means that $\lambda(G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1) < 0$.

Then, we have:

$$\delta > \max_{i=1,\dots,r} \left(\frac{\lambda_{min}(P_1 L_i C_i [S_{ii}^T P_2 + P_2 S_{ii} + (s - \varepsilon)Q_2]^{-1} (L_i C_i)^T P_1)}{\lambda_{max}(G_{ii}^T P_1 + P_1 G_{ii} + (s - \varepsilon)Q_1)} \right) = \delta_1 \qquad (3.64)$$

For the second term (3.62b), we have

$$U \doteq \left(M_{ij} + M_{ji} \right)^T P + P \left(M_{ij} + M_{ji} \right) - 2\varepsilon Q$$

$$= \begin{pmatrix} \left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1 & P_1 (L_i C_j + L_j C_i) \\ (L_i C_j + L_j C_i)^T P_1 & \delta[\left(S_{ij} + S_{ji} \right)^T P_2 + P_2 \left(S_{ij} + S_{ji} \right) - 2\varepsilon Q_2] \end{pmatrix}$$

To get $U \leq 0$, we have to verify also that:

- $\left(S_{ij} + S_{ji} \right)^T P_2 + P_2 \left(S_{ij} + S_{ji} \right) - 2\varepsilon Q_2 \leq 0$, which is verified by (3.60).
- $\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1 \leq 0$, which is verified by (3.60).
- $\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1 \leq \frac{1}{\delta} P_1 (L_i C_j + L_j C_i)[\left(S_{ij} + S_{ji} \right)^T P_2 + P_2 \left(S_{ij} + S_{ji} \right) - 2\varepsilon Q_2]^{-1} (L_i C_j + L_j C_i)^T P_1$

$$\Rightarrow \quad \lambda_{max}[\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1] \leq \frac{1}{\delta} \lambda_{min}[P_1 (L_i C_j + L_j C_i)[\left(S_{ij} + S_{ji} \right)^T P_2 + P_2 \left(S_{ij} + S_{ji} \right) - 2\varepsilon Q_2]^{-1} (L_i C_j + L_j C_i)^T P_1].$$

From the last condition, we get also: $\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1 \leq 0$

Or equivalently, $\lambda[\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1] \leq 0$

Then,

$$\delta > \max_{i<j=1,\dots,r} \left(\frac{\lambda_{min}[P_1 (L_i C_j + L_j C_i)[\left(S_{ij} + S_{ji} \right)^T P_2 + P_2 \left(S_{ij} + S_{ji} \right) - 2\varepsilon Q_2]^{-1} (L_i C_j + L_j C_i)^T P_1]}{\lambda_{max}[\left(G_{ij} + G_{ji} \right)^T P_1 + P_1 \left(G_{ij} + G_{ji} \right) - 2\varepsilon Q_1]} \right) \doteq \delta_2$$

$$(3.65)$$

From (3.64) and (3.65), to have $\dot{V}(x) < 0$, it is required that: $\delta > \max(\delta_1, \delta_2)$.

We linearize the BMIs in (3.60) by using the congruence transformation by P_1^{-1} and the following variables changes:

$X_1 = P_1^{-1}$, $M_i = K_i X_1$, $Y_1 = P_1^{-1} Q_1 P_1^{-1}$ and $N_i = P_2 L_i$.

Then, we obtain the problem of Generalized Eigenvalues (GEVP) in X_1, Y_1, Q_2, P_2, N_i and ε and the variable to find are given by:

$P_1 = X_1^{-1}$, $Q_1 = P_1 Y_1 P_1$, $K_i = M_i P_1$ and $L_i = P_2^{-1} N_i$.

The positivity of the states of the system and their estimates requires the positivity of the augmented system $\tilde{x}(t) = [\hat{x}^T(t) \ e^T(t)]^T$.

We have then that $M_{ij} = \begin{pmatrix} A_i - B_i K_j & L_i C_j \\ 0 & A_i - L_i C_j \end{pmatrix}$ is Metzler.

$$\Rightarrow \begin{cases} A_i - B_i K_j \text{ is Metzler} \\ A_i - L_i C_j \text{ is Metzler} \\ L_i C_j \geqslant 0 \end{cases} \tag{3.66}$$

Using Definition 1.9, there exist positive scalars $\beta_1 \geq 0$ and $\beta_2 \geq 0$ such that:

$A_i - B_i K_j + \beta_1 I \geqslant 0$ and $A_i - L_i C_j + \beta_2 I \geqslant 0$.

By multiplying the first inequality by P_1 and the second by P_2, we get:

$P_1(G_{ij} + \beta_1 I) \geqslant 0$ and $P_2(S_{ij} + \beta_2 I) \geqslant 0$.

This ends the proof.■

3.3.1.2. Positive Interval Observer for Autonomous Positive T-S systems

In this section, we consider the positive observer design for the positive T-S system (3.52) by considering interval bounds of the system matrices $A_i \in [\underline{A}_i, \overline{A}_i]$ and $C \in [\underline{C}, \overline{C}]$ that fulfill: $\underline{A}_i \in \mathbb{R}^{n_x \times n_x}$ is Metzler and $\underline{C} \geqslant 0 \in \mathbb{R}^{n_y \times n_x}$.

We will adopt the following observer:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(z(t)) \left(G_i \hat{x}(t) + L_i y(t) \right) \tag{3.67}$$

where $G_i \in \mathbb{R}^{n_x \times n_x}$ and $L_i \in \mathbb{R}^{n_x \times n_y}$ are the observer matrices to be identified. The following assumption will be used in this section:

Assumption 3.1. The matrices $\overline{A}_i$ are Hurwitz, $i = 1, \dots, r$.

Define the error as $e(t) = x(t) - \hat{x}(t)$, then, the augmented observing system is given by:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) \begin{pmatrix} A_i & 0 \\ A_i - L_jC - G_j & G \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \tag{3.68}$$

In fact, the nonnegativity of the error $e(t)$ is compulsory for the design of interval observers. That is why, we have to design Metzler G_j and $L_j \geqslant 0$ of the positive observer (3.67) such that the augmented system (3.86) is positive and asymptotically stable for any $A_i \in [\underline{A}_i, \overline{A}_i]$ and $C \in [\underline{C}, \overline{C}]$.

The following theorem provides the conditions of synthesis of a positive interval observer for autonomous positive interval T-S systems.

Theorem 3.6. (Zaidi et al.)

There exists a positive observer of the form (3.67) for system (3.68) if and only if there exist matrices $P = diag(p_{11}, \dots, p_{nn}) > 0$, $Q = diag(q_{11}, \dots, q_{nn}) > 0$, $V_j \geqslant 0$, Metzler W_j, $j = 1, \dots, r$ such that the following LMIs hold:

$$\begin{pmatrix} P\overline{A}_i + \overline{A}_i^T P & \overline{A}_i^T Q - \underline{C}^T V_j^T - W_j^T \\ Q\overline{A}_i - V_j \underline{C} - W_j & W_j + W_j^T \end{pmatrix} < 0 \tag{3.69}$$

$$Q\underline{A}_i - V_j \overline{C} - W_j \geqslant 0 \tag{3.70}$$

Under these conditions, desired observer matrices can be obtained:

$$G_j = Q^{-1}W_j, \ L_j = Q^{-1}V_j \tag{3.71}$$

Proof:

Sufficiency:

From (3.69), we obtain that $W_j \neq 0$, $j = 1, \dots, r$. Therefore, the obtained G_j are Metzler, and $L_j \geqslant 0$, since Q^{-1} is diagonally strictly positive.

It follows from (3.70) and (3.71) that:

$$Q(\underline{A}_i - L_j\overline{C} - G_j) \geqslant 0 \text{ which implies: } \underline{A}_i - L_j\overline{C} - G_j \geqslant 0 \tag{3.72}$$

For any $A_i \in [\underline{A}_i, \overline{A}_i]$ and $C \in [\underline{C}, \overline{C}]$, it is obvious that

$$\underline{A}_i \leqslant A_i \leqslant \overline{A}_i \text{ and } L_j\underline{C} \leqslant L_jC \leqslant L_j\overline{C} \tag{3.73}$$

Combining (3.72) and (3.73) yields that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$ and $C \in [\underline{C}, \overline{C}]$,

$$A_i - L_jC - G_j \geqslant \underline{A}_i - L_j\overline{C} - G_j \geqslant 0 \tag{3.74}$$

which shows that the augmented system (3.68) is positive.

From (3.69) and (3.71), we have:

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \overline{A}_i & 0 \\ \overline{A}_i - L_j\underline{C} - G_j & G_j \end{pmatrix} + \begin{pmatrix} \overline{A}_i & 0 \\ \overline{A}_i - L_j\underline{C} - G_j & G_j \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} < 0 \qquad (3.75)$$

which implies:

$$\mu\left(\begin{bmatrix} \overline{A}_i & 0 \\ \overline{A}_i - L_j\underline{C} - G_j & G_j \end{bmatrix}\right) < 0 \qquad (3.76)$$

From (3.74), we obtain that, for any $A_i \in [\underline{A}_i , \overline{A}_i]$ and $C \in [\underline{C} , \overline{C}]$,

$$\begin{pmatrix} A_i & 0 \\ A_i - L_jC - G_j & G_j \end{pmatrix} \preccurlyeq \begin{pmatrix} \overline{A}_i & 0 \\ \overline{A}_i - L_j\underline{C} - G_j & G_j \end{pmatrix} \qquad (3.77)$$

Then, combining (3.75)-(3.77), we get that for any $A_i \in [\underline{A}_i , \overline{A}_i]$ and $C \in [\underline{C} , \overline{C}]$,

$$\mu\left(\begin{bmatrix} A_i & 0 \\ A_i - L_jC - G_j & G_j \end{bmatrix}\right) < 0 \qquad (3.78)$$

which means that the augmented system (3.68) is asymptotically stable, for any $A_i \in [\underline{A}_i , \overline{A}_i]$ and $C \in [\underline{C} , \overline{C}]$. This proves the sufficiency.

Necessity:

Suppose that there exist G_j and L_j such that the observer (3.67) is positive, i.e., G_j are Metzler and $L_j \geqslant 0$, and the augmented system (3.68) is positive and asymptotically stable, for any $A_i \in [\underline{A}_i , \overline{A}_i]$ and $C \in [\underline{C} , \overline{C}]$. Then, we get that there exist matrices $P = diag(p_1, \dots, p_n) > 0$ and $Q = diag(q_1, \dots, q_n) > 0$ such that: $\forall i,j = 1, \dots, r$, (3.75) is satisfied.

By setting: $V_j = QL_j$ and $W_j = QG_j$ \qquad (3.79)

We obviously obtain, due to the diagonal strict positivity of Q_j, that $V_j \geqslant 0$ and W_j is Metzler. Substituting (3.79) in (3.75), we further obtain (3.69).

Since the augmented system (3.68) is positive for any $A_i \in [\underline{A}_i , \overline{A}_i]$ and $C \in [\underline{C} , \overline{C}]$, we obtain that:

$$\begin{pmatrix} \underline{A}_i & 0 \\ \underline{A}_i - L_j\overline{C} - G_j & G_j \end{pmatrix}$$ is Metzler.

This implies that $\underline{A}_i - L\overline{C} - G \succcurlyeq 0$. By the positivity of Q and (3.79), we further have: $Q\underline{A}_i - V_j\overline{C} - W_j \succcurlyeq 0$, which is equivalent to (3.70), which proves the necessity. ∎

We propose now an approach to synthesize an observer-based state-feedback controller for positive T-S systems.

3.3.1.3. Positive Interval T-S Observer-based Controller (Second approach)

In this section, we consider the observer design for the interval positive T-S system with bounded matrices (3.52) where $A_i \in [\underline{A}_i\ ,\ \overline{A}_i]$, $B_i \in [\underline{B}_i\ ,\overline{B}_i]$ and $C_i \in [\underline{C}_i\ ,\overline{C}_i]$, $\forall\ i = 1, \dots, r$, that fulfill: $\underline{A}_i \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{B}_j \succcurlyeq 0 \in \mathbb{R}^{n_x \times n_u}$, $\underline{C}_j \succcurlyeq 0 \in \mathbb{R}^{n_y \times n_x}$.

Lemma 3.3. (Zaidi et al, 2015)

The interval T-S system (3.52) is positive and asymptotically stable if the following system is positive and asymptotically stable:

$$
\begin{cases}
\dot{x}(t) = \displaystyle\sum_{i=1}^{r} h_i(z(t))(\underline{A}_i x(t) + \underline{B}_j u(t)) \\
y(t) = \displaystyle\sum_{i=1}^{r} h_i(z(t))\underline{C}_j x(t)
\end{cases}
\tag{3.80}
$$

Proof:

Suppose that the system (3.80) is positive and asymptotically stable, which means that there exists a diagonal matrix $P > 0$ satisfying the following LMI:

$$P\underline{A}_i + \underline{A}_i^T P < 0\ , \forall\ i = 1, \dots, r \tag{3.81}$$

so that $\mu(\underline{A}_i) < 0$. We also have that $\underline{A}_i \preccurlyeq A_i$: as $\underline{A}_i$ is Metzler then A_i is Metzler and $\mu(\underline{A}_i) \leq \mu(A_i) < 0$, which means that there exists a diagonal matrix $P > 0$ satisfying: $PA_i + A_i^T P < 0, \forall\ i = 1, \dots, r$ (3.82)

So, the system is asymptotically stable.

Moreover, since $\underline{A}_i$ is Metzler, $\underline{B}_j \succcurlyeq 0$ and $C_i \succcurlyeq 0$, we deduce that $A_i \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $B_i \succcurlyeq 0$ and $C_i \succcurlyeq 0$, which means that system (3.52) is positive. This ends the proof. ∎

We consider the observer given in (3.67) and the state-feedback control law is

given by the following expression:

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big) K_i \hat{x}(t) \tag{3.83}$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$ are the controller matrices to be determined.

Then, combining (3.52) and (3.83) gives the closed-loop system:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) (A_i x(t) + B_i K_j \hat{x}(t)) \tag{3.84}$$

The error $e(t)$ is defined by $e(t) = x(t) - \hat{x}(t)$.

If we choose $[x^T(t) \quad e^T(t)]^T$ as the new augmented state variable, then the new augmented closed-loop system is:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i K_j - G_j & G_j - B_i K_j \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \tag{3.85}$$

The main objective is to guarantee the asymptotic stability of the augmented system and the nonnegativity of the state $x(t)$ and the error $e(t)$.

We provide a necessary condition for the existence of a continuous-time observer-based controller.

Theorem 3.8. (Zaidi et al, 2015)

If there exists a state-feedback controller (3.83) that stabilizes system (3.52), using the observer (3.67), with a positive augmented system (3.106), then the following inequalities with respect to Metzler G_j, $L_j \geqslant 0$ and $K_j \leqslant 0$, $j = 1, \dots, r$, have a solution: $\forall\, i,j = 1, \dots, r$

$$trace\big(\underline{A}_i + G_j + (\overline{B}_i - \underline{B}_i) K_j\big) < 0 , \tag{3.86}$$

$$[\overline{A}_i + \underline{B}_i K_j]_{lm} \geq 0 , 1 \leq l \neq m \leq n , \tag{3.87}$$

$$[G_j - \overline{B}_i K_j]_{lm} \geq 0 , 1 \leq l \neq m \leq n, \tag{3.88}$$

$$\overline{A}_i - L_j \underline{C}_j + \underline{B}_i K_j - G_j \geqslant 0 \tag{3.89}$$

Proof:

This proof is a parallel extension of the proof of Theorem 3.3.∎

In the following theorem, we provide sufficient conditions for the existence of an observer-based controller.

Theorem 3.9. (Zaidi et al, 2015)

There exists a solution to the problem of existence of a continuous-time observer-based controller (3.67) if, for a positive scalar ε, there exist matrices $P = diag[P_1 \; P_2] > 0$, a Metzler matrix G_j, $L_j \geqslant 0$ and $K_j \leqslant 0$, $j = 1, \ldots, r$, such that:
$\forall \; i, j = 1, \ldots, r$

$$\begin{pmatrix} \mathcal{A}_i^T P + P\mathcal{A}_i - \varepsilon\mathcal{B}_i\mathcal{B}_i^T P - \varepsilon P\mathcal{B}_i^T\mathcal{B}_i & P\mathcal{B}_i + \mathcal{C}_i^T\mathcal{K}_j^T \\ * & -I \end{pmatrix} < 0 \tag{3.90}$$

$$[\underline{A}_i + \overline{B}_i K_j]_{lm} \geq 0 \,, 1 \leq l \neq m \leq n \tag{3.91}$$

$$[G_j - \underline{B}_j K_j]_{lm} \geq 0 \,, 1 \leq l \neq m \leq n \tag{3.92}$$

$$\underline{A}_i - L_j\overline{C}_i + \overline{B}_i K_j - G_j \geqslant 0, \tag{3.93}$$

where:

$$\mathcal{A} = \begin{pmatrix} \overline{A}_i & 0 \\ \overline{A}_i & 0 \end{pmatrix}, \; \mathcal{B} = \begin{pmatrix} 0 & \underline{B}_i - \overline{B}_i & 0 & \underline{B}_i \\ 0 & \underline{B}_i - \overline{B}_i & -I & \underline{B}_i \end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix} G_j & L_j & 0 & 0 \\ K_j & 0 & 0 & 0 \\ 0 & 0 & G_j & L_j \\ 0 & 0 & K_j & 0 \end{pmatrix}, \; \mathcal{C} = \begin{pmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C}_i & 0 \end{pmatrix} \tag{3.94}$$

Proof:

This proof is a parallel extension of the proof of Theorem 3.4.∎

3.3.2. OBSERVER DESIGN OF POSITIVE T-S SYSTEMS WITH UNMEASURABLE PREMISE VARIABLES

In the following, we consider the synthesis problem of positive T-S systems with unmeasurable premise variables, allowing the development of a system model with activation functions that depend on the state of the system. Recently, observer-based-control design for T-S models has attracted much attention because it leads to a suitable solution for the control of complex systems that have unmeasurable state variables (Bergsten & Palm, 2000), (Kau et al, 2007), (Sala & Arino, 2007), (Ting, 2005), (Zhang et al, 2008) and (Chen & Saif, 2007).

To our knowledge, the T-S with UDV structure is not well studied in the literature, which motivates our work in this direction, while having an idea on its features and advantages once compared to the T-S structure with measurable premise variables. We develop, in this section, estimation methods for positive T-S systems with unmeasurable premise variables. The first method is based on the $\mathcal{L}_2$ performance and rewriting the system as a positive uncertain one. Then, we develop an approach to design positive interval observer-based controllers.

3.3.2.1. Positive L_2 observer design for positive T-S systems

We consider the class of positive T-S continuous-time models given in (3.52).

We suppose that $A_i = [a_{jk}] \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $B_i = [b_{jk}] \in \mathbb{R}^{n_x \times n_u} \geqslant 0$ and $C = [c_{jk}] \in \mathbb{R}^{n_y \times n_x} \geqslant 0$ are given system matrices.

As a result, we assume that the fuzzy weighting functions depend on the estimated state $h_i(\hat{x}(t))$. The T-S model with unmeasurable variables (3.52) can be reduced to:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(\hat{x}(t))\,(A_i x(t) + B_i\,u(t)) + w(t) \\ y(t) = C x(t) \end{cases} \tag{3.95}$$

where:

$$w(t) = \sum_{i=1}^{r} (h_i(x(t)) - h_i(\hat{x}(t)))\,(A_i x(t) + B_i u(t)) \tag{3.96}$$

Using Lemma 2.4, we have that:

$$\sum_{i=1}^{r} (h_i(x) - h_i(\hat{x})) X_i = \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \Delta X_{ij} \tag{3.97}$$

where $X_i \in \{A_i, B_i, C_i\}$ and ΔX_{ij} is defined by:

$$\Delta X_{ij} = X_i - X_j \tag{3.98}$$

Then, system (3.95) can be transformed into the following system:

$$\begin{cases} \dot{x}(t) = \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \left((A_j + \Delta A_{ij}) x(t) + (B_j + \Delta B_{ij}) u(t) \right) \\ y(t) = C\,x(t) \end{cases} \tag{3.99}$$

where ΔX_{ij} are known constant matrices defined in (3.98).

Our aim is to design a positive observer which guarantees simultaneously the state estimation and the nonnegativity of their estimates.

We consider the T-S observer represented as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i\left(\hat{x}(t)\right)\left(A_i\hat{x}(t) + B_i u(t) + L_i\left(y(t) - \hat{y}(t)\right)\right) \\ \hat{y}(t) = C\,\hat{x}(t) \end{cases} \tag{3.100}$$

where:

$\hat{x}(t)$ and $\hat{y}(t)$ denote the estimations of $x(t)$ and $y(t)$, respectively,

$L_i \in \mathbb{R}^{n_x \times n_y}$ are the observer gains to be determined,

By considering the state estimation error as: $e(t) = x(t) - \hat{x}(t)$, the dynamics of the state estimation error become:

$$\dot{e}(t) = \sum_{i,j=1}^{r} h_i(x)h_j(\hat{x})\left(\left(A_j - L_jC\right)e(t) + \Delta A_{ij}\,x(t) + \Delta B_{ij}\,u(t)\right) \tag{3.101}$$

We define the augmented state $\bar{\bar{x}} = [e^T \ x^T]^T$. Using the expressions in (3.100) and (3.101), the dynamics of the augmented state are given by the following augmented system:

$$\dot{\bar{\bar{x}}}(t) = \sum_{i,j=1}^{r} h_i(x)h_j(\hat{x})(R_{ij}\,\bar{\bar{x}}(t) + S_{ij}\,u(t)) \tag{3.102}$$

$$\partial(t) = H\,\bar{\bar{x}}(t) \tag{3.103}$$

where:

$$R_{ij} = \begin{bmatrix} A_j - L_jC & \Delta A_{ij} \\ 0 & A_j + \Delta A_{ij} \end{bmatrix}, S_{ij} = \begin{bmatrix} \Delta B_{ij} \\ B_j + \Delta B_{ij} \end{bmatrix}, \text{ and } H = [I \quad 0] \tag{3.104}$$

Thus, the objective of this study is to determine the observer gains L_i for the augmented model (3.102)-(3.104) such that the T-S augmented system (3.102) is asymptotically stable and positive, while attenuating the effect of the input $u(t)$ on $\partial(t)$.

Theorem 3.10. (Zaidi et al, 2013c)

The system (3.121) is stable and positive and the L_2 gain from $u(t)$ to $\partial(t)$ is bounded, if there exist symmetric matrices $P_1 > 0$ and $P_2 > 0$, matrices K_j and a nonnegative scalar γ, such that the following conditions hold:

$$\begin{bmatrix} X_{1j} & W_{ij} & Y_{ij} \\ W_{ij}^T & X_{2ij} & Z_{ij} \\ Y_{ij}^T & Z_{ij}^T & -\gamma^2 I \end{bmatrix} < 0 \,, \qquad \forall i,j = 1,\dots,r \tag{3.105}$$

where

$$X_{1j} = A_j^T P_1 + P_1 A_j - K_j C - C^T K_j^T + I \tag{3.106}$$

$$X_{2ij} = (A_j + \Delta A_{ij})^T P_2 + P_2 (A_j + \Delta A_{ij}) \tag{3.107}$$

$$W_{ij} = P_1 \Delta A_{ij} \tag{3.108}$$

$$Y_{ij} = P_1 \Delta B_{ij} \tag{3.109}$$

$$Z_{ij} = P_2 (B_j + \Delta B_{ij}) \tag{3.110}$$

$$\begin{cases} A_i(x,y) + \Delta A_{ij}(x,y) \geqslant 0 & , \forall\ 1 \leq x \neq y \leq n_x \\ A_i(x,y) - \displaystyle\sum_{w=1}^{n_y} L_i(x,w) C(w,y) \geqslant 0 & , \forall\ 1 \leq x \neq y \leq n_x \\ \displaystyle\sum_{w=1}^{n_y} L_i(x,w) C(w,y) \geqslant 0 & , \forall\ 1 \leq x,y \leq n_x \\ B_i(x,z) \geqslant 0 & , \forall\ 1 \leq x \leq n_x \ , \forall\ 1 \leq z \leq n_u \\ B_i(x,z) + \Delta B_{ij}(x,z) \geqslant 0 \end{cases} \tag{3.111}$$

and gains of the observer are derived from

$$L_j = P_1^{-1} K_j \tag{3.112}$$

The guaranteed attenuation level is γ.

Proof :

In order to make the augmented model (3.102) asymptotically stable, let us consider the Lyapunov function:

$$V(\bar{\bar{x}}(t)) = \bar{\bar{x}}(t)^T P \bar{\bar{x}}(t) \,, \qquad P = P^T \tag{3.113}$$

Its derivative with respect to time is given by:

$$\dot{V}(\bar{\bar{x}}(t)) = \dot{\bar{\bar{x}}}(t)^T P \bar{\bar{x}}(t) + \bar{\bar{x}}(t)^T P \dot{\bar{\bar{x}}}(t) \tag{3.114}$$

By substituting $\dot{\bar{\bar{x}}}(t)$ of (3.102) in (3.114), we obtain:

$$\dot{V}(\bar{\bar{x}}(t)) = \sum_{i,j=1}^{r} h_i(x(t)) h_j(\hat{x}(t))$$

$$\left(\bar{\bar{x}}(t)^T \left(R_{ij}^T P + P R_{ij} \right) \bar{\bar{x}}(t) + \bar{\bar{x}}(t)^T P S_{ij} u(t) + u(t)^T S_{ij}^T P \bar{\bar{x}}(t) \right) \tag{3.115}$$

Our goal is to attenuate the effect of the input $u(t)$ on $\partial(t)$. Thus, in order to guarantee the stability of (3.102) and the boundedness of the transfer from $u(t)$ to $\partial(t)$:

$$\frac{\|\partial(t)\|_2}{\|u(t)\|_2} < \gamma, \quad \|u(t)\|_2 \neq 0, \quad \gamma > 0, \tag{3.116}$$

Consider the following criterion:

$$\dot{V}(\bar{\bar{x}}(t)) + \partial(t)^T\partial(t) - \gamma^2 u(t)^T u(t) < 0 \tag{3.117}$$

Substituting (3.115) and (3.103) in (3.117), we obtain:

$$\sum_{i,j=1}^{r} h_i(x)h_j(\hat{x}) \left(\bar{x}(t)^T \left(R_{ij}{}^T P + PR_{ij} \right) \bar{x}(t) + \bar{x}(t)^T PS_{ij}u(t) + u(t)^T S_{ij}{}^T P\bar{x}(t) \right)$$
$$+\bar{\bar{x}}(t)^T H^T H \, \bar{\bar{x}}(t) - \gamma^2 u(t)^T u(t) < 0 \tag{3.118}$$

Taking into account the convex sum property of the weighting functions, we are allowed to write:

$$\sum_{i,j=1}^{r} h_i(x)h_j(\hat{x}) \, (\bar{\bar{x}}(t)^T \left(R_{ij}{}^T P + PR_{ij} \right) \bar{\bar{x}}(t)$$
$$+\bar{\bar{x}}(t)^T PS_{ij}u(t) + u(t)^T S_{ij}{}^T P\bar{\bar{x}}(t) + \bar{\bar{x}}(t)^T H^T H \, \bar{\bar{x}}(t) - \gamma^2 u(t)^T u(t)) < 0 \tag{3.119}$$

which can be reformulated in the following way :

$$\sum_{i,j=1}^{r} h_i(x)h_j(\hat{x}) \, \tilde{s}(t)^T T_{ij}\tilde{s}(t) < 0 \tag{3.120}$$

where

$$T_{ij} = \begin{bmatrix} R_{ij}{}^T P + PR_{ij} + H^T H & PS_{ij} \\ S_{ij}{}^T P & -\gamma^2 I \end{bmatrix}, \, \tilde{s}(t) = \begin{bmatrix} \bar{\bar{x}}(t) \\ u(t) \end{bmatrix}$$

A sufficient condition for (3.119) to hold is, that $\forall \, i,j = 1, \dots, r$

$$\begin{bmatrix} R_{ij}{}^T P + PR_{ij} + H^T H & PS_{ij} \\ S_{ij}{}^T P & -\gamma^2 I \end{bmatrix} < 0 \tag{3.121}$$

In order to facilitate the calculation of the observer gains, the matrix variable $(P > 0)$ is chosen to be diagonal with respect to appropriate matrix blocks:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{3.122}$$

Using the definitions of R_{ij} and S_{ij} given in (3.104), and the change of variables:

$$K_j = P_1 L_j \tag{3.123}$$

We obtain then the LMI conditions expressed in (3.105).

In order to prove the positivity of the system, we resort to the following augmented system:

$$\dot{\aleph}(t) = \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \left(\bar{A}_{ij} \, \aleph(t) + \bar{B}_{ij} u(t) \right) \tag{3.124}$$

where :

$$\aleph(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} , \bar{A}_{ij} = \begin{bmatrix} A_j + \Delta A_{ij} & 0 \\ L_j C & A_j - L_j C \end{bmatrix} \text{ and } \bar{B}_{ij} = \begin{bmatrix} B_j + \Delta B_{ij} \\ B_j \end{bmatrix} \tag{3.125}$$

We have to prove that: $\forall \, i,j = 1, \dots, r$

$$\begin{cases} \bar{A}_{ij} \text{ is Metzler} \\ \bar{B}_{ij} > 0 \end{cases} \tag{3.126}$$

which means that: $\forall \, i,j = 1, \dots, r$

$$\begin{cases} A_j + \Delta A_{ij} \text{ is Metzler} \\ A_j - L_j C \quad \text{is Metzler} \\ L_j C \succcurlyeq 0 \\ B_j \succcurlyeq 0 \\ B_j + \Delta B_{ij} \succcurlyeq 0 \end{cases} \tag{3.127}$$

which leads to the following inequalities:

$$\begin{cases} A_i(x,y) + \Delta A_{ij}(x,y) \succcurlyeq 0 & , \forall \, 1 \le x \ne y \le n_x \\ A_i(x,y) - \sum_{w=1}^{n_y} L_i(x,w) C(w,y) \succcurlyeq 0 & , \forall \, 1 \le x \ne y \le n_x \\ \sum_{w=1}^{n_y} L_i(x,w) C(w,y) \succcurlyeq 0 & , \forall \, 1 \le x, y \le n_x \\ B_i(x,z) \succcurlyeq 0 & , \forall \, 1 \le x \le n_x \ , \forall \, 1 \le z \le n_u \\ B_i(x,z) + \Delta B_{ij}(x,z) \succcurlyeq 0 \end{cases} \tag{3.128}$$

This completes the proof.■

3.3.2.2. Illustrative examples

- **Example 3.1**

Let us consider the T-S system (3.52) where $r = 2$ and the matrices of the system are defined by:

$$A_1 = \begin{bmatrix} 5 & 6 & 5.5 \\ 2 & 4.5 & 4 \\ 3.5 & 5 & 4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2.5 & 2 & 3 \\ 2 & 1 & 1.5 \\ 4 & 2.5 & 3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}, \ B_2 = \begin{bmatrix} 2 \\ 1.5 \\ 0.5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The membership functions are:

$$M^1 = \frac{1 - \tanh(x_1(t))}{2}, M^2 = \frac{1 + \tanh(x_1(t))}{2} \tag{3.129}$$

A solution of the LMIs of Theorem 3.10. with $\gamma = 1.5$, gives the following Lyapunov matrices:

$$P_1 = \begin{bmatrix} 0.8370 & 0.0056 & 0.1423 \\ 0.0056 & 0.1423 & 0.2513 \\ 0.1423 & 0.2513 & 0.7123 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.5321 & 0.0214 & 0.1512 \\ 0.0214 & 0.1512 & 0.0431 \\ 0.1512 & 0.0431 & 0.2843 \end{bmatrix}$$

From the solution of the LMIs of the Theorem 3.10, the controller gains K_1 and K_2, and the observer ones L_1 and L_2 can be calculated:

$$K_1 = \begin{bmatrix} -1.8553 & 5.5081 \\ 0.4236 & 0.7834 \\ -0.1475 & 3.6641 \end{bmatrix}, \ K_2 = \begin{bmatrix} -2.2407 & 3.6933 \\ -0.1417 & 0.8102 \\ -1.4583 & 3.3784 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} -1.854 & 5.642 \\ 7.325 & -4.802 \\ -2.421 & 5.711 \end{bmatrix}, \ L_2 = \begin{bmatrix} -2.125 & 3.501 \\ 5.182 & -4.205 \\ -3.451 & 5.527 \end{bmatrix}$$

Figure 3.4 shows the evolution of the Lyapunov functions, starting from the initial condition $x(0) = [0.4 \ 0.4 \ 0.4]^T$.
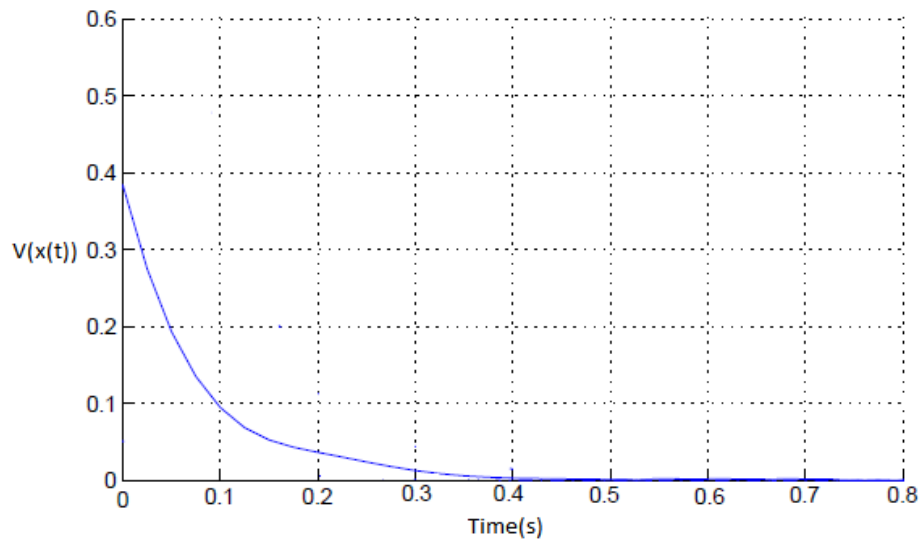
Figure 3. 4. Evolution of the Lyapunov function $V(x(t))$ in Example 3.1
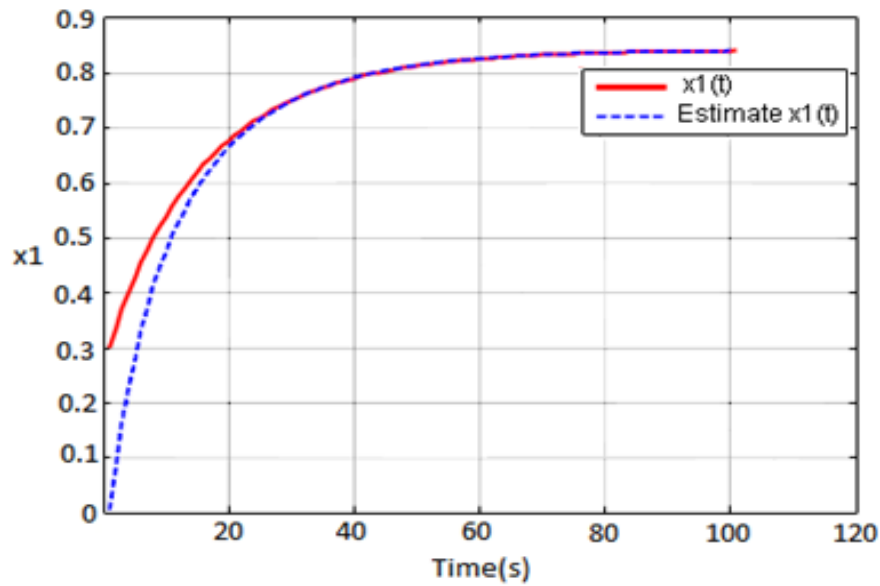
Figure 3. 5. Evolution of the state $x_1(t)$ and its estimation $\hat{x}_1(t)$ in Example 3.1
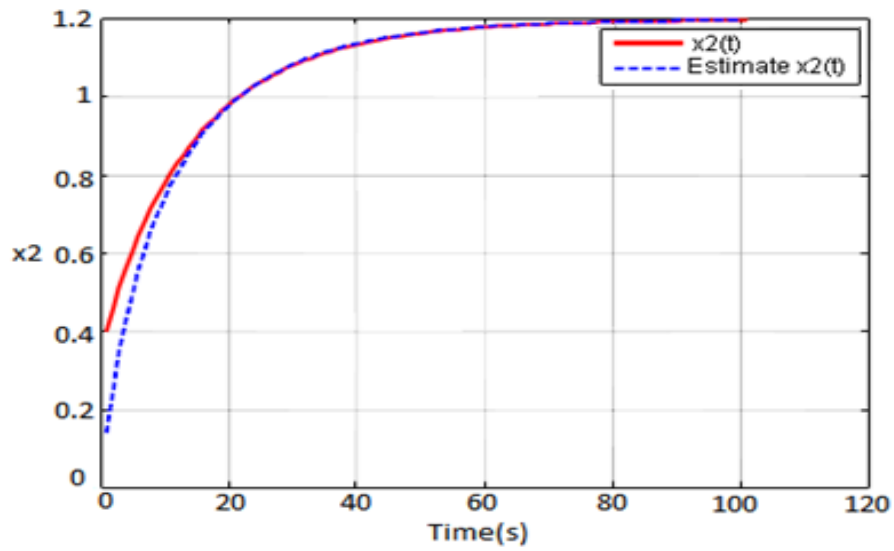
Figure 3. 6. Evolution of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$ in Example 3.1
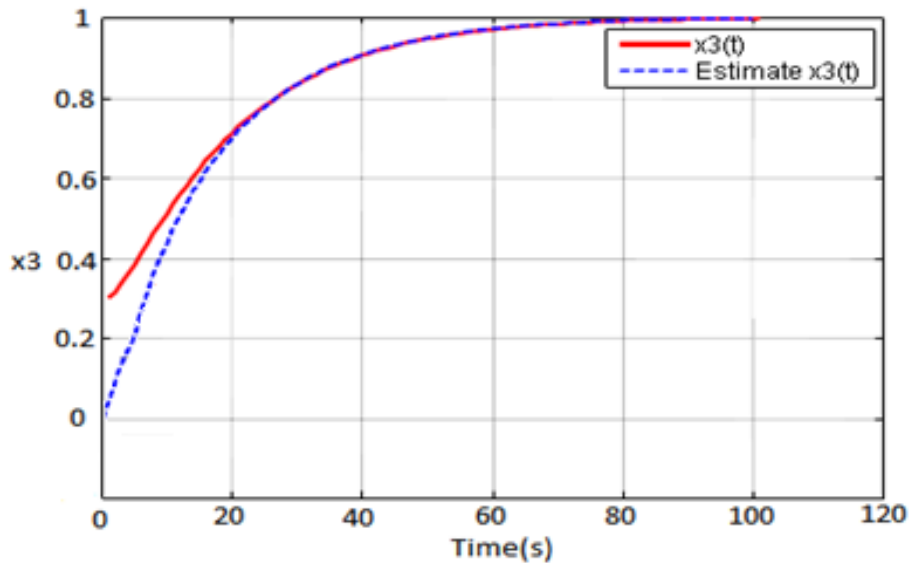
Figure 3. 7. Evolution of the state $x_3(t)$ and its estimation $\hat{x}_3(t)$ in Example 3.1
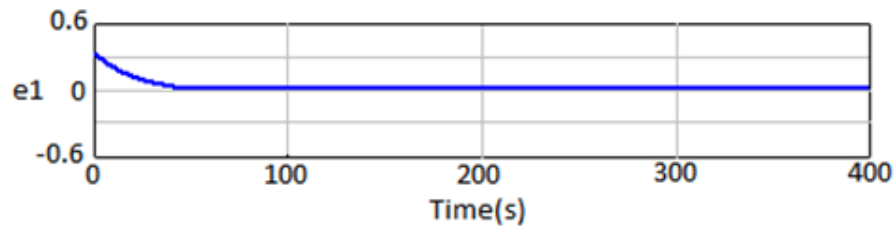
Figure 3. 8. Evolution of the estimation error $e_1(t)$ for $x(0) = [0.3\ 0.4\ 0.3]^T$ in Example 3.1

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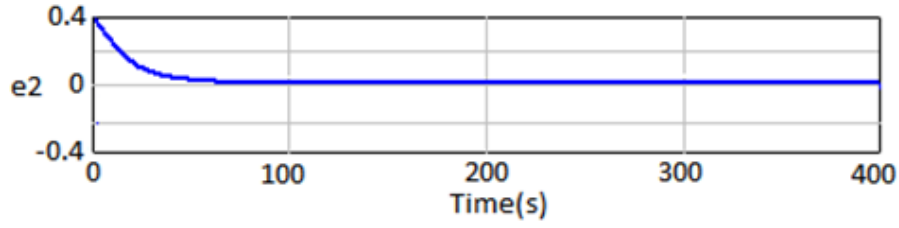
Figure 3. 9. Evolution of the estimation error $e_2(t)$ for $x(0) = [0.3\ 0.4\ 0.3]^T$ in Example 3.1
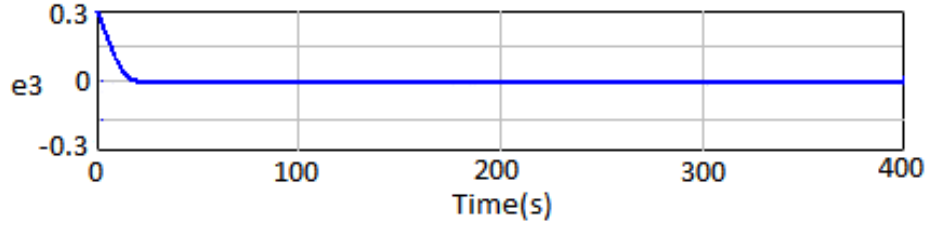
Figure 3. 10. Evolution of the estimation error $e_3(t)$ for $x(0) = [0.3\ 0.4\ 0.3]^T$ in Example 3.1

From Figures 3.5, 3.6 and 3.7, that represent the evolution of the system states and their estimates, we can see that the states and their estimations are always nonnegative, which shows the effectiveness of the proposed approach. In addition, by examining the trajectories of the estimation errors in figures 3.8, 3.9 and 3.10, we can observe the rapid convergence, showing the performance of the designed observer.

- Example 3.2 : Positive electrical circuit

Consider the electrical circuit shown on Figure 3.11. (Kaczorek, 2012) with known values for the resistances R_1, R_2 and R_3, inductances L_1, L_2 and L_3 and voltage sources v_1 and v_2.
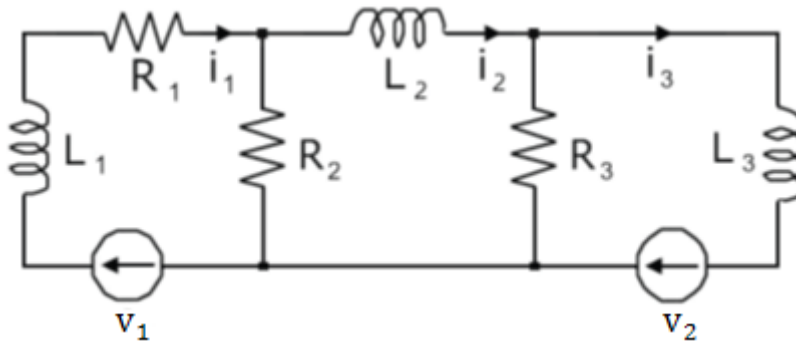
Figure 3. 11. Electrical Circuit (Kaczorek, 2012)

Using the Kirchhoff laws, we obtain the following equations that represent the system:

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3.130}$$

where:

$$A = \begin{bmatrix} -\dfrac{R_1 + R_2}{L_1} & \dfrac{R_2}{L_1} & 0 \\[2ex] \dfrac{R_2}{L_2} & -\dfrac{R_2 + R_3}{L_2} & \dfrac{R_3}{L_2} \\[2ex] 0 & \dfrac{R_3}{L_3} & -\dfrac{R_3}{L_3} \end{bmatrix}, B = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\[2ex] 0 & 0 \\[2ex] 0 & \dfrac{1}{L_3} \end{bmatrix} \tag{3.131}$$

This electrical circuit is then a positive system for all values of R_1, R_2 and R_3 and all nonzero L_1, L_2 and L_3, since A is a Metzler matrix and $B \geqslant 0$. Thus, we suppose in this example that R_1, R_2, R_3, L_1, L_2 and L_3 are positive, so $\det A \neq 0$ and the system is stable.

We suppose that:

❏ $R_1 = R_2 = 30 \, \Omega$ $(RL.\,1)$

❏ $L_1 = L_2 = L_3 = 4 \, H$ $(RL.\,2)$

We establish the following relationship between the resistance R_3 and the temperature:

$$R_3(\theta) = R_0\big(1 + \alpha(\theta - \theta_{ini})\big) \tag{3.132}$$

We fix the following values from the literature:

$R_0 = 40 \, \Omega$

$\alpha = 4.\,10^{-3}[K^{-1}]$

$\theta_{ini} = 20^{\circ} \, C$

It is assumed that the operating temperature of R_3 varies from $50^{\circ}C$ to $80^{\circ}C$, so:

$R_3(50^{\circ}) = 44.8 \, \Omega$

$R_3(80^{\circ}) = 49.6 \, \Omega$

Replacing the values of parameters in (3.131), we get a T-S model defined by the following matrices:

- For $R_3(50º) = 44.8\,\Omega$,

$$A_1 = \begin{bmatrix} -19.9 & 7.5 & 0 \\ 7.5 & -19.9 & 12.4 \\ 0 & 12.4 & -12.4 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 0.25 \end{bmatrix},$$

- For $R_3(80º) = 49.6\,\Omega$,

$$A_2 = \begin{bmatrix} -18.7 & 7.5 & 0 \\ 7.5 & -18.7 & 11.2 \\ 0 & 11.2 & -11.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 0.25 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Using the Theorem 3.10, for $\gamma = 2.12$, we obtain the following controller and observer gains:

$$K_1 = \begin{bmatrix} -2.129 & 1.085 & 2.392 \\ 1.782 & 1.725 & 1.965 \end{bmatrix}, \; K_2 = \begin{bmatrix} -2.129 & 1.085 & 2.392 \\ 1.782 & 1.725 & 1.965 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} -1.712 & 7.523 \\ 8.124 & -3.521 \\ -2.725 & 8.252 \end{bmatrix}, \; L_2 = \begin{bmatrix} -3.219 & 8.315 \\ 9.104 & -4.725 \\ -1.627 & 7.819 \end{bmatrix}$$
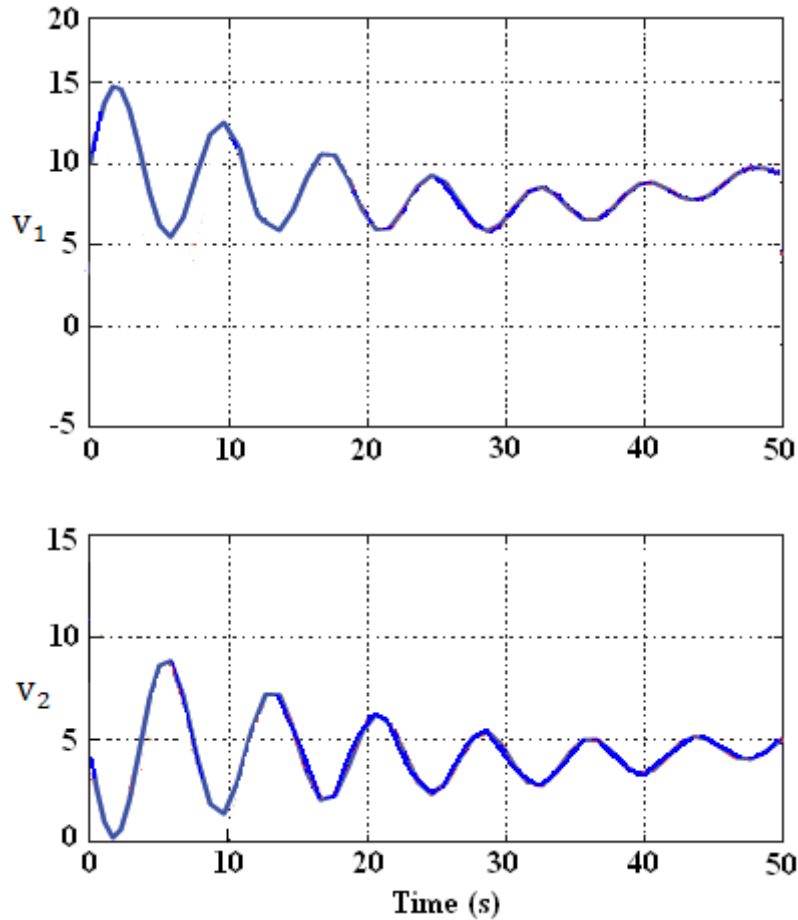
Figure 3. 12. Evolution of the voltage sources v_1 and v_2 in Example 3.2

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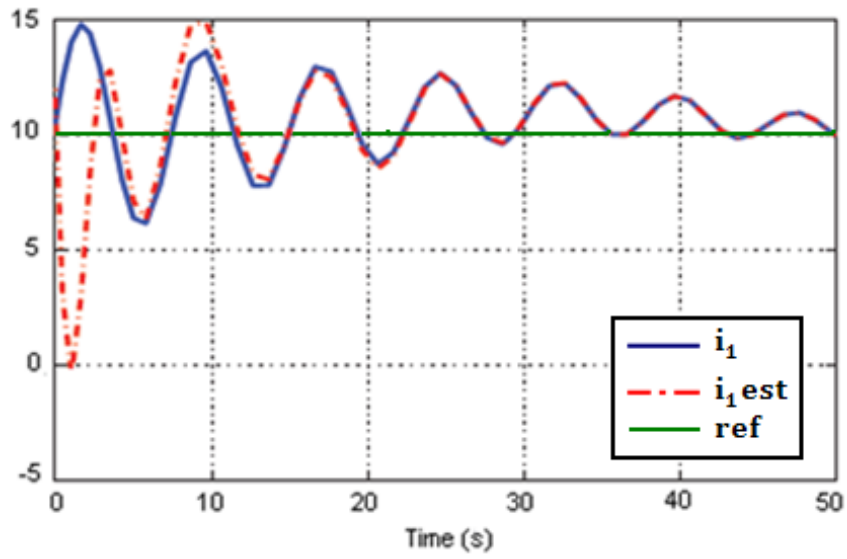
Figure 3. 13. The evolution of the current $i_1(t)$ in Example 3.2
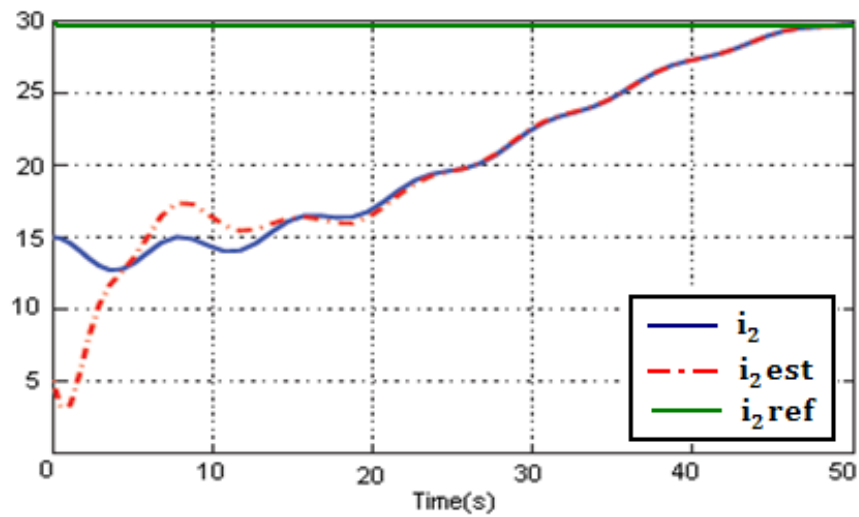
Figure 3. 14. The evolution of the current $i_2(t)$ in Example 3.2
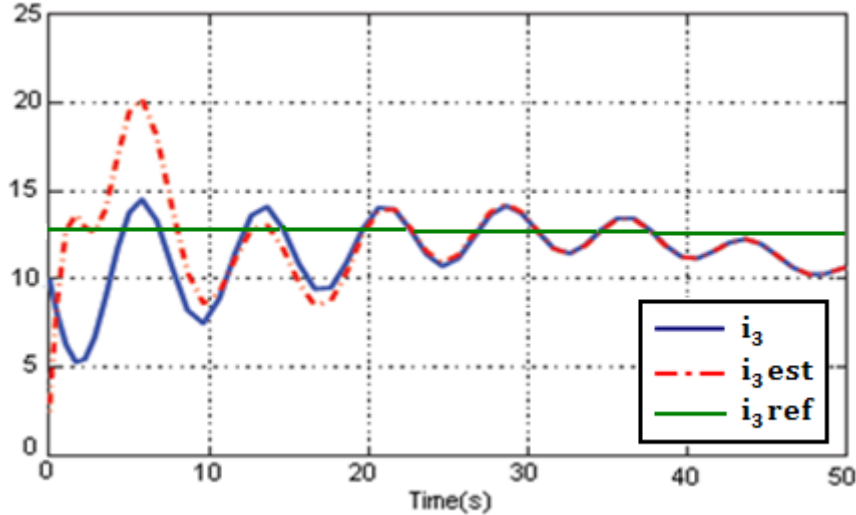
Figure 3. 15. The evolution of the current $i_3(t)$ in Example 3.2

The proposed system (3.130)-(3.131) was simulated, using the observer given by (3.100), in an ambient temperature equal to 61ºC, taking into account the values of the resistances and inductances given in (RL.1) and (RL.2). The voltage sources $v_1(t)$ and $v_2(t)$ are illustrated in Figure 3.12. Figures 3.13, 3.14 and 3.15 plot the evolution of the currents of the circuit $i_1(t)$, $i_2(t)$ and $i_3(t)$, their estimations and their references: we can see that the T-S system and the T-S observer are stable. Added to that, the inputs of the system, the states and their estimations always remain nonnegative, which proves the effectiveness of the proposed approach.

We calculate, from the simulation, the value of the relative attenuation level

$$\gamma = \frac{\|e(t)\|_2}{\|v(t)\|_2} = \frac{\sqrt{|i_1 - \hat{\imath}_1|^2 + |i_2 - \hat{\imath}_2|^2 + |i_3 - \hat{\imath}_3|^2}}{\sqrt{|v_1|^2 + |v_2|^2}} = 2.106.$$

3.3.2.3. Positive Interval Observer-Based Controller design for Positive T-S systems with unmeasurable premise variables

In this section, we propose an approach of designing a positive interval observer-based controller, where the premise variables are unmeasurable. We provide a necessary condition for the existence of solutions to the problem of existence of an observer-based controller in this case. Otherwise, we further study sufficient conditions and the corresponding synthesis approach for this problem.

After mathematical manipulation, we are allowed to transform system (3.52) into the following system (Zaidi et al, 2013c):

$$\begin{cases} \dot{x}(t) = \sum_{i,j=1}^{r} h_i(x(t))h_j(\hat{x}(t))\big((A_i + \Delta A_{ij})x(t) + (B_i + \Delta B_{ij})u(t)\big) \\ y(t) = C\,x(t) \end{cases} \tag{3.133}$$

where $\Delta X_{ij} = X_i - X_j$, with $X_i = \{A_i, A_{1i}, B_i\}$.

The overall observer-based controller under consideration is of the form:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\hat{x}(t))(G_i\,\hat{x}(t) + L_i y(t)) \tag{3.134}$$

where $G_i \in \mathbb{R}^{n_x \times n_x}$ and $L_i \in \mathbb{R}^{n_x \times n_y}$, $i = 1, \dots, r$, are the observer matrices to be determined.

The static state-feedback control law is given by the following expression:

$$u(t) = \sum_{i=1}^{r} h_i(\hat{x}(t))K_i\hat{x}(t) \tag{3.135}$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$ are the controller matrices to be determined.

Then, the closed-loop system is written as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i,j,k=1}^{r} h_i(x)h_j(\hat{x})h_k(\hat{x})\big((A_i + \Delta A_{ij})x(t) + (B + \Delta B_{ij})K_k\hat{x}(t)\big) \\ y(t) = C\,x(t) \end{cases} \tag{3.136}$$

Choosing $[x^T(t) \ \ e^T(t)]^T$ as the new augmented state variable, the new augmented system is:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \sum_{i,j,k=1}^{r} h_i(x(t))\, h_j(\hat{x}(t))h_k(\hat{x}(t))A_x^{ijk} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \tag{3.137}$$

where

$$A_x^{ijk} = \begin{pmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k & -(B_i + \Delta B_{ij})K_k \\ A_i + \Delta A_{ij} - L_iC + (B_i + \Delta B_{ij})K_k - G_i & G_i - (B_i + \Delta B_{ij})K_k \end{pmatrix} \tag{3.138}$$

Theorem 3.11. (Zaidi et al, 2014b)

For a given positive T-S system with unmeasurable premise variables in (3.133), if there exist diagonal matrices X_1, X_2, Q_1 and Q_2, matrices W_{1i}, W_{2i}, W_{11i}, W_{12i}, L_i, K_k, $\forall i,j,k = 1, \dots, r$ and a scalar $\sigma > 0$ such that :

$$\begin{pmatrix} S_{ijk} & T_{ijk} \\ * & U_{ijk} \end{pmatrix} < 0 \tag{3.139}$$

$$\begin{pmatrix} -Q + \gamma^2 I & P \\ P & -I \end{pmatrix} < 0 \tag{3.140}$$

and

$$\begin{pmatrix} L_{ijk} & M_{ijk} \\ N_{ijk} & O_{ijk} \end{pmatrix} \geqslant 0, \tag{3.141}$$

where

$S_{ijk} = (A_i + \Delta A_{ij})X_1 + X_1(A_i + \Delta A_{ij})^T + (B_i + \Delta B_{ij})Y_{1k} + Y_{1k}{}^T(B_i + \Delta B_{ij})^T + R_1$

$T_{ijk} = X_1(A_i + \Delta A_{ij})^T - (B_i + \Delta B_{ij})Y_{2k} - C^T Z_i{}^T + Y_{1k}{}^T(B_i + \Delta B_{ij})^T - W_{1i}{}^T$

$U_{ijk} = -(B_i + \Delta B_{ij})Y_{2k} - Y_{2k}{}^T(B_i + \Delta B_{ij})^T + R_1$

$L_{ijk} = (A_i + \Delta A_{ij})X_1 + (B_i + \Delta B_{ij})Y_{1k} + \sigma X_1$

$M_{ijk} = -(B_i + \Delta B_{ij})Y_{2k}$

$N_{ijk} = (A_i + \Delta A_{ij})X_1 - Z_i C + (B_i + \Delta B_{ij})Y_{1k} - W_{1i}$

$O_{ijk} = -(B_i + \Delta B_{ij})Y_{2k} + \sigma X_2$

then, the augmented system (3.137) is asymptotically stable, while remaining positive.

Under these conditions, the observer and controller gain matrices may be obtained from

$K_k = Y_{1k}X_1^{-1} = Y_{2k}X_2^{-1}, L_i = Z_i V_1^{-1}, G_i = W_{1i}X_1^{-1},$ *where V_1 fulfills $CX_1 = V_1 C$.*

Proof:

Assume that there exist, $\forall i, k = 1, \dots, r$, G_i, L_i and K_k such that (3.137) is positive and asymptotically stable, then, there exist diagonal matrices P and Q in the following forms $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} > 0$, $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} > 0$ and a positive scalar γ, such that the following LMIs hold :

$$PA_x^{ijk} + A_x^{ijk^T}P + Q < 0, \qquad i, j, k = 1, \dots, r \tag{3.142}$$

$$\begin{pmatrix} -Q + \gamma^2 I & P \\ P & -I \end{pmatrix} < 0 \tag{3.143}$$

Multiplying each LMI on the left by $diag(P^{-1}, P^{-1})$, we get:

$$A_x^{ijk} + P^{-1}A_x^{ijk^T}P + P^{-1}Q < 0 \qquad (3.144)$$

Then, multiplying on the right by $diag(P^{-1}, P^{-1})$, we obtain:

$$A_x^{ijk}P^{-1} + P^{-1}A_x^{ijk^T} + P^{-1}QP^{-1} < 0 \qquad (3.145)$$

Taking the following change of variables: $X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$, $Y_{1k} = K_k X_1$, $Y_{2k} = K_k X_2$ and $V_1 C = C X_1$, leads to the following LMIs:

$$\begin{pmatrix} \mathfrak{A}^{ijk} & \mathfrak{B}^{ijk} \\ * & \mathfrak{C}^{ijk} \end{pmatrix} < 0 \qquad (3.146)$$

where

$\mathfrak{A}^{ijk} = (A_i + \Delta A_{ij})X_1 + X_1(A_i + \Delta A_{ij})^T + (B_i + \Delta B_{ij})Y_{1k} + Y_{1k}{}^T(B_i + \Delta B_{ij})^T + X_1 Q_1 X_1$,

$\mathfrak{B}^{ijk} = X_1(A_i + \Delta A_{ij})^T - (B_i + \Delta B_{ij})Y_{2k} - C^T V_1{}^T L_i{}^T + Y_{1k}{}^T(B_i + \Delta B_{ij})^T - X_1 G_i{}^T$,

$\mathfrak{C}^{ijk} = -(B_i + \Delta B_{ij})Y_2 - Y_2{}^T(B_i + \Delta B_{ij})^T + X_2 G_i{}^T + X_2 Q_2 X_2$

, $\forall\, i, j, k = 1, \dots, r$.

Considering that $Z_i = L_i V_1$, $W_{1i} = G_i X_1$, $W_{2i} = G_i X_2$, $R_1 = X_1 Q_1 X_1$ and $R_2 = X_2 Q_2 X_2$, we get (3.139).

In order to guarantee the positivity of the augmented system (1.137), we have to prove that all A_x^{ijk} are Metzler, or equivalently, that there exists a positive scalar γ such that $A_x^{ijk} + \sigma \mathbb{I} > 0$. Multiplying on the right by $diag(X_1, X_2)$, we get (3.140).

Once these LMIs are programmed and solved, we can obtain the observer and controller gain matrices:

$K_k = Y_{2k} X_2^{-1}$, $L_i = Z_i V_1^{-1}$ and $G_i = W_{1i} X_1^{-1}$.

Therefore, if there exist diagonal matrices P and Q, matrices G_i, L_i and K_k such that (3.139), (3.140) and (3.141) are satisfied, then, the closed-loop system (3.136) is positive and asymptotically stable.∎

- **Proposals of interval observer-based controller design**

We consider now that system (3.133) is subject to some uncertainties on the state, the input and the output, where: $\Delta X_{ij} = X_i - X_j$, with : $X_i = \{A_i, A_{1i}, B_i\}$ are the interval uncertainties of the system.

$\forall\, i = 1, \ldots, r, \; A_i \in [\underline{A_i}\,,\,\overline{A_i}], \Delta A_i \in [\underline{\Delta A_i}\,,\,\overline{\Delta A_i}], \; B_i \in [\underline{B_i}\,,\,\overline{B_i}], \; \Delta B_i \in [\underline{\Delta B_i}\,,\,\Delta \overline{B_i}]$ and $C \in [\underline{C}\,,\,\overline{C}]$ are unknown constant matrices with known bounds, that fulfill $\underline{A_i} \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{\Delta A_i} \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{B_i} \succcurlyeq 0 \; \in \mathbb{R}^{n_x \times n_u}$, $\underline{\Delta B_i} \succcurlyeq 0$ $\in \mathbb{R}^{n_x \times n_u}, \underline{C} \succcurlyeq 0 \in \mathbb{R}^{n_y \times n_x}$.

We provide a necessary condition for the existence of solutions to the problem of existence of an observer-based controller for positive interval T-S systems. Otherwise, we further study sufficient conditions for the corresponding synthesis problem.

Theorem 3.12. (Zaidi et al, 2015)

If there exists a static state-feedback controller (3.135) that stabilizes system (3.133), using the observer (3.134) with a positive augmented system (3.137), then the following inequalities with respect to Metzler G_j, $L_j \succcurlyeq 0$ and $K_k \preccurlyeq 0$, $j = 1, \ldots, r$, have a solution: $\forall\, i, j = 1, \ldots, r$

$$trace\left(\underline{A_i} + \underline{\Delta A_{ij}} + G_j + (\overline{B_i} + \overline{\Delta B_i} - \underline{B_i} - \underline{\Delta B_i})K_k\right) < 0\,, \tag{3.147}$$

$$[\overline{A_i} + \overline{\Delta A_i} + (\underline{B_i} + \underline{\Delta B_{ij}})K_k]_{lm} \geq 0\,, 1 \leq l \neq m \leq n\,, \tag{3.148}$$

$$[G_j - (\overline{B_i} + \overline{\Delta B_i})K_k]_{lm} \geq 0\,, 1 \leq l \neq m \leq n, \tag{3.149}$$

$$\overline{A_i} + \overline{\Delta A_i} - L_j\underline{C} + (\underline{B_i} + \underline{\Delta B_{ij}})K_k - G_j \succcurlyeq 0 \tag{3.150}$$

Proof:

The proof is analogical to the proof of Theorem 3.8, taking into account the interval uncertainties of the system (3.133). ∎

The following theorem provides sufficient conditions and the corresponding synthesis approach for this problem.

Theorem 3.13. (Zaidi et al, 2015)

For a positive scalar ε, there exists a solution to the problem of existence of an observer-based controller (3.135) for the positive interval T-S system (3.133) if, $\forall\, i, j, k = 1, \ldots, r$, there exist matrices $P = diag[P_1, P_2] > 0$, matrices $L_i \succcurlyeq 0$, $K_k \preccurlyeq 0$ and Metzler matrices G_i such that:

$$\begin{pmatrix} \mathcal{A}_{ij}^T P + P\mathcal{A}_{ij} - \varepsilon\mathcal{B}_{ij}\mathcal{B}_{ij}^T P - \varepsilon P\mathcal{B}_{ij}^T\mathcal{B}_{ij} + \varepsilon^2\mathcal{B}_{ij}\mathcal{B}_{ij}^T & * \\ \mathcal{B}_{ij}^T P + \mathcal{K}_{ik}\mathcal{C} & -I \end{pmatrix} < 0 \tag{3.151}$$

$$[\underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k]_{lm} \geq 0 \, , 1 \leq l \neq m \leq n \tag{3.152}$$

$$[G_i - (\underline{B}_i + \underline{\Delta B}_{ij})K_k]_{lm} \geq 0 \, , 1 \leq l \neq m \leq n \tag{3.153}$$

$$[\underline{A}_i + \underline{\Delta A}_{ij} - L_i(\overline{C} + \overline{\Delta C}) + (\overline{B}_i + \overline{\Delta B}_{ij})K_k - G_i]_{lm} \geq 0, 1 \leq l \neq m \leq n \tag{3.154}$$

where

$$\mathcal{A}_{ij} = \begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} & 0 \\ \overline{A}_i & 0 \end{pmatrix}, \mathcal{B}_{ij} = \begin{pmatrix} 0 & \underline{B}_i + \underline{\Delta B}_{ij} - (\overline{B}_i + \overline{\Delta B}_{ij}) & 0 & \underline{B}_i + \underline{\Delta B}_{ij} \\ 0 & \underline{B}_i + \underline{\Delta B}_{ij} - (\overline{B}_i + \overline{\Delta B}_{ij}) & -I & \underline{B}_i + \underline{\Delta B}_{ij} \end{pmatrix}$$

$$\mathcal{K}_{ik} = \begin{pmatrix} G_i & L_i & 0 & 0 \\ K_k & 0 & 0 & 0 \\ 0 & 0 & G_i & L_i \\ 0 & 0 & K_k & 0 \end{pmatrix}, \mathcal{C} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C} & 0 \end{bmatrix} \tag{3.155}$$

Proof:

It follows from (3.152) that $\underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k$ is Metzler. Combining this with $K \preccurlyeq 0$ yields that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C \in [\underline{C}, \overline{C}]$:

$$A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k \succcurlyeq \underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k \text{ is Metzler} \tag{3.156}$$

In addition, from G_i being Metzler and $K_k \preccurlyeq 0$, we obtain that, for any $B_i \in [\underline{B}_i, \overline{B}_i]$:

$$-(B_i + \Delta B_{ij})K_k \succcurlyeq 0 \tag{3.157}$$

and

$$G_i - (B_i + \Delta B_{ij})K_k \text{ is Metzler} \tag{3.158}$$

We have also:

$$A_i + \Delta A_{ij} - L_i(C + \Delta C) + (B_i + \Delta B_{ij})K_k - G_i$$

$$\succcurlyeq \underline{A}_i + \underline{\Delta A}_{ij} - L_i(\overline{C} + \overline{\Delta C}) + (\overline{B}_i + \overline{\Delta B}_{ij})K_k - G_i \tag{3.159}$$

Therefore, from (3.156)-(3.159), we have that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C \in [\underline{C}, \overline{C}]$, the augmented system (3.137) is positive.

It follows from (3.151), by Schur complement, that $\forall \, i, j, k = 1, \dots, r,$

$$\mathcal{A}_{ij}{}^T P + P\mathcal{A}_{ij} - \varepsilon \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T P - \varepsilon P \mathcal{B}_{ij}{}^T \mathcal{B}_{ij} + \varepsilon^2 \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T$$

$$+ \left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ik}\mathcal{C}\right)^T \left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ik}\mathcal{C}\right) < 0 \tag{3.160}$$

Taking into account the following relationship:

$$PBB^T P - \varepsilon BB^T P - \varepsilon PB^T B + \varepsilon^2 BB^T = (PB - \varepsilon B)(B^T P - \varepsilon B^T) \geq 0 \tag{3.161}$$

We obtain the following inequality:

$$\mathcal{A}_{ij}{}^T P + P\mathcal{A}_{ij} - P\mathcal{B}_{ij}\mathcal{B}_{ij}{}^T P + \left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ik}\mathcal{C}\right)^T\left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ik}\mathcal{C}\right) < 0 \qquad (3.162)$$

Rewriting (3.162) yields that:

$$(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C})^T P + P\left(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C}\right) + \mathcal{C}^T\mathcal{K}_{ik}{}^T\mathcal{K}_{ik}\,\mathcal{C} < 0 \qquad (3.163)$$

which implies that:

$$(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C})^T P + P\left(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C}\right) < 0 \qquad (3.164)$$

Then, we get that:

$$\mu\left(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C}\right) < 0. \qquad (3.165)$$

Some algebraic manipulations lead to the following equivalence:

$$\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{ik}\mathcal{C} = \begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k & -(\overline{B}_i + \overline{\Delta B}_{ij})K_k \\ \overline{A}_i + \overline{\Delta A}_{ij} - L_i\underline{C} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k - G_i & G_i - (\overline{B}_i + \overline{\Delta B}_{ij})K_k \end{pmatrix} \qquad (3.166)$$

In addition, it is easy to show that:

$$\begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k & -(\overline{B}_i + \overline{\Delta B}_{ij})K_k \\ \overline{A}_i + \overline{\Delta A}_{ij} - L_i\underline{C} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k - G_i & G_i - (\overline{B}_i + \overline{\Delta B}_{ij})K_k \end{pmatrix}$$
$$\succcurlyeq \begin{pmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k & -(B_i + \Delta B_{ij})K_k \\ A_i + \Delta A_{ij} - L_iC + (B_i + \Delta B_{ij})K_k - G_i & G_i - (B_i + \Delta B_{ij})K_k \end{pmatrix} \qquad (3.167)$$

Therefore, by combining (3.165)-(3.167) and using Lemma 3.1, we obtain that:

$$\mu\left(\begin{bmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k & -(B_i + \Delta B_{ij})K_k \\ A_i + \Delta A_{ij} - L_iC + (B_i + \Delta B_{ij})K_k - G_i & G_i - (B_i + \Delta B_{ij})K_k \end{bmatrix}\right) < 0 \qquad (3.168)$$

which means that the augmented system (3.137) is asymptotically stable for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C \in [\underline{C}, \overline{C}]$.

This completes the proof. ∎

3.3.2.4. Illustrative example: Two-tanks Hydraulic System

We consider, as an application, a process composed of two linked tanks (Zhang & Ding, 2005), (Benzaouia & El Hajjaji, 2011).

The two-tank system considered in this section is described by Figure 3.16. It consists of two cylinders interconnected by a pipe. Two pumps supply water, and

outflows are located at the bottom of each tank. The control objective is to manipulate the flow rates, so that water levels in the tanks are regulated, following some given reference signals.
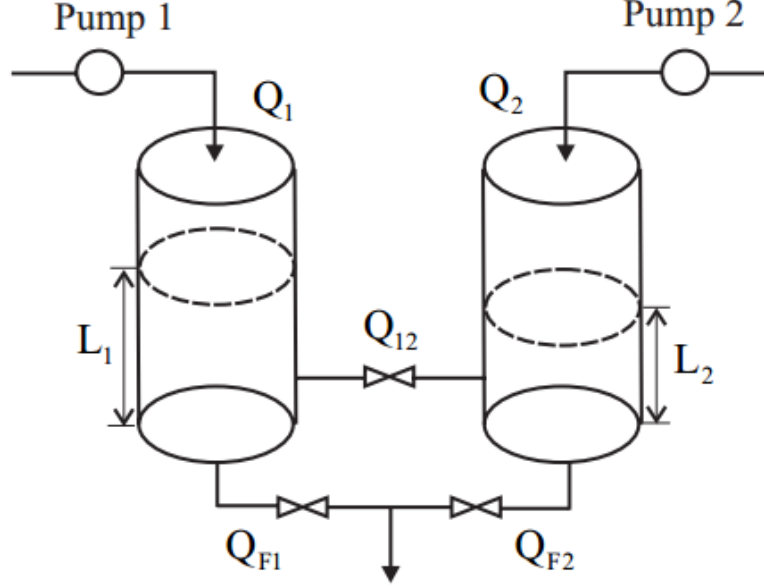
Figure 3. 16. Structure of the two-tank hydraulic system (Zhang & Ding, 2005)

The corresponding nonlinear model can be described as a T-S system as follows:

$$\begin{cases} \dot{x}(t) = A(z_1, z_2)\, x(t) + Bu(t) \\ y(t) = C\, x(t) \end{cases} \tag{3.169}$$

We consider that $z_1 \in [a_1; b_1]$ and $z_2 \in [a_2; b_2]$, so we get the following four rules:

$$\begin{cases} \text{If } z_1 \text{ is } a_1 \text{ and } z_2 \text{ is } a_2 \text{ then } A(z_1, z_2) = A_1 \\ \text{If } z_1 \text{ is } a_1 \text{ and } z_2 \text{ is } b_2 \text{ then } A(z_1, z_2) = A_2 \\ \text{If } z_1 \text{ is } b_1 \text{ and } z_2 \text{ is } a_2 \text{ then } A(z_1, z_2) = A_3 \\ \text{If } z_1 \text{ is } b_1 \text{ and } z_2 \text{ is } b_2 \text{ then } A(z_1, z_2) = A_4 \end{cases} \tag{3.170}$$

where the obtained matrices A_i of the subsystems are:

$$A_1 = \begin{pmatrix} -R_1 a_1 - \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1{}^2 - a_2{}^2|}} & \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1{}^2 - a_2{}^2|}} \\[4ex] \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1{}^2 - a_2{}^2|}} & -R_2 a_2 - \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1{}^2 - a_2{}^2|}} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -R_1 a_1 - \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1{}^2 - b_2{}^2|}} & \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1{}^2 - b_2{}^2|}} \\[4ex] \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1{}^2 - b_2{}^2|}} & -R_2 a_2 - \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1{}^2 - b_2{}^2|}} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -R_1 b_1 - \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1{}^2 - a_2{}^2|}} & \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1{}^2 - a_2{}^2|}} \\ \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1{}^2 - a_2{}^2|}} & -R_2 a_2 - \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1{}^2 - a_2{}^2|}} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -R_1 b_1 - \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1{}^2 - b_2{}^2|}} & \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1{}^2 - b_2{}^2|}} \\ \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1{}^2 - b_2{}^2|}} & -R_2 b_2 - \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1{}^2 - b_2{}^2|}} \end{pmatrix}$$

with R_1, R_2 and R_{12} are uncertain parameters of the hydraulic system.

The membership functions are given by:

$$h_1(t) = f_{11}(t)\, f_{21}(t); \quad h_2(t) = f_{11}(t)\, f_{22}(t);$$

$$h_3(t) = f_{12}(t)\, f_{21}(t); \quad h_4(t) = f_{12}(t)\, f_{22}(t);$$

where: $f_{i1}(t) = \dfrac{z_i(t) - b_i}{a_i - b_i}$ and $f_{i2}(t) = 1 - f_{i1}(t) = \dfrac{a_i - z_i(t)}{a_i - b_i}$; $i = 1,2$.

The overall obtained T-S model with delay is given by:

$$\begin{cases} \dot{x}(t) = \displaystyle\sum_{i=1}^{4} h_i(x(t))(A_i x(t) + B u(t)) \\ y(t) = C\, x(t) \end{cases} \tag{3.171}$$

The objective is to design an interval observer-based controller which ensures the stabilization and the estimation of the system associated to the real plant, in which matrices $A_i \in [\underline{A_i}, \overline{A_i}]$ are Metzler, $\forall\, i = 1,2$ and matrices B and C are nonnegative.

The bounding matrices are:

$$\underline{A}_1 = \begin{pmatrix} -0.3878 & 0.1476 \\ 0.1476 & -0.4163 \end{pmatrix}; \overline{A}_1 = \begin{pmatrix} -0.2541 & 0.2532 \\ 0.3546 & -0.1653 \end{pmatrix}$$

$$\underline{A}_2 = \begin{pmatrix} -0.3935 & 0.1533 \\ 0.1533 & -0.4067 \end{pmatrix}; \overline{A}_2 = \begin{pmatrix} -0.1250 & 0.4016 \\ 0.5233 & -0.3908 \end{pmatrix}$$

$$\underline{A}_3 = \begin{pmatrix} -0.8881 & 0.2442 \\ 0.2442 & -0.5128 \end{pmatrix}; \overline{A}_3 = \begin{pmatrix} -0.4623 & 0.5233 \\ 0.6301 & -0.2752 \end{pmatrix}$$

$$\underline{A}_4 = \begin{pmatrix} -0.8669 & 0.2229 \\ 0.2229 & -0.4763 \end{pmatrix}; \overline{A}_4 = \begin{pmatrix} -0.5301 & 0.6781 \\ 0.5252 & -0.3046 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Applying Theorems 3.12 and 3.13, we get the following observer and controller gains :

$$K_1 = \begin{pmatrix} -0.1352 & -0.2413 \\ -0.5312 & -0.6341 \end{pmatrix} ; K_2 = \begin{pmatrix} -0.2152 & -0.8542 \\ -0.5264 & -0.7112 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} -1.0352 & -0.9523 \\ -0.8621 & -1.0652 \end{pmatrix} ; K_4 = \begin{pmatrix} -1.2588 & -0.6322 \\ -1.8522 & 0.4205 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 0.0481 \\ 0.0516 \end{pmatrix} ; L_2 = \begin{pmatrix} 0.0483 \\ 0.0476 \end{pmatrix} ; L_3 = \begin{pmatrix} 0.1560 \\ 0.0783 \end{pmatrix} ; L_4 = \begin{pmatrix} 0.1658 \\ 0.0807 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0.2966 & -0.2107 \\ -0.2156 & 0.3053 \end{pmatrix} ; G_2 = \begin{pmatrix} 0.3015 & -0.2136 \\ -0.2179 & 0.3024 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0.6334 & -0.3443 \\ -0.3496 & 0.3648 \end{pmatrix} ; G_4 = \begin{pmatrix} 0.6048 & -0.3224 \\ -0.3285 & 0.3423 \end{pmatrix}$$

Some simulation results using the proposed observer-based controller are presented in figures 3.17 to 3.19 for the given system matrices. We can observe that the evolution of the real state vector $x(t)$ as well as that of the estimated state vector $\hat{x}(t)$ is always in the positive orthant. Moreover, the upper and lower estimated states are nonnegative and converge to the real value. These properties can be seen in Figures 3.18 and 3.19, that plot the state evolutions from given initial conditions. These facts show the effectiveness of the proposed approach.
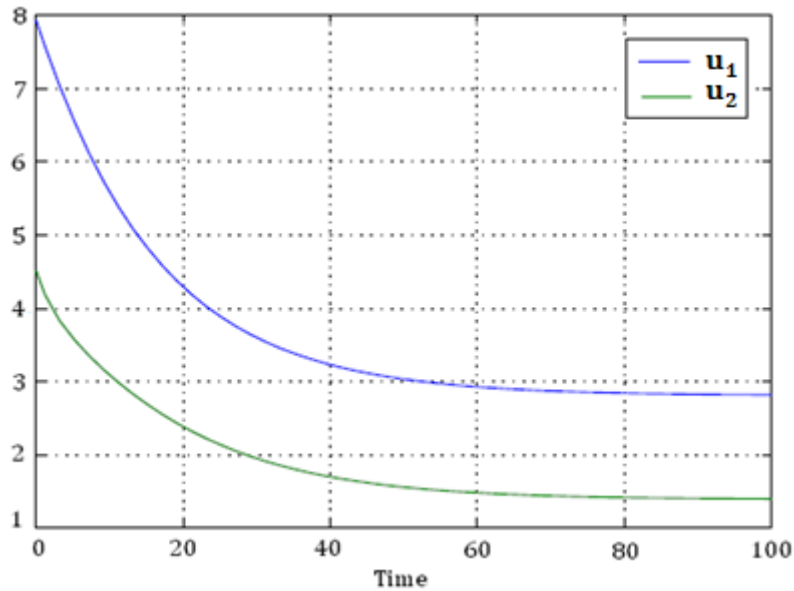
Figure 3. 17. The evolution of the pump flows
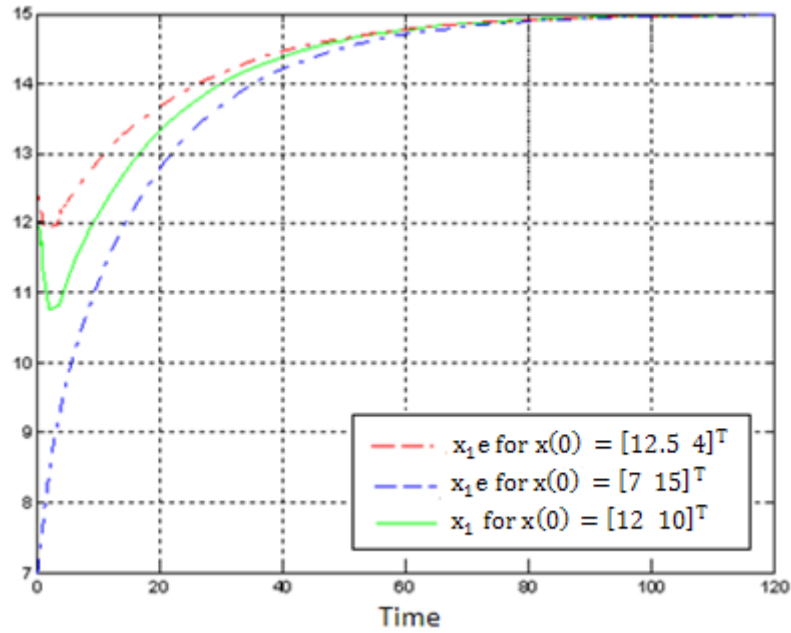
Figure 3. 18. Evolution of the state $x_1(t)$ and its estimation $\hat{x}_1(t)$
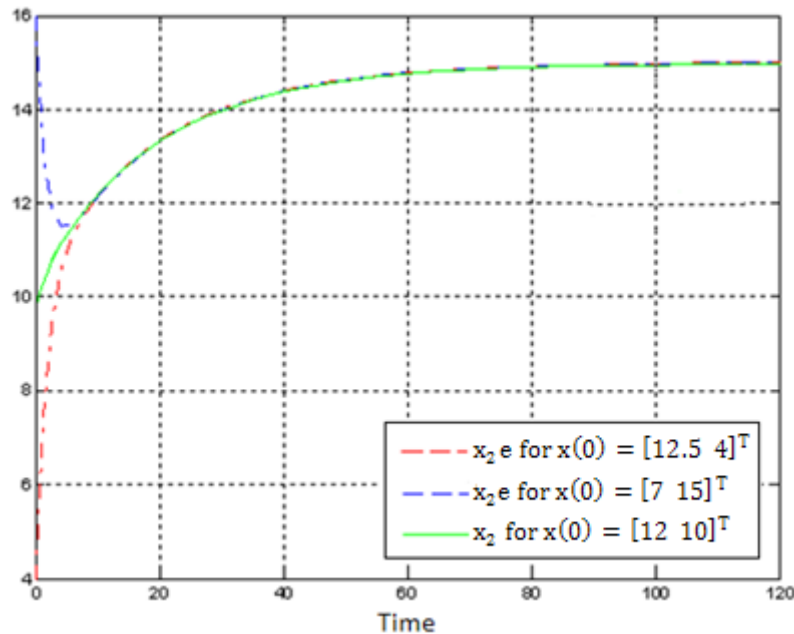
Figure 3. 19. Evolution of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$

3.4.   Conclusion

This chapter presented the results of observer-based controller design for positive linear and positive T-S systems, which may be with measurable and unmeasurable decision variables. Regarding positive T-S systems with measurable premise variables, we have developed two techniques. The first one is based on the synthesis of an observer using the PDC technique and the representation of

the state estimation error that has been stabilized. Using quadratic stabilization techniques for the considered augmented system can guarantee the convergence of the state estimation error, taking into account the positivity constraints of the augmented system. The second approach is based on the representation of the positive asymptotic stabilization of the considered augmented system, by providing first of all a necessary condition for the existence of such observers for this class of systems. Next, sufficient conditions have been expressed through LMI formulation.

As for positive T-S systems with unmeasurable premise variables, two approaches have been developed: The first one concentrates on L_2 performance and is based on rewriting the positive T-S system in the form of an uncertain system. L_2 techniques have then been applied in order to improve the estimation quality and to guarantee a better convergence of the state estimation error while remaining positive. The second method is based on designing positive interval observer-based controllers, taking into account the uncertainties of the system. The bilinearity problems of the resulting LMI formulation have been discussed and solved.

Several examples have been used for the validation of the developed results: First, numerical examples, an electrical circuit application and a practical two-tank-system, which have been dealt with to show the effectiveness of the previously cited methods.

Chapter IV

Stability and Stabilization of Positive time-delay systems

4.1. Introduction

The phenomenon of delay appears naturally in the modeling of many processes in various fields such as physics, mechanics, biology, ecology, engineering, telecommunications, etc. Indeed, the presence of delays in the system may lead to instabilities or to a very poor performance. These systems are described by differential equations whose development depends not only on the value of variables at the current time, but also on a part of its history, that is to say the values at each time. In effect, it is reasonable to consider the delay as a universal characteristic of the interaction between man and nature, in the field of engineering sciences. Indeed, we are interested in positive time-delay systems whose states remain nonnegative. The constraint of the nonnegativity of the states is compulsory in practical fields, such as chemical, physical and biological processes involve quantities that have intrinsically constant and nonnegative signs. Therefore, the study of positive time-delay linear and T-S systems is a very interesting area of research where several works are developed over the past two decades (Benzaouia et al, 2014), (Benzaouia & Oubah, 2014), (Benzaouia & El Hajjaji, 2014), (Benzaouia et al, 2011), (Benzaouia & Oubah, 2014), (Benzaouia & El Hajjaji, 2011), etc. We are meant to study positive time-delay systems that maintain positivity and stability against unknown delay factors and/or interval uncertainties. As demonstrated in (Bolajraf, 2012), it was proved that asymptotic stability of a positive system is independent of the delay sizes. Moreover, checking the asymptotic stability of a positive system can be easily done by analyzing a specific Metzler matrix, connected to its dynamic that is independent of the delay sizes (Shu et al, 2008).

In this chapter, we develop the asymptotic stability and the positivity conditions for constant and variable delays. We will also deal with the robust α-stability notion that guarantees a specific decay rate with the presence of uncertainties on the system. In addition, necessary and sufficient conditions are provided for the stabilization of positive interval linear and T-S systems with time-delay by means of decomposed state-feedback laws that can be chosen with or without memory.

Numerical examples are presented to illustrate the effectiveness of the developed methods.

4.1.1. ASYMPTOTIC STABILITY OF POSITIVE LINEAR TIME-DELAY SYSTEMS

In this section, we propose a method for the analysis of the stability of positive time-delay systems. In a first step, we study the problem for constant delays that will be extended in a second step to the case of variable and multiple delays.

4.1.1.1. Case of a single constant delay

We consider the time-delay system as follows:

$$\dot{x}(t) = A x(t) + A_1 x(t - \tau) \tag{4.1}$$

with the following condition:

$$x(t) = \varphi(t) \in \mathbb{R}_+^{n_x}, t \in [-\tau, 0] \tag{4.2}$$

Inspired by the works developed by Saadni and Mehdi (Saadni & Mehdi, 2004) and (Saadni, 2006), we deduce the conditions of stability and positivity of the system (4.2), summarized in the following theorem.

Theorem 4.1. (Zaidi et al, 2015b)

Suppose that A is Metzler, $A_1 \succcurlyeq 0$ and $0 \leq \tau \leq \overline{\tau}$. If there exist symmetric and positive definite matrices P, Q and W and matrices γ_i, $i = 1, \dots, 4$, X, Y and Z such that:

$$X - Y < 0 \tag{4.3}$$

$$\begin{pmatrix} Z & \overline{\tau}Y \\ \overline{\tau}Y^T & \overline{\tau}X \end{pmatrix} \geq 0 \tag{4.4}$$

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ 0 & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A \;\; A_1 \;\; 0 \;\; -I) \right\} \leq 0 \tag{4.5}$$

are satisfied with:

$$\chi_0 = \overline{\tau}^2 W, \; \chi_1 = Q + \overline{\tau}(Z + Y + Y^T), \; \chi_2 = Q \; and \; \chi_3 = \overline{\tau}Y$$

Then, system (4.2) with a constant delay is positive and asymptotically stable.

Proof:

For the analysis of the stability of this class of systems, we adopt the following Lyapunov candidate function (Saadni, 2006) defined by:

$$V\big(x(t)\big) = V_0\big(x(t)\big) + V_1\big(x(t)\big) + V_2\big(x(t)\big) \tag{4.6}$$

such that the functionals $V_0\big(x(t)\big)$, $V_1\big(x(t)\big)$ and $V_2\big(x(t)\big)$ are given by:

$$V_0\big(x(t)\big) = x^T(t)Px(t) \tag{4.7}$$

$$V_1\big(x(t)\big) = \int_{t-\tau}^{0} x^T(s)Qx(s)\,ds \tag{4.8}$$

$$V_2(x(t)) = \int_{t-\tau}^{t}\int_{s}^{t} \dot{x}^T(z)(\overline{\tau}W)\dot{x}(z)dzds \tag{4.9}$$

where P, Q and W are symmetric positive definite matrices.

The derivative of the functional $V_2(x(t))$ along the trajectory of the system (4.9) is given by:

$$\dot{V}_2\big(x(t)\big) = \overline{\tau}^2\dot{x}^T(t)Wx(t) - \int_{t-\tau}^{t}\dot{x}^T(s)(\overline{\tau}W)\dot{x}(s)ds \tag{4.10}$$

Thus, the derivative of the candidate Lyapunov functional (4.6) is written:

$$\begin{aligned}
\dot{V}\big(x(t)\big) &= \dot{V}_0\big(x(t)\big) + \dot{V}_1\big(x(t)\big) + \dot{V}_2\big(x(t)\big)\\
&= x^T(t)(A^TP + PA)x(t) + x^T(t)PA_1x(t-\tau) + x^T(t-\tau)A_1^TPx(t)\\
&\quad + \overline{\tau}^2\big(x^T(t)A^TWAx(t) + x^T(t)A^TWA_1x(t-\tau) + x^T(t-\tau)A_1^TWAx(t)\\
&\quad + x^T(t-\tau)A_1^TWA_1x(t-\tau)\big) + [x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau)]\\
&\quad - \int_{t-\tau}^{t}\dot{x}^T(s)(\overline{\tau}W)\dot{x}(s)ds
\end{aligned} \tag{4.11}$$

For matrices $X > 0$, Y and Z satisfying:

$$\begin{pmatrix} Z & \overline{\tau}Y \\ \overline{\tau}Y^T & \overline{\tau}X \end{pmatrix} \geq 0 \tag{4.12}$$

We get:

$$\int_{t-\tau}^{t} (x^T(t)\ \dot{x}^T(s)) \begin{pmatrix} Z & \overline{\tau}Y \\ \overline{\tau}Y^T & \overline{\tau}X \end{pmatrix}\begin{pmatrix} x(t) \\ \dot{x}(s) \end{pmatrix} ds \geq 0 \tag{4.13}$$

From (4.13), we can easily deduce that:

$$-\int_{t-\tau}^{t} \dot{x}^T(s)(\overline{\tau}W)\dot{x}(s)ds \le \int_{t-\tau}^{t} \left(x^T(t)\ \dot{x}^T(s)\right)\begin{pmatrix} Z & \overline{\tau}Y \\ \overline{\tau}Y^T & \overline{\tau}X \end{pmatrix}\begin{pmatrix} x(t) \\ \dot{x}(s) \end{pmatrix}ds \qquad (4.14)$$

If there exist symmetric positive definite matrices W and X, the expression (4.11) can be written as follows:

$$
\begin{aligned}
\dot{V}\big(x(t)\big) = \ & x^T(t)(A^T P + PA)x(t) + x^T(t)PA_1 x(t-\tau) + x^T(t-\tau)A_1^T P x(t) \\
& + \overline{\tau}^2\big(x^T(t)A^T WAx(t) + x^T(t)A^T WA_1 x(t-\tau) + x^T(t-\tau)A_1^T WAx(t) \\
& + x^T(t-\tau)A_1^T WA_1 x(t-\tau)\big) - \int_{t-\tau}^{t} \dot{x}^T(s)(\overline{\tau}W)\dot{x}(s)ds \\
& - \int_{t-\tau}^{t} \dot{x}^T(s)\overline{\tau}(W-X)\dot{x}(s)ds \qquad (4.15)
\end{aligned}
$$

Using (4.14), $\dot{V}(x(t))$ can be bounded as follows:

$$
\begin{aligned}
\dot{V}\big(x(t)\big) \le \ & x^T(t)(A^T P + PA)x(t) + x^T(t)PA_1 x(t-\tau) + x^T(t-\tau)A_1^T P x(t) \\
& + \overline{\tau}^2\big(x^T(t)A^T WAx(t) + x^T(t)A^T WA_1 x(t-\tau) + x^T(t-\tau)A_1^T WAx(t) \\
& + x^T(t-\tau)A_1^T WA_1 x(t-\tau)\big) \\
& + \int_{t-\tau}^{t} \left(x^T(t)\ \dot{x}^T(t)\right)\begin{pmatrix} Z & \overline{\tau}Y \\ \overline{\tau}Y^T & \overline{\tau}X \end{pmatrix}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}ds \\
& - \int_{t-\tau}^{t} \dot{x}^T(s)\overline{\tau}(W-X)\dot{x}(s)ds \qquad (4.16)
\end{aligned}
$$

After manipulation, we have:

$$\dot{V}\big(x(t)\big) \le \left(x^T(t)\ x^T(t-\tau)\right)\phi \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix} - \int_{t-\tau}^{t} \dot{x}^T(s)\overline{\tau}(W-X)\dot{x}(s)ds \qquad (4.17)$$

with:

$$
\phi = \begin{pmatrix} A^T P + PA + Q & PA_1 \\ * & -Q \end{pmatrix} + \begin{pmatrix} A^T \\ A_1^T \end{pmatrix}(\overline{\tau}^2 W)(A\ A_1)
$$

$$
+ \begin{pmatrix} \overline{\tau}Z + \overline{\tau}Y + \overline{\tau}Y^T & -\overline{\tau}Y \\ * & 0 \end{pmatrix} \qquad (4.18)
$$

Thus, to ensure the negativity of $\dot{V}(x(t))$, it is sufficient that: $\phi < 0$ and $X - W < 0$.

Using the Schur complement in Lemma B.1 (Annex), the matrix ϕ given by (4.18) becomes:

$$\phi = \begin{pmatrix} A^T P + PA + \chi_1 & PA_1 - \chi_3 & A^T \chi_0 \\ * & -\chi_2 & A_1^T \chi_0 \\ * & * & -\chi_0 \end{pmatrix} \qquad (4.19)$$

where the expressions of χ_0, χ_1, χ_2 and χ_3 are defined by:

$$\chi_0 = \bar{\tau}^2 W, \; \chi_1 = Q + \bar{\tau}(Z + Y + Y^T), \; \chi_2 = Q \text{ and } \chi_3 = \bar{\tau} Y.$$

The expression in ϕ may be rewritten in the following form:

$$\phi = \begin{pmatrix} \chi_1 & -\chi_3 & 0 \\ * & -\chi_2 & 0 \\ * & * & -\chi_0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} P \\ 0 \\ \chi_0 \end{pmatrix} (A \;\; A_1 \;\; 0) \right\} \leq 0 \qquad (4.20)$$

By applying Lemma B.3 and Lemma B.4 (Annex), the inequality (4.20) becomes:

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} (A \;\; A_1 \;\; 0 \;\; -I) \right\} \leq 0 \qquad (4.21)$$

Positivity of system (4.2) is guaranteed by the fact that A is Metzler and $A_1 \succcurlyeq 0$. This ends the proof. ∎

In the next section, we propose a generalization of the theorem 4.1. in the case of a variable and multiple delay positive system.

4.1.1.2. Case of variable and multiple delays

Consider the following system with multiple and variable delays:

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{p} A_j x\left(t - \tau_j(t)\right) \qquad (4.22)$$

where $\tau_j(t)$, $j = 1, \dots, p$ are variable delays such that:

$$0 \leq \tau_j(t) \leq \bar{h}_j, \; \dot{\tau}_j(t) \leq d_j < 1$$

where $\dot{\tau}_j(t)$ and d_j, $j = 1, \dots, p$ are given positive scalars.

We denote also: $\bar{h} = \max(\bar{h}_1, \dots, \bar{h}_p)$.

System (4.22) can be rewritten in the following compact form:

$$\dot{x}(t) = Ax(t) + A_\tau x_\tau(t) \qquad (4.23)$$

with

$$x_\tau(t)^T(t) = (x(t - \tau_1(t))^T \dots x(t - \tau_p(t))^T)^T$$

$$A_\tau = (A_1 \dots A_p).$$

Inspired by the results of (Saadni, 2006), we establish the conditions ensuring the stability and positivity of the system (4.23) which are summarized in the following Theorem.

Theorem 4.2.

Suppose that A is Metzler, $A_j \geqslant 0$, $0 \leq \tau_j(t) \leq \overline{h}_j$, $\dot{\tau}_j(t) \leq d_j < 1$, $j = 1, \dots, p$. If there exist symmetric and positive definite matrices P, Q_j and W_j and matrices γ_i, $i = 1, \dots, 4$, X_j, Y_j and Z_j such that:

$$X_j - W_j < 0 \tag{4.24}$$

$$\begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \geq 0 \tag{4.25}$$

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ 0 & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A \ A_\tau \ 0 \ -I) \right\} < 0 \tag{4.26}$$

are satisfied with:

$$\chi_0 = \sum_{j=1}^{p} \overline{h}_j^2 W_j$$

$$\chi_1 = \sum_{j=1}^{p} [Q_j + (1 - d_j)\overline{h}_j(Z_j + Y_j + Y_j^T)]$$

$$\chi_2 = diag((1 - d_1)Q_1, \dots, (1 - d_p)Q_p)$$

$$\chi_3 = ((1 - d_1)\overline{h}_1 Y_1, \dots, (1 - d_p)\overline{h}_p Y_p)$$

Then, system (4.23) with a variable multiple delay is positive and asymptotically stable.

Proof:

For the analysis of the stability of this class of systems, we adopt the following Lyapunov candidate function defined by:

$$V(x(t)) = V_0(x(t)) + V_1(x(t)) + V_2(x(t)) \tag{4.27}$$

such that the functionals $V_0\big(x(t)\big)$, $V_1\big(x(t)\big)$ and $V_2\big(x(t)\big)$ are given by:

$$V_0\big(x(t)\big) = x^T(t)Px(t) \tag{4.28}$$

$$V_1\big(x(t)\big) = \sum_{j=1}^{p} \int_{t-\tau_j(t)}^{0} x^T(s)Q_j x(s)\, ds \tag{4.29}$$

$$V_2(x(t)) = \sum_{j=1}^{p} \int_{t-\overline{h}_j}^{t} \int_{s}^{t} \dot{x}^T(z)\big(\overline{h}_j W_j\big)\dot{x}(z)dzds \tag{4.30}$$

where P, Q_j and W_j are symmetric positive definite matrices.

The derivative of each functional along the trajectory of the system (4.23) is given by:

$$\dot{V}_0\big(x(t)\big) = 2\dot{x}^T(t)Px(t) = 2[Ax(t) + A_\tau x_\tau(t)]^T Px(t)$$

$$= x^T(t)[A^T P + PA]x(t) + x^T(t)PA_\tau x_\tau(t) + x_\tau(t)^T PA_\tau x_\tau(t) \tag{4.31}$$

$$\dot{V}_1\big(x(t)\big) = \sum_{j=1}^{p} [x^T(t)Q_j x(t) - (1 - \dot{\tau}_j(t))x^T(t - \tau_j(t))Q_j x(t - \tau_j(t))] \tag{4.32}$$

$$\dot{V}_2\big(x(t)\big) = \sum_{j=1}^{p} \overline{h}_j^2 \dot{x}^T(t)W_j\dot{x}(t) - (1 - \dot{\tau}_j(t))\int_{t-\tau_j(t)}^{t} \dot{x}^T(s)\big(\overline{h}_j W_j\big)\dot{x}(s)ds \tag{4.33}$$

For matrices $X_j > 0$, Y_j and Z_j satisfying:

$$\begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \geq 0 \tag{4.34}$$

We get:

$$\int_{t-\tau_j(t)}^{t} (x^T(t)\ \dot{x}^T(s)) \begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(s) \end{pmatrix} ds \geq 0 \tag{4.35}$$

From (4.35), we can easily deduce that:

$$-\int_{t-\tau_j(t)}^{t} \dot{x}^T(s)\big(\overline{h}_j W_j\big)\dot{x}(s)ds \leq \int_{t-\tau}^{t} (x^T(t)\ \dot{x}^T(s)) \begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(s) \end{pmatrix} ds \tag{4.36}$$

If there exist symmetric positive definite matrices W_j and X_j and after manipulation, the expression of $\dot{V}\big(x(t)\big)$ can be bounded as follows:

$$\dot{V}(x(t)) \leq x^T(t)\left[A^TP + PA\right.$$

$$+ \sum_{j=1}^{p}\left(\overline{h}_j^2 A^T W_j A + Q_j + (1-d_j)\left(\overline{h}_j(Z_j + Y_j + Y_j^T)\right)\right)\bigg] x(t)$$

$$+ x^T(t)\left[\sum_{j=1}^{p}(PA_j - (1-d_j)\overline{h}_j Y_j + \sum_{n=1}^{p}\overline{h}_n^2 A^T W_n A_j)x\left(t - \tau_j(t)\right)\right]$$

$$+ \left[\sum_{j=1}^{p} x\left(t - \tau_j(t)\right)^T\left(A_j^T P - (1-d_j)\overline{h}_j Y_j^T + \sum_{n=1}^{p}\overline{h}_n^2 W_n A\right)\right] x(t)$$

$$- \sum_{j=1}^{p} x\left(t - \tau_j(t)\right)^T (1-d_j)Q_j x\left(t - \tau_j(t)\right)$$

$$+ \sum_{j,k,n}^{p}\overline{h}_n^2 x(t - \tau_k(t))^T A_k^T W_j A_n x(t - \tau_n(t))$$

$$- \sum_{j=1}^{p}(1-d_j)\int_{t-\tau_j(t)}^{t}\dot{x}^T(s)\left(\overline{h}_j(W_j - X_j)\right)\dot{x}(s)ds \qquad (4.37)$$

After manipulation, we have:

$$\dot{V}(x(t)) \leq \rho^T(t)M\rho(t) - \sum_{i=1}^{p}\int_{t-\tau_j(t)}^{t}\dot{x}^T(s)\left(\overline{h}_j(W_j - X_j)\right)\dot{x}(s)ds \qquad (4.38)$$

with:

$$\rho^T(t) = (x^T(t) \quad x^T(t - \tau_1(t)) \dots x^T\left(t - \tau_p(t)\right))$$

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$$

where:

$$M_{11} = A^T P + PA + A^T \chi_0 A + \chi_1$$

$$M_{12} = PA_\tau + A^T \chi_0 A_\tau - \chi_3$$

$$M_{22} = A_\tau^T \chi_0 A_\tau - \chi_2$$

$$\chi_0 = \sum_{j=1}^{p}\overline{h}_j^2 W_j$$

$$\chi_1 = \sum_{j=1}^{p} [Q_j + (1 - d_j)\overline{h}_j(Z_j + Y_j + Y_j^T)]$$

$$\chi_2 = diag((1 - d_1)Q_1, \dots, (1 - d_p)Q_p)$$

$$\chi_3 = ((1 - d_1)\overline{h}_1 Y_1, \dots, (1 - d_p)\overline{h}_p Y_p)$$

Analogically with the proof of theorem 4.1, we deduce (4.26).

Positivity of system (4.23) is guaranteed by the fact that A is Metzler and $A_1 \succcurlyeq 0$. This ends the proof. ∎

4.1.2. STABILIZATION OF POSITIVE LINEAR TIME-DELAY SYSTEMS

After determining the criteria for stability of positive linear systems with delays, in this section, we will develop an approach for the design of control laws to stabilize the closed-loop time-delay system. In the study of stabilization, we may need to develop conditions which may or may not depend on the size of the delay. Asymptotic positive stabilization conditions by state-feedback control for a class of positive systems with constant, variable, single or multiple, with or without uncertainties are formulated. An extension for the exponential stabilization of positive time-delay systems is established. Examples of numerical simulation are presented in order to illustrate the efficiency of these methods.

4.1.2.1. State-feedback Stabilization of Positive Linear time-delay systems

We deal in this paragraph with the problem of stabilization of positive time-delay systems with state-feedback control. Thereafter, we determine the stabilization conditions for systems in the nominal case while we will use the decomposition technique of the controller gains to guarantee the stabilization and the positivity of the considered class of systems.

We consider the multiple delay system defined by:

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{p} A_j x\left(t - \tau_j(t)\right) + Bu(t) \tag{4.39}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input. The matrices $A \in \mathbb{R}^{n_x \times n_x}$ Metzler, $B \in \mathbb{R}^{n_x \times n_u} \succcurlyeq 0$ represent the nominal system without delay,

$A_j \in \mathbb{R}^{n_x \times n_x} \succcurlyeq 0$ is the delay matrix of the system and $\tau_j(t)$, $j = 1, ..., p$ are variable delays such that:

$$0 \leq \tau_j(t) \leq \overline{h}_j, \; \dot{\tau}_j(t) \leq d_j < 1 \tag{4.40}$$

The stabilization of this class of systems is to determine the control law $u(t)$ ensuring the stability of the closed-loop system for this type of delay, taking into account the positivity constraints of the system.

System (4.39) can be rewritten in the following compact form:

$$\dot{x}(t) = Ax(t) + A_\tau x_\tau(t) + Bu(t) \tag{4.41}$$

with

$$x_\tau(t)^T = (x(t - \tau_1(t))^T ... x(t - \tau_p(t))^T)^T$$

$$A_\tau = (A_1 ... A_p)$$

We suppose then that the following assumptions are checked out:

Assumption 4.1.

- The pair (A, B) is controllable
- The pair $(A + A_\tau, B)$ is controllable

We apply to the system (4.39) a state-feedback control law of the form:

$$u(t) = Kx(t), \; K \in \mathbb{R}^{n_u \times n_x} \tag{4.42}$$

We denote $A_{cl} = A + BK$, the system (4.39) becomes:

$$\dot{x}(t) = A_{cl}x(t) + \sum_{j=1}^{p} A_j \, x(t - \tau_j(t)) \tag{4.43}$$

The system (4.43) can be rewritten in its compact form as follows:

$$\dot{x}(t) = A_{cl}x(t) + A_\tau x_\tau(t) \tag{4.44}$$

We are allowed to decompose the system matrices A_{cl} and A_τ as follows:

$$A_{cl} = A + BK = A + BK_r + B(K - K_r) = A_r + BK_0 \tag{4.45}$$

$$A_\tau = A_\tau + BK_{dr} - BK_{dr} = A_{dr} - BK_{dr} \tag{4.46}$$

where $A_r = A + BK_r$, $A_{dr} = A_\tau + BK_{dr}$, $K_0 = K - K_r$.

In the following and based on the results for the stability developed in Theorem 4.2, we propose a solution for the positive state-feedback stabilization problem in the case of systems with multiple and variable delays (4.43). The following theorem provides a solution to the positive state-feedback stabilization problem to the class of the considered system (4.43).

Theorem 4.3. (Zaidi et al, 2015b)

Suppose that A is Metzler, $B \succcurlyeq 0$, $A_j \succcurlyeq 0$, $0 \le \tau_j(t) \le \overline{h}_j$, $\dot{\tau}_j(t) \le d_j < 1$, $j = 1, \dots, p$. The system (4.43) is positive and asymptotically stable, if there exist symmetric and positive definite matrices P, Q_j and W_j, matrices X_j, Z_j, Y_j, $j = 1, \dots, p$, matrices γ_i, $i = 1, \dots, 4$, L, E and a positive scalar β such that the following conditions:

$$X_j - Y_j < 0 \tag{4.47}$$

$$\begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \ge 0 \tag{4.48}$$

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym \left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_r \ A_{dr} \ 0 \ -I \ B) \right\}$$

$$+ Sym \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} ((L - EK_r) \ -EK_{dr} \ 0 \ 0 \ -E) \right\} < 0 \tag{4.49}$$

$$AE + BL + \beta E \succcurlyeq 0 \tag{4.50}$$

are satisfied. The stabilizing state-feedback is given by:

$$K = E^{-1}L \tag{4.51}$$

with

$$\chi_0 = \sum_{j=1}^{p} \overline{h}_j^2 W_j$$

$$\chi_1 = \sum_{j=1}^{p} [Q_j + (1 - d_j)\overline{h}_j(Z_j + Y_j + Y_j^T)]$$

$$\chi_2 = diag((1 - d_1)Q_1, \ldots, (1 - d_p)Q_p)$$

$$\chi_3 = ((1 - d_1)\overline{h}_1 Y_1, \ldots, (1 - d_p)\overline{h}_p Y_p)$$

Proof:

By replacing A by A_{cl} in the asymptotic stability condition (4.26) of Theorem 4.2, we get:

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ 0 & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_{cl} \ \ A_\tau \ \ 0 \ -I) \right\} < 0 \tag{4.52}$$

Using the decomposition of A_{cl} and A_τ given in (4.45) and (4.46), the inequality (4.52) becomes:

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_r \ \ A_{dr} \ \ 0 \ -I) \right\}$$

$$+ Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (B(K - K_r) \ \ -BK_{dr} \ \ 0 \ \ 0) \right\} < 0 \tag{4.53}$$

Denote that:

$$\Omega = \begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_r \ \ A_{dr} \ \ 0 \ -I) \right\} \tag{4.54}$$

We choose the gains K_r and K_{dr} in order to maintain the term Ω negative. We can apply Lemma B.3 (Annex), LMI (4.53) is equivalent to:

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_r \ \ A_{dr} \ \ 0 \ -I \ \ B) \right\}$$

$$+Sym\left\{\begin{pmatrix}0\\0\\0\\0\\I\end{pmatrix}E\left((K-K_r)-K_{dr}\quad 0\quad 0\quad -I\right)\right\}\leq 0 \qquad (4.55)$$

By imposing $L = EK$, we get the inequality (4.49).

As for positivity, we have to guarantee that: A_{cl} is Metzler.

Following Remark 1.1., this implies that: $A + BK + \beta I \geqslant 0$ (4.56)

Then, by multiplying the inequality (4.56) on the left by matrix E, we get (4.50).

This completes the proof.∎

In the following, we address the problem of stabilizing positive time-delay systems with a decomposed memory state-feedback law. The decomposition of the controller gains has to improve the stabilization techniques of this class of these systems and to maintain the positivity of the states in a better way.

4.1.2.2. Stabilization of Positive Linear time-delay systems with a decomposed memory state-feedback control

We consider the following delayed system with a single and variable time-delay:

$$\dot{x}(t) = Ax(t) + A_1 x\big(t - \tau(t)\big) + Bu(t) \qquad (4.57)$$

$$x(t) = \varphi(t),\ t \in \left[-\overline{h}, 0\right] \qquad (4.58)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input. The matrices $A \in \mathbb{R}^{n_x \times n_x}$ Metzler, $B \in \mathbb{R}^{n_x \times n_u} \geqslant 0$ represent the nominal system without delay, $A_1 \in \mathbb{R}^{n_x \times n_x} \geqslant 0$ is the delay matrix of the system, $\tau(t)$ is a varying time-delay such that:

$$0 \leq \tau(t) \leq \overline{h}\ ,\ \dot{\tau}(t) \leq d < 1 \qquad (4.59)$$

In this section, we aim to design for system (4.57) a memory state-feedback controller of the form:

$$u(t) = Kx(t) + K_1 x(t - \tau(t)),\ K,\ K_1 \in \mathbb{R}^{n_u \times n_x}, \qquad (4.60)$$

To solve this, we use the fact that for any matrices K and K_1, there exist nonnegative matrices K^-, K^+, K_1^- and K_1^+, such that:

$$K = K^+ - K^-, K_1 = K_1^+ - K_1^-, \qquad (4.61)$$

Using this fact, we rewrite the state-feedback controller (4.60) in the following decomposed form:

$$u(t) = (K^+ - K^-)x(t) + (K_1^+ - K_1^-)x(t - \tau_i) \tag{4.62}$$

Then, the corresponding closed-loop system becomes:

$$\dot{x}(t) = (A + BK^+ - BK^-)x(t) + (A_1 + BK_1^+ - BK_1^-)x(t - \tau(t)) \tag{4.63}$$

This expression will be used to develop the conditions of stabilization: Design a memory state-feedback controller (4.62) such that the closed-loop system (4.63) is positive and asymptotically stable.

Next, we will provide conditions on the matrices A, A_1, B such that there exist matrices K^-, K^+, K_1^- and $K_1^+ \in \mathbb{R}_+^{n_u \times n_x}$, satisfying:

-Positivity of the closed-loop system

$$\begin{cases} A + BK^+ - BK^- \text{ are Metzler} \\ A_1 + BK_1^+ - BK_1^- \geqslant 0 \end{cases} \tag{4.64}$$

-Stability of the closed-loop system:

$$(A + BK^+ - BK^-)x(t) + (A_1 + BK_1^+ - BK_1^-)x(t - \tau(t)) \text{ is stable} \tag{4.65}$$

Moreover, we will show that the asymptotic stability of linear time-delay systems is independent of their delays. In the following, a new approach is investigated to establish a theorem guaranteeing the stability and positivity of system (4.63) with the memory control law given in (4.62).

For this, we consider the following assumption:

Assumption 4.2.

- The pairs (A, B) are controllable.

- The pairs $(A + A_1, B)$ are controllable.

First of all, we rewrite system (4.63) in the following form:

$$\dot{x}(t) = A_c x(t) + A_{c1} x(t - \tau(t)) \tag{4.66}$$

where:

$$A_c = A + BK^+ - BK^- \text{ and } A_{c1} = A_1 + BK_1^+ - BK_1^- \tag{4.67}$$

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We are allowed to decompose the system matrices A_c and A_{c1} as follows:

$$A_c = A + BK^+ - BK^- = A + BK_{r1} - B(K^+ - K^- - K_{r1}) = A_{r1} + BR_1 \qquad (4.68)$$

$$A_{c1} = A_1 + B(K_1^+ - K_1^-) \qquad (4.69)$$

where

$$A_{r1} = A + BK_{r1}, \; R_1 = K^+ - K^- - K_{r1}.$$

The following theorem provides a solution to the positive state-feedback stabilization problem to the class of the considered system.

Theorem 4.4. (Zaidi et al, 2015b)

Suppose that A is Metzler, $B \geqslant 0$, $A_j \geqslant 0$, $0 \leq \tau(t) \leq \bar{h}$, $\dot{\tau}(t) \leq d < 1$. The closed-loop system (4.63) is positive and asymptotically stable, if there exist symmetric and positive definite matrices P, Q and W, matrices X, Z, Y, matrices γ_i, $i = 1, \dots, 4$, L, E and a positive scalar β such that the following conditions:

$$X - Y < 0 \qquad (4.70)$$

$$\begin{pmatrix} Z & \bar{h}Y \\ \bar{h}Y^T & \bar{h}X \end{pmatrix} \geq 0 \qquad (4.71)$$

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_{r1} \; A_1 \; 0 \; -I \; B) \right\}$$

$$+ Sym\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} ((L^+ - L^- - EK_{r1}) \; (L_1^+ - L_1^-) \; 0 \; 0 \; -I) \right\} < 0 \qquad (4.72)$$

$$AE + B(L^+ - L^-) + \beta E \geqslant 0 \qquad (4.73)$$

$$A_1 E + B(L_1^+ - L_1^-) \geqslant 0 \qquad (4.74)$$

are satisfied. The stabilizing state-feedback gains are given by:

$$L^+ = EK^+, \; L^- = EK^-, \; L_1^+ = EK_1^+ \; and \; L_1^- = EK_1^- \qquad (4.75)$$

with

$$\chi_0 = \bar{h}^2 W, \; \chi_1 = Q + \bar{\tau}(Z + Y + Y^T), \; \chi_2 = Q \; and \; \chi_3 = \bar{h}Y$$

Proof:

By replacing A_c by A_{c1} in the asymptotic stability condition (4.26) of Theorem 4.2, we get:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ 0 & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_c \ \ A_{c1} \ \ 0 \ \ -I) \right\} < 0
\tag{4.76}
$$

Using the decomposition of A_c and A_{c1} given in (4.95) and (4.96), the inequality (4.76) becomes:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_{r1} \ \ A_1 \ \ 0 \ \ -I) \right\}
$$

$$
+ Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (B(K^+ - K^- - K_{r1}) \ \ B(K_1^+ - K_1^-) \ \ 0 \ \ 0) \right\} < 0
\tag{4.77}
$$

Denote that:

$$
\Omega = \begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_r \ \ A_1 \ \ 0 \ \ -I) \right\}
\tag{4.78}
$$

We choose the gains K^-, K^+, K_1^- and K_1^+ in order to maintain the term Ω negative. We can apply Lemma B.3 (Annex), LMI (4.77) is equivalent to:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_r \ \ A_1 \ \ 0 \ \ -I \ \ B) \right\}
$$

$$
+ Sym\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} E\left((K^+ - K^- - K_{r1}) \ \ (K_1^+ - K_1^-) \ \ 0 \ \ 0 \ \ -I \right) \right\} \leq 0
\tag{4.79}
$$

By imposing $L^+ = EK^+$, $L^- = EK^-$, $L_1^+ = EK_1^+$ and $L_1^- = EK_1^-$, we get the inequality (4.72).

As for positivity, we have to guarantee that:

$$\begin{cases} A_1 + BK^+ - BK^- \text{ is Metzler} \\ A_1 + BK_1^+ - BK_1^- \geqslant 0 \end{cases}$$

(4.80)

Or equivalently that:

$$\begin{cases} A_1 + BK^+ - BK^- + \beta I \geqslant 0 \\ A_1 + BK_1^+ - BK_1^- \geqslant 0 \end{cases}$$

(4.81)

Then, by multiplying each inequality of (4.81) on the left by matrix E, using and the variable changes $L^+ = EK^+$, $L^- = EK^-$, $L_1^+ = EK_1^+$ and $L_1^- = EK_1^-$, we obtain (4.73) and (4.74). ∎

4.2. Stabilization of Positive Takagi-Sugeno systems with time-delay

4.2.1. INTRODUCTORY REMARKS

In this section, we are meant to consider an additional problem usually found in dynamical systems: the nonnegativity of the states. The study of positive Takagi-Sugeno systems has a great importance in practical fields because many chemical, physical and biological processes require quantities that have intrinsically constant and nonnegative signs: the concentrations of substances, the levels of liquids, etc, are always nonnegative (Rami & Tadeo, 2006), (Rami & Tadeo, 2007), (Benzaouia & Tadeo, 2010) and (Benzaouia & Tadeo, 2008). The design of stabilizing control laws for positive linear and T-S time –delay systems has been studied by (Benzaouia & El Hajjaji, 2011), (Benzaouia & Oubah, 2014), (Benzaouia & Hajjaji, 2014), (Benzaouia et al, 2014), etc.

We are interested, in this section, in the stability analysis and stabilization of positive T-S systems with time-delay.

4.2.2. STABILIZATION OF POSITIVE T-S TIME-DELAY SYSTEMS WITH A DECOMPOSED STATE-FEEDBACK CONTROL

4.2.2.1. Asymptotic Stabilization of Positive T-S systems with time-delay with decomposed state-feedback controller

Consider a a T-S time-delay model described by:

Rule i: If z_1 is F_i^1 and...and z_p is F_i^p Then

$$\dot{x}(t) = A_i x(t) + A_{1i} x\big(t - \tau(t)\big) + B_i u(t)$$

(4.82)

$$x(t) = \varphi(t), t \in \left[-\overline{h}, 0\right]$$

where $z_j(x(t))$ and $\alpha_{ij}(i = 1, \dots, r, j = 1, \dots, p)$ are the premise variables and the fuzzy sets, $\varphi(t)$ is the initial conditions; $x(t) \in \mathbb{R}^{n_x}$ is the state and $u(t) \in \mathbb{R}^{n_u}$ is the control input. r is the number of IF-THEN rules. The time delay $\tau(t)$ is a time-varying continuous function such that

$$0 \leq \tau(t) \leq \overline{h}, \ \dot{\tau}(t) \leq d < 1 \tag{4.83}$$

Then, the overall T-S system can be inferred as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\Big(A_i x(t) + A_{1i} x\big(t - \tau(t)\big) + B_i u(t)\Big) \tag{4.84}$$

In this section, we design for system (4.84) a decomposed Takagi-Sugeno state-feedback controller of the form

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big) K_i x(t) \tag{4.85}$$

For any matrices K_i, $i = 1, \dots, r$, it is obvious that there exist nonnegative matrices K_i^-, K_i^+ such that:

$$K_i = K_i^+ - K_i^-, i = 1, \dots, r \tag{4.86}$$

Using this fact, we rewrite the state-feedback controller (4.85) in the following form:

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)(K_i^+ - K_i^-)x(t) \tag{4.87}$$

Then, the corresponding closed-loop system becomes:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) \Big((A_i + B_i K_j^+ - B_i K_j^-)x(t) + A_{1i}x(t - \tau(t))\Big) \tag{4.88}$$

Thereafter, we develop the conditions of the stabilization problem: Design of a robust state-feedback controller (4.87) such that the closed-loop system (4.88) is positive and globally asymptotically stable.

For this, we consider the following assumption.

Assumption 4.3.

- The pair (A_i, B_i) is controllable.

- The pair $(A_i + A_{1i}, B_i)$ is controllable.

Next, we will provide necessary and sufficient conditions on the matrices A_i, B_i such that there exist matrices $K_i^+, K_i^-, \in \mathbb{R}_+^{n_u \times n_x}$, $i = 1, \dots, r$ satisfying:

- Positivity of the closed-loop system:

$$\begin{cases} A_i + B_i K_j^+ - B_i K_j^- \text{ are Metzler}, \forall i, j = 1, \dots, r \\ A_{1i} \succcurlyeq 0, \forall i = 1, \dots, r \end{cases}$$

- Stability of the closed-loop system:

$$\left(A_i + B_i K_j^+ - B_i K_j^-\right) x(t) + A_{1i} x(t - \tau(t)) \text{ is stable.}$$

In the following theorem, sufficient conditions are given to guarantee the positivity and asymptotic stability of the closed-loop system.

Theorem 4.5.

The closed-loop system (4.88) is positive and asymptotically stable with the control law (4.87) if the following LMI problem in the variables $\beta \in \mathbb{R}_+$, diagonal P, Y_i^+, $Y_i^- \in \mathbb{R}_+^{n_u \times n_x}$ is feasible:

$$\begin{cases} A_i P + P A_i^T + B_i Y_i^+ + Y_i^{+^T} B_i - B_i Y_i^- - Y_i^{-^T} B_i < 0 \\ A_i P + B_i Y_i^+ - B_i Y_i^- + \beta I \succcurlyeq 0 \\ A_{1i} \succcurlyeq 0 \\ P > 0 \end{cases} \qquad (4.89)$$

Moreover, the gain matrices K_i^+ and K_i^-, $i = 1, \dots, r$ can be computed as follows:

$$K_i^+ = Y_i^+ P^{-1}, K_i^- = Y_i^- P^{-1} \qquad (4.90)$$

where P, Y_i^+, Y_i^- are any feasible solution to the above LMI problem (4.88).

Proof:

The proof is deduced from the generalization of that of (Bolajraf, 2012) for positive time-delay T-S systems.

4.2.2.2. Robust α-stabilization of positive T-S systems with time-delay with decomposed state-feedback controller

We consider the T-S time-delay system (4.84) where the matrices A_i, A_{1i} and B_i are uncertain with known bounds $\underline{A}_i$, $\overline{A}_i$, $\underline{A}_{1i}$, $\overline{A}_{1i}$ $\underline{B}_i$ and $\overline{B}_i$, $i = 1, \dots, r$, such that: $\underline{A}_i \leq A_i \leq \overline{A}_i$, $\underline{A}_{1i} \leq A_{1i} \leq \overline{A}_{1i}$, $\underline{B}_i \leq B_i \leq \overline{B}_i$.

We may use some results regarding the positivity and α-stability of the autonomous case of system (4.84) which can resulted by exploiting the monotonicity property below, with respect to the dynamical matrices of system (4.84).

Lemma 4.3.

Consider the two following autonomous time-delay systems:

$$
\begin{cases}
\dot{x}_1(t) = A^1 x_1(t) + \sum_{i=1}^{p} A_i^1 x_1(t - \tau_i) \\
\dot{x}_2(t) = A^2 x_2(t) + \sum_{i=1}^{p} A_i^2 x_2(t - \tau_i) \\
0 \preccurlyeq x_1(\theta) \preccurlyeq x_2(\theta), \theta \in [-\tau, 0], \tau = \max_{1 \leq i \leq p} \tau_i
\end{cases}
\tag{4.91}
$$

Then, if A^1 is Metzler and the matrices A_i^1 are nonnegative matrices, we have:

$$0 \preccurlyeq x_1(t) \preccurlyeq x_2(t), \forall t \geqslant 0 \tag{4.92}$$

Proof:

This lemma is an extension of Theorem 1.17 in the case of positive T-S systems. ∎

In this section, we design for system (4.84) a T-S memory state-feedback controller of the form

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\big(K_i x(t) + K_{1i} x(t - \tau(t))\big) \tag{4.93}$$

In order to reduce the conservatism, we decomposed the state-feedback gain independently of the sign of these feedback gains as follows; There exist nonnegative matrices K_j^-, K_j^+, K_{1j}^- and K_{1j}^+, $j = 1, \dots, r$ such that:

$$K_j = K_j^+ - K_j^-, K_{1j} = K_{1j}^+ - K_{1j}^-, j = 1, \dots, r j = 1, \dots, r \tag{4.94}$$

Using this fact, we rewrite the state-feedback controller (4.93) in the following form:

$$u(t) = \sum_{i=1}^{r} h_i\big(z(t)\big)\big((K_i^+ - K_i^-)x(t) + (K_{1i}^+ - K_{1i}^-)x(t - \tau(t))\big) \qquad (4.95)$$

So, the corresponding closed-loop T-S system becomes:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z)h_j(z)\Big((A_i + B_i K_j^+ - B_i K_j^-)x(t) + \big(A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^-\big)x(t - \tau(t))\Big)$$

$$(4.96)$$

Therefore, we develop the conditions of the stabilization problem: Design of a robust memory state-feedback controller (4.95) such that the closed-loop system (4.96) is positive and α-stable.

Next, we will provide sufficient conditions on the matrices A_i, A_{1i}, B_i such that there exist matrices K_j^+, K_j^-, K_{1j}^+ and $K_{1j}^- \in \mathbb{R}_+^{n_u \times n_x}$, $i,j = 1, \dots, r$ satisfying:

- Positivity of the closed-loop system:

$$\begin{cases} A_i + B_i K_j^+ - B_i K_j^- \text{ are Metzler}, \forall i,j = 1, \dots, r \\ A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^- \geqslant 0, \forall i,j = 1, \dots, r \end{cases}$$

- Stability of the closed-loop system:

$$\sum_{i,j=1}^{r} h_i(z(t))h_j(z(t)) \Big((A_i + B_i K_j^+ - B_i K_j^-)x(t) + \big(A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^-\big)x(t - \tau(t))\Big)$$

is α-stable.

In the following theorem, we establish then a theorem that guarantees the α-stability and positivity of system (4.96):

Theorem 4.6. (Zaidi et al, 2015a)

For a specific decay rate $\alpha > 0$, the closed-loop T-S system (4.96) is positive and α-stable with the control law (4.95) if the following LMI problem in the variables $\beta \in \mathbb{R}_+$, $d \in \mathbb{R}^n$, $Y_i^+, Y_i^-, Y_j^+, Y_j^- \in \mathbb{R}_+^{n_u \times n_x}$ is feasible, $\forall i,j = 1, \dots, r$

$$\begin{cases} \overline{A}_{0i}\,diag(d) + diag(d)\overline{A}_{0i}^T + \overline{B}_i Y_{0j}^+ - \underline{B}_i Y_{0j}^- + \big(Y_{0j}^+\big)^T \overline{B}_i^T - \big(Y_{0j}^-\big)^T \underline{B}_i^T < 0 \\ \underline{A}_i\,diag(d) + \underline{B}_i Y_j^+ - \overline{B}_i Y_j^- + \beta I \geqslant 0 \\ \underline{A}_{1i}\,diag(d) + \overline{B}_i Y_{1j}^+ - \underline{B}_i Y_{1j}^- \geqslant 0 \\ d > 0 \end{cases} \qquad (4.97)$$

where

$A_{0i} = A_i + \alpha I + e^{\alpha \bar{h}} A_{1i}$, $Y_{0j}^+ = Y_j^+ + e^{\alpha \bar{h}} Y_{1j}^+$ and $Y_{0j}^- = Y_j^- + e^{\alpha \bar{h}} Y_{1j}^-$.

Moreover, the $\boldsymbol{\alpha}$-stabilizing gain matrices K_j and K_{1j}, $j = 1, ..., r$ can be computed as follows:

$$\begin{cases} K_j = \left(Y_j^+ + Y_j^-\right) diag(d)^{-1} \\ K_{1j} = \left(Y_{1j}^+ + Y_{1j}^-\right) diag(d)^{-1} \end{cases} \tag{4.98}$$

where d, Y_j^+, Y_j^-, Y_{1j}^+, Y_{1j}^- are any feasible solution to the above LMI problem (4.97).

Proof:

Firstly, we have to consider that the closed-loop T-S system (4.96) is positive and $\boldsymbol{\alpha}$-stable by taking into account that $\forall i, j = 1, ..., r$

$$\begin{cases} \left(A_i + B_i K_j^+ - B_i K_j^-\right) x(t) + \left(A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^-\right) x(t - \tau(t)) \text{ is } \alpha\text{-stable} \\ A_i + B_i K_j^+ - B_i K_j^- \text{ is Metzler}, \forall i, j = 1, ..., r \\ A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^- \succcurlyeq 0, \forall i, j = 1, ..., r \end{cases}$$

$$\tag{4.99}$$

with K_j^+, K_j^-, K_{1j}^+ and K_{1j}^- defined in (4.94).

By applying Corollary 1.2, we have that $A_i + B_i K_j^+ - B_i K_j^-$ is $\boldsymbol{\alpha}$-stable if and only if there exists $diag(d) \succ 0$ such that:

$$\left(A_{0i} + B_i K_j^+ - B_i K_j^-\right) diag(d) + diag(d) \left(A_{0i} + B_i K_j^+ - B_i K_j^-\right)^T + 2\alpha diag(d) < 0$$

By recalling that $K_j^+ = Y_j^+ diag(d)^{-1}$, $K_j^- = Y_j^- diag(d)^{-1}$, $K_{1j}^+ = Y_{1j}^+ diag(d)^{-1}$ and $K_{1j}^- = Y_{1j}^- diag(d)^{-1}$, we can obtain from the first inequality :

$$A_{0i} diag(d) + diag(d) A_{0i}^T + B_i Y_{0j}^+ - B_i Y_{0j}^- + \left(Y_{0j}^+\right)^T B_i^T - \left(Y_{0j}^-\right)^T B_i^T < 0 \tag{4.100}$$

where:

$A_{0i} = A_i + \alpha I + e^{\alpha \bar{h}} A_{1i}$, $Y_{0j}^+ = Y_j^+ + e^{\alpha \bar{h}} Y_{1j}^+$, $Y_{0j}^- = Y_j^- + e^{\alpha \bar{h}} Y_{1j}^-$

$K_j^+ = Y_j^+ diag(d)^{-1}$, $K_j^- = Y_j^- diag(d)^{-1}$
$K_{1j}^+ = Y_{1j}^+ diag(d)^{-1}$, $K_{1j}^- = Y_{1j}^- diag(d)^{-1}$

By using Lemma 4.3 of monotonicity, for any upper system

$$\dot{x}^u = \sum_{i,j=1}^{r} h_i(z) h_j(z)$$

$$\left(\left(\overline{A}_i + \overline{B}_i K_j^+ - \underline{B}_i K_j^- \right) x^u + \left(\overline{A}_{1i} + \overline{B}_i K_{1j}^+ - \underline{B}_i K_{1j}^- \right) x^u (t - \tau(t)) \right) \tag{4.101}$$

we have $0 \leqslant x(t) \leqslant x^u(t)$. Then, if this upper system is α-stable, we have that $0 \leq \lim_{t \to \infty} e^{\alpha t} x(t) \leq lim_{t \to \infty} e^{\alpha t} x^u(t) = 0$. We conclude then that system (4.96) is α-stable.

The second inequality of (4.97) and Remark 1.1 yield that:

$$A_i + B_i K_j^+ - B_i K_j^- + \beta_i I \geqslant 0 \tag{4.102}$$

Then, since $\underline{A}_i \leqslant A_i \leqslant \overline{A}_i$, $\underline{B}_i \leqslant B_i \leqslant \overline{B}_i$, $K_j^+ = Y_j^+ diag(d)^{-1}$ and $K_j^- = Y_j^- diag(d)^{-1}$, we get $\underline{A}_i diag(d) + \underline{B}_i Y_j^+ - \overline{B}_i Y_j^- + \beta I \geqslant 0$. The same argument is valid for the third inequality of (4.97).

The reverse implication can be deduced from simple matrix manipulation, so the proof is complete.■

4.2.2.3. Illustrative Example: Hydraulic two-tank-system

In order to show the interest of this study, we consider a process composed of two hydraulic tanks linked to each other (Zhang & Ding, 2005), (Benzaouia & El Hajjaji, 2011).

In a simplified form, this system can be described by the following balance equations (Benzaouia & El Hajjaji, 2011):

$$\dot{x}(t) = A(z_1, z_2) x(t) + B u(t) \tag{4.103}$$

where z_1 and z_2 are the levels in each tank and the system matrices are:

$$B = I_2$$

$$A(z_1, z_2) = \begin{pmatrix} -R_1 z_1 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \\ \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & -R_2 z_2 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \end{pmatrix} \tag{4.104}$$

which is Metzler, with R_1, R_2 and R_{12} positive physical constants, as shown in (Benzaouia & El Hajjaji, 2011).

However, this system is known to have transport delays, so system (4.103) is replaced here by a T-S time-delay system given by:

$$\dot{x}(t) = \hat{A}(z_1, z_2) x(t) + \hat{A}_\tau(z_1, z_2) x(t - \tau(t)) + B u(t) \tag{4.105}$$

where:

$$\hat{A} = \begin{pmatrix} -R_1 z_1 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & 0 \\[4ex] 0 & -R_2 z_2 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \end{pmatrix} \tag{4.106}$$

and

$$\hat{A}_\tau = \begin{pmatrix} 0 & \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \\[4ex] \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & 0 \end{pmatrix} \tag{4.107}$$

By considering that $z_i \in [a_i; b_i]$, the four following rules are taken into account:

IF $z_1(t)$ is a_1 and $z_2(t)$ is a_2, THEN

$$\hat{A}(z_1, z_2) = \hat{A}(a_1, a_2) \doteq \hat{A}_1 = \begin{pmatrix} -R_1 a_1 - \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & 0 \\[4ex] 0 & -R_2 a_2 - \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \end{pmatrix}$$

$$\hat{A}_\tau(z_1, z_2) = \hat{A}_\tau(a_1, a_2) \doteq \hat{A}_{\tau 1} = \begin{pmatrix} 0 & \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \\[4ex] \dfrac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & 0 \end{pmatrix}$$

IF $z_1(t)$ is a_1 and $z_2(t)$ is b_2, THEN

$$\hat{A}(z_1, z_2) = \hat{A}(a_1, b_2) \doteq \hat{A}_2 = \begin{pmatrix} -R_1 a_1 - \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & 0 \\[4ex] 0 & -R_2 b_2 - \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \end{pmatrix}$$

$$\hat{A}_\tau(z_1, z_2) = \hat{A}_\tau(a_1, b_2) \doteq \hat{A}_{\tau 2} = \begin{pmatrix} 0 & \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \\[4ex] \dfrac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & 0 \end{pmatrix}$$

IF $z_1(t)$ is b_1 and $z_2(t)$ is a_2, THEN

$$\hat{A}(z_1, z_2) = \hat{A}(b_1, a_2) \doteq \hat{A}_3 = \begin{pmatrix} -R_1 b_1 - \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & 0 \\ 0 & -R_2 a_2 - \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \end{pmatrix}$$

$$\hat{A}_\tau(z_1, z_2) = \hat{A}_\tau(b_1, a_2) \doteq \hat{A}_{\tau 3} = \begin{pmatrix} 0 & \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \\ \dfrac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & 0 \end{pmatrix}$$

IF $z_1(t)$ is b_1 and $z_2(t)$ is b_2, THEN

$$\hat{A}(z_1, z_2) = \hat{A}(b_1, b_2) \doteq \hat{A}_4 = \begin{pmatrix} -R_1 b_1 - \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & 0 \\ 0 & -R_2 b_2 - \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \end{pmatrix}$$

$$\hat{A}_\tau(z_1, z_2) = \hat{A}_\tau(b_1, b_2) \doteq \hat{A}_{\tau 4} = \begin{pmatrix} 0 & \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \\ \dfrac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & 0 \end{pmatrix}$$

The membership functions are given by:

$$h_1(t) = f_{11}(t) f_{21}(t), \ h_2(t) = f_{11}(t) f_{22}(t),$$

$$h_3(t) = f_{12}(t) f_{21}(t), \ h_4(t) = f_{12}(t) f_{22}(t)$$

with $f_{i1}(t) = \dfrac{z_i(t) - b_i}{a_i - b_i}$, $f_{i2}(t) = 1 - f_{i1}(t) = \dfrac{a_i - z_i(t)}{a_i - b_i}$, $i = 1,2$.

For calculation, we fix the parameters and their uncertainties as follows:

$R_1 = R_2 = 0.95 \pm 0.02, \ R_{12} = 0.65 \pm 0.03, \ a_1 = 0.2236, \ b_1 = 0.4472$

$a_2 = 0.2582, \ b_2 = 0.4082$.

$\tau(t) = 6 + 3\sin(t), \ \overline{h} = 9$.

In order to α-stabilize the T-S time-delay system (4.105) while imposing positivity in closed-loop, we solved the LMIs of Theorem 4.6. If these LMIs are feasible, one can compute the required controllers gains K_i and the corresponding Lyapunov function $P = diag(d)$. Using the LMI TOOLBOX in Matlab, the

conditions (4.97) are feasible when, for example, $\alpha = 2$ and $\beta = 3$. The obtained solutions are as follows:

$$diag(d) = \begin{pmatrix} 0.0214 & 0 \\ 0 & 0.0132 \end{pmatrix}$$

$$Y_1^- = \begin{pmatrix} 0.0041 & 0.0052 \\ 0.0034 & 0.0059 \end{pmatrix}, Y_1^+ = \begin{pmatrix} 0.0024 & 0.0064 \\ 0.0091 & 0.0022 \end{pmatrix}$$

$$Y_2^- = \begin{pmatrix} 0.0018 & 0.0031 \\ 0.0019 & 0.0022 \end{pmatrix}, Y_2^+ = \begin{pmatrix} 0.0039 & 0.0016 \\ 0.0066 & 0.0023 \end{pmatrix}$$

$$Y_3^- = \begin{pmatrix} 0.0072 & 0.0380 \\ 0.0045 & 0.0056 \end{pmatrix}, Y_3^+ = \begin{pmatrix} 0.0021 & 0.0036 \\ 0.0055 & 0.0023 \end{pmatrix}$$

$$Y_4^- = \begin{pmatrix} 0.0215 & 0.0012 \\ 0.0073 & 0.0026 \end{pmatrix}, Y_4^+ = \begin{pmatrix} 0.0027 & 0.0064 \\ 0.0091 & 0.0052 \end{pmatrix}$$

$$Y_{11}^- = \begin{pmatrix} 0.0011 & 0.0081 \\ 0.0094 & 0.0064 \end{pmatrix}, Y_{11}^+ = \begin{pmatrix} 0.0032 & 0.0049 \\ 0.0015 & 0.0042 \end{pmatrix}$$

$$Y_{12}^- = \begin{pmatrix} 0.0015 & 0.0045 \\ 0.0034 & 0.0082 \end{pmatrix}, Y_{12}^+ = \begin{pmatrix} 0.0031 & 0.0083 \\ 0.0064 & 0.0045 \end{pmatrix}$$

$$Y_{13}^- = \begin{pmatrix} 0.0067 & 0.0085 \\ 0.0013 & 0.0052 \end{pmatrix}, Y_{13}^+ = \begin{pmatrix} 0.0064 & 0.0521 \\ 0.0022 & 0.0351 \end{pmatrix}$$

$$Y_{14}^- = \begin{pmatrix} 0.0051 & 0.0078 \\ 0.0034 & 0.0070 \end{pmatrix}, Y_{14}^+ = \begin{pmatrix} 0.0062 & 0.0023 \\ 0.0071 & 0.0075 \end{pmatrix}$$

Then, the following gain matrices can be calculated:

$$K_1 = \begin{pmatrix} 0.3037 & 0.8788 \\ 0.5841 & 0.6136 \end{pmatrix}, K_2 = \begin{pmatrix} 0.2664 & 0.3561 \\ 0.3972 & 0.3409 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 0.4346 & 3.1515 \\ 0.4673 & 0.5985 \end{pmatrix}, K_4 = \begin{pmatrix} 1.1308 & 0.5758 \\ 0.7664 & 0.5909 \end{pmatrix}$$

$$K_{11} = \begin{pmatrix} 0.2009 & 0.9848 \\ 0.5093 & 0.8030 \end{pmatrix}, K_{12} = \begin{pmatrix} 0.2150 & 0.9697 \\ 0.4579 & 0.9621 \end{pmatrix}$$

$$K_{13} = \begin{pmatrix} 0.6121 & 4.5909 \\ 0.1636 & 3.0530 \end{pmatrix}, K_{14} = \begin{pmatrix} 0.5280 & 0.7652 \\ 0.4907 & 1.0985 \end{pmatrix}$$
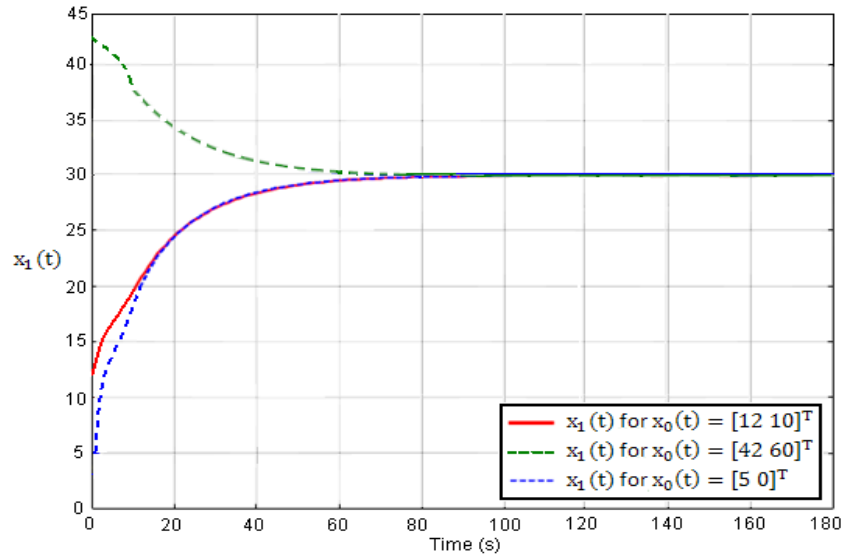
Figure 4. 1. State trajectories of $x_1(t)$ for different initial conditions, $\forall t \in [-9,0]$
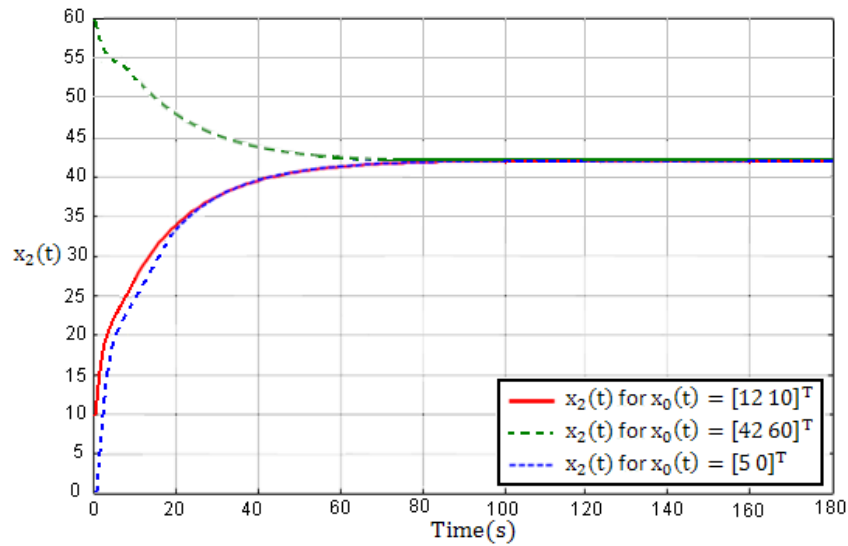
Figure 4. 2. State trajectories of $x_2(t)$ for different initial conditions, $\forall t \in [-9,0]$
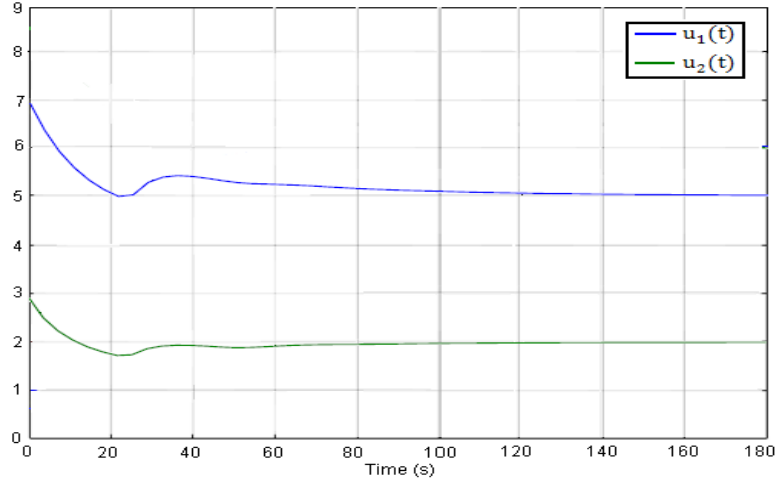
Figure 4. 3. The evolution of the two pump flows $u_1(t)$ and $u_2(t)$

Figures 4.1 and 4.2 represent the evolutions of $x_1(t)$ and $x_2(t)$ respectively starting from different initial conditions $[12\ 10]^T$, $[42\ 60]^T$ and $[5\ 0]^T$ for $t \in [-9; 0]$. The desired reference $y_r = [30; 42]^T$ is reached while the state always remains nonnegative. Figure 4.3 plots the evolution of inputs of the considered hydraulic system: the pump flows $u_1(t)$ and $u_2(t)$.

4.2.2.4. Asymptotic Stabilization of Positive time-delay T-S systems with decomposed memory state-feedback controller

In this section, we establish new conditions for the design of a decomposed memory state-feedback control which guarantees the stability and positivity of positive T-S systems with multiple and time-varying delays.

Consider a T-S time-delay model described by (4.84).

We recall that we aim to design for system (4.84) a memory and decomposed T-S state-feedback controller of the form (4.93)

Then, the corresponding closed-loop system becomes:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))$$

$$\left(\left(A_i + B_i K_j^+ - B_i K_j^-\right)x(t) + \left(A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^-\right)x(t - \tau_i(t))\right) \qquad (4.108)$$

under the initial condition: $x(t) = \varphi(t) \in \mathbb{R}_+^{n_x}, t \in \left[-\overline{h}, 0\right]$.

Next, we provide sufficient conditions on the matrices A_i, A_{1i} and B_i such that there exist matrices K_j^+, K_j^-, K_{1j}^+ and $K_{1j}^- \in \mathbb{R}_+^{n_u \times n_x}$, $i, j = 1, \dots, r$ satisfying:

-Positivity of the closed-loop system: $\forall\, i, j = 1, \dots, r$

$$\begin{cases} A_i + B_i K_j^+ - B_i K_j^- \text{ are Metzler} \\ A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^- \geqslant 0 \end{cases} \tag{4.109}$$

-Stability of the closed-loop system: $\forall i, j = 1, \dots, r$

$$\sum_{i,j=1}^{r} h_i(z) h_j(z) \left(\left(A_i + B_i K_j^+ - B_i K_j^- \right) x + \left(A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^- \right) x(t - \tau_i(t)) \right) \tag{4.110}$$

In the following, necessary and sufficient conditions are developed for positivity and asymptotic stability of the closed-loop system (4.108).

First of all, Assumption 4.3 has to be guaranteed.

Then, we rewrite system (4.108) in the following form:

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) \left(A_{cij} x(t) + A_{c1ij} x(t - \tau_j(t)) \right) \tag{4.111}$$

where: $A_{cij} = A_i + B_i K_j^+ - B_i K_j^-$ and $A_{c1ij} = A_{1i} + B_i K_{1j}^+ - B_i K_{1j}^-$

Equivalently, system (4.111) may be given in a compact form as follows:

$$\dot{x}(t) = A_c x(t) + A_{c1} x_{\tau j}(t) \tag{4.112}$$

where:

$$A_c = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) A_{cij} \tag{4.113}$$

$$A_{c1} = \sum_{i,j=1}^{r} h_i(z(t)) h_j(z(t)) A_{c1ij} \tag{4.114}$$

$$x_{\tau j}(t) = x\big(t - \tau_j(t)\big) \tag{4.115}$$

Therefore, conditions ensuring the asymptotic stability and positivity of the system (4.108) are based on the works of (Saadni, 2006) and summarized in the following Theorem.

Theorem 4.7. (Zaidi et al, 2015b)

Suppose that $0 \leq \tau_j(t) \leq \overline{h}_j$, $\dot{\tau}_j(t) \leq d_j < 1$, $j = 1, \dots, r$. The system (4.108) is positive and asymptotically stable, if there exist symmetric and positive definite matrices P, Q_j and W_j, matrices X_j, Z_j, Y_j, $j = 1, \dots, r$, matrices γ_i, $i = 1, \dots, 4$, L, E and a positive scalar β such that the following conditions:

$$X_j - W_j < 0 \tag{4.116}$$

$$\begin{pmatrix} Z_j & \overline{h}_j Y_j \\ \overline{h}_j Y_j^T & \overline{h}_j X_j \end{pmatrix} \geq 0 \tag{4.117}$$

$$\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_i \ A_{1i} \ 0 \ -I \ B_i) \right\}$$

$$+ Sym\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \left((L_j^+ - L_j^-) \ (L_{1j}^+ - L_{1j}^-) \ 0 \ 0 \ -I \right) \right\} \leq 0 \tag{4.118}$$

$$A_k E + B_k L_j^+ - B_k L_j^- + \beta E \succcurlyeq 0, k = 1, \dots, r \tag{4.119}$$

$$A_{1k} E + B_k \left(L_{1j}^+ - L_{1j}^- \right) \succcurlyeq 0, k = 1, \dots, r \tag{4.120}$$

are satisfied. The stabilizing state-feedback gains are given by:

$$L^+ = EK^+, \ L^- = EK^-, \ L_1^+ = EK_1^+ \ and \ L_1^- = EK_1^- \tag{4.121}$$

with

$$\chi_0 = \overline{h}_j^2 W_j, \ \chi_1 = [Q_j + (1 - d_j)\overline{h}_j(Z_j + Y_j + Y_j^T)]$$

$$\chi_2 = diag\left((1 - d_1)Q_1, \dots, (1 - d_r)Q_r \right)$$

$$\chi_3 = \left((1 - d_1)\overline{h}_1 Y_1, \dots, (1 - d_r)\overline{h}_r Y_r \right)$$

Moreover, the state-feedback gains for the stabilizing controller (4.93) are given by:

$$K_j^+ = E^{-1}L_j^+, \ K_j^- = E^{-1}L_j^-, \ K_{1j}^+ = E^{-1}L_{1j}^+, K_{1j}^- = E^{-1}L_{1j}^- \tag{4.122}$$

Proof:

By replacing A_{cij} by A_{c1ij} in the asymptotic stability condition (4.26) of Theorem 4.2, we get:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_{cij} \ A_{c1ij} \ 0 \ -I) \right\} < 0 \tag{4.123}
$$

Using the decomposition of A_{cij} and A_{c1ij}, the inequality (4.123) becomes:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_i \ A_{1i} \ 0 \ -I) \right\}
$$

$$
+ Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (B_i(K_j^+ - K_j^-) \ B_i(K_{1j}^+ - K_{1j}^-) \ 0 \ 0) \right\} < 0 \tag{4.124}
$$

Denote that:

$$
\Omega = \begin{pmatrix} \chi_1 & -\chi_3 & 0 & P \\ * & -\chi_2 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 \\ * & * & \chi_0^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} (A_i \ A_{1i} \ 0 \ -I) \right\} \tag{4.125}
$$

We choose the gains K^-, K^+, K_1^- and K_1^+ in order to maintain the term Ω negative. Applying then Lemma B.2 (Annex), LMI (4.123) is equivalent to:

$$
\begin{pmatrix} \chi_1 & -\chi_3 & 0 & P & 0 \\ 0 & -\chi_2 & 0 & 0 & 0 \\ * & * & -\chi_0 & \chi_0 & 0 \\ * & * & \chi_0^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ 0 \end{pmatrix} (A_i \ A_{1i} \ 0 \ -I \ B) \right\}
$$

$$
+ Sym\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} E \left((K_j^+ - K_j^-) \ (K_{1j}^+ - K_{1j}^-) \ 0 \ 0 \ -I \right) \right\} \leq 0 \tag{4.126}
$$

By imposing $L_j^+ = EK_j^+$, $L_j^- = EK_j^-$, $L_{1j}^+ = EK_{1j}^+$ and $L_{1j}^- = EK_{1j}^-$, we get the inequality (4.118).

As for positivity, we have to guarantee that: $\forall j, k = 1, ..., r$

$$
\begin{cases} A_k + B_k K_j^+ - B_k K_j^- \ are \ Metzler \\ A_{1k} + B_k K_{1j}^+ - B_k K_{1j}^- \geqslant 0 \end{cases} \tag{4.127}
$$

Or equivalently that:

$$
\begin{cases} A_k + B_k K_j^+ - B_k K_j^- + \beta I \geqslant 0 \\ A_{1k} + B_i K_{1k}^+ - B_i K_{1k}^- \geqslant 0 \end{cases} \tag{4.128}
$$

Then, by multiplying each inequality of (2.128) on the left by matrix E and using the changes of variable $L_j^+ = EK_j^+$, $L_j^- = EK_j^-$, $L_{1j}^+ = EK_{1j}^+$ and $L_{1j}^- = EK_{1j}^-$, we obtain (4.119) and (4.120). ∎

Remark 4.4.

We can deduce, from the LMIs of Theorem 4.7, the asymptotic stability of the positive time-delay T-S system (4.108) does not depend on its delays $\tau_i(t)$, $i = 1, \ldots, r$, but only on their upper bound $\overline{h}$ and the upper bounds of its derivatives d_i, which make the conditions less conservative than the stability conditions of Theorem 4.6 and reduces the complexity of the calculation by reducing the number of the LMIs to solve.

4.2.2.5. Illustrative Example

Consider the following T-S time-delay system:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(z(t))\big(A_i x(t) + A_{1i} x(t - \tau_i) + B_i u(t)\big) \tag{4.129}$$

where:

$A_1 = \begin{pmatrix} -0.4 & 0 \\ 0 & -0.6 \end{pmatrix}$, $A_2 = \begin{pmatrix} -0.7 & 0 \\ 0.2 & -0.5 \end{pmatrix}$,

$A_{11} = \begin{pmatrix} 0.04 & 0.08 \\ 0 & 0.01 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 0.02 & 0.08 \\ 0.01 & 0 \end{pmatrix}$, $B_1 = B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$h_1\big(x_1(t)\big) = \dfrac{1}{1 + exp(-2x_1(t))}$

$h_2\big(x_1(t)\big) = 1 - h_1\big(x_1(t)\big)$

$\tau_1(t) = 0.25 + 0.14\,sin(t)$, $\tau_2(t) = 0.24 + 0.12\,cos(t)$, $\overline{h} = 0.39$.

with the control law given by:

$$u(t) = \sum_{i=1}^{2} h_i(z(t))\big((K_i^+ - K_i^-)x(t) + (K_{1i}^+ - K_{1i}^-)x(t - \tau_i(t))\big) \tag{4.130}$$

Based on Theorem 4.7, we can design a memory controller to stabilize system (4.108); for example, for $\beta = 0.19$, solving the LMIs of Theorem 4.7, we obtain the following feasible solution:

$G = \begin{pmatrix} 0.2241 & 14.6231 \\ 26.3412 & 57.6282 \end{pmatrix}$

$L_1^+ = (13.1011; 71.5970)$, $L_1^- = (0.3784; 12.0896)$

$L_2^+ = (14.6444; 70.0948)$, $L_2^- = (0.9826; 19.8142)$

$L_{11}^+ = (5.8267; 47.8068)$, $L_{11}^- = (13.4859; 62.8412)$

$L_{12}^+ = (8.2651; 43.0504)$, $L_{12}^- = (12.7541; 59.8755)$

Then, the following gain matrices can be calculated

$K_1^+ = (0.7843; 0.8839)$, $K_1^- = (0.4163; 0.0195)$

$K_2^+ = (0.4864; 0.9940)$, $K_2^- = (0.6262; 0.0576)$

$K_{11}^+ = (0.9759; 0.3835)$, $K_{11}^- = (0.3808; 0.9164)$

$K_{12}^+ = (0.4116; 0.5589)$, $K_{12}^- = (0.3776; 0.8664)$

Figures 4.4, 4.5 and 4.6 represent, respectively, the trajectories of the states $x_1(t)$, $x_2(t)$ and the input $u(t)$ for an upper bound of the time-delays $\overline{h} = 0.39$ under two initial conditions $x_0(t)$ for $t \in [-0.39; 0]$.

We can see from Figures 4.4. and Figure 4.5, that the system states always remain in the positive orthant, so do not obtain negative states, for different nonnegative inputs $u_1(t)$ and $u_2(t)$ whose evolutions are illustrated in Figure 4.6. In addition, we can see the delay impact on the evolution of the stabilized states of the system and the stabilization achieved using the decomposed controller.
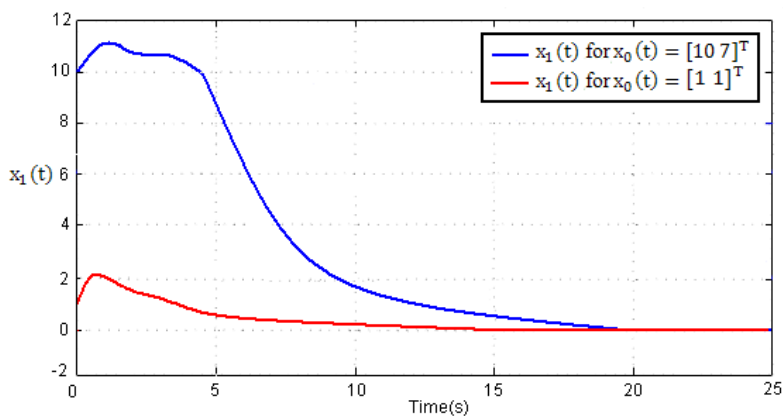
Figure 4. 4. State trajectory of $x_1(t)$
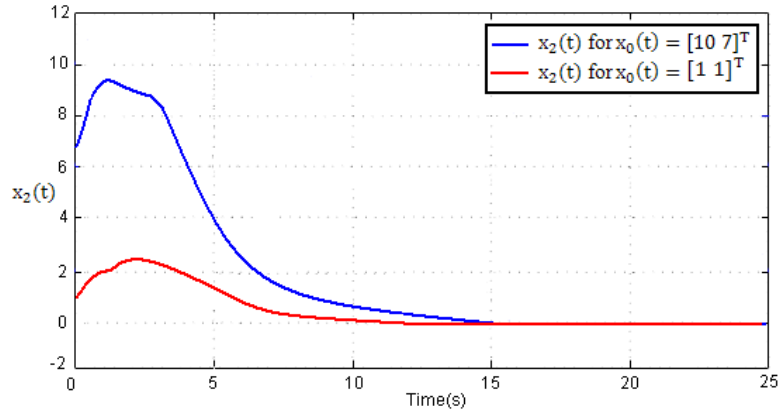
Figure 4. 5. State trajectory of $x_2(t)$
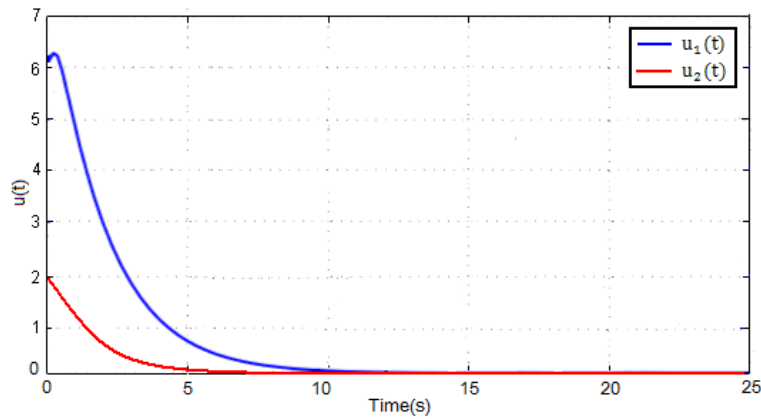
Figure 4. 6. Evolution of the system inputs $u_1(t)$ and $u_2(t)$

4.3. Conclusion

This chapter presents some approaches to the stabilization of positive time-delay linear and T-S systems in nominal and uncertain cases. Asymptotic and exponential stability conditions have been first established for these types of systems, depending on the nature of delays: single, multiple, constant or time-varying. Then, we dealt with the asymptotic stabilization and α-stabilization problem using state-feedback control. Conservatism has been reduced using the decomposition techniques of the designed controller gains and using memory state-feedback control. All conditions are established in terms of Linear Matrix Inequalities (LMIs). Both numerical and practical examples of comparison have been proposed and simulation results have been illustrated in order to show the effectiveness of the methods.

Chapter V

Observers and Controllers for Positive systems with time-delay

5.1. Introduction

In this chapter, we make a proposal for designing observer-based controllers for positive interval linear and T-S systems with time-delay. In this context, LMIs conditions are established by considering interval uncertainties. A positive interval observer is interesting since it can provide lower and upper estimates on the unmeasurable positive states. Also, it can guarantee the stability and positivity of the estimation error of the system. Firstly, we treat the problem of the design of positive observers for positive linear time-delay systems, with and without interval uncertainties. Secondly, we consider the positive observer-based controller design for positive interval linear systems with time-delay. Finally, we extend the previous works for positive interval T-S systems with time-delay, taking into account the nature of the premise variables of the system (measurable or unmeasurable). Numerical and practical examples have been illustrated to prove the effectiveness of the different proposed approaches.

5.2. Positive Observer design for Positive Linear systems with time-delay

5.2.1. POSITIVE OBSERVER DESIGN FOR POSITIVE AUTONOMOUS LINEAR SYSTEMS WITH TIME-DELAY

In this section, we investigate the design of positive observers for positive time-delay linear systems. Conditions on the existence of positive observers are established, with the desired observer matrices constructed through the solution of LMIs. This problem has been treated by proposing a solution using LP techniques that reveal less efficient than LMI formulation (Li & Lam, 2012), (Rami et al, 2013).

Thus, we consider now the observer design of the following linear positive system with time-delay:

$$\begin{cases} \dot{x}(t) = A\,x(t) + A_1 x\big(t - \tau(t)\big) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in \big[-\overline{h}, 0\big] \end{cases} \tag{5.1}$$

where:

$x(t) \in \mathbb{R}^{n_x}$ is the state vector,

$A \in \mathbb{R}^{n_x \times n_x}$ is a Metzler matrix, $A_1 \in \mathbb{R}^{n_x \times n_x} \succcurlyeq 0$ and $C \in \mathbb{R}^{n_y \times n_x} \succcurlyeq 0$

$\tau(t) \in \mathbb{R}$ is the time-varying delay that is considered as a bounded continuous function as follows:

$$\begin{cases} 0 \leq \tau(t) \leq \overline{h} \\ \dot{\tau}(t) \leq d < 1 \end{cases} \qquad (5.2)$$

The Luenberger observer structure used in (Hof, 1998) is not suitable for such systems. For this reason, a more general observer structure will be adopted, of the following form:

$$\dot{\hat{x}}(t) = G \, \hat{x}(t) + G_1 \hat{x}\big(t - h(t)\big) + L \, y(t) \qquad (5.3)$$

where $G \in \mathbb{R}^{n_x \times n_x}$, $G_1 \in \mathbb{R}^{n_x \times n_x}$ and $L \in \mathbb{R}^{n_x \times n_y}$ are the observer matrices to be determined. Throughout this section, we make the following assumption:

Assumption 5.1. The matrix A is Hurwitz.

Define the error signal as $e(t) = x(t) - \hat{x}(t)$ \qquad (5.4)

We define the augmented state $\tilde{x} = [e^T \quad x^T]^T$. Using (5.1) and (5.3), the dynamics of this augmented state are given by the following augmented system:

$$\dot{\tilde{x}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \mathbb{A}_x \, \tilde{x}(t) + \mathbb{A}_h \, \tilde{x}(t - h(t)) \qquad (5.5)$$

where $\mathbb{A}_x = \begin{pmatrix} A & 0 \\ A - LC - G & G \end{pmatrix}$ and $\mathbb{A}_h = \begin{pmatrix} A_1 & 0 \\ A_1 - G_1 & G_1 \end{pmatrix}$

Based on this, we provide the conditions of design of a positive observer, in the following theorem:

Theorem 5.1. (Zaidi et al.)

For given scalars $\beta \geq 0$ and $d < 1$, the augmented system (5.5) is asymptotically stable and positive if there exist diagonal positive definite matrices P_1, P_2, Q_1 and Q_2, matrices $Y \geqslant 0, Z_1 \geqslant 0$ and Z Metzler such that the following conditions are satisfied:

$$\begin{pmatrix} \Omega_{11} & I & \Omega_{12} & 0 & A_1{}^T P_1 & \Omega_{13} \\ I & -2I + Q_1 & 0 & 0 & 0 & Z_1{}^T \\ \Omega_{21} & 0 & Z^T + Z & I & 0 & 0 \\ 0 & 0 & I & -2I + Q_2 & 0 & 0 \\ P_1 A_1 & 0 & 0 & 0 & -(1-d)Q_1 & 0 \\ \Omega_{31} & Z_1 & 0 & 0 & 0 & -(1-d)Q_2 \end{pmatrix} < 0 \quad (5.6)$$

$$\begin{pmatrix} P_1 A & 0 \\ P_2 A - YC - Z & P_2 G \end{pmatrix} + \beta I \geqslant 0 \qquad (5.7)$$

$$\begin{pmatrix} P_1 A_1 & 0 \\ P_2 A_1 - Z_1 & Z_1 \end{pmatrix} \geqslant 0 \qquad (5.8)$$

where:

$$\Omega_{11} = A^T P_1 + P_1 A$$

$$\Omega_{12} = \Omega_{21}^T = A^T P_2 - C^T Y^T - Z^T$$

$$\Omega_{13} = \Omega_{13}{}^T = A_1{}^T P_2 - Z_1{}^T$$

Under these conditions, the desired observer matrices are obtained as follows:

$$G = P_2{}^{-1} Z \qquad (5.9)$$

$$G_1 = P_2{}^{-1} Z_1 \qquad (5.10)$$

$$L = P_2{}^{-1} Y \qquad (5.11)$$

Proof:

In order to make the augmented model (5.5) asymptotically stable, let us consider the Lyapunov–Krasovskii function:

$$V(\tilde{x}(t)) = V_0(\tilde{x}(t)) + V_1(\tilde{x}(t)) \qquad (5.12)$$

where

$$V_0(\tilde{x}(t)) = \tilde{x}(t)^T P \tilde{x}(t), \qquad P = diag[P_1\ P_2] > 0 \qquad (5.13)$$

$$V_1(\tilde{x}(t)) = \int_{t-h(t)}^{t} \tilde{x}(t)^T Q\ \tilde{x}(t)\ ds, \qquad Q = diag[Q_1\ Q_2] > 0 \qquad (5.14)$$

Its derivative with respect to time is given by:

$$\dot{V}(\tilde{x}(t)) = \dot{\tilde{x}}(t)^T P \tilde{x}(t) + \tilde{x}(t)^T P \dot{\tilde{x}}(t) + \tilde{x}(t)^T Q\ \tilde{x}(t)$$

$$- \left(1 - \dot{h}(t)\right) \tilde{x}\left(t - \tau(t)\right)^T Q\ \tilde{x}(t) \qquad (5.15)$$

Firstly, our goal is to guarantee the stability of the augmented system. The augmented time-delay system (5.5) is asymptotically stable if and only if:

$$\dot{V}(\tilde{x}(t)) < 0 \qquad (5.16)$$

Then, we obtain the following inequality:

$$\tilde{x}(t)^T \left(\mathbb{A}_x{}^T P + P\ \mathbb{A}_x + Q\right) \tilde{x}(t) + \tilde{x}\left(t - \tau(t)\right)^T \mathbb{A}_h{}^T P\ \tilde{x}(t)$$

$$+ \tilde{x}(t)^T P\ \mathbb{A}_h\ \tilde{x}\left(t - \tau(t)\right) - \left(1 - \dot{\tau}(t)\right) \tilde{x}\left(t - \tau(t)\right)^T Q\ \tilde{x}\left(t - \tau(t)\right) < 0 \qquad (5.17)$$

Since, $-\big(1 - \dot{\tau}(t)\big) \leq -(1 - d) < 0$ \hfill (5.18)

Then, using Schur Complement (Annex B.2), the expression (5.17) becomes formulated in the following LMI:

$$\begin{pmatrix} \mathbb{A}_x{}^T P + P\,\mathbb{A}_x + Q & \mathbb{A}_h{}^T P \\ P\mathbb{A}_h & -(1-d)Q \end{pmatrix} < 0 \hfill (5.19)$$

In order to facilitate the calculation of the observer gains L, G and G_1, the matrix variables $P > 0$ and $Q > 0$ are chosen to be diagonal with respect to appropriate matrix blocks :

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \hfill (5.20)$$

By substituting (5.20) into (5.19) and using the expressions of $\mathbb{A}_x$ and $\mathbb{A}_h$ given in (5.5), we obtain:

$$\begin{pmatrix} S_{11} & S_{12} & A_1{}^T P_1 & S_{13} \\ S_{12}^T & S_{22} & 0 & G_1{}^T P_2 \\ P_1 A_1 & 0 & -(1-d)Q_1 & 0 \\ S_{13}^T & P_2 G_1 & 0 & -(1-d)Q_2 \end{pmatrix} < 0 \hfill (5.21)$$

where:

$$S_{11} = A^T P_1 + P_1 A + Q_1$$

$$S_{12} = A^T P_2 - C^T L^T P_2 - G^T P_2$$

$$S_{22} = G^T P_2 + P_2 G + Q_2$$

$$S_{13} = A_1{}^T P_2 - G_1{}^T P_2$$

We replace $Y = P_2 L$, $Z = P_2 G$ and $Z_1 = P_2 G_1$ from (5.9)-(5.11).

By applying the Schur complement, S_{11} is equivalent to the following LMI:

$$\begin{bmatrix} A^T P_1 + P_1 A & I \\ I & -Q_1{}^{-1} \end{bmatrix} < 0 \hfill (5.22)$$

By applying the Schur complement to S_{22} , we get the following LMI:

$$\begin{bmatrix} Z + Z^T & I \\ I & -Q_2{}^{-1} \end{bmatrix} < 0 \hfill (5.23)$$

The parameters P_1, P_2, Q_1, Q_2, $G = P_2{}^{-1}Z$, $G_1 = P_2{}^{-1}Z_1$ and $L = P_2{}^{-1}Y$ are obtained by solving LMIs in (5.6).

Secondly, the positivity of the augmented system (5.5) has to be taken into account. Therefore, we have to prove that $\mathbb{A}_x$ is Metzler and $\mathbb{A}_h \succcurlyeq 0$.

We get then the following inequality from Definition 1.9 and Remark 1.1:

$$\begin{cases} P\,\mathbb{A}_x + \beta\,I \geqslant 0 \\ P\mathbb{A}_h \geqslant 0 \end{cases} , \tag{5.24}$$

This implies that for the augmented system to be positive:

$$\begin{cases} P_1 A + \beta\,I_n \geqslant 0 \\ P_2 A - YC - Z \geqslant 0 \\ Z + \beta\,I_n \geqslant 0 \\ P_1 A_1 \geqslant 0 \\ P_2 A_1 - Z_1 \geqslant 0 \\ Z_1 \geqslant 0 \end{cases} \tag{5.25}$$

Finally, we get the following set of constraints for the augmented system to be positive:

$$\begin{cases} P_1 A + \beta\,I_n \geqslant 0 \\ P_2 A - YC - Z \geqslant 0 \\ Z + \beta\,I_n \geqslant 0 \\ A_1 \geqslant 0 \\ G_1 \geqslant 0 \\ P_2 A_1 - Z_1 \geqslant 0 \end{cases} \tag{5.26}$$

This concludes that the augmented system given in (5.5) is asymptotically stable and positive, which completes the proof.■

5.2.2. Positive Observer design for Positive Interval Linear systems with time-delay

In this section, we consider the observer design of the following uncertain interval positive system with time-delay:

$$\begin{cases} \dot{x}(t) = A\,x(t) + A_1 x\big(t - \tau(t)\big) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \geqslant 0, \forall\, t \in \big[-\overline{h}, 0\big] \end{cases} \tag{5.27}$$

where:

$x(t) \in \mathbb{R}^{n_x}$ is the state vector;

$A \in \mathbb{R}^{n_x \times n_x}$ is a Metzler matrix, $A_1 \in \mathbb{R}^{n_x \times n_x} \geqslant 0$ and $C \in \mathbb{R}^{n_y \times n_x} \geqslant 0$;

$A \in [\underline{A}\,,\,\overline{A}]$, $A_1 \in [\underline{A}_1, \overline{A}_1]$, $C \in [\underline{C}\,,\overline{C}]$, with $\underline{A}\,,\overline{A}$, $\underline{A}_1$, $\overline{A}_1$, $\underline{C}$ and $\overline{C}$ known constant matrices;

$\tau(t) \in \mathbb{R}$ is the time-varying delay, a continuous function given by (5.2).

By considering the proposed interval positive system with time-delay given in (5.27) and the observer structure proposed in (5.3), necessary and sufficient stability and positivity conditions are presented in the following theorem:

Theorem 5.2. (Zaidi et al.)

For given scalars $\beta \geq 0$ and $d < 1$, a positive robust observer of the form (5.3) exists for the uncertain system (5.27) if and only if there exist diagonal matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, and matrices $Y \succcurlyeq 0$, $Z_1 \succcurlyeq 0$ and Z Metzler, such that the following conditions hold:

$$
\begin{pmatrix}
\bar{A}^T P_1 + P_1 \bar{A} & I & \bar{A}^T P_2 - \underline{C}^T Y^T - Z^T & 0 & A_1^T P_1 & A_1^T P_2 - Z_1 \\
I & -Q_1^{-1} & 0 & 0 & 0 & Z_1^T \\
P_2 \overline{A} - Y\underline{C} - Z & 0 & Z^T + Z & I & 0 & 0 \\
0 & 0 & I & -Q_2^{-1} & 0 & 0 \\
P_1 A_1 & 0 & 0 & 0 & -(1-d)Q_1 & 0 \\
P_2 A_1 - Z_1 & Z_1 & 0 & 0 & 0 & -(1-d)Q_2
\end{pmatrix}
$$
$$< 0 \tag{5.28}$$

$$
\begin{pmatrix}
P_1 \underline{A} & 0 \\
P_2 \underline{A} - Y\overline{C} - Z & P_2 G
\end{pmatrix} + \beta I \succcurlyeq 0 \tag{5.29}
$$

$$
\begin{pmatrix}
P_1 \underline{A}_1 & 0 \\
P_2 \underline{A}_1 - Z_1 & Z_1
\end{pmatrix} \succcurlyeq 0 \tag{5.30}
$$

Under these conditions, the desired observer matrices may be obtained by:

$$L = P_2^{-1} Y \ , G = P_2^{-1} Z \ and \ G_1 = P_2^{-1} Z_1 \tag{5.31}$$

Proof:

Sufficiency:

From (5.29), we obtain that $Y \neq 0$. P_2^{-1} is diagonally strictly positive. Therefore, the obtained G is Metzler and $L \succcurlyeq 0$. $Z_1 \neq 0$, then $G_1 \succcurlyeq 0$.

It follows from (5.29), (5.30) and (5.31) that:

$$P_2\left(\underline{A} - L\overline{C} - G\right) \succcurlyeq 0 \tag{5.32}$$

which implies that: $\underline{A} - L\overline{C} - G \succcurlyeq 0$ (5.33)

For any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$, as $L \geq 0$, it is obvious that:

$$\underline{A} \preccurlyeq A \preccurlyeq \overline{A} \ and \ L\underline{C} \preccurlyeq LC \preccurlyeq L\overline{C} \tag{5.34}$$

Combining (5.33) and (5.34), we get for any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$:

$$A - LC - G \succcurlyeq \underline{A} - L\overline{C} - G \succcurlyeq 0 \tag{5.35}$$

with A and G are Metzler matrices. Thus, we conclude then that the matrix $\mathbb{A}_x$ defined in (5.5) is Metzler.

Moreover, we have $A_1 \succcurlyeq 0$ and $G_1 \succcurlyeq 0$. Thus, it follows from (5.30) that

$$P_2 A_1 - Z_1 = P_2(A_1 - G_1) \succcurlyeq P_2(\underline{A}_1 - G_1) = P_2 \underline{A}_1 - Z_1 \tag{5.36}$$

which implies that $A_1 - G_1 \succcurlyeq 0$ \hfill (5.37)

Then, the matrix $\mathbb{A}_h$ is nonnegative.

The facts that $\mathbb{A}_x$ is Metzler and $\mathbb{A}_h$ is nonnegative show that the augmented system (5.5) is a positive system.

From (5.28), we obtain that:

$$\begin{pmatrix} \overline{\mathbb{A}}_x^{\ T} P + P\, \overline{\mathbb{A}}_x + Q & \overline{\mathbb{A}}_h^{\ T} P \\ * & -(1-d)Q \end{pmatrix} < 0 \tag{5.38}$$

where:

$$\overline{\mathbb{A}}_x = \begin{pmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{pmatrix} \text{ and } \overline{\mathbb{A}}_h = \begin{pmatrix} \overline{A}_1 & 0 \\ \overline{A}_1 - G_1 & G_1 \end{pmatrix}$$

Then, we obtain the following inequality:

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{pmatrix} + \begin{pmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < 0$$

$$\tag{5.39}$$

We get then the fact that:

$$\mu\left(\begin{bmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{bmatrix}\right) < 0 \tag{5.40}$$

Using (5.35), we get obviously that for any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$,

$$\begin{bmatrix} A & 0 \\ A - LC - G & G \end{bmatrix} \preccurlyeq \begin{bmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{bmatrix} \tag{5.41}$$

Then, combining (35), (39), (40) and by Lemma 3.2, we deduce that, for any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$,

$$\mu\left(\begin{bmatrix} A & 0 \\ A - LC - G & G \end{bmatrix}\right) < 0 \tag{5.42}$$

which means that the augmented system (5.5) is asymptotically stable within the bounds of variation of A and C.

Necessity:

Suppose that there exist G, G_1 and L such that the observer given in (5.3) is positive, i.e, G is Metzler, $L \geqslant 0$ and $G_1 \geqslant 0$ and the augmented system (5.5) is a positive asymptotically stable system for any $A \in [\underline{A}, \overline{A}]$, $A_1 \in [\underline{A}_1, \overline{A}_1]$ and $C \in [\underline{C}, \overline{C}]$.

Then, there exist diagonal positive matrices P_1, P_2, Q_1 and Q_2 such that:

$$\begin{pmatrix} \overline{\mathbb{A}}_x{}^T P + P \overline{\mathbb{A}}_x + Q & \overline{\mathbb{A}}_h{}^T P \\ * & -(1-d)Q \end{pmatrix} < 0 \tag{5.43}$$

Then, the following inequality is obviously given:

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{pmatrix} + \begin{pmatrix} \overline{A} & 0 \\ \overline{A} - L\underline{C} - G & G \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < 0 \tag{5.44}$$

and

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \overline{A}_1 & 0 \\ \overline{A}_1 - G_1 & G_1 \end{pmatrix} < 0 \tag{5.45}$$

We have:

$$Y = P_2 L, \ Z = P_2 G \text{ and } Z_1 = P_2 G_1 \tag{5.46}$$

We easily note, from the diagonal strict positivity of P_2, that Z is Metzler, $Y \geqslant 0$ and $Z_1 \geqslant 0$. If we substitute (5.46) in (5.44) and (5.45), we get the inequality given in (5.28).

Since the augmented delayed system (5.5) is positive for any $A \in [\underline{A}, \overline{A}]$ and $C \in [\underline{C}, \overline{C}]$, we obtain that:

$$\begin{pmatrix} \underline{A} & 0 \\ \underline{A} - L\overline{C} - G & G \end{pmatrix}$$ is Metzler, which implies that $\underline{A} - L\overline{C} - G \geqslant 0$. From the positivity of P_2 and (4.46), we further obtain that: $P_2\underline{A} - Y\overline{C} - Z \geqslant 0$.

With the same manner, we get from the inequality given in (4.45) that:

$$P_2 A_1 - Z_1 \geqslant 0 .$$

which proves the necessity.

This completes the proof.∎

5.2.3. ILLUSTRATIVE EXAMPLE

This numerical example consists of a two-compartmental model of a biological system (Petrás & Magin, 2011) that does involve delays, as shown in Figure 5.1.
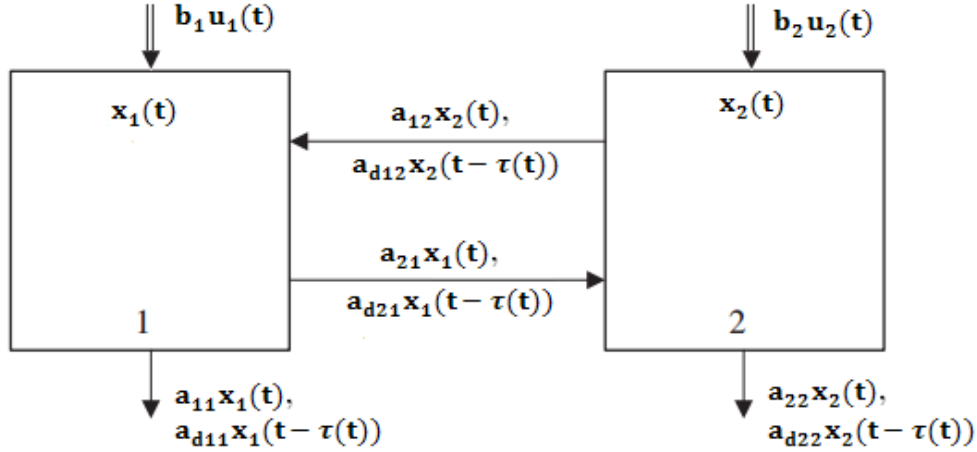
Figure 5. 1. Two-compartment model with the definition of model parameters

This system corresponds to the following time-delay system in the following form (5.27) where:

$$A = \begin{bmatrix} -a_{11} - a_{21} & a_{12} \\ a_{21} & -a_{22} - a_{12} \end{bmatrix}, A_1 = \begin{bmatrix} a_{d11} & a_{d12} \\ a_{d21} & a_{d22} \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

Assume that the estimated parameters for this model are:

$a_{11} = 1.8195 \pm 0.0431$, $a_{21} = 1.6510 \pm 0.0630$

$a_{12} = 2.1565 \pm 0.0237$, $a_{22} = 1.2050 \pm 0.1010$

$a_{d11} = 0.0412 \pm 0.0021$, $a_{d12} = 0.2145 \pm 0.0035$

$a_{d21} = 0.1214 \pm 0.0035$, $a_{d22} = 0.0852 \pm 0.0036$

$b_1 = 0.1$, $b_2 = 0.2$, $c_1 = c_2 = 1 \pm 0.05$

Then, by using the MATLAB LMI Toolbox, it can be seen that the conditions in Theorem 5.2. are feasible, for $d = 0.5$ and $\beta = 2$, with the following solution:

$$G = \begin{bmatrix} -4.1239 & 1.1952 \\ 0.6454 & -5.1407 \end{bmatrix}, G_1 = \begin{bmatrix} 0.0352 & 0.1523 \\ 0.0935 & 0.0741 \end{bmatrix}, L = \begin{bmatrix} 0.1951 & 0.4552 \\ 0.2892 & 0.8733 \end{bmatrix}$$
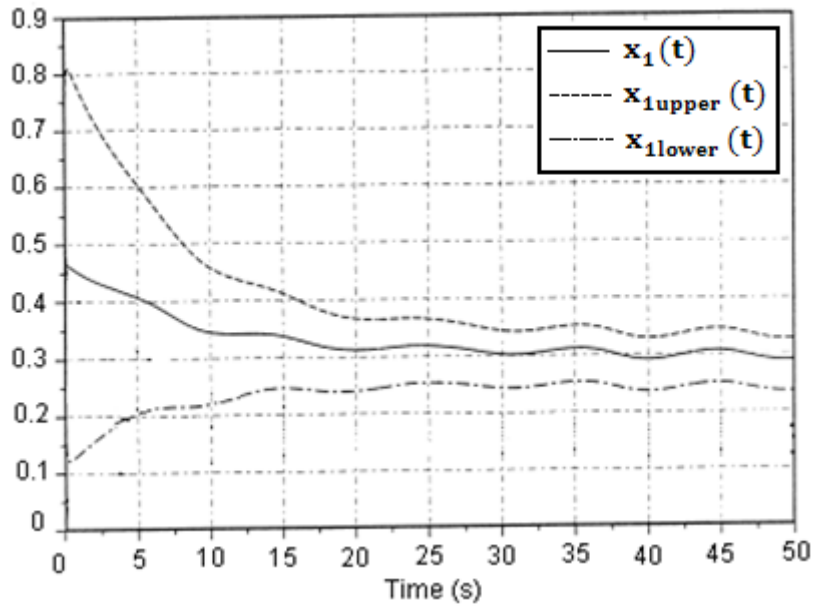
Figure 5. 2. The evolution of the interval estimates of $x_1(t)$
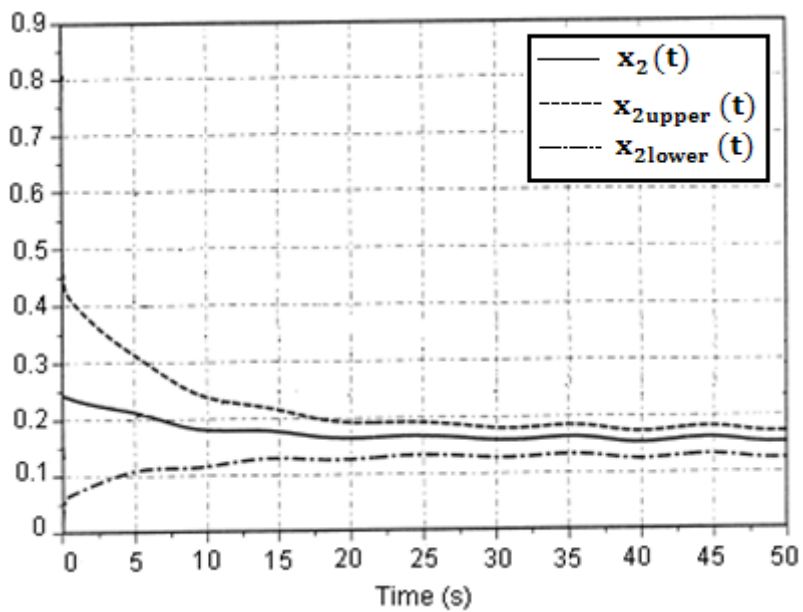
Figure 5. 3. The evolution of the interval estimates of $x_2(t)$

If we denote $x_{lower}(t)$ the evolution of the lower bound on the state from the initial condition $\underline{x}_0$ and $x_{upper}(t)$ the evolution of the upper bound on the state from the initial condition $\overline{x}_0$, then, the evolution of the estimated state $\hat{x}(t)$ will always be between the lower and upper states $x_{lower}(t)$ and $x_{upper}(t)$. Moreover, these states always remain nonnegative. These properties can be seen in Figures 5.2 and 5.3, that represent the state evolutions from the initial conditions $(\overline{x}_0 = [0.81\ 0.45]^T$ and $\underline{x}_0 = [0.12\ 0.05]^T)$. These facts show the effectiveness of the proposed approach.

5.3. Positive Observer-Based Controller design for Positive Interval Linear systems with time-delay

5.3.1. THEORETICAL APPROACH

We consider the interval positive time-delay system:

$$\begin{cases} \dot{x}(t) = A\,x(t) + A_1 x\big(t - \tau(t)\big) + B\,u(t) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \geqslant 0, \forall\, t \in \big[-\overline{h}, 0\big] \end{cases} \tag{5.47}$$

where:

$x(t) \in \mathbb{R}^{n_x}$ is the state vector,

$u(t) \in \mathbb{R}^{n_u}$ is the control vector,

$A \in [\underline{A}\,,\,\overline{A}], A_1 \in \big[\underline{A_1}, \overline{A_1}\big], B \in [\underline{B}\,,\overline{B}]$ and $C \in [\underline{C}\,,\overline{C}]$ are unknown constant matrices with known bounds, that fulfill $\underline{A} \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{A_1} \in \mathbb{R}^{n_x \times n_x} \geqslant 0$, $\underline{B} \geqslant 0 \in \mathbb{R}^{n_x \times n_u}$ and $\underline{C} \geqslant 0 \in \mathbb{R}^{n_y \times n_x}$.

$\tau(t) \in \mathbb{R}$ is the time-varying delay: a differentiable continuous function, satisfying the following the conditions given in (5.2).

The following lemma provides conditions that guarantee the asymptotic stability and the positivity of the interval system (5.47).

Lemma 5.1. (Zaidi et al.)

If there exists a diagonal matrix $R = R^T > 0$ and a matrix $S = S^T > 0$ satisfying the following LMI:

$$\begin{pmatrix} R\overline{A} + \overline{A}^T R + S & R\overline{A_1} \\ * & -(1-d)S \end{pmatrix} < 0 \tag{5.48}$$

then, the interval system (5.47) is positive and asymptotically stable.

Proof:

We have $A \leqslant \overline{A}$, $A_1 \leqslant \overline{A_1}$ and $R > 0$ with R diagonal; then, $RA \leqslant R\overline{A}$ and $RA_1 \leqslant R\overline{A_1}$. Moreover, we have: $RA + A^T R + S \leqslant R\overline{A} + \overline{A}^T R + S$ for any matrix S, which means that:

$$\begin{pmatrix} RA + A^T R + S & RA_1 \\ * & -(1-d)S \end{pmatrix} \leqslant \begin{pmatrix} R\overline{A} + \overline{A}^T R + S & R\overline{A_1} \\ * & -(1-d)S \end{pmatrix} \tag{5.49}$$

From (5.48) and (5.49) and using (4.12), we deduce that the existence of a diagonal matrix $R = R^T > 0$ and a matrix $S = S^T > 0$ satisfying LMI (5.48) implies the feasibility of the following LMI with the same matrices R and S:

$$\begin{pmatrix} RA + A^T R + S & RA_1 \\ * & -(1-d)S \end{pmatrix} < 0 \tag{5.50}$$

Then, the system (5.47) is asymptotically stable.

Moreover, system (5.47) is defined by its matrices that fulfill $A \in [\underline{A}, \overline{A}]$, $A_1 \in [\underline{A_1}, \overline{A_1}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$. Since $\underline{A} \in \mathbb{R}^{n \times n}$ is Metzler, $\underline{A_1} \geqslant 0$, $\underline{B} \geqslant 0$ and $\underline{C} \geqslant 0$, we deduce that $A \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $A_1 \geqslant 0$, $B \geqslant 0$ and $C \geqslant 0$, which means that system (5.47) is positive.

Both of the previous facts conclude that system (5.47) is positive and asymptotically stable. ∎

We use the following observer defined by:

$$\dot{\hat{x}}(t) = G\,\hat{x}(t) + G_1 \hat{x}(t - \tau(t)) + L\,y(t) \tag{5.51}$$

where $G \in \mathbb{R}^{n_x \times n_x}$, $G_1 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_x \times n_y}$ are the observer matrices to be determined.

The static state-feedback control law used is given by the following expression:

$$u(t) = K\hat{x}(t) \tag{5.52}$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is the controller matrix to be determined.

Then, the closed-loop system is written as follows:

$$\begin{cases} \dot{x}(t) = A\,x(t) + A_1 x(t - \tau(t)) + B\,K\hat{x}(t) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \geqslant 0, \forall\, t \in [-\overline{h}, 0] \\ \hat{x}(t) = \vartheta(t) \geqslant 0, \forall\, t \in [-\overline{h}, 0] \end{cases} \tag{5.53}$$

To ensure the positivity of the state estimation, the key lies in the nonnegativity of the error, which is defined by:

$$e(t) = x(t) - \hat{x}(t) \tag{5.54}$$

If we choose $[x^T(t)\ \ e^T(t)]^T$ as the new augmented state variable, then the augmented system will become:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = A_x \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} + A_h \begin{pmatrix} x(t - h(t)) \\ e(t - h(t)) \end{pmatrix} \tag{5.55}$$

where:

$$A_x = \begin{pmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{pmatrix} \tag{5.56}$$

and

$$A_h = \begin{pmatrix} A_1 & 0 \\ A_1 - G_1 & G_1 \end{pmatrix} \tag{5.57}$$

The problem to be solved is to find G, G_1, L and K such that the augmented system (5.55) is positive and asymptotically stable. Meanwhile, to guarantee the nonnegativity of the estimate $\hat{x}(t)$, it is obvious to require that G is Metzler and L is nonnegative. The following theorem provides a necessary condition for the existence of solutions to the problem of a continuous-time observer-based controller.

Theorem 5.3. (Zaidi et al.)

If there exists a static state-feedback controller (5.52) that stabilizes system (5.47), using the observer (5.51) with a positive augmented system (5.55), then the following inequalities with respect to Metzler G, $G_1 \geqslant 0$, $L \geqslant 0$ and $K \leqslant 0$ have a solution:

$$trace\left(\underline{A} + G + (\overline{B} - \underline{B})K\right) < 0 \tag{5.58}$$

$$[\overline{A} + \underline{B}K]_{ij} \geq 0 \,, 1 \leq i \neq j \leq n \tag{5.59}$$

$$[G - \overline{B}K]_{ij} \geq 0 \,, 1 \leq i \neq j \leq n \tag{5.60}$$

$$\overline{A} - L\underline{C} + \underline{B}K - G \geqslant 0 \tag{5.61}$$

$$\overline{A}_1 - G_1 \geqslant 0 \tag{5.62}$$

Proof:

If there exists such observer-based controller, then, from (5.55), we have that

$$\mu \left(\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix} \right) < 0 \tag{5.63}$$

and $\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix}$ is Metzler $\tag{5.64}$

As we have that:

$$\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix} \preccurlyeq \begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix}$$

$$\preccurlyeq \begin{bmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{bmatrix} \tag{5.65}$$

Then, we deduce that

$$\mu\left(\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix}\right) < 0 \tag{5.66}$$

and

$$\begin{bmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{bmatrix} \text{ is Metzler} \tag{5.67}$$

It follows from (5.66) that

$$trace\left(\begin{bmatrix} \underline{A} + \overline{B}K & -\underline{B}K \\ \underline{A} - L\overline{C} + \overline{B}K - G & G - \underline{B}K \end{bmatrix}\right) < 0 \tag{5.68}$$

which is equivalent to (5.58). Moreover, it is obvious that (5.67) is equivalent to (5.59), (5.60) and (5.61). Moreover, the fact that $A_{hh} \succcurlyeq 0$ implies (5.62), which completes the proof.■

Thus, we now study sufficient conditions and the corresponding synthesis approach for this problem in this theorem which guarantees the asymptotic stability and the positivity of the augmented system (5.47).

Theorem 5.4. (Zaidi et al.)

There exists an observer-based controller (5.51)-(5.52) for the system (5.47) that provides asymptotic stability and positivity of the augmented system (5.55) if there exists a positive scalar ε and matrices $P = diag[P_1\ P_2] > 0$, $Q = diag[Q_1\ Q_2] > 0$, a Metzler matrix G, $G_1 \succcurlyeq 0$, $L \succcurlyeq 0$ and $K \preccurlyeq 0$ such that:

$$\begin{pmatrix} \mathbb{A} & A_h^T P & P\mathcal{B} + \mathcal{C}^T\mathcal{K}^T & \varepsilon\mathcal{B} \\ * & -(1-d)Q & 0 & 0 \\ * & 0 & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \tag{5.70}$$

$$[\underline{A} + \overline{B}K]_{ij} \geq 0 , 1 \leq i \neq j \leq n \tag{5.71}$$

$$[G - \underline{B}K]_{ij} \geq 0 , 1 \leq i \neq j \leq n \tag{5.72}$$

$$\underline{A} - L\overline{C} + \overline{B}K - G \succcurlyeq 0, \tag{5.73}$$

$$\underline{A_1} - G_1 \geqslant 0 \tag{5.74}$$

where:

$$\mathbb{A} = \mathcal{A}^T P + P\mathcal{A} - \varepsilon \mathcal{B}\mathcal{B}^T P - \varepsilon P \mathcal{B}^T \mathcal{B} + Q$$

$$\mathcal{A} = \begin{pmatrix} \overline{A} & 0 \\ \underline{A} & 0 \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} 0 & \underline{B} - \overline{B} & 0 & \underline{B} \\ 0 & \underline{B} - \overline{B} & -I & \underline{B} \end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix} G & L & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & G & L \\ 0 & 0 & K & 0 \end{pmatrix} \mathcal{C} = \begin{pmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C} & 0 \end{pmatrix} \tag{5.75}$$

Proof:

It follows from (5.71) that $\underline{A} + \overline{B}K$ is Metzler. Combining this with $K \leqslant 0$ and $L \geqslant 0$ yields, for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$, that:

$$A + BK \geqslant \underline{A} + \overline{B}K \text{ is Metzler} \tag{5.76}$$

$$A - LC + BK - G \geqslant \underline{A} - L\overline{C} + \overline{B}K - G \geqslant 0 \tag{5.77}$$

In addition, from G being Metzler and $K \leqslant 0$, we obtain that, for any $B \in [\underline{B}, \overline{B}]$:
$$-BK \geqslant 0 \tag{5.78}$$

$$G - BK \text{ is Metzler} \tag{5.79}$$

Therefore, from (5.76)-(5.79) and the fact that $A_h \geqslant 0$, we have that, for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$, the augmented system (5.55) is positive.

It follows from (5.70), by the Schur complement, that:

$$\begin{pmatrix} \aleph_1 & A_h^T P \\ P A_h & -(1-d)Q \end{pmatrix} < 0 \tag{5.80}$$

where:

$$\aleph_1 = \mathcal{A}^T P + P\mathcal{A} - \varepsilon \mathcal{B}\mathcal{B}^T P - \varepsilon P \mathcal{B}^T \mathcal{B} + \varepsilon^2 \mathcal{B}\mathcal{B}^T + (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C})^T (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C}) + Q$$

We have then:

$$\mathcal{A}^T P + P\mathcal{A} - \varepsilon \mathcal{B}\mathcal{B}^T P - \varepsilon P \mathcal{B}^T \mathcal{B} + \varepsilon^2 \mathcal{B}\mathcal{B}^T + (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C})^T (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C}) + Q < 0 \tag{5.81}$$

Taking into account the following relationship:

$$PBB^T P - \varepsilon BB^T P - \varepsilon PB^T B + \varepsilon^2 BB^T = (PB \ -\varepsilon\, B)(B^T P - \varepsilon B^T) \geq 0 \tag{5.82}$$

we obtain, from (5.81) the following inequality:

$$\mathcal{A}^T P + P\mathcal{A} - P\mathcal{B}\mathcal{B}^T P + (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C})^T (\mathcal{B}^T P + \mathcal{K}\,\mathcal{C}) + Q \ < 0 \qquad (5.83)$$

Rewriting (5.83) yields that:

$$(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) + \mathcal{C}^T \mathcal{K}^T \mathcal{K}\,\mathcal{C} + Q < 0 \qquad (5.84)$$

which implies that:

$$(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) + Q < 0 \qquad (5.85)$$

Therefore, we get: $\mu(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) < 0$ \qquad (5.86)

Replacing $\aleph_1$ with (5.85) in (5.80), we get:

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}) + Q & A_{hh}^T P \\ PA_{hh} & -(1-d)Q \end{pmatrix} < 0 \qquad (5.87)$$

Some algebraic manipulations, using (5.75), lead to:

$$\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C} = \begin{pmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{pmatrix} \qquad (5.88)$$

In addition, it is easy to show that:

$$\begin{pmatrix} \overline{A} + \underline{B}K & -\overline{B}K \\ \overline{A} - L\underline{C} + \underline{B}K - G & G - \overline{B}K \end{pmatrix} \succcurlyeq \begin{pmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{pmatrix} \qquad (5.89)$$

Therefore, by combining (5.86)-(5.89), we obtain that:

$$\mu\left(\begin{bmatrix} A + BK & -BK \\ A - LC + BK - G & G - BK \end{bmatrix} \right) < 0 \qquad (5.90)$$

which means that (5.55) is asymptotically stable for any $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$ and $C \in [\underline{C}, \overline{C}]$.

This completes the proof. ∎

5.3.2. ILLUSTRATIVE EXAMPLE

This numerical example is based on a compartmental system that involves delays, as shown in Figure 5.1 where the system parameters are:

$$A = \begin{bmatrix} -(a_{11} + a_{21}) & a_{12} \\ a_{21} & -(a_{22} + a_{12}) \end{bmatrix},\ A_1 = \begin{bmatrix} a_{d11} & a_{d12} \\ a_{d21} & a_{d22} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = [c_{11}\quad 0]$$

The elements of the bounded matrices $\overline{A}$, $\underline{A}$, $\overline{C}$ and $\underline{C}$ are obtained from:

$a_{11} = 1.2 \pm 0.042,\ a_{12} = 0.3 \pm 0.061,\ a_{21} = 0.5 \pm 0.045,\ a_{22} = 0.6 \pm 0.140,$

$a_{d11} = 0.05 \pm 0.005$, $a_{d12} = 0.01 \pm 0.005$,

$a_{d21} = 0.08 \pm 0.005$, $a_{d22} = 0.06 \pm 0.005$, $c_{11} = 1 \pm 0.1$.

$\tau_1(t) = 0.25 + 0.14\ sin(t)$, bounded by $\overline{h} = 0.39$.

Then, it can be seen that the conditions in Theorems 5.3 and 5.4 are feasible, for $\varepsilon = 15$, $\overline{h} = 0.39$ and $d = 0.5$, with the following solution, obtained by using the Scilab 5.3.3 LMI toolbox:

$$P_1 = \begin{bmatrix} 240.8184 & -214.7411 \\ -214.7411 & 222.8628 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 151.2631 & -201.4101 \\ -165.0750 & 101.6504 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 347.6394 & -256.5724 \\ -256.5724 & 205.6603 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 82.9465 & -21.2129 \\ -21.2129 & 79.4133 \end{bmatrix}$$

$$G = \begin{bmatrix} -12.3163 & 5.8404 \\ 6.3600 & -11.5066 \end{bmatrix}, G_1 = \begin{bmatrix} 0.0250 & 0.0049 \\ 0.0400 & 0.0300 \end{bmatrix}$$

$$L = \begin{bmatrix} 6.5852 \\ 15.8514 \end{bmatrix}, K = \begin{bmatrix} -0.2824 & -0.1117 \\ -1.6278 & -0.6251 \end{bmatrix}$$

The simulations presented in Figures 5.4 to 5.6 show that the real state vector $x(t)$, as well as the estimated state vector $\hat{x}(t)$, are nonnegative and converge. These properties can be seen in Figures 5.4 and 5.5, which represent the state evolutions from given initial conditions $x_0 = [6.7\ 8]^T$ and $\hat{x}_0 = [5\ 4]^T$, for $t \in [-0.39\ 0]$, when $A = \overline{A}$, $B = \overline{B}$, $C = \overline{C}$, $A_1 = \overline{A}_1$ and $\overline{h} = 0.39$. It is possible to see, from Figure 5.6, that the estimation errors are nonnegative. These facts show the effectiveness of the proposed approach.
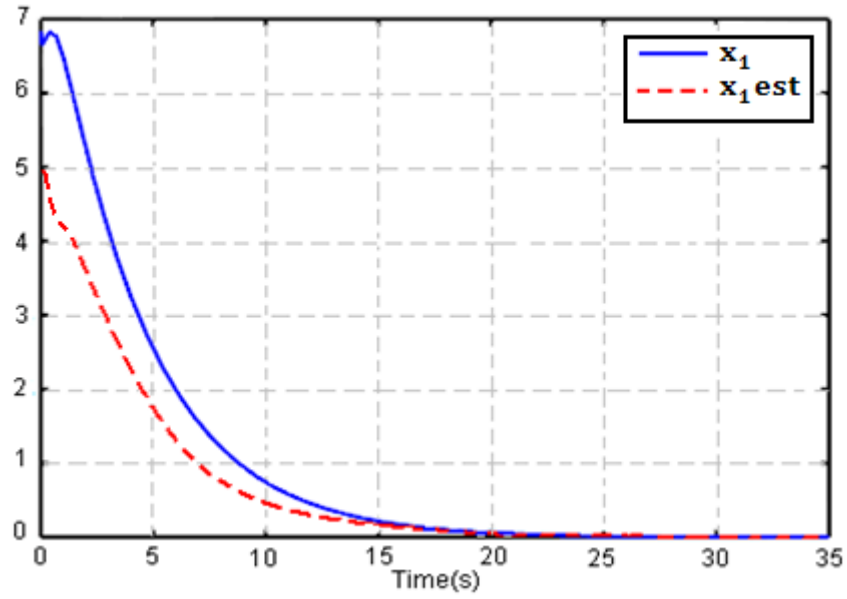
Figure 5. 4. Evolution of the state $x_1(t)$, and its estimation $\hat{x}_1(t)$ for $\hat{x}_0 = [5\ 4]^T$
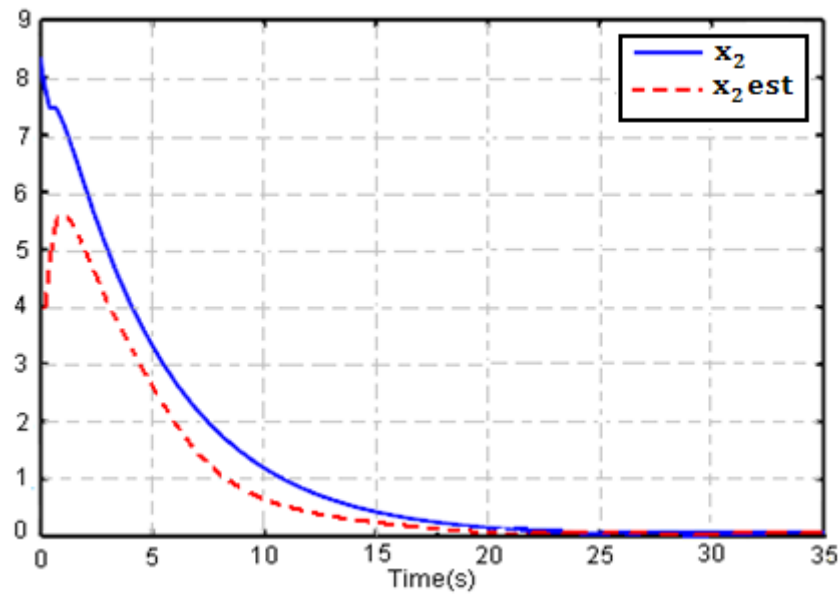
Figure 5. 5. Evolution of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$ for $\hat{x}_0 = [5\ 4]^T$
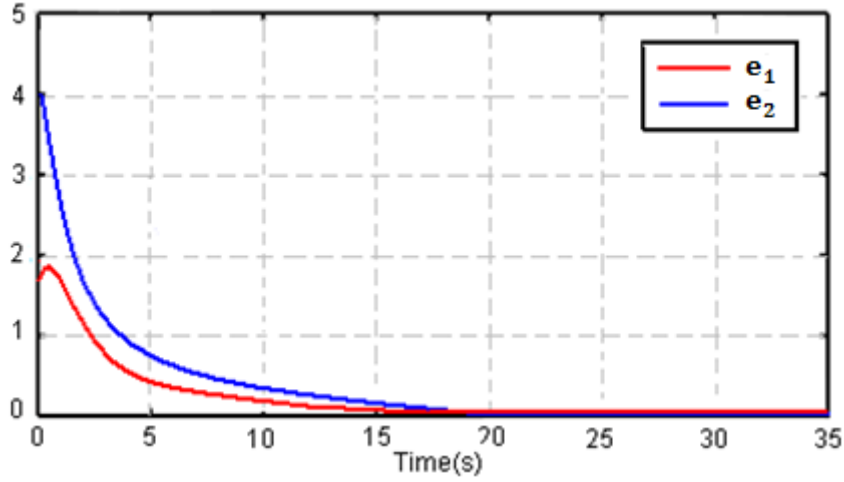
Figure 5. 6. Evolution of the estimation errors $e_1(t)$ and $e_2(t)$ for $x_0 = [6.7\ 8]^T$ and $\hat{x}_0 = [5\ 4]^T$

5.4. Positive Observer-Based Controller design for Positive Interval T-S systems with time-delay

In this section, we are interested in the synthesis of positive observer-based controllers for positive interval T-S systems with time-delay. We will study both cases: when the decision variables of the system are measurable and when they are unmeasurable.

5.4.1. CASE 1: MEASURABLE PREMISE VARIABLES

We consider the observer design for the following interval positive time-delay T-S system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i\, x(t) + A_{1i}x(t - \tau_i(t)) + B_i\, u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(z(t))\, C_i\, x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in \left[-\overline{h}, 0\right] \end{cases} \tag{5.91}$$

where:

$x(t) \in \mathbb{R}^{n_x}$ is the state vector,

$u(t) \in \mathbb{R}^{n_u}$ is the control vector,

$h_i(z(t))$ are fuzzy weighting functions satisfying (1.4),

$A_i \in [\underline{A}_i \,, \, \overline{A}_i], A_{1i} \in [\underline{A}_{1i}, \overline{A}_{1i}], B_i \in [\underline{B}_i \,, \overline{B}_i]$ and $C_i \in [\underline{C}_i \,, \overline{C}_i]$ are unknown constant matrices with known bounds, that fulfill:

$\underline{A}_i \in \mathbb{R}^{n_x \times n_x}$ is Metzler, $\underline{A}_{1i} \in \mathbb{R}^{n_x \times n_x} \geqslant 0$, $\underline{B}_i \geqslant 0 \in \mathbb{R}^{n_x \times n_u}$, $\underline{C}_i \geqslant 0 \in \mathbb{R}^{n_y \times n_x}$ and

the time-delays $\tau_i(t)$ are time-varying continuous functions that satisfy:

$$0 \leq \tau_i(t) \leq \overline{h}_i < \infty, \, \dot{\tau}_i(t) \leq d_i < 1, \, i = 1, \dots, r, \tag{5.92}$$

$$\overline{h} = \max_{1 \leq i \leq r} \overline{h}_i \tag{5.93}$$

Lemma 5.2. (Zaidi et al.)

If there exists a diagonal matrix $R = R^T > 0$ and a matrix $S = S^T > 0$ satisfying the following LMI: $\forall \, i = 1, \dots, r$

$$\begin{pmatrix} R\overline{A}_i + \overline{A}_i^T R + S & R\overline{A}_{1i} \\ * & -(1-d)S \end{pmatrix} < 0 \tag{5.94}$$

then, the interval system (5.91) is positive and asymptotically stable.

Proof:

Suppose that there exists a diagonal matrix $R = R^T > 0$ and a matrix $S = S^T > 0$ satisfying the LMI (5.94).

We have $\forall \, i = 1, \dots, r$, $A_i \leqslant \overline{A}_i$ and $A_{1i} \leqslant \overline{A}_{1i}$ with $R > 0$, and diagonal; then, $RA_i \leqslant R\overline{A}_i$ and $RA_{1i} \leqslant R\overline{A}_{1i}$. Moreover, we have: $RA_i + A_i^T R + S \leqslant R\overline{A}_i + \overline{A}_i^T R + S$ for any matrix S, which means that:

$$\begin{pmatrix} RA_i + A_i^T R + S & RA_{1i} \\ * & -(1-d)S \end{pmatrix} \leqslant \begin{pmatrix} R\overline{A}_i + \overline{A}_i^T R + S & R\overline{A}_{1i} \\ * & -(1-d)S \end{pmatrix} \tag{5.95}$$

From (5.94) and (5.95) and using (4.12), we deduce that the existence of a diagonal matrix $R = R^T > 0$ and a matrix $S = S^T > 0$ satisfying LMI (5.94) implies the following LMI with the same matrices R and S:

$$\begin{pmatrix} RA_i + A_i^T R + S & RA_{1i} \\ * & -(1-d)S \end{pmatrix} < 0 \tag{5.96}$$

Then, the system (5.91) is stable.

Moreover, system (5.91) is defined by its matrices given by $A_i \in [\underline{A}_i \,, \, \overline{A}_i], A_{1i} \in [\underline{A}_{1i}, \overline{A}_{1i}], B_i \in [\underline{B}_i \,, \overline{B}_i]$ and $C_i \in [\underline{C}_i \,, \overline{C}_i]$. Since $\underline{A}_i \in \mathbb{R}^{n \times n}$ is Metzler, $\underline{A}_{1i} \geqslant 0$, $\underline{B}_i \geqslant 0$ and $\underline{C} \geqslant 0$, we deduce that $A_i \in \mathbb{R}^{n \times n}$ is Metzler, $A_{1i} \geqslant 0$, $B_i \geqslant 0$ and

$C_i \succcurlyeq 0$, which means that system (5.91) is positive.

Both of the previous facts conclude that system (5.91) is positive and asymptotically stable. ∎

We will use the following state observer, which allows the states of model (5.91) to be estimated.

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(z(t))(G_i\,\hat{x}(t) + G_{1i}\,\hat{x}(t - \tau_i(t)) + L_i y(t)) \tag{5.97}$$

where $G_i \in \mathbb{R}^{n_x \times n_x}$, $G_{1i} \in \mathbb{R}^{n_x \times n_x}$, $L_i \in \mathbb{R}^{n_x \times n_y}$ are the state observer matrices to be determined.

The static state-feedback control law is given by the following expression:

$$u(t) = \sum_{i=1}^{r} h_i(z(t))\,K_i\,\hat{x}(t) \tag{5.98}$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$ are the controller matrices to be determined.

Then, the closed-loop system is written as follows:

$$\begin{cases} \dot{x}(t) = \displaystyle\sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\left(A_i\,x(t) + A_{1i}x(t - \tau_i(t)) + B_i\,K_j\,\hat{x}(t)\right) \\ y(t) = \displaystyle\sum_{i=1}^{r} h_i(z(t))\,C_i\,x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\,t \in \left[-\overline{h}, 0\right] \\ \hat{x}(t) = \vartheta(t) \succcurlyeq 0, \forall\,t \in \left[-\overline{h}, 0\right] \end{cases} \tag{5.99}$$

To guarantee the positivity of the system, the key lies once again in the nonnegativity of the error signal, which is defined by (5.54).

$$e(t) = x(t) - \hat{x}(t) \tag{5.100}$$

If we choose $[x^T(t) \quad e^T(t)]^T$ as the new augmented state variable, then the augmented closed-loop system is:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\left(A_{Xij}\begin{pmatrix} x(t) \\ e(t) \end{pmatrix} + A_{Hij}\begin{pmatrix} x(t - h(t)) \\ e(t - h(t)) \end{pmatrix}\right) \tag{5.101}$$

with

$$A_{Xij} = \begin{pmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i\,K_j - G_j & G_j - B_i\,K_j \end{pmatrix} \tag{5.102}$$

and

$$A_{Hij} = \begin{pmatrix} A_{1i} & 0 \\ A_{1i} - G_{1j} & G_{1j} \end{pmatrix}$$ (5.103)

The problem to be solved is to find G_j, G_{1j}, L_j and K_j such that the augmented system (5.101) is positive and asymptotically stable. The following theorem provides a necessary condition for the existence of solutions to the problem of a continuous-time observer-based controller.

Theorem 5.5. (Zaidi et al.)

If there exists a static state-feedback controller (5.98) that stabilizes system (5.99), using the observer (5.97) with a positive augmented system (5.101), then the following inequalities with respect to Metzler G_j, $G_{1j} \geqslant 0$, $L_j \geqslant 0$ and $K_j \leqslant 0$ have a solution, $\forall i, j = 1, \ldots, r$:

$$trace\left(\underline{A}_i + G_j + (\overline{B}_i - \underline{B}_i)\right) < 0$$ (5.104)

$$[\overline{A}_i + \underline{B}_i K_j]_{lm} \geq 0 \,,\, 1 \leq l \neq m \leq n$$ (5.105)

$$[G_j - \overline{B}_i K_j]_{lm} \geq 0 \,,\, 1 \leq l \neq m \leq n$$ (5.106)

$$\overline{A}_i - L_j \underline{C}_i + \underline{B}_i K_j - G_j \geqslant 0$$ (5.107)

$$\overline{A}_{1i} - G_{1j} \geqslant 0$$ (5.108)

Proof:

If there exists such observer-based controller, then, from (5.102), we have that $\forall\, i, j = 1, \ldots, r$,

$$\mu\left(\begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i K_j - G_j & G_j - B_i K_j \end{bmatrix}\right) < 0$$ (5.109)

and $\begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i K_j - G_j & G_j - B_i K_j \end{bmatrix}$ is Metzler (5.110)

As we have that:

$$\begin{bmatrix} \underline{A}_i + \overline{B}_i K_j & -\underline{B}_i K_j \\ \underline{A}_i - L_j \overline{C}_i + \overline{B}_i K_j - G_j & G_j - \underline{B}_i K_j \end{bmatrix} \preccurlyeq \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i K_j - G_j & G_j - B_i K_j \end{bmatrix}$$

$$\preccurlyeq \begin{bmatrix} \overline{A}_i + \underline{B}_i K_j & -\underline{B}_i K_j \\ \overline{A}_i - L_j \underline{C}_i + \underline{B}_i K_j - G_j & G_j - \overline{B}_i K_j \end{bmatrix}$$ (5.111)

Then, we deduce that

$$\mu\left(\begin{bmatrix} \underline{A}_i + \overline{B}_i K_j & -\underline{B}_i K_j \\ \underline{A}_i - L_j \overline{C}_i + \overline{B}_i K_j - G_j & G_j - \underline{B}_i K_j \end{bmatrix}\right) < 0 \qquad (5.112)$$

and

$$\begin{bmatrix} \overline{A}_i + \underline{B}_i K_j & -\underline{B}_i K_j \\ \overline{A}_i - L_j \underline{C}_i + \underline{B}_i K_j - G_j & G_j - \overline{B}_i K_j \end{bmatrix} \text{ is Metzler} \qquad (5.113)$$

It follows from (5.112) that

$$trace\left(\begin{bmatrix} \underline{A}_i + \overline{B}_i K_j & -\underline{B}_i K_j \\ \underline{A}_i - L_j \overline{C}_i + \overline{B}_i K_j - G_j & G_j - \underline{B}_i K_j \end{bmatrix}\right) < 0 \qquad (5.114)$$

which is equivalent to (5.104). Moreover, it is obvious that (5.113) is equivalent to (5.105), (5.106) and (5.107). Moreover, the fact that $A_{Hij} \succcurlyeq 0$ implies (5.108), which completes the proof.∎

Thus, we now study sufficient conditions in the following theorem that consists of guaranteeing the asymptotic stability and the positivity of the augmented system (5.101).

Theorem 5.6. (Zaidi et al.)

There exists an observer-based controller (5.97)-(5.98) for the system (5.99) that provides stability and positivity of the augmented system (5.101) if there exists a positive scalar ε and matrices $P = diag[P_1 \ P_2] > 0$, $Q = diag[Q_1 \ Q_2] > 0$, Metzler $G_j, G_{1j} \succcurlyeq 0, L_j \succcurlyeq 0$ and $K_j \preccurlyeq 0$, such that $\forall i,j = 1, \dots, r$:

$$\begin{pmatrix} \mathbb{A}_i & A_{Hij}^T P & P\mathcal{B}_i + \mathcal{C}_i^T \mathcal{K}_j^T & \varepsilon \mathcal{B}_i \\ * & -(1-d)Q & 0 & 0 \\ * & 0 & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \qquad (5.115)$$

$$[\underline{A}_i + \overline{B}_i K_j]_{lm} \geq 0 \, , 1 \leq l \neq m \leq n \qquad (5.116)$$

$$[G_j - \underline{B}_j K_j]_{lm} \geq 0 \, , 1 \leq l \neq m \leq n \qquad (5.117)$$

$$\underline{A}_i - L_j \overline{C}_i + \overline{B}_i K_j - G_j \succcurlyeq 0, \qquad (5.118)$$

$$\underline{A}_{1i} - G_{1j} \succcurlyeq 0 \qquad (5.119)$$

where:

$$\mathbb{A}_i = \mathcal{A}_i^T P + P\mathcal{A}_i - \varepsilon \mathcal{B}_i \mathcal{B}_i^T P - \varepsilon P \mathcal{B}_i^T \mathcal{B}_i + Q$$

$$\mathcal{A}_i = \begin{pmatrix} \overline{A}_i & 0 \\ \underline{A}_i & 0 \end{pmatrix}, \ \mathcal{B}_i = \begin{pmatrix} 0 & \underline{B}_i - \overline{B}_i & 0 & \underline{B}_i \\ 0 & \underline{B}_i - \overline{B}_i & -I & \underline{B}_i \end{pmatrix}$$

$$\mathcal{K}_j = \begin{pmatrix} G_j & L_j & 0 & 0 \\ K_j & 0 & 0 & 0 \\ 0 & 0 & G_j & L_j \\ 0 & 0 & K_j & 0 \end{pmatrix}, \mathcal{C}_i = \begin{pmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C}_i & 0 \end{pmatrix} \qquad (5.120)$$

Proof:

It follows from (5.116) that $\forall i, j = 1, \dots, r$, $\underline{A}_i + \overline{B}_i K_j$ is Metzler. Combining this with $K_j \leqslant 0$ and $L_j \geqslant 0$ yields, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C_i \in [\underline{C}_i, \overline{C}_i]$, we get: $A_i + B_i K_j \geqslant \underline{A}_i + \overline{B}_i K_j$ is Metzler $\qquad (5.121)$

and $A_i - L_j C_i + B_i K_j - G_j \geqslant \underline{A}_i - L_j \overline{C}_i + \overline{B}_i K_j - G_j \geqslant 0$ $\qquad (5.122)$

In addition, from G_j being Metzler and $K_j \leqslant 0$, we obtain that, for any $B_i \in [\underline{B}_i, \overline{B}_i]$: $-B_i K_j \geqslant 0$ $\qquad (5.123)$

and $G_j - B_i K_j$ is Metzler $\qquad (5.124)$

Therefore, from (6.121)-(5.124) and the fact that $A_{Hij} \geqslant 0$, we have that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $A_{1i} \in [\underline{A}_{1i}, \overline{A}_{1i}]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C_i \in [\underline{C}_i, \overline{C}_i]$, the augmented system (5.101) is positive.

It follows from (5.115), by the Schur complement, that:

$$\begin{pmatrix} \aleph_{ij} & A_{Hij}^T P \\ * & -(1-d)Q \end{pmatrix} < 0 \qquad (5.125)$$

where:

$$\aleph_{ij} = \mathcal{A}_i^T P + P\mathcal{A}_i - \varepsilon \mathcal{B}_i \mathcal{B}_i^T P - \varepsilon P \mathcal{B}_i^T \mathcal{B}_i + \varepsilon^2 \mathcal{B}_i \mathcal{B}_i^T + \left(\mathcal{B}_i^T P + \mathcal{K}_j \mathcal{C}_i \right)^T \left(\mathcal{B}_i^T P + \mathcal{K}_j \mathcal{C}_i \right)$$
$$+Q$$

We have then:

$$\mathcal{A}_i^T P + P\mathcal{A}_i - \varepsilon \mathcal{B}_i \mathcal{B}_i^T P - \varepsilon P \mathcal{B}_i^T \mathcal{B}_i + \varepsilon^2 \mathcal{B}_i \mathcal{B}_i^T + \left(\mathcal{B}_i^T P + \mathcal{K}_j \mathcal{C}_i \right)^T \left(\mathcal{B}_i^T P + \mathcal{K}_j \mathcal{C}_i \right)$$
$$+Q < 0 \qquad (5.126)$$

Taking into account the following relationship:

$$P\mathcal{B}_i \mathcal{B}_i^T P - \varepsilon \mathcal{B}_i \mathcal{B}_i^T P - \varepsilon P \mathcal{B}_i^T \mathcal{B}_i + \varepsilon^2 \mathcal{B}_i \mathcal{B}_i^T = (P\mathcal{B}_i - \varepsilon \mathcal{B}_i)(\mathcal{B}_i^T P - \varepsilon \mathcal{B}_i^T) \geq 0 \quad (5.127)$$

we obtain the following inequality:

$$\mathcal{A}_i^T P + P\mathcal{A}_i - P\mathcal{B}_i\mathcal{B}_i^T P + \left(\mathcal{B}_i^T P + \mathcal{K}_j\,C_i\right)^T\left(\mathcal{B}_i^T P + \mathcal{K}_j\,C_i\right) + Q\ < 0 \tag{5.128}$$

Rewriting (5.128) yields that:

$$(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i)^T P + P\left(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i\right) + C_i^T\mathcal{K}_j^T\mathcal{K}_j\,C_i + Q < 0 \tag{5.129}$$

which implies that:

$$(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i)^T P + P\left(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i\right) + Q < 0 \tag{5.128}$$

Therefore, we get: $\mu\left(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i\right) < 0$ \hfill (5.129)

Replacing $\aleph_{ij}$ with (5.1128) in (5.125), we get:

$$\begin{pmatrix} \left(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i\right)^T P + P\left(\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i\right) + Q & * \\ PA_{Hij} & -(1-d)Q \end{pmatrix} < 0 \tag{5.130}$$

Some algebraic manipulations lead to:

$$\mathcal{A}_i + \mathcal{B}_i\mathcal{K}_j\,C_i = \begin{pmatrix} \overline{A}_i + \underline{B}_i K_j & -\underline{B}_i K_j \\ \overline{A}_i - L_j\underline{C}_i + \underline{B}_i K_j - G_j & G_j - \overline{B}_i K_j \end{pmatrix} \tag{5.131}$$

In addition, it is easy to show that:

$$\begin{pmatrix} \overline{A}_i + \underline{B}_i K_j & -\underline{B}_i K_j \\ \overline{A}_i - L_j\underline{C}_i + \underline{B}_i K_j - G_j & G_j - \overline{B}_i K_j \end{pmatrix} \succcurlyeq \begin{pmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i\,K_j - G_j & G_j - B_i\,K_j \end{pmatrix} \tag{5.132}$$

Therefore, by combining (5.130)-(5.132) and using Lemma 3.1, we obtain that:

$$\mu\left(\begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ A_i - L_j C_i + B_i\,K_j - G_j & G_j - B_i\,K_j \end{bmatrix}\right) < 0 \tag{5.132}$$

which means that (5.101) is asymptotically stable for any $A_i \in [\underline{A}_i\,,\ \overline{A}_i]$, $B_i \in [\underline{B}_i\,,\overline{B}_i]$ and $C_i \in [\underline{C}_i\,,\overline{C}_i]$.

This completes the proof.$\blacksquare$

5.4.2. CASE 2: UNMEASURABLE PREMISE VARIABLES

We consider the observer design for the following positive interval time-delay T-S system (5.91) with unmeasurable premise variables

$$\begin{cases} \dot{x}(t) = \displaystyle\sum_{i=1}^{r} h_i(x(t))(A_i x(t) + A_{1i}x\left(t - \tau_i(t)\right) + B_i u(t)) \\ y(t) = \displaystyle\sum_{i=1}^{r} h_i(x(t))\,C_i\,x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in \left[-\overline{h}, 0\right] \end{cases} \tag{5.133}$$

where $h_i(x(t))$ are fuzzy weighting functions that depend on the state of the system and satisfy (1.4)

Remark 5.1.

In order to simplify the mathematical notations, we consider a single delay in the model. However, the proposed technique can be easily generalized to systems with multiple delays (Following, for example, (Rami & Tadeo, 2007)).

Taking into account that the state is estimated, the T-S model with unmeasurable variables (5.91) can be reduced to (Zaidi et al, 2013c):

$$
\begin{cases}
\dot{x}(t) = \displaystyle\sum_{i=1}^{r} h_i(\hat{x}(t)) \left(A_i \hat{x}(t) + B_i u(t)\right) + w(t) \\
y(t) = \displaystyle\sum_{i=1}^{r} h_i(x(t)) C_i\, x(t) \\
x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in \left[-\bar{h}, 0\right]
\end{cases}
\tag{5.134}
$$

where :

$$
w(t) = \sum_{i=1}^{r} \left(h_i(x(t)) - h_i(\hat{x}(t)) \right) \left(A_i x(t) + B_i u(t)\right)
\tag{5.135}
$$

Using Lemma 2.4, we have :

$$
\sum_{i=1}^{r} \left(h_i(x) - h_i(\hat{x}) \right) X_i = \sum_{i,j=1}^{r} h_i(x) h_j(\hat{x}) \Delta X_{ij}
\tag{5.136}
$$

where $X_i \in \{A_i, A_{1i}, B_i\}$ and ΔX_{ij} is defined by:

$$
\Delta X_{ij} = X_i - X_j
\tag{5.137}
$$

System (5.134) can then be transformed into the following system (Zaidi et al, 2013c):

$$
\begin{cases}
\dot{x}(t) = \displaystyle\sum_{i,j=1}^{r} h_i(x(t)) h_j(\hat{x}(t)) \left((A_i + \Delta A_{ij}) x(t) + (A_{1i} + \Delta A_{1ij}) x\big(t - \tau_i(t)\big) + (B_i + \Delta B_{ij}) u(t) \right) \\
y(t) = \displaystyle\sum_{i=1}^{r} h_i(x(t)) C_i\, x(t) \\
x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in \left[-\bar{h}, 0\right]
\end{cases}
$$

$$
\tag{5.138}
$$

where: $\Delta X_{ij} = X_i - X_j$, with $X_i = \{A_i, A_{1i}, B_i\}$.

The overall observer-based controller under consideration has the following form:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\hat{x}(t))(G_i\,\hat{x}(t) + G_{1i}\,\hat{x}(t - \tau(t)) + L_i y(t)) \tag{5.139}$$

where $G_i \in \mathbb{R}^{n_x \times n_x}$, $G_{1i} \in \mathbb{R}^{n_x \times n_x}$, $L_i \in \mathbb{R}^{n_x \times n_y}$ and $K_i \in \mathbb{R}^{n_u \times n_x}$, $\forall\, i = 1, \dots, r$, are the observer matrices to be determined.

The static state-feedback control law is given by the following expression:

$$u(t) = \sum_{i=1}^{r} h_i(\hat{x}(t)) K_i \hat{x}(t) \tag{5.140}$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$ are the controller matrices to be determined.

Then, the closed-loop system is written as follows:

$$\begin{cases} \dot{x}(t) = \displaystyle\sum_{i,j,k=1}^{r} h_i(x) h_j(\hat{x}) h_k(\hat{x}) \left((A_i + \Delta A_{ij}) x(t) + (A_{1i} + \Delta A_{1ij}) x(t - \tau_i(t)) + (B + \Delta B_{ij}) K_k \hat{x}(t) \right) \\ y(t) = \displaystyle\sum_{i=1}^{r} h_i(x(t))\, C_i\, x(t) \\ x(t) = \varphi(t) \geqslant 0, \forall\, t \in \left[-\overline{h}, 0 \right] \\ \hat{x}(t) = \vartheta(t) \geqslant 0, \forall\, t \in \left[-\overline{h}, 0 \right] \end{cases} \tag{5.141}$$

The key lies in the error, which is defined by:

$$e(t) = x(t) - \hat{x}(t). \tag{5.142}$$

If we choose $[x^T(t) \quad e^T(t)]^T$ as the new augmented state variable, then the new closed-loop system will become:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \sum_{i,j,k=1}^{r} h_i(x(t))\, h_j(\hat{x}(t)) h_k(\hat{x}(t)) \left(A_X^{ijk} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} + A_H^{ij} \begin{pmatrix} x(t - h(t)) \\ e(t - h(t)) \end{pmatrix} \right) \tag{5.143}$$

where:

$$A_X^{ijk} = \begin{pmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij}) K_k & -(B_i + \Delta B_{ij}) K_k \\ A_i + \Delta A_{ij} - L_j C_i + (B_i + \Delta B_{ij}) K_k - G_j & G_j - (B_i + \Delta B_{ij}) K_k \end{pmatrix}$$

and

$$A_H^{ij} = \begin{pmatrix} A_{1i} + \Delta A_{1ij} & 0 \\ A_{1i} + \Delta A_{1ij} - G_{1j} & G_{1j} \end{pmatrix}$$

The following theorem provides necessary conditions for the existence of an observer-based controller (5.139) for positive time-delay T-S systems with unmeasurable premise variables.

Theorem 5.7. (Zaidi et al.)

For a given interval positive closed-loop T-S system in (5.141), if there exist diagonal matrices X_1, X_2, Q_1 *and* Q_2, *matrices* W_{1j}, W_{2j}, W_{11j}, W_{12j}, L_i, K_k, $\forall\, i, j, k = 1, \ldots, r$ *and a scalar* $\gamma > 0$ *such that :*

$$\begin{pmatrix} S_{ijk} & T_{ijk} & (A_{1i} + \Delta A_{1ij})X_1 & 0 \\ * & U_{ijk} & (A_{1i} + \Delta A_{1ij})X_1 - W_{11j} & W_{12j} \\ * & * & -(1-d)R_1 & 0 \\ * & * & * & -(1-d)R_2 \end{pmatrix} < 0 \qquad (5.144)$$

and

$$\begin{pmatrix} L_{ijk} & M_{ijk} \\ N_{ijk} & O_{ijk} \end{pmatrix} \geqslant 0 \qquad (5.145)$$

where:

$S_{ijk} = (A_i + \Delta A_{ij})X_1 + X_1(A_i + \Delta A_{ij})^T + (B_i + \Delta B_{ij})Y_{1k} + Y_{1k}{}^T(B_i + \Delta B_{ij})^T + R_1$

$T_{ijk} = X_1(A_i + \Delta A_{ij})^T - (B_i + \Delta B_{ij})Y_{2k} - C_i^T Z_j^T + Y_{1k}{}^T(B_i + \Delta B_{ij})^T - W_{1j}{}^T$

$U_{ijk} = -(B_i + \Delta B_{ij})Y_{2k} - Y_{2k}{}^T(B_i + \Delta B_{ij})^T + W_{2j}{}^T + R_1$

$L_{ijk} = (A_i + \Delta A_{ij})X_1 + (B_i + \Delta B_{ij})Y_{1k} + \gamma X_1$

$M_{ijk} = -(B_i + \Delta B_{ij})Y_{2k}$

$N_{ijk} = (A_i + \Delta A_{ij})X_1 - Z_j C_i + (B_i + \Delta B_{ij})Y_{1k} - W_{1j}$

$O_{ijk} = W_{2j} - (B_i + \Delta B_{ij})Y_{2k} + \gamma X_2$

$X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$, $R_1 = X_1 Q_1^{-1} X_1$, $R_2 = X_2 Q_2^{-1} X_2$,

then, the augmented system (5.143) is asymptotically stable, while remaining positive.

Under these conditions, the desired observer and controller gain matrices may be obtained from

$K_k = Y_{1k}X_1^{-1}$, $L_j = Z_j V_1^{-1}$, $G_j = W_{1j}X_1^{-1}$, $G_{1j} = W_{11j}X_1^{-1}$,

where V_1 fulfills $C_i X_1 = V_1 C_i$.

Proof:

Assume that $\forall\, i, j, k = 1, \ldots, r$, there exist, G_i, G_{1i}, L_i and K_k such that (5.143) is positive and asymptotically stable, then, there exist diagonal matrices P and Q in the following forms $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} > 0$ and $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} > 0$ such that the following LMIs hold :

$$\begin{pmatrix} PA_X^{ijk} + A_X^{ijk\,T} P + Q & PA_H^{ij} \\ * & -(1-d)Q \end{pmatrix} < 0 \tag{5.146}$$

Multiplying each LMI on the left by $diag(P^{-1}, P^{-1})$, we get:

$$\begin{pmatrix} A_X^{ijk} + P^{-1}A_X^{ijk\,T} P + P^{-1}Q & A_H^{ij} \\ * & -(1-d)P^{-1}Q \end{pmatrix} < 0 \tag{5.147}$$

Then, multiplying on the right by $diag(P^{-1}, P^{-1})$, we obtain:

$$\begin{pmatrix} A_X^{ijk}P^{-1} + P^{-1}A_X^{ijk\,T} + P^{-1}QP^{-1} & A_H^{ij}P^{-1} \\ * & -(1-d)P^{-1}QP^{-1} \end{pmatrix} < 0 \tag{5.148}$$

Taking the following change of variables: $X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$, $Y_{1k} = K_k X_1$, $Y_{2k} = K_k X_2$ and $V_1 C_i = C_i X_1$, leads to the following LMIs:

$$\begin{pmatrix} a^{ijk} & b^{ijk} & (A_{1i} + \Delta A_{1ij})X_1 & 0 \\ * & c^{ijk} & (A_{1i} + \Delta A_{1ij})X_1 - G_{1j}X_2 & G_{1j}X_2 \\ * & * & -(1-d)X_1 Q_1 X_1 & 0 \\ * & * & * & -(1-d)X_2 Q_2 X_2 \end{pmatrix} < 0 \tag{5.149}$$

where

$a^{ijk} = (A_i + \Delta A_{ij})X_1 + X_1(A_i + \Delta A_{ij})^T + (B_i + \Delta B_{ij})Y_{1k} + Y_{1k}^T(B_i + \Delta B_{ij})^T$

$+ X_1 Q_1 X_1$

$b^{ijk} = X_1(A_i + \Delta A_{ij})^T - (B_i + \Delta B_{ij})Y_{2k} - C_i^T V_1^T L_j^T + Y_{1k}^T(B_i + \Delta B_{ij})^T - X_1 G_i^T$

$c^{ijk} = -(B_i + \Delta B_{ij})Y_2 - Y_2^T(B_i + \Delta B_{ij})^T + X_2 G_i^T + X_2 Q_2 X_2$

$\forall\, i, j, k = 1, \ldots, r.$

Considering that $Z_j = L_j V_1$, $W_{1j} = G_j X_1$, $W_{2j} = G_j X_2$, $W_{11j} = G_{1j}X_1$, $W_{12j} = G_{1j}$, $R_1 = X_1 Q_1 X_1$ and $R_2 = X_2 Q_2 X_2$, we get:

$$\begin{pmatrix} S_{ijk} & T_{ijk} & (A_{1i} + \Delta A_{1ij})X_1 & 0 \\ * & U_{ijk} & (A_{1i} + \Delta A_{1ij})X_1 - W_{11j} & W_{12j} \\ * & * & -(1-d)R_1 & 0 \\ * & * & * & -(1-d)R_2 \end{pmatrix} < 0 \qquad (5.150)$$

In order to guarantee the positivity of the augmented system (5.143), we have to prove that all A_X^{ijk} are Metzler, or equivalently, that there exists a positive scalar γ such that $A_X^{ijk} + \gamma I \geqslant 0$. Multiplying on the right by $diag(X_1, X_2)$, we get (5.145).

Once these LMIs are programmed and solved, we can obtain the observer and controller gain matrices:

$K_k = Y_{2k}X_2{}^{-1}$, $L_j = Z_j V_1{}^{-1}$, $G_j = W_{1j}X_1{}^{-1}$ and $G_{1j} = W_{11j}X_1{}^{-1}$.

Therefore, if there exist diagonal matrices P and Q, matrices G_j, G_{1j}, L_j and K_k such that (5.144) and (5.145) are satisfied, then, the augmented system (5.143) is positive and asymptotically stable. ∎

Otherwise, we further study sufficient conditions and the corresponding synthesis approach for this problem in the following theorem.

Theorem 5.8. (Zaidi et al.)

For a positive scalar ε, there exists a solution to the problem of continuous-time general observer-based controller for the closed-loop system (5.141) if $\forall\, i, j, k = 1, \dots, r$, there exist matrices $P = diag[P_1\ P_2] > 0$, $Q = diag[Q_1\ Q_2] > 0$, matrices $G_{1j} \geqslant 0$, $L_j \geqslant 0$, $K_k \leqslant 0$ and Metzler matrices G_j such that:

$$\begin{pmatrix} \beth_{ij} & \left(A_H^{ij}\right)^T P & P\mathcal{B}_{ij} + \mathcal{C}_i^T \mathcal{K}_{jk}{}^T & \varepsilon\mathcal{B}_{ij} \\ * & -(1-d)Q & 0 & 0 \\ * & 0 & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \qquad (5.151)$$

$[\underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k]_{lm} \geq 0\,, 1 \leq l \neq m \leq n \qquad (5.152)$

$[G_j - (\underline{B}_i + \underline{\Delta B}_{ij})K_k]_{lm} \geq 0\,, 1 \leq l \neq m \leq n \qquad (5.153)$

$\underline{A}_i + \underline{\Delta A}_{ij} - L_j\overline{C}_i + (\overline{B}_i + \overline{\Delta B}_{ij})K_k - G_i \geqslant 0 \qquad (5.154)$

where:

$\beth_{ij} = \mathcal{A}_{ij}{}^T P + P\mathcal{A}_{ij} - \varepsilon\mathcal{B}_{ij}\mathcal{B}_{ij}{}^T P - \varepsilon P\mathcal{B}_{ij}{}^T \mathcal{B}_{ij} + Q$

$$\mathcal{A}_{ij} = \begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} & 0 \\ \overline{A}_i & 0 \end{pmatrix}, \ \mathcal{B}_{ij} = \begin{pmatrix} 0 & \underline{B}_i + \underline{\Delta B}_{ij} - (\overline{B}_i + \overline{\Delta B}_{ij}) & 0 & \underline{B}_i + \underline{\Delta B}_{ij} \\ 0 & \underline{B}_i + \underline{\Delta B}_{ij} - (\overline{B}_i + \overline{\Delta B}_{ij}) & -I & \underline{B}_i + \underline{\Delta B}_{ij} \end{pmatrix}$$

$$\mathcal{K}_{jk} = \begin{pmatrix} G_j & L_j & 0 & 0 \\ K_k & 0 & 0 & 0 \\ 0 & 0 & G_j & L_j \\ 0 & 0 & K_k & 0 \end{pmatrix}, \ \mathcal{C}_i = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C}_i & 0 \end{bmatrix} \tag{5.155}$$

Proof:

It follows from (5.152) that $\forall \ i, j, k = 1, \dots, r$, $\underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k$ is Metzler. Combining this with $K \leqslant 0$ yields that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C_i \in [\underline{C}_i, \overline{C}_i]$:

$$A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k \geqslant \underline{A}_i + \underline{\Delta A}_{ij} + (\overline{B}_i + \overline{\Delta B}_{ij})K_k \text{ is Metzler} \tag{5.156}$$

From (5.153), we have: $G_j - (\underline{B}_i + \underline{\Delta B}_{ij})K_k$, then, for any $B_i \in [\underline{B}_i, \overline{B}_i]$:

$$G_j - (B_i + \Delta B_{ij})K_k \geqslant G_j - (\underline{B}_i + \underline{\Delta B}_{ij})K_k \text{ is Metzler} \tag{5.157}$$

We have also: $\underline{A}_i + \underline{\Delta A}_{ij} - L_j\overline{C}_i + (\overline{B}_i + \overline{\Delta B}_{ij})K_k - G_i \geqslant 0$. Since $A_i + \Delta A_{ij} - L_j C_i + (B_i + \Delta B_{ij})K_k - G_i \geqslant \underline{A}_i + \underline{\Delta A}_{ij} - L_j\overline{C}_i + (\overline{B}_i + \overline{\Delta B}_{ij})K_k - G_i$ (5.158)

In addition, from G_i being Metzler and $K_k \leqslant 0$, we obtain that, for any $B_i \in [\underline{B}_i, \overline{B}_i]$: $-(B_i + \Delta B_{ij})K_k \geqslant 0$ (5.159)

Therefore, from (5.154)-(5.157), we have that, for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C_i \in [\underline{C}_i, \overline{C}_i]$, the augmented system (5.143) is positive.

It follows from (5.151), by Schur complement, that

$$\begin{pmatrix} N_{ijk} & \left(A_H^{ij}\right)^T P \\ * & -(1-d)Q \end{pmatrix} < 0 \tag{5.160}$$

where:

$$N_{ijk} = \mathcal{A}_{ij}{}^T P + P\mathcal{A}_{ij} - \varepsilon \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T P - \varepsilon P\mathcal{B}_{ij}{}^T \mathcal{B}_{ij} + \varepsilon^2 \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T$$

$$+\left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{jk}\mathcal{C}_i\right)^T \left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{jk}\mathcal{C}_i\right) + Q \tag{5.161}$$

We have then:

$$\mathcal{A}_{ij}{}^T P + P\mathcal{A}_{ij} - \varepsilon \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T P - \varepsilon P\mathcal{B}_{ij}{}^T \mathcal{B}_{ij} + \varepsilon^2 \mathcal{B}_{ij}\mathcal{B}_{ij}{}^T$$

$$+\left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ij}\mathcal{C}_i\right)^T \left(\mathcal{B}_{ij}{}^T P + \mathcal{K}_{ij}\mathcal{C}_i\right) + Q < 0 \tag{5.162}$$

Taking into account the following relationship:

$$PBB^TP - \varepsilon BB^TP - \varepsilon PB^TB + \varepsilon^2 BB^T = (PB - \varepsilon B)(B^TP - \varepsilon B^T) \geq 0. \qquad (5.163)$$

we obtain the following inequality:

$$\mathcal{A}_{ij}{}^TP + P\mathcal{A}_{ij} - P\mathcal{B}_{ij}\mathcal{B}_{ij}{}^TP + \left(\mathcal{B}_{ij}{}^TP + \mathcal{K}_{jk}\mathcal{C}_i\right)^T\left(\mathcal{B}_{ij}{}^TP + \mathcal{K}_{jk}\mathcal{C}_i\right) + Q < 0 \quad (5.164)$$

Rewriting (5.164) yields that:

$$(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i)^TP + P(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i) + \mathcal{C}_i^T\mathcal{K}_{jk}{}^T\mathcal{K}_{jk}\,\mathcal{C}_i + Q < 0 \qquad (5.165)$$

which implies that:

$$(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i)^TP + P(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i) + Q < 0 \qquad (5.166)$$

From (5.160), we get:

$$\begin{pmatrix} (\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i)^TP + P(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i) + Q & * \\ PA_H^{ij} & -(1-d)Q \end{pmatrix} < 0 \qquad (5.167)$$

Therefore, we get:

$$\mu(\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i) < 0 \qquad (5.168)$$

Some algebraic manipulations lead to the following equivalence:

$$\mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_{jk}\mathcal{C}_i = \begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k & -(\overline{B}_i + \overline{\Delta B}_{ij})K_k \\ \overline{A}_i + \overline{\Delta A}_{ij} - L_j\underline{C}_i + (\underline{B}_i + \underline{\Delta B}_{ij})K_k - G_j & G_j - (\overline{B}_i + \overline{\Delta B}_{ij})K_k \end{pmatrix}$$

In addition, it is easy to show that:

$$\begin{pmatrix} \overline{A}_i + \overline{\Delta A}_{ij} + (\underline{B}_i + \underline{\Delta B}_{ij})K_k & -(\overline{B}_i + \overline{\Delta B}_{ij})K_k \\ \overline{A}_i + \overline{\Delta A}_{ij} - L_j\underline{C}_i + (\underline{B}_i + \underline{\Delta B}_{ij})K_k - G_j & G_j - (\overline{B}_i + \overline{\Delta B}_{ij})K_k \end{pmatrix}$$

$$\succcurlyeq \begin{pmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k & -(B_i + \Delta B_{ij})K_k \\ A_i + \Delta A_{ij} - L_jC_i + (B_i + \Delta B_{ij})K_k - G_j & G_j - (B_i + \Delta B_{ij})K_k \end{pmatrix} \qquad (5.169)$$

Therefore, by combining (5.167)-(5.169) and using Lemma 3.2, we obtain that:

$$\mu\left(\begin{bmatrix} A_i + \Delta A_{ij} + (B_i + \Delta B_{ij})K_k & -(B_i + \Delta B_{ij})K_k \\ A_i + \Delta A_{ij} - L_jC_i + (B_i + \Delta B_{ij})K_k - G_j & G_j - (B_i + \Delta B_{ij})K_k \end{bmatrix}\right) < 0 \qquad (5.170)$$

which means that (5.143) is asymptotically stable for any $A_i \in [\underline{A}_i, \overline{A}_i]$, $B_i \in [\underline{B}_i, \overline{B}_i]$ and $C_i \in [\underline{C}_i, \overline{C}_i]$.

This completes the proof. ∎

5.4.3. ILLUSTRATIVE EXAMPLE: HYDRAULIC TWO-TANK SYSTEM

We consider, as an application, the process composed of two linked tanks stated in Section 3.3.2.4. (Benzaouia & Hajjaji, 2011), (Zhang & Ding, 2005).

Now, we synthesized an observer-based controller for the positive interval T-S model with unmeasurable premise variables (5.91), in order guarantee a good estimation for the state variables of the system.

We can write the time-delay model as follows:

$$\begin{cases} \dot{x}(t) = \hat{A}(z_1, z_2)x(t) + \hat{A}_\tau(z_1, z_2)x(t - \tau(t)) + Bu(t) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in [-\overline{h}, 0] \end{cases} \tag{5.171}$$

where:

$$\hat{A} = \begin{pmatrix} -R_1 z_1 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & 0 \\ 0 & -R_2 z_2 - \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \end{pmatrix} \tag{5.172}$$

and

$$\hat{A}_\tau = \begin{pmatrix} 0 & \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} \\ \dfrac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2|}} & 0 \end{pmatrix} \tag{5.173}$$

with $\tau(t) \in \mathbb{R}$ a time-varying delay, considered as a continuous function, satisfying

$$\begin{cases} 0 \leq \tau(t) \leq \overline{h} \\ \dot{\tau}(t) \leq d < 1 \end{cases} \tag{5.174}$$

and expressed by:

$$\tau(t) = \frac{0.5}{1 + t}, \forall\, t > 0 \tag{5.175}$$

with R_1, R_2 and R_{12} positive physical constants, as shown in (Benzaouia & El Hajjaji, 2011).

So, $\overline{h} = 0.5$ and $d = 0.5$.

We consider that the variables are bounded as follows:

$$a_1 \leq z_1 \leq b_1 \tag{5.176}$$

$$a_2 \leq z_2 \leq b_2 \tag{5.177}$$

so we get the following four rules:

$$\begin{cases} \text{If } z_1 \text{ is } a_1 \text{ and } z_2 \text{ is } a_2 \text{ then } A(z_1, z_2) = A_1 \\ \text{If } z_1 \text{ is } a_1 \text{ and } z_2 \text{ is } b_2 \text{ then } A(z_1, z_2) = A_2 \\ \text{If } z_1 \text{ is } b_1 \text{ and } z_2 \text{ is } a_2 \text{ then } A(z_1, z_2) = A_3 \\ \text{If } z_1 \text{ is } b_1 \text{ and } z_2 \text{ is } b_2 \text{ then } A(z_1, z_2) = A_4 \end{cases} \tag{5.178}$$

The membership functions are given by

$$h_1(t) = f_{11}(t)\, f_{21}(t); \quad h_2(t) = f_{11}(t)\, f_{22}(t);$$

$$h_3(t) = f_{12}(t)\, f_{21}(t); \quad h_4(t) = f_{12}(t)\, f_{22}(t);$$

where

$$f_{11}(t) = \frac{z_1(t) - b_1}{a_1 - b_1} \quad \text{and} \quad f_{12}(t) = 1 - f_{11}(t) = \frac{a_1 - z_1(t)}{a_1 - b_1} \tag{5.179}$$

$$f_{21}(t) = \frac{z_2(t) - b_2}{a_2 - b_2} \quad \text{and} \quad f_{12}(t) = 1 - f_{21}(t) = \frac{a_2 - z_2(t)}{a_2 - b_2} \tag{5.180}$$

The resulting matrices A_i of the subsystems are:

$$\hat{A}_1 = \begin{pmatrix} -R_1 a_1 - \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & 0 \\ 0 & -R_2 a_2 - \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \end{pmatrix}, \quad \hat{A}_{\tau 1} = \begin{pmatrix} 0 & \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \\ \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & 0 \end{pmatrix}$$

$$\hat{A}_2 = \begin{pmatrix} -R_1 a_1 - \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & 0 \\ 0 & -R_2 b_2 - \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \end{pmatrix}, \quad \hat{A}_{\tau 2} = \begin{pmatrix} 0 & \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \\ \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & 0 \end{pmatrix}$$

$$\hat{A}_3 = \begin{pmatrix} -R_1 b_1 - \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & 0 \\ 0 & -R_2 a_2 - \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \end{pmatrix}, \quad \hat{A}_{\tau 3} = \begin{pmatrix} 0 & \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \\ \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & 0 \end{pmatrix}$$

$$\hat{A}_4 = \begin{pmatrix} -R_1 b_1 - \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & 0 \\ 0 & -R_2 b_2 - \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \end{pmatrix}, \quad \hat{A}_{\tau 4} = \begin{pmatrix} 0 & \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \\ \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & 0 \end{pmatrix}$$

The overall obtained T-S time-delay model with interval uncertainties is then given by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{4} h_i(x(t))((\hat{A}_i + \Delta\hat{A}_{ij})x(t) + (\hat{A}_{\tau i} + \Delta\hat{A}_{\tau ij})x(t - \tau(t)) + Bu(t)) \\ y(t) = C\,x(t) \\ x(t) = \varphi(t) \succcurlyeq 0, \forall\, t \in [-\overline{\tau}, 0] \end{cases} \qquad (5.181)$$

For calculation, we fix the parameters and their uncertainties as follows:

$R_1 = R_2 = 0.95 \pm 0.02, \;\; R_{12} = 0.65 \pm 0.03,$

$a_1 = 0.2236 \pm 0.011, \; b_1 = 0.4472 \pm 0.021$

$a_2 = 0.2582 \pm 0.016, \; b_2 = 0.4082 \pm 0.015$

The objective is to design an interval observer-based controller which ensures the stabilization and the estimation of the system associated to the real plant, in which matrices $\hat{A}_i + \Delta\hat{A}_{ij}$ are Metzler and matrices $\hat{A}_{\tau i} + \Delta\hat{A}_{\tau ij}$, B and C are nonnegative, $\forall\, i,j = 1, \ldots, r$.

We have:

$\underline{\hat{A}}_1 = \begin{pmatrix} -0.3878 & 0 \\ 0 & -0.4163 \end{pmatrix}; \overline{\hat{A}}_1 = \begin{pmatrix} -0.2541 & 0 \\ 0 & -0.1653 \end{pmatrix}$

$\underline{\hat{A}}_2 = \begin{pmatrix} -0.3935 & 0 \\ 0 & -0.4067 \end{pmatrix}; \overline{\hat{A}}_2 = \begin{pmatrix} -0.1250 & 0 \\ 0 & -0.3908 \end{pmatrix}$

$\underline{\hat{A}}_3 = \begin{pmatrix} -0.8881 & 0 \\ 0 & -0.5128 \end{pmatrix}; \overline{\hat{A}}_3 = \begin{pmatrix} -0.4623 & 0 \\ 0 & -0.2752 \end{pmatrix}$

$\underline{\hat{A}}_4 = \begin{pmatrix} -0.8669 & 0 \\ 0.2229 & -0.4763 \end{pmatrix}; \overline{\hat{A}}_4 = \begin{pmatrix} -0.5301 & 0 \\ 0 & -0.3046 \end{pmatrix}$

$\underline{\hat{A}}_{\tau 1} = \begin{pmatrix} 0 & 0.0164 \\ 0.0164 & 0 \end{pmatrix}; \overline{\hat{A}}_{\tau 1} = \begin{pmatrix} 0 & 0.0737 \\ 0.0292 & 0 \end{pmatrix}$

$\underline{\hat{A}}_{\tau 2} = \begin{pmatrix} 0 & 0.0170 \\ 0.0170 & 0 \end{pmatrix}; \overline{\hat{A}}_{\tau 2} = \begin{pmatrix} 0 & 0.0571 \\ 0.0258 & 0 \end{pmatrix}$

$\underline{\hat{A}}_{\tau 3} = \begin{pmatrix} 0 & 0.0271 \\ 0.0271 & 0 \end{pmatrix}; \overline{\hat{A}}_{\tau 3} = \begin{pmatrix} 0 & 0.0386 \\ 0.0443 & 0 \end{pmatrix}$

$\underline{\hat{A}}_{\tau 4} = \begin{pmatrix} 0 & 0.0248 \\ 0.0248 & 0 \end{pmatrix}; \overline{\hat{A}}_{\tau 4} = \begin{pmatrix} 0 & 0.0892 \\ 0.0732 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; C = (1 \quad 0)$

where $\underline{\hat{A}}_i$, $\overline{\hat{A}}_i$, $\underline{\hat{A}}_{\tau i}$ and $\overline{\hat{A}}_{\tau i}$ correspond respectively to the state and delay matrices of the interval system (5.181), $\forall\, i = 1, \ldots, 4$.

Applying Theorem 5.8. to the system (5.181), we get the following gain matrices:

$$K_1 = \begin{pmatrix} -0.1352 & -0.2413 \\ -0.5312 & -0.6341 \end{pmatrix} ; K_2 = \begin{pmatrix} -0.2152 & -0.8542 \\ -0.5264 & -0.7112 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} -1.0352 & -0.9523 \\ -0.8621 & -1.0652 \end{pmatrix} ; K_4 = \begin{pmatrix} -1.2588 & -0.6322 \\ -1.8522 & 0.4205 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 0.0481 \\ 0.0516 \end{pmatrix} ; L_2 = \begin{pmatrix} 0.0483 \\ 0.0476 \end{pmatrix} ;$$

$$L_3 = \begin{pmatrix} 0.1560 \\ 0.0783 \end{pmatrix} ; L_4 = \begin{pmatrix} 0.1658 \\ 0.0807 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0.2966 & -0.2107 \\ -0.2156 & 0.3053 \end{pmatrix} ; G_2 = \begin{pmatrix} 0.3015 & -0.2136 \\ -0.2179 & 0.3024 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0.6334 & -0.3443 \\ -0.3496 & 0.3648 \end{pmatrix} ; G_4 = \begin{pmatrix} 0.6048 & -0.3224 \\ -0.3285 & 0.3423 \end{pmatrix}$$

$$G_{11} = \begin{pmatrix} 0.0249 & 0.0050 \\ 0.0399 & 0.0299 \end{pmatrix} ; G_{12} = \begin{pmatrix} 0.0158 & 0.0124 \\ 0.0321 & 0.0359 \end{pmatrix}$$

$$G_{13} = \begin{pmatrix} 0.1135 & 0.0421 \\ 0.0366 & 0.0236 \end{pmatrix} ; G_{14} = \begin{pmatrix} 0.0249 & 0.0912 \\ 0.0152 & 0.0648 \end{pmatrix}$$

Some simulation results using the proposed observer-based controller are presented in figures 5.7 to 5.9 for the system matrices with smaller and bigger components. The desired reference to reach here is $y_r = [30\ 30]^T$. We can observe that the evolutions of the real state vector $x(t)$ and of the estimation $\hat{x}(t)$ are always in the positive orthant, for given initial conditions $x(t) = [10\ 13]^T$, $\hat{x}(t) = [42\ 35]^T$ and $\hat{x}(t) = [3\ 5]^T$, $t \in [-0.5\ 0]$. Moreover, the upper and lower estimated states are nonnegative and converge to the real value. These properties can be seen in Figures 5.8 and 5.9 that plot the state evolutions from the given initial conditions. These facts show the effectiveness of the proposed approach.
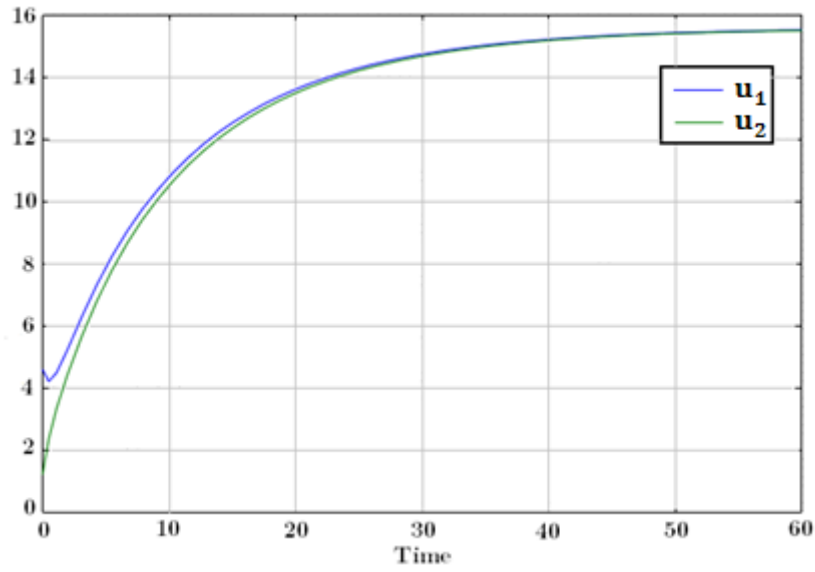
Figure 5. 7. Evolution of the two pump flows $u_1(t)$ and $u_2(t)$
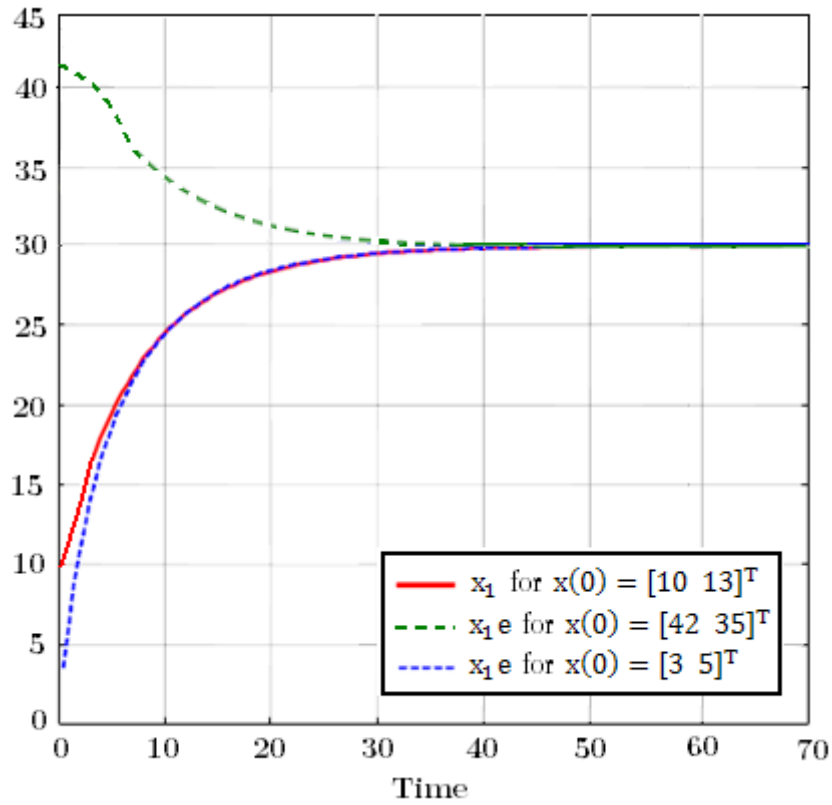
Figure 5. 8. Evolution of the state $x_1(t)$ and its estimation $\hat{x}_1(t)$
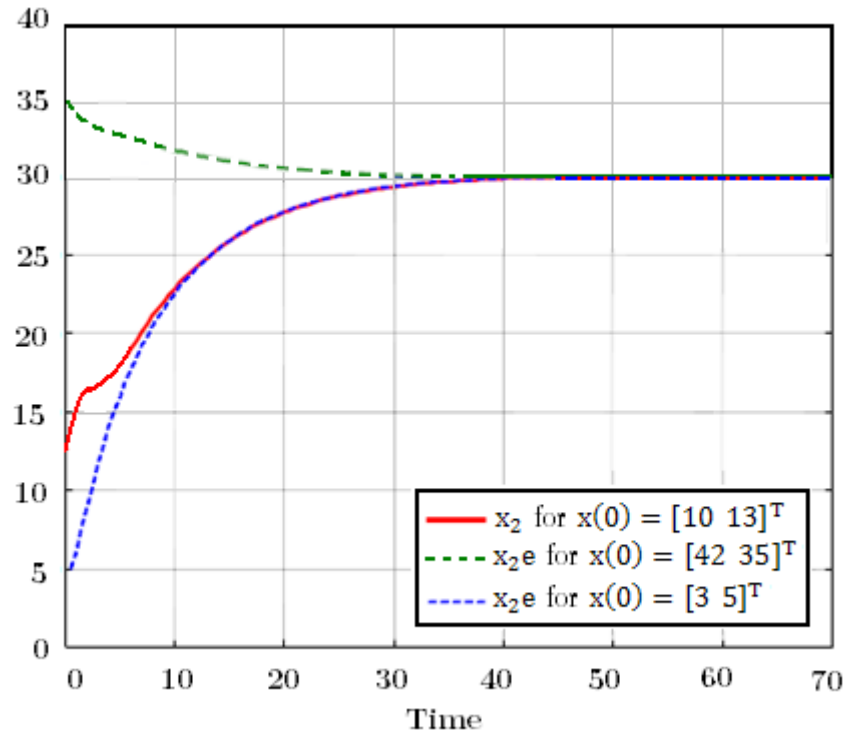
Figure 5. 9. Evolution of the state $x_2(t)$ and its estimation $\hat{x}_2(t)$

5.5. Conclusion

In this section, we have dealt with the problem of observer-based control design for positive time-delay systems. Firstly, we have designed positive observers for positive interval linear time-delay systems; necessary and sufficient conditions have been established, taking into account the positivity constraints. Secondly, observer-based controllers have been synthesized for this class of systems. Necessary conditions have been established in order to verify the existence of an observer-based controller for the considered closed-loop system. Once satisfied, we study the sufficient conditions and the corresponding synthesis for this problem. Then, we extend these approaches for positive interval time-delay T-S systems. For this, we consider two cases: when the decision variables are measurable and when the decision variables are unmeasurable.

Finally, illustrative results of numerical and practical examples have been given to show the effectiveness of these approaches.

Conclusions and Perspectives

6.1. Summary and Contributions

This thesis firstly presents systematic procedures to design robust controllers and observer-based controllers for positive linear and T-S systems, with/without time-delay and with/without interval uncertainties. These approaches are based on the resolution of optimization problems, expressed as LMIs (Linear Matrix Inequalities). Furthermore, the proposed methodologies are extended to constrained design problems. These techniques have been applied on numerical and practical examples from literature, giving satisfactory results.

More precisely, the main contributions of the first part of this work lead to establish approaches to stabilize and to solve the problem of designing decomposed state-feedback controllers for positive linear and T-S systems, which can be subject to interval uncertainties. We have also presented the robust α-stability notion that guarantees a specified decay rate in the presence of possible uncertainties on the considered types of systems. Moreover, necessary and sufficient conditions are given for the stabilization of positive T-S systems, taking into account two cases: when the decision variables of the system are measurable or not. This may be provided by means of state-feedback laws that can be chosen with or without memory. Our contribution is mainly focused in the field of robust estimation through designing positive observer-based controllers and positive interval observer-based controllers for these types of systems, which can have interval uncertainties.

In this thesis, the theoretical developments improving interval estimations have been given and their efficiency has been demonstrated through various examples with significant uncertainties. It has been shown that the existence of robust interval positive observer-based controllers can be expressed in terms of LMI conditions, so they can be easily computed. The second part of this dissertation is devoted to the stability analysis, stabilization and observer-based control design of positive time-delay positive linear and T-S systems.

We consider in this thesis different types of delays: constant, variable, single and multiple. Techniques of stabilization and observer-based control of these systems have been established, corresponding to the considered time-delay and expressed in terms of LMI conditions.

Illustrative examples have shown the performance of the proposed approaches and we have proposed some practical applications: electrical systems and hydraulic systems.

6.2. Future Lines of Research

The work presented in this dissertation is an open research line and inspires to several interesting problems in the fields of control and estimation. Other interesting issues will be investigated such as static and dynamic output-feedback controller design for positive T-S systems with and without delay. Moreover, we tend for applying the approaches of this thesis on photovoltaic systems.

In addition, there exist more extensions under study. Among them, we can cite:

- Positive descriptor design for positive T-S systems with time-varying delays.
- Diagnosis and faults detection for positive T-S systems with time-varying delay.
- Positive multi-dimensional systems.
- Positive LPV systems.
- Implementation of the results to more practical systems.

Mostly, in the area of faults detection and isolation, interval observers and interval observer-based controllers using decomposed control laws can have a great importance. This can open the doors for multiple lines of future research.

Also, interval observation can provide useful estimates for robust control, where the control strategy and the interval estimates complement each other in order to deal with the uncertainties of the system and to guarantee a better level of performance.

Annex: Important Lemmas

B.1. Schur Complement

Let us consider three matrices $Q(x)$, $S(x)$ and $R(x)$, affine with respect to the variable x. Matrices $Q(x)$ and $R(x)$ are symmetric. The LMI:

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0 \Leftrightarrow \begin{cases} R(x) > 0 \\ Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \end{cases} \qquad (B.1.1)$$

B.2. (Zhou & Khargonedkar, 1988)

Consider two matrices X and Y of appropriate dimensions. The following inequality is always applicable for every matrix $Q = Q^T > 0$:

$$XY^T + YX^T \leq XQX^T + YQ^{-1}Y^T \qquad (B.1.2)$$

B.3. (Kacem, 2009)

Consider ϕ, a, b matrices with appropriate dimensions. The following hypotheses are equivalent:

a-The LMI

$$\begin{pmatrix} \phi & a \\ a^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (b^T \quad -I) \right\} < 0 \qquad (B.1.3)$$

 is feasible for the variables F_1 and F_2.

b-Matrices ϕ, a and b verify the condition: $\phi + ab^T + ba^T < 0$.

B.4. (Kacem, 2009)

Consider ϕ, a, b matrices with appropriate dimensions. The following hypotheses are equivalent:

a- ϕ, a and b verify:

$$\begin{cases} \phi < 0 \\ \phi + ab^T + ba^T < 0 \end{cases} \qquad (B.1.4)$$

b-The following LMI:

$$\begin{pmatrix} \phi & a + bG^T \\ a^T + Gb^T & -G - G^T \end{pmatrix} = \begin{pmatrix} \phi & a \\ a^T & 0 \end{pmatrix} + Sym\left\{ \begin{pmatrix} 0 \\ I \end{pmatrix} G(b^T \quad -I) \right\} < 0 \qquad (B.1.5)$$

is feasible for the variable G.

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Objetivos, metodología y resultados generales del trabajo

El objetivo principal de esta tesis es el desarrollo de procedimientos sistemáticos para el diseño de controladores robustos y controladores basados en observadores para los sistemas lineales y de Takagi-Sugeno, con/sin retraso, con/sin incertidumbres de intervalo.

La metodología de diseño se basa en el planteamiento del diseño como un problema de resolución de problemas de optimización, expresados como LMIs (Linear Matrix Inequalities). Por otra parte, las metodologías propuestas se extienden a los problemas de diseño con restricciones. Estas técnicas se han aplicado en ejemplos numéricos y prácticos de la literatura, dando resultados satisfactorios.

Más precisamente, las principales aportaciones de la primera parte de este trabajo son las propuestas para establecer enfoques para analizar la estabilidad y para resolver el problema del diseño de controladores descompuestos de realimentación del estado para sistemas lineales y de Takagi-Sugeno positivos con incertidumbres de intervalo. También se presenta la noción de α-estabilidad robusta, que garantiza una tasa de atenuación especificada en la presencia de posibles incertidumbres sobre los tipos de sistemas considerados. Además, se dan las condiciones necesarias y suficientes para la estabilización de los sistemas de Takagi-Sugeno positivos, teniendo en cuenta si las variables de decisión del sistema son medibles o no. Esto puede ser previsto, mediante el uso de las leyes de realimentación del estado, que se pueden elegir con memoria o no.

Entonces, se han presentado nuevos resultados en el campo de la estimación robusta mediante el diseño de controladores basados en observadores positivos y controladores basados en observadores de intervalo positivos. Nuestra contribución se centra principalmente en la estimación y la estabilización de las variables de estado para estos tipos de sistemas, que pueden tener incertidumbres de intervalo. Debe mencionarse que los desarrollos teóricos que mejoran las estimaciones de intervalo se han demostrado a través de diversos ejemplos con incertidumbres significativas.

Se ha demostrado que la existencia de controladores basados en los observadores robustos positivos de intervalo se puede expresar en términos de condiciones de LMIs, por lo que puede ser calculada fácilmente.

La segunda parte de esta tesis está dedicada al análisis de la estabilidad, la estabilización y el diseño de control basado en observador con retraso para sistemas lineales y de Takagi-Sugeno positivos. Se consideran diferentes tipos de retrasos: constantes, variables, simples y múltiples. Técnicas de estabilización y control basado en observador de estos sistemas se han establecido, correspondientes al tiempo de retraso considerado y expresadas en términos de condiciones de LMIs.

Ejemplos ilustrativos demuestran el rendimiento de los enfoques propuestos y nos han propuesto algunas aplicaciones prácticas: sistemas eléctricos y sistemas hidráulicos.

Resumen Detallado por Capítulos de Principales Resultados

Capítulo 1:

En el capítulo 1 empezamos por establecer el contexto y presentar el material de referencia para el trabajo presentado más adelante. Definimos diversos conceptos y resultados que se utilizarán en las siguientes secciones. En primer lugar, se da una visión general sobre el enfoque de modelado de sistemas de Takagi-Sugeno (T-S), las técnicas de estabilización y de estimación de este tipo de sistemas. En segundo lugar, los conceptos básicos de clasificación y el modelado de sistemas T-S con retraso. A continuación, nos concentramos en la estabilidad y la estabilización de los sistemas de T-S con retraso. Más tarde, se presentara el diseño de observadores y controladores basados en observadores. Una parte de los resultados se referirá a los sistemas positivos y sus propiedades; el resto de los resultados que se presentan en este capítulo son resultados bien conocidos en la literatura. A continuación, nos centramos en la positividad de los sistemas lineales y sus propiedades que se han utilizado con éxito en los últimos trabajos. Clases de sistemas de T-S con retraso, que serán utilizados en varias ocasiones en los capítulos siguientes, se analizan en el contexto de garantizar su estabilidad y la positividad y la construcción de observadores positivos de intervalo. La α-estabilidad se discutirá a fondo en los diferentes capítulos de la tesis; por lo tanto, se discuten las propiedades básicas de estabilidad.

Capítulo 2

En el capítulo 2, estamos interesados en el análisis de la estabilidad y la estabilización de los sistemas dinámicos que sólo implican estados no negativos. En primer lugar, presentamos información básica acerca de los sistemas T-S positivos y sus propiedades estructurales. Estos sistemas son particulares y deben satisfacer una restricción inherente de signo: los componentes de estado del sistema deben seguir siendo negativos incluso cuando comienzan por valores iniciales negativas. Por otra parte, se dedica a introducir el concepto de la estabilidad asintótica y la α-estabilidad para los sistemas no lineales positivos, específicamente sistemas de T-S, donde los principales enfoques de estabilidad y estabilización difieren según el tipo de las variables de decisión asociados a cada modelo: si son medibles o no. Por otra parte, estamos interesados en la α-estabilización robusta de los sistemas diseñados, teniendo en cuenta simultáneamente su positividad. Leyes de control descompuestas para los sistemas de intervalo positivos se establecen a continuación.

Capítulo 3

En el capítulo 3, se establecen condiciones LMIs (Linear Matrix Inequalities) para sintetizar los controladores basados en intervalos de observación para los sistemas lineales y de Takagi-Sugeno. Este enfoque es muy importante, ya que puede proporcionar estimaciones inferiores y superiores de los estados no medibles, incluso en la presencia de incertidumbres en el sistema y sus mediciones. Por lo tanto, se añaden restricciones a los observadores Luenberger de intervalo en presencia de medidas corruptas (posiblemente con entradas desconocidas). Esto se puede hacer mediante la minimización de una cota adecuada sobre los errores de intervalo, resolviéndose a través de un problema de optimización LMI.

Capítulo 4

En el Capítulo 4, se estudian los sistemas con retraso que mantienen la positividad y la estabilidad contra un factor de retardo desconocido y/o incertidumbres de intervalo. Se introducen resultados de la estabilidad de los sistemas lineales con retraso positivos. También se presenta la noción de la α-estabilidad robusta que garantiza una tasa de atenuación específica contra posibles incertidumbres sobre el sistema. Además, se proporcionan las condiciones necesarias y suficientes para la estabilización de los sistemas de intervalo positivas

con retrasos simples y múltiples, constantes y variables a través de las leyes del retroalimentación del estado que se puede elegir con o sin memoria.

Capítulo 5

En el capítulo 5, primero se diseñam observadores positivos para los sistemas lineales de intervalo con retraso positivos en el caso autónomo; condiciones necesarias y suficientes se han sido establecidas y expresadas en términos de LMIs, teniendo en cuenta las limitaciones de positividad del sistema aumentado. En segundo lugar, los controladores basados en observadores se sintetizan. Condiciones necesarias se han formulado para comprobar la existencia de cualquier solución al problema de control basado en observador de tiempo continuo. Una vez satisfecho, se estudian las condiciones suficientes y la síntesis correspondiente para este problema. Por otra parte, las extensiones de de estos enfoques se han aplicado a los sistemas positivos de Takagi-Sugeno con retraso variable. Se consideran dos casos para resolver el problema del diseño positivo de control basado en observador: cuando las variables de decisión son medibles y cuando no lo son.

Finalmente, los resultados ilustrativos de ejemplos numéricos y prácticos se han dado para mostrar la eficacia de estos enfoques.

En la conclusión, resumimos los resultados obtenidos y concluimos la tesis resumiendo posibles direcciones para extender esos resultados.