Allowing agents to be imprecise: A proposal using multiple linguistic terms

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Abstract

In this paper we propose a decision-making procedure where the agents judge the alternatives through linguistic terms such as 'very good', 'good', 'acceptable', etc. If the agents are not confident about their opinions, they can use a linguistic expression formed by several consecutive linguistic terms. To obtain a ranking on the set of alternatives, the method consists of three different stages. The first stage looks for the alternatives in which the overall opinion is closer to the ideal assessment. The overall opinion is developed by a distance-based process among the individual assessments. The next two stages form a tie-breaking process. Firstly by using a dispersion index based on the Gini coefficient, and secondly by taking into account the number of best-assessments. The main characteristics of the proposed decision-making procedure are analyzed.
Keywords: Social Choice; voting systems; linguistic assessments; Majority Judgment; imprecision.

1. Introduction

Social Choice Theory shows that there is no voting system that is able to rank and choose alternatives in a completely acceptable way. In this regard, the well-known *Arrow's impossibility theorem* [1] shows, with absolute certainty,

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that there is no voting system that simultaneously satisfies several desirable properties[1].

Arrow's pessimistic result imposes the limits of preference aggregation, but it does not stop the search for a procedure that fulfills some properties at the expense of others. One escape route for Arrow's impossibility theorem consists in allowing the agents to show their opinions not in a strictly ordinal way, but through numerical or linguistic assessments. A short review of one of the most known linguistic decision procedures is detailed below, as well as some of their extensions. Moreover, we provide an introduction to the issue of the imprecision of the agents, upon which our process is based. Finally, we summarize the proposal of the paper.

1.1. Majority Judgment

Balinski and Laraki [2, 3] have proposed a voting system called *Majority Judgment* (MJ) which tries to avoid the unsatisfactory result of the Arrow theorem and allows the voters to assess the alternatives through linguistic terms, such as 'excellent', 'very good', 'good', etc., instead of ordering the alternatives by rank. Among all the individual assessments given by the voters, MJ chooses the median as the collective assessment. Balinski and Laraki also describe a tie-breaking process which compares the number of assessments above the collective assessment with those below it.

These authors also carried out an experimental analysis of MJ [4] in Orsay during the 2007 French presidential election. In that paper the authors show some interesting properties of MJ and they argue that this voting system is easily implemented and avoids the need for a second round[2], typical of French presidential elections.

Desirable properties and advantages have been attributed to MJ compared to the classical Arrow framework of preferences aggregation. Among these advantages is the possibility that voters show their opinions more faithfully and properly than in the conventional voting systems.

Besides MJ, other decision-making procedures in which the agents assess the alternatives through linguistic terms can be found in the literature. For instance in García-Lapresta [10] a general voting system that generalizes the *simple majority* through linguistic preferences is designed and studied. Similarly, in García-Lapresta *et al.* [11, 13] a system which generalizes the *Borda rule* [5] is studied.

1.2. Majority Judgment extensions

It is worth pointing out that some authors have shown several paradoxes and inconsistencies of MJ (see Felsenthal and Machover [9], Smith [27], García-

[1] Any voting rule that generates a collective weak order from every profile of weak orders, and satisfies independence of irrelevant alternatives and unanimity is necessarily dictatorial, insofar as there are at least three alternatives and three agents.

[2] If there is no candidate with more than half of the votes, the second round consists of voting on the two candidates with most votes in the first round.

Lapresta and Martínez-Panero [12] and Nurmi [22], among others).

In order to reduce some of the drawbacks produced by MJ in small committees, García-Lapresta and Martínez-Panero [12] developed a proposal in which the linguistic information is aggregated by means of *centered OWA operators* (Yager [34]), and a 2-*tuple fuzzy linguistic representation* (Herrera and Martínez [17]). Another way of thinking was proposed by Zahid [37] who combined MJ with the *Borda Count* [5] in order to avoid some other inconveniences of MJ.

Moreover, in Falcó and García-Lapresta [6, 7] an extension of MJ, based on the distances between the linguistic terms is proposed. These distances are induced by the parameterized family of Minkowski metrics and allow us to treat the problem in a more flexible way. The extension carried out in Falcó and García-Lapresta [6] chooses as the collective assessment, a linguistic term that minimizes the total distance to all the individual assessments (this would be the median, whenever the Manhattan metric is used). In addition, a method of choosing a unique collective assessment, in the case that several assessments fulfil the requirement, is provided. The contribution of Falcó and García-Lapresta [6, 7] is also a refinement of the tie-breaking process that not only counts the number of assessments above and below the collective assessment (the median in MJ), but which also takes into account the specific assessments (above and below the collective assessment) by measuring the distances between them and the collective assessment.

1.3. Imprecise assessments

According to Zimmer [38], people generally prefer to handle the imprecision with linguistic terms rather than with numbers. Usually opinions are imprecise, therefore, trying to represent them by using a precise term is meaningless. In addition, Wallsten *et al.* [31] have shown empirically how most people are more comfortable using words rather than numbers to describe probabilities. As a result of this evidence and reflection, the program *computing with words* has been developed, where the agents express themselves through linguistic terms instead of numbers (see Kacprzyk and Zadrożny [19] and Zadeh [35, 36], among others).

Although the use of linguistic information brings the design of decision-making procedures closer to the imprecision that agents face when judging the alternatives, occasionally, the agents may be unconfident about which linguistic term to use. For this reason, it is interesting to allow the agents to judge in a more imprecise way, giving them the option of assessing several consecutive linguistic terms. For other papers regarding this issue, see Tang and Zheng [28], Ma *et al.* [21] and Rodríguez *et al.* [23].

Our proposal concerning the imprecision is based on an adaptation of the *absolute order of magnitude spaces* introduced in Travé-Massuyès and Dague [29], and Travé-Massuyès and Piera [30]; more specifically in the extensions devised by Roselló *et al.* [24, 25, 26].

The authors of this paper have previously made an attempt to deal with imprecise assessments which was also based on the absolute order of magnitude spaces (Falcó *et al.* [8]). In that paper, they used a system of penalization for the use of a linguistic expression (a linguistic expression is the combination of several consecutive linguistic terms). The penalization function worked roughly as follows: the more linguistic terms an agent uses, the more that agent should be penalized.

1.4. Our proposal

In this paper we set up a decision-making procedure in which agents can express their assessments of the alternatives using a linguistic term from a predetermined linguistic scale. If they are not confident about which term to use, they can use a linguistic expression created by several consecutive linguistic terms.

We have assumed that the linguistic scale is uniform and symmetrically distributed. Thus, the distance between consecutive linguistic terms is assumed to be the same for all the agents. We have also considered that agents show their true opinions and they do not act strategically to favor or penalize any alternative.

The procedure is divided into several stages which will be presented in Section 3. Initially, the overall opinion for an alternative is calculated by finding the set of linguistic expressions that minimize the sum of distances to every expression given by the agents, for said alternative. Taking into account this information, the alternatives are ordered by the proximity of their overall opinions to the "ideal" assessment. Thus, the closer an overall opinion is to the highest linguistic term, the better the alternative would be considered. Since ties among different alternatives may appear, we present a tie-breaking process. The tie-breaking process is constructed using a dispersion index based on the Gini coefficient. The less dispersion there is among agents' assessments, the more preferred this alternative will be considered to be. After this stage, some alternatives can still be in a tie, therefore a further refinement is presented for the tie-breaking process. If the distance to the "ideal" assessment as well as the dispersion are the same, the number of highest assessments are counted. If there is still a tie, the number of second highest assessments are counted, and so on.

The paper is organized as follows. Section 2 is devoted to introducing the notation and concepts needed in the rest of the paper. Section 3 includes the proposed decision-making procedure. In Section 4, some properties of the process are analyzed. Section 5 contains the concluding remarks of this paper.

2. Notation and basic notions

Let $I = \{1, \dots, m\}$, with $m \geq 2$, be a set of agents and let $X = \{x_1, \dots, x_n\}$, with $n \geq 2$, be the set of alternatives which are evaluated. Every agent assesses a linguistic term for each candidate within a linguistic ordered scale $L = \{l_1, \dots, l_g\}$, where $l_1 < l_2 < \cdots < l_g$. The *granularity* of L is its cardinality $\#L = g \geq 2$. The linguistic scale is balanced and equispaced between consecutive terms. The terms on L can be linguistic terms as 'excellent', 'very good', 'good', etc.

A binary relation $\succcurlyeq$ on a set $A \neq \emptyset$ is a *weak order* (or *complete preorder*) if it is complete ($a \succcurlyeq b$ or $b \succcurlyeq a$, for all $a, b \in A$) and transitive (if $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$, for all $a, b, c \in A$). On the other hand, a *linear order* on $A \neq \emptyset$ is an antisymmetric[3] weak order on A. Given a weak or linear order $\succcurlyeq$ on $A \neq \emptyset$, the asymmetric and symmetric parts of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively, i.e., $a \succ b$ if not $b \succcurlyeq a$, and $a \sim b$ if $a \succcurlyeq b$ and $b \succcurlyeq a$. $W(A)$ denotes the set of weak orders on A.

Given $\succcurlyeq \in W(A)$, the inverse of $\succcurlyeq$ is defined as $x_i \succcurlyeq^{-1} x_j \Leftrightarrow x_j \succcurlyeq x_i$.

2.1. Linguistic expressions

Based on the *absolute order of magnitude spaces* introduced by Travé-Massuyès and Piera [30], we define the *set of linguistic expressions* as follows

$$\mathbb{L} = \{[l_h, l_k] \mid l_h, l_k \in L\,,\ 1 \leq h \leq k \leq g\},$$

where $[l_h, l_k] = \{l_h, l_{h+1}, \ldots, l_k\}$ and $[l_1, l_g] = \{l_1, \ldots, l_g\}$. Given that $[l_h, l_h] = \{l_h\}$, this linguistic expression can be replaced by the linguistic term l_h. In this way, $L \subset \mathbb{L}$.

Example 1. Consider the set of linguistic terms $L = \{l_1, l_2, l_3, l_4, l_5\}$ with the meanings given in Table 1.

l_1	l_2	l_3	l_4	l_5
very bad	bad	acceptable	good	very good

Table 1: Meaning of the linguistic terms.

Each linguistic expression has a meaning on its own. For instance, $[l_2, l_4]$ means 'between bad and good', $[l_4, l_5]$ means 'between good and very good', or 'at least good', etc.

For an interpretation of the linguistic expressions based on context-free grammar, see Rodríguez *et al.* [23].

Adopting the treatment introduced in Roselló *et al.* [26], the set of all the linguistic expressions can be represented by a graph $G_{\mathbb{L}}$. In the graph, the lowest layer represents the linguistic terms $l_h \in L \subset \mathbb{L}$, the second layer represents the linguistic expressions created by two consecutive linguistic terms $[l_h, l_{h+1}]$, the third layer represents the linguistic expressions created by three consecutive linguistic terms $[l_h, l_{h+2}]$, and so on up to last layer where we represent the linguistic expression $[l_1, l_g]$. As a result, the higher an element is, the more imprecise it becomes. The vertices in $G_{\mathbb{L}}$ are the elements of $\mathbb{L}$ and the edges $\mathcal{E} - \mathcal{F}$, where $\mathcal{E} = [l_h, l_k]$ and $\mathcal{F} = [l_h, l_{k+1}]$, or $\mathcal{E} = [l_h, l_k]$ and $\mathcal{F} = [l_{h+1}, l_k]$. The graph representation of Example 1 is included in Fig. 1.

[3] $\succcurlyeq$ is antisymmetric if for all $a, b \in A$ such that $a \neq b$ it holds $a \succ b$ or $b \succ a$.

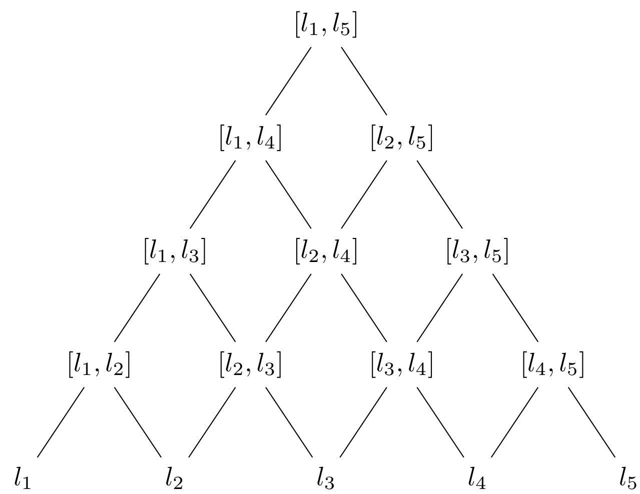

Figure 1: Graph representation of the linguistic expressions for $g = 5$.

When a voter is confident about his opinion on an alternative, he might use a linguistic term $l_h \in L$. Whereas if he is unconfident about his opinion, he might use a linguistic expression $[l_h, l_k] \in \mathbb{L}$, with $h < k$. For a more extensive treatment, see Roselló *et al.* [24, 25, 26].

Let us note that all the computations in $\mathbb{L}$ can be done in $\mathbb{Z}^2$ by means of the injection $\psi : \mathbb{L} \longrightarrow \mathbb{Z}^2$, defined as $\psi([l_h, l_k]) = (k-1, h-1)$. Through the function ψ we can represent a linguistic expression as a point in the plane. This function allows us to work in an easier computational setting. For example, $l_h \in L$ is identified with $(h-1, h-1) \in \mathbb{Z}^2$, and the linguistic expression $[l_3, l_5] \in \mathbb{L}$ is identified with the point $(4, 2) \in \mathbb{Z}^2$ (see Fig. 2).

2.2. Distances between linguistic expressions

The distance between two linguistic expressions $\mathcal{E}, \mathcal{F} \in \mathbb{L}$ is defined as the geodesic distance in the graph $G_{\mathbb{L}}$ between their associated vertices and it is denoted by $d(\mathcal{E}, \mathcal{F})$. The geodesic distance between two vertices in a graph is the number of edges in one of the shortest paths connecting them.

Remark 1. Taking into account the injection $\psi : \mathbb{L} \longrightarrow \mathbb{Z}^2$, the distance between two linguistic expressions $\mathcal{E}$ and $\mathcal{F}$ can be computed in $\mathbb{Z}^2$ as the *Manhattan distance*[4] between the corresponding points $\psi(\mathcal{E})$ and $\psi(\mathcal{F})$:

$$d(\mathcal{E}, \mathcal{F}) = d_M(\psi(\mathcal{E}), \psi(\mathcal{F})). \tag{1}$$

[4]The Manhattan distance in $\mathbb{R}^q$ is the function $d_M : \mathbb{R}^q \times \mathbb{R}^q \longrightarrow \mathbb{R}$ defined as $d_M((a_1, \ldots, a_q), (b_1, \ldots, b_q)) = \sum_{k=1}^{q} |a_k - b_k|$.

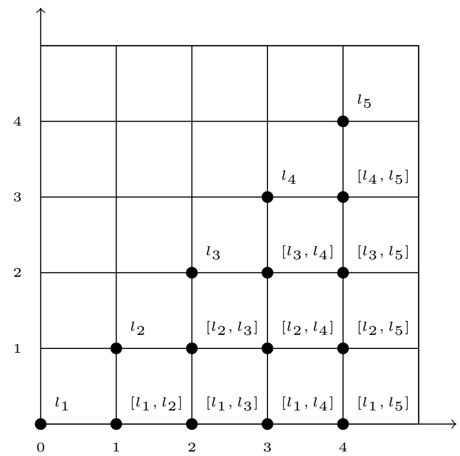

Figure 2: The injection from $\mathbb{L}$ into $\mathbb{Z}^2$ for $g = 5$.

Example 2. The distance between the linguistic expressions $\mathcal{E} = [l_1, l_3]$ and $\mathcal{F} = \{l_4\}$ in $\mathbb{L}$, for $g = 5$, is the length of the shortest path from one vertex to the other, $d(\mathcal{E}, \mathcal{F}) = 4$: from vertex $[l_1, l_3]$ to vertex $[l_2, l_3]$, from vertex $[l_2, l_3]$ to vertex l_3, from l_3 to $[l_3, l_4]$ and, finally, from $[l_3, l_4]$ to l_4. This path is not unique, but it is one of those shortest paths (see Fig. 3). Or, by means of $\mathbb{Z}^2$ as:

$$d(\mathcal{E}, \mathcal{F}) = d_M(\psi(\mathcal{E}), \psi(\mathcal{F})) = d_M((2, 0), (3, 3)) = |2 - 3| + |0 - 3| = 4.$$

2.3. The potential

We are now going to introduce the potential of a linguistic expression with respect to a vector of linguistic expressions (and also with respect to a subset of linguistic expressions). It is defined as the sum of distances between the linguistic expression and the components of the vector (the elements of the subset). It will be very useful to shorten the notation.

Definition 1. *Given* $\mathcal{E} \in \mathbb{L}$ *and* $\boldsymbol{y} = (y_1, \ldots, y_q) \in \mathbb{L}^q$ *, the* potential of $\mathcal{E}$ with respect to $\boldsymbol{y}$ *is defined as*

$$\Phi(\mathcal{E}, \boldsymbol{y}) = \sum_{k=1}^{q} d(\mathcal{E}, y_k).$$

Definition 2. *Given* $\mathcal{E} \in \mathbb{L}$ *and* $F \subseteq \mathbb{L}$, *the* potential of $\mathcal{E}$ with respect to F *is defined as*

$$\Phi(\mathcal{E}, F) = \sum_{\mathcal{F} \in F} d(\mathcal{E}, \mathcal{F}).$$

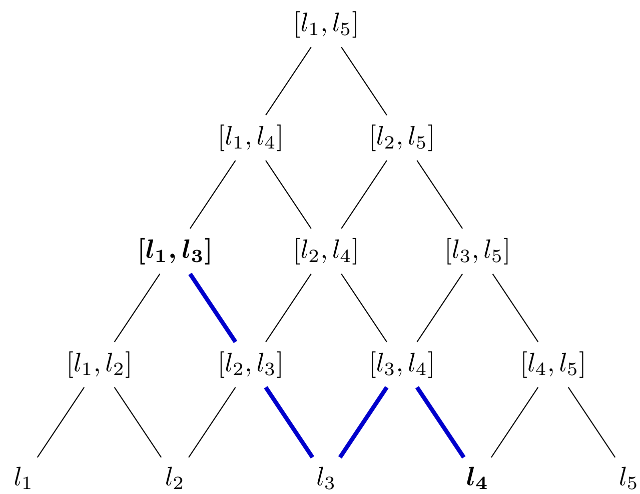

Figure 3: Graph representation distance between $[l_1, l_3]$ and l_4 for Example 2.

2.4. The overall opinion

The Fermat point of a triangle is a point such that the total distance from the three vertices of the triangle to the point is the minimum possible (see [33]). This Fermat point was generalized in the so-called geometric median or Fermat-Weber point by Weber [32], and it is the point that minimizes the sum of distances to the sample points in an Euclidean space. Based on these ideas, we now introduce a similar notion in the setting of linguistic expressions.

A *profile* V is a matrix (v_i^p) consisting of m rows and n columns of linguistic expressions, where the element $v_i^p \in \mathbb{L}$ represents the linguistic assessment given by the voter $p \in I$ to the alternative $x_i \in X$. The set of all possible profiles is denoted by $\mathbb{V}$. We denote by $\boldsymbol{v}_i = (v_i^1, \ldots, v_i^m) \in \mathbb{L}^m$ the *assessments vector* of x_i. Similarly, $\boldsymbol{v}^p = (v_1^p, \ldots, v_n^p) \in \mathbb{L}^n$ denotes the *assessments vector* of agent p for all the alternatives. Then,

$$V = \begin{pmatrix} v_1^1 & \cdots & v_i^1 & \cdots & v_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^p & \cdots & v_i^p & \cdots & v_n^p \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^m & \cdots & v_i^m & \cdots & v_n^m \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}^1 \\ \vdots \\ \boldsymbol{v}^m \end{pmatrix} = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n)\,.$$

Definition 3. *Given a profile $V \in \mathbb{V}$ and the assessments vector of x_i, $\boldsymbol{v}_i \in \mathbb{L}^m$, the* Fermat set *of x_i is defined as*

$$F_i^V = \arg\min_{\mathcal{E} \in \mathbb{L}} \Phi(\mathcal{E}, \boldsymbol{v}_i).$$

In other words,

$$\mathcal{E} \in F_i^V \ \Leftrightarrow \ \forall \mathcal{F} \in \mathbb{L} \ \ \Phi(\mathcal{E}, \boldsymbol{v}_i) \leq \Phi(\mathcal{F}, \boldsymbol{v}_i)$$

and, equivalently,

$$\mathcal{E} \in F_i^V \;\Leftrightarrow\; \forall \mathcal{F} \in \mathbb{L} \;\; \sum_{p=1}^{m} d(\mathcal{E}, v_i^p) \leq \sum_{p=1}^{m} d(\mathcal{F}, v_i^p).$$

For the sake of simplicity, the superindex V is omitted whenever it is clear from the context.

The Fermat set F_i contains all the linguistic expressions that minimize the sum of the distances to all the assessments for x_i. This set somehow represents the *overall opinion* of x_i, and it may contain more than one linguistic expression. Notice that a linguistic expression can be in the Fermat set although it does not belong to the assessments vector $\boldsymbol{v}_i$.

Example 3. Consider $X = \{x_1, x_2\}$, $I = \{1, 2, 3\}$ and $g = 5$ with the assessments given in Table 2 and Fig. 4.

	x_1	x_2
1	l_4	l_3
2	$[l_3, l_4]$	$[l_3, l_4]$
3	$[l_2, l_4]$	$[l_3, l_5]$

Table 2: Agents' assessments in Example 3.

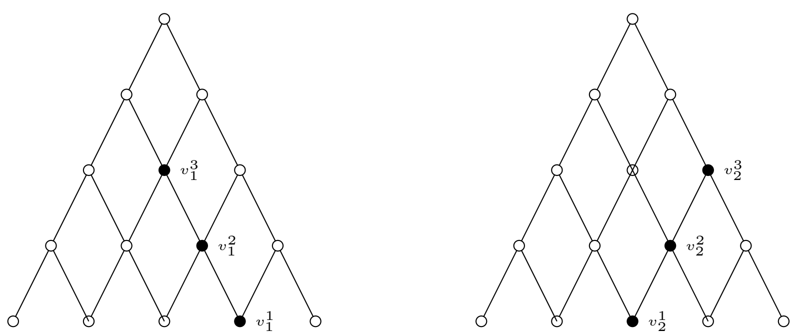

Figure 4: Agents's assessments for alternatives x_1 (left) and x_2 (right).

Looking the results given in Tables 3 and 4, where we can see all the potentials for every possible linguistic expression in $\mathbb{L}$, in both cases the linguistic expression with the minimal potential is $[l_3, l_4]$. Then, $F_1 = F_2 = \{[l_3, l_4]\}$.

3. The decision-making procedure

The proposal is divided in several stages:

$\mathcal{E}$	$\Phi(\mathcal{E}, \boldsymbol{v}_1)$
l_1	15
$[l_1, l_2]$	12
l_2	9
$[l_1, l_3]$	9
$[l_2, l_3]$	6
$[l_1, l_4]$	6
l_3	5
$[l_2, l_4]$	3
$[l_1, l_5]$	9
$\boldsymbol{[l_3, l_4]}$	**2**
$[l_2, l_5]$	6
l_4	3
$[l_3, l_5]$	5
$[l_4, l_5]$	6
l_5	9

Table 3: Potentials with respect to x_1.

$\mathcal{E}$	$\Phi(\mathcal{E}, \boldsymbol{v}_2)$
l_1	15
$[l_1, l_2]$	12
l_2	9
$[l_1, l_3]$	9
$[l_2, l_3]$	6
$[l_1, l_4]$	8
l_3	3
$[l_2, l_4]$	5
$[l_1, l_5]$	9
$\boldsymbol{[l_3, l_4]}$	**2**
$[l_2, l_5]$	6
l_4	5
$[l_3, l_5]$	3
$[l_4, l_5]$	6
l_5	9

Table 4: Potentials with respect to x_2.

1. We calculate the distances from each Fermat set F_i to the linguistic term l_g, for all the alternatives. Then, the alternatives are ordered according to their proximity to l_g. The closer to l_g, the better.
2. If ties are present after the first stage, we break them through the dispersion of the individual assessments. The lower the dispersion is, the better.
3. If still some alternatives are in a draw, then we look for the number of assessments in the best linguistic expression, then the second-best one, then the third-best one, and so on.

In next subsections we explain all these stages in depth. After every step, we show how to apply the procedure through the results in the Example 3.

3.1. Closeness to the "ideal" assessment

For every alternative $x_i \in X$ we calculate its overall opinion by means of the Fermat set F_i presented in the previous section. Once the overall opinions of the alternatives have been obtained, we compare it with the best possible result an alternative can get. The linguistic term l_g is always the highest assessment one alternative can achieve, hence the "ideal" assessment (the closer to the "ideal", the better).

Given $\mathcal{E} \in \mathbb{L}$ and $F \subseteq \mathbb{L}$, we denote by $\bar{d}(\mathcal{E}, F)$ the average distance between $\mathcal{E}$ and the elements of F:

$$\bar{d}(\mathcal{E}, F) = \frac{\sum_{\mathcal{F} \in F} d(\mathcal{E}, \mathcal{F})}{\# F}\,.$$

Definition 4. *Given $V \in \mathbb{V}$, the binary relation $\succcurlyeq_1^V$ on X is defined as*

$$x_i \succcurlyeq_1^V x_j \;\Leftrightarrow\; \bar{d}(l_g, F_i) \leq \bar{d}(l_g, F_j),$$

i.e.,

$$x_i \succcurlyeq_1^V x_j \;\Leftrightarrow\; \frac{\Phi\left(l_g, F_i\right)}{\#F_i} \leq \frac{\Phi\left(l_g, F_j\right)}{\#F_j} \;\Leftrightarrow\; \frac{\displaystyle\sum_{\mathcal{E} \in F_i} d(l_g, \mathcal{E})}{\#F_i} \leq \frac{\displaystyle\sum_{\mathcal{E} \in F_j} d(l_g, \mathcal{E})}{\#F_j}.$$

For the sake of simplicity, the superindex V is omitted whenever it is clear from the context and we will denote the binary relation simply as $\succcurlyeq_1$.

Clearly, $\succcurlyeq_1$ is a weak order on X.

This order seems natural considering that the closer the assessments are in average to the linguistic term l_g (and thereby with a minimal average distance), the better the alternative is.

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) is a multi-criteria decision analysis method, originally developed by Hwang and Yoon [18]. As TOPSIS suggests, the chosen alternative should have the shortest geometric distance from the positive ideal solution and the longest geometric distance from the negative ideal solution.

The following results establishes how the minimum average distance to the best linguistic assessment implies also the maximum average distance to the worst linguistic assessment.

Lemma 1. *For all $x_i, x_j \in X$ and $V \in \mathbb{V}$ it holds*

$$x_i \succcurlyeq_1^V x_j \;\Leftrightarrow\; \bar{d}(l_1, F_i) \geq \bar{d}(l_1, F_j).$$

PROOF: Since the maximum distance in the graph is $d(l_1, l_g) = 2g - 2$, it is easy to see that $d(l_g, \mathcal{E}) = 2g - 2 - d(l_1, \mathcal{E})$, for every $\mathcal{E} \in \mathbb{L}$. Then, for a subset $E \subseteq \mathbb{L}$

$$\sum_{\mathcal{E} \in E} d(l_g, \mathcal{E}) = \#E \cdot (2g - 2) - \sum_{\mathcal{E} \in E} d(l_1, \mathcal{E}).$$

Dividing by the subset cardinality,

$$\frac{\displaystyle\sum_{\mathcal{E} \in E} d(l_g, \mathcal{E})}{\#E} = \frac{\#E \cdot (2g - 2)}{\#E} - \frac{\displaystyle\sum_{\mathcal{E} \in E} d(l_1, \mathcal{E})}{\#E},$$

or, what it is the same,

$$\bar{d}(l_g, E) = 2g - 2 - \bar{d}(l_1, E).$$

Consequently, applying this results on the Fermat sets

$$\begin{aligned} \bar{d}(l_g, F_i) \leq \bar{d}(l_g, F_j) \quad &\Leftrightarrow \quad 2g - 2 - \bar{d}(l_1, F_i) \leq 2g - 2 - \bar{d}(l_1, F_j) \\ &\Leftrightarrow \quad \bar{d}(l_1, F_i) \geq \bar{d}(l_1, F_j). \blacksquare \end{aligned}$$

As $\bar{d}(l_1, F_i) \geq \bar{d}(l_1, F_j) \;\Leftrightarrow\; \bar{d}(l_g, F_i) \leq \bar{d}(l_g, F_j)$, the relation $\succcurlyeq_1^V$ follows TOPSIS idea. Our ranking prefers the alternative whose overall opinion is closer to the "ideal" assessment and, simultaneously, prefers the alternative whose overall opinion is further of the "worst" assessment.

Example 4. Coming back to Example 3, the distance to the best assessment would be the same in both alternatives:

$$\bar{d}(l_g, F_1) = d(l_5, [l_3, l_4]) = 3 = \bar{d}(l_g, F_2).$$

The first order does not provide a ranking between both alternatives so, the result is $x_1 \sim_1 x_2$.

In this case, both overall opinions are the same[5] and also are their distances to the "ideal" assessment. Thus, it is necessary to introduce another step to break the ties among alternatives. We propose a method based on the Gini coefficient [15] and the consensus measures introduced by García-Lapresta and Pérez-Román in [14].

3.2. Dispersion of the agents' assessments

Definition 5. *Given $\boldsymbol{v}_i \in \mathbb{L}^m$, the* dispersion index *of the agents assessing the alternative $x_i \in X$ is defined as*

$$\delta_i = \frac{\displaystyle\sum_{p=1}^{m} \Phi\left(v_i^p, \boldsymbol{v}_i\right)}{2 \cdot (g-1) \cdot m \cdot (m-1)},$$

or in terms of distances,

$$\delta_i = \frac{\displaystyle\sum_{q=1}^{m} \sum_{p=1}^{m} d\left(v_i^p, v_i^q\right)}{2 \cdot (g-1) \cdot m \cdot (m-1)}.$$

The dispersion is calculated through the sum of the potentials of every agent with respect to the alternative x_i. In terms of distances first we calculate for every agent the distance from his assessment to the other agents' assessments. Then, we sum the result for every agent. The denominator role is to normalize the result.

Proposition 1. *The dispersion index δ_i verifies the following properties:*

1. $0 \leq \delta_i \leq 1$.

[5] Notice how different overall opinions (or Fermat sets) can also provide a tie among alternatives. For instance, $F_1 = \{[l_3, l_4]\}$ and $F_2 = \{[l_2, l_4], [l_3, l_4], [l_2, l_5], [l_3, l_5]\}$ are two possible Fermat sets which have the same distance to the best assessment. In such a way, $\bar{d}(l_5, F_1) = \frac{2+3+3+4}{4} = 3 = \bar{d}(l_5, F_2)$. Then, $x_1 \sim_1 x_2$.

2. $\delta_i = 1 \Leftrightarrow m = 2$ *and one of the following conditions holds*
   (a) $v_i^1 = l_1$ *and* $v_i^2 = l_g$
   (b) $v_i^1 = l_g$ *and* $v_i^2 = l_1$.
3. $\delta_i = 0 \Leftrightarrow v_i^1 = v_i^2 = \cdots = v_i^m$.

PROOF:

1. Since $d(l_1, l_g) = 2 \cdot (g-1)$ is the maximum distance between elements of $\mathbb{L}$, and $m \cdot (m-1)$ is the number of eventually non-zero terms in the numerator of the formula above, then the quotient is between 0 and 1.
2. Taking into account (1) and
$$\delta_i = 1 \Leftrightarrow \sum_{p \in I} \Phi\left(v_i^p, \boldsymbol{v}_i\right) = 2 \cdot (g-1) \cdot m \cdot (m-1),$$
it is easy to see that if $m > 2$, the last equality is not possible. Moreover, if $m = 2$, then $\delta_i = 1$ if and only if ($v_i^1 = l_1$ and $v_i^2 = l_g$) or ($v_i^1 = l_g$ and $v_i^2 = l_1$).
3. By
$$\delta_i = 0 \Leftrightarrow \sum_{p \in I} \Phi\left(v_i^p, \boldsymbol{v}_i\right) = 0,$$
and $\Phi\left(v_i^p, \boldsymbol{v}_i\right) \geq 0$ for all $p \in I$, $\delta_i = 0$ if and only if $\Phi\left(v_i^p, \boldsymbol{v}_i\right) = 0$ for all $p \in I$, i.e., $v_i^1 = \cdots = v_i^m$. ■

Given two alternatives $x_i, x_j \in X$ such that the distances from their overall opinions to the highest assessment are the same, we will prefer that alternative with the greater agreement taking into account agents' assessments.

Definition 6. *Given $V \in \mathbb{V}$, the binary relation $\succcurlyeq_2^V$ is defined as*
$$x_i \succcurlyeq_2^V x_j \Leftrightarrow \delta_i \leq \delta_j.$$
For the sake of simplicity, the superindex V is omitted whenever it is clear from the context and we will denote the binary relation simply as $\succcurlyeq_2$.

Clearly, $\succcurlyeq_2$ is a weak order on X. This order represents the idea of how an alternative should be preferred if the level of agreement among the agents is high.

Example 5. Following with Example 3,
$$\delta_1 = \frac{\Phi\left(l_4, \boldsymbol{v}_1\right) + \Phi\left([l_3, l_4], \boldsymbol{v}_1\right) + \Phi\left([l_2, l_4], \boldsymbol{v}_1\right)}{2 \cdot (5-1) \cdot 3 \cdot (3-1)} = \frac{3+2+3}{2 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{6},$$
$$\delta_2 = \frac{\Phi\left(l_3, \boldsymbol{v}_2\right) + \Phi\left([l_3, l_4], \boldsymbol{v}_2\right) + \Phi\left([l_3, l_5], \boldsymbol{v}_2\right)}{2 \cdot (5-1) \cdot 3 \cdot (3-1)} = \frac{3+2+3}{2 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{6}.$$
Since $\delta_1 = \delta_2$, then $x_1 \sim_2 x_2$. Again, both alternatives are in a tie.

If there are two alternatives $x_i, x_j \in X$ in a tie, that is $x_i \sim_1 x_j$ and $x_i \sim_2 x_j$, because $\bar{d}(l_g, F_i) = \bar{d}(l_g, F_j)$ and $\delta_i = \delta_j$, we will consider a sequential process where the number of best assessments, second-best assessments, etc. obtained by the tied alternatives are taken into account.

3.3. Number of best assessments

The idea of best assessment, second-best assessment, etc. assumes that an order within the set $\mathbb{L}$ exists. In this way, we consider an order where we prefer linguistic expressions closer to the ideal assessment, and, if two elements of $\mathbb{L}$ have the same distance to l_g, we prefer the more precise one (with less number of linguistic terms).

Definition 7. *The binary relation* $\succcurlyeq_{\mathbb{L}}$ *on* $\mathbb{L}$ *is defined as* $\mathcal{E} \succcurlyeq_{\mathbb{L}} \mathcal{F}$ *if one of the following conditions holds*

1. $d(\mathcal{E}, l_g) < d(\mathcal{F}, l_g)$.
2. $d(\mathcal{E}, l_g) = d(\mathcal{F}, l_g)$ *and* $\#\mathcal{E} \leq \#\mathcal{F}$.

Proposition 2. *The binary relation* $\succcurlyeq_{\mathbb{L}}$ *is a linear order.*

Proof: Clearly, $\succcurlyeq_{\mathbb{L}}$ is a weak order. For proving antisymmetry, consider $\mathcal{E} = [l_h, l_k]$ and $\mathcal{F} = [l_r, l_s] \in \mathbb{L}$ such that $\mathcal{E} \sim_{\mathbb{L}} \mathcal{F}$, i.e., $d(l_g, \mathcal{E}) = d(l_g, \mathcal{F})$ and $\#\mathcal{E} = \#\mathcal{F}$. Since $d(l_g, \mathcal{E}) = 2g-(h+k)$, $d(l_g, \mathcal{F}) = 2g-(r+s)$, $\#\mathcal{E} = k-h+1$ and $\#\mathcal{F} = s-r+1$, we have $s = k$ and $r = h$, hence, $\mathcal{E} = \mathcal{F}$. ■

As an example, for $g = 5$, the linguistic expressions of $\mathbb{L}$ are ordered as follows (see also Fig. 5):

$$\begin{aligned} & l_5 \succ_{\mathbb{L}} [l_4, l_5] \succ_{\mathbb{L}} l_4 \succ_{\mathbb{L}} [l_3, l_5] \succ_{\mathbb{L}} [l_3, l_4] \succ_{\mathbb{L}} [l_2, l_5] \succ_{\mathbb{L}} l_3 \succ_{\mathbb{L}} [l_2, l_4] \succ_{\mathbb{L}} \\ & \succ_{\mathbb{L}} [l_1, l_5] \succ_{\mathbb{L}} [l_2, l_3] \succ_{\mathbb{L}} [l_1, l_4] \succ_{\mathbb{L}} l_2 \succ_{\mathbb{L}} [l_1, l_3] \succ_{\mathbb{L}} [l_1, l_2] \succ_{\mathbb{L}} l_1. \end{aligned}$$

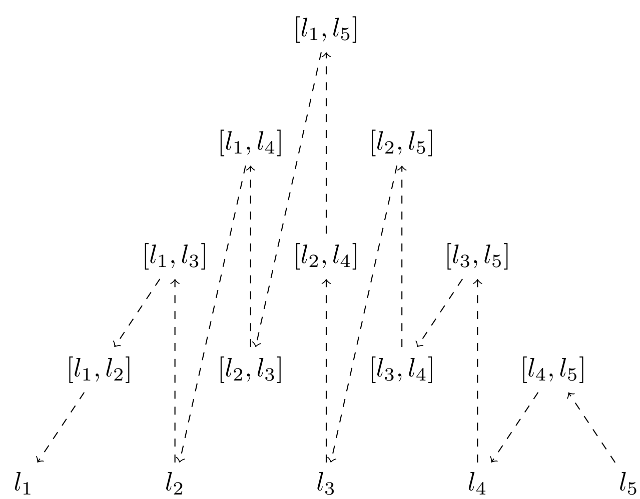

Figure 5: Linear order in $\mathbb{L}$ for $g = 5$.

The last step to break the tie between x_i and x_j is counting how many agents assessed the alternatives with the linguistic term l_g: if this amount of agents is bigger for x_i than for x_j, then we will say that $x_i \succcurlyeq_3^V x_j$. If we are still in a tie, then we will count how many agents assessed the alternatives with the linguistic expression $[l_{g-1}, l_g]$, and so on. It is summarized in the next definition.

Definition 8. *Given $V \in \mathbb{V}$, the binary relations $\succcurlyeq_3^V$, $\succcurlyeq_4^V$, ... are defined as*

$$
\begin{aligned}
&x_i \succcurlyeq_3^V x_j \;\Leftrightarrow\; \#\{p \in I \mid v_i^p = l_g\} \geq \#\{q \in I \mid v_j^q = l_g\},\\
&x_i \succcurlyeq_4^V x_j \;\Leftrightarrow\; \#\{p \in I \mid v_i^p = [l_{g-1}, l_g]\} \geq \#\{q \in I \mid v_j^q = [l_{g-1}, l_g]\},\\
&\ldots
\end{aligned}
$$

For the sake of simplicity, the superindex V is omitted whenever it is clear from the context and we will denote the binary relations simply as $\succcurlyeq_3$, $\succcurlyeq_4$, etc.

Clearly, $\succcurlyeq_3$, $\succcurlyeq_4$, etc. are weak orders on X. Summarizing, the process is conducted by the lexicographic weak order $\succcurlyeq^V$ (as until now, the subindex will be omitted if there is no possibility of confusion) on X defined as $x_i \succcurlyeq^V x_j$ if and only if one of the following conditions holds

1. $x_i \succ_1 x_j$.
2. $x_i \sim_1 x_j$ and $x_i \succ_2 x_j$.
3. $x_i \sim_1 x_j$, $x_i \sim_2 x_j$ and $x_i \succ_3 x_j$.
4. $x_i \sim_1 x_j$, $x_i \sim_2 x_j$, $x_i \sim_3 x_j$ and $x_i \succ_4 x_j$.
5. ...

First, we look for the alternative with an overall opinion closer to the ideal assessment. If there are alternatives with the same distance, we look for the alternative with smaller dispersion. If still some alternatives have the same result, we would look for the one with the bigger number of “best” assessments.

Example 6. The tie-breaking process applied to Example 3 is as follows:
$x_1 \sim_3 x_2$: $\#\{p \in I \mid v_1^p = l_5\} = 0 = \#\{p \in I \mid v_2^p = l_5\}$.
$x_1 \sim_4 x_2$: $\#\{p \in I \mid v_1^p = [l_4, l_5]\} = 0 = \#\{p \in I \mid v_2^p = [l_4, l_5]\}$.
$x_1 \succ_5 x_2$: $\#\{p \in I \mid v_1^p = l_4\} = 1 > 0 = \#\{p \in I \mid v_2^p = l_4\}$.
Consequently, $x_1 \succ x_2$.

3.4. An illustrative example

As mentioned before, it is common for agents to be uncertain about which assessments assign to the alternatives, and sometimes they are more comfortable using a linguistic expression than a single linguistic term. However, usually agents are forced to assess a single linguistic term. Example 7 shows how taking into account the imprecision of the agents can lead to different results than when they are forced to be precise.

Alternative	v_i^1			v_i^2			v_i^3			v_i^4		
	$\mathbb{L}$		L	$\mathbb{L}$		L	$\mathbb{L}$		L	$\mathbb{L}$		L
x_1	l_3	$\rightarrow$	l_3	$[l_2, l_4]$	$\rightarrow$	l_3	$[l_3, l_5]$	$\rightarrow$	l_4	$[l_3, l_5]$	$\rightarrow$	l_4
x_2	$[l_2, l_4]$	$\rightarrow$	l_3	$[l_2, l_5]$	$\rightarrow$	l_4	$[l_2, l_5]$	$\rightarrow$	l_4	l_5	$\rightarrow$	l_5

Table 5: Agents' assessments in the Example 7.

Example 7. Consider $I = \{1, \ldots, 4\}$, $L = \{l_1, \ldots, l_5\}$ and $X = \{x_1, x_2\}$. Table 5 shows two assessments for each agent: their sincere assessments which consist of linguistic expressions ($\mathbb{L}$ column), and the linguistic terms they assess when they are required to be precise (L column).

If we evaluate the opinions in which the agents assess only one linguistic term, we obtain that the overall opinions for both alternatives are $F_1 = \{l_3, [l_3, l_4], l_4\}$ and $F_2 = \{l_4\}$. Since

$$\bar{d}(l_5, F_1) = \frac{4+3+2}{3} = 3 > 2 = \frac{2}{1} = \bar{d}(l_5, F_2),$$

we have $x_2 \succ_1 x_1$, and then, $x_2 \succ x_1$.

If we now calculate the results when considering the sincere linguistic expressions given by the agents in $\mathbb{L}$, we obtain that the overall opinions for both alternatives are $F_1 = \{[l_3, l_4], [l_3, l_5]\}$ and $F_2 = \{[l_2, l_5]\}$. Since

$$\bar{d}(l_5, F_1) = \frac{3+2}{2} = 2.5 < 3 = \frac{3}{1} = \bar{d}(l_5, F_2),$$

contrary to the outcome provided by the single-linguistic-term assessments, we now have $x_1 \succ_1 x_2$, and then, $x_1 \succ x_2$.

3.5. Relationship with Majority Judgment

In this subsection we are going to show the relationship between MJ and the decision-making procedure presented in this paper. Since MJ does not use multiple linguistic terms, we will consider the special case where all voters assess a simple linguistic term to each alternative, i.e., $\boldsymbol{v}_i = (v_i^1, \ldots, v_i^m) \in L^m$ for every $i \in \{1, \ldots, n\}$. Let us denote

$$N_h(x_i) = \#\{p \in I \mid v_i^p = l_h\},$$

for every $h \in \{1, \ldots, g\}$.

Lemma 2. *If m is odd and $l_M \in L$ is the median of the data distribution given by $\boldsymbol{v}_i$, then*

$$\sum_{h=1}^{k} N_h(x_i) > \frac{m-1}{2} \geq \sum_{h=k+1}^{g} N_h(x_i),$$

for every $k \geq M$.

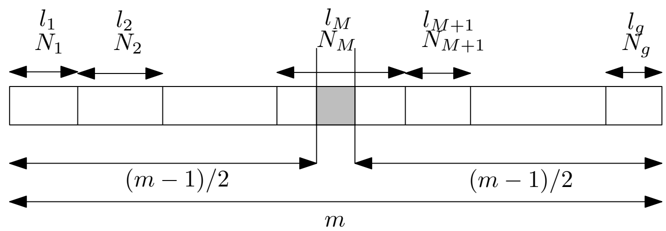

Figure 6: Lemma 2.

Proof: Trivial by the definition of median (see Fig.6).■

Using the definition of N_h and the expression of d in terms of $\mathbb{Z}^2$, the potential of a linguistic term with respect to $\boldsymbol{v}_i$ can be written as

$$\Phi(l_h, \boldsymbol{v}_i) = 2\sum_{k=1}^{g} N_k(x_i)|h-k|.$$

Lemma 3. *If m is odd and l_M is the median of the data distribution given by $\boldsymbol{v}_i$, then*

$$\Phi(l_M, \boldsymbol{v}_i) < \Phi(l_{M+1}, \boldsymbol{v}_i) < \cdots < \Phi(l_g, \boldsymbol{v}_i).$$

Proof: Since

$$\Phi(l_h, \boldsymbol{v}_i) - \Phi(l_{h+1}, \boldsymbol{v}_i) = 2\sum_{k=1}^{g} N_k(x_i)(|h-k| - |h-k+1|).$$

and

$$|h-k| - |h-k+1| = \begin{cases} -1, & \text{if } k \le h, \\ 1, & \text{if } k > h, \end{cases}$$

we have

$$\Phi(l_h, \boldsymbol{v}_i) - \Phi(l_{h+1}, \boldsymbol{v}_i) = -2\sum_{k=1}^{h} N_k(x_i) + 2\sum_{k=h+1}^{g} N_k(x_i).$$

If $h \ge M$, by Lemma 2, we have

$$\Phi(l_h, \boldsymbol{v}_i) - \Phi(l_{h+1}, \boldsymbol{v}_i) \le 0.■$$

The previous lemma can be generalized.

Lemma 4. *For every $h \in \mathbb{N}$ and every non-negative integer h' satisfying $h + h' \le g-1$, it holds*

$$\Phi([l_h, l_{h+h'}], \boldsymbol{v}_i) \le \Phi([l_h, l_{h+h'+1}], \boldsymbol{v}_i).$$

PROOF: Taking into account

$$\Phi([l_h, l_{h+h'}], \boldsymbol{v}_i) - \Phi([l_h, l_{h+h'+1}], \boldsymbol{v}_i) = \sum_{k=1}^{g} N_k(x_i)(|h+h'-k| - |h+h'-k+1|)$$

and from a similar reasoning as in Lemma 3, the result is proven. ■

Proposition 3. *If l_M is the median of the data distribution given by $\boldsymbol{v}_i$ and l_M is in $\boldsymbol{v}_i$, then $F_i = \{l_M\}$.*

PROOF: By Lemmas 3 and 4, it holds

$$\Phi(l_M, \boldsymbol{v}_i) \leq \Phi(l_h, \boldsymbol{v}_i) \leq \cdots \leq \Phi([l_h, l_{h+h'}], \boldsymbol{v}_i),$$

where $h = 1, \ldots, g$ and h' is an integer number satisfying $h + h' \leq g$. Then, by the definition of F_i, the proposition is proven. ■

Proposition 3 proves that in this special case we get the same representative linguistic term of MJ. The next results are devoted to study the case where m is even.

Lemma 5. *If m is even and l_a and l_b are the linguistic terms at positions $\frac{m}{2}$ and $\frac{m}{2}+1$ in the data distribution given by $\boldsymbol{v}_i$, respectively, then $\Phi(l_a, \boldsymbol{v}_i) = \Phi(l_b, \boldsymbol{v}_i)$.*

PROOF: By the definition of l_a and l_b, we have $\displaystyle\sum_{h=1}^{a} N_h(x_i) = \sum_{h=b}^{g} N_h(x_i) = \frac{m}{2}$. On the other hand,

$$\Phi(l_a, \boldsymbol{v}_i) - \Phi(l_b, \boldsymbol{v}_i) = 2\sum_{h=1}^{g} N_h(x_i)(|h-a| + |h-b|).$$

Since

$$|h-a| - |h-b| = \begin{cases} a-b, & \text{if } h \leq a, \\ -(a-b), & \text{if } h \geq b, \end{cases}$$

then, we have

$$\Phi(l_a, \boldsymbol{v}_i) - \Phi(l_b, \boldsymbol{v}_i) = 2\sum_{h=1}^{a} N_h(x_i)(a-b) - 2\sum_{h=b}^{g} N_h(x_i)(a-b) = 0. \blacksquare$$

Lemma 6. *If $P = (a, 0)$, $Q = (0, b)$, $R = (c_1, c_2) \in \mathbb{Z}^2$, with $0 \leq c_1 \leq a$ and $0 \leq c_2 \leq b$, then $d_M(P, R) + d_M(R, Q) = d_M(P, Q)$.*

PROOF: By definition of d_M, we have $d_M(P, Q) = a + b$. On the other hand, $d_M(P, R) = |a - c_1| + c_2 = a - c_1 + c_2$ and $d_M(R, Q) = |b - c_2| + c_1 = b - c_2 + c_1$, so $d_M(P, R) + d_M(R, Q) = d_M(P, Q)$. ■

The next proposition shows that if all voters assess simple linguistic terms and m is even, then F_i has not to be a singleton of $\mathbb{L}$ and the arbitrariness of choosing the median disappears. See Fig. 7, where the grey zone denotes F_i.

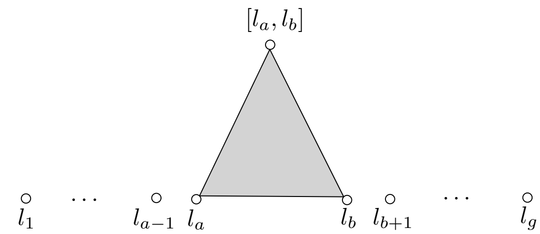

Figure 7: An illustration of Proposition 4.

Proposition 4. *If m is even and l_a and l_b are the linguistic terms at positions $\frac{m}{2}$ and $\frac{m}{2}+1$ in the data distribution given by $\boldsymbol{v}_i$, respectively, then*

$$F_i = \{[l_h, l_k] \mid a \leq h \leq k \leq b\}.$$

PROOF: If $a \leq h \leq k \leq b$, by Lemmas 5 and 6 we have $\Phi(l_a, \boldsymbol{v}_i) = \Phi(l_b, \boldsymbol{v}_i) = \Phi(l_h, \boldsymbol{v}_i)$. Let us write $F = \{[l_h, l_k] \mid a \leq h \leq k \leq b\}$. Since m is even, the data distribution given by $\boldsymbol{v}_i$ includes the median $l_M \in F$. From a similar reasoning as in Proposition 3, it is proven that $F_i = F$. ■

Example 8. This example shows the advantages of the symmetry of sets F_i when all voters use simple linguistic terms. Consider that two candidates x_1 and x_2 are graded by four voters with the assessments given in Table 6.

	x_1	x_2
1	l_2	l_2
2	l_2	l_3
3	l_5	l_3
4	l_5	l_3

Table 6: Agents' assessments in Example 8.

The overall opinion of x_1 is

$$F_1 = \{l_2, l_3, l_4, l_5, [l_2, l_3], [l_3, l_4], [l_4, l_5], [l_2, l_4], [l_3, l_5], [l_2, l_5]\},$$

and the overall opinion of x_2 is $F_2 = \{l_3\}$. Since $\bar{d}(l_5, F_1) = 3 < 4 = \bar{d}(l_5, F_2)$, then $x_1 \succ_1 x_2$. However, MJ declares x_2 as the winner, in spite of x_1 is better graded by the agents than x_2.

4. Properties

In this section, we introduce some interesting properties that are satisfied by our decision rule.

Definition 9. *A* decision rule *is a mapping $\varphi : \mathbb{V} \longrightarrow W(X)$. The weak order $\varphi(V)$ will be denoted by $\succcurlyeq^V$.*

In order to introduce some properties that decision rules can satisfy, we need some pieces of notation and basic notions.

Given a permutation π on I and a profile $V = (v_i^p) \in \mathbb{V}$, we denote $\pi(V) = \left(v_i^{\pi(p)}\right)$.

Given a permutation σ on $\{1, \ldots, n\}$ and a profile $V = (v_i^p) \in \mathbb{V}$, we denote $\sigma(V) = \left(v_{\sigma(i)}^p\right)$, and $\sigma\left(\succcurlyeq^V\right)$ is the order $\succcurlyeq^V$ but taking into account the new names of the alternatives.

The inverse of a linguistic expression $\mathcal{E} = [l_h, l_k] \in \mathbb{L}$ is defined as

$$\mathcal{E}^{-1} = [l_{g-k+1}, l_{g-h+1}].$$

For $g = 5$, with the meanings given in Table 1, the inverse of $[l_2, l_3]$ ('between bad and acceptable') is $[l_2, l_3]^{-1} = [l_{5-3+1}, l_{5-2+1}] = [l_3, l_4]$ ('between acceptable and good'); similarly, the inverse of l_1 ('very bad') is $(l_1)^{-1} = l_5$ ('very good').

Given a profile $V = (v_i^p) \in \mathbb{V}$, its inverse is defined as $V^{-1} = \left((v_i^p)^{-1}\right)$.

Before introducing in a formal way the properties involved in the result of the section, we give a rough idea of them.

Anonymity means that if the order of the agents are changed, then the order over the alternatives should be the same.

Neutrality means that if the names of the alternatives are changed, then the new order should be the same but with the name of the alternatives changed in the same way.

Reversal symmetry means that if all the assessments are reversed, then the outcome is also reversed.

Definition 10. *Let $\varphi : \mathbb{V} \longrightarrow W(X)$ be a decision rule.*

- φ *satisfies* Anonymity *if for every permutation π on I and every $V \in \mathbb{V}$ it holds $\varphi(\pi(V)) = \varphi(V)$, i.e.,*

$$x_i \succcurlyeq^{\pi(V)} x_j \;\Leftrightarrow\; x_i \succcurlyeq^V x_j\,,$$

  for all $x_i, x_j \in X$.

- φ *satisfies* Neutrality *if for every permutation σ on $\{1, \ldots, n\}$ and every $V \in \mathbb{V}$ it holds $\varphi(\sigma(V)) = \varphi(V)$, i.e.,*

$$x_{\sigma(i)}\, \sigma\left(\succcurlyeq^V\right) x_{\sigma(j)} \;\Leftrightarrow\; x_i \succcurlyeq^V x_j\,,$$

  for all $x_i, x_j \in X$.

- φ *satisfies* Reversal Symmetry *if for every $V \in \mathbb{V}$ it holds $\varphi(V^{-1}) = (\varphi(V))^{-1}$, i.e.,*

$$x_i \succcurlyeq^{V^{-1}} x_j \;\Leftrightarrow\; x_j \succcurlyeq^V x_i,$$

  for all $x_i, x_j \in X$.

Now, two new properties related with a non-constant number of agents or alternatives are going to be presented.

Given $V = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n) \in \mathbb{V}$ and $t \in \{1, \ldots, n\}$, with $U = V - \boldsymbol{v}_t$ we denote the reduced profile where the individual assessments over x_t are removed, i.e., $U = V - \boldsymbol{v}_t = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_{t-1} \ \boldsymbol{v}_{t+1} \ \cdots \ \boldsymbol{v}_n)$, and $\boldsymbol{u}^p = (u_1^p, \ldots, u_{t-1}^p, u_{t+1}^p, \ldots, u_n^p)$ for every $p \in I$.

Given $V = (\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n) \in \mathbb{V}$ and $\lambda \in \mathbb{N}$, with λV we denote the replicated profile of λ copies of V, defined as $\lambda V = \left(\boldsymbol{v}_1 \overset{(\lambda \text{ times})}{\cdots} \boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n \overset{(\lambda \text{ times})}{\cdots} \boldsymbol{v}_n\right)$.

Independence means that if an alternative is removed, then the order between other alternatives should remain the same.

Invariance for Replications means that if a profile is replicated, then the final order should be the same than in the original profile.

Definition 11. *Let $\varphi : \mathbb{V} \longrightarrow W(X)$ be a decision rule.*

- *A decision rule φ satisfies* Independence *if for all $x_i, x_j, x_t \in X$ and $U, V \in \mathbb{V}$ such that $U = V - \boldsymbol{v}_t$ it holds*

$$x_i \succcurlyeq^V x_j \ \Rightarrow \ x_i \succcurlyeq^U x_j.$$

- *φ satisfies* Invariance for Replications *if for all $x_i, x_j \in X$, $V \in \mathbb{V}$ and $\lambda \in \mathbb{N}$ it holds $\varphi(\lambda V) = \varphi(V)$, i.e.,*

$$x_i \succcurlyeq^{\lambda V} x_j \ \Leftrightarrow \ x_i \succcurlyeq^V x_j.$$

In order to justify the properties of our decision rule, some technical results are needed. They are presented in the following lemmas.

Lemma 7. *For all $\mathcal{E}, \mathcal{F} \in \mathbb{L}$ it holds $d\left(\mathcal{E}^{-1}, \mathcal{F}^{-1}\right) = d(\mathcal{E}, \mathcal{F})$.*

Proof: If $\mathcal{E} = [l_h, l_k]$ and $\mathcal{F} = [l_r, l_s]$, we have

$$\begin{aligned}
d(\mathcal{E}^{-1}, \mathcal{F}^{-1}) &= d_M(\psi(\mathcal{E}^{-1}), \psi(\mathcal{F}^{-1})) = \\
&= d_M((g-h, g-k), (g-r, g-s)) = |r-h| + |s-k| \\
&= |k-s| + |h-r| = d_M((k-1, h-1), (s-1, r-1)) \\
&= d_M(\psi(\mathcal{E}), \psi(\mathcal{F})) = d(\mathcal{E}, \mathcal{F}). \blacksquare
\end{aligned}$$

Lemma 8. *For every $V \in \mathbb{V}$ it holds $\mathcal{E} \in F_i^V \ \Leftrightarrow \ \mathcal{E}^{-1} \in F_i^{V^{-1}}$.*

Proof: By Lemma 7, we have

$$\sum_{p=1}^{m} d(\mathcal{E}, v_i^p) = \sum_{p=1}^{m} d(\mathcal{E}^{-1}, (v_i^p)^{-1}).$$

Substituting in the definition of Fermat set, we have

$$\begin{aligned}\mathcal{E} \in F_i^V \ \Leftrightarrow \ \forall \mathcal{F} \in \mathbb{L} \ \sum_{p=1}^{m} d(\mathcal{E}, v_i^p) &= \sum_{p=1}^{m} d(\mathcal{E}^{-1}, (v_i^p)^{-1}) \leq \\ \leq \sum_{p=1}^{m} d(\mathcal{F}, v_i^p) &= \sum_{p=1}^{m} d(\mathcal{F}^{-1}, (v_i^p)^{-1}).\end{aligned}$$

Consequently,

$$\mathcal{E} \in F_i^V \ \Leftrightarrow \ \forall \mathcal{F} \in \mathbb{L} \ \ \sum_{p=1}^{m} d(\mathcal{E}^{-1}, (v_i^p)^{-1}) \leq \sum_{p=1}^{m} d(\mathcal{F}, (v_i^p)^{-1}) \ \Leftrightarrow \ \mathcal{E}^{-1} \in F_i^{V^{-1}}. \blacksquare$$

Lemma 9. *For all* $x_i \in X$, $V \in \mathbb{V}$ *and* $\lambda \in \mathbb{N}$ *it holds* $F_i^{\lambda V} = F_i^V$.

Proof: $\mathcal{E} \in F_i^{\lambda V}$ is equivalent to

$$\sum_{p=1}^{m} d(\mathcal{E}, v_i^p) + \overset{(\lambda \text{ times})}{\cdots} + \sum_{p=1}^{m} d(\mathcal{E}, v_i^p) \leq \sum_{p=1}^{m} d(\mathcal{F}, v_i^p) + \overset{(\lambda \text{ times})}{\cdots} + \sum_{p=1}^{m} d(\mathcal{F}, v_i^p),$$

for every $\mathcal{F} \in \mathbb{L}$, and then to $\sum_{p=1}^{m} d(\mathcal{E}, v_i^p) \leq \sum_{p=1}^{m} d(\mathcal{F}, v_i^p)$, for every $\mathcal{F} \in \mathbb{L}$, i.e., to $\mathcal{E} \in F_i^V$. $\blacksquare$

Theorem 1. *The decision rule* $\varphi_1 : \mathbb{V} \longrightarrow W(X)$ *defined as* $\varphi_1(V) = \succcurlyeq_1^V$ *satisfies Anonymity, Neutrality, Reversal Symmetry, Independence and Invariance for Replications.*

Proof:

The proof of Anonymity, Neutrality and Independence is straightforward.

For proving Reversal Symmetry, we have to justify that for all $x_i, x_j \in X$ it holds

$$x_i \succcurlyeq_1^V x_j \ \Leftrightarrow \ x_j \succcurlyeq_1^{V^{-1}} x_i.$$

By definition,

$$x_i \succcurlyeq_1^V x_j \ \Leftrightarrow \ \frac{\displaystyle\sum_{\mathcal{E} \in F_i} d(l_g, \mathcal{E})}{\# F_i} \leq \frac{\displaystyle\sum_{\mathcal{E} \in F_j} d(l_g, \mathcal{E})}{\# F_j}.$$

We know by Lemma 7 and Lemma 8 that

$$\frac{\displaystyle\sum_{\mathcal{E} \in F_i} d(l_g, \mathcal{E})}{\# F_i} = \frac{\displaystyle\sum_{\mathcal{E}^{-1} \in F_i^{-1}} d((l_g)^{-1}, \mathcal{E}^{-1})}{\# F_i^{-1}}.$$

Taking into account the fact that $(l_g)^{-1} = l_1$, we have

$$x_i \succcurlyeq_1^V x_j \Leftrightarrow \frac{\sum\limits_{\mathcal{E} \in F_i^{-1}} d(l_1, \mathcal{E})}{\# F_i^{-1}} \leq \frac{\sum\limits_{\mathcal{E} \in F_j^{-1}} d(l_1, \mathcal{E})}{\# F_j^{-1}}.$$

And finally, taking into account $d(l_g, \mathcal{E}) = 2g - 2 - d(l_1, \mathcal{E})$ and Lemma 1, we have

$$x_i \succcurlyeq_1^V x_j \Leftrightarrow \frac{\sum\limits_{\mathcal{E} \in F_i^{-1}} d(l_g, \mathcal{E})}{\# F_i^{-1}} \geq \frac{\sum\limits_{\mathcal{E} \in F_j^{-1}} d(l_g, \mathcal{E})}{\# F_j^{-1}} \Leftrightarrow x_j \succcurlyeq_1^{V^{-1}} x_i. \blacksquare$$

By Independence and Lemma 9, Invariance for Replications is satisfied. ■

5. Concluding remarks

The introduction of the new Majority Judgment voting system by Balinski and Laraki [2, 3] can be considered to be divergent to the classical Arrow approach [1]. Although in both cases, individual opinions are aggregated in order to obtain a collective weak order on the set of alternatives, the information provided by the agents is different. In the Arrow approach, agents are required to order the alternatives by rank of preference. However, in Majority Judgment, a common language composed by a small number of linguistic terms is used by the agents for assessing the candidates one by one.

Imagine two agents rank three alternatives A, B and C in the same order, for instance B C A. However, it is possible that the opinions of these agents could be different, for instance the first one could think that B is very good, C is good and A is acceptable while the second agent thinks that B is good, C is bad and A is very bad. Within the Arrow framework, this information is not considered and the opinions of both agents are taken as being equal.

It is worth mentioning that not only voters but also experts are not always confident about their opinions when they have to declare them using the fixed terms of a finite scale[6]. In this paper we have taken the same starting point as that of Majority Judgment except that we allow agents to be imprecise in their assessments by using several consecutive linguistic terms, if necessary.

Our proposal is essentially different to that of Majority Judgment which is based on the median of the individual assessments and a tie-breaking process[7].

[6]For instance, the reviewers of some scientific journals have to select a recommendation for a paper between the following four modalities: 'accept', 'minor revision', 'major revision' or 'rejection'. Sometimes, after choosing one of the four possibilities, some reviewers write 'between major revision and rejection' or similar sentences in the notes for the editors.

[7]We should note that our proposal coincides with Majority Judgment in the first stage when an odd number of agents assess the alternatives using linguistic terms. In this sense, our proposal can be considered to be an extension of Majority Judgment.

We have proposed a distance-based aggregation procedure in which the alternatives are ordered according to the distances between the overall opinion and the highest possible assessment. Although our procedure generates fewer ties than Majority Judgment, they can still appear. So, we have also proposed a tie-breaking process, first by taking into account the dispersion of individual assessments, and subsequently by considering the number of best assessments, etc.

Sometimes social choice theorists advocate the principle that voters should easily understand how the voting rules work. Surely the simplest and most popular voting rule is plurality, where each agent votes for his favorite candidate and the winner(s) is/are the candidate(s) who obtain a greater number of votes. In spite of its popularity, plurality can be considered to be, in practice, the worst voting rule (see Laslier [20]). Clearly, simplicity is not the most important feature of a voting rule.

In group decision-making, it is not essential that the decision processes are simple, rather that they are flexible and that they consistently manage the information provided by the agents for generating the collective decisions. Our proposal is more complicated than plurality and other basic voting rules, but it is more faithful to the individual opinions. The properties we have proven within the social choice framework provide initial support to our proposal. The study of other properties and some comparative analyses with other group decision-making procedures, specially with Majority Judgment, would be pertinent as further research.

In numerous real decision problems, experts have to assess the alternatives through scales with more number of terms in the positive side of the scale than in the negative one. In this way, it is important to note that Herrera *et al.* [16] analyze this problem and provide a methodology based on linguistic hierarchies and the 2-tuple fuzzy linguistic representation (see Herrera and Martínez [17]). We leave for further research to deal with unbalanced linguistic term sets within the metric-based approach we have developed in the present paper. Another worthy problem to be addressed in future research is the one where the distances between consecutive linguistic terms are not constant, irrespectively of the scale being balanced or not.

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