GEOMETRY OF FAST MAGNETOSONIC RAYS, WAVEFRONTS AND SHOCK WAVES

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Abstract

Fast magnetosonic waves in a two-dimensional plasma are studied in the geometrical optics approximation. The geometry of rays and wavefronts influences decisively the formation and ulterior evolution of shock waves. It is shown that the curvature of the curve where rays start and the angle between rays and wavefronts are the main parameters governing a wide variety of possible outcomes.

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1 Introduction

While the model of ideal magnetohydrodynamics represents the simplest description of the evolution of a neutral plasma, and both its weaknesses and its strengths are well known, the long term behavior of solutions is anything but easy to predict. In common with all nonlinear hyperbolic systems, shocks may develop and indeed do in many physically relevant situations, but their location and later evolution are extremely complex problems. However, for high frequency perturbations the methods of nonlinear geometric optics provide a more amenable analytical approach. While its philosophy is highly classical [1, 2], rigorous mathematical justifications are more recent and in fact continue to this day [3–5]. There exists a vast bibliography for this technique and its applications [6–8], e.g. in elasticity [9], fluid dynamics [10, 11] and ideal MHD [12, 13].

The most desirable case occurs when dealing with waves of a single phase. When one admits superposition of waves whose phases are different solutions of the eikonal equation, resonance may occur [14, 15] and the waves interact in unsuspected ways. Methods to deal with particular cases have been successfully applied e.g. to the two-dimensional Euler equation [5]. We will assume a single phase and make use of two excellent survey articles [16, 17]. Even in this case most specific results assume dependence on a single spatial variable (although the system itself may be multidimensional). This way rays are straight and parallel lines and there is no trouble with their intersection. We wish to analyze a genuinely multidimensional case, keeping the remaining data as simple as possible; thus we consider propagation of fast magnetohydrodynamic waves in a static plane equilibrium: density, pressure and magnetic field are constant. Rays are straight lines and their angle with wavefronts is constant along each ray. Nevertheless, setting the location of the initial perturbation along an arbitrary curve in the plane, we allow for rays to converge generating caustics, and the wavefront normal also differ among different rays. The crucial parameters are precisely the curvature of the original curve, and also the variation along it of the angle between the static magnetic field and the normal. A very lengthy calculation shows that the first order term for the asymptotic expansion of the solution satisfies a differential equation along the rays which may be reduced to the Burgers equation, whose behavior is well understood. In particular the time of shock formation and the ulterior evolution of the shock wave are widely available in the literature e.g. in [7, 8] and specially in [18]. However, the necessary changes both of variables and functions to reduce our problem to a Burgers form depend on the sign and relative size of the equilibrium quantities, plus the data in the original curve, so the admirable universality of the Burgers solution (which tends always to an N-wave) gives rise to a surprising variety of possible outcomes for the velocity, the shock strength and the overall shape of this wave. A final word of caution related to the intrinsic limitations of nonlinear geometrical optics. The evolution of the shock along each ray is governed by the Rankine-Hugoniot relations, but there is no guarantee that the final solution, transported through different rays, will satisfy also the Rankine-Hugoniot relations in the transverse direction to these rays. To achieve this further constraints in the original values would be necessary. Although the wavelength of our solutions is small, when rays approach one another interference occurs, which is not covered by geometrical optics; obviously for diverging rays the approximation is excellent.

2 Geometry of rays

Consider a quasilinear hyperbolic system, written in the Einstein notation

$$\frac{\partial \mathbf{u}}{\partial t} + A_j(\mathbf{x}, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{C}(\mathbf{x}, \mathbf{u}) = \mathbf{0}, \tag{1}$$

For any spatial vector $\mathbf{k}$ and equilibrium state $\mathbf{u}_0$ take a fixed eigenvalue $\Lambda(\mathbf{k})$,

$$\det(\Lambda(\mathbf{k})I + A_j(\mathbf{x}, \mathbf{u}_0)k_j) = 0. \tag{2}$$

The eikonal equation associated to this eigenvalue is

$$\frac{\partial \phi}{\partial t} = \Lambda(\nabla \phi), \tag{3}$$

and ϕ is the phase. In our case the system will be the ideal MHD one, and we choose for Λ the fast magnetosonic frequency (see e.g. [19]). If $\mathbf{u}_0$ corresponds to a static state with pressure P, density ρ and magnetic field $\mathbf{B}$,

$$\Lambda(\mathbf{k})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)|\mathbf{k}|^2 + \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)^2 |\mathbf{k}|^4 - 4\frac{\partial P}{\partial \rho}\frac{(\mathbf{B}\cdot\mathbf{k})^2}{\rho}|\mathbf{k}|^2\right]^{1/2}. \tag{4}$$

Rays are solutions of the system

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \nabla_{\mathbf{k}}\Lambda(\mathbf{x}, \mathbf{k}) \\ \frac{d\mathbf{k}}{dt} &= -\nabla_{\mathbf{x}}\Lambda(\mathbf{x}, \mathbf{k}). \end{aligned} \tag{5}$$

The phase is constant along rays,

$$\frac{d}{dt}(\phi(t, \mathbf{x}(t))) = 0. \tag{6}$$

Often one takes a normalized vector $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$ and uses the frequency

$$c(\mathbf{n}) = \frac{\Lambda(\mathbf{k})}{|\mathbf{k}|}. \tag{7}$$

Equations (5) for the plane may be written in terms of c, $\mathbf{n}$ and its orthogonal $\mathbf{n}^\perp$, chosen so that $\{\mathbf{n}, \mathbf{n}^\perp\}$ form an orthonormal positive system:

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + (\mathbf{n}^\perp \cdot \nabla_{\mathbf{n}} c)\mathbf{n}^\perp, \tag{8}$$

$$\frac{d\mathbf{n}}{dt} = -(\mathbf{n}^\perp \cdot \nabla_{\mathbf{x}} c)\mathbf{n}^\perp. \tag{9}$$

For static equilibria, the fast magnetosonic frequency $c(\mathbf{n})$ satisfies

$$c(\mathbf{n})^2 = \frac{1}{2}\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right) + \frac{1}{2}\left[\left(\frac{\partial P}{\partial \rho} + \frac{B^2}{\rho}\right)^2 - 4\frac{\partial P}{\partial \rho}\frac{(\mathbf{B}\cdot\mathbf{n})^2}{\rho}\right]^{1/2}. \tag{10}$$

This equation may be written in terms of the speed of sound $c_s^2 = \partial P/\partial\rho$, Alfvén speed $c_A^2 = B^2/\rho$, and the angle θ that forms the magnetic field $\mathbf{B}$ with $\mathbf{n}$:

$$c(\mathbf{n})^2 = \frac{1}{2}(c_s^2 + c_A^2) + \frac{1}{2}\left[(c_s^2 + c_A^2)^2 - 4c_s^2 c_A^2 \cos^2\theta\right]^{1/2}. \tag{11}$$

From now on we will consider a static ideal MHD equilibrium where both magnetic field and density are constant in space. In this case $c(\mathbf{n})$ does not depend on $\mathbf{x}$, so that we find from (9) that $\mathbf{n}$ (and $\mathbf{n}^\perp$) are constant along the ray; and since both coefficients in (8) are constant, rays are straight lines. Denoting by $\mathbf{b}$ the unit magnetic field, by our definition of θ

$$\begin{aligned} \mathbf{n} &= \cos\theta\,\mathbf{b} + \sin\theta\,\mathbf{b}^\perp \\ \mathbf{n}^\perp &= -\sin\theta\,\mathbf{b} + \cos\theta\,\mathbf{b}^\perp, \end{aligned} \tag{12}$$

which implies

$$\frac{d\mathbf{n}}{d\theta} = (-\sin\theta)\,\mathbf{b} + (\cos\theta)\,\mathbf{b}^\perp = \mathbf{n}^\perp, \tag{13}$$

so that, writing as in (11) c as a function of θ (all the rest being constants), and denoting by c' the derivative of c with respect to θ,

$$\frac{dc}{d\theta} = c'(\theta) = \frac{d\mathbf{n}}{d\theta}\cdot\nabla_{\mathbf{n}} c = \mathbf{n}^\perp\cdot\nabla_{\mathbf{n}} c, \tag{14}$$

so that (8) may be written

$$\frac{d\mathbf{x}}{dt} = c\mathbf{n} + c'\mathbf{n}^\perp. \tag{15}$$

We see from (5) that $\mathbf{k} = \mathbf{k}_0$ is constant along the ray, and since $\nabla\phi = \mathbf{k}_0$, this is also constant along the ray, as well as

$$\frac{\partial\phi}{\partial t} = -c|\mathbf{k}_0|. \tag{16}$$

Let us fix a single ray, and call $\boldsymbol{\alpha} = c\mathbf{n} + c'\mathbf{n}^\perp$. Then the ray is given by

$$\mathbf{x}(t) = \boldsymbol{\alpha} t + \mathbf{x}(0). \tag{17}$$

Choosing as parameter the arc length s instead of t so that we may reserve this for the time,

$$\mathbf{x}(s) = \frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}s + \mathbf{x}(0). \tag{18}$$

Since

$$\frac{d}{dt}\phi(t,s(t)) = \frac{\partial \phi}{\partial t} + |\boldsymbol{\alpha}|\frac{\partial \phi}{\partial s} = 0, \tag{19}$$

we find the simple expression for the phase in a single ray

$$\phi(t,s) = c\frac{|\mathbf{k}_0|}{|\boldsymbol{\alpha}|}(s - |\boldsymbol{\alpha}|t) + \text{const.} \tag{20}$$

Abbreviating $|\boldsymbol{\alpha}| = \alpha$, and labeling $\phi(0,0) = 0$, we may write

$$\phi(t,s) = A(s - \alpha t). \tag{21}$$

To set the ray geometry appropriate to our problem, we start from a curve $\mathbf{g}$ parametrized by the arc length ξ, $\xi \in (\xi_0 - \epsilon, \xi_0 + \epsilon)$, and consider rays orthogonal to this curve. Let us therefore take a normal unitary vector $\mathbf{T}$, chosen so that $\mathbf{g}'(\xi) = \mathbf{T}^{\perp}(\xi)$, and $\mathbf{T}$, $\mathbf{T}^{\perp}$ form a positive orthonormal system. Thus the ray starting at $\mathbf{g}(\xi)$ may be parametrized by the arc length s as $s \to \mathbf{g}(\xi) + s\mathbf{T}(\xi)$. It is easy to see that the transported curves $\mathbf{g}_s : \xi \to \mathbf{g}(\xi) + s\mathbf{T}(\xi)$ form with the rays a family of orthogonal curves for as long as there is not self-intersection. While (ξ, s) form a global family of orthogonal coordinates for the area covered by these curves, the fact that ξ is not the arc length in the curve $\mathbf{g}_s$ makes us to choose r, the arc length on this curve, starting at $(r = 0)$ (i.e. $\xi = \xi_0$), as new variable. We see that $r = r(\xi, s)$. Since $\mathbf{T}^{\perp}$ is the tangent vector to $\mathbf{g}_s$ and $\mathbf{T}$ is minus the normal vector,

$$\frac{d\mathbf{T}^{\perp}}{dr} = -\kappa \mathbf{T} \qquad \frac{d\mathbf{T}}{dr} = \kappa \mathbf{T}^{\perp}, \tag{22}$$

where $\kappa = \kappa(r,s)$ represents the curvature of $\mathbf{g}_s$. Let us find $\kappa(r,s)$ in terms of the curvature of the original curve $\kappa(r,0)$. We have

$$\begin{aligned}\mathbf{g}_s' &: r \to \mathbf{g}'(r) + s\mathbf{T}'(r) = \mathbf{T}^{\perp}(r) + s\kappa(r,0)\mathbf{T}^{\perp}(r) \\ \mathbf{g}_s'' &: r \to -\kappa(r,0)\mathbf{T}(r) + s\frac{\partial \kappa}{\partial r}(r,0)\mathbf{T}^{\perp}(r) - s\kappa(r,0)^2\mathbf{T}(r).\end{aligned} \tag{23}$$

For any plane curve such as $\mathbf{g}_s$ we have the formula

$$\kappa = \frac{(\mathbf{g}_s' \times \mathbf{g}_s'') \cdot \hat{z}}{|\mathbf{g}_s'|^3}. \tag{24}$$

Thus we find

$$\begin{aligned}\kappa(r,s) &= \frac{(1 + s\kappa(r,0))(-\kappa(r,0) - s\kappa(r,0)^2)(\mathbf{T}^{\perp} \times \mathbf{T}) \cdot \hat{z}}{|1 + s\kappa(r,0)|^3} \\ &= \frac{\kappa(r,0)}{1 + s\kappa(r,0)}.\end{aligned} \tag{25}$$

The sign of $1 + s\kappa(r, 0)$ is always positive: obviously for $\kappa \geq 0$, which corresponds to rays spreading from the (convex) curve, and for $\kappa < 0$ because at $s = -1/\kappa(r, 0)$ rays cut themselves and a caustic appears. These rays emanate from the immediate vicinity of $r = 0$, $s = 0$, not from some other portion of the curve. (22) may be therefore written as

$$\frac{d\mathbf{T}^{\perp}}{dr} = -\frac{\kappa(r,0)}{1+s\kappa(r,0)}\mathbf{T} \qquad \frac{d\mathbf{T}}{dr} = \frac{\kappa(r,0)}{1+s\kappa(r,0)}\mathbf{T}^{\perp}. \tag{26}$$

Let us consider now the normal vector to the wavefront $\mathbf{n}$ as a function of (r, s). As a function of (ξ, s) it only depends on ξ, but $r = r(\xi, s)$. Since $\mathbf{n} \cdot \mathbf{n} = 1$, there exists a scalar function $\lambda(r, s)$ such that

$$\frac{d\mathbf{n}}{dr} = \lambda\mathbf{n}^{\perp}, \quad \frac{d\mathbf{n}^{\perp}}{dr} = -\lambda\mathbf{n}. \tag{27}$$

For any function constant along the rays, $h = h(\xi)$, we may find dh/dr as follows: since r is the arc length of the curve $\mathbf{g}_s$, whose derivative is found in (23),

$$\frac{d\xi}{dr} = \frac{1}{1+s\kappa(r,0)}, \tag{28}$$

we have

$$\frac{dh}{dr}(r,s) = \frac{1}{1+s\kappa(r,0)}\frac{dh}{d\xi}(r,s) = \frac{1}{1+s\kappa(r,0)}\frac{dh}{dr}(r,0). \tag{29}$$

In particular

$$\lambda(r,s) = \frac{\lambda(r,0)}{1+s\kappa(r,0)}. \tag{30}$$

λ may be related to two angles: to the angle θ formed by the magnetic field and the normal, and to the angle ψ formed by the normal and the ray. Start from (12) and recall that $\mathbf{b}$ is constant everywhere. Thus

$$\frac{d\mathbf{n}}{dr} = -\sin\theta\left(\frac{d\theta}{dr}\right)\mathbf{b} + \cos\theta\left(\frac{d\theta}{dr}\right)\mathbf{b}^{\perp} = \left(\frac{d\theta}{dr}\right)\mathbf{n}^{\perp}, \tag{31}$$

so that

$$\lambda(r,s) = \frac{d\theta}{dr}(r,s) = \frac{1}{1+s\kappa(r,0)}\frac{d\theta}{dr}(r,0). \tag{32}$$

On the other hand, writing

$$\begin{aligned} \mathbf{n} &= \cos\psi\,\mathbf{T} + \sin\psi\,\mathbf{T}^{\perp} \\ \mathbf{n}^{\perp} &= -\sin\psi\,\mathbf{T} + \cos\psi\,\mathbf{T}^{\perp}, \end{aligned} \tag{33}$$

we get

$$\frac{d\mathbf{n}}{dr}(r,s) = -\sin\psi(r,s)\left(\frac{\partial\psi}{\partial r}(r,s)+\kappa(r,s)\right)\mathbf{T} + \cos\psi(r,s)\left(\frac{\partial\psi}{\partial r}(r,s)+\kappa(r,s)\right)\mathbf{T}^{\perp}. \tag{34}$$

therefore

$$\lambda(r,s) = \frac{\partial\psi}{\partial r}(r,s)+\kappa(r,s) = \frac{1}{1+s\kappa(r,0)}\left(\frac{\partial\psi}{\partial r}(r,0)+\kappa(r,0)\right). \tag{35}$$

We may find easily ψ from (15). Since $\mathbf{T} = (\cos\psi)\mathbf{n} - (\sin\psi)\mathbf{n}^{\perp}$,

$$\cos\psi = \frac{c}{\sqrt{c^2+c'^2}}, \quad \sin\psi = -\frac{c'}{\sqrt{c^2+c'^2}}. \tag{36}$$

Expression (35) shows that by choosing appropriately the variation of $\mathbf{n}$ with respect to r in the original curve, we may give to λ any sign.

3 Weakly nonlinear geometrical optics

We return to (1) for the ideal MHD system in the plane and take a left and right eigenvectors of the matrix in (2). That is,

$$\mathbf{L}\cdot(cI+n_jA_j(\mathbf{u}_0)) = (cI+n_jA_j(\mathbf{u}_0))\cdot\mathbf{R} = \mathbf{0}. \tag{37}$$

Let ϕ be the phase associated to c and to any initial values we choose. Let $\epsilon \ll 1$ and $\zeta = \phi/\epsilon$. It is shown in [13] that the first order term in the asymptotic expansion of the solution

$$\mathbf{v} = \epsilon\mathbf{v}_1(t,\mathbf{x},\zeta) + \epsilon^2\mathbf{v}_2(t,\mathbf{x},\zeta) + \dots, \tag{38}$$

has the form

$$\mathbf{v}_1(t,\mathbf{x},\zeta) = w(t,\mathbf{x},\zeta)\mathbf{R}(\mathbf{x},\mathbf{u}_0). \tag{39}$$

Moreover, defining

$$q_0 = \mathbf{L}\cdot(\mathbf{R}\cdot\nabla_u A_j(\mathbf{u}_0)(\mathbf{R}))\frac{\partial\tau}{\partial x_j}, \tag{40}$$

$$p_0 = \mathbf{L}\cdot\left[\frac{\partial\mathbf{R}}{\partial t} + A_j(\mathbf{u}_0)\frac{\partial\mathbf{R}}{\partial x_j} + \nabla_u\mathbf{C}(\mathbf{u}_0)\cdot\mathbf{R} + (\mathbf{R}\cdot\nabla_u A_j(\mathbf{u}_0))\frac{\partial\mathbf{u}_0}{\partial x_j}\right] \tag{41}$$

$$q = q_0(\mathbf{L}\cdot\mathbf{R})^{-1}, \quad p = p_0(\mathbf{L}\cdot\mathbf{R})^{-1}, \tag{42}$$

where $\nabla_u A_j = R_k \partial A_j / \partial u_k$, w satisfies

$$\frac{\partial w}{\partial t} + \mathbf{L} \cdot \left((A_j(\mathbf{u}_0) \cdot \mathbf{R}) \frac{\partial w}{\partial x_j} \right) + pw + qw \frac{\partial w}{\partial \zeta} = 0. \tag{43}$$

The first two terms represent the derivative along the ray,

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \alpha \frac{\partial w}{\partial s}. \tag{44}$$

In [16, 17] it is shown why for initial conditions of compact support in ζ the asymptotic expansion is valid, there is no resonance and the approximation remains valid even after the shock is formed. The terms $\mathbf{L} \cdot \mathbf{R}$ and q_0 may be obtained independently of any chosen frame: their value may be found in [12] for a general static equilibrium. Its precise form does not need to concern us for our case, except to notice that for constant equilibria they are both positive constants, and therefore so is q. The term p_0 is the hardest one. Let us write the vector $\mathbf{u}$ as

$$\mathbf{u} = (B_1, B_2, v_1, v_2, \rho), \tag{45}$$

where $\mathbf{B}$ represents the magnetic field, $\mathbf{v}$ the velocity and ρ the density (see e.g. [19]). Let us take in $\mathbb{R}^5$ the basis given by

$$\begin{aligned}
(\mathbf{n}; \mathbf{0}; 0) &= (n_1, n_2, 0, 0, 0) \\
(\mathbf{n}^\perp; \mathbf{0}; 0) &= (-n_2, n_1, 0, 0, 0) \\
(\mathbf{0}; \mathbf{n}; 0) &= (0, 0, n_1, n_2, 0) \\
(\mathbf{0}; \mathbf{n}^\perp; 0) &= (0, 0, -n_2, n_1, 0) \\
(\mathbf{0}; \mathbf{0}; 1) &= (0, 0, 0, 0, 1).
\end{aligned} \tag{46}$$

The matrices A_j may be written as

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{n^\perp} & -B_n & 0 \\ 0 & B_{n^\perp}/\rho & 0 & 0 & P_\rho/\rho \\ 0 & -B_n/\rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \end{bmatrix}, \tag{47}$$

and

$$A_{n^\perp} = \begin{bmatrix} 0 & 0 & -B_{n^\perp} & B_n & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -B_{n^\perp}/\rho & 0 & 0 & 0 & 0 \\ B_n/\rho & 0 & 0 & 0 & P_\rho/\rho \\ 0 & 0 & 0 & \rho & 0 \end{bmatrix}. \tag{48}$$

and the vectors $\mathbf{R}$ and $\mathbf{L}$ are, up to multiplicative constants,

$$\mathbf{R} = \left(0, \frac{\rho c^2 B_{n^\perp}}{\rho c^2 - B_n^2}, c, \frac{-cB_n B_{n^\perp}}{\rho c^2 - B_n^2}, \rho\right), \tag{49}$$

$$\mathbf{L} = \left(0, \frac{\rho c^2 B_{n^\perp}}{\rho c^2 - B_n^2}, \rho c, \frac{-\rho c B_n B_{n^\perp}}{\rho c^2 - B_n^2}, P_\rho\right). \tag{50}$$

All those terms may also be set in terms of the angle θ,

$$A_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B\sin\theta & -B\cos\theta & 0 \\ 0 & (B\sin\theta)/\rho & 0 & 0 & P_\rho/\rho \\ 0 & -(B\cos\theta)/\rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \end{bmatrix}, \tag{51}$$

and

$$A_{n^\perp} = \begin{bmatrix} 0 & 0 & -B\sin\theta & B\cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -(B\sin\theta)/\rho & 0 & 0 & 0 & 0 \\ (B\cos\theta)/\rho & 0 & 0 & 0 & P_\rho/\rho \\ 0 & 0 & 0 & \rho & 0 \end{bmatrix}. \tag{52}$$

Also

$$\mathbf{R} = \left(0, \frac{\rho c^2 B\sin\theta}{\rho c^2 - B^2\cos^2\theta}, c, \frac{-cB^2\sin\theta\cos\theta}{\rho c^2 - B^2\cos^2\theta}, \rho\right), \tag{53}$$

$$\mathbf{L} = \left(0, \frac{\rho c^2 B\sin\theta}{\rho c^2 - B^2\cos^2\theta}, \rho c, \frac{-\rho c B^2\sin\theta\cos\theta}{\rho c^2 - B^2\cos^2\theta}, P_\rho\right). \tag{54}$$

Let us study p_0. Since $\mathbf{u}_0$ is constant, the only non vanishing term is

$$p_0 = \mathbf{L}\cdot\left(A_j(\mathbf{u}_0)\frac{\partial\mathbf{R}}{\partial x_j}\right). \tag{55}$$

It may be set in any orthonormal base: thus

$$\begin{aligned} p_0 &= \mathbf{L}\cdot\left(A_n\nabla_{\mathbf{n}}\mathbf{R} + A_{n^\perp}\nabla_{\mathbf{n}^\perp}\mathbf{R}\right) \\ &= \mathbf{L}\cdot\left(A_T\nabla_{\mathbf{T}}\mathbf{R} + A_{T^\perp}\nabla_{\mathbf{T}^\perp}\mathbf{R}\right). \end{aligned} \tag{56}$$

The last expression is useful because since $\mathbf{R}$ depends only on θ, it is constant along the ray, so

$$\nabla_{\mathbf{T}}\mathbf{R} = \mathbf{0}, \quad \nabla_{\mathbf{T}^\perp}\mathbf{R} = \frac{\partial\mathbf{R}}{\partial r}. \tag{57}$$

On the other hand, by (33),

$$\nabla_{\mathbf{n}}\mathbf{R} = \cos\psi \nabla_{\mathbf{T}}\mathbf{R} + \sin\psi \nabla_{\mathbf{T}^\perp}\mathbf{R} = \sin\psi \frac{\partial \mathbf{R}}{\partial r}$$
$$\nabla_{\mathbf{n}^\perp}\mathbf{R} = -\sin\psi \nabla_{\mathbf{T}}\mathbf{R} + \cos\psi \nabla_{\mathbf{T}^\perp}\mathbf{R} = \cos\psi \frac{\partial \mathbf{R}}{\partial r}, \tag{58}$$

which means

$$p_0 = \mathbf{L} \cdot (\sin\psi A_n + \cos\psi A_{n^\perp}) \frac{\partial \mathbf{R}}{\partial r} = \frac{1}{\sqrt{c^2 + c'^2}} \mathbf{L} \cdot (-c' A_n + c A_{n^\perp}) \frac{\partial \mathbf{R}}{\partial r}. \tag{59}$$

Expression (53) may be written as

$$\mathbf{R} = \left(\frac{\rho c^2 B \sin\theta}{\rho c^2 - B^2 \cos^2\theta} \mathbf{n}^\perp; c\mathbf{n} + \frac{-cB^2 \sin\theta \cos\theta}{\rho c^2 - B^2 \cos^2\theta} \mathbf{n}^\perp; \rho \right). \tag{60}$$

Using (27) and (32), we find

$$\frac{\partial \mathbf{R}}{\partial r} = \lambda \mathbf{z}(\theta), \tag{61}$$

for a certain vector $\mathbf{z}$ obtained by differentiating the components of $\mathbf{R}$ with respect to θ, and using the known values of the derivatives of $\mathbf{n}$ and $\mathbf{n}^\perp$ with respect to r. Thus

$$p(r,s) = \lambda(r,s)\mu(\theta) = \frac{\lambda(r,0)}{1 + s\kappa(r,0)} \mu(\theta), \tag{62}$$

where

$$\mu = \frac{1}{(\mathbf{L} \cdot \mathbf{R})\sqrt{c^2 + c'^2}} \mathbf{L} \cdot (-c' A_n + c A_{n^\perp}) \cdot \mathbf{z}. \tag{63}$$

The expression of μ in terms of the equilibrium functions is enormously complex, but the fact is that it depends only on θ and therefore it is constant along the ray. From now on we may consider only the ray $r = 0$ so that the only variable in (62) is s. To reduce (43) to a practicable form we change variables:

$$(s,t) \to (s', \zeta),$$
$$s' = s, \qquad \zeta = \frac{A}{\epsilon}(s - \alpha t). \tag{64}$$

Thus

$$\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial s} = \frac{1}{\alpha} \frac{\partial}{\partial s'}. \tag{65}$$

Calling again $s' = s$, (43) may be written

$$\frac{\partial w}{\partial s} + \alpha p w + \alpha q w \frac{\partial w}{\partial \zeta} = 0. \tag{66}$$

From now on, we will use

$$\beta = \alpha\mu\lambda(0,0), \quad \alpha q = \gamma. \tag{67}$$

4 Shock wave evolution

To reduce (66) to the Burgers form, one takes the new function

$$\sigma(s,\zeta) = w(s,\zeta)\exp\left(\int_0^s \alpha p(s_1)\,ds_1\right), \tag{68}$$

and then one changes s to ℓ, given by

$$\ell(s) = \int_0^s \alpha q \exp\left(\int_0^{s_1} \alpha p(s_2)\,ds_2\right)\,ds_1. \tag{69}$$

Notice that $\sigma(0,\zeta) = w(0,\zeta)$. σ satisfies

$$\frac{\partial\sigma}{\partial\ell} + \sigma\frac{\partial\sigma}{\partial\zeta} = 0. \tag{70}$$

We will assume that $w(0,\zeta)$ has compact support, say contained in $[\zeta_0,\zeta_1]$. This means

$$\zeta_0 \leq \frac{A}{\epsilon}(s-\alpha t) \leq \zeta_1, \tag{71}$$

and it may be achieved in several ways: if we wish to set the initial condition at $t=0$, s must be allowed to vary in the portion of the ray $\epsilon\zeta_0/A \leq s \leq \epsilon\zeta_1/A$; or if the initial condition must be set at the initial curve $s=0$, then it is defined along the time interval $-\epsilon\zeta_1/(\alpha A) \leq t \leq -\epsilon\zeta_0/(\alpha A)$. Other combinations are possible.

(68) and (69) may be written for our particular p as

$$\sigma(s,\zeta) = w(s,\zeta)\exp\left(\int_0^s \frac{\beta}{1+\kappa s_1}\,ds_1\right), \tag{72}$$

$$\ell(s) = \int_0^s \gamma\exp\left(\int_0^{s_1} \frac{\beta}{1+\kappa s_2}\,ds_2\right)\,ds_1. \tag{73}$$

Thus there are the following possibilities:

$a1)$ For $\kappa \neq 0$, $\sigma(s,\zeta) = w(s,\zeta)(1+\kappa s)^{\beta/\kappa}$. (74)

$a2)$ For $\kappa = 0$, $\sigma(s,\zeta) = w(s,\zeta)e^{\beta s}$. (75)

$b1)$ For $\kappa \neq 0$, $\beta/\kappa \neq -1$, $\ell(s) = \dfrac{\gamma}{\beta+\kappa}\left[(1+\kappa s)^{1+\beta/\kappa} - 1\right]$. (76)

$b2)$ For $\kappa \neq 0$, $\beta/\kappa = -1$, $\ell(s) = \dfrac{\gamma}{\kappa}\ln(1+\kappa s)$. (77)

$b3)$ For $\kappa = 0$, $\beta \neq 0$, $\ell(s) = \gamma\dfrac{e^{\beta s}-1}{\beta}$. (78)

$b4)$ For $\beta = 0$, $\ell(s) = \gamma s$. (79)

Let us consider first the formation time of the shock wave. It is well known (see e.g. [18]) that for the initial condition $\sigma_0(\zeta) = \sigma(0,\zeta)$, the shock occurs precisely at

$$\ell_b = -\left(\inf_{\zeta\in\mathbb{R}} \frac{d\sigma_0}{d\zeta}\right)^{-1}, \tag{80}$$

provided this infimum is negative; otherwise, there exists a rarefaction wave and no shock. If σ_0 possesses compact support, σ remains within this support until the shock forms. To see if this ℓ_b may be reached by a length s_b within the ray, we must study cases $b1$)-$b4$). Recall that for $\kappa < 0$, the ray is limited by $1/|\kappa|$. We will write on the left the interval of values of s and on the right the one of $\ell(s)$.

$$c1) \quad \kappa > 0, \quad \frac{\beta}{\kappa} + 1 > 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,\infty). \tag{81}$$

$$c2) \quad \kappa < 0, \quad \frac{\beta}{\kappa} + 1 > 0, \quad s \in [0,1/|\kappa|), \quad \ell(s) \in [0,\gamma/(|\kappa| - \beta)). \tag{82}$$

$$c3) \quad \kappa > 0, \quad \frac{\beta}{\kappa} + 1 < 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,\gamma/(|\kappa| + \beta)). \tag{83}$$

$$c4) \quad \kappa < 0, \quad \frac{\beta}{\kappa} + 1 < 0, \quad s \in [0,1/|\kappa|), \quad \ell(s) \in [0,\infty). \tag{84}$$

$$c5) \quad \kappa > 0, \quad \frac{\beta}{\kappa} + 1 = 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,\infty). \tag{85}$$

$$c6) \quad \kappa < 0, \quad \frac{\beta}{\kappa} + 1 = 0, \quad s \in [0,1/|\kappa|), \quad \ell(s) \in [0,\infty). \tag{86}$$

$$c7) \quad \kappa = 0, \quad \beta > 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,\infty). \tag{87}$$

$$c8) \quad \kappa = 0, \quad \beta < 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,1/|\beta|). \tag{88}$$

$$c9) \quad \beta = 0, \quad s \in [0,\infty), \quad \ell(s) \in [0,\infty). \tag{89}$$

Thus there is guarantee that shock will form except in cases $c2$, $c3$ and $c8$. For those the value ℓ_b given in (80) must lie within the interval in the right hand side to achieve this. Notice that there is no correspondence between short rays and absence of shocks: in $c3$, the ray goes to infinity, but there are not shocks for all compressive initial conditions; whereas in $c6$ the ray is short, but there is always shock for compressive initial conditions.

Let us turn now to the evolution of the shock one this is formed. Referring again to the admirable exposition in [18], pp.136ss, we assume σ_0 to have compact support and define

$$M_0 = -2 \inf_{\zeta \in \mathbb{R}} \int_{-\infty}^{\zeta} \sigma(0, \zeta_1)\, d\zeta_1 \geq 0$$
$$M_1 = 2 \sup_{\zeta \in \mathbb{R}} \int_{\zeta}^{\infty} \sigma(0, \zeta_1)\, d\zeta_1 \geq 0. \tag{90}$$

Let $N(\ell, \zeta)$ be the triangular function (N-wave)

$$N(\ell, \zeta) = \frac{\zeta}{\ell} \qquad \text{for} \quad -\sqrt{M_0 \ell} < \zeta < \sqrt{M_1 \ell}, \tag{91}$$

and 0 otherwise. Then there exists a constant C such that

$$\int_{-\infty}^{\infty} |\sigma(\ell, \zeta) - N(\ell, \zeta)|\, d\zeta \leq \frac{C}{\ell^{1/2}}. \tag{92}$$

Often the convergence is both fast and more accurate than this mere convergence in integral suggests. Notice that if the initial condition is a positive hump ($M_0 = 0$) the wave only propagates to the right; if negative, to the left. When $\sigma_0 = w_0$ has several maxima, shock superposition occurs, but nevertheless the limit is precisely the N-wave. The jump of N at the ends of the interval (the force of the shock) is respectively $-\sqrt{M_0/\ell}$, $\sqrt{M_1/\ell}$. When writing the N-wave in terms of (t, s), we get

$$N(t, s) = \frac{A(s - \alpha t)}{\epsilon \ell(s)}, \qquad -\epsilon\sqrt{M_0 \ell(s)} < A(s - \alpha t) < \epsilon\sqrt{M_1 \ell(s)}, \tag{93}$$

0 otherwise. If in addition, $\kappa < 0$, the condition $s < 1/|\kappa|$ must be added. Notice that the center of the N-wave corresponds to $s = \alpha t$, i.e. it travels along the ray at speed α. For a fixed s, the time where the wave travels along s is given by

$$\frac{s}{\alpha} - \frac{\epsilon}{\alpha A}\sqrt{M_1 \ell(s)} \leq t \leq \frac{s}{\alpha} + \frac{\epsilon}{\alpha A}\sqrt{M_0 \ell(s)}. \tag{94}$$

Recall that this is a valid asymptotic expression when $\ell \to \infty$, so it cannot be used in the cases $c2$ (82), $c3$ (83) or $c8$ (88). Still, there is a rich array of possibilities for the wave to behave. As an example, when $\kappa < 0$, $1 + \beta/\kappa < 0$, as s approaches its limit $1/|\kappa|$, $\ell(s)$ tends to infinity, the wave speed decreases so much that it lingers indefinitely, although the force of the shock tends to vanish.

Remember that in addition to the departure of the triangular form due to the factor $\ell(s)$, from (74) and (75) the function N must be multiplied either by $(1 + \kappa s)^{-\beta/\kappa}$ (for $\kappa \neq 0$) or by $e^{-\beta s}$ (for $\kappa = 0$). The possible behavior of

w is very diverse according to the signs of the different parameters. Take for example $\kappa > 0$, $1 + \beta/\kappa > 0$. Then w for s (and therefore $\ell(s)$) large as

$$\frac{A(\beta + \kappa)(s - \alpha t)}{\gamma(1 + \kappa s)^{\beta/\kappa} \left[(1 + \kappa s)^{\beta+\kappa+1} - 1\right]}, \tag{95}$$

which, when moving with the wave (i.e. keeping $s - \alpha t$ constant) decreases with s when $2\beta/\kappa + 1 > 0$, but increases if $2\beta/\kappa + 1 < 0$, i.e. $-\kappa < \beta < -\kappa/2$. Although the possible shape of the limit shock wave seems so manifold as to be almost unmanageable, at least this fact reminds us of how sensitive is the fast wave to small amounts of variation in the original data: in this case, curvature of the data curve, plus the angles between the wavefront, the equilibrium magnetic field and the ray direction.

5 Conclusions

It is well known that solutions of nonlinear hyperbolic systems may evolve into shock waves, and also that both the shock formation and its later evolution are phenomena of bewildering complexity. We consider one of the simplest examples: the equations of ideal magnetohydrodynamics in the plane and the evolution of high frequency fast magnetosonic waves in a stationary constant state. This problem may be studied with the methods of nonlinear geometrical optics, which have been applied successfully in many instances of weak nonlinearity. In our case rays are straight lines and the wavefront normals are constant along each ray, but rays may converge giving rise to caustics, and the angle with the magnetic field varies from one ray to another. The equations satisfied by the first order solution yield a differential system along each ray, involving both derivatives with respect to the ray and to the phase of the solution. This equation may be reduce to the Burgers one, which is so well known that we may predict both the time of formation of the shock and the later evolution of this, to the so-called N-wave. However, the changes necessary for this reduction depend on the geometrical parameters of the equilibrium and the initial state in such a sensitive way that when recovering the original solution as a function of time and space we find a bewildering array of possibilities: the ray may be finite (ending in a caustic) or infinite, the shock may form or not, the shock wave may slow or increase its velocity, and its force may increase or decrease. While lacking in universality, these results show the intrinsic complexity of the problem. There is also a caveat: geometrical optics works well as long as the wavelength is shorter than the typical dimensions of the physical setting, but when they fail to do so (e.g. when rays collide) it fails; in particular, there is no interference if the solution depends only on what happens along each ray. As a consequence, while our solutions satisfy the Rankine-Hugoniot relations along the ray, they in general do not transversally to them. This, however, does not affect our conclusion about the sensitivity of fast shock waves with respect to initial conditions.

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