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TESIS DOCTORAL:

Applications of regular variation and proximate orders to ultraholomorphic classes, asymptotic expansions and multisummability

Presentada por Javier Jiménez Garrido para optar al grado de
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Dirigida por:
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Que le consta que el trabajo es original e inédito, y que autoriza su presentación.

Y para que conste a los efectos oportunos, firma la presente en Valladolid a quince de diciembre de dos mil diecisiete.

Fdo.: Javier Sanz Gil

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Javier Jiménez Garrido

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Introducción

El principal objetivo de esta memoria es dar respuesta a varias preguntas abiertas relativas a las clases ultraholomorfas de tipo Carleman-Roumieu de funciones, definidas en sectores de la superficie de Riemann del logaritmo mediante restricciones para el crecimiento de sus derivadas dadas en términos de una sucesión de números reales positivos. La motivación de estos problemas surge del estudio de algunas propiedades que aparecen a la hora de trabajar con un proceso de sumabilidad de series de potencias formales en este contexto y de la construcción de la nueva herramienta de multisumabilidad correspondiente. La solución que se presenta aquí depende fuertemente de las teorías clásicas de variación regular y de órdenes aproximados, que están estrechamente relacionadas. En los siguientes párrafos se describe el origen y desarrollo de estos ingredientes esenciales.

El primer tema básico que nos interesa es el estudio de series divergentes a través de los desarrollos asintóticos. El comienzo de la manipulación sistemática de series divergentes, atribuido normalmente a L. Euler, data del siglo XVIII. Las empleó principalmente para la aproximación de constantes como e y π. Sin embargo, durante el siglo XIX las series divergentes fueron, a grandes rasgos, excluidas de las matemáticas. La causa principal de este hecho fue la definición rigurosa y general de la suma de una serie (convergente) proporcionada por A. L. Cauchy, que rápidamente se convirtió en la estándar. En este sentido, podemos citar a G. H. Hardy quien, en su libro [35, p. 5] de 1949, cuando está debatiendo acerca de la definición apropiada de la suma de una serie divergente, apunta lo siguiente:

> “it does not occur to a modern mathematician that a collection of mathematical symbols should have a ‘meaning’ until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, ‘by X we mean Y’. There are reservations to be made, but it is broadly true to say that mathematicians before Cauchy asked not ‘How shall we define $1-1+1-...$?’ but ‘What is $1-1+1-...$?’ and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.”

En 1886, H. Poincaré renovó el interés matemático en el uso de series de potencias formales (normalmente divergentes) introduciendo la noción de desarrollo asintótico para resolver diversos problemas de física matemática y de la mecánica celeste. Los desarrollos asintóticos, en el sentido de Poincaré, son una especie de desarrollo de Taylor que proporciona aproximaciones sucesivas: una función f compleja y holomorfa en un sector $S=\{z\in\mathbb{C};0<|z|<r,a<\arg(z)<b\}$, admite a la serie de potencias formal con coeficientes complejos $\hat{f}=\sum_{p=0}^{\infty}a_pz^p$ como su desarrollo asintótico (uniforme) en el origen si para todo $p\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ existe una constante positiva C_p tal que para cada $z\in S$ se tiene que

$$\big|f(z)-\sum_{n=0}^{p-1}a_nz^n\big|\le C_p|z|^p, \tag{1}$$

en cuyo caso escribimos $f \in \tilde{\mathcal{A}}(S)$. En este contexto, es natural considerar la aplicación de Borel asintótica $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}(S) \to \mathbb{C}[[z]]$ que envía cada función f en su desarrollo asintótico $\hat{f}$.

En 1916 J. F. Ritt mostró que esta aplicación es sobreyectiva para todo sector S, mientras que no es nunca inyectiva (dado un sector bisecado por la dirección 0 la exponencial $\exp(-z^{-\alpha})$, para una elección adecuada de $\alpha > 0$, es una función plana, i.e., asintóticamente nula, y no trivial). Por tanto, dada una serie de potencias formal $\hat{f}$ y un sector S es en general inútil intentar asignarle una suma de una forma correcta, en el sentido de que no hay una única función en S asintótica a $\hat{f}$.

Durante los años 1970 se produjeron determinantes y originales avances, en este sentido, con los trabajos de J. P. Ramis, en los cuales se observa que, aunque las series de potencias formales son a menudo divergentes, bajo condiciones bastante generales el ritmo de crecimiento de sus coeficientes no es arbitrario. De hecho, un resultado notorio de E. Maillet [66] de 1903 establece que para toda solución $\hat{f} = \sum_{p\geq 0} a_p z^p$ de una ecuación diferencial analítica existen $C, A, k > 0$ tales que $|a_p| \leq CA^p(p!)^{1/k}$ para todo $p \in \mathbb{N}_0$. Inspirado por esto, J. P. Ramis introdujo la noción y los rudimentos de la $k-$sumabilidad, que se apoya en resultados clásicos de G. N. Watson y R. Nevannlina y generaliza el método de sumabilidad de Borel. Sus desarrollos se basan en una modificación de los desarrollos de Poincaré donde el crecimiento de la constante C_p en (1) se expresa de forma explícita como $C_p = CA^p(p!)^{1/k}$ para ciertas constantes $A, C > 0$, lo que conlleva estimaciones del mismo tipo para los coeficientes a_p de $\hat{f}$. La sucesión $\mathbb{M}_{1/k} = (p!^{1/k})_{p\in\mathbb{N}_0}$ es la sucesión Gevrey de orden $1/k$, decimos que f es asintótica $1/k-$Gevrey a $\hat{f}$ (denotado $f \in \tilde{\mathcal{A}}_{\mathbb{M}_{1/k}}(S)$) y , debido a las estimaciones que satisfacen sus coeficientes, se dice que $\hat{f}$ es una serie $1/k-$Gevrey ($\hat{f} \in \mathbb{C}[[z]]_{\mathbb{M}_{1/k}}$). La aplicación de Borel, definida en este caso de $\tilde{\mathcal{A}}_{\mathbb{M}_{1/k}}(S)$ a $\mathbb{C}[[z]]_{\mathbb{M}_{1/k}}$, es sobreyectiva si y sólo si la apertura del sector S es menor o igual que π/k (Teorema de Borel-Ritt-Gevrey), y es inyectiva si y sólo si la apertura es mayor que π/k (Lema de Watson), ver Sección 3.2.

Este último hecho permite dar la definición de una serie de potencias formal $k-$sumable en una dirección d como aquella en la imagen de la aplicación de Borel para un sector S suficientemente amplio y bisecado por la dirección d, a la cual se le puede asignar una $k-$suma (la única función holomorfa en S asintótica a ella). J. P. Ramis probó, de forma puramente teórica (no explícita), que toda solución formal en un punto singular irregular de un sistema lineal de ecuaciones diferenciales ordinarias meromorfo en el dominio complejo puede escribirse como ciertas funciones conocidas multiplicadas por un producto de series formales, cada una de las cuales es $k-$sumable (i.e., $k-$sumable en toda dirección excepto un número finito de ellas) para algún nivel k que depende de la serie. El carácter no constructivo de la prueba fue solventado mediante el uso de una herramienta más potente, la acelerosumabilidad, introducida por J. Ecalle [27] y que, en el caso de involucrar solamente a un número finito de niveles Gevrey, se denomina multisumabilidad (en el sentido de iteración de procesos elementales de $k-$sumabilidad). De hecho, en 1991 W. Balser, B. L. J. Braaksma, J. P. Ramis and Y. Sibuya [9] (ver también [7, 73]) probaron la multisumabilidad de las soluciones formales en un punto singular de las ecuaciones diferenciales lineales meromorfas y B. L. J. Braaksma [19] (diversas pruebas se pueden consultar en [6, 85]) extendió este resultado para ecuaciones no lineales en 1992, lo que permite en cada caso calcular soluciones concretas a partir de las formales. Se ha mostrado que esta técnica se aplica con éxito a multitud de situaciones relacionadas con el estudio de series de potencias formales que son solución en un punto singular de ecuaciones en derivadas parciales, así como problemas de perturbación singular (ver la introducción del Capítulo 4 para más detalles).

No obstante, pueden aparecer series de potencias formales que no son Gevrey en diferentes tipos de ecuaciones que no pueden ser diferenciales ordinarias (a tenor del resultado de

B. L. J. Braaksma). Por ejemplo, V. Thilliez ha probado ciertos resultados en estas clases más generales para soluciones formales de ecuaciones algebraicas en [97]. Así mismo, G. K. Immink en [40, 41] ha obtenido algunos resultados de sumabilidad para soluciones formales de ecuaciones en diferencias cuyos coeficientes crecen al ritmo marcado por la sucesión $(p!\log(p+e)^{-p})_{p\in\mathbb{N}_0}$, pertenecientes al llamado nivel 1^+. Más recientemente, S. Malek [70] ha estudiado ciertas ecuaciones diferenciales-en diferencias no lineales singularmente perturbadas de paso pequeño cuyas soluciones formales con respecto al parámetro de perturbación pueden descomponerse como la suma de dos series formales, una Gevrey de orden 1 y la otra de nivel 1^+, un fenómeno que ya se había observado para ecuaciones en diferencias [20].

Todos estos ejemplos muestran que es interesante proporcionar una herramienta para un tratamiento general de la sumabilidad que extienda las potentes teorías de $k-$sumabilidad y multisumabildad, y que nos permita trabajar con desarrollos asintóticos donde las estimaciones en (1) estén dadas por una constante C_p de la forma $C_p = CA^pM_p$ para ciertos $A, C > 0$ y para una sucesión $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ de números reales positivos apropiada. Resulta que los coeficientes de la serie formal cuyas sumas parciales aparecen en (1), para esta elección de C_p, están controlados por $\mathbb{M}$ del mismo modo. La correspondiente clase de series de potencias formales se denotará por $\mathbb{C}[[z]]_{\mathbb{M}}$.

Esta tarea, el tratamiento común de la sumabilidad, requiere de dos tipos de resultados o técnicas:

(i) El conocimiento de la inyectividad y la sobreyectividad de la aplicación de Borel en este contexto general. Obsérvese que una versión análoga del Lema de Watson debería obtenerse para tener una definición adecuada de sumabilidad.

(ii) La construcción de núcleos integrales que nos permitan generalizar las transformadas de Laplace y Borel (analíticas y formales), mediante las que poder dar una expresión explícita de la suma de una serie $\mathbb{M}-$sumable en una dirección. Además, diferentes niveles correspondientes a sucesiones distintas deberían poder combinarse, del mismo modo que los distintos métodos de $k-$sumabilidad producen la multisumabilidad. Esto nos conduce a ser capaces de trabajar con núcleos de convolución y aceleración, como los desarrollados por W. Balser en [7].

Como se explica a continuación, algunas partes del planteamiento previo habían sido resueltas cuando me incorporé al grupo de investigación en el que he realizado mi doctorado, otras se desarrollarán en esta memoria y el resto es trabajo en curso que será comentado hasta cierto punto en las conclusiones.

Para abordar el primer problema, debemos enfatizar que pueden considerarse tres clases ultraholomorfas de funciones en un sector S de la superficie de Riemann del logaritmo y que están íntimamente relacionadas: la clase $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ de funciones holomorfas con desarrollo uniforme en S, verificando (1) para la elección previa de C_p; la clase $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ formada por las funciones holomorfas con desarrollo no uniforme en S, lo que quiere decir que (1) se verifica para $C_p(T) = C_TA_T^pM_p$ en todo subsector propio y acotado T de S (en lugar de uniformemente en S), donde $C_T, A_T > 0$ dependen del subsector; y, finalmente, la clase $\mathcal{A}_{\mathbb{M}}(S)$ de funciones con derivadas acotadas, para las cuales existe $A = A(f) > 0$ tal que

$$\sup_{z\in S,\, p\in\mathbb{N}_0} \frac{|f^{(p)}(z)|}{A^p p! M_p} < \infty.$$

Anterior al estudio de las clases ultraholomorfas es el de las ultradiferenciables, que presentamos seguidamente. Se trata de un hecho notorio que una función $f : [a, b] \to \mathbb{C}$ es real analítica si y sólo si existe una constante $A > 0$ tal que

$$\sup_{x\in[a,b],\, p\in\mathbb{N}_0} \frac{|f^{(p)}(x)|}{A^p p!} < \infty.$$

Más aún, una función real analítica está determinada por el valor de sus derivadas en un punto del intervalo. En 1901, É. Borel mostró la existencia de clases de funciones indefinidamente derivables (no analíticas), i.e., contenidas en $\mathcal{C}^\infty([a,b])$, que heredan la propiedad de unicidad, y a las que él llamó clases casianalíticas. En 1912, en un intento por formalizar este estudio e inspirado por un trabajo de E. Holmgren sobre la ecuación del calor, J. Hadamard propuso considerar las clases $\mathcal{E}_{\mathbb{M}}([a,b])$ de funciones indefinidamente derivables en $[a,b]$ tales que existe $A > 0$ para el cual se tiene que

$$\sup_{x\in[a,b],\, p\in\mathbb{N}_0} \frac{|f^{(p)}(x)|}{A^p p! M_p} < \infty. \tag{2}$$

Una de estas clases $\mathcal{E}_{\mathbb{M}}([a,b])$ es casianalítica si y sólo si siempre que un elemento f de la clase verifica que $f^{(p)}(x_0) = 0$ para todo $p \in \mathbb{N}_0$ y para algún $x_0 \in [a,b]$, se tiene que f es idénticamente nula en $[a,b]$ (avisamos al lector que las notaciones aquí presentes difieren de las utilizadas en los trabajos clásicos a los que nos referimos, ver la Observación 3.1.11). Con este convenio, la formulación del problema es más sencilla: ¿para qué sucesiones $\mathbb{M}$ la clase $\mathcal{E}_{\mathbb{M}}([a,b])$ es casianalítica? Estas clases para la sucesión $(p!^{1/k})_{p\in\mathbb{N}_0}$ aparecen en un trabajo de M. Gevrey de 1918, de ahí su nombre. En 1921 A. Denjoy presentó una condición suficiente y T. Carleman dió una solución completa al problema de casianaliticidad en 1923. Por tanto, este resultado se conoce hoy en día como Teorema de Denjoy-Carleman (véase [38, Th. 1.3.8]), y las clases $\mathcal{E}_{\mathbb{M}}$, que se denominan a menudo clases ultradiferenciables de Carleman, se sitúan entre la clase de funciones reales analíticas y la clase de funciones indefinidamente diferenciables siempre que la sucesión $\mathbb{M}$ verifique $\inf_{p\in\mathbb{N}_0}(M_p)^{1/p} > 0$. Además, si la sucesión $(p!M_p)_{p\in\mathbb{N}_0}$ es logarítmicamente convexa (i.e., la gráfica de la poligonal que une los puntos $(p, \log(p!M_p))$ es convexa), el teorema establece que $\mathcal{E}_{\mathbb{M}}([a,b])$ es casianalítica si y solo si

$$\sum_{p=0}^{\infty} \frac{M_p}{(p+1)M_{p+1}} = \infty.$$

Vale la pena mencionar que, en 1940, A. Gorny y H. Cartan mostraron que la hipótesis sobre la convexidad logarítmica no es restrictiva. Por ejemplo, las clases Gevrey son no casianalíticas para todo $k > 0$ (véase el trabajo panorámico de V. Thilliez [94] sobre casianaliticidad).

Las clases casianalíticas y no casianalíticas han sido ampliamente examinadas en las últimas décadas; la importancia de las no casianalíticas reside en el hecho de que su dual topológico es más grande que el espacio de distribuciones, así que se pueden obtener soluciones más débiles de ciertas clases de ecuaciones en derivadas parciales. Con respecto a su topología natural los espacios anteriores, denominados de tipo Roumieu, son espacios de Hausdorff (LB), límite inductivo de espacios de Banach, mientras que si se pide que (2) se cumpla para todo $A > 0$ tenemos los espacios de tipo Beurling cuya topología es más manejable al tratarse de espacios de Fréchet.

Naturalmente, la aplicación de Borel se puede considerar en este contexto, enviando a una función $f \in \mathcal{C}^\infty([-1,1])$ en la serie de potencias formal construida a partir de su sucesión de derivadas en cero, $\sum_{p=0}^{\infty}(f^{(p)}(0)/p!)z^p \in \mathbb{C}[[z]]$. En 1895, É. Borel probó que esta aplicación es siempre sobreyectiva, por lo que tiene sentido preguntarse acerca de la sobreyectividad de esta aplicación cuando nos restringimos a una clase ultradiferenciable, $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([a,b])\to\mathbb{C}[[z]]_{\mathbb{M}}$. Tras T. Carleman, quien mostró en 1923 que la sobreyectividad no se da para clases casianalíticas que contienen estrictamente a la clase de funciones analíticas, la respuesta completa fue obtenida por H.-J.Petzsche en 1988 (con algunas imprecisiones corregidas en J. Schmets and M. Valdivia [91] en 2000): Si $(p!M_p)_{p\in\mathbb{N}_0}$ es logarítmicamente convexa, entonces $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([a,b])\to\mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva si y solo si $\mathbb{M}$ es fuertemente no casianalítica, es decir, existe $B>0$ tal que

$$\sum_{q=p}^{\infty}\frac{M_q}{(q+1)M_{q+1}}\le B\frac{M_p}{M_{p+1}},\qquad p\in\mathbb{N}_0.$$

En resumen, mientras que se han caracterizado por completo la inyectividad y la sobreyectividad de la aplicación de Borel para clases ultradiferenciables, el problema para clases ultraholomorfas, especialmente en lo que se refiere a la sobreyectividad, dista mucho de estar resuelto íntegramente.

La inyectividad para las clases $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ y $\mathcal{A}_{\mathbb{M}}(S)$ fue completamente resuelto por S. Mandelbrojt y B. Rodríguez-Salinas en los años 1950 (véase la Sección 3.2).

En lo relativo a la sobreyectividad sólo había disponibles informaciones parciales. Aparte del anteriormente mencionado Teorema de Borel-Ritt-Gevrey de 1978, y mediante la aplicación de técnicas del marco ultradiferenciable, V. Thilliez probó en 1995 para la clase Gevrey $\mathcal{A}_{\mathbb{M}_\alpha}(S)$ que se tiene sobreyectividad si y sólo si la apertura del sector es estrictamente menor que $\pi\alpha$. En el año 2000, J. Schmets y M. Valdivia dieron los primeros resultados para una *sucesión peso* $\mathbb{M}$, es decir, logarítmicamente convexa tal que su *sucesión de cocientes* de términos consecutivos $\mathbf{m}=(m_p=M_{p+1}/M_p)_{p\in\mathbb{N}_0}$ tiende a infinito. Su enfoque se basa en la consideración de ciertas clases de funciones ultradiferenciables no canónicas y obtienen, para sucesiones peso verificando la propiedad de ser cerrada por derivación, esto es, existe $A>0$ tal que $M_{p+1}\le A^{p+1}M_p$ para todo $p\in\mathbb{N}_0$, una caracterización de la existencia de operadores de extensión lineales y continuos de $\mathbb{C}[[z]]_{\mathbb{M}}$ en $\mathcal{A}_{\mathbb{M}}(S)$ para cualquier sector S, lo que es mucho más exigente que la sobreyectividad, así que de sus resultados sólo pueden deducirse informaciones parciales. En 2003, V. Thilliez define la noción de sucesión fuertemente regular, i.e., logarítmicamente convexa, fuertemente no casianalítica que, además, satisface la condición de crecimiento moderado, es decir, que existe $A>0$ tal que $M_{p+q}\le A^{p+q}M_pM_q$ para todos $p,q\in\mathbb{N}_0$. Del mismo modo, introduce el índice $\gamma(\mathbb{M})$ y prueba que si la amplitud del sector es estrictamente más pequeña que $\pi\gamma(\mathbb{M})$ entonces $\tilde{\mathcal{B}}:\mathcal{A}_{\mathbb{M}}(S)\to\mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva y no inyectiva (véase la Section 3.3 para más detalles y referencias). Sin embargo, incluso para sucesiones fuertemente regulares los resultados precedentes para clases ultraholomorfas no son enteramente satisfactorios, dado que las equivalencias establecidas en el Teorema de Borel-Ritt-Gevrey y en el Lema de Watson para el caso Gevrey sólo son ahora implicaciones en una dirección.

Los resultados de S. Mandelbrojt y B. Rodríguez-Salinas sugieren la consideración de un índice de crecimiento $\omega(\mathbb{M})$, inicialmente definido por J. Sanz [88] para sucesiones fuertemente regulares, que separa las amplitudes para las cuales las tres clases antes mencionadas son o no son casianalíticas. No obstante, quedaba abierta en general la casianalíticidad de la clase $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ para sectores de amplitud $\pi\omega(\mathbb{M})$.

La primera solución, aunque parcial, a esta situación depende del concepto de orden aproximado, disponible desde los años 1920 y extremadamente útil en la teoría de crecimiento de funciones enteras, y de ciertos resultado de L. S. Maergoiz [65] de 2001 relacionados con el mismo: si definimos las funciones auxiliares $\omega_{\mathbb{M}}(t)=\sup_{p\in\mathbb{N}_0}\log(t^p/M_p)$ y $d_{\mathbb{M}}(t):=\log(\omega_{\mathbb{M}}(t))/\log(t)$ asociadas a $\mathbb{M}$, se mostró en [88] que, siempre que $d_{\mathbb{M}}(t)$ sea un orden aproximado no nulo, se pueden construir funciones planas no triviales en sectores de amplitud óptima y además se pueden dar versiones generalizadas del Lema de Watson y del Teorema de Borel-Ritt-Gevrey. La prueba de la sobreyectividad se basa en el uso de una transformada de Laplace truncada cuyos núcleos están dados por los resultados de Maergoiz (véase la Subsección 1.2.2) sobre existencia de funciones analíticas en sectores cuyo crecimiento en la parte central del sector se puede precisar en términos de $\omega_{\mathbb{M}}(t)$. Se ha observado que para que los argumentos anteriores funcionen, no es necesario que $d_{\mathbb{M}}$ sea un orden aproximado, sino que basta con que esté suficientemente cerca de uno $\rho(t)$, lo que quiere decir que existen constantes $A, B > 0$ tales que

$$A \leq \log(t)\big(d_{\mathbb{M}}(t) - \rho(t)\big) \leq B \qquad \text{para } t \text{ suficientemente grande.} \tag{3}$$

Esta propiedad se reformulará diciendo que $\mathbb{M}$ admite un orden aproximado.

En lo que respecta al elemento (ii) del plan establecido anteriormente, y siguiendo los métodos de sumabilidad de momentos desarrollados por W. Balser [7] en el caso Gevrey, A. Lastra, S. Malek and J. Sanz [60] han descrito recientemente la correspondiente teoría de $\mathbb{M}-$sumabilidad. La pieza clave es la construcción de los núcleos de $\mathbb{M}-$sumabilidad, con sus respectivas transformadas analíticas y formales, en términos de las cuales se puede reconstruir la $\mathbb{M}-$suma de una serie formal de potencias $\mathbb{M}-$sumable en una dirección (véase la Sección 4.1). La existencia de estos núcleos, bajo condiciones bastante sencillas, está de nuevo garantizada por los resultados de L. S. Maergoiz y, por tanto, depende de la posibilidad de asociar a $\mathbb{M}$ un orden aproximado. Sin embargo, la combinación de los métodos de sumabilidad de momentos correspondientes a sucesiones diferentes (no equivalentes pero comparables, como se explicará a continuación) era una tarea pendiente.

Consecuentemente, al iniciarse esta investigación se pretendía resolver los siguientes problemas:

(A) Caracterizar las sucesiones $\mathbb{M}$ tales que $d_{\mathbb{M}}$ es un orden aproximado no nulo.

(B) Caracterizar las sucesiones $\mathbb{M}$ que admiten un orden aproximado no nulo. Para estas sucesiones, el método de $\mathbb{M}-$sumabilidad está disponible.

(C) Determinar si los índices $\gamma(\mathbb{M})$ y $\omega(\mathbb{M})$ coinciden siempre para sucesiones fuertemente regulares, como es el caso para las sucesiones que aparecen en las aplicaciones.

(D) Decidir si la aplicación de Borel es o no inyectiva en el caso que queda por resolver, es decir, para el espacio $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ y para un sector S de amplitud $\pi\omega(\mathbb{M})$ cuando $\mathbb{M}$ no admite un orden aproximado.

(E) Mejorar el conocimiento acerca de la sobreyectividad de la aplicación de Borel en clases ultraholomorfas. Si $\gamma(\mathbb{M})$ y $\omega(\mathbb{M})$ no son siempre coincidentes, será especialmente interesante determinar cuál de los dos separa las amplitudes de sobreyectividad de las de no sobreyectividad, dado que los resultados conocidos previamente no permiten llegar a una conclusión.

(F) Llegar, tan lejos como sea posible, en el estudio de la multisumabilidad en este contexto general. Aplicar estas técnicas al estudio de las soluciones formales de ecuaciones en diferencias como las estudiadas por G. K. Immink, u otro tipo de ecuaciones.

En este punto, comenzaremos a describir los resultados obtenidos en esta memoria y su organización.

El Capítulo 1, de naturaleza preparatoria, contiene en su primera sección todas las definiciones preliminares necesarias y un breve resumen sobre las propiedades de las sucesiones que aparecen cuando se consideran clases ultraholomorfas y ultradiferenciables. Se presentarán también detalladamente la función asociada $\omega_{\mathbb{M}}$ y los índices de crecimiento $\gamma(\mathbb{M})$ y $\omega(\mathbb{M})$.

Mientras se intentaba dar respuesta a los tres primeros elementos de la lista anterior, se observó que el concepto de variación regular y sus extensiones aparecían una y otra vez relacionados con nuestros problemas. Como se deducirá de los desarrollos presentados en los capítulos segundo y tercero, resultan ser de hecho fundamentales en la solución de los mismos.

En 1930 J. Karamata inició la disciplina de la variación regular y la aplicó a problemas tauberianos, como el Teorema de Hardy-Littlewood-Karamata. Sus ideas fueron desarrolladas por sus colaboradores y alumnos de la 'Escuela Yugoslava' en las siguientes décadas. Una función medible $f : [a, \infty) \to (0, \infty)$, con $a \geq 0$, es de *variación regular* si

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0, \infty) \tag{4}$$

para todo $\lambda \in (0, \infty)$. La teoría de variación regular garantiza que existe $\rho \in \mathbb{R}$ tal que $g(\lambda) = \lambda^\rho$ y la convergencia es uniforme para λ en los conjuntos compactos de $(0, \infty)$ (véase la Sección 1.2). Esta disciplina se popularizó en los años 1970 gracias a su uso en teoría de la probabilidad, impulsado por los trabajos de W. Feller y L. de Haan. Sin embargo, nosotros estamos especialmente interesados en su aplicación al análisis complejo, donde aparece estrechamente relacionada con el concepto de orden aproximado, que proviene del estudio del crecimiento de funciones enteras.

En algunas ocasiones esta teoría puede ser muy restrictiva y se han dado diversas generalizaciones de la misma. En esta memoria, además de la variación regular, se considerará la llamada O-variación regular donde el lim en (4) se sustituye por dos condiciones con lim sup y lim inf en su lugar. Esta noción ya fue considerada por Karamata, pero fue difundida gracias a W. Matuszewska que estableció su conexión con los espacios de Orlicz en 1964. Además, caracterizó esta noción en términos de dos índices, conocidos hoy en día como índices de Matuszewska. Debido a la naturaleza de este trabajo estamos particularmente interesados en las versiones discretas de estos conceptos. La extensión de la variación regular para sucesiones de números reales positivos fue llevada a cabo por R. Bojanić y E. Seneta en 1973, y la de la O-variación regular ha sido proporcionada por D. Djurčić y V. Božin en 1997. En la segunda sección de este primer capítulo se resumirán de forma concisa pero completa estos elementos característicos de la teoría de Karamata.

El propósito principal del segundo capítulo es la descripción de las relaciones existentes entre las nociones presentadas en el primero. La primera sección se centra en la noción de O-variación regular. Ciertas condiciones, como son el crecimiento moderado o la no casianaliticidad fuerte, que se asumen a menudo para la sucesión $\mathbb{M}$ con el objetivo de que la clase correspondiente tenga ciertas propiedades, pueden reformularse en términos de la O-variación regular. Merece ser mencionado que el índice de Thilliez $\gamma(\mathbb{M})$ y el índice de Sanz $\omega(\mathbb{M})$, introducidos de forma independiente, resultan tener una representación adecuada en estos términos. Para establecer la conexión con las sucesiones logarítmicamente convexas, será necesario expresar estas condiciones por medio de propiedades de casimonotonía y considerar los órdenes $\mu(\mathbf{m})$ y $\rho(\mathbf{m})$ y los índices de Matuszewska $\beta(\mathbf{m})$ y $\alpha(\mathbf{m})$ para su sucesión de cocientes $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$. Esta relación queda reflejada en los dos siguientes resultados (Teorema 2.1.16 y Proposición 2.1.18).

Teorema. Sea $\mathbb{M}$ una sucesión de números reales positivos con sucesión de cocientes $\mathbf{m}$. Entonces

$$\gamma(\mathbb{M}) = \beta(\mathbf{m}), \qquad \omega(\mathbb{M}) = \mu(\mathbf{m}).$$

Proposición. Sea $\mathbb{M}$ una sucesión de números reales positivos con sucesión de cocientes $\mathbf{m}$. Supongamos que $(p!M_p)_{p\in\mathbb{N}_0}$ is logarítmicamente convexa, entonces

(i) $\mathbb{M}$ tiene crecimiento moderado si y sólo si $\alpha(\mathbf{m}) < \infty$ si y sólo si $\mathbf{m}$ es de O-variación regular.

(ii) $\mathbb{M}$ es fuertemente no casianalítica si y sólo si $\beta(\mathbf{m}) > 0$.

En consecuencia, existe una conexión estrecha entre la O-variación regular de $\mathbf{m}$ y la regularidad fuerte de $\mathbb{M}$. Como resultado colateral, se obtendrán varias definiciones equivalentes de estos índices y órdenes, que se emplearán en el Capítulo 3 al lidiar con la sobreyectividad de la aplicación de Borel, la cual motivó este estudio. Se proporcionará un gran número de detalles, más de los necesarios para el problema de sobreyectividad, con el objetivo de dar una visión completa del problema. En particular veremos que $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$ (lo que ya se sabía con anterioridad), pero la información más relevante deducida de la O-variación regular es que en general para sucesiones fuertemente regulares estos índices son distintos. Esto no ha sido sencillo de mostrar, dado que la mayoría de las sucesiones fuertemente regulares que aparecen en las aplicaciones admiten un orden aproximado no nulo, que es una condición mucho más fuerte que implica, en particular, que $\omega(\mathbb{M}) = \gamma(\mathbb{M})$. El Ejemplo 2.2.26, al final de este capítulo, muestra cómo se pueden construir sucesiones fuertemente regulares con valores arbitrarios de estos índices, $0 < \gamma(\mathbb{M}) < \omega(\mathbb{M}) < \infty$. Por tanto, el problema (C) está resuelto.

Para terminar la sección, se explica la relación de estas nociones con la función asociada $\omega_{\mathbb{M}}$ y la función de conteo $\nu_{\mathbf{m}}$ en el Teorema 2.1.30 y en la Proposición 2.1.38, de donde se deduce la construción de una sucesión dual.

En la segunda sección de este capítulo, el papel protagonista lo tienen los órdenes aproximados. Se explora la relación entre la variación regular, los órdenes aproximados y las sucesiones peso, lo que, como se mencionó anteriormente, es crucial para la disponibilidad de la teoría de $\mathbb{M}-$sumabilidad. Se obtendrá una caracterización de las sucesiones para las cuales $d_{\mathbb{M}} = \log(\omega_{\mathbb{M}}(t))/\log(t)$ es un orden aproximado no nulo, que era la pregunta abierta (A) satisfactoriamente respondida en el Teorema 2.2.6 en términos de la variación regular de $\mathbf{m}$, resumida a continuación:

Teorema. Sea $\mathbb{M}$ una sucesión peso. Son equivalentes:

(i) $d_{\mathbb{M}}(t)$ es un orden aproximado con $\lim_{t\to\infty} d_{\mathbb{M}}(t) \in (0,\infty)$,

(ii) $\mathbf{m}$ es de variación regular con índice positivo.

En el caso de que cualquiera de estos supuestos se verifique, el valor del índice en (ii) es $\omega(\mathbb{M})$ y el valor del límite en (i) es $1/\omega(\mathbb{M})$.

Para sucesiones que admiten un orden aproximado, en el sentido de (3), se obtendrá una fórmula de representación que soluciona el problema en (B). Del mismo modo, dado que sucesiones equivalentes definen las mismas clases ultraholomorfas, resulta que la admisibilidad de un orden aproximado por parte de $\mathbb{M}$ es una condición natural en el sentido de que es equivalente a la existencia de una sucesión $\mathbb{L}$ equivalente a $\mathbb{M}$ y tal que $d_{\mathbb{L}}$ es un orden aproximado no nulo. La prueba de estos resultados se establece en el Teorema 2.2.19 que sigue:

Teorema. Sea $\mathbb{M}$ una sucesión peso, entonces son equivalentes:

(i) existe una sucesión peso $\mathbb{L}$ equivalente a $\mathbb{M}$ (i.e., existen constantes $A, B > 0$ tales que $A^p L_p \leq M_p \leq B^p L_p$ para todo $p \in \mathbb{N}_0$) tal que $d_{\mathbb{L}}(t)$ es un orden aproximado no nulo,

(ii) $\mathbb{M}$ admite un orden aproximado no nulo,

(iii) existen $\omega \in (0,\infty)$ y sucesiones acotadas de números reales $(b_p)_{p\in\mathbb{N}}$, $(\eta_p)_{p\in\mathbb{N}}$ tales que $(\eta_p)_{p\in\mathbb{N}}$ converge hacia ω y podemos escribir

$$m_p = \exp\left(b_{p+1} + \sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

En el caso de que cualquiera de los anteriores sea válido, $\lim_{t\to\infty} d_{\mathbb{L}}(t) = 1/\omega = 1/\omega(\mathbb{M})$.

Además se obtendrá una nueva caracterización de las sucesiones de variación regular en la Proposición 2.2.3 y se mostrará cómo se pueden construir sucesiones con buen comportamiento a partir de los órdenes aproximados en el Teorema 2.2.14. Finalmente, en la Subsección 2.2.5 y gracias a una mejor comprensión de las propiedades involucradas, se presentarán varios ejemplos que exhiben diferentes comportamientos patológicos, incluyendo el Ejemplo 2.2.26 antes mencionado.

Si, en lugar de la admisibilidad de un orden aproximado, se piden condiciones más débiles para la sucesión $\mathbb{M}$, por ejemplo y con la notación de la fórmula de representación de arriba, si $(\eta_p)_{p\in\mathbb{N}}$ es solamente acotada, lo que equivale a decir que $\mathbf{m}$ es de O-variación regular, no se sabe cómo puede ser reproducido el método de $\mathbb{M}-$sumabilidad. No obstante, y este es el problema que se plantea en (D) y (E), es natural preguntarse sobre la inyectividad y la sobreyectividad de la aplicación de Borel. El tercer capítulo está dedicado al estudio de estas cuestiones para las tres clases ultraholomorfas que comentamos anteriormente. Tras introducir la notación básica en la primera sección, se examinará la inyectividad para la cual, como se ha expuesto antes, casi toda la información era conocida. Resulta que los órdenes aproximados nos proporcionan una solución definitiva al problema de inyectividad: incluso si $\mathbb{M}$ no admite un orden aproximado, siempre podemos controlar la función $d_{\mathbb{M}}$ por arriba por un orden aproximado, y un uso adecuado de la variación regular de las funciones de Maergoiz asociadas a este orden permite construir funciones planas en sectores de amplitud óptima. Por tanto, la cuestión en (D) queda respondida, ver Teorema 3.2.15. Aquí el índice $\omega(\mathbb{M})$ muestra su carácter divisorio. Esta sección termina, ayudados por los resultados de casianaliticidad, con la prueba en el Teorema 3.2.16 de que la aplicación de Borel no es nunca biyectiva.

La última sección se centra en el problema de la sobreyectividad. Se obtendrán resultados parciales para sucesiones peso, por ejemplo se mostrará que si hay sobreyectividad para cualquier amplitud entonces $\gamma(\mathbb{M}) > 0$ o, en otras palabras, $\mathbb{M}$ debe ser fuertemente no casianalítica. Sin embargo, los principales avances se han producido para sucesiones fuertemente regulares. En el Corolario 3.3.18, si S_γ es el sector no acotado de amplitud $\pi\gamma$ y bisecado por la dirección $d = 0$ de la superficie de Riemann del logaritmo, se probará lo siguiente:

Corolario. Sea $\mathbb{M}$ fuertemente regular y $t \in \mathbb{R}$, $t > 0$. Cada una de estas afirmaciones implica la siguiente:

(i) $t < \gamma(\mathbb{M})$,

(ii) la aplicación de Borel $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva,

(iii) la aplicación de Borel $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva,

(iv) la aplicación de Borel $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva,

(v) para todo $\xi \in \mathbb{I}$ con $\xi < t$, la aplicación de Borel $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\xi) \to \mathbb{C}[[z]]_{\mathbb{M}}$ es sobreyectiva,

(vi) $t \le \gamma(\mathbb{M})$.

Por consiguiente, $\gamma(\mathbb{M})$ se muestra como el valor límite adecuado para la sobreyectividad, lo que responde a la pregunta (E). Por último, se dan ciertas informaciones en el caso de que $\mathbb{M}$ sea todavía más regular.

En el Capítulo 4 volvemos a nuestro último problema (F) de la lista. Obsérvese que ahora ya conocemos para qué sucesiones el método de $\mathbb{M}-$sumabilidad está disponible. En la primera sección se recuerdan los elementos característicos de esta teoría de $\mathbb{M}-$sumabilidad. Ahora estamos interesados en la extensión de la multisumabilidad a este contexto general. Comenzaremos con una discusión de carácter preliminar que afecta a la necesidad, para que el problema tenga sentido, de que las sucesiones, que definen los métodos de sumabilidad que van a combinarse, sean comparables y no equivalentes, lo que se estudiará en la Subsección 4.2.1. Tras establecer las propiedades básicas de las sucesiones producto y cociente de dos sucesiones peso, se obtendrá el Teorema Tauberiano 4.2.14:

Teorema. Sean $\mathbb{L}$ y $\mathbb{M}$ sucesiones peso tales que $\mathbb{L}$ admite un orden no nulo aproximado, $\mathbb{M}/\mathbb{L}$ es logarítmicamente convexa y $\omega(\mathbb{L}) < \omega(\mathbb{M})$. Si $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{L}}$ y $\hat{f}$ es $\mathbb{M}-$sumable en todas las direcciones salvo un número finito (mod 2π), entonces $\hat{f}$ es convergente.

Gracias a este resultado, es posible dar una definición consistente de multisumabilidad: una serie de potencias formal se dirá multisumable si se puede dividir en una suma finita de series de potencias formales $\hat{f}_j$, cada una de las cuales es sumable para la correspondiente sucesión $\mathbb{M}_j$ que admite un orden aproximado. Nuestro objetivo es determinar el procedimiento para reconstruir de forma explícita su suma. En los Teoremas 4.3.21 y 4.3.25, se construirán los núcleos de sumabilidad para las sucesiones cociente y producto, con ellos se podrá reconstruir la multisuma como se muestra en el Teorema 4.3.31. Debemos mencionar que este estudio no se ha completado aún, especialmente en lo que respecta a las posibles aplicaciones, pero posponemos estos comentarios sobre este trabajo en curso para las conclusiones.

El último capítulo de esta memoria no estaba inicialmente programado, pero su inclusión es natural una vez que las técnicas de los órdenes aproximados y la variación regular se han incorporado. En él se trata la propagación de las estimaciones $\mathbb{M}-$asintóticas en una dirección de funciones, holomorfas y asintóticamente acotadas, a toda una región sectorial, donde $\mathbb{M}$ es una sucesión peso que admite un orden aproximado no nulo. El resultado principal es el siguiente, Teorema 5.3.1:

Teorema. Dado $\gamma > 0$, supongamos que f es holomorfa en una región sectorial G de amplitud $\pi\gamma$ y bisecada por la dirección 0, f está acotada en todo subsector T propio y acotado de G, y admite a $\hat{f} \in \mathbb{C}[[z]]$ como su $\mathbb{M}-$desarrollo asintótico en una dirección $\theta \in (-\pi\gamma/2, \pi\gamma/2)$. Entonces, $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ y f admite a $\hat{f}$ como su $\mathbb{M}-$desarrollo asintótico en G.

Este teorema generaliza un resultado de A. Fruchard y C. Zhang [29] de 1999 para desarrollos Gevrey. Como en el caso Gevrey, las pruebas de estos resultados dependen, por un lado, de una versión adecuada del clásico Teorema de Phragmén-Lindelöf, ver Lema 5.1.6, y por otro lado, de la disponibilidad de versiones apropiadas del Lema de Watson y el Teorema de Borel-Ritt-Gevrey.

Introduction

The main objective of this dissertation is to answer several open questions related to Carleman-Roumieu ultraholomorphic classes of functions, defined in sectors of the Riemann surface of the logarithm by imposing constraints for their derivatives' growth in terms of a sequence of positive real numbers. These problems were motivated by the study of some properties involved in the work with a summability procedure of formal power series in this context, and by the introduction of the corresponding new tool of multisummability. The solutions provided here will heavily rest on the classical, and closely related, theories of regular variation and proximate orders. The foundations and development of these diverse essential ingredients will be described in the following paragraphs.

The first and basic topic which we are interested in is the study of divergent series through asymptotic expansions. L. Euler is commonly credited with starting the systematic manipulation of divergent series in the 18th century. He was concerned with their application for the approximation of the values of constants such as e and π. However, during the 19th century and for some time after this, divergent series were, roughly speaking, excluded from mathematics. This was mainly due to the fact that A. L. Cauchy gave a rigorous and general definition of the sum of a (convergent) series which quickly became the standard one. In this sense, we may cite G. H. Hardy that, in his book [35, p. 5] in 1949, when discussing about the appropriate definition of the sum of a divergent series, notes the following:

> "it does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by X we mean Y'. There are reservations to be made, but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we define $1-1+1-...$?' but 'What is $1-1+1-...$?' and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal."

In 1886, H. Poincaré boosted again the mathematical interest in formal (usually divergent) power series introducing the notion of asymptotic expansion in order to solve several problems of mathematical physics and celestial mechanics. The asymptotic expansions, in the sense of Poincaré, are kind of Taylor expansions which provide successive approximations: a complex function f, holomorphic on a sector $S=\{z\in\mathbb{C};0<|z|<r,a<\arg(z)<b\}$, admits the complex formal power series $\hat{f}=\sum_{p=0}^{\infty}a_pz^p$ as its (uniform) asymptotic expansion at the origin if for every $p\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ there exists a positive constant C_p such that for every $z\in S$ one has

$$\Big|f(z)-\sum_{n=0}^{p-1}a_nz^n\Big|\le C_p|z|^p, \tag{1}$$

and we write $f \in \tilde{\mathcal{A}}(S)$. In this context it is natural to consider the asymptotic Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}(S) \to \mathbb{C}[[z]]$ sending a function f into its asymptotic expansion $\hat{f}$.

In 1916, J. F. Ritt showed that this map is surjective for any sector S, while it is never injective (given a sector bisected by direction 0, the exponential $\exp(-z^{-\alpha})$, $\alpha > 0$, is a nontrivial flat, i.e., asymptotically null, function for a suitable choice of α). Hence, given a formal power series $\hat{f}$ and a sector S, it is in general hopeless to try to assign a well-defined sum to it, in the sense that there is not a unique holomorphic function in S asymptotic to $\hat{f}$.

Crucial and original advances were produced in this sense during the 1970's with the works of J. P. Ramis. He noted that, although the formal power series solutions to differential equations are frequently divergent, under fairly general conditions the rate of growth of their coefficients is not arbitrary. Indeed, a remarkable result of E. Maillet [66] in 1903 states that for any solution $\hat{f} = \sum_{p\geq 0} a_p z^p$ of an analytic differential equation there will exist $C, A, k > 0$ such that $|a_p| \leq CA^p(p!)^{1/k}$ for every $p \in \mathbb{N}_0$. Inspired by this fact, Ramis introduces and structures the notion of $k-$summability, that rests on classical results by G. N. Watson and R. Nevannlina and generalizes Borel's summability method. His developments are based on a modification of Poincaré's asymptotic expansion where the growth of the constant C_p in (1) is made explicit in the form $C_p = CA^p(p!)^{1/k}$ for some $A, C > 0$, what entails the same kind of estimates for the coefficients a_p in $\hat{f}$. The sequence $\mathbb{M}_{1/k} = (p!^{1/k})_{p\in\mathbb{N}_0}$ is the Gevrey sequence of order $1/k$, f is said to be $1/k-$Gevrey asymptotic to $\hat{f}$ (denoted by $f \in \tilde{\mathcal{A}}_{\mathbb{M}_{1/k}}(S)$), and $\hat{f}$, because of the estimates satisfied by its coefficients, is said to be a $1/k-$Gevrey series ($\hat{f} \in \mathbb{C}[[z]]_{\mathbb{M}_{1/k}}$). The Borel map, defined in this case from $\tilde{\mathcal{A}}_{\mathbb{M}_{1/k}}(S)$ to $\mathbb{C}[[z]]_{\mathbb{M}_{1/k}}$, is surjective if and only if the opening of the sector S is smaller than or equal to π/k (Borel-Ritt-Gevrey Theorem), and it is injective if and only if the opening is greater than π/k (Watson's Lemma), see Section 3.2.

This last fact enables the definition of $k-$summable power series in a direction d as those in the image of the Borel map for a wide enough sector S bisected by d, to which a $k-$sum (the unique holomorphic function in S asymptotic to it) is assigned. J. P. Ramis proved, by a purely theoretical (not explicit) method, that every formal solution to a linear system of meromorphic ordinary differential equations in the complex domain at an irregular singular point can be written as some known functions times a finite product of formal power series, each of which is $k-$summable (i.e., $k-$summable in every direction except for a finite number of them) for some level k depending on the series. The non-constructive character of the proof was solved by the introduction of a more powerful tool, accelerosummability, due to J. Ecalle [27] and which, in the case involving only a finite number of Gevrey levels, is named multisummability (in a sense, an iteration of elementary $k-$summability procedures). Indeed, in 1991 W. Balser, B. L. J. Braaksma, J. P. Ramis and Y. Sibuya [9] (see also [7, 73]) proved the multisummability of the formal solutions of linear meromorphic differential equations at a singular point, and B. L. J. Braaksma [19] (for different proofs, see [6, 85]) extended this result for nonlinear equations in 1992, which allows in every case to compute actual solutions from formal ones. This technique has also been proven to apply successfully to a plethora of situations concerning the study of formal power series solutions at a singular point for partial differential equations, as well as for singular perturbation problems (see the introduction to Chapter 4 for further references).

However, nonGevrey formal power series solutions may appear for different kinds of equations, which must not be ordinary differential equations (according to the aforementioned result by B. L. J. Braaksma). For example, V. Thilliez has proven some results on formal solutions within these more general classes for algebraic equations in [97]. Also, G. K. Immink in [40, 41] has obtained some results on summability for formal solutions of difference equations whose coefficients grow at the rate specified by the sequence $(p!\log(p+e)^{-p})_{p\in\mathbb{N}_0}$, belonging to the so-called level 1^+. More recently, S. Malek [70] has studied some singularly perturbed small step size difference-differential nonlinear equations whose formal solutions with respect to the perturbation parameter can be decomposed as sums of two formal series, one with Gevrey order 1, the other of 1^+ level, a phenomenon already observed for difference equations [20].

All these examples made it interesting to provide the tools for a general, common treatment of summability, extending the powerful theory of $k-$summability and multisummability, and which were able to deal with asymptotic expansions whose estimates in (1) correspond to a constant C_p of the form $C_p = CA^pM_p$ for some $A, C > 0$ for a suitable sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive real numbers. It turns out that the coefficients of the formal power series whose partial sums appear in (1) for this choice of C_p are also controlled in the same way by $\mathbb{M}$. The corresponding class of formal power series is denoted by $\mathbb{C}[[z]]_{\mathbb{M}}$.

This task, the common treatment of summability, requires two main types of results or techniques:

(i) The knowledge of injectivity and surjectivity results for the Borel map in this general context. Observe that an analogue of Watson's Lemma should be obtained for a proper definition of summability.

(ii) The construction of integral kernels for generalized Laplace and Borel (both formal and analytic) transforms which allow one to obtain an explicit expression for the sum of an $\mathbb{M}-$summable series in a direction. Moreover, different levels corresponding to distinct sequences should be combined, in the same way as different $k-$summability methods produce multisummability. This amounts to being able to deal with convolution and acceleration kernels, as developed by W. Balser in [7].

As we will explain now, some parts of the previous program had been already achieved when I joined the research team in which my PhD has been developed, some other parts will be carried out in this dissertation, and the rest are work in progress and will be commented on to some extent in the conclusions.

In order to tackle the first problem, one should emphasize that one may consider three closely related, so-called ultraholomorphic classes of functions in a sector S of the Riemann surface of the logarithm: the class $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ of holomorphic functions with uniform asymptotic expansion in S, satisfying (1) for the above choice of C_p; the class $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ consisting of holomorphic functions with nonuniform asymptotic expansion in S, meaning that (1) holds for $C_p(T) = C_TA_T^pM_p$ on every proper bounded subsector T of S (instead of uniformly on S), where $C_T, A_T > 0$ depend on the subsector; and, finally, the class $\mathcal{A}_{\mathbb{M}}(S)$ of functions with bounded derivatives and for which there exists $A = A(f) > 0$ such that

$$\sup_{z\in S,\, p\in\mathbb{N}_0} \frac{|f^{(p)}(z)|}{A^pp!M_p} < \infty.$$

Much older than the study of the ultraholomorphic classes is that of the ultradifferentiable ones, which we introduce now. It is well-known that a function $f : [a, b] \to \mathbb{C}$ is real analytic if and only if there exists a constant $A > 0$ such that

$$\sup_{x\in[a,b],\, p\in\mathbb{N}_0} \frac{|f^{(p)}(x)|}{A^pp!} < \infty.$$

Moreover, a real analytic function is determined by the values of its derivatives at a point of the interval. In 1901, É. Borel showed the existence of classes of (nonanalytic) smooth functions, i.e., contained in $\mathcal{C}^\infty([a,b])$, which inherit that uniqueness property, what he called *quasianalytic* classes. In 1912, in order to formalize this study and inspired by a work of E. Holmgren for the heat equation, J. Hadamard proposed the consideration of the classes $\mathcal{E}_{\mathbb{M}}([a,b])$ of smooth functions in $[a,b]$ such that there exists $A>0$ for which we have

$$\sup_{x\in[a,b],\, p\in\mathbb{N}_0} \frac{|f^{(p)}(x)|}{A^p p! M_p} < \infty. \tag{2}$$

Such a class $\mathcal{E}_{\mathbb{M}}([a,b])$ is said to be quasianalytic if and only if whenever an element f in this class satisfies $f^{(p)}(x_0)=0$ for all $p\in\mathbb{N}_0$ and for some $x_0\in[a,b]$, then f identically vanishes on $[a,b]$ (we warn the reader that our notations differ from those in the classical works, see Remark 3.1.11). With this conventions the formulation of the problem became simple: for which sequences $\mathbb{M}$ the class $\mathcal{E}_{\mathbb{M}}([a,b])$ is quasianalytic? These classes for the sequence $(p!^{1/k})_{p\in\mathbb{N}_0}$ appear in a work of M. Gevrey in 1918, hence their name. In 1921 A. Denjoy presented a sufficient condition, and T. Carleman gave a complete solution of the problem of quasianalyticity in 1923. Hence, this result is nowadays called Denjoy-Carleman Theorem (see [38, Th. 1.3.8]), and the classes $\mathcal{E}_{\mathbb{M}}$ are frequently named Carleman ultradifferentiable classes, lying between the classes of real analytic and of smooth functions as soon as the sequence $\mathbb{M}$ is assumed to satisfy $\inf_{p\in\mathbb{N}_0}(M_p)^{1/p}>0$. In addition, if the sequence $(p!M_p)_{p\in\mathbb{N}_0}$ is logarithmically convex (i.e., the graph of the polygonal curve joining the points $(p,\log(p!M_p))$ is convex), the theorem states that $\mathcal{E}_{\mathbb{M}}([a,b])$ is quasianalytic if and only if

$$\sum_{p=0}^{\infty} \frac{M_p}{(p+1)M_{p+1}} = \infty.$$

It is worthy to mention that, in 1940, A. Gorny and H. Cartan showed that the logarithmic convexity assumption is not restrictive. For example, Gevrey classes are nonquasianalytic for all $k>0$ (see the panoramic work about quasianalytic classes of V. Thilliez [94]).

Quasianalytic and nonquasianalytic classes have been broadly analyzed in the past decades; the importance of the nonquasianalytic classes lies in the fact that their topological dual is bigger than the space of distributions, so one may obtain 'weaker' solutions for some classes of partial differential equations. Regarding their natural topological structure, the former spaces, called of Roumieu type, are Hausdorff (LB)-spaces, inductive limit of Banach spaces, whereas if we require that (2) holds for every $A>0$ we will have Beurling type spaces whose topology is nicer, they are Fréchet spaces.

Of course, one may also consider the Borel map in this context, sending a smooth function $f\in\mathcal{C}^\infty([-1,1])$ into the formal power series constructed by the sequence of its derivatives at zero, $\sum_{p=0}^{\infty}(f^{(p)}(0)/p!)z^p\in\mathbb{C}[[z]]$. In 1895, É. Borel proved that this map is surjective, and it makes sense to wonder about the surjectivity of its restriction to an ultradifferentiable class, $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([a,b])\to\mathbb{C}[[z]]_{\mathbb{M}}$. After T. Carleman, who showed in 1923 that surjectivity is never the case for quasianalytic classes strictly containing the class of analytic functions, the complete answer was achieved by H.-J.Petzsche in 1988 (with some inaccurate statements corrected by J. Schmets and M. Valdivia [91] in 2000): if $(p!M_p)_{p\in\mathbb{N}_0}$ is logarithmically convex, then $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([a,b])\to\mathbb{C}[[z]]_{\mathbb{M}}$ is surjective if and only if $\mathbb{M}$ is strongly nonquasianalytic, that is, there exists $B>0$ such that

$$\sum_{q=p}^{\infty} \frac{M_q}{(q+1)M_{q+1}} \le B\frac{M_p}{M_{p+1}}, \qquad p\in\mathbb{N}_0.$$

Summing up, while injectivity and surjectivity of the Borel map for ultradifferentiable classes have been fully characterized, the problem for ultraholomorphic classes, specially in the case of surjectivity, was far from being completely solved.

The injectivity for the classes $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ and $\mathcal{A}_{\mathbb{M}}(S)$ was completely solved by S. Mandelbrojt and B. Rodríguez-Salinas in the 1950's (see Section 3.2).

For the surjectivity only partial informations were available. After the aforementioned Borel-Ritt-Gevrey Theorem in 1978, and by applying techniques from the ultradifferentiable setting, V. Thilliez proved in 1995 that for the Gevrey class $\mathcal{A}_{\mathbb{M}_\alpha}(S)$ one has surjectivity if and only if the opening of the sector is strictly smaller than $\pi\alpha$. In 2000 J. Schmets and M. Valdivia gave the first results for a *weight sequence* $\mathbb{M}$, that is, logarithmically convex sequence such that its *sequence of quotients* of consecutive terms $\mathbf{m} = (m_p = M_{p+1}/M_p)_{p\in\mathbb{N}_0}$ tends to infinity. Their approach is based on the consideration of some nonclassical ultradifferentiable classes, and they obtained, for weight sequences satisfying the property derivation closedness, namely there exists $A>0$ such that $M_{p+1} \le A^{p+1}M_p$ for every $p\in\mathbb{N}_0$, a characterization for the existence of linear and continuous global extension from $\mathbb{C}[[z]]_{\mathbb{M}}$ to $\mathcal{A}_{\mathbb{M}}(S)$ for any sector S, which is much more demanding than surjectivity, so only partial information can be inferred from their results. In 2003, V. Thilliez defined the notion of strongly regular sequence, i.e., logarithmically convex, strongly nonquasianalytic sequences that, in addition, satisfy the moderate growth condition, that is, there exists $A>0$ such that $M_{p+q}\le A^{p+q}M_pM_q$ for every $p,q\in\mathbb{N}_0$. Moreover, he introduced the index $\gamma(\mathbb{M})$ and showed that if the opening of the sector is strictly smaller than $\pi\gamma(\mathbb{M})$ then $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective and not injective (see Section 3.3 for further details and references). However, even for strongly regular sequences the preceding results for ultraholomorphic classes are not fully satisfactory, since the equivalences stated in Borel-Ritt-Gevrey Theorem and Watson's Lemma for the Gevrey case are now only one-side implications.

The results of S. Mandelbrojt and B. Rodríguez-Salinas suggested the introduction of a growth index $\omega(\mathbb{M})$, initially given by J. Sanz [88] for strongly regular sequences $\mathbb{M}$, which puts apart the openings of quasianalyticity from those of nonquasianalyticity for the three ultraholomorphic classes considered. Nevertheless, in general it remained open the quasianalyticity of the class $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ for sectors of opening $\pi\omega(\mathbb{M})$.

A first and partial solution to this situation relies on the concept of proximate order, available since the 1920s and extremely useful in the theory of growth of entire functions, and on some related results of L. S. Maergoiz [65] in 2001: if we define the auxiliary functions $\omega_{\mathbb{M}}(t) = \sup_{p\in\mathbb{N}_0}\log(t^p/M_p)$ and $d_{\mathbb{M}}(t) := \log(\omega_{\mathbb{M}}(t))/\log(t)$ associated with $\mathbb{M}$, it was shown in [88] that, whenever $d_{\mathbb{M}}(t)$ is a nonzero proximate order, one is able to produce nontrivial flat functions in sectors of optimal opening, and generalized versions of Watson's Lemma and Borel-Ritt-Gevrey Theorem are available. The proof of the surjectivity rests on a truncated Laplace transform technique with kernels provided by the results of Maergoiz (see Subsection 1.2.2) on the existence of suitable analytic functions in sectors whose growth in the central part of the sector is accurately given by $\omega_{\mathbb{M}}(t)$. Moreover, one may note that, for the previous arguments to work, $d_{\mathbb{M}}$ need not be a proximate order, but rather be close enough to a proximate order $\rho(t)$ in the sense that there exist constants $A,B>0$ such that

$$A \le \log(t)\big(d_{\mathbb{M}}(t) - \rho(t)\big) \le B \qquad \text{for } t \text{ large enough.} \tag{3}$$

This fact will be rephrased by saying that $\mathbb{M}$ admits a proximate order.

Regarding the program in (ii) above, and following the technique of moment summability methods developed by W. Balser [7] in the Gevrey case, A. Lastra, S. Malek and J. Sanz [60] have recently put forward the corresponding $\mathbb{M}-$summability theory. The main point is the introduction of kernels of $\mathbb{M}-$summability, and the associated formal and analytic transforms, in terms of which to reconstruct the sums of $\mathbb{M}-$summable formal power series in a direction (see Section 4.1). The existence of such kernels, under fairly mild assumptions, is again guaranteed by the results of L. S. Maergoiz and so depends on the possibility of associating $\mathbb{M}$ with a proximate order. However, the combination of summability methods corresponding to different (nonequivalent but comparable, as it will be explained later) sequences was left as a pending task.

So, the problems we had in mind when this research was initiated were the following:

(A) Characterize the sequences $\mathbb{M}$ such that $d_{\mathbb{M}}$ is a nonzero proximate order.

(B) Characterize the sequences $\mathbb{M}$ that admit a nonzero proximate order. For these sequences, the $\mathbb{M}-$summability technique is available.

(C) Determine whether the indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ are always equal for strongly regular sequences, as it happened to be the case for the sequences appearing in the applications.

(D) Decide about the injectivity of the Borel map in the only unsolved case, the space $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ for a sector S of opening $\pi\omega(\mathbb{M})$ in case $\mathbb{M}$ does not admit a nonzero proximate order.

(E) Improve our knowledge about the surjectivity of the Borel map in general ultraholomorphic classes. In case $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ are not always equal, it is specially interesting to determine which of them puts apart the values of surjectivity from those of nonsurjectivity, since previously known results do not lead to a conclusion.

(F) Proceed, as far as possible, in the study of multisummability in this general context. If reasonable, apply the technique to the formal solutions of some class of difference equations, as those studied by G. K. Immink, or other types of equations.

At this point we start describing the results obtained in this dissertation and how they are organized.

Chapter 1, of a preparatory nature, contains in its first section all the preliminary definitions needed and a brief overview of the most common properties for sequences that appear in the consideration of ultraholomorphic and ultradifferentiable classes. The associated function $\omega_{\mathbb{M}}$ and the growth indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ will also be presented in detail.

While preparing for addressing the first three items in the previous list, we found that the concept of regular variation and its extensions appeared once and again related to our problems. As it will be inferred from the developments presented in the second and third chapter, it enters crucially in the solution of these problems.

The subject of regular variation was initiated by J. Karamata in 1930, who made use of it in Tauberian Theorems, like the Hardy-Littlewood-Karamata Theorem. His ideas were developed by his collaborators and pupils from the 'Yugoslavian School' in the following decades. A measurable function $f:[a,\infty)\to(0,\infty)$, with $a\geq 0$, is *regularly varying* if

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=g(\lambda)\in(0,\infty) \tag{4}$$

for every $\lambda\in(0,\infty)$. The theory of regular variation ensures that there exists $\rho\in\mathbb{R}$ such that $g(\lambda)=\lambda^{\rho}$ and the convergence is uniform for λ in the compact sets of $(0,\infty)$ (see Section 1.2). This subject was popularized in the 1970's by its applications to probability theory, stimulated by the contributions of W. Feller and L. de Haan. However, we will be specially interested in its application to complex analysis, where it appears tightly connected to the notion of proximate order whose definition was motivated by the study of the growth of entire functions.

In some occasions this theory is too limited and several generalizations have been provided. In this dissertation, apart from regular variation, we will concentrate on the so-called O-regular variation, where the lim in (4) is substituted by two conditions with lim sup and lim inf instead. This notion was also considered by J. Karamata, but it was spread by W. Matuszewska thanks to its relation with Orlicz spaces in 1964. She characterized it in terms of two indices commonly known as Matuszewska indices. In virtue of the nature of this work we will be specially interested in the discrete versions of these concepts. The extension of regular variation for sequences of positive real numbers was carried out by R. Bojanić and E. Seneta in 1973, and the O-regularly varying version was provided by D. Djurčić and V. Božin in 1997. In the second section of the first chapter the elementary features of Karamata theory are summarized in a concise but complete form.

The main purpose of the second chapter is the description of the existing relations among the notions presented in the first one. The first section is centered on the notion of O-regular variation. Some of the conditions, for instance, moderate growth and strongly nonquasianalyticity, frequently assumed for the sequence $\mathbb{M}$ in order to have suitable properties for the corresponding class, can be restated in terms of O-regular variation. It deserves a specific mention that Thilliez's index $\gamma(\mathbb{M})$ and Sanz's index $\omega(\mathbb{M})$, independently introduced, will be proved to have an adequate representation in the classical theory of O-regular variation. In order to establish its connection to logarithmically convex sequences, we will need to express these conditions by means of almost monotonicity properties, and to introduce the orders $\mu(\mathbf{m})$ and $\rho(\mathbf{m})$ and the Matuszewska indices $\beta(\mathbf{m})$ and $\alpha(\mathbf{m})$ for its sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$. The following two results (Theorem 2.1.16 and Proposition 2.1.18) illustrate this relation.

Theorem. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m}$. Then

$$\gamma(\mathbb{M}) = \beta(\mathbf{m}), \qquad \omega(\mathbb{M}) = \mu(\mathbf{m}).$$

Proposition. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m}$. Assume that $(p!M_p)_{p\in\mathbb{N}_0}$ is logarithmically convex, then

(i) $\mathbb{M}$ has moderate growth if and only if $\alpha(\mathbf{m}) < \infty$ if and only if $\mathbf{m}$ is O-regularly varying.

(ii) $\mathbb{M}$ is strongly nonquasianalytic if and only if $\beta(\mathbf{m}) > 0$.

Consequently, there is a tight connection between the O-regular variation of $\mathbf{m}$ and the strong regularity of $\mathbb{M}$. As a by-product, equivalent descriptions of those indices and orders are obtained, which will be employed in Chapter 3 when dealing with the surjectivity of the Borel problem, which indeed motivated the study. A considerable number of details, more than needed for the surjectivity issue, will be provided in order to exhibit a complete vision of the subject. In particular, we always have that $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$ (as it was already known before noting this link), but the most revealing feature deduced from O-regular variation will be that, in general, they are distinct for strongly regular sequences. This has been not easy to show, since most of the strongly regular sequences appearing in the applications admit a nonzero proximate order, which is a stronger condition that in particular implies that $\omega(\mathbb{M}) = \gamma(\mathbb{M})$. Example 2.2.26, at the end of the chapter, shows how to construct a strongly regular sequence with arbitrarily prescribed values of these two indices, $0 < \gamma(\mathbb{M}) < \omega(\mathbb{M}) < \infty$. Hence, problem (C) is solved.

At the end of the section, the link of these notions with the associated function $\omega_{\mathbb{M}}$ and the counting function $\nu_{\mathbf{m}}$ is explained in Theorem 2.1.30 and Proposition 2.1.38, from which the construction of a dual sequence is derived.

In the second section of this chapter, the leading role is played by proximate orders. The relation between regularly varying sequences, proximate orders and weight sequences is explored, which, as mentioned before, is crucial for the availability of the $\mathbb{M}-$summability theory. We will obtain a characterization of the sequences for which $d_{\mathbb{M}} = \log(\omega_{\mathbb{M}}(t))/\log(t)$ is a nonzero proximate order, which was the open question (A) and has been successfully answered in Theorem 2.2.6 in terms of the regular variation of $\mathbf{m}$, summarized as follows:

Theorem. Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(i) $d_{\mathbb{M}}(t)$ is a proximate order with $\lim_{t\to\infty} d_{\mathbb{M}}(t) \in (0,\infty)$,

(ii) $\mathbf{m}$ is regularly varying with a positive index of regular variation.

In case any of these statements holds, the value of the index mentioned in (ii) is $\omega(\mathbb{M})$ and the limit in (i) is $1/\omega(\mathbb{M})$.

Regarding the sequences admitting a nonzero proximate order, in the sense of (3), we will get a representation formula for them so solving the problem in (B). Furthermore, since equivalent sequences define the same ultraholomorphic classes, it will turn out that the admissibility of a proximate order by $\mathbb{M}$ is a natural condition in the sense that it is equivalent to the existence of a sequence $\mathbb{L}$ equivalent to $\mathbb{M}$ and such that $d_{\mathbb{L}}$ is a nonzero proximate order. The proof of these facts is presented in Theorem 2.2.19, that states:

Theorem. Let $\mathbb{M}$ be a weight sequence, then the following conditions are equivalent:

(i) There exists a weight sequence $\mathbb{L}$ equivalent to $\mathbb{M}$ (i.e., there exist constants $A, B > 0$ such that $A^p L_p \leq M_p \leq B^p L_p$ for all $p \in \mathbb{N}_0$) such that $d_{\mathbb{L}}(t)$ is a nonzero proximate order,

(ii) $\mathbb{M}$ admits a nonzero proximate order,

(iii) There exist $\omega \in (0,\infty)$ and bounded sequences of real numbers $(b_p)_{p\in\mathbb{N}}$, $(\eta_p)_{p\in\mathbb{N}}$ such that $(\eta_p)_{p\in\mathbb{N}}$ converges to ω and we can write

$$m_p = \exp\left(b_{p+1} + \sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

In case the previous holds, $\lim_{t\to\infty} d_{\mathbb{L}}(t) = 1/\omega = 1/\omega(\mathbb{M})$.

In addition, a new characterization of regularly varying sequences will be obtained in Proposition 2.2.3 and it will also be shown how one can construct well-behaved weight sequences from proximate orders in Theorem 2.2.14. Finally, in Subsection 2.2.5 and thanks to our improved understanding of the properties involved, several examples will be provided exhibiting different pathological behaviors, including Example 2.2.26 mentioned before.

If, instead of admitting a proximate order, weaker conditions are asked for $\mathbb{M}$, for instance and with the notation of the above representation formula, if $(\eta_p)_{p\in\mathbb{N}}$ is only bounded, which is equivalent to the O-regular variation of $\mathbf{m}$, it is not known how the $\mathbb{M}-$summability method can be replicated. However, and this was the problem posed in (D) and (E), it is natural to ask oneself about the injectivity and surjectivity of the Borel map. The third chapter is devoted to the study of these questions in the three ultraholomorphic classes of functions previously considered. After introducing the basic notation in the first section, we analyze the injectivity for which, as commented above, nearly all the information was already known. It turns out that proximate orders again provide the definitive solution for the injectivity problem: even if $\mathbb{M}$ does not admit a nonzero proximate order, one can always control $d_{\mathbb{M}}$ by a nonzero proximate order from above, and a suitable use of the regular variation of the functions of Maergoiz associated with this proximate order allows one to construct flat functions in sectors of optimal opening.

So, the question in (D) is answered, see Theorem 3.2.15. Here the index $\omega(\mathbb{M})$ shows its dividing character. This section ends, helped by the quasianalyticity results, showing in Theorem 3.2.16 that the Borel map is never bijective.

The last section is centered on the surjectivity problem. Some partial results will be obtained for weight sequences, for example it is shown that for arbitrary weight sequences, surjectivity for any opening requires $\gamma(\mathbb{M}) > 0$ or, in other words, $\mathbb{M}$ has to be strongly nonquasianalytic. However, the main advances are for strongly regular sequences. In Corollary 3.3.18, if S_γ is the unbounded sector of opening $\pi\gamma$ and bisecting direction $d = 0$ of the Riemann surface of the logarithm, it will be proved the following:

Corollary. Let $\mathbb{M}$ be a strongly regular sequence, and let $t \in \mathbb{R}$, $t > 0$. Each assertion implies the following one:

(i) $t < \gamma(\mathbb{M})$,

(ii) the Borel map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(iii) the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(iv) the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(v) for every $\xi \in \mathbb{I}$ with $\xi < t$, the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\xi) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(vi) $t \leq \gamma(\mathbb{M})$.

Hence, $\gamma(\mathbb{M})$ is shown to be the suitable limiting value for surjectivity and this answers the previous question (E). Finally, some information is specified in case the sequence is even more regular.

In Chapter 4 we turn to last problem, (F), in the previous list. Observe that we know now for which sequences the $\mathbb{M}-$summability method is available. The first section recalls the most import features of $\mathbb{M}-$summability theory. So, we are now interested in the extension of multisummability to this general context. A preliminary discussion concerns the necessity, for the problem to make sense, that the sequences, which define the summability methods to be merged, are comparable and nonequivalent, what will be studied in Subsection 4.2.1. After establishing the basic properties of the quotient and product sequences of two weight sequences, the Tauberian Theorem 4.2.14 will be obtained:

Theorem. Let $\mathbb{L}$ and $\mathbb{M}$ be weight sequences such that $\mathbb{L}$ admits a nonzero proximate order, $\mathbb{M}/\mathbb{L}$ is logarithmically convex and $\omega(\mathbb{L}) < \omega(\mathbb{M})$. If $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{L}}$ and $\hat{f}$ is $\mathbb{M}-$summable in all directions except a finite set (mod 2π), then $\hat{f}$ is convergent.

Thanks to this result, a consistent definition of multisummability can be given: a formal power series will be said to be multisummable if it can be split into the sum of finitely many formal power series $\hat{f}_j$, each of them summable for a corresponding sequence $\mathbb{M}_j$ admitting a nonzero proximate order. Our objective is to devise a procedure for the explicit reconstruction of its sum. In Theorems 4.3.21 and 4.3.25, the summability kernels for the quotient and product sequences of two sequences will be built, with them we will be able to construct the multisum as it is shown in Theorem 4.3.31. We should mention that the study of multisummability has not been completed, specially what pertains to some of its possible applications, but we will postpone the comments on this work in progress to the conclusions.

The last chapter in this dissertation was not initially scheduled, but its inclusion is natural once the techniques of proximate order and regular variation have been incorporated. It deals with the propagation of $\mathbb{M}-$asymptotics in a direction for holomorphic and asymptotically bounded functions to the whole sectorial region, where $\mathbb{M}$ is a weight sequence admitting a nonzero proximate order. The main result is the following, Theorem 5.3.1:

Theorem. Given $\gamma > 0$, suppose f is holomorphic in a sectorial region G of opening $\pi\gamma$ and bisected by direction 0, f is bounded in every proper and bounded subsector T of G and it admits $\hat{f} \in \mathbb{C}[[z]]$ as its $\mathbb{M}-$asymptotic expansion in a direction $\theta \in (-\pi\gamma/2, \pi\gamma/2)$. Then, $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ and f admits $\hat{f}$ as its $\mathbb{M}-$asymptotic expansion in G.

This generalizes a result by A. Fruchard and C. Zhang [29] in 1999 for Gevrey asymptotic expansions. As in the Gevrey version, the proofs of the results rest, on one hand, on a suitable version of the classical Phragmén-Lindelöf Theorem, Lemma 5.1.6, here obtained for functions whose growth in a sector is specified by a nonzero proximate order; and, on the other hand, on the available versions of the Watson Lemma and Borel-Ritt-Gevrey Theorem.

Chapter 1

Preliminaries

1.1 Logarithmically convex sequences

The classes of functions and formal power series considered in this dissertation are defined by growth restrictions of their derivatives or of their coefficients, respectively. These restrictions will be expressed in terms of a sequence of positive real numbers that will be assumed to satisfy suitable conditions depending on the problem. In this first section, these conditions and their immediate consequences will be presented. Most of the information is taken from the classical works of S. Mandelbrojt [72] and H. Komatsu [52], which we refer to for further details.

1.1.1 Definition and properties

In what follows, $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ always stands for a sequence of positive real numbers, and we always impose that $M_0 = 1$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$. The names of the conditions given by V. Thilliez and, for the convenience of the reader, the corresponding descriptive acronyms employed by G. Schindl [90] have been used.

Definition 1.1.1. We say that:

(i) $\mathbb{M}$ is *logarithmically convex* (for short, (lc)) if

$$M_p^2 \leq M_{p-1}M_{p+1}, \qquad p \in \mathbb{N}.$$

(ii) $\mathbb{M}$ is of or has *moderate growth* (briefly, (mg)) whenever there exists $A > 0$ such that

$$M_{p+q} \leq A^{p+q} M_p M_q, \qquad p, q \in \mathbb{N}_0.$$

(iii) $\mathbb{M}$ satisfies the *strong nonquasianalyticity condition* (for short, (snq)) if there exists $B > 0$ such that

$$\sum_{q=p}^{\infty} \frac{M_q}{(q+1)M_{q+1}} \leq B \frac{M_p}{M_{p+1}}, \qquad p \in \mathbb{N}_0.$$

According to V. Thilliez [95], if $\mathbb{M}$ is (lc), has (mg) and satisfies (snq), we say that $\mathbb{M}$ is a *strongly regular* sequence.

Definition 1.1.2. For a sequence $\mathbb{M}$ we define *the sequence of quotients* $\mathbf{m} = (m_p)_{p\in\mathbb{N}_0}$ by

$$m_p := \frac{M_{p+1}}{M_p} \qquad p \in \mathbb{N}_0.$$

Remark 1.1.3. The properties (lc) and (snq) can be easily stated in terms of the sequence of quotients and, as we will see in Lemmas 1.1.9 and 2.1.3, the same holds for (mg). Moreover, observe that for every $p \in \mathbb{N}$ one has

$$M_p = \frac{M_p}{M_{p-1}} \frac{M_{p-1}}{M_{p-2}} \cdots \frac{M_2}{M_1} \frac{M_1}{M_0} = m_{p-1} m_{p-2} \cdots m_1 m_0. \tag{1.1}$$

So, one may recover the sequence $\mathbb{M}$ (with $M_0 = 1$) once $\mathbf{m}$ is known, and hence the knowledge of one of the sequences amounts to that of the other. Sequences of quotients of sequences $\mathbb{M}$, $\mathbb{L}$, etc. will be denoted by lowercase letters $\mathbf{m}$, $\boldsymbol{l}$ and so on. Whenever some statement refers to a sequence denoted by a lowercase letter such as $\mathbf{m}$, it will be understood that we are dealing with a sequence of quotients (of the sequence $\mathbb{M}$ given by (1.1)).

Example 1.1.4. We mention some interesting examples. In particular, those in (i) and (iii) appear in the applications of summability theory to the study of formal power series solutions for different kinds of equations.

(i) The sequences $\mathbb{M}_{\alpha,\beta} := \big(p!^{\alpha} \prod_{m=0}^{p} \log^{\beta}(e+m)\big)_{p\in\mathbb{N}_0}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, are strongly regular (in case $\beta < 0$, the first terms of the sequence have to be suitably modified in order to ensure (lc), see Remark 1.1.19). In case $\beta = 0$, we have the best known example of strongly regular sequence, $\mathbb{M}_{\alpha} := \mathbb{M}_{\alpha,0} = (p!^{\alpha})_{p\in\mathbb{N}_0}$, called the *Gevrey sequence of order* α.

(ii) The sequence $\mathbb{M}_{0,\beta} := (\prod_{m=0}^{p} \log^{\beta}(e+m))_{p\in\mathbb{N}_0}$, with $\beta > 0$, is (lc), (mg) and $\mathbf{m}$ tends to infinity, but (snq) is not satisfied.

(iii) For $q > 1$, $\mathbb{M}_q := (q^{p^2})_{p\in\mathbb{N}_0}$ is (lc) and (snq), but not (mg).

Some results remain valid, however, when (mg) and (snq) are replaced by the following weaker conditions:

Definition 1.1.5. Let $\mathbb{M}$ be a sequence, we say that

(i) $\mathbb{M}$ is *stable under differential operators* or satisfies *derivation closedness condition* (briefly, (dc)) if there exists $D > 0$ such that

$$M_{p+1} \le D^{p+1} M_p, \qquad p \in \mathbb{N}_0.$$

(ii) $\mathbb{M}$ is *nonquasianalytic* (for short, (nq)) if

$$\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} < \infty.$$

The following properties are easy consequences of the definitions.

Lemma 1.1.6. For every sequence $\mathbb{M}$ we have the following properties:

(i) If $\mathbb{M}$ has moderate growth then $\mathbb{M}$ is stable under differential operators.

(ii) If $\mathbb{M}$ is strongly nonquasianalytic then $\mathbb{M}$ is nonquasianalytic.

Lemma 1.1.7. For every sequence $\mathbb{M}$ the following holds:

(i) $\mathbb{M}$ is logarithmically convex if and only if $\mathbf{m}$ is nondecreasing.

(ii) If $\mathbb{M}$ is logarithmically convex and nonquasianalytic, then $\lim_{p\to\infty} m_p = \infty$.

(iii) If $\lim_{p\to\infty} m_p = \infty$, then it exists $p_0 \in \mathbb{N}$ such that $M_p \leq M_{p+1}$ for every $p \geq p_0$.

(iv) If $\mathbb{M}$ is logarithmically convex, then $M_p M_l \leq M_{p+l}$ for every $p, l \in \mathbb{N}_0$.

(v) If $\mathbb{M}$ is logarithmically convex, then $(M_p)^{1/p} \leq m_{p-1}$ for every $p \in \mathbb{N}$.

(vi) If $\mathbb{M}$ is logarithmically convex, then $((M_p)^{1/p})_{p\in\mathbb{N}}$ is nondecreasing.

(vii) If $\mathbb{M}$ is logarithmically convex, $\lim_{p\to\infty}(M_p)^{1/p} = \infty$ if and only if $\lim_{p\to\infty} m_p = \infty$.

Proof. (i) Note that $\mathbb{M}$ is logarithmically convex if and only if $M_p/M_{p-1} \leq M_{p+1}/M_p$ for every $p \in \mathbb{N}$. Since $m_p = M_{p+1}/M_p$, $\mathbb{M}$ is logarithmically convex if and only if $m_{p-1} \leq m_p$ for every $p \in \mathbb{N}$.

(ii) Since $\mathbb{M}$ is logarithmically convex, $\mathbf{m}$ is nondecreasing (by (i)). If we suppose that $\mathbf{m}$ is bounded, i.e., it exists $C > 0$ such that $m_p \leq C$ for every $p \in \mathbb{N}$, we see that

$$\sum_{l=0}^{\infty} \frac{M_l}{(l+1)M_{l+1}} = \sum_{l=0}^{\infty} \frac{1}{(l+1)m_l} \geq \sum_{l=0}^{\infty} \frac{1}{(l+1)C} = \infty.$$

This is impossible if $\mathbb{M}$ is nonquasianalytic, so $\mathbf{m}$ is unbounded and nondecreasing, and we conclude that $\lim_{p\to\infty} m_p = \infty$.

(iii) If $\lim_{p\to\infty} m_p = \infty$, it exists $p_0 \in \mathbb{N}$ such that $m_p \geq 1$ for every $p \geq p_0$ that implies $M_{p+1} \geq M_p$ for every $p \geq p_0$.

(iv) We fix $p \in \mathbb{N}_0$ and apply an induction argument on ℓ. The statement holds for $\ell = 0$, because $M_0 = 1$. Assume that it is valid for some value of ℓ, using the induction hypothesis and that $\mathbf{m}$ is nondecreasing (by (i)) we have

$$M_p M_{\ell+1} = M_p M_\ell \frac{M_{\ell+1}}{M_\ell} \leq M_{p+\ell} m_\ell \leq M_{p+\ell} m_{p+\ell} = M_{p+\ell+1}.$$

(v) We observe that $M_p = m_0 m_1 \cdots m_{p-2} m_{p-1}$ and, since $\mathbf{m}$ is nondecreasing (by (i)), we have $M_p \leq (m_{p-1})^p$ for every $p \in \mathbb{N}$.

(vi) By (v), we deduce that

$$M_p^{(p+1)/p} = M_p (M_p)^{1/p} \leq M_p m_{p-1} \leq M_p m_p = M_{p+1}, \qquad p \in \mathbb{N}.$$

(vii) By (v), if $\lim_{p\to\infty}(M_p)^{1/p} = \infty$, then $\lim_{p\to\infty} m_p = \infty$. If $\lim_{p\to\infty} m_p = \infty$, since

$$\lim_{p\to\infty} \frac{\log(M_{p+1}) - \log(M_p)}{(p+1) - p} = \lim_{p\to\infty} \log(m_p) = \infty,$$

we deduce by the Stolz's criterion that $\lim_{p\to\infty}(M_p)^{1/p} = \infty$.

□

Along this document, we may use the basic properties of the last lemma, specially (i) and (ii), without mentioning.

Definition 1.1.8. We say that a sequence $\mathbb{M}$ is a *weight sequence* if it is logarithmically convex and $\lim_{p\to\infty}(M_p)^{1/p} = \infty$ or, equivalently by the lemma above, if $\mathbf{m}$ is nondecreasing and $\lim_{p\to\infty} m_p = \infty$.

Under the assumption of logarithmic convexity H.-J. Petzsche and D. Vogt gave the following characterization of the moderate growth condition in terms of $\mathbf{m}$. For the sake of clarity and completeness, since there is an index shift in their definition of the quotient sequence, the proof has been included.

Lemma 1.1.9 ([78], Lemma 5.3)**.** Let $\mathbb{M}$ be a logarithmically convex sequence. Then the following statements are equivalent:

(i) $\mathbb{M}$ has moderate growth,

(ii) $\sup_{p\in\mathbb{N}}(m_p/M_p^{1/p}) < \infty$,

(iii) $\sup_{p\in\mathbb{N}}(m_{2p}/m_p) < \infty$,

(iv) $\sup_{p\in\mathbb{N}}\left(M_{2p}/M_p^2\right)^{1/p} < \infty$.

Proof. (i)$\Rightarrow$ (ii) From the logarithmic convexity, we know that $\mathbf{m}$ is nondecreasing and for all $p \in \mathbb{N}$, we have

$$(m_p)^p \leq m_p m_{p+1} \dots m_{2p-1} = \frac{M_{2p}}{M_p},$$

and, applying the (mg) condition with $p = l$, we show that it exists $A > 1$ such that $(m_p)^p \leq M_{2p}/M_p \leq A^{2p} M_p$ for all $p \in \mathbb{N}$.
(ii) $\Rightarrow$ (iii) By the logarithmic convexity, if we assume that (ii) is true, we see that there exists $H > 0$ such that

$$m_{2p}^{2p} \leq H^{2p} M_{2p} = H^{2p} m_0 m_1 \dots m_{2p-1} \leq H^{2p}(m_p)^p(m_{2p})^p, \qquad p \in \mathbb{N}.$$

(iii)$\Rightarrow$ (iv) First, we will show that

$$\sup_{p\in\mathbb{N}}(m_{2p}/m_{p-1}) < \infty. \tag{1.2}$$

Using (iii), we see that it exists $H > 1$ such that

$$\frac{m_{2p}}{m_{p-1}} = \frac{m_{2p}}{m_p}\frac{m_p}{m_{p-1}} \leq H\frac{m_p}{m_{p-1}}, \qquad p \in \mathbb{N}.$$

We observe that for $p \geq 2$, we have $2p - 2 \geq p$. Using the logarithmic convexity and applying (iii) again, we show that

$$\frac{m_{2p}}{m_{p-1}} \leq H\frac{m_p}{m_{p-1}} \leq H\frac{m_{2p-2}}{m_{p-1}} \leq H^2, \qquad p \geq 2.$$

Finally, taking $C := \max(H^2, m_2/m_0)$, we see that $\sup_{p\in\mathbb{N}}(m_{2p}/m_{p-1}) \leq C$. Now, using the logarithmic convexity, we have that

$$M_{2p} = m_0 m_1 \dots m_{2p-2} m_{2p-1} \leq m_0 m_2^2 m_4^2 \dots m_{2(p-1)}^2 m_{2p}^2 \frac{1}{m_{2p}}, \qquad p \in \mathbb{N}.$$

Applying (1.2) and the logarithmic convexity again, we obtain

$$M_{2p} \leq C^{2p} m_0^2 m_1^2 m_2^2 \dots m_{p-2}^2 m_{p-1}^2 \frac{m_0}{m_{2p}} \leq C^{2p} M_p^2, \qquad p \in \mathbb{N}.$$

(iv) $\Rightarrow$ (i) We fix $p, \ell \in \mathbb{N}_0$. Firstly, if one or both of them are equal to 0, since $M_0 = 1$, condition (mg) holds for $A = 1$. Secondly, if $p+\ell = 2k$ with $k \in \mathbb{N}$, by (iv), we deduce that it exists $H > 1$ such that

$$M_{p+\ell} = M_{2k} \leq H^k M_k^2 = \left(\sqrt{H}\right)^{2k} M_p M_\ell \frac{M_k}{M_p}\frac{M_k}{M_\ell}.$$

We suppose $p \leq k \leq \ell$ (the proof is the same if $\ell < k < p$). Then, by the logarithmic convexity, we see that

$$\frac{M_k}{M_p}\frac{M_k}{M_\ell} = m_p m_{p+1}\dots m_{k-1}\frac{1}{m_k m_{k+1}\dots m_{\ell-1}} \leq \frac{m_k^{k-p}}{m_k^{\ell-k}} = m_k^0 = 1.$$

Finally, if $p+\ell = 2k-1$ with $k \geq 2$, then one of the values ℓ or p is odd and the other is even. Without lost of generality, we suppose ℓ is odd and so $p \geq 2$. Using the property Lemma 1.1.7.(iv) and applying the last statement for p and $\ell+1$, we see that it exists $A > 1$ such that

$$M_{2k-1} \leq \frac{M_{2k}}{M_1} \leq \frac{A^{2k} M_p M_{\ell+1}}{M_1}.$$

Applying again the first part for 1 and ℓ, whose sum $\ell+1$ is even, we see that $M_{\ell+1} \leq A^{\ell+1} M_\ell M_1$. So we have

$$M_{p+\ell} = M_{2k-1} \leq A^{\ell+p+1} M_p A^{\ell+1} M_\ell \leq (A^2)^{\ell+p} M_p M_\ell.$$

□

Remark 1.1.10. By a slight modification of this proof, given $k \in \mathbb{N}$, $k \geq 2$, we can see that the following conditions are equivalent:

(i) $\mathbb{M}$ has moderate growth,

(ii) $\sup_{p\in\mathbb{N}}(m_{kp}/m_p) < \infty$,

(iii) $\sup_{p\in\mathbb{N}} \left(M_{kp}/M_p^k\right)^{1/p} < \infty$.

As an immediate consequence of Lemma 1.1.7.(v) and Lemma 1.1.9.(ii), V. Thilliez gave the following result.

Lemma 1.1.11 ([95])**.** Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a (lc) sequence satisfying (mg) condition for a constant $A > 0$ appearing in Definition 1.1.1.(ii). Then,

$$m_p \leq A^2 M_p^{1/p} \leq A^2 m_p \qquad \text{for every } p \in \mathbb{N}. \tag{1.3}$$

1.1.2 Equivalent and comparable sequences

The nature of the classes of functions defined in terms of sequences (see Section 3.1) leads us to consider the notions of equivalent and comparable sequences.

Definition 1.1.12. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences, we say that $\mathbb{M}$ *is smaller than* $\mathbb{L}$ if it exists $C > 0$ such that

$$M_p \leq C^p L_p, \qquad p \in \mathbb{N}_0,$$

or, equivalently, if

$$\sup_{p\in\mathbb{N}} \left(\frac{M_p}{L_p}\right)^{1/p} < \infty,$$

and we write $\mathbb{M} \precsim \mathbb{L}$. We call $\mathbb{M}$ and $\mathbb{L}$ *comparable* if $\mathbb{M} \precsim \mathbb{L}$ or $\mathbb{L} \precsim \mathbb{M}$ holds. If both conditions hold, we say that $\mathbb{M}$ *is equivalent to* $\mathbb{L}$, and we write $\mathbb{M} \approx \mathbb{L}$.

From (1.3) we observe that if $\mathbb{M}$ is a (lc) sequence satisfying (mg) condition, then $\mathbb{M}$ and $(m_p^p)_{p\in\mathbb{N}_0}$ are equivalent.

Remark 1.1.13. Since equivalent sequences will turn out to define the same class of functions or series (see Remarks 3.1.4 and 3.1.7), we are particularly interested in comparable but not equivalent sequences. In particular, if $\mathbb{M} \precsim \mathbb{L}$ and $\mathbb{M} \not\approx \mathbb{L}$, we observe that

$$\inf_{p\in\mathbb{N}} \left(\frac{M_p}{L_p}\right)^{1/p} = 0 \qquad \text{and} \qquad \sup_{p\in\mathbb{N}} \left(\frac{M_p}{L_p}\right)^{1/p} < \infty,$$

or, equivalently, if

$$\liminf_{p\to\infty} \left(\frac{M_p}{L_p}\right)^{1/p} = 0 \qquad \text{and} \qquad \limsup_{p\to\infty} \left(\frac{M_p}{L_p}\right)^{1/p} < \infty.$$

Consequently, $\mathbb{M}$ and $\mathbb{L}$ are *noncomparable* if

$$\inf_{p\in\mathbb{N}} \left(\frac{M_p}{L_p}\right)^{1/p} = 0 \qquad \text{and} \qquad \sup_{p\in\mathbb{N}} \left(\frac{M_p}{L_p}\right)^{1/p} = \infty,$$

or, equivalently, if

$$\liminf_{p\to\infty} \left(\frac{M_p}{L_p}\right)^{1/p} = 0 \qquad \text{and} \qquad \limsup_{p\to\infty} \left(\frac{M_p}{L_p}\right)^{1/p} = \infty.$$

In the next definitions and results we take into account the conventions adopted in Remark 1.1.3.

Definition 1.1.14. Let $\mathbf{m}$ and $\boldsymbol{\ell}$ be sequences of positive real numbers, we say that $\boldsymbol{m}$ *is bounded from above by* $\boldsymbol{\ell}$ if it exists $c > 0$ such that

$$m_p \le c\,\ell_p, \qquad p \in \mathbb{N}_0,$$

or, equivalently, if

$$\sup_{p\in\mathbb{N}_0} \frac{m_p}{\ell_p} < \infty,$$

and we write $\mathbf{m} \preceq \boldsymbol{\ell}$. The sequence $\mathbf{m}$ is said to be *similar to* $\boldsymbol{\ell}$ if $\mathbf{m} \preceq \boldsymbol{\ell}$ and $\boldsymbol{\ell} \preceq \mathbf{m}$ and we write $\mathbf{m} \simeq \boldsymbol{\ell}$.

Proposition 1.1.15. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences, $\mathbf{m}$ and $\boldsymbol{\ell}$ the sequences of quotients associated with $\mathbb{M}$ and $\mathbb{L}$, respectively. If $\mathbf{m} \preceq \boldsymbol{\ell}$, then $\mathbb{M} \precsim \mathbb{L}$. Consequently, if $\mathbf{m} \simeq \boldsymbol{\ell}$ then $\mathbb{M} \approx \mathbb{L}$.

Proof. There exists $c > 0$ such that $m_p \le c\,\ell_p$ for every $p \in \mathbb{N}_0$. Writing $M_p = m_0 m_1 \dots m_{p-1}$ and $L_p = \ell_0\ell_1 \dots \ell_{p-1}$, we see that

$$M_p = m_0 m_1 \dots m_{p-2} m_{p-1} \le c\ell_0 c\ell_1 \dots c\ell_{p-2} c\ell_{p-1} = c^p L_p,$$

for every $p \in \mathbb{N}_0$, then $\mathbb{M} \precsim \mathbb{L}$. □

The last proposition shows that the notion of comparability for the sequences of quotients is stronger than the former one. Under suitable assumptions, the converse implication can be obtained.

Proposition 1.1.16. Let $\mathbb{M}$ and $\mathbb{L}$ be (lc) sequences, $\mathbf{m}$ and $\boldsymbol{\ell}$ the corresponding sequences of quotients. We suppose that $\mathbb{M}$ has moderate growth. If $\mathbb{M} \precsim \mathbb{L}$, then $\mathbf{m} \preceq \boldsymbol{\ell}$.

Proof. By Lemma 1.1.7.(v), we see that $(L_p)^{1/p} \leq \ell_p$ and, by Lemma 1.1.9.(ii), $m_p \leq A^2(M_p)^{1/p}$ for some $A > 0$ and for every $p \in \mathbb{N}$. If one has $\mathbb{M} \precsim \mathbb{L}$, then there exists $C > 0$ such that $M_p \leq C^p L_p$ for every $p \in \mathbb{N}_0$ and we conclude that

$$m_p \leq A^2(M_p)^{1/p} \leq A^2 C(L_p)^{1/p} \leq A^2 C\ell_p,$$

for every $p \in \mathbb{N}$, then $\mathbf{m} \preceq \boldsymbol{\ell}$. □

We study the stability under equivalence of the properties in Definition 1.1.1.

Proposition 1.1.17. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences. If $\mathbb{M} \approx \mathbb{L}$ and $\mathbb{M}$ has (mg), then $\mathbb{L}$ also has (mg).

Proof. Since $\mathbb{L} \approx \mathbb{M}$, there exists $C > 1$ such that $C^{-p}L_p \leq M_p \leq C^p L_p$ for every $p \in \mathbb{N}_0$. Using the moderate growth of $\mathbb{M}$, we see that

$$L_{p+q} \leq C^{q+p} M_{q+p} \leq (AC)^{q+p} M_p M_q \leq (AC)^{q+p} C^p C^q L_q L_p = (AC^2)^{p+q} L_q L_p,$$

for every $p, q \in \mathbb{N}_0$, then $\mathbb{L}$ satisfies (mg) condition. □

Proposition 1.1.18. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences. If $\mathbf{m} \simeq \boldsymbol{\ell}$ and $\mathbb{M}$ is (snq), then $\mathbb{L}$ also is (snq).

Proof. Since $\boldsymbol{\ell} \simeq \mathbf{m}$, there exists $c > 1$ such that $c^{-1}\,\ell_p \leq m_p \leq c\,\ell_p$ for every $p \in \mathbb{N}_0$. Using the strong nonquasianalyticity of $\mathbb{M}$, we have

$$\sum_{k=p}^{\infty} \frac{L_k}{(k+1)L_{k+1}} \leq c \sum_{k=p}^{\infty} \frac{M_k}{(k+1)M_{k+1}} \leq cB\frac{M_p}{M_{p+1}} \leq c^2 B \frac{L_p}{L_{p+1}},$$

for every $p \in \mathbb{N}_0$, then $\mathbb{L}$ also satisfies (snq) condition. □

Remark 1.1.19. If $\mathbb{M}$ is a sequence such that $(m_p)_{p \geq p_0}$ is nondecreasing for some $p_0 \in \mathbb{N}$, i.e., $\mathbf{m}$ is eventually nondecreasing, we define $\ell_p = m_{p_0}$ for $p < p_0$ and $\ell_p = m_p$ for $p \geq p_0$, then $\boldsymbol{\ell}$ is nondecreasing ($\mathbb{L}$ is (lc)) and $\boldsymbol{\ell} \simeq \mathbf{m}$. Moreover, whenever $\mathbb{M}$ is (mg) or (snq), by Propositions 1.1.15, 1.1.17 and 1.1.18 we have that $\mathbb{L}$ is also (mg) or (snq).

Using the Propositions 1.1.16, 1.1.17 and 1.1.18, above we easily deduce that:

Proposition 1.1.20. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences, $\mathbf{m}$ and $\boldsymbol{\ell}$ the corresponding sequences of quotients. We suppose that $\mathbb{M}$ and $\mathbb{L}$ are (lc) and one of them has (mg). If $\mathbb{M} \approx \mathbb{L}$, then $\mathbf{m} \simeq \boldsymbol{\ell}$.

In particular, if $\mathbb{M}$ is strongly regular and $\mathbb{L}$ is (lc), then $\mathbf{m} \preceq \boldsymbol{\ell}$ if and only if $\mathbb{M} \precsim \mathbb{L}$, and $\mathbf{m} \simeq \boldsymbol{\ell}$ if and only if $\mathbb{M} \approx \mathbb{L}$. Consequently, if $\mathbb{M} \approx \mathbb{L}$, then $\mathbb{L}$ is also strongly regular.

Remark 1.1.21. Logarithmic convexity is not stable for either $\approx$ or $\simeq$. Regarding (snq), apart from Proposition 1.1.18, one may deduce that, for weight sequences, (snq) is stable for $\approx$ by using [77, Th. 3.4], restated with our notation in Theorem 3.3.4. In this result of H.-J. Petszche, the stability of the condition (γ_1) for $\approx$ (see Remark 2.1.23 for the connection between (γ_1) and (snq)) is indirectly deduced, to the best of our knowledge there is no direct proof of this fact.

1.1.3 Associated functions

In the classical study of classes of functions defined in terms of a sequence of positive real numbers, for instance see S. Mandelbrojt [72] and H. Komatsu [52], the importance of the functions $\omega_{\mathbb{M}}(t)$ and $h_{\mathbb{M}}(t)$ considered below has been illustrated.

For any sequence $\mathbb{M}$ we can consider the map $\omega_{\mathbb{M}}:[0,\infty)\to\mathbb{R}$, defined by

$$\omega_{\mathbb{M}}(t):=\sup_{p\in\mathbb{N}_0}\log\big(\frac{t^p}{M_p}\big),\quad t>0;\qquad \omega_{\mathbb{M}}(0)=0. \tag{1.4}$$

If $\mathbb{M}$ is a weight sequence, i.e., (lc) and such that $\mathbf{m}$ tends to infinity, we can show that $\omega_{\mathbb{M}}$ is a nondecreasing continuous map in $[0,\infty)$ with $\lim_{t\to\infty}\omega_{\mathbb{M}}(t)=\infty$. Indeed,

$$\omega_{\mathbb{M}}(t)=\begin{cases} p\log t-\log(M_p) & \text{if } t\in[m_{p-1},m_p),\ p=1,2,\dots,\\ 0 & \text{if } t\in[0,m_0).\end{cases} \tag{1.5}$$

and one can easily check that $\omega_{\mathbb{M}}$ is convex in $\log t$, i.e., the map $t\mapsto\omega_{\mathbb{M}}(e^t)$ is convex in $\mathbb{R}$. We also observe that

$$\omega_{\mathbb{M}}(m_p)=\log\left(\frac{m_p^p}{M_p}\right),\qquad p\in\mathbb{N}_0. \tag{1.6}$$

Alternatively, we can consider the map $h_{\mathbb{M}}:[0,\infty)\to\mathbb{R}$, given by

$$h_{\mathbb{M}}(t):=\inf_{p\in\mathbb{N}_0}M_pt^p=\exp\big(-\omega_{\mathbb{M}}(1/t)\big),\quad t>0;\qquad h_{\mathbb{M}}(0)=0,$$

which turns out to be, for a weight sequence, a nondecreasing continuous map in $[0,\infty)$ onto $[0,1]$. In fact,

$$h_{\mathbb{M}}(t)=\begin{cases} M_pt^p & \text{if } t\in\big[\frac{1}{m_p},\frac{1}{m_{p-1}}\big),\ p=1,2,\dots,\\ 1 & \text{if } t\geq 1/m_0.\end{cases}$$

If $\mathbb{L}$ is another weight sequence such that $\mathbb{L}\approx\mathbb{M}$, it is straightforward to check that there exist $A,B>0$ such that

$$\omega_{\mathbb{M}}(At)\leq\omega_{\mathbb{L}}(t)\leq\omega_{\mathbb{M}}(Bt),\qquad t\geq 0, \tag{1.7}$$

or, equivalently, $H,L>0$ such that

$$h_{\mathbb{M}}(Lt)\leq h_{\mathbb{L}}(t)\leq h_{\mathbb{M}}(Ht),\qquad t\geq 0.$$

Example 1.1.22. The following information with respect to the associated function $\omega_{\mathbb{M}}(t)$ for the sequences appearing in Example 1.1.4 can be given.

(i) We recall that $\mathbb{M}_{\alpha,\beta}=\big(p!^{\alpha}\prod_{m=0}^{p}\log^{\beta}(e+m)\big)_{p\in\mathbb{N}_0}$ and we have that there exist positive constants A,B such that:

$$At^{1/\alpha}\log(t)^{\beta/\alpha}\leq\omega_{\mathbb{M}_{\alpha,\beta}}(t)\leq Bt^{1/\alpha}\log(t)^{\beta/\alpha},\qquad t \text{ large enough}$$

(see [98, Example 1.2.2]). In case $\beta=0$, i.e., for the Gevrey sequence of order α, $At^{1/\alpha}\leq\omega_{\mathbb{M}_{\alpha}}(t)\leq Bt^{1/\alpha}$ for t large enough.

(ii) For $q>1$, $\mathbb{M}_q=(q^{p^2})_{p\in\mathbb{N}_0}$ we can show that there exist positive constants A,B such that:

$$A\log(t)^2\leq\omega_{\mathbb{M}_q}(t)\leq B\log(t)^2,\qquad t \text{ large enough}$$

(see [17, Example 21]).

One may consider the *logarithmically convex minorant sequence* $\mathbb{M}^{(lc)}$ of a sequence $\mathbb{M}$, that is, the (lc) sequence such that $M_p^{(lc)} \le M_p$ for all $p \in \mathbb{N}_0$, and for every other (lc) sequence $\mathbb{L}$ with $L_p \le M_p$ for every $p \in \mathbb{N}_0$ we have that $L_p \le M_p^{(lc)}$ for all $p \in \mathbb{N}_0$. The associated function is related to this minorant in the following sense.

Proposition 1.1.23 ([72] p. 17 and [52] Prop. 3.2.)**.** Let $\mathbb{M}$ be a sequence with $\liminf M_p^{1/p} > 0$. Then, we have that $M_p^{(lc)} = \sup_{t>0} t^p/e^{\omega_{\mathbb{M}}(t)}$ for all $p \in \mathbb{N}_0$. Consequently, $\mathbb{M}$ is (lc) if and only if

$$M_p = \sup_{t>0} \frac{t^p}{e^{\omega_{\mathbb{M}}(t)}} = \sup_{t>0} t^p h_{\mathbb{M}}(1/t), \qquad p \in \mathbb{N}_0.$$

In particular, this representation is valid for weight sequences.

Some of the conditions for sequences in Section 1.1.1 can be described in terms of the associated function, see Subsection 2.1.4. In particular, the next characterization of (mg) condition plays a fundamental role in many of our arguments, it already appears in the work of H. Komatsu [52, Prop. 3.6] and of V. Thilliez [95].

Lemma 1.1.24. Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. The following are equivalent:

(i) $\mathbb{M}$ has (mg),

(ii) For every real number with $s \ge 1$, there exists $\rho(s) \ge 1$ (only depending on s and $\mathbb{M}$) such that
$$h_{\mathbb{M}}(t) \le (h_{\mathbb{M}}(\rho(s)t))^s \qquad \text{for } t \ge 0,$$
or, equivalently, that
$$s\omega_{\mathbb{M}}(t) \le \omega_{\mathbb{M}}(\rho(s)t) \qquad \text{for } t \ge 0. \tag{1.8}$$

(iii) There exist $H \ge 1$ and $t_0 > 0$ (only depending on $\mathbb{M}$) such that
$$h_{\mathbb{M}}(t) \le (h_{\mathbb{M}}(Ht))^2 \qquad \text{for } t \le 1/t_0,$$
or, equivalently, that
$$2\omega_{\mathbb{M}}(t) \le \omega_{\mathbb{M}}(Ht) \qquad \text{for } t \ge t_0.$$

Proof. (i) $\Rightarrow$ (ii) Given $s \ge 1$ we take $k \in \mathbb{N}$ such that $k > s$. By (mg) condition, there exists $A > 0$, depending on k and $\mathbb{M}$ (see Remark 1.1.10), such that $M_{kp} \le A^{kp} M_p^k$ for every $p \in \mathbb{N}_0$. We deduce that

$$h_{\mathbb{M}}(t) = \inf_{p\in\mathbb{N}_0} t^p M_p \le \inf_{p\in\mathbb{N}_0} t^{kp} M_{kp} \le \inf_{p\in\mathbb{N}_0} (tA)^{kp} M_p^k = (h_{\mathbb{M}}(At))^k,$$

for every $t \ge 0$. Since $h_{\mathbb{M}}(t) \in [0,1]$, $(h_{\mathbb{M}}(At))^k \le (h_{\mathbb{M}}(At))^s$ for all $t \ge 0$, then (ii) is satisfied with $\rho(s) = A$.
(ii) $\Rightarrow$ (iii) Immediate.
(iii) $\Rightarrow$ (i) Since $\mathbb{M}$ is (lc), applying Proposition 1.1.23, we see that $M_p = \sup_{t>0}(t^p h_{\mathbb{M}}(1/t))$ for all $p \in \mathbb{N}_0$. By (iii), we obtain that

$$\begin{aligned} M_{2p} &= \sup_{t>0} t^{2p} h_{\mathbb{M}}(1/t) = \max\Big(\sup_{0<t<t_0}(t^{2p}h_{\mathbb{M}}(1/t)), \sup_{t\ge t_0}(t^{2p}h_{\mathbb{M}}(1/t))\Big) \\ &\le \max\Big(t_0^{2p}, \sup_{t\ge t_0}(t^{2p}h_{\mathbb{M}}(H/t)^2)\Big) \le \max(t_0^{2p}, H^{2p}M_p^2) \end{aligned}$$

for all $p \in \mathbb{N}_0$. Since $M_p \ge 1$, for p large enough, we can choose $A \ge 1$ such that $M_{2p} \le A^{2p} M_p^2$ for every $p \in \mathbb{N}_0$. Then, by Lemma 1.1.9, we deduce that $\mathbb{M}$ satisfies (mg).

□

1.1.4 Growth indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$

The growth index $\gamma(\mathbb{M})$ was defined and considered by V. Thilliez [95, Sect. 1.3] in the study of ultraholomorphic classes of functions. The original definition was given for strongly regular sequences and $\gamma > 0$, but one can consider it for any sequence $\mathbb{M}$ and $\gamma \in \mathbb{R}$.

Definition 1.1.25. Let $\mathbb{M}$ be a sequence and $\gamma \in \mathbb{R}$. We say $\mathbb{M}$ satisfies property (P_γ) if there exists a sequence of real numbers $\boldsymbol{\ell} = (\ell_p)_{p\in\mathbb{N}_0}$ such that:

(i) $\mathbf{m} \simeq \boldsymbol{\ell}$, that is, there is a constant $a \geq 1$ such that $a^{-1}m_p \leq \ell_p \leq am_p$, for all $p \in \mathbb{N}_0$,

(ii) $((p+1)^{-\gamma}\ell_p)_{p\in\mathbb{N}_0}$ is nondecreasing.

If (P_γ) is satisfied, then $(P_{\gamma'})$ is satisfied for $\gamma' \leq \gamma$. It is natural to consider its *growth index* $\gamma(\mathbb{M})$ defined by

$$\gamma(\mathbb{M}) := \sup\{\gamma \in \mathbb{R} : (P_\gamma) \text{ is fulfilled}\}.$$

Remark 1.1.26. Thanks to the property described above, we are allowed to use the classical conventions $\inf\emptyset = \sup\mathbb{R} = \infty$ and $\inf\mathbb{R} = \sup\emptyset = -\infty$.

For the study of the injectivity of the asymptotic Borel map (see Section 3.2 for further details), J. Sanz [88] defined the *growth index* $\omega(\mathbb{M})$.

Definition 1.1.27. Let $\mathbb{M}$ be a sequence. We define its index $\omega(\mathbb{M})$ by

$$\omega(\mathbb{M}) := \liminf_{p\to\infty} \frac{\log m_p}{\log p}.$$

By definition, the value of $\gamma(\mathbb{M})$ and of $\omega(\mathbb{M})$ is stable for $\simeq$. For weight sequences with (mg), in particular for strongly regular sequences, these values are also stable for $\approx$, thanks to the equivalence between $\approx$ and $\simeq$ (see Proposition 1.1.20). In Section 2.1, we will eventually show the stability under $\approx$ for arbitrary weight sequences.

Regarding the relation between $\omega(\mathbb{M})$ and $\gamma(\mathbb{M})$, J. Sanz [88, Prop. 3.7], using the properties of $\gamma(\mathbb{M})$ described in [95, Sect. 1.3], stated the following result for strongly regular sequences, which also holds for any sequence.

Proposition 1.1.28. For every sequence $\mathbb{M}$ one has $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$.

Proof. If $\gamma < \gamma(\mathbb{M})$, then $\mathbb{M}$ satisfies (P_γ). By (P_γ), for every $p \in \mathbb{N}_0$, we observe that

$$m_0 \leq a\ell_0 \leq a\frac{\ell_p}{(p+1)^\gamma} \leq a^2\frac{m_p}{(p+1)^\gamma}.$$

Consequently, $a^{-2}m_0(p+1)^\gamma \leq m_p$ for every $p \in \mathbb{N}_0$ and we deduce that

$$\omega(\mathbb{M}) = \liminf_{p\to\infty} \frac{\log m_p}{\log p} \geq \liminf_{p\to\infty} \frac{\log(a^{-2}m_0(p+1)^\gamma)}{\log p} = \gamma,$$

and we conclude that $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$. □

The regularity of the sequence entails properties for the corresponding indices.

Lemma 1.1.29. If $\mathbb{M}$ is (lc), then $\gamma(\mathbb{M}), \omega(\mathbb{M}) \in [0, \infty]$.

Proof. Since $\mathbf{m}$ is nondecreasing, (P_0) is satisfied then $\gamma(\mathbb{M}) \geq 0$ and we also have

$$\frac{\log m_p}{\log p} \geq \frac{\log m_1}{\log p}$$

for all $p \in \mathbb{N}$ then $\omega(\mathbb{M}) \geq 0$. □

From this last lemma, we deduce also that for (lc) sequences the original definition of $\gamma(\mathbb{M})$ given by V. Thilliez, where the supremum is taken only for $\gamma > 0$, coincides with the general one considered in this subsection. In [88, 95], it has been shown that if $\mathbb{M}$ is strongly regular then $\gamma(\mathbb{M}), \omega(\mathbb{M}) \in (0, \infty)$ (see also Remark 2.1.19). However, there are sequences that are not strongly regular such that $\gamma(\mathbb{M}), \omega(\mathbb{M}) \in (0, \infty)$ (see Remark 2.2.27). These properties of the indices will be obtained in Section 2.1, where the relation between O-regularly varying sequences and (lc) sequences is presented, as an easy consequence of Theorem 2.1.16.

Example 1.1.30. For the sequences appearing in Example 1.1.4, one may prove (see Example 2.1.20) that

(i) For $\alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$ we have that $\gamma(\mathbb{M}_{\alpha,\beta}) = \omega(\mathbb{M}_{\alpha,\beta}) = \alpha$.

(ii) For $(q^{p^2})_{p\in\mathbb{N}_0}$ with $q > 1$, $\gamma((q^{p^2})_{p\in\mathbb{N}_0}) = \omega((q^{p^2})_{p\in\mathbb{N}_0}) = \infty$.

Most of the classical examples of strongly regular sequences satisfy that $\omega(\mathbb{M}) = \gamma(\mathbb{M})$. Moreover, in Section 2.2 we will show that the values of the indices coincide for a large class of sequences. However, it is possible to construct a strongly regular sequence for which the values are different, arbitrarily chosen, positive real numbers (see Example 2.2.26).

1.2 Regular variation, O-regular variation and proximate orders

In the next chapter, the relations between the sequences, the associated functions and the notions of regular variation, O-regular variation and proximate orders will be studied. This section is devoted to the description of these concepts and their fundamental properties.

1.2.1 Regularly varying functions

First, we will recall the notion of regular variation introduced in 1930 by J. Karamata ([49, 50]), although partial treatments may be found in the works of E. Landau [56], G. Valiron [102], G. Pólya [79] and others (see the historical survey [14]). Several applications of this concept have been shown in analytic number theory, complex analysis and, specially, in probability. The proofs of most of the results in this subsection are gathered in the books of E. Seneta [92] and N. H. Bingham, C. M. Goldie and J. L. Teugels [13].

Definition 1.2.1. A measurable function $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, is *regularly varying* if

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0, \infty), \tag{1.9}$$

for every $\lambda \in (0, \infty)$.

There are three main results regarding the notion of regular variation, the continuous version of these results is due to J. Karamata [49] and the measurable one was given by J. Korevaar, T. van Aardenne-Ehrenfest and N.G. de Bruijn [55].

Theorem 1.2.2 ([13], Th. 1.4.1, Characterization Theorem)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. If f is regularly varying, then there exists $\rho \in \mathbb{R}$ such that the function $g(\lambda)$ in (1.9) is equal to λ^ρ.

In this case, ρ is called the *index of regular variation of* f, we write $f \in R_\rho$ and $RV := \cup_{\rho>0} R_\rho$.

If $\rho = 0$, then f is said to be *slowly varying*. We have that $f \in R_\rho$ if and only if $f(x) = x^\rho \ell(x)$ for some $\ell \in R_0$.

Consequently, the behavior of a regularly varying function at ∞ is in some sense similar to the behavior of a power-like function. For $n \in \mathbb{N}$, $\log_n x$ denotes the $n-$th iteration of the logarithm. Given $n_i \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}$ for $i = 0, 1 \dots, k$, the classical example of a regularly varying function is

$$f(x) = x^{\alpha_0} (\log_{n_1} x)^{\alpha_1} (\log_{n_2} x)^{\alpha_2} \cdots (\log_{n_k} x)^{\alpha_k}.$$

Theorem 1.2.3 ([13], Th. 1.5.2, Uniform Convergence Theorem)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. If $f \in R_\rho$, then for every $b_1, b_2 \in (0,\infty)$ with $b_1 \leq b_2$ we have that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho$$

uniformly for $\lambda \in [b_1, b_2]$.

Theorem 1.2.4 ([13], Th. 1.3.1 and Th. 1.4.1, Representation Theorem)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. Then $f \in R_\rho$ if and only if there exist $A \geq a$ and measurable and bounded functions $c, \eta : [A,\infty) \to \mathbb{R}$ with $\lim_{x\to\infty} c(x) = c \in \mathbb{R}$ and $\lim_{x\to\infty} \eta(x) = \rho \in \mathbb{R}$ such that

$$f(x) = \exp\left(c(x) + \int_A^x \eta(u) \frac{du}{u} \right) \qquad x \geq A.$$

The value of A is unimportant because f, c and η can be redefined in finite intervals preserving the regular variation, then A can be chosen equal to 0, 1 or a as appropriate. Moreover, this representation is not unique because, for instance, one may take

$$\tilde{c}(x) = c(x) + \frac{1}{x} - \frac{1}{A}, \qquad \tilde{\eta}(x) = \eta(x) + \frac{1}{x}.$$

Remark 1.2.5. As an immediate consequence of the Representation Theorem, if $f \in R_\rho$ we deduce that

$$\rho = \lim_{x\to\infty} \eta(x) = \lim_{x\to\infty} \frac{\log f(x)}{\log x}.$$

Finally, we can see that regular variation is preserved for the classical equivalence.

Remark 1.2.6. Let $f, g : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be measurable functions. Assume that f and g are *equivalent, in the classical sense, at* ∞, that is,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

we write $f \sim g$. It is plain to check that $f \in R_\rho$ if and only if $g \in R_\rho$.

1.2.2 Proximate orders and smooth variation

In order to study quasianalyticity of the ultraholomorphic classes (see Section 3.2) and to construct kernels of summability in this context (see Section 4.1) we need to introduce the notion of proximate order, appearing in the theory of growth of entire functions and developed, among others, by G. Valiron [103], B. Ja. Levin [63] and A. A. Goldberg and I. V. Ostrosvkii [32]. In this dissertation, a prominent role is played by the results of L. S. Maergoiz [65]. The concept of proximate order, its elementary properties, its relation to regular variation and the main results of L. S. Maergoiz are presented in this subsection.

Definition 1.2.7. We say a real function $\rho(t)$, defined on (c,∞) for some $c\geq 0$, is a *proximate order*, if the following hold:

(A) ρ is continuous and piecewise continuously differentiable in (c,∞) (meaning that it is differentiable except possibly at a sequence of points, tending to infinity, at any of which it is continuous and has finite but distinct lateral derivatives),

(B) $\rho(t)\geq 0$ for every $t>c$,

(C) $\lim_{t\to\infty}\rho(t)=\rho<\infty$,

(D) $\lim_{t\to\infty} t\rho'(t)\log t=0$.

In case the value ρ in (C) is positive (respectively, is 0), we say $\rho(t)$ is a *nonzero* (resp. *zero*) proximate order.

Remark 1.2.8. If $\rho(t)$ is a proximate order with limit ρ at infinity, for every $\varepsilon>0$ there exists $t_\varepsilon>1$ such that

$$t^{\rho-\varepsilon}<t^{\rho(t)}<t^{\rho+\varepsilon},\qquad t>t_\varepsilon.$$

Definition 1.2.9. Two proximate orders $\rho_1(t)$ and $\rho_2(t)$ are said to be *equivalent* if

$$\lim_{t\to\infty}[\rho_1(t)-\rho_2(t)]\log t=0.$$

For the functions $V_1(t)=t^{\rho_1(t)}$ and $V_2(t)=t^{\rho_2(t)}$, this precisely means that

$$\lim_{t\to\infty}\frac{V_1(t)}{V_2(t)}=\lim_{t\to\infty}\frac{t^{\rho_1(t)}}{t^{\rho_2(t)}}=1,$$

that is, $V_1\sim V_2$.

Remark 1.2.10. If $\rho_1(t)$ and $\rho_2(t)$ are equivalent and $\lim_{t\to\infty}\rho_1(t)=\rho$, then

$$\lim_{t\to\infty}\rho_2(t)=\rho.$$

Example 1.2.11. The following are examples of proximate orders, defined in suitable intervals (c,∞):

(i) $\rho_{\alpha,\beta}(t)=\dfrac{1}{\alpha}-\dfrac{\beta}{\alpha}\dfrac{\log(\log t)}{\log t}$, $\alpha>0$, $\beta\in\mathbb{R}$.

(ii) $\rho(t)=\rho+\dfrac{1}{t^{\gamma}}$, $\rho\geq 0$, $\gamma>0$.

(iii) $\rho(t)=\rho+\dfrac{1}{\log^{\gamma}(t)}$, $\rho\geq 0$, $\gamma>0$.

An example of a function verifying all the conditions except (D) is $\rho(t) = \rho + \sin(t)/t$.

There is a basic connection between regular variation and proximate orders.

Lemma 1.2.12 ([63], Sect. I.12, p.32). Let $\rho(t)$ be a proximate order with $\lim_{t\to\infty} \rho(t) = \rho$. Then, the function $V(t) = t^{\rho(t)} \in R_\rho$.

The relation is even stronger, that is, it is also possible to go from regularly varying functions of positive index to nonzero proximate orders. Even more general, we can associate with a regularly varying function a smooth function of the class defined below and considered by A. A. Balkema, J. L. Geluk and L. de Haan [4]. Several of the next results will not be used in the forthcoming sections and have been included to make the reader aware of the deep connection between proximate orders and regular variation.

Definition 1.2.13. A function $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, *varies smoothly* with *index* $\rho \in \mathbb{R}$, if $f \in \mathcal{C}^\infty((a, \infty))$ and the function $h(x) = \log(f(e^x))$ satisfies

$$\lim_{x\to\infty} h'(x) = \rho, \qquad \lim_{x\to\infty} h^{(n)}(x) = 0, \quad n \geq 2. \tag{1.10}$$

In this case, we write $f \in SR_\rho$.

If $f \in SR_\rho$, it turns out that the function $\eta(x) := h'(\log(x)) = xf'(x)/f(x)$ tends to ρ as x tend to ∞ and

$$f(x) = \exp\left(\log(f(a)) + \int_a^x \eta(u)\frac{du}{u}\right),$$

and by the Representation Theorem 1.2.4, we deduce that $f \in R_\rho$, i.e., $SR_\rho \subseteq R_\rho$. Furthermore, one may check (see [4, Lemma 9]) that (1.10) is equivalent to the fact that

$$\lim_{x\to\infty} \frac{x^n f^{(n)}(x)}{f(x)} = \rho(\rho - 1)\ldots(\rho - n + 1), \qquad n \in \mathbb{N}.$$

The interest of smoothly varying functions is that every function in R_ρ can be approximated by a function in SR_ρ.

Theorem 1.2.14 ([13], Th. 1.8.2). Let $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, be a measurable function. If $f \in R_\rho$, then there exist $f_1, f_2 \in SR_\rho$ with $f_1 \sim f_2$, equivalent in the classical sense, such that $f_1(x) \leq f(x) \leq f_2(x)$ for x large enough. In particular, if $f \in R_\rho$ there exists $g \in SR_\rho$ such that $g \sim f$.

We observe that if $f \in SR_\rho$, defining $\rho_f(x) := \log(f(x))/\log(x) \in \mathcal{C}^\infty$ for x large enough and, since $f \in R_\rho$, by Remark 1.2.5, we have that $\lim_{x\to\infty} \rho_f(x) = \rho$. Finally, we notice that

$$\lim_{x\to\infty} x\rho_f'(x)\log(x) = \lim_{x\to\infty} \left(\frac{xf'(x)}{f(x)} - \rho_f(x)\right) = 0.$$

Consequently, if $\rho > 0$, $\rho_f(x)$ is also positive for x large enough and we deduce that $\rho_f(x)$ is a proximate order. As an immediate consequence of the previous result and this construction, one may also approximate regularly varying functions by proximate orders.

Proposition 1.2.15 ([13], Prop. 7.4.1). Let $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, be a measurable function and $\rho > 0$. Then $f \in R_\rho$ if and only if there exists a proximate order $\rho(x)$ with $\lim_{t\to\infty} \rho(t) = \rho$ such that $f(x) \sim x^{\rho(x)}$.

If condition (B) is removed from the definition of proximate order the previous result is valid for $\rho \in \mathbb{R}$. Then, as it was mentioned in [13, p. 311], "it is a matter of indifference whether one uses the language of regular variation or of proximate orders. Incidentally, from an historical point of view it seems that Valiron may well be credited with initiating the subject of regular variation."

Moreover, condition (A) is imposed for essentially traditional reasons, because Lemma 1.2.12 and Theorem 1.2.14 show, up to asymptotic equivalence, that smoothness for $\rho(x)$ can be assumed. In the same direction, one may study if stronger regularity conditions for $\rho(x)$ could be guaranteed. In particular, G. Valiron [102] showed that, always up to asymptotic equivalence, the function $x^{\rho(x)}$ has an analytic continuation to a sector in the complex plane containing the positive real axis (see [13, Th. 7.4.3]). For our purposes, we will use the extension constructed by L.S. Maergoiz in Theorem 1.2.16 below. For an arbitrary sector bisected by the positive real axis, it provides holomorphic functions whose restriction to $(0,\infty)$ is real, has a growth at infinity specified by a prescribed proximate and satisfies several regularity properties.

In order to state this result, we need to consider unbounded sectors of the Riemann surface of the logarithm $\mathcal{R}$

$$S(d,\gamma) := \{z \in \mathcal{R} : |\arg(z) - d| < \frac{\gamma\pi}{2}\},$$

with *bisecting direction* $d \in \mathbb{R}$ and *opening* $\gamma\pi$ ($\gamma > 0$). If $d = 0$, we write $S_\gamma := S(0,\gamma)$.

Theorem 1.2.16 ([65], Th. 2.4)**.** Let $\rho(t)$ be a nonzero proximate order and $\rho = \lim_{t\to\infty} \rho(t)$. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:

(I) For every $z \in S_\gamma$,

$$\lim_{t\to\infty} \frac{V(zt)}{V(t)} = z^\rho,$$

uniformly in the compact sets of S_γ.

(II) $\overline{V(z)} = V(\overline{z})$ for every $z \in S_\gamma$ (where, for $z = (|z|, \arg(z))$, we put $\overline{z} = (|z|, -\arg(z))$).

(III) $V(t)$ is positive in $(0,\infty)$, strictly increasing and $\lim_{t\to 0} V(t) = 0$.

(IV) The function $r \in \mathbb{R} \mapsto V(e^r)$ is strictly convex (i.e. V is strictly convex relative to $\log(r)$).

(V) The function $\log(V(t))$ is strictly concave in $(0,\infty)$.

(VI) The function $\rho_V(t) := \log(V(t))/\log(t)$, $t > 0$, is a proximate order equivalent to $\rho(t)$, that is,

$$\lim_{t\to\infty} V(t)/t^{\rho(t)} = \lim_{t\to\infty} t^{\rho_V(t)}/t^{\rho(t)} = 1.$$

This result motivates the following definition.

Definition 1.2.17. Given $\gamma > 0$ and $\rho(t)$ a nonzero proximate order, $MF(\gamma,\rho(t))$ denotes the set of Maergoiz functions V defined in S_γ and satisfying the conditions (I)-(VI) of Theorem 1.2.16.

Remark 1.2.18. Suppose $\rho(t)$ ($t \geq c \geq 0$) is a nonzero proximate order. Then the function $V(t) = t^{\rho(t)}$ is strictly increasing for $t > R$, where R is large enough. The inverse function $t = U(s)$, defined for every $s > V(R)$, has the property that the function $\rho^*(s) := \log(U(s))/\log(s)$ is a proximate order and $\rho^*(s)$ tends to $1/\rho$ as s tends to ∞ (see [65, Property 1.8]). This $\rho^*(s)$ is called the *proximate order conjugate to* $\rho(t)$. Note that, by Lemma 1.2.12, the function U is regularly varying.

This conjugate proximate order can be also extended, up to equivalence, to an analytic function.

Theorem 1.2.19 ([65], Th. 2.6)**.** Let $\rho(t)$ be a nonzero proximate order, $\gamma > 0$ and $V \in MF(\gamma,\rho(t))$. Let $t = U(s)$, defined for all $s > 0$, be the function inverse to $s = V(t)$, for every $t \in (0,\infty)$, and let $\rho^*(s)$ be the proximate order conjugate to $\rho(t)$. Then $\ln U(s)/\ln s$ is a proximate order equivalent to $\rho^*(s)$, and the function $U(s)$ admits an analytic continuation to a function $U(W)$ in a domain $T \subseteq S_{\rho\gamma}$ symetric relative to the real axis and such that for $\beta < \gamma$ there exists $R_\beta > 0$ such that the domain T contains $S_{\rho\beta} \cap \{|z| > R_\beta\}$. Furthermore, the function U verifies, in its domain, the properties (I)-(VI) in Theorem 1.2.16 of the functions of the class $MF(\rho\gamma, \rho^*(s))$.

In Section 2.2, the possibility of associating with a weight sequence a nonzero proximate order, consequently also a Maergoiz function, is characterized. These functions are used in the study of the injectivity of the Borel map in Section 3.2 are an essential part of the asymptotic problems considered in Chapter 4. For the classical problem that motivates the introduction of the notion of proximate order, that is, the study of the growth of entire functions, the reader is referred to [63, Sect. I.12], [32, Section 2.2] and [13, Sect. 7.4], in the last one the solution is stated through regular variation (see also Theorem 3.2.12).

1.2.3 O-regularly varying functions

For some of our purposes, the theory of regular variation is too restrictive and one may ask what remains valid if we replace lim by lim sup and lim inf in (1.9). This extension of the class of regularly varying functions, was defined by J. Karamata [51], V. G. Avakumović [3] and considered by W. Matuszewska [74] and W. Feller [28]. Although we refer to the book of N. H. Bingham, C. M. Goldie, and J. L. Teugels [13] for the proofs, the first complete study was done by S. Aljančić and I. D. Arandjelović [1] in 1977. In this subsection, it is shown that this weaker notion preserves several desirable properties.

Definition 1.2.20. A measurable function $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, is *O-regularly varying* if

$$0 < f_{\mathrm{low}}(\lambda) := \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)} \leq f^{\mathrm{up}}(\lambda) := \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} < \infty \tag{1.11}$$

for every $\lambda \geq 1$, and we write $f \in ORV$.

Remark 1.2.21. We observe that $f_{\mathrm{low}}(\lambda) = 1/f^{\mathrm{up}}(1/\lambda)$ for every $\lambda \geq 1$. Consequently, if $f \in ORV$, then (1.11) holds for every $\lambda \in (0,\infty)$ and we deduce that $RV \subseteq ORV$. Moreover, $f \in ORV$ if and only if

$$f^{\mathrm{up}}(\lambda) = \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} < \infty$$

for every $\lambda \in (0,\infty)$.

In this general context, the index ρ of regular variation is split into two values, the Matuszewska indices, defined for any positive function.

Definition 1.2.22. For a function $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, its *upper Matuszewska index* $\alpha(f)$ is defined by

$$\alpha(f) := \inf\left\{\alpha \in \mathbb{R};\ \exists C_\alpha > 0 \text{ s.t. } \forall \Lambda > 1,\ \limsup_{x\to\infty} \sup_{\lambda\in[1,\Lambda]} \frac{f(\lambda x)}{\lambda^\alpha f(x)} \leq C_\alpha \right\}$$

and its *lower Matuszewska index* $\beta(f)$ by

$$\beta(f) := \sup\left\{\beta \in \mathbb{R};\ \exists D_\beta > 0 \text{ s.t. } \forall \Lambda > 1,\ \liminf_{x\to\infty} \inf_{\lambda\in[1,\Lambda]} \frac{f(\lambda x)}{\lambda^\beta f(x)} \geq D_\beta, \right\}$$

with the conventions in Remark 1.1.26.

We always have that $\beta(f) \leq \alpha(f)$. The finiteness of these indices characterizes O-regular variation and the analogous version of the three main theorems of regular variation is available.

Theorem 1.2.23 ([13], Th. 2.1.7, Characterization Theorem for ORV)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. Then

$$f \text{ is O-regularly varying if and only if } \beta(f) > -\infty \text{ and } \alpha(f) < \infty.$$

Theorem 1.2.24 ([13], Th. 2.0.7, Uniform Convergence Theorem for ORV)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. If $f \in ORV$, then for every $\Lambda > 1$ we have that

$$0 < \liminf_{x\to\infty} \inf_{\lambda\in[1,\Lambda]} \frac{f(\lambda x)}{f(x)} \leq \limsup_{x\to\infty} \sup_{\lambda\in[1,\Lambda]} \frac{f(\lambda x)}{f(x)} < \infty.$$

Theorem 1.2.25 ([13], Th. 2.2.7, Representation Theorem for ORV)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. Then $f \in ORV$ if and only if there exist $A \geq a$ and measurable and bounded functions $d, \xi : [A,\infty) \to \mathbb{R}$ such that

$$f(x) = \exp\left(d(x) + \int_A^x \xi(u)\frac{du}{u}\right), \qquad x \geq A. \tag{1.12}$$

This representation is not unique and for every α, β satisfying $\beta < \beta(f) \leq \alpha(f) < \alpha$, representations exist with the function ξ taking values only in $[\beta, \alpha]$.

We can give several alternative definitions of the indices $\alpha(f)$ and $\beta(f)$.

Theorem 1.2.26 ([13], Th. 2.1.5 , Coro. 2.1.6 and Th. 2.1.7)**.** Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a measurable function. If $\beta(f) > -\infty$ or $\alpha(f) < \infty$, then

$$\alpha(f) = \lim_{\lambda\to\infty} \frac{\log f^{\text{up}}(\lambda)}{\log \lambda} = \inf_{\lambda>1} \frac{\log f^{\text{up}}(\lambda)}{\log \lambda},$$

$$\beta(f) = \lim_{\lambda\to\infty} \frac{\log f_{\text{low}}(\lambda)}{\log \lambda} = \sup_{\lambda>1} \frac{\log f_{\text{low}}(\lambda)}{\log \lambda}.$$

Please note that, in particular, last theorem states that if $\beta(f) > -\infty$ the formula for $\alpha(f)$ holds, and the one for $\beta(f)$ is valid if $\alpha(f) < \infty$. For instance, if f is nondecreasing $\beta(f) \geq 0$ and both formulas hold.

The second equivalent definition is a concise characterization in terms of almost increasing and almost decreasing properties and it is valid for any positive function.

Definition 1.2.27. Let $f : [a,\infty) \to (0,\infty)$, with $a \geq 0$, be a function. We say that f is *almost increasing* if there exists $M > 0$ such that

$$f(x) \leq Mf(y), \qquad y \geq x \geq a,$$

and f is said to be *almost decreasing* if there exists $m > 0$ such that

$$f(x) \geq mf(y), \qquad y \geq x \geq a.$$

Theorem 1.2.28 ([13], Th. 2.2.2)**.** Let $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, be a function. Then

$$\alpha(f) = \inf\{\alpha \in \mathbb{R};\ x^{-\alpha} f(x) \text{ almost decreasing}\},$$

$$\beta(f) = \sup\{\beta \in \mathbb{R};\ x^{-\beta} f(x) \text{ almost increasing}\}.$$

Finally, the third alternative definition is a consequence of the Representation Theorem for ORV.

Theorem 1.2.29 ([1], Th. 3)**.** Let $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, be a measurable function. If $f \in ORV$, then

$$\alpha(f) = \inf_{\xi}\{\limsup_{x\to\infty} \xi(x)\},$$

$$\beta(f) = \sup_{\xi}\{\liminf_{x\to\infty} \xi(x)\}.$$

where the sup and inf are taken over all measurable and bounded functions ξ for which there exists d measurable and bounded such that (1.12) holds.

Other definitions of α and β are available if one assumes that f is O-regularly varying and locally integrable on $[a, \infty)$ (see [1]).

These indices can be compared to the classical upper and lower order of a function.

Definition 1.2.30. For a function $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, its *upper order* $\rho(f)$ (often shorted to *order*) and *lower order* $\mu(f)$ are defined by

$$\mu(f) := \liminf_{x\to\infty} \frac{\log f(x)}{\log x}, \qquad \rho(f) := \limsup_{x\to\infty} \frac{\log f(x)}{\log x}.$$

It is possible to define a class of functions, that contains ORV, attending to the finiteness of these orders. In the recent work of M. Cadena, M. Kratz and E. Omey [21], a generalization of the main theorems of regular variation and O-regular variation has been shown for this class.

Similarly to Remark 1.2.6, O-regular variation has some stability property.

Remark 1.2.31. If $f, g : [a, \infty) \to (0, \infty)$, with $a \geq 0$, are measurable functions with

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty, \tag{1.13}$$

it is plain to check that $\beta(f) = \beta(g)$, $\mu(f) = \mu(g)$, $\rho(f) = \rho(g)$ and $\alpha(f) = \alpha(g)$. Consequently, $f \in ORV$ if and only if $g \in ORV$. In particular, if $f \in RV$ then $g \in ORV$ and $\beta(g) = \mu(g) = \rho(g) = \alpha(g)$.

Proposition 1.2.32 ([13], Prop. 2.2.5)**.** Let $f : [a, \infty) \to (0, \infty)$, with $a \geq 0$, be a function. The orders and Matuszewska indices of f are related by

$$\beta(f) \leq \mu(f) \leq \rho(f) \leq \alpha(f).$$

As an easy consequence of the Representation Theorem 1.2.4, if $f \in R_\rho$, then $\beta(f) = \mu(f) = \rho(f) = \alpha(f) = \rho$. However, the converse is not true as it shows the next example.

Example 1.2.33 ([13], Prop. 2.2.8)**.** The function $f : [1, \infty) \to (0, \infty)$ given by

$$f(x) = \begin{cases} f(e^{2^j}) \exp((\log(x) - 2^j)^{1/2}) & \text{if} \quad x \in (e^{2^j}, e^{2^{j+1}}],\ j = 0, 1, 2, \ldots, \\ 1 & \text{if} \quad x \in [1, e]. \end{cases}$$

satisfies the following properties:

(i) f is nondecreasing and continuous,

(ii) $1 \le f_{\text{low}}(\lambda) \le f^{\text{up}}(\lambda) = \exp((\log(\lambda))^{1/2})$ for every $\lambda > 1$,

(iii) $f \in ORV$ and $\beta(f) = \mu(f) = \rho(f) = \alpha(f) = 0$,

(iv) There do not exist $A \ge a$ and measurable and bounded functions $d, \xi : [A, \infty) \to \mathbb{R}$ with $\lim_{x\to\infty} \xi(x) = 0$ such that

$$f(x) = \exp\left(d(x) + \int_A^x \xi(u)\frac{du}{u}\right), \qquad x \ge A.$$

Consequently, by Theorem 1.2.4, f is not RV and it does not exists $g \in RV$ such that (1.13) holds. Regarding Theorem 1.2.25 and Theorem 1.2.29 it shows that neither $\beta(f)$ nor $\alpha(f)$ is in general attainable for the representation.

We refer the reader to the cited proposition for the proof.

In Section 2.1, the relation between the orders, Matuszewska indices and O-regular variation with growth properties for sequences is studied. In this context, this final result, comparing the O-regular variation of a function and its derivative, will be used to connect the associated function and the counting function of a weight sequence $\mathbb{M}$ (see Subsection 2.1.4).

Theorem 1.2.34 ([13], Th. 2.6.1, Coro. 2.6.2)**.** Let $f : [X, \infty) \to (0, \infty)$ be a locally integrable function. We define $F(x) := \int_X^x f(t)/t dt$. Then,

(i) if $\alpha(f) < \infty$, then $\limsup_{x\to\infty} f(x)/F(x) < \infty$.

(ii) if $\beta(f) > 0$, then $\liminf_{x\to\infty} f(x)/F(x) > 0$.

(iii) we have that $\alpha(F) \le \limsup_{x\to\infty} f(x)/F(x)$.

(iv) we have that $\beta(F) \ge \liminf_{x\to\infty} f(x)/F(x)$.

Moreover, we have that

$$0 < \liminf_{x\to\infty} \frac{f(x)}{F(x)} \le \limsup_{x\to\infty} \frac{f(x)}{F(x)} < \infty.$$

if and only if $\alpha(f) < \infty$ and $\beta(f) > 0$. In this case, $\alpha(F) = \alpha(f)$ and $\beta(F) = \beta(f)$.

1.2.4 Regularly varying sequences

In 1973, R. Bojanić and E. Seneta [16] show that, under a suitable adaptation, one may consider regularly varying sequences satisfying similar properties to the ones of regularly varying functions. Even if all the results in this subsection, except the last one, were shown by R. Bojanić and E. Seneta, we refer to [13] for the proofs, as in the previous sections. In the next chapter, this notion will be used to characterize those sequences which can be attached to a proximate order. This characterization will be given in terms of the sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ of $\mathbb{M}$, then the notation in this subsection has been chosen according to the considerations in Remark 1.1.3.

Definition 1.2.35. A sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ of positive numbers is said to be *regularly varying* if

$$\lim_{p\to\infty} \frac{a_{\lfloor \lambda p \rfloor}}{a_p} = g(\lambda) \in (0, \infty), \tag{1.14}$$

for every $\lambda \in (0, \infty)$.

Theorem 1.2.36 ([13], Th. 1.9.5, Characterization Theorem for regularly varying sequences)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a regularly varying sequence of positive numbers. Then there exists $\rho \in \mathbb{R}$ such that the function $g(\lambda)$ in (1.14) is equal to λ^ρ.

In this case, ρ is called the *index of regular variation* of $\boldsymbol{a}$.

The next theorem makes possible to apply the results about the theory of regularly varying functions to regularly varying sequences.

Theorem 1.2.37 ([13], Th. 1.9.5)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. The sequence $\boldsymbol{a}$ is regularly varying if and only if the function $f_{\boldsymbol{a}}(x) = a_{\lfloor x\rfloor}$ for $x \geq 1$ is regularly varying.

From this embedding result, we deduce that the convergence of the limit in (1.14) is uniform in the compact sets of $(0,\infty)$ and we see that regularly varying sequences also admit a very convenient representation.

Theorem 1.2.38 ([13], Th. 1.9.7, Representation Theorem for regularly varying sequences)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a regularly varying sequence of positive numbers of index ρ. There exist sequences of real numbers $(c_p)_{p\in\mathbb{N}}$ and $(\eta_p)_{p\in\mathbb{N}}$, converging to $c \in \mathbb{R}$ and ρ, respectively, such that

$$a_p = \exp\left(c_p + \sum_{j=1}^{p} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}.$$

Conversely, such a representation for a sequence $(a_p)_{p\in\mathbb{N}}$ implies that it is regularly varying of index ρ.

As it happens for functions, this notion is stable for the classical equivalence.

Remark 1.2.39. Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ and $\boldsymbol{b} = (b_p)_{p\in\mathbb{N}}$ be sequences of positive numbers. Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *equivalent in the classical sense* , that is,

$$\lim_{p\to\infty} \frac{a_p}{b_p} = 1,$$

we write $\boldsymbol{a} \sim \boldsymbol{b}$. It is plain to check that $\boldsymbol{a}$ is a regularly varying sequence of index ρ if and only if $\boldsymbol{b}$ also is. Please note that if $\boldsymbol{a} \sim \boldsymbol{b}$, then there exists $c > 0$ such that $c^{-1}b_p \leq a_p \leq cb_p$ for every $p \in \mathbb{N}$, that is, $\boldsymbol{a} \simeq \boldsymbol{b}$.

In Section 2.2, we will need to deal with sequences defined for $p \in \mathbb{N}_0$. As it is shown below, there is no problem with this approach since regular variation is stable for index shifts.

Lemma 1.2.40 ([13], Lemma 1.9.6.)**.** If $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ is a regularly varying sequence, then

$$\lim_{p\to\infty} a_{p+1}/a_p = 1.$$

Consequently, the sequence $\boldsymbol{a}$ is regularly varying of index ρ if and only if the shifted sequence $\boldsymbol{s_a} := (s_p = a_{p+1})_{p\in\mathbb{N}}$ is regularly varying of index ρ.

The next theorem is the discrete version of Proposition 1.2.15. We construct a 'smooth' sequence $\boldsymbol{b} = (b_p)_{p\in\mathbb{N}}$ from a regularly varying sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$, where the condition $\lim_{x\to\infty} xf'(x)/f(x) = \rho$ is replaced by

$$\lim_{p\to\infty} \frac{p(b_{p+1} - b_p)}{b_p} = \rho.$$

Theorem 1.2.41 ([13], Th. 1.9.8)**.** Let $\rho \in \mathbb{R}$ be given. A sequence of positive real numbers $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ is regularly varying with index ρ if and only if it exists a sequence of positive real numbers $\boldsymbol{b} = (b_p)_{p\in\mathbb{N}}$ such that

(i) $\lim_{p\to\infty} b_p/a_p = 1$, i.e., $\boldsymbol{a} \sim \boldsymbol{b}$.

(ii)
$$\frac{b_{p+1}}{b_p} = 1 + \frac{\omega}{p} + o\left(\frac{1}{p}\right) \qquad \text{as } p \to \infty, \tag{1.15}$$

and we know by Remark 1.2.39 that $\boldsymbol{b}$ is also a regularly varying sequence of index ρ.

Example 2.2.21 at the end of Chapter 2 shows that (1.15) does not hold in general for regularly varying sequences.

Finally, we should mention that the theory of regularly varying sequences becomes much simpler if one considers only monotone sequences. The following theorem of L. de Haan [33] shows that if we have monotonicity, we only need to prove (1.14) for two suitable integer values of λ.

Theorem 1.2.42 ([33], Th. 1.1.2)**.** A monotone sequence of positive real numbers $(a_p)_{p\in\mathbb{N}_0}$ is regularly varying if there exist positive integers $\ell_1, \ell_2 \geq 2$ with $\log(\ell_1)/\log(\ell_2)$ irrational such that for some real number ρ,
$$\lim_{p\to\infty} \frac{a_{\ell_j p}}{a_p} = \ell_j^\rho, \qquad j = 1, 2.$$

This property is not true if the monotonicity hypothesis is removed as it has been proved by J. Galambos and E. Seneta [30]. As it will be shown in Section 2.2, this is specially useful when dealing with the quotient sequence $\mathbf{m}$ of a (lc) sequence $\mathbb{M}$.

1.2.5 O-regularly varying sequences

The extension of the notion of O-regular variation for sequences was stated by S. Aljančić [2] and detailed by D. Djurčić and V. Božin [25] in 1997. We introduce the basic elements of this concept that will be required in the next chapter. For the definition, the characterization of O-regularly varying functions given in Remark 1.2.21 has been taken into account.

Definition 1.2.43. A sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ of positive numbers is said to be *O-regularly varying* if
$$\limsup_{p\to\infty} \frac{a_{\lfloor \lambda p \rfloor}}{a_p} < \infty,$$
for every $\lambda \in (0, \infty)$.

Note that if $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ is regularly varying then it is O-regularly varying. As for regular variation, O-regularly varying sequences are embeddable as O-regularly varying step function.

Theorem 1.2.44 ([25], Th. 1)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. The sequence $\boldsymbol{a}$ is O-regularly varying if and only if the function $f_{\boldsymbol{a}}(x) = a_{\lfloor x \rfloor}$ for $x \geq 1$ is O-regularly varying.

From this result, we obtain the Uniform Convergence and Representation Theorems.

Theorem 1.2.45 ([25], Th. 2, Uniform Convergence Theorem for O-regularly varying sequences)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be an O-regularly varying sequence of positive numbers and $a, b \in (0, \infty)$ with $a \leq b$. Then
$$\limsup_{p\to\infty} \sup_{\lambda\in[a,b]} \frac{a_{\lfloor \lambda p \rfloor}}{a_p} < \infty.$$

Theorem 1.2.46 ([25], Th. 3, Representation Theorem for O-regularly varying sequences)**.** Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be an O-regularly varying sequence of positive numbers. Then there exist bounded sequences of real numbers $(d_p)_{p\in\mathbb{N}}$ and $(\xi_p)_{p\in\mathbb{N}}$ such that

$$a_p = \exp\left(d_p + \sum_{j=1}^{p} \frac{\xi_j}{j}\right), \quad p \in \mathbb{N}.$$

Conversely, such a representation for a sequence $(a_p)_{p\in\mathbb{N}}$ implies that it is O-regularly varying.

In Section 2.1, the relation between weight functions and ORV sequences is established. In that context, this Representation Theorem will play a key role in the construction of pathological examples (see Example 2.2.26).

One may notice the stability of the notion of O-regular variation for sequences under $\simeq$.

Remark 1.2.47. Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ and $\boldsymbol{b} = (b_p)_{p\in\mathbb{N}}$ be sequences of positive numbers with $\boldsymbol{a} \simeq \boldsymbol{b}$. It is plain to check that $\boldsymbol{a}$ is O-regularly varying sequence if and only if $\boldsymbol{b}$ also is.

Finally, it naturally arises the question about the possibility of considering the Matuszewska indices and the orders for sequences. This question was not posed by D. Djurčić and V. Božin and will be the main topic of Subsection 2.1.2.

Chapter 2

Logarithmically convex sequences, O-regular variation and nonzero proximate orders

The main objective of this chapter is to describe the connection between weight sequences and the notions of proximate orders, regular and O-regular variation. In the first section, it will be shown that the growth properties and indices of Section 1.1 can be represented in terms of upper and lower orders and the Matuszewska indices. In the second section, we will restrict to the study of those weight sequences which it is possible to associate a nonzero proximate order with, characterizing this crucial point for the success in putting forward a satisfactory summability theory in the general context.

2.1 Logarithmically convex sequences and O-regular variation

The results presented below revolve around the notion of O-regular variation. First, basic properties of weight sequences are described in different ways, for instance, in terms of almost monotonicity properties from which the connection with O-regular variation is inferred. Simultaneously in the second subsection, Matuszewska indices, upper and lower orders for sequences are formalized together with the proof of some distinctive features. In third place, the ingredients of the previous subsections will be combined, stating this way qualitative growth properties in terms of quantitative values, orders and indices, which are related to the (independently defined) indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ (see Subsection 1.1.4), measuring the opening of the regions for which the Borel map is surjective or injective, respectively (see Chapter 3). In subsection four, the interaction of the preceding concepts with the associated function $\omega_{\mathbb{M}}$, considered in Subsection 1.1.3, is illustrated. Finally, dual and bidual sequences are constructed giving a possible explanation for some open questions regarding the essence of indices and orders.

2.1.1 Strongly nonquasianalyticity and moderate growth characterizations

This subsection is primarily devoted to the study of (snq) and (mg) conditions. Characterizations of these properties will be obtained merging slightly improved versions of some classical results leading, in the forthcoming subsections, to the notion of O-regular variation. For this purpose, we need to introduce almost increasing and almost decreasing concepts for sequences, analogous to the ones for functions (see Definition 1.2.27).

Definition 2.1.1. Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. We say that $\boldsymbol{a}$ is *almost increasing* if there exists $M > 0$ such that

$$a_p \le M a_q, \qquad \text{for all} \quad p, q \in \mathbb{N} \quad \text{with} \quad q \ge p \ge 1,$$

and $\boldsymbol{a}$ is said to be *almost decreasing* if there exists $m > 0$ such that

$$a_p \ge m a_q, \qquad \text{for all} \quad p, q \in \mathbb{N} \quad \text{with} \quad q \ge p \ge 1.$$

We directly see that if $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ is nondecreasing (resp. nonincreasing) then it is almost increasing (resp. almost decreasing) with $M = 1$ (resp. $m = 1$). It is also plain to check that these notions are stable for $\simeq$.

The first result characterizes (snq) property in five different forms. It is obtained by the combination of results of R. Meise and B. A. Taylor [75, Prop. 1.3], K. N. Bari and S. B. Stečkin [12, Lemma 2], H.-J. Petszche [77, Prop. 1.1, Coro. 1.3.(a)] and S. Tikhonov [99, Lemma 4.5]. The proof is included because some implications are not a direct application of their theorems, for instance some of them were proved for functions and in others it is assumed that the sequence $\mathbb{M}$ is (lc), which is not necessary.

Proposition 2.1.2. Let $\mathbb{M}$ be a sequence such that the sequence $\widehat{\mathbb{M}}$, given by $\widehat{M_p} := p!M_p$, $p \in \mathbb{N}_0$, is logarithmically convex. Then, the following statements are equivalent:

(i) the sequence $\mathbb{M}$ satisfies (snq), that is, there exists $B > 0$ such that

$$\sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} \le \frac{B}{m_p}, \qquad p \in \mathbb{N}_0,$$

(ii) there exists a logarithmically convex sequence $\mathbb{H}$ such that $\mathbf{h} \simeq \mathbf{m}$ and

$$\inf_{p\ge1} \frac{h_{2p}}{h_p} > 1,$$

(iii) we have that $\lim_{k\to\infty} \liminf_{p\to\infty} \dfrac{m_{kp}}{m_p} = \infty$,

(iv) there exists $k \in \mathbb{N}$, $k \ge 2$, such that

$$\liminf_{p\to\infty} \frac{m_{kp}}{m_p} > 1, \tag{2.1}$$

(v) there exists $\varepsilon > 0$ such that $(m_p/p^{\varepsilon})_{p\in\mathbb{N}}$ is almost increasing,

(vi) for every $\theta \in (0,1)$ there exists $k \in \mathbb{N}$, $k \ge 2$, such that for every $p \in \mathbb{N}$ we have that $m_p \le \theta\, m_{kp}$.

Proof. (i)⇒(ii) For every $p \in \mathbb{N}_0$, we define the sequence

$$t_p := \frac{1}{m_p} + \sum_{\ell\ge p} \frac{1}{(\ell+1)m_\ell}.$$

Since $\widehat{\mathbf{m}} = (\widehat{m}_p = (p+1)m_p)_{p\in\mathbb{N}_0}$ is nondecreasing (see Lemma 1.1.7.(i)), the sequence $(t_p)_{p\in\mathbb{N}_0}$ is nonincreasing because

$$\begin{aligned} t_p &= \frac{1}{m_p} + \frac{1}{(p+1)m_p} + \sum_{\ell=p+1}^{\infty} \frac{1}{(\ell+1)m_\ell} = \frac{p+2}{\widehat{m}_p} + \sum_{\ell=p+1}^{\infty} \frac{1}{(\ell+1)m_\ell} \\ &\geq \frac{p+2}{\widehat{m}_{p+1}} + \sum_{\ell=p+1}^{\infty} \frac{1}{(\ell+1)m_\ell} = t_{p+1}, \qquad \text{for all} \quad p\in\mathbb{N}_0. \end{aligned}$$

Using that $\mathbb{M}$ satisfies (i), i.e., (snq), for every $p\in\mathbb{N}_0$ we observe that

$$\begin{aligned} m_p t_p &= 1 + m_p \sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} \leq 1 + B, \\ m_p t_p &= 1 + m_p \sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} \geq 1 + 0. \end{aligned}$$

Hence there exists $C := 1 + B > 1$ such that $C^{-1}m_p \leq (t_p)^{-1} \leq m_p$ for each $p\in\mathbb{N}_0$ and we define $b_p := 1/t_p$ for every $p\in\mathbb{N}_0$. With the conventions in Remark 1.1.3, we have that $\mathbf{b} \simeq \mathbf{m}$, then, by Proposition 1.1.18, $\mathbb{B}$ satisfies (snq) and, since $(t_p)_{p\in\mathbb{N}_0}$ is nonincreasing, we have that $\mathbb{B}$ is (lc). For every $p\in\mathbb{N}_0$, we consider

$$s_p := \frac{1}{b_p} + \sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)b_\ell}.$$

Since $\mathbb{B}$ is (lc), we have that $\widehat{\boldsymbol{b}} = (\widehat{b}_p := (p+1)b_p)_{p\in\mathbb{N}_0}$ is nondecreasing and, as above, we see that $(s_p)_{p\in\mathbb{N}_0}$ is nonincreasing. We define $h_p := 1/s_p$ for every $p\in\mathbb{N}_0$, then $\mathbb{H}$ is (lc). Since $\mathbb{B}$ satisfies (snq), proceeding as before, we see that $\mathbf{h} \simeq \mathbf{b}$ and, consequently, $\mathbf{h} \simeq \mathbf{m}$. Moreover, since $\boldsymbol{b}$ is nondecreasing, for every $p\in\mathbb{N}$ we observe that

$$\frac{h_{2p}}{h_p} = \frac{1/s_{2p}}{1/s_p} = \frac{(1/b_p) + \sum_{\ell\geq p}(1/(\ell+1)b_\ell)}{(1/b_{2p}) + \sum_{\ell\geq 2p}(1/(\ell+1)b_\ell)} \geq 1 + \frac{\sum_{\ell=p}^{2p-1}(1/(\ell+1)b_\ell)}{(1/b_{2p}) + \sum_{\ell\geq 2p}(1/(\ell+1)b_\ell)}.$$

Applying again that $\boldsymbol{b}$ is nondecreasing, for every $p\in\mathbb{N}$ we see that

$$\frac{h_{2p}}{h_p} \geq 1 + \frac{p/(2pb_{2p})}{s_{2p}} = 1 + \frac{1}{2b_{2p}s_{2p}}.$$

Since $\mathbb{B}$ satisfies (snq), as we previously did, we can show that there exists $D > 0$ such that for every $p\in\mathbb{N}$ we have $b_p s_p \leq D$ and we conclude that

$$\frac{h_{2p}}{h_p} \geq 1 + \frac{1}{2D}.$$

(ii)⇒(iii) First we will show that

$$\lim_{k\to\infty} \liminf_{p\to\infty} \frac{h_{kp}}{h_p} = \infty. \tag{2.2}$$

By (ii), there exists $\gamma > 1$ such that $h_{2p}/h_p > \gamma$ for all $p\in\mathbb{N}$ and we deduce that $h_{2^n p}/h_p > \gamma^n$ for every $p, n\in\mathbb{N}$. Therefore, for every $n\in\mathbb{N}$ we have that

$$\liminf_{p\to\infty} \frac{h_{2^n p}}{h_p} \geq \gamma^n.$$

Given $M>0$, there exists $n_0\in\mathbb{N}$ such that $\gamma^{n_0}>M$. Using that $\mathbb{H}$ is (lc) (by (ii)), for every $k\geq 2^{n_0}$ we see that $h_{kp}\geq h_{2^{n_0}p}$ for every $p\in\mathbb{N}$ and we deduce that

$$\liminf_{p\to\infty}\frac{h_{kp}}{h_p}\geq\liminf_{p\to\infty}\frac{h_{2^{n_0}p}}{h_p}\geq\gamma^{n_0}>M,$$

i.e., (2.2) is valid. By (ii), $\mathbf{h}\simeq\mathbf{m}$, that is, there exists $c>1$ such that $c^{-1}h_p\leq m_p\leq ch_p$ for every $p\in\mathbb{N}_0$. Then, for every $k,p\in\mathbb{N}$ we have that

$$\frac{h_{kp}}{c^2\,h_p}\leq\frac{m_{kp}}{m_p}.$$

Since (2.2) holds, we conclude that

$$\lim_{k\to\infty}\liminf_{p\to\infty}\frac{m_{kp}}{m_p}=\infty.$$

(iii)⇒(iv) Immediate.
(iv)⇒(v) We deduce from (iv) that there exists $\varepsilon>0$ such that

$$\liminf_{p\to\infty}m_{kp}/m_p>k^{\varepsilon}.$$

Then, there exists $p_0\in\mathbb{N}$ such that $m_{kp}/k^{\varepsilon}>m_p$ for every $p\geq p_0$. For every $q,\ell\in\mathbb{N}$ with $q\geq\ell\geq p_0$ there exists $n\in\mathbb{N}_0$ such that $k^n\ell\leq q<k^{n+1}\ell$, so we have that

$$\frac{m_\ell}{\ell^\varepsilon}\leq\frac{m_{k\ell}}{(k\ell)^\varepsilon}\leq\frac{m_{k^n\ell}}{(k^n\ell)^\varepsilon}.$$

Since $\widehat{\mathbf{m}}$ is nondecreasing, for $q\geq\ell\geq p_0$ with $k^n\ell\leq q<k^{n+1}\ell$ we deduce that

$$\frac{m_\ell}{\ell^\varepsilon}\leq\frac{m_{k^n\ell}}{(k^n\ell)^\varepsilon}\leq\frac{(q+1)m_q}{(k^n\ell+1)(k^n\ell)^\varepsilon}=\frac{m_q}{q^\varepsilon}\,\frac{(q+1)q^\varepsilon}{(k^n\ell+1)(k^n\ell)^\varepsilon}\leq\frac{m_q}{q^\varepsilon}k^\varepsilon k.$$

We denote by

$$A:=\max_{1\leq\ell\leq q\leq p_0}\frac{m_\ell}{m_q}\frac{q^\varepsilon}{\ell^\varepsilon}>1,\qquad C:=Ak^\varepsilon k.$$

If $q\geq\ell\geq p_0$ or if $\ell\leq q\leq p_0$, we see that $m_\ell q^\varepsilon\leq C\,m_q\ell^\varepsilon$, and if $\ell\leq p_0\leq q$ we observe that

$$\frac{m_\ell}{\ell^\varepsilon}\leq A\frac{m_{p_0}}{p_0^\varepsilon}\leq C\frac{m_q}{q^\varepsilon},$$

and we conclude that $(m_p/p^\varepsilon)_{p\in\mathbb{N}}$ is almost increasing.
(v)⇒(vi) Since $(m_p/p^\varepsilon)_{p\in\mathbb{N}}$ is almost increasing, there exists $C>1$ such that for every $q,p\in\mathbb{N}$ with $q\geq p$ we have that

$$\frac{m_p}{p^\varepsilon}\leq C\frac{m_q}{q^\varepsilon}.$$

We fix $\theta\in(0,1)$, we take $k\in\mathbb{N}$ such that $k>(C/\theta)^{1/\varepsilon}$ with $k\geq 2$ and for every $p\in\mathbb{N}$ we have that

$$m_p\leq C\frac{m_{kp}}{k^\varepsilon}<\theta\,m_{kp}.$$

(vi)⇒ (i) For $\theta=1/2$ there exists $k\in\mathbb{N}$, $k\geq 2$, such that $2m_p\leq m_{kp}$ for every $n,p\in\mathbb{N}$ we deduce that

$$2^n m_p\leq m_{k^n p}.\tag{2.3}$$

Since $\widehat{\mathbf{m}}$ is nondecreasing, for every $p \in \mathbb{N}$ we have that

$$\sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} = \sum_{n=0}^{\infty} \sum_{\ell=k^n p}^{k^{n+1}p-1} \frac{1}{(\ell+1)m_\ell} \leq \sum_{n=0}^{\infty} \frac{k^n p(k-1)}{(k^n p+1)m_{k^n p}} \leq (k-1)\sum_{n=0}^{\infty} \frac{1}{m_{k^n p}}.$$

Applying (2.3), for all $p \in \mathbb{N}$ we deduce that

$$\sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} \leq \frac{k-1}{m_p} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{2(k-1)}{m_p}.$$

It only remains the case $p = 0$, by taking $B = \max(2(k-1), 1+2(k-1)m_0/m_1)$, we deduce that

$$\sum_{\ell=p}^{\infty} \frac{1}{(\ell+1)m_\ell} \leq \frac{B}{m_p}$$

for every $p \in \mathbb{N}_0$. □

If $\mathbb{M}$ is (lc), $\widehat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ is also (lc). However, the opposite is not true in general. The ultradifferentiable classes of functions are frequently defined only assuming that $\widehat{\mathbb{M}}$ is (lc), for further details see Remark 3.1.11. This weaker condition for $\mathbb{M}$ will be considered along this dissertation in order to be able to bring our results and the ones in the ultradifferentiable setting together.

Applying Lemma 1.1.9, we can give a characterization of (mg) condition in terms of the sequence of quotients of a sequence $\mathbb{M}$ such that $\widehat{\mathbb{M}}$ is (lc).

Lemma 2.1.3. Let $\mathbb{M}$ be a sequence such that the sequence $\widehat{\mathbb{M}}$ is (lc). The following are equivalent:

(i) $\mathbb{M}$ has (mg),

(ii) $\displaystyle\sup_{p\in\mathbb{N}_0} \frac{m_{2p}}{m_p} < \infty$,

(iii) there exists $\gamma > 0$ and $k \in \mathbb{N}$, $k \geq 2$, such that

$$\limsup_{p\to\infty} \frac{m_{kp}}{m_p} < k^\gamma. \tag{2.4}$$

Proof. (i)⇒(ii) Since $\mathbb{M}$ has (mg) and $(p+q)! \leq 2^{p+q}p!q!$ for all $p, q \in \mathbb{N}_0$ we observe that

$$\widehat{M}_{p+q} = (p+q)!M_{p+q} \leq 2^{p+q}p!q!A^{p+q}M_pM_q = (2A)^{p+q}\widehat{M}_q\widehat{M}_p,$$

that is, $\widehat{\mathbb{M}}$ also is (mg). Using that $\widehat{\mathbb{M}}$ is (lc) and Lemma 1.1.9, we obtain that $\sup_{p\in\mathbb{N}_0} \widehat{m}_{2p}/\widehat{m}_p$ is finite and we conclude that

$$\sup_{p\in\mathbb{N}_0} \frac{m_{2p}}{m_p} = \sup_{p\in\mathbb{N}_0} \frac{\widehat{m}_{2p}}{\widehat{m}_p}\frac{p+1}{2p+1} \leq \sup_{p\in\mathbb{N}_0} \frac{\widehat{m}_{2p}}{\widehat{m}_p} < \infty.$$

(ii)⇒(iii) Immediate, taking $k = 2$ and $\gamma > \log(\sup_{p\in\mathbb{N}_0}(m_{2p}/m_p))/\log(2)$.
(iii)⇒(i) From (iii), it follows that $\sup_{p\in\mathbb{N}_0}(m_{kp}/m_p)$ is finite. Since $\widehat{\mathbb{M}}$ is (lc), we see that

$$\sup_{p\in\mathbb{N}_0} \frac{\widehat{m}_{2p}}{\widehat{m}_p} \leq \sup_{p\in\mathbb{N}_0} \frac{\widehat{m}_{kp}}{\widehat{m}_p} = \sup_{p\in\mathbb{N}_0} \frac{m_{kp}}{m_p}\frac{kp+1}{p+1} < \infty.$$

Hence, we can apply Lemma 1.1.9 to the (lc) sequence $\widehat{\mathbb{M}}$ and we deduce that $\widehat{\mathbb{M}}$ has (mg). Since $(p+q)! \geq p!q!$ we conclude that $\mathbb{M}$ has also (mg). □

Following the ideas in K. N. Bari and S. B. Stečkin [12, Lemma 3], we can give five equivalent conditions, analogous to the ones for (snq) that, according to the last lemma condition (iii), are tightly connected to (mg). The limiting value -1 appears because $\widehat{\mathbb{M}}$ is (lc), under weaker conditions a more general version of this result might be given in the same direction of the auxiliary Lemma 2.1.21 that extends Proposition 2.1.2.

Proposition 2.1.4. Let $\mathbb{M}$ be a sequence such that the sequence $\widehat{\mathbb{M}}$ is (lc). For every $\gamma > -1$, the following statements are equivalent:

(i) there exists $B > 0$ such that

$$\sum_{\ell=0}^{p} \frac{(\ell+1)^\gamma}{(\ell+1)m_\ell} \leq \frac{B(p+1)^\gamma}{m_p} \qquad \text{for all} \quad p \in \mathbb{N}_0,$$

(ii) there exists a sequence $\mathbb{H}$ such that $((p+1)^{-\gamma}h_p)_{p\in\mathbb{N}_0}$ is nonincreasing, $\mathbf{h} \simeq \mathbf{m}$ and

$$\sup_{p\geq 1} \frac{h_{2p}}{h_p} < 2^\gamma, \tag{2.5}$$

(iii) we have that $\lim_{k\to\infty} \limsup_{p\to\infty} \dfrac{m_{kp}}{k^\gamma m_p} = 0$,

(iv) there exists $k \in \mathbb{N}$, $k \geq 2$, such that

$$\limsup_{p\to\infty} \frac{m_{kp}}{m_p} < k^\gamma,$$

(v) there exists $\varepsilon > 0$ such that $(m_p/p^{\gamma-\varepsilon})_{p\in\mathbb{N}}$ is almost decreasing,

(vi) For every $\theta \in (0,1)$ there exists $k \in \mathbb{N}$, $k \geq 2$, such that for every $p \in \mathbb{N}$ we have that $m_{kp} < \theta k^\gamma m_p$.

Proof. (i)$\Rightarrow$(ii) First, we consider the auxiliary sequence $(\alpha_p)_{p\in\mathbb{N}_0}$ given by

$$\alpha_p := \frac{(p+2)^{\gamma+1} - (p+1)^{\gamma+1}}{(p+2)^\gamma}, \qquad p \in \mathbb{N}_0.$$

We observe that $\lim_{p\to\infty} \alpha_p = \gamma + 1 > 0$, we take $D > \max(2^{\gamma+1}(\gamma+1)/(2^{\gamma+1}-1), \sup_{p\in\mathbb{N}} \alpha_p)$ and for every $p \in \mathbb{N}_0$ we define the sequence

$$t_p := \frac{1}{(p+1)^\gamma} \sum_{\ell=0}^{p} \frac{(\ell+1)^\gamma}{(\ell+1)m_\ell} - \frac{1}{Dm_p}.$$

Using that $\mathbf{m}$ satisfies (i), for every $p \in \mathbb{N}_0$ we have that

$$m_p t_p = \frac{m_p}{(p+1)^\gamma} \sum_{\ell=0}^{p} \frac{(\ell+1)^\gamma}{(\ell+1)m_\ell} - \frac{1}{D} \leq B,$$

and, since $\widehat{\mathbf{m}}$ is nondecreasing and $\gamma > -1$, for every $p \in \mathbb{N}$ we see that

$$\begin{aligned} m_p t_p &\geq \frac{1}{(p+1)^{\gamma+1}} \sum_{\ell=0}^{p} (\ell+1)^\gamma - \frac{1}{D} \geq \frac{1}{(p+1)^{\gamma+1}} \int_1^{p+1} x^\gamma dx - \frac{1}{D} \\ &\geq \frac{1}{(\gamma+1)} - \frac{1}{(\gamma+1)(p+1)^{\gamma+1}} - \frac{1}{D} \geq \frac{D(2^{\gamma+1}-1) - (\gamma+1)2^{\gamma+1}}{(\gamma+1)2^{\gamma+1}D} > 0. \end{aligned}$$

Consequently, if $b_p := 1/t_p$ for every $p \in \mathbb{N}_0$, we have that $\mathbf{b} \simeq \mathbf{m}$. It follows that $\boldsymbol{b}$ also satisfies (i) for a constant $C \geq B$. We will verify that $((p+1)^{-\gamma} b_p)_{p\in\mathbb{N}_0}$ is nonincreasing or, equivalently, that $((p+1)^{\gamma} t_p)_{p\in\mathbb{N}_0}$ is nondecreasing. Since $\widehat{\mathbf{m}}$ is nondecreasing, for all $p \in \mathbb{N}_0$

$$(p+1)^{\gamma} t_p = \sum_{\ell=0}^{p} \frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} - \frac{(p+1)^{\gamma}}{Dm_p} \leq \sum_{\ell=0}^{p+1} \frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} - \frac{(p+2)^{\gamma}}{(p+2)m_{p+1}} - \frac{(p+1)^{\gamma}(p+1)}{D(p+2)m_{p+1}}.$$

For all $p \in \mathbb{N}_0$, by the definition of D, $D \geq \alpha_p$ which leads to $(p+2)^{\gamma} D + (p+1)^{\gamma+1} \geq (p+2)^{\gamma+1}$ and we deduce that

$$(p+1)^{\gamma} t_p \leq \sum_{\ell=0}^{p+1} \frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} - \frac{(p+2)^{\gamma} D + (p+1)^{\gamma+1}}{D(p+2)m_{p+1}} \leq (p+2)^{\gamma} t_{p+1},$$

as desired. From the sequence $\mathbf{b}$, for every $p \in \mathbb{N}_0$ we construct

$$s_p := \frac{1}{(p+1)^{\gamma}} \sum_{\ell=0}^{p} \frac{(\ell+1)^{\gamma}}{(\ell+1)b_\ell} + \frac{1}{b_p}.$$

Since $\mathbf{b}$ also satisfies (i), for every $p \in \mathbb{N}_0$ we observe that

$$1 \leq b_p s_p = 1 + \frac{b_p}{(p+1)^{\gamma}} \sum_{\ell=0}^{p} \frac{(\ell+1)^{\gamma}}{(\ell+1)b_\ell} \leq 1 + C,$$

then if $h_p := 1/s_p$ for every $p \in \mathbb{N}_0$, we have that $\boldsymbol{h} \simeq \boldsymbol{b} \simeq \mathbf{m}$. Using that $((p+1)^{-\gamma} b_p)_{p\in\mathbb{N}_0}$ is nonincreasing, we notice that

$$\begin{aligned}(p+1)^{\gamma} s_p =& \sum_{\ell=0}^{p+1} \frac{(\ell+1)^{\gamma}}{(\ell+1)b_\ell} - \frac{(p+2)^{\gamma}}{(p+2)b_{p+1}} + \frac{(p+1)^{\gamma}}{b_p} \\ \leq& \sum_{\ell=0}^{p+1} \frac{(\ell+1)^{\gamma}}{(\ell+1)b_\ell} + \frac{(p+2)^{\gamma}}{b_{p+1}} \frac{(p+1)}{p+2} \leq (p+2)^{\gamma} s_{p+1}, \quad p \in \mathbb{N}_0\end{aligned}$$

Hence $((p+1)^{-\gamma} h_p)_{p\in\mathbb{N}_0}$ is nonincreasing. Finally, we verify that $\boldsymbol{h}$ satisfies (2.5). Applying again that $((p+1)^{-\gamma} b_p)_{p\in\mathbb{N}_0}$ is nonincreasing, we see that

$$\begin{aligned}\frac{h_{2p}}{h_p} =& \frac{1/s_{2p}}{1/s_p} = \frac{(p+1)^{-\gamma} \sum_{\ell=0}^{p} (\ell+1)^{\gamma-1}(b_\ell)^{-1} + (b_p)^{-1}}{(2p+1)^{-\gamma} \sum_{\ell=0}^{2p} (\ell+1)^{\gamma-1}(b_\ell)^{-1} + (b_{2p})^{-1}} \\ \leq& \frac{(2p+1)^{\gamma}}{(p+1)^{\gamma}} \left(1 - \frac{\sum_{\ell=p+1}^{2p} (\ell+1)^{\gamma-1}(b_\ell)^{-1}}{\sum_{\ell=0}^{2p} (\ell+1)^{\gamma-1}(b_\ell)^{-1} + (2p+1)^{\gamma}(b_{2p})^{-1}}\right).\end{aligned}$$

Once more, since $((p+1)^{-\gamma} b_p)_{p\in\mathbb{N}_0}$ is nonincreasing, for every $p \in \mathbb{N}$ we have that

$$\sum_{\ell=p+1}^{2p} \frac{(\ell+1)^{-1}}{(\ell+1)^{-\gamma}(b_\ell)} \geq \frac{(2p+1)^{-1} p}{(p+1)^{-\gamma}(b_p)},$$

and, by the definition of s_{2p}, we deduce that

$$\frac{h_{2p}}{h_p} \leq \frac{(2p+1)^{\gamma}}{(p+1)^{\gamma}} \left(1 - \frac{(p+1)^{\gamma} p}{b_p s_{2p} (2p+1)^{\gamma+1}}\right) \leq E\left(1 - \frac{1}{(b_p/b_{2p}) b_{2p} s_{2p} 3E}\right).$$

where $E = \max(1, 2^\gamma)$. Using that $\boldsymbol{b} \simeq \mathbf{m}$ and that $\widehat{\mathbf{m}}$ is nondecreasing, we know that there exists a constant $a \geq 1$ such that for all $p \in \mathbb{N}$

$$\frac{b_p}{b_{2p}} \leq a^2 \frac{m_p}{m_{2p}} \leq a^2 \frac{2p+1}{p+1} \leq 2a^2.$$

Finally, since $b_p s_p \leq 1 + C$ for every $p \in \mathbb{N}$ and $3E > 1$, we conclude that

$$\frac{h_{2p}}{h_p} \leq E\left(1 - \frac{1}{2a^2(1+C)3E}\right) < E.$$

(ii)⇒(iii) First, we will show that

$$\lim_{k\to\infty} \limsup_{p\to\infty} \frac{h_{kp}}{k^\gamma h_p} = 0. \tag{2.6}$$

By (ii), there exists $\varepsilon > 0$ such that $h_{2p}/h_p < 2^{\gamma-\varepsilon}$, so we deduce that $h_{2^n p}/h_p < 2^{(\gamma-\varepsilon)n}$ for every $p, n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ we have that

$$\limsup_{p\to\infty} \frac{h_{2^n p}}{h_p} \leq 2^{n(\gamma-\varepsilon)}.$$

Given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $(2^{-\varepsilon})^{n_0} \max(2^{-\gamma}, 2^\gamma) < \delta$. We take $k \in \mathbb{N}$, $k \geq 2^{n_0}$, there exists $n \geq n_0$ such that $2^n \leq k < 2^{n+1}$. Using that $((p+1)^{-\gamma} h_p)_{p\in\mathbb{N}_0}$ is nonincreasing, we see that

$$\frac{h_{kp}}{k^\gamma h_p} \leq \frac{h_{2^n p}(kp+1)^\gamma}{k^\gamma h_p (2^n p+1)^\gamma} \leq \frac{h_{2^n p}}{2^{n\gamma} h_p} \max(2^{-\gamma}, 2^\gamma).$$

Consequently, we deduce that

$$\limsup_{p\to\infty} \frac{h_{kp}}{k^\gamma h_p} \leq \limsup_{p\to\infty} \frac{h_{2^n p}}{2^{n\gamma} h_p} \max(2^{-\gamma}, 2^\gamma) < \delta, \qquad k \geq 2^{n_0},$$

so (2.6) is valid. By (ii), $\mathbf{h} \simeq \mathbf{m}$, that is, there exists $c > 1$ such that $c^{-1} h_p \leq m_p \leq c h_p$ for every $p \in \mathbb{N}_0$. Then, for every $k, p \in \mathbb{N}$ we have that

$$\frac{m_{kp}}{k^\gamma m_p} \leq \frac{c^2\, h_{kp}}{k^\gamma h_p}.$$

Since (2.6) holds, we conclude that $\lim_{k\to\infty} \limsup_{p\to\infty} m_{kp}/(k^\gamma m_p) = 0$.
(iii)⇒(iv) Immediate.
(iv)⇒(v) There exists $p_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $m_{kp} < k^{\gamma-\varepsilon} m_p$ for every $p \geq p_0$. By iterating this inequality, for every $j \in \mathbb{N}$ we obtain that

$$\frac{m_{k^j p}}{m_p} < k^{j(\gamma-\varepsilon)}, \qquad p \geq p_0.$$

For $q \geq p \geq p_0$, there exists $j \in \mathbb{N}_0$ such that $k^j p \leq q < k^{j+1} p$ and, using that $\widehat{\mathbf{m}}$ is nondecreasing, we see that

$$\frac{m_q}{q^{\gamma-\varepsilon}} \leq \frac{m_{k^{j+1}p}(k^{j+1}p+1)}{q^{\gamma-\varepsilon}(q+1)} \leq \frac{k^{(j+1)(\gamma-\varepsilon)}}{q^{\gamma-\varepsilon}} \frac{(k^{j+1}p+1)}{(q+1)} m_p \leq \max(k^{\gamma-\varepsilon}, 1) k \frac{m_p}{p^{\gamma-\varepsilon}}.$$

We define $D := \sup_{1\leq p\leq q\leq p_0} \{m_q p^{\gamma-\varepsilon} (q^{\gamma-\varepsilon} m_p)^{-1}\} \geq 1$ and $E := Dk \max(k^{\gamma-\varepsilon}, 1)$, it follows that

$$\frac{m_q}{q^{\gamma-\varepsilon}} \leq E \frac{m_p}{p^{\gamma-\varepsilon}}, \qquad q \geq p \geq 1.$$

(v)⇒(vi) Since $(m_p/p^{\gamma-\varepsilon})_{p\in\mathbb{N}}$ is almost decreasing, there exists $E>0$ such that for every $k,p\in\mathbb{N}$ we have that

$$\frac{m_{kp}}{(kp)^{\gamma-\varepsilon}} \leq E\frac{m_p}{p^{\gamma-\varepsilon}}.$$

Then for every $\theta\in(0,1)$ we take k large enough such that $E<\theta k^{\varepsilon}$. Consequently, for every $p\in\mathbb{N}$ we see that

$$m_{kp} \leq \frac{E}{k^{\varepsilon}}k^{\gamma}m_p < \theta k^{\gamma}m_p.$$

(vi)⇒(i) For $\theta\in(0,1)$ there exists $k\in\mathbb{N}$ with $k\geq 2$ such that for every $q\in\mathbb{N}$ we have that $m_{kq}<\theta k^{\gamma}\,m_q$. Then for any $s,j\in\mathbb{N}_0$ with $s\geq j$ we see that

$$m_{k^{s+1}} < (\theta k^{\gamma})^{s+1-j}m_{k^j}. \tag{2.7}$$

For all $p\in\mathbb{N}$ there exists $s\in\mathbb{N}_0$ such that $k^s\leq p<k^{s+1}$. Since $\widehat{\mathbf{m}}$ is nondecreasing, for all $p\in\mathbb{N}$ we have that

$$\begin{aligned}\sum_{\ell=1}^{p}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} &\leq \sum_{\ell=1}^{k^{s+1}-1}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} = \sum_{j=0}^{s}\sum_{\ell=k^j}^{k^{j+1}-1}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell}\\ &\leq \sum_{j=0}^{s}\frac{1}{(k^j+1)m_{k^j}}\sum_{\ell=k^j}^{k^{j+1}-1}(\ell+1)^{\gamma} \leq \sum_{j=0}^{s}\frac{k^j(k-1)(k^j)^{\gamma}\max(1,k^{\gamma})}{(k^j+1)m_{k^j}}.\end{aligned}$$

Then by (2.7), for every $p\in\mathbb{N}$ we get

$$\sum_{\ell=1}^{p}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} \leq \max(1,k^{\gamma})(k-1)\sum_{j=0}^{s}\frac{k^{j\gamma}(\theta k^{\gamma})^{s+1-j}}{m_{k^{s+1}}} \leq \frac{(k-1)\max(1,k^{\gamma})k^{(s+1)\gamma}}{m_{k^{s+1}}}\sum_{j=0}^{s}\theta^{s+1-j}.$$

Since $\widehat{\mathbf{m}}$ is nondecreasing and $k^s\leq p<k^{s+1}$, we deduce that

$$\sum_{\ell=1}^{p}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} \leq \frac{(k-1)\max(1,k^{2\gamma})(p+1)^{\gamma}}{(1-\theta)m_p}\frac{(k^{s+1}+1)}{(p+1)} \leq C\frac{(p+1)^{\gamma}}{m_p}.$$

with $C:=(k-1)\max(1,k^{2\gamma})k(1-\theta)^{-1}$. We need to add the term for $\ell=0$. By (2.7), we see that $m_{k^{s+1}}<k^{\gamma(s+1)}m_1$ for all $s\in\mathbb{N}_0$. As before, using that $\widehat{\mathbf{m}}$ is nondecreasing, for every $k^s\leq p<k^{s+1}$ we obtain that

$$\frac{1}{m_1} \leq \frac{k^{\gamma(s+1)}}{m_{k^{s+1}}} \leq \frac{k^{\gamma(s+1)}k}{m_p} \leq k\max(1,k^{\gamma})\frac{(p+1)^{\gamma}}{m_p}.$$

Taking $M=\max\{k\max(1,k^{\gamma}),C\}$, for all $p\in\mathbb{N}$ we conclude that

$$\sum_{\ell=0}^{p}\frac{(\ell+1)^{\gamma}}{(\ell+1)m_\ell} \leq \frac{1}{m_0}+M\frac{(p+1)^{\gamma}}{m_p} \leq \left(\frac{m_1}{m_0}+1\right)M\frac{(p+1)^{\gamma}}{m_p}.$$

Since for $p=0$ (i) trivially holds, we conclude that (i) is valid for every $p\in\mathbb{N}_0$. □

Remark 2.1.5. If condition (iv) in Proposition 2.1.4 holds for some $\gamma>-1$ then it exists $-1<\gamma'<\gamma$ such that (iv) is also true for γ'. Then conditions (i)-(vi) are also valid for γ' and the set of $\gamma>-1$ such that any of these conditions is satisfied is open.

Assume that (2.4) is satisfied for some $k_0 \in \mathbb{N}$. Then it is straightforward to check (2.4) is also satisfied, suitably enlarging the value of γ, for all $k \geq k_0$ applying the trivial estimation deduced for k_0^n, with $n \in \mathbb{N}$ such that $k_0^n \geq k$. Consequently, since the previous results can also be applied to a (lc) sequence $\mathbb{M}$, it is possible to characterize strong regularity of a sequence $\mathbb{M}$ in terms of the sequence of quotients using (2.1) and (2.4).

Corollary 2.1.6. Let $\mathbb{M}$ be a sequence of positive numbers with $M_0 = 1$. The following are equivalent:

(i) $\mathbb{M}$ is strongly regular,

(ii) $\mathbf{m}$ is nondecreasing and there exists $k \in \mathbb{N}$, $k \geq 2$, such that

$$1 < \liminf_{p\to\infty} \frac{m_{kp}}{m_p} \leq \limsup_{p\to\infty} \frac{m_{kp}}{m_p} < \infty.$$

This corollary points out the connection between these properties and the notion of O-regular variation (see Definition 1.2.20 and Definition 1.2.43). The study of this relation will be the main aim of the subsequent subsections.

Remark 2.1.7. Moreover, this characterization allows us to easily verify if a sequence is or not strongly regular. For example:

(i) We consider the sequences $\mathbb{M}_{\alpha,\beta} = (p!^\alpha \Pi_{m=0}^p \log^\beta(e+m))_{p\in\mathbb{N}_0}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$. Since $\mathbf{m} = ((p+1)^\alpha \log^\beta(e+p+1))_{p\in\mathbb{N}_0}$ for $\beta \geq 0$ is nondecreasing $\mathbb{M}_{\alpha,\beta}$ is (lc). For $\beta < 0$ since $m_p = (p+1)^\alpha \log^\beta(e+p+1)$ is eventually nondecreasing, we can modify the first terms according to Remark 1.1.19 and change the sequence for a (lc) one. We observe that

$$\lim_{p\to\infty} \frac{m_{2p}}{m_p} = 2^\alpha.$$

By Lemma 1.1.9, we deduce that $\mathbb{M}_{\alpha,\beta}$ has (mg) and, by Proposition 2.1.2, we have that $\mathbb{M}_{\alpha,\beta}$ is (snq).

(ii) For the sequence $\mathbb{M}_{0,\beta} = (\Pi_{m=0}^p \log^\beta(e+m))_{p\in\mathbb{N}_0}$ with $\beta > 0$, we also have that it is (lc) and we see that

$$\lim_{p\to\infty} \frac{m_{kp}}{m_p} = 1 \qquad \text{for all} \quad k \in \mathbb{N}, \quad k \geq 2.$$

By Lemma 1.1.9 we deduce that $\mathbb{M}_{0,\beta}$ has (mg) and, by Proposition 2.1.2, we have that $\mathbb{M}_{0,\beta}$ does not satisfy (snq).

(iii) Finally, $\mathbb{M} = (q^{p^2})_{p\in\mathbb{N}_0}$ with $q > 1$ is (lc) because $\mathbf{m} = (q^{2p+1})_{p\in\mathbb{N}_0}$ is nondecreasing. We have that

$$\lim_{p\to\infty} \frac{m_{kp}}{m_p} = \lim_{p\to\infty} q^{2p(k-1)} = \infty, \qquad \text{for every} \quad k \in \mathbb{N}, \quad k \geq 2.$$

By Lemma 1.1.9 we deduce that $\mathbb{M}$ has not (mg) and, by Proposition 2.1.2, we have that $\mathbb{M}$ is (snq).

2.1.2 Orders and Matuszewska indices for sequences

The work of D. Djurčić and V. Božin [25] deals with the definition of O-regularly varying sequences and the proof of the fundamental theorems (see Subsection 1.2.5). Even if some information can be inferred from their paper, to the best of our knowledge, the notions of orders and Matuszewska indices for sequences have not been considered. In this subsection, a possible formalization of these concepts is proposed, providing a simple description, analyzing their behavior under elementary sequence transformations and showing some stability properties.

The regular variation and the O-regular variation of a sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ is equivalent to the regular variation and, respectively, the O-regular variation of the function $f_{\boldsymbol{a}}(x) = a_{\lfloor x \rfloor}$ which suggests the next definition.

Definition 2.1.8. Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. We define its *upper Matuszewska index* $\alpha(\boldsymbol{a})$, its *lower Matuszewska index* $\beta(\boldsymbol{a})$, its *upper order* $\rho(\boldsymbol{a})$ and its *lower order* $\mu(\boldsymbol{a})$ by

$$\alpha(\boldsymbol{a}) := \alpha(f_{\boldsymbol{a}}), \quad \beta(\boldsymbol{a}) := \beta(f_{\boldsymbol{a}}), \quad \rho(\boldsymbol{a}) := \rho(f_{\boldsymbol{a}}), \quad \mu(\boldsymbol{a}) := \mu(f_{\boldsymbol{a}}),$$

where $f_{\boldsymbol{a}}(x) = a_{\lfloor x \rfloor}$ for all $x \geq 1$.

Remark 2.1.9. By Proposition 1.2.32 , it immediately follows that

$$\beta(\boldsymbol{a}) \leq \mu(\boldsymbol{a}) \leq \rho(\boldsymbol{a}) \leq \alpha(\boldsymbol{a}).$$

and, using Theorems 1.2.23 and 1.2.44, we see that $\boldsymbol{a}$ is O-regularly varying if and only if $\beta(\boldsymbol{a}) > -\infty$ and $\alpha(\boldsymbol{a}) < \infty$.

Thanks to the almost increasing and almost decreasing notions for sequences defined in the previous subsection, it is possible to skip the step function $f_{\boldsymbol{a}}$ and give a simple characterization of these indices and orders only in terms of the sequence $\boldsymbol{a}$.

Proposition 2.1.10. Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. We have that

$$\begin{aligned}
\alpha(\boldsymbol{a}) &= \inf\{\alpha \in \mathbb{R};\ (a_p/p^{\alpha})_{p\in\mathbb{N}} \text{ is almost decreasing}\},\\
\beta(\boldsymbol{a}) &= \sup\{\beta \in \mathbb{R};\ (a_p/p^{\beta})_{p\in\mathbb{N}} \text{ is almost increasing}\},\\
\rho(\boldsymbol{a}) &= \limsup_{p\to\infty} \frac{\log(a_p)}{\log(p)}, \qquad \mu(\boldsymbol{a}) = \liminf_{p\to\infty} \frac{\log(a_p)}{\log(p)}.
\end{aligned}$$

Proof. By Theorem 1.2.28, we have that

$$\alpha(\boldsymbol{a}) = \alpha(f_{\boldsymbol{a}}) = \inf\{\alpha \in \mathbb{R};\ x^{-\alpha} a_{\lfloor x \rfloor} \text{ almost decreasing}\},$$

$$\beta(\boldsymbol{a}) = \beta(f_{\boldsymbol{a}}) = \sup\{\beta \in \mathbb{R};\ x^{-\beta} a_{\lfloor x \rfloor} \text{ almost increasing}\}.$$

If $x^{-\gamma} a_{\lfloor x \rfloor}$ is almost decreasing or almost increasing, it follows immediately that $(p^{-\gamma}a_p)_{p\in\mathbb{N}}$ is almost decreasing or almost increasing, respectively. Conversely, if $(p^{-\gamma}a_p)_{p\in\mathbb{N}}$ is almost decreasing or almost increasing and we take any $y \geq x \geq 1$, then $\lfloor y \rfloor \geq \lfloor x \rfloor \geq 1$ and we have that

$$\frac{a_{\lfloor x \rfloor}}{x^{\gamma}} = \frac{(\lfloor x \rfloor)^{\gamma}}{x^{\gamma}} \frac{a_{\lfloor x \rfloor}}{(\lfloor x \rfloor)^{\gamma}} \geq m \frac{(\lfloor x \rfloor)^{\gamma}}{x^{\gamma}} \frac{a_{\lfloor y \rfloor}}{(\lfloor y \rfloor)^{\gamma}} \geq \frac{m}{2} \frac{a_{\lfloor y \rfloor}}{y^{\gamma}},$$

$$\frac{a_{\lfloor x \rfloor}}{x^{\gamma}} \leq \frac{a_{\lfloor x \rfloor}}{(\lfloor x \rfloor)^{\gamma}} \leq M \frac{a_{\lfloor y \rfloor}}{(\lfloor y \rfloor)^{\gamma}} = M \frac{y^{\gamma}}{(\lfloor y \rfloor)^{\gamma}} \frac{a_{\lfloor y \rfloor}}{y^{\gamma}} \leq 2M \frac{a_{\lfloor y \rfloor}}{y^{\gamma}},$$

where M and m are the positive constants of the almost monotonicity of $(p^{-\gamma}a_p)_{p\in\mathbb{N}}$. Consequently, the equality for the Matuszewska indices holds. On the other hand, we have that

$$\frac{\log(a_{\lfloor x\rfloor})}{\log(x)} \leq \frac{\log(a_{\lfloor x\rfloor})}{\log(\lfloor x\rfloor)} \leq \frac{\log(x)}{\log(\lfloor x\rfloor)}\frac{\log(a_{\lfloor x\rfloor})}{\log(x)}, \quad x\geq 1.$$

Since $\lim_{x\to\infty}\log(x)/\log(\lfloor x\rfloor)=1$, we conclude that

$$\rho(\boldsymbol{a}) = \limsup_{x\to\infty}\frac{\log(a_{\lfloor x\rfloor})}{\log(x)} = \limsup_{x\to\infty}\frac{\log(a_{\lfloor x\rfloor})}{\log(\lfloor x\rfloor)} = \limsup_{p\to\infty}\frac{\log(a_p)}{\log(p)},$$
$$\mu(\boldsymbol{a}) = \liminf_{x\to\infty}\frac{\log(a_{\lfloor x\rfloor})}{\log(x)} = \liminf_{x\to\infty}\frac{\log(a_{\lfloor x\rfloor})}{\log(\lfloor x\rfloor)} = \liminf_{p\to\infty}\frac{\log(a_p)}{\log(p)}.$$

□

When applying ramification arguments in the classes of functions defined in terms of a given sequence $\mathbb{M}$, transformations of $\mathbb{M}$ will appear. Using this last characterization result, the indices for the transforms and the original sequence can be compared as indicated below.

Proposition 2.1.11. Let $\boldsymbol{a}=(a_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers. For any $r\in\mathbb{R}\backslash\{0\}$, we have that

$$\alpha(\boldsymbol{a}^r)=r\alpha(\boldsymbol{a}),\quad \beta(\boldsymbol{a}^r)=r\beta(\boldsymbol{a}),\quad \rho(\boldsymbol{a}^r)=r\rho(\boldsymbol{a}),\quad \mu(\boldsymbol{a}^r)=r\mu(\boldsymbol{a}),$$

where $\boldsymbol{a}^r:=(a_p^r)_{p\in\mathbb{N}}$ and we also obtain that

$$\alpha(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\alpha(\boldsymbol{a}),\quad \beta(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\beta(\boldsymbol{a}),\quad \rho(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\rho(\boldsymbol{a}),\quad \mu(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\mu(\boldsymbol{a}),$$

where $\boldsymbol{g}_r:=(p^r)_{p\in\mathbb{N}}$ and $\boldsymbol{g}_r\cdot\boldsymbol{a}=(p^ra_p)_{p\in\mathbb{N}}$.

Proof. For every $\alpha\in\mathbb{R}$, we observe that $(a_pp^{-\gamma})_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing) if and only if $(a_p^rp^{-\gamma r})_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing). By Proposition 2.1.10, we deduce that $\alpha(\boldsymbol{a}^r)=r\alpha(\boldsymbol{a})$ and $\beta(\boldsymbol{a}^r)=r\beta(\boldsymbol{a})$. Similarly, $(a_pp^{-\gamma})_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing) if and only if $(p^ra_pp^{-\gamma-r})_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing), then $\alpha(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\alpha(\boldsymbol{a})$ and $\beta(\boldsymbol{g}_r\cdot\boldsymbol{a})=r+\beta(\boldsymbol{a})$.

Employing the representation given by Proposition 2.1.10 of μ and ρ, we conclude that

$$\rho(\boldsymbol{a}^r) = \limsup_{p\to\infty}\frac{r\log(a_p)}{\log p} = r\rho(\boldsymbol{a}), \qquad \rho(\boldsymbol{g}_r\cdot\boldsymbol{a}) = \limsup_{p\to\infty}\frac{r\log(p)+\log(a_p)}{\log p} = r+\rho(\boldsymbol{a}),$$
$$\mu(\boldsymbol{a}^r) = \liminf_{p\to\infty}\frac{r\log(a_p)}{\log p} = r\mu(\boldsymbol{a}), \qquad \mu(\boldsymbol{g}_r\cdot\boldsymbol{a}) = \liminf_{p\to\infty}\frac{r\log(p)+\log(a_p)}{\log p} = r+\mu(\boldsymbol{a}).$$

□

In the context of ultraholomorphic classes, it is always possible to switch $\mathbb{M}$ for an equivalent sequence. Since O-regular variation is stable for $\simeq$ (see Remark 1.2.47), it is unavoidable to ask if the same happens for the orders and the Matuszewska indices.

Lemma 2.1.12. Let $\boldsymbol{a}=(a_p)_{p\in\mathbb{N}}$ and $\boldsymbol{b}=(b_p)_{p\in\mathbb{N}}$ be sequences of positive numbers with $\boldsymbol{a}\simeq\boldsymbol{b}$. Then, we see that

$$\alpha(\boldsymbol{a})=\alpha(\boldsymbol{b}),\quad \beta(\boldsymbol{a})=\beta(\boldsymbol{b}),\quad \rho(\boldsymbol{a})=\rho(\boldsymbol{b}),\quad \mu(\boldsymbol{a})=\mu(\boldsymbol{b}).$$

Proof. For every $r \in \mathbb{R}\backslash\{0\}$, we observe that $\boldsymbol{a} \simeq \boldsymbol{b}$ implies that $\boldsymbol{g}_r\boldsymbol{a} \simeq \boldsymbol{g}_r\boldsymbol{b}$, where $\boldsymbol{g}_r = (p^r)_{p\in\mathbb{N}}$. It is plain to check that almost monotonicity is kept for $\simeq$ by suitably enlarging the corresponding constant and we conclude, using Proposition 2.1.10, that $\alpha(\boldsymbol{a}) = \alpha(\boldsymbol{b})$ and $\beta(\boldsymbol{a}) = \beta(\boldsymbol{b})$. Since $\boldsymbol{a} \simeq \boldsymbol{b}$, there exists $c > 1$ such that $a_p c^{-1} \le b_p \le c$ for every $p \in \mathbb{N}$ and we have that

$$\frac{\log(a_p) - \log(c)}{\log(p)} \le \frac{\log(b_p)}{\log(p)} \le \frac{\log(a_p) + \log(c)}{\log(p)}, \qquad p \in \mathbb{N}.$$

Taking lim sup and lim inf in these inequalities, by Proposition 2.1.10, we get $\rho(\boldsymbol{a}) = \rho(\boldsymbol{b})$ and $\mu(\boldsymbol{a}) = \mu(\boldsymbol{b})$. □

In the next subsection, these results will be applied for the sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ of $\mathbb{M}$, then the stability of those indices under $\simeq$ is a first approach but the appropriate question is the stability under $\approx$. A partial but sufficient solution is given at the end of the current section (see Remarks 2.1.24 and 2.1.32).

Since the sequence $\mathbf{m}$ is defined for $p \in \mathbb{N}_0$, it also naturally arises the question of the stability of these values and the notion of O-regular variation for index shifts.

Lemma 2.1.13. For any sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ and the corresponding shifted sequence $\boldsymbol{s_a} := (a_{p+1})_{p\in\mathbb{N}}$ we have that

$$\alpha(\boldsymbol{a}) = \alpha(\boldsymbol{s_a}), \quad \beta(\boldsymbol{a}) = \beta(\boldsymbol{s_a}), \quad \rho(\boldsymbol{a}) = \rho(\boldsymbol{s_a}), \quad \mu(\boldsymbol{a}) = \mu(\boldsymbol{s_a}).$$

Consequently, by Remark 2.1.9, $\boldsymbol{a}$ is O-regularly varying if and only if $\boldsymbol{s_a}$ also is.

Proof. We observe that for every $\alpha \ge 0$ we have that $p^\alpha \le (p+1)^\alpha \le 2^\alpha p^\alpha$, then for any $\gamma \in \mathbb{R}$ we see that $(p^\gamma a_{p+1})_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing) if and only if $(p^\gamma a_p)_{p\in\mathbb{N}}$ is almost decreasing (resp. almost increasing). Hence $\alpha(\boldsymbol{a}) = \alpha(\boldsymbol{s_a})$ and $\beta(\boldsymbol{a}) = \beta(\boldsymbol{s_a})$.

We also observe that
$$\frac{\log(a_{p+1})}{\log(p)} = \frac{\log(a_{p+1})}{\log(p+1)}\frac{\log(p+1)}{\log(p)}$$
and, since $\lim_{p\to\infty} \log(p+1)/\log(p) = 1$, we conclude that $\rho(\boldsymbol{a}) = \rho(\boldsymbol{s_a})$ and $\mu(\boldsymbol{a}) = \mu(\boldsymbol{s_a})$. □

Remark 2.1.14. If the sequence $\boldsymbol{a}$ is regularly varying of index $\omega \in \mathbb{R}$, by Theorem 1.2.37, the step function $\boldsymbol{f_a}$ is also regularly varying of index ω, we deduce that

$$\beta(\boldsymbol{a}) = \mu(\boldsymbol{a}) = \rho(\boldsymbol{a}) = \alpha(\boldsymbol{a}) = \omega.$$

The opposite is not true in general, see Examples 2.2.22 and 2.2.23 at the end of next section.

2.1.3 Logarithmically convex sequences, growth indices and O-regular variation

Almost monotonicity notions appear in the last two subsections: characterizing (snq) and (mg) conditions and in the definition of the Matuszewska indices for sequences. This fact is an obvious hint of the relation between growth properties for weight sequences and O-regular variation which will be settled in this subsection. Therefore, several equivalent definitions of the Matuszewska indices are deduced. Finally, the true nature of Thilliez's and Sanz's growth indices, $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$, is revealed. These growth indices, which were independently defined and, as it will be shown in Chapter 3, characterize the injectivity and surjectivity of the Borel map, coincide with $\beta(\mathbf{m})$ and $\mu(\mathbf{m})$, respectively.

The sequence of quotients $\mathbf{m}$ of a sequence $\mathbb{M}$, with the conventions in Remark 1.1.3, is defined for $p \in \mathbb{N}_0$, subsequently, a index shift problem appears. Please note that, as some authors [17, 52, 77, 78, 91] have done, it is possible to consider a different definition of the sequences of quotients that will make the results in this section more friendly, but other parts in the forthcoming sections will become troublesome. Moreover, taking into account Lemma 1.2.40 and Lemma 2.1.13, the study of the regular variation, O-regular variation, orders and Matuszewska indices of $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ is equivalent to the study of the same features for the shifted sequence $\boldsymbol{s}_{\mathbf{m}} = (m_p)_{p\in\mathbb{N}}$. Hence we will be able to deal with both approaches at once using one or another sequence, as appropriate.

The central connection between logarithmic convexity and O-regular variation can be formulated as follows.

Proposition 2.1.15. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$. For any $\gamma \in \mathbb{R}$, we have that

(i) if there exists $\boldsymbol{t} = (t_p)_{p\in\mathbb{N}_0}$ nondecreasing such that $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0} \simeq \boldsymbol{t}$, then $\beta(\mathbf{m}) \geq \gamma$.

(ii) if $\beta(\mathbf{m}) > \gamma$, then there exists $\boldsymbol{t} = (t_p)_{p\in\mathbb{N}_0}$ nondecreasing such that $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0} \simeq \boldsymbol{t}$.

(iii) if $\widehat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ is (lc), then $\beta(\mathbf{m}) \geq -1$.

(iv) if $\mathbb{M}$ is (lc), then $\beta(\mathbf{m}) \geq 0$.

Proof. (i) If there exists $\boldsymbol{t} = (t_p)_{p\in\mathbb{N}_0}$ nondecreasing such that $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0} \simeq \boldsymbol{t}$, then there exists $c > 1$ such that

$$\frac{m_p}{(p+1)^\gamma} \leq ct_p \leq ct_q \leq c^2\frac{m_q}{(q+1)^\gamma}, \qquad q \geq p, \quad q,p \in \mathbb{N}_0.$$

Consequently, $(p^{-\gamma}m_{p-1})_{p\in\mathbb{N}}$ is almost increasing then, by Proposition 2.1.10, $\beta(\mathbf{m}) \geq \gamma$.
(ii) By Proposition 2.1.10, $(p^{-\gamma}m_{p-1})_{p\in\mathbb{N}}$ is almost increasing, then there exists $M \geq 1$ such that

$$(p+1)^{-\gamma}m_p \leq M(q+1)^{-\gamma}m_q, \qquad \text{for all} \quad q \geq p, \quad q,p \in \mathbb{N}_0.$$

We define $t_p := \inf_{s\geq p}((s+1)^{-\gamma}m_s)$ for every $p \in \mathbb{N}_0$. For every $q,p \in \mathbb{N}_0$ with $q \geq p$, we check that

(1) $t_p = \inf_{s\geq p}((s+1)^{-\gamma}m_s) \leq \inf_{s\geq q}((s+1)^{-\gamma}m_s) = t_q$.

(2) $M^{-1}(p+1)^{-\gamma}m_p \leq \inf_{s\geq p}((s+1)^{-\gamma}m_s) = t_p \leq (p+1)^{-\gamma}m_p$.

Then $\boldsymbol{t}$ is nondecreasing and $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0} \simeq \boldsymbol{t}$.
Finally, (iii) and (iv) follow immediately from (i). □

According to Lemma 2.1.13 and Proposition 2.1.10, the lower order $\mu(\mathbf{m})$ and Sanz's growth index $\omega(\mathbb{M})$, see Definition 1.1.27, coincide for any sequence $\mathbb{M}$. The relation between $\gamma(\mathbb{M})$ and Matuszewska indices can be deduced from the previous result. We have included a weaker version of it, for strongly regular sequences and $\gamma > 0$, in [43, Prop. 4.15] where the connection with O-regular variation was unknown.

Theorem 2.1.16. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$. Then

$$\gamma(\mathbb{M}) = \beta(\mathbf{m}), \qquad \omega(\mathbb{M}) = \mu(\mathbf{m}).$$

Proof. From Lemma 2.1.13 and Proposition 2.1.10, we have that

$$\mu(\mathbf{m}) = \mu(\boldsymbol{s}_{\mathbf{m}}) = \liminf_{p\to\infty} \frac{\log(m_p)}{\log(p)} = \omega(\mathbb{M}),$$

where $\boldsymbol{s}_{\mathbf{m}} = (m_p)_{p\in\mathbb{N}}$ is the shifted sequence.

If $\gamma(\mathbb{M}) > \gamma$, we have that $\mathbb{M}$ satisfies (P_γ), then there exists a sequence $\boldsymbol{\ell}$ such that $\mathbf{m} \simeq \boldsymbol{\ell}$ and $((p+1)^{-\gamma}\ell_p)_{p\in\mathbb{N}_0}$ is nondecreasing or, equivalently, there exists $\boldsymbol{t} = ((p+1)^{-\gamma}\ell_p)_{p\in\mathbb{N}_0}$ nondecreasing with $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0} \simeq \boldsymbol{t}$. Then, by Proposition 2.1.15.(i), $\beta(\mathbf{m}) \geq \gamma$.

Conversely, if $\beta(\mathbf{m}) > \gamma$, from the relation between $\boldsymbol{\ell}$ and $\boldsymbol{t}$ and using Proposition 2.1.15.(ii), we deduce that $\gamma(\mathbb{M}) \geq \gamma$. □

Remark 2.1.17. The result above shows that the growth index $\gamma(\mathbb{M})$ can also be defined by

$$\gamma(\mathbb{M}) = \sup\{\gamma \in \mathbb{R} : \text{the sequence } \big((p+1)^{-\gamma}m_p\big)_{p\in\mathbb{N}_0} \text{ is almost increasing}\},$$

or, as it will be shown in Proposition 2.1.22, using Lemma 2.1.13 by

$$\gamma(\mathbb{M}) = \sup\{\gamma \in \mathbb{R} : \text{the sequence } \big(p^{-\gamma}m_p\big)_{p\in\mathbb{N}} \text{ is almost increasing}\}.$$

Combining the previous results with Proposition 2.1.2 and Proposition 2.1.4, a simple connection of (snq) and (mg) with the Matuszewska indices is provided.

Proposition 2.1.18. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$. Assume that $\widehat{\mathbb{M}}$ is (lc), then

(i) $\mathbb{M}$ has (mg) if and only if $\alpha(\mathbf{m}) < \infty$ if and only if $\mathbf{m}$ is O-regularly varying.

(ii) $\mathbb{M}$ has (snq) if and only if $\beta(\mathbf{m}) > 0$.

Proof. (i) By Lemma 2.1.3, $\mathbb{M}$ has (mg) if and only if there exists $\alpha > 0$ and $k \in \mathbb{N}$ with $k \geq 2$ such that $\limsup_{p\to\infty} m_{kp}/m_p < k^\alpha$ and, by Proposition 2.1.4, this happens if and only if there exists $\alpha > 0$ and $\varepsilon > 0$ such that $(p^{-\alpha+\varepsilon}m_p)_{p\in\mathbb{N}}$ is almost decreasing, or, equivalently, $\alpha(\boldsymbol{s}_{\mathbf{m}}) < \infty$ where $\boldsymbol{s}_{\mathbf{m}} = (m_p)_{p\in\mathbb{N}}$ is the shifted sequence that, by Lemma 2.1.13, is the same as saying that $\alpha(\mathbf{m}) < \infty$.

Finally, since $\widehat{\mathbb{M}}$ is (lc), by Proposition 2.1.15.(iii), $\beta(\mathbf{m}) \geq -1 > -\infty$. Consequently, by Remark 2.1.9, $\alpha(\mathbf{m}) < \infty$ if and only if $\mathbf{m}$ is ORV.

(ii) By Proposition 2.1.2, $\mathbb{M}$ is (snq) if and only if there exists $\varepsilon > 0$ such that $(p^{-\varepsilon}m_p)_{p\in\mathbb{N}}$ is almost increasing or, equivalently by Lemma 2.1.13, $\beta(\mathbf{m}) > 0$.

□

Remark 2.1.19. From the previous results and taking into account Remark 2.1.9, Lemma 1.1.29 can be generalized. Concretely, we have that if $\widehat{\mathbb{M}}$ is (lc), then

$$-1 \leq \beta(\mathbf{m}) = \gamma(\mathbb{M}) \leq \mu(\mathbf{m}) = \omega(\mathbb{M}) \leq \rho(\mathbf{m}) \leq \alpha(\mathbf{m}) \leq \infty,$$

and if $\mathbb{M}$ is (lc), then

$$0 \leq \beta(\mathbf{m}) = \gamma(\mathbb{M}) \leq \mu(\mathbf{m}) = \omega(\mathbb{M}) \leq \rho(\mathbf{m}) \leq \alpha(\mathbf{m}) \leq \infty,$$

Moreover, we can extend the characterization of strongly regular sequences in Corollary 2.1.6, i.e., the following are equivalent:

(i) $\mathbb{M}$ is strongly regular,

(iii) $\mathbf{m}$ is nondecreasing, $\alpha(\mathbf{m}) < \infty$ and $\beta(\mathbf{m}) > 0$

(iv) $\mathbf{m}$ is nondecreasing, O-regularly varying and $\beta(\mathbf{m}) > 0$.

In particular, for strongly regular sequences, $\beta(\mathbf{m}) = \gamma(\mathbb{M})$, $\mu(\mathbf{m}) = \omega(\mathbb{M})$, $\rho(\mathbf{m})$, $\alpha(\mathbf{m}) \in (0,\infty)$. In the context of O-regular variation, it is usual to name the class of positive functions having the Matuszewska indices in certain range: if $\alpha(f) < \infty$, f is said to be of *bounded increase* and if $\beta(f) > 0$, f is said to be of *positive increase*. The same terminology can be adopted for sequences, then a nondecreasing sequence $\mathbf{m}$ is strongly regular if and only if it is of bounded and positive increase.

Example 2.1.20. For all the examples of weight sequences considered until now in this dissertation and for most of the ones appearing in the applications the sequence of quotients is either regularly varying or rapidly varying (see [13, Sect. 2.4, Rapid variation]). The first case occurs for $\mathbb{M}_{\alpha,\beta} = (n!^{\alpha}\Pi_{m=0}^{n} \log^{\beta}(e+m))_{n\in\mathbb{N}_0}$ with $\alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$ where for all $\lambda \in (0,\infty)$ we have that

$$\lim_{p\to\infty} \frac{m_{\lfloor \lambda p \rfloor}}{m_p} = \lim_{p\to\infty} \frac{(\lfloor \lambda p \rfloor + 1)^{\alpha}(\log(e+\lfloor \lambda p \rfloor +1))^{\beta}}{(p+1)^{\alpha}(\log(e+p+1))^{\beta}} = \lambda^{\alpha},$$

then as it happens for all the regularly varying sequences, by Remark 2.1.14, we deduce that

$$\gamma(\mathbb{M}) = \beta(\mathbf{m}) = \omega(\mathbb{M}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = \alpha.$$

The second case appears when dealing with the $q-$Gevrey sequence $(q^{p^2})_{p\in\mathbb{N}_0}$ with $q > 1$ where a similar computation leads to $\beta(\mathbf{m}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = \infty$. Consequently, for these classical examples the existence of different orders and indices remains hidden.

However, this is not the general case, using the Representation Theorem at the end of this chapter we construct strongly regular sequences with the following properties:

(i) In Examples 2.2.21, 2.2.22 and 2.2.23 all the indices and orders coincide with 3/2 in the first case in which the sequence is regularly varying and with 1 in the other two where the sequences are only O-regularly varying.

(ii) For the Example 2.2.24 we have that $\beta(\mathbf{m}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = 1$ and $\alpha(\mathbf{m}) = 2$.

(iii) In the Example 2.2.26 we see that $\beta(\mathbf{m}) = 2$, $\mu(\mathbf{m}) = 5/2$, $\rho(\mathbf{m}) = 11/4$ and $\alpha(\mathbf{m}) = 3$.

Regarding other examples we have found in the literature, a careful computation leads to the following conclusion:

(i) for Example 3.3 in [57], where the sequences $\mathbf{m} = (m_p)_{p\in\mathbb{N}_0}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are defined inductively by

$$m_p := \begin{cases} c_k^3, & \text{for all} \quad c_k \le p \le (c_k)^{3/2} =: d_k - 1, \\ p^4/d_k^2, & \text{for all} \quad d_k \le p \le (d_k)^2 =: c_{k+1} - 1, \end{cases}$$

taking $c_1 = 1$ and $m_0 = 1$, we can show that $\beta(\mathbf{m}) = 0$, $\mu(\mathbf{m}) = 2$, $\rho(\mathbf{m}) = 3$ and $\alpha(\mathbf{m}) = 4$.

(ii) for Example 21 in [17], $M_p := \exp(p^s)$ for $s \in (1,2]$ and $p \in \mathbb{N}_0$ as in the $q-$Gevrey case, we have that $\beta(\mathbf{m}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = \infty$.

(iii) for Example 25 in [17], $M_p := ((p+1)\log_s(e_s+p))^p$ for $s,p\in\mathbb{N}$ where $e_s := \exp(e_{s-1})$, $e_0 = 1$, $\log_s = \log(\log_{s-1}(x))$ and $\log_0(x) = x$, as in the first case, the regular variation entails $\beta(\mathbf{m}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = 1$.

It is possible to generalize Proposition 2.1.18, that has been stated individually for its relevance, by giving several alternative definitions of $\beta(\mathbf{m})$ and $\alpha(\mathbf{m})$, but for this purpose we need the next auxiliary lemma that extends Proposition 2.1.2 in the direction of Proposition 2.1.4 but going one step further.

Lemma 2.1.21. Let $\mathbf{m}$ be a sequence such that $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0}$ is nondecreasing. Then for every $\beta>\gamma$, the following are equivalent:

(i) $\lim_{k\to\infty}\liminf_{p\to\infty}\dfrac{m_{kp}}{k^\beta m_p} = \infty$,

(ii) there exists $k\in\mathbb{N}$, $k\geq 2$ such that

$$\liminf_{p\to\infty}\frac{m_{kp}}{m_p} > k^\beta,$$

(iii) there exists $\varepsilon>0$ such that $(p^{-\beta-\varepsilon}m_p)_{p\in\mathbb{N}}$ is almost increasing,

(iv) there exist $\delta>0$ and $A>0$ such that

$$\sum_{k=p}^{\infty}\frac{(k+1)^{\beta+\delta}}{(k+1)m_k} \leq \frac{A(p+1)^{\beta+\delta}}{m_p}, \qquad p\in\mathbb{N}_0.$$

Proof. (i) $\Rightarrow$ (ii) Immediate. (ii) $\Rightarrow$ (iii) There exists $\varepsilon>0$ such that $m_{kp}/m_p > k^{\beta+\varepsilon}$ for every $p\geq p_0\geq 1$. Then, for every $q\geq p\geq p_0$ there exists $n\in\mathbb{N}_0$ such that $k^np\leq q<k^{n+1}p$. Using that $((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0}$ is nondecreasing and iterating the previous inequality we get

$$\frac{m_q}{q^{\beta+\varepsilon}} \geq \frac{(q+1)^\gamma m_{k^np}}{q^{\beta+\varepsilon}(k^np+1)^\gamma} > \frac{(q+1)^\gamma m_p k^{n(\beta+\varepsilon)}}{q^{\beta+\varepsilon}(k^np+1)^\gamma} \qquad \text{for all} \quad q\geq p\geq p_0.$$

Since $k^np\leq q<k^{n+1}p$, we see that $1\geq (k^np+1)/(q+1)\geq k^{-1}$ and $1\geq k^np/q\geq k^{-1}$. Then

$$\frac{m_q}{q^{\beta+\varepsilon}} \geq \min(1,k^{-\beta-\varepsilon})\min(1,k^\gamma)\frac{m_p}{p^{\beta+\varepsilon}} \qquad \text{for all} \quad q\geq p\geq p_0.$$

Then, by suitably enlarging the constant as in the proof of Proposition 2.1.2, we see that $(p^{-\beta-\varepsilon}m_p)_{p\in\mathbb{N}}$ is almost increasing.
(iii) $\Rightarrow$ (iv) As in the proof of Lemma 2.1.13, one can show that$((p+1)^{-\beta-\varepsilon}m_p)_{p\in\mathbb{N}_0}$ is almost increasing. For every $p\in\mathbb{N}_0$, for suitable $M>0$, we see that

$$\sum_{k=p}^{\infty}\frac{(k+1)^{\beta+\varepsilon/2}}{(k+1)m_k} \leq M\frac{(p+1)^{\beta+\varepsilon}}{m_p}\int_{p+1}^{\infty}\frac{dx}{x^{1+\varepsilon/2}} \leq \frac{2M(p+1)^{\beta+\varepsilon/2}}{\varepsilon m_p}.$$

We conclude by choosing $\delta := \varepsilon/2 > 0$ and $A := 2M/\varepsilon$.
(iv) $\Rightarrow$ (i) Using that $((q+1)^{-\gamma}m_q)_{p\in\mathbb{N}_0}$ is nondecreasing, for $q,p\in\mathbb{N}_0$ with $q\geq p$ we have that

$$A\frac{(p+1)^{\beta+\delta}}{m_p} \geq \sum_{k=p}^{\infty}\frac{(k+1)^{\beta+\delta}}{(k+1)m_k} \geq \sum_{k=p}^{q}\frac{(k+1)^{\beta-\gamma+\delta-1}}{(k+1)^{-\gamma}m_k} \geq \frac{1}{(q+1)^{-\gamma}m_q}\int_{p+1}^{q+1}x^{\beta-\gamma+\delta-1}dx.$$

Since $\beta-\gamma+\delta>0$, we get

$$A\frac{(p+1)^{\beta+\delta}}{m_p}\geq\frac{(q+1)^{\beta+\delta}}{(\beta-\gamma+\delta)m_q}\left(1-\left(\frac{p+1}{q+1}\right)^{\beta-\gamma+\delta}\right).$$

For any $k\in\mathbb{N}$, $k\geq 2$, taking $q=kp$ we observe that

$$\liminf_{p\to\infty}\frac{m_{kp}}{m_p}\geq\frac{k^{\beta+\delta}}{A(\beta-\gamma+\delta)}\left(1-\frac{1}{k^{\beta-\gamma+\delta}}\right),$$

which implies (i). □

From the previous results, concerning the characterizations of (snq) and (mg) and the properties of the Matuszewska indices, several equivalent representations of $\alpha(\mathbf{m})$ and $\beta(\mathbf{m})$ are obtained, similar to the ones in the work of S. Aljančić and I. D. Arandjelović [1] for O-regularly varying functions. Although weaker conditions on $\mathbb{M}$ might be assumed, the proposition is stated in a quite general form which includes the situation when $\widehat{\mathbb{M}}$ or $\mathbb{M}$ are (lc), providing flexible and practical conclusions for the applications.

Proposition 2.1.22. Let $\mathbb{M}$ be a sequence of positive real numbers with sequence of quotients $\mathbf{m}=(m_{p-1})_{p\in\mathbb{N}}$. We have that

(i) $\alpha(\mathbf{m})=\inf\{\alpha\in\mathbb{R};\ (m_p/p^{\alpha})_{p\in\mathbb{N}}$ is almost decreasing$\}$,

(ii) $\beta(\mathbf{m})=\sup\{\beta\in\mathbb{R};\ (m_p/p^{\beta})_{p\in\mathbb{N}}$ is almost increasing$\}$,

If there exists $\gamma\in\mathbb{R}$ and a nondecreasing sequence $\boldsymbol{\ell}$ such that $\boldsymbol{\ell}\simeq((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0}$, then $\beta(\mathbf{m})\geq\gamma$ and we have that

(iii) $\alpha(\mathbf{m})=\inf\{\alpha>\gamma;\ \lim_{k\to\infty}\limsup_{p\to\infty}\dfrac{m_{kp}}{k^{\alpha}m_p}=0\}$,

(iv) $\alpha(\mathbf{m})=\inf\{\alpha>\gamma;\ \exists\,k\in\mathbb{N},\ k\geq 2;\ \limsup_{p\to\infty}\dfrac{m_{kp}}{m_p}<k^{\alpha}\}$,

(v) $\alpha(\mathbf{m})=\inf\{\alpha>\gamma;\ \exists\,A>0;\ \displaystyle\sum_{\ell=0}^{p}\frac{(\ell+1)^{\alpha}}{(\ell+1)m_\ell}\leq\frac{A(p+1)^{\alpha}}{m_p},\quad p\in\mathbb{N}_0\}$,

(vi) $\alpha(\mathbf{m})=\inf\{\alpha>\gamma;\ \exists\,A>0;\ \displaystyle\sum_{\ell=0}^{p}\frac{1}{((\ell+1)^{-\gamma}m_\ell)^{1/(\alpha-\gamma)}}\leq\frac{A(p+1)}{((p+1)^{-\gamma}m_p)^{1/(\alpha-\gamma)}},\ p\in\mathbb{N}_0\}$,

(vii) $\beta(\mathbf{m})=\sup\{\beta>\gamma;\ \lim_{k\to\infty}\liminf_{p\to\infty}\dfrac{m_{kp}}{k^{\beta}m_p}=\infty\}$,

(viii) $\beta(\mathbf{m})=\sup\{\beta>\gamma;\ \exists\,k\in\mathbb{N},\ k\geq 2;\ \liminf_{p\to\infty}\dfrac{m_{kp}}{m_p}>k^{\beta}\}$,

(ix) $\beta(\mathbf{m})=\sup\{\beta>\gamma;\exists\,A>0;\ \displaystyle\sum_{\ell=p}^{\infty}\frac{(\ell+1)^{\beta}}{(\ell+1)m_\ell}\leq\frac{A(p+1)^{\beta}}{m_p},\quad p\in\mathbb{N}_0\}$,

(x) $\beta(\mathbf{m})=\sup\{\beta>\gamma;\exists\,A>0;\ \displaystyle\sum_{\ell=p}^{\infty}\frac{1}{((\ell+1)^{-\gamma}m_\ell)^{1/(\beta-\gamma)}}\leq\frac{A(p+1)}{((p+1)^{-\gamma}m_p)^{1/(\beta-\gamma)}},\ p\in\mathbb{N}_0\}$.

Proof. The expressions in (i) and (ii) are immediately deduced from Lemma 2.1.13 since $\beta(\mathbf{m}) = \beta(\boldsymbol{s}_{\mathbf{m}})$ and $\alpha(\mathbf{m}) = \alpha(\boldsymbol{s}_{\mathbf{m}})$ where $\boldsymbol{s}_{\mathbf{m}} = (m_p)_{p\in\mathbb{N}}$ is the shifted sequence.

We observe that $\gamma \leq \beta(\mathbf{m}) \leq \alpha(\mathbf{m}) < \alpha$, then $\alpha-\gamma > 0$. By Proposition 2.1.11, we have that $\alpha(\mathbf{m}) < \alpha$ if and only if $\alpha(\boldsymbol{g}_{-\gamma}\mathbf{m}) < \alpha-\gamma$ where $\boldsymbol{g}_{-\gamma} = (p^{-\gamma})_{p\in\mathbb{N}}$ and $\boldsymbol{g}_{-\gamma}\cdot\mathbf{m} = ((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0}$ and, by Lemma 2.1.12, this happens if and only if $\alpha(\boldsymbol{\ell}) < \alpha-\gamma$ or, equivalently, if there exists $\varepsilon \in (0, \alpha-\gamma)$ such that $\alpha(\boldsymbol{\ell}) < \alpha-\gamma-\varepsilon$. Summarizing and using (i), $\alpha(\mathbf{m}) < \alpha$ if and only if $(p^{\gamma-\alpha+\varepsilon}\ell_p)_{p\in\mathbb{N}}$ is almost decreasing. We observe that the sequence

$$t_p := (p+1)^{-1}\ell_p, \qquad p \in \mathbb{N}_0,$$

satisfies that $((p+1)t_p)_{p\in\mathbb{N}_0} = \boldsymbol{\ell}$ is nondecreasing and $(p^{\gamma-\alpha+\varepsilon+1}t_p)_{p\in\mathbb{N}}$ is almost decreasing. Since $\alpha-\gamma-1 > -1$, applying Proposition 2.1.4 for $\boldsymbol{t}$, the following are equivalent

(1) $\alpha(\mathbf{m}) < \alpha$,

(2) There exists $A > 0$ such that

$$\sum_{k=0}^{p} \frac{(k+1)^{\alpha-\gamma}}{(k+1)\ell_k} = \sum_{k=0}^{p} \frac{(k+1)^{\alpha-\gamma-1}}{(k+1)t_k} \leq \frac{A(p+1)^{\alpha-\gamma-1}}{t_p} = \frac{A(p+1)^{\alpha-\gamma}}{\ell_p} \qquad p \in \mathbb{N}_0,$$

(3) $\displaystyle\lim_{k\to\infty}\limsup_{p\to\infty} \frac{\ell_{kp}}{k^{\alpha-\gamma}\ell_p} = \lim_{k\to\infty}\limsup_{p\to\infty} \frac{t_{kp}}{k^{\alpha-\gamma-1}t_p} = 0$,

(4) There exists $k \in \mathbb{N}$, $k \geq 2$, such that

$$\limsup_{p\to\infty} \frac{\ell_{kp}}{k\ell_p} = \limsup_{p\to\infty} \frac{t_{kp}}{t_p} < k^{\alpha-\gamma-1}.$$

Since $\boldsymbol{\ell} \simeq \boldsymbol{g}_{-\gamma}\mathbf{m}$, it follows that (iii) holds and, by suitably enlarging A and k, (iv) and (v) are also valid.

As before, $\alpha(\mathbf{m}) < \alpha$ if and only if there exists $\varepsilon \in (0, \alpha-\gamma)$ such that $(p^{\gamma-\alpha+\varepsilon}\ell_p)_{p\in\mathbb{N}}$ is almost decreasing. Equivalently, we observe that the sequence

$$u_p := (p+1)^{-1}(\ell_p)^{1/(\alpha-\gamma)}, \qquad p \in \mathbb{N}_0,$$

satisfies that $((p+1)u_p)_{p\in\mathbb{N}_0}$ is nondecreasing and there exists $\delta \in (0,1)$ such that $(p^{\delta}u_p)_{p\in\mathbb{N}}$ is almost decreasing. By Proposition 2.1.4 applied to $\boldsymbol{u}$, $\alpha(\mathbf{m}) < \alpha$ if and only if there exists $A > 0$ such that

$$\sum_{k=0}^{p} \frac{1}{(\ell_k)^{1/(\alpha-\gamma)}} = \sum_{k=0}^{p} \frac{1}{(k+1)u_k} \leq \frac{A}{u_p} = \frac{A(p+1)}{(\ell_p)^{1/(\alpha-\gamma)}}$$

Then, since $\boldsymbol{\ell} \simeq \boldsymbol{g}_{-\gamma}\mathbf{m}$, enlarging A, (vi) holds.

For any $\beta > \gamma$, with the same reasoning, we see that $\beta < \beta(\mathbf{m})$ if and only if there exists $\varepsilon > 0$ such that $(p^{\gamma-\beta-\varepsilon}\ell_p)_{p\in\mathbb{N}}$ is almost increasing. Since $\beta-\gamma > 0$, applying Lemma 2.1.21 to the nondecreasing sequence $\boldsymbol{\ell}$ and then using that $\boldsymbol{\ell} \simeq \boldsymbol{g}_{-\gamma}\mathbf{m}$, the following are equivalent:

(1) $\displaystyle\lim_{k\to\infty}\liminf_{p\to\infty} \frac{m_{kp}}{k^{\beta}m_p} = \infty$,

(2) There exists $k \in \mathbb{N}$, $k \geq 2$ such that

$$\liminf_{p\to\infty} \frac{m_{kp}}{m_p} > k^{\beta},$$

(3) $\beta < \beta(\mathbf{m})$,

(4) There exist $\delta > 0$ and $A > 0$ such that

$$\sum_{k=p}^{\infty} \frac{(k+1)^{\beta+\delta}}{(k+1)m_k} \leq \frac{A(p+1)^{\beta+\delta}}{m_p}, \quad p \in \mathbb{N}_0.$$

Hence (vii), (viii) and (ix) are true.

Analogous to (vi), we apply Proposition 2.1.2 to the sequence $v_p := (\ell_p)^{1/(\beta-\gamma)}(p+1)^{-1}$ for $p \in \mathbb{N}_0$, using that $\boldsymbol{\ell} \simeq \boldsymbol{g}_{-\gamma}\mathbf{m}$ we obtain (x).

□

Remark 2.1.23. We are specially interested in the (lc) sequence case, particularly when $\widehat{\mathbf{m}} = (pm_{p-1})_{p\in\mathbb{N}}$ or $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ are nondecreasing, then $\gamma = -1$ or $\gamma = 0$, respectively (see Remark 3.1.11 for further information about both situations). In the first case, (x) can be expressed as follows

$$\beta(\mathbf{m}) = \sup\{\beta > -1; \exists\, A > 0, \text{ s.t. } \sum_{\ell=p}^{\infty} \frac{1}{(\widehat{m}_\ell)^{1/(\beta+1)}} \leq \frac{A(p+1)}{(\widehat{m}_p)^{1/(\beta+1)}}, \quad p \in \mathbb{N}_0\},$$

The reader may notice the connection with condition (γ_1) of H.-J. Petzsche [77] for $\widehat{\mathbf{m}}$ by taking $\beta = 0$, that is, if there exists $A > 0$ such that

$$(\gamma_1) \qquad \sum_{\ell=p}^{\infty} \frac{1}{\widehat{m}_\ell} \leq \frac{A(p+1)}{\widehat{m}_p}, \qquad p \in \mathbb{N}_0,$$

and condition $(\gamma_{\beta+1})$ of J. Schmets, M. Valdivia [91] for $\widehat{\mathbf{m}}$ by taking $\beta \in \mathbb{N}_0$, that is, if there exists $A > 0$ such that

$$(\gamma_{\beta+1}) \qquad \sum_{\ell=p}^{\infty} \frac{1}{(\widehat{m}_\ell)^{1/(\beta+1)}} \leq \frac{A(p+1)}{(\widehat{m}_p)^{1/(\beta+1)}}, \qquad p \in \mathbb{N}_0,$$

that will be used in Section 3.3 when studying the surjectivity of the Borel map. For $\widehat{\mathbf{m}}$ nondecreasing, extending this condition for $\beta \in \mathbb{R}$, $\beta > -1$, one can, equivalently, write

$$\beta(\mathbf{m}) = \sup\{\beta > -1;\ \widehat{\mathbf{m}} \text{ satisfies } (\gamma_{\beta+1})\},$$

and for $\mathbf{m}$ nondecreasing,

$$\beta(\mathbf{m}) = \sup\{\beta > 0;\ \mathbf{m} \text{ satisfies } (\gamma_\beta)\}.$$

It is plain to check that $\widehat{\mathbf{m}}$ satisfies (γ_1) if and only if $\mathbb{M}$ satisfies (snq). Moreover, from Proposition 2.1.11 we know that $\beta(\widehat{\mathbf{m}}) = 1 + \beta(\mathbf{m})$ and, using Proposition 2.1.18, for a nondecreasing sequence $\widehat{\mathbf{m}}$ we see that

$$\beta(\widehat{\mathbf{m}}) > 1 \quad \text{if and only if} \quad \widehat{\mathbf{m}} \text{ satisfies } (\gamma_1).$$

Assuming again that $\widehat{\mathbf{m}}$ is nondecreasing, by Proposition 2.1.22.(viii), we obtain that

$$\beta(\widehat{\mathbf{m}}) > 0 \text{ if and only if there exists } k \in \mathbb{N},\ k \geq 2, \text{ such that } \liminf_{p\to\infty} \widehat{m}_{kp}/\widehat{m}_p > 1.$$

It is worthy to mention that, due to the previously commented index shift, the last condition also appears in the literature as: there exists $k \in \mathbb{N}$, $k \geq 2$, such that $\liminf_{p\to\infty} \widehat{m}_{kp-1}/\widehat{m}_{p-1} > 1$. Thanks to Lemma 2.1.13, we know both are equivalent.

Finally, using Proposition 2.1.18, $\alpha(\widehat{\mathbf{m}}) = \alpha(\mathbf{m}) + 1$ and that $\mathbb{M}$ has (mg) if and only if $\widehat{\mathbb{M}}$ also has (see the proof of Lemma 2.1.3), for a nondecreasing sequence $\widehat{\mathbf{m}}$, we obtain that

$$\alpha(\widehat{\mathbf{m}}) < \infty \text{ if and only if } \widehat{\mathbb{M}} \text{ has (mg).}$$

In particular, the classical result of J. Bonet, R. Meise and S.N. Melikhov [17, Th. 14], stated in a more general framework in [90, Sect. 6] by G. Schindl, can be translated into the following form: the ultradifferentiable space defined by a weight sequence $\widehat{\mathbb{M}}$ satisfying $\beta(\widehat{\mathbf{m}}) > 0$ can be defined in terms of a weight function ω (see Remark 2.1.33), measuring the decay properties of the Fourier transform of the functions in the space, if and only if $\alpha(\widehat{\mathbf{m}}) < \infty$.

Taking into account the stability of (γ_1) property under $\approx$, we can also obtain the stability of the index $\gamma(\mathbb{M}) = \beta(\mathbf{m})$ for weight sequences generalizing Lemma 2.1.12 where only stability for $\simeq$ was proved.

Remark 2.1.24. Let $\mathbb{M}$ and $\mathbb{L}$ be sequences such that $\widehat{\mathbb{M}}$ and $\widehat{\mathbb{L}}$ are weight sequences, i.e., (lc) and such that $\widehat{\mathbf{m}}$ and $\widehat{\boldsymbol{\ell}}$ tend to infinity, with $\widehat{\mathbb{M}} \approx \widehat{\mathbb{L}}$. Then $\beta(\widehat{\mathbf{m}}), \beta(\widehat{\boldsymbol{\ell}}) \geq 0$ and, by Proposition 2.1.11, we have that $\beta(\widehat{\mathbf{m}}) > \gamma > 0$ if and only if $\beta(\widehat{\mathbf{m}}^{1/\gamma}) > 1$ where $\widehat{\mathbf{m}}^{1/\gamma} = ((m_p(p+1))^{1/\gamma})_{p\in\mathbb{N}_0}$. Since $\widehat{\mathbf{m}}^{1/\gamma}$ is nondecreasing, by the previous remark, $\beta(\widehat{\mathbf{m}}^{1/\gamma}) > 1$ if and only if $\widehat{\mathbf{m}}^{1/\gamma}$ satisfies (γ_1). Using that $\widehat{\mathbb{M}}^{1/\gamma}$ and $\widehat{\mathbb{L}}^{1/\gamma}$ are weight sequences with $\widehat{\mathbb{M}}^{1/\gamma} \approx \widehat{\mathbb{L}}^{1/\gamma}$ and that H.-J. Petszche has proved the stability of (γ_1) for weight sequences (see [77, Th. 3.4]), we see that $\beta(\widehat{\mathbf{m}}) > \gamma > 0$ if and only if $\widehat{\boldsymbol{\ell}}^{1/\gamma}$ satisfies (γ_1) which, with the same reasoning, is equivalent to $\beta(\widehat{\boldsymbol{\ell}}) > \gamma > 0$. Then

$$\gamma(\widehat{\mathbb{M}}) = \beta(\widehat{\mathbf{m}}) = \beta(\widehat{\boldsymbol{\ell}}) = \gamma(\widehat{\mathbb{L}}).$$

Moreover, again by Proposition 2.1.11 and Theorem 2.1.16, we deduce that

$$\gamma(\mathbb{M}) = \gamma(\widehat{\mathbb{M}}) - 1 = \gamma(\widehat{\mathbb{L}}) - 1 = \gamma(\mathbb{L}).$$

In particular, if $\mathbb{M}$ and $\mathbb{L}$ are weight sequences with $\mathbb{M} \approx \mathbb{L}$, the same is true for $\widehat{\mathbb{M}}$ and $\widehat{\mathbb{L}}$ and the last equality also holds.

Similarly but more directly, the stability for $\approx$ of the value $\alpha(\mathbf{m})$ is obtained. Because if $\mathbb{M}$ and $\mathbb{L}$ are weight sequences with $\mathbb{M} \approx \mathbb{L}$, by Proposition 2.1.18, $\alpha(\mathbf{m}) = \infty$ if and only if $\mathbb{M}$ is not (mg), or equivalently, by Proposition 1.1.17, $\mathbb{L}$ is not (mg), that is, $\alpha(\boldsymbol{\ell}) = \infty$, so $\alpha(\mathbf{m}) = \alpha(\boldsymbol{\ell})$. If $\alpha(\mathbf{m}) < \infty$ or $\alpha(\boldsymbol{\ell}) < \infty$, by Proposition 1.1.20 $\approx$ is equivalent to $\simeq$, using Proposition 2.1.12 we conclude that $\alpha(\mathbf{m}) = \alpha(\boldsymbol{\ell})$. Furthermore, this implies that O-regular variation for the sequence of quotients of a weight sequences is also stable for $\approx$.

Recalling the definition of exponent of convergence of a sequence and how it may be computed, a characterization for the lower order μ, similar to (vi) and (x) in Proposition 2.1.22, can be established, getting as a byproduct the relation with (nq) property.

Proposition 2.1.25 ([37], p. 65). Let $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$ be a nondecreasing sequence of positive real numbers. The *exponent of convergence* of $\boldsymbol{a}$ is defined as

$$\lambda(\boldsymbol{a}) := \inf\{\mu > 0 : \sum_{p=1}^{\infty} \frac{1}{a_p^{\mu}} \text{ converges}\}$$

(as in Remark 1.1.26, if the previous set is empty, we put $\lambda(\boldsymbol{a})=\infty$). Then, one has

$$\lambda(\boldsymbol{a})=\limsup_{p\to\infty}\frac{\log(p)}{\log(a_p)}.$$

Proposition 2.1.26. If there exists $\gamma\in\mathbb{R}$ and a nondecreasing sequence $\boldsymbol{\ell}$ such that $\boldsymbol{\ell}\simeq((p+1)^{-\gamma}m_p)_{p\in\mathbb{N}_0}$ then $\mu(\mathbf{m})\geq\beta(\mathbf{m})\geq\gamma$ and we have that

$$\mu(\mathbf{m})=\sup\{\mu>\gamma;\quad\sum_{\ell=0}^{\infty}\frac{1}{((\ell+1)^{-\gamma}m_\ell)^{1/(\mu-\gamma)}}<\infty\}.$$

In particular, if $\widehat{\mathbb{M}}$ is (lc), then $\gamma=-1$ and

(i) if $\mathbb{M}$ satisfies (nq) (see Definition 1.1.5), then $\mu(\mathbf{m})\geq 0$,

(ii) if $\mu(\mathbf{m})>0$, then $\mathbb{M}$ satisfies (nq).

Proof. By Proposition 2.1.25, the inverse of the exponent of convergence of the nondecreasing sequence $\boldsymbol{\ell}=(\ell_{p-1})_{p\in\mathbb{N}}$ is given by

$$\frac{1}{\lambda(\boldsymbol{\ell})}=\liminf_{p\to\infty}\frac{\log(\ell_{p-1})}{\log(p)}=\mu(\boldsymbol{\ell})=-\gamma+\mu(\mathbf{m})$$

applying Proposition 2.1.11 and Lemma 2.1.12 for the last equality, because $\boldsymbol{\ell}\simeq\boldsymbol{g}_{-\gamma}\mathbf{m}$. Then, using again that $\boldsymbol{\ell}\simeq\boldsymbol{g}_{-\gamma}\mathbf{m}$, we see that

$$\begin{aligned}\mu(\mathbf{m})=&\sup\{\lambda>0;\sum_{p=0}^{\infty}1/(\ell_p)^{1/\lambda}<\infty\}+\gamma=\sup\{\mu>\gamma;\sum_{p=0}^{\infty}1/(\ell_p)^{1/(\mu-\gamma)}<\infty\}\\=&\sup\{\mu>\gamma;\sum_{p=0}^{\infty}1/((p+1)^{-\gamma}m_p)^{1/(\mu-\gamma)}<\infty\}.\end{aligned}$$

□

There is not a straightforward extension of this characterization, similar to (vi) and (x) in Proposition 2.1.22, for the upper order $\rho(\mathbf{m})$ (see also Remark 2.1.31).

2.1.4 O-regular variation of the associated function

Departing from a weight sequence $\mathbb{M}$, this subsection is devoted to the study of the properties of orders and Matuszewska indices of the associated function $\omega_{\mathbb{M}}$ and the counting function. As it happens for the sequences, these values characterize several classical properties of these functions. However, only the necessary statements for our aim will be shown (see Remark 2.1.33). The connection between the indices of $\mathbb{M}$ and $\omega_{\mathbb{M}}$ (see Theorem 2.1.30, which we have partially stated in [43, Th. 3.2]) is the central point of this subsection.

We start by recalling the following definitions and facts, mainly taken from the book of A. A. Goldberg and I. V. Ostrovskii [32].

Definition 2.1.27. Given a weight sequence $\mathbb{M}$, i.e., (lc) and such that $\mathbf{m}$ tends to infinity, we consider the *counting function* for the sequence of quotients $\mathbf{m}$, $\nu_{\mathbf{m}}:(0,\infty)\to\mathbb{N}_0$ given by

$$\nu_{\mathbf{m}}(t):=\#\{j\in\mathbb{N}_0:m_j\leq t\}=\max\{j\in\mathbb{N}:m_{j-1}\leq t\}.$$

For a weight sequence $\mathbb{M}$, using (1.5) we recover the classical relation, which can be also found in [72], between $\nu_{\mathbf{m}}$ and the associated function $\omega_{\mathbb{M}}$. One has that

$$\omega_{\mathbb{M}}(t) = \int_0^t \frac{\nu_{\mathbf{m}}(r)}{r}\,dr, \qquad t>0, \tag{2.8}$$

which allows us to write

$$\omega_{\mathbb{M}}(t) = \nu_{\mathbf{m}}(t)\log(t) - \log(M_{\nu_{\mathbf{m}}(t)}), \quad t>0;\ \omega'_{\mathbb{M}}(t) = \frac{\nu_{\mathbf{m}}(t)}{t}, \quad t>0,\ t\neq m_p,\ p\in\mathbb{N}_0. \tag{2.9}$$

The associated function $\omega_{\mathbb{M}}(t)$ of a weight sequence $\mathbb{M}$ is continuous, so measurable, and positive in $[X,\infty)$ with $X > m_0$. Since the counting function $\nu_{\mathbf{m}}(t)$ is nondecreasing, the same holds. Consequently, we can consider their Matuszewska indices $\beta(\omega_{\mathbb{M}})$, $\beta(\nu_{\mathbf{m}})$, $\alpha(\omega_{\mathbb{M}})$, $\alpha(\nu_{\mathbf{m}})$ and their upper and lower orders $\mu(\omega_{\mathbb{M}})$, $\mu(\nu_{\mathbf{m}})$, $\rho(\omega_{\mathbb{M}})$, $\rho(\nu_{\mathbf{m}})$. Please note that if $\omega_{\mathbb{M}}$ (or $\nu_{\mathbf{m}}$) is regularly varying of index ρ, the corresponding indices of $\omega_{\mathbb{M}}$ (or of $\nu_{\mathbf{m}}$) are all equal to ρ. Using the monotonicity of these functions, we easily obtain the analogue version of Proposition 2.1.18.

Proposition 2.1.28. Let $\mathbb{M}$ be a weight sequence, then

(i) $\beta(\omega_{\mathbb{M}})$, $\beta(\nu_{\mathbf{m}})$, $\alpha(\omega_{\mathbb{M}})$, $\alpha(\nu_{\mathbf{m}})$, $\mu(\omega_{\mathbb{M}})$, $\mu(\nu_{\mathbf{m}})$, $\rho(\omega_{\mathbb{M}})$, $\rho(\nu_{\mathbf{m}}) \in [0,\infty]$.

(ii) $\nu_{\mathbf{m}} \in ORV$ if and only if $\alpha(\nu_{\mathbf{m}}) < \infty$ if and only if $\nu_{\mathbf{m}}(2t) = O(\nu_{\mathbf{m}}(t))$.

(iii) $\omega_{\mathbb{M}} \in ORV$ if and only if $\alpha(\omega_{\mathbb{M}}) < \infty$ if and only if $\omega_{\mathbb{M}}(2t) = O(\omega_{\mathbb{M}}(t))$.

(iv) $\beta(\nu_{\mathbf{m}}) > 0$ if and only if there exists $H \geq 1$ such that $\liminf_{t\to\infty} \nu_{\mathbf{m}}(Ht)/\nu_{\mathbf{m}}(t) > 1$.

(v) $\beta(\omega_{\mathbb{M}}) > 0$ if and only if there exists $H \geq 1$ such that $\liminf_{t\to\infty} \omega_{\mathbb{M}}(Ht)/\omega_{\mathbb{M}}(t) > 1$.

Proof. (i) By Theorem 1.2.28, since $\omega_{\mathbb{M}}$ and $\nu_{\mathbf{m}}$ are nondecreasing, $\beta(\omega_{\mathbb{M}}), \beta(\nu_{\mathbf{m}}) \in [0,\infty]$, from Proposition 1.2.32, (i) is valid.
(ii) By (i) and by Theorem 1.2.23, $\nu_{\mathbf{m}} \in ORV$ if and only if $\alpha(\nu_{\mathbf{m}}) < \infty$.

By Theorem 1.2.28, if $\alpha(\nu_{\mathbf{m}}) < \infty$, then there exists $\alpha > 0$ such that $\nu_{\mathbf{m}}(t)t^{-\alpha}$ is almost decreasing. Hence there exists $c > 0$ such that $\nu_{\mathbf{m}}(t)t^{-\alpha} \geq c\nu_{\mathbf{m}}(2t)(2t)^{-\alpha}$ for $t \geq X$. Subsequently, $\nu_{\mathbf{m}}(2t) = O(\nu_{\mathbf{m}}(t))$.

Conversely, if $\nu_{\mathbf{m}}(2t) = O(\nu_{\mathbf{m}}(t))$ there exist $\alpha, t_0 > 0$ such that $\nu_{\mathbf{m}}(2t) \leq 2^\alpha \nu_{\mathbf{m}}(t)$ for $t \geq t_0$. Then, for $s \geq t \geq t_0$ there exists $j \in \mathbb{N}_0$ such that $s \in [2^j t, 2^{j+1} t]$ and iterating the last inequality we see that

$$\frac{\nu_{\mathbf{m}}(s)}{s^\alpha} \leq \frac{\nu_{\mathbf{m}}(2^{j+1}t)}{(2^j t)^\alpha} \leq \frac{(2^{j+1})^\alpha \nu_{\mathbf{m}}(t)}{(2^j t)^\alpha} = 2^\alpha \frac{\nu_{\mathbf{m}}(t)}{t^\alpha}.$$

By suitably choosing $A \geq 2^\alpha$, we obtain that $\nu_{\mathbf{m}}(t)t^{-\alpha}$ is almost decreasing for $t \geq X$.
(iii) Analogous to (ii).
(iv) By Theorem 1.2.28, $\beta(\nu_{\mathbf{m}}) > 0$ if and only if there exists $\varepsilon > 0$ such that $\nu_{\mathbf{m}}(t)t^{-\varepsilon}$ is almost increasing. Then, if $\beta(\nu_{\mathbf{m}}) > 0$, there exists $c > 0$ such that for any $H \geq 1$ we have that

$$\nu_{\mathbf{m}}(Ht) \geq (Ht)^\varepsilon c \frac{\nu_{\mathbf{m}}(t)}{t^\varepsilon} = H^\varepsilon c \nu_{\mathbf{m}}(t), \qquad t \geq X.$$

We take H such that $H^\varepsilon c > 1$, then $\liminf_{t\to\infty} \nu_{\mathbf{m}}(Ht)/\nu_{\mathbf{m}}(t) > 1$. Reciprocally, if

$$\liminf_{t\to\infty} \nu_{\mathbf{m}}(Ht)/\nu_{\mathbf{m}}(t) > 1,$$

there exist $\varepsilon, t_0 > 0$ such that $\nu_{\mathbf{m}}(Ht) \geq H^{\varepsilon}\nu_{\mathbf{m}}(t)$ for $t \geq t_0$. Reasoning as in (ii), we conclude that $\nu_{\mathbf{m}}(t)t^{-\varepsilon}$ is almost increasing.
(v) Analogous to (iv).

□

Since $\nu_{\mathbf{m}} : [m_0, \infty) \to (0, \infty)$ is a locally integrable function, an easy consequence of (2.8) and Theorem 1.2.34 is the following:

Theorem 2.1.29. Let $\mathbb{M}$ be a weight sequence. Then the following are equivalent:

(i) $0 < \liminf_{t\to\infty} \dfrac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \leq \limsup_{t\to\infty} \dfrac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} < \infty$,

(ii) $\beta(\nu_{\mathbf{m}}) > 0$ and $\alpha(\nu_{\mathbf{m}}) < \infty$,

(iii) $\beta(\omega_{\mathbb{M}}) > 0$ and $\alpha(\omega_{\mathbb{M}}) < \infty$.

In this case, $\beta(\omega_{\mathbb{M}}) = \beta(\nu_{\mathbf{m}})$ and $\alpha(\omega_{\mathbb{M}}) = \alpha(\nu_{\mathbf{m}})$.

Proof. (i) $\Leftrightarrow$ (ii) Immediate from Theorem 1.2.34.
(i) and (ii) $\Rightarrow$ (iii) Again by Theorem 1.2.34, we have that $\beta(\omega_{\mathbb{M}}) = \beta(\nu_{\mathbf{m}}) > 0$ and $\alpha(\omega_{\mathbb{M}}) = \alpha(\nu_{\mathbf{m}}) < \infty$.
(iii) $\Rightarrow$ (i) From (2.8) and the monotonicity of $\nu_{\mathbf{m}}(t)$ for every $t > 0$ we get

$$\omega_{\mathbb{M}}(et) = \int_0^{et} \nu_{\mathbf{m}}(u)\frac{du}{u} \geq \int_t^{et} \nu_{\mathbf{m}}(u)\frac{du}{u} \geq \nu_{\mathbf{m}}(t).$$

Since $\alpha(\omega_{\mathbb{M}}) < \infty$, by definition, this means that

$$\limsup_{t\to\infty} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \leq \limsup_{t\to\infty} \frac{\omega_{\mathbb{M}}(et)}{\omega_{\mathbb{M}}(t)} < \infty.$$

Using that $\beta(\omega_{\mathbb{M}}) > 0$, again by definition, there exist $\beta, C > 0$ such that

$$\liminf_{t\to\infty} \frac{\omega_{\mathbb{M}}(\lambda t)}{\omega_{\mathbb{M}}(t)} \geq C\lambda^{\beta}, \qquad \text{for every} \quad \lambda > 0.$$

Taking $\lambda > (2/C)^{1/\beta}$, we see that $2\omega_{\mathbb{M}}(t) \leq \omega_{\mathbb{M}}(\lambda t)$ for $t \geq t_0$. By Lemma 1.1.24, we have that $\mathbb{M}$ has (mg), then, by Lemma 1.1.9, this implies that there exists $A > 1$ such that

$$\sup_{p\in\mathbb{N}} \frac{m_p}{M_p^{1/p}} \leq A < \infty.$$

Hence, for every $t \geq m_0$, there exists $p \in \mathbb{N}_0$ such that $t \in [m_p, m_{p+1})$ and we have that

$$\omega_{\mathbb{M}}(t) \leq \omega_{\mathbb{M}}(m_{p+1}) = (p+1)\log\left(\frac{m_{p+1}}{M_{p+1}^{1/(p+1)}}\right) \leq (p+1)\log(A) = \nu_{\mathbf{m}}(t)\log(A).$$

Consequently, we conclude that

$$\liminf_{t\to\infty} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \geq (\log(A))^{-1} > 0.$$

□

Specially relevant for our purposes, regarding the construction of proximate orders from weight sequences, is the next result that connects the upper order of $\omega_{\mathbb{M}}$ and $\nu_{\mathbb{M}}$ with the lower one of $\mathbf{m}$. In the following subsection, a more general dual relation between $\mathbf{m}$ and $\nu_{\mathbf{m}}$ is established.

Theorem 2.1.30. Let $\mathbb{M}$ be a weight sequence, then

$$\rho(\omega_{\mathbb{M}}) = \limsup_{t\to\infty} \frac{\log \omega_{\mathbb{M}}(t)}{\log t} = \rho(\nu_{\mathbf{m}}) = \limsup_{p\to\infty} \frac{\log(p)}{\log(m_p)} = \frac{1}{\mu(\mathbf{m})} = \frac{1}{\omega(\mathbb{M})}$$

(where the last quotient is understood as 0 if $\omega(\mathbb{M}) = \infty$, and as ∞ if $\omega(\mathbb{M}) = 0$). Moreover,

$$\mu(\omega_{\mathbb{M}}) = \liminf_{t\to\infty} \frac{\log \omega_{\mathbb{M}}(t)}{\log t} = \liminf_{t\to\infty} \frac{\log \nu_{\mathbf{m}}(t)}{\log t} = \mu(\nu_{\mathbf{m}}).$$

Proof. The first expression for $\rho(\omega_{\mathbb{M}})$ came from the very definition of the order because for $t > m_0$, $\omega_{\mathbb{M}}(t) > 0$. For the second expression of $\rho(\omega_{\mathbb{M}})$, we take into account the link given in (2.8) between $\omega_{\mathbb{M}}(t)$ and the counting function $\nu_{\mathbf{m}}(t)$. Since both functions are positive and nondecreasing for $t > m_0$, one may apply [32, Ch. 2, Th. 1.1], in which the following classical chain of inequalities is stated

$$\nu_{\mathbf{m}}(t) \le \omega_{\mathbb{M}}(et) \le \nu_{\mathbf{m}}(et)\log(et/m_0), \qquad t > m_0, \tag{2.10}$$

to deduce that the upper order of $\omega_{\mathbb{M}}(t)$ equals that of $\nu_{\mathbf{m}}(t)$.

Now, from [32, Ch. 2, Th. 1.8] we know that the upper order of $\nu_{\mathbf{m}}(t)$ is in turn the exponent of convergence of $\mathbf{m}$ (see Proposition 2.1.25), we conclude that $\rho(\nu_{\mathbf{m}}) = \lambda(\mathbf{m}) = 1/\mu(\mathbf{m})$ using Proposition 2.1.26, with $\gamma = 0$.

The last expression can be obtained as the first one, that is, from (2.10) but taking lim inf instead of lim sup. □

Remark 2.1.31. The main difficulty regarding the connection between $\mu(\omega_{\mathbb{M}})$, $\mu(\nu_{\mathbf{m}})$ and $\rho(\mathbf{m})$ is that there is not an extension of the notion of exponent of convergence in order to provide an analogous result to Proposition 2.1.26 for $\rho(\mathbf{m})$. This problem will be skipped in the next subsection by passing to a dual sequence (see Theorem 2.1.43).

Remark 2.1.32. We observe that if $\mathbb{M}$ and $\mathbb{L}$ are weight sequences with $\mathbb{M} \approx \mathbb{L}$ then, as in (1.7), there exists $A \ge 1$ such that

$$\omega_{\mathbb{L}}(A^{-1}t) \le \omega_{\mathbb{M}}(t) \le \omega_{\mathbb{L}}(At), \qquad t > 0,$$

and we can show that $\rho(\omega_{\mathbb{M}}) = \rho(\omega_{\mathbb{L}})$. As an easy consequence of the last theorem and Theorem 2.1.16, we have that

$$\omega(\mathbb{M}) = \mu(\mathbf{m}) = \mu(\boldsymbol{\ell}) = \omega(\mathbb{L}).$$

Together with Remark 2.1.24, this means that one can extend Lemma 2.1.12, that is, we have stability for $\approx$, for the two relevant indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$, as it will be shown in the next chapter, in the study of the asymptotic Borel map.

Remark 2.1.33. Ultraholomorphic and ultradifferentiable classes can also be defined in terms of a *weight function*, that is, a function $\omega : [0,\infty) \to [0,\infty)$ continuous, nondecreasing with $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$. For these functions Matuszewska indices and orders can be considered, in the same way we have done for the associated and the counting function. These values characterize growth conditions of the function $\omega(t)$, like the ones in Proposition 2.1.28 or others like the strong nonquasianalyticity [75, condition (ε)], that, similarly to the sequence case, describe elementary properties of the corresponding spaces. Since these classes will be out of the study in this dissertation, the reader is referred to our works [45, 47].

2.1.5 Dual sequence

In the next chapter, it will be shown that the value of the index $\omega(\mathbb{M})$ characterizes the injectivity of the Borel map for weight sequences and $\gamma(\mathbb{M})$ characterizes the surjectivity for strongly regular sequences. As we have seen, Theorem 2.1.16, these growth indices coincide with the lower order $\mu(\mathbf{m})$ and the lower Matuszewska index $\beta(\mathbf{m})$, respectively. One may naturally ask what the upper order $\rho(\mathbf{m})$ or the upper Matuszewska index $\alpha(\mathbf{m})$ stand for. In this subsection, a possible interpretation of their meaning is given by constructing a dual sequence $\mathbb{D}$ of a weight sequence $\mathbb{M}$ such that $\omega(\mathbb{D}) = 1/\rho(\mathbf{m})$ and $\gamma(\mathbb{D}) = 1/\alpha(\mathbf{m})$. This dual construction will be employed in Subsection 2.2.3 in order to associate a regularly varying sequence to a proximate order.

In the preceding subsections, it has been shown that the regular variation, the O-regular variation and the Matuszewska indices of a sequence $\boldsymbol{a}$ are characterized in terms of the step function $f_{\boldsymbol{a}}$ (see Theorem 1.2.37, Theorem 1.2.44 and Proposition 2.1.10). One might think when the opposite is true, i.e., when a function f can be described by its values at the natural numbers. The answer to this question will be used in the construction of the dual sequence through the counting function and in the examples at the end of the chapter. Starting with a nondecreasing function f, one can prove the following:

Lemma 2.1.34. Let $f : [N,+\infty) \to (0,+\infty)$, with $N \in \mathbb{N}$, be a nondecreasing function. For $p \in \mathbb{N}_0$, we define a sequence $a_p := f(p)$ for $p \geq N$ and $a_p := f(N)$ for $p < N$. Then, the following are equivalent:

(i) there exists $C \geq 1$ such that
$$a_{p+1} \leq C a_p, \quad p \in \mathbb{N}_0, \tag{2.11}$$

(ii) the function f satisfies
$$\sup_{x\geq N} \frac{f(x+1)}{f(x)} < +\infty. \tag{2.12}$$

Whenever any of the previous equivalent conditions holds, we have that there exists a constant $C \geq 1$ such that
$$C^{-1} f_{\boldsymbol{a}}(x) \leq f(x) \leq C f_{\boldsymbol{a}}(x) \qquad x \geq N, \tag{2.13}$$
where $f_{\boldsymbol{a}}(x) = a_{\lfloor x\rfloor}$. Similarly, the following are equivalent:

(iii) $\displaystyle\lim_{p\to\infty} \frac{a_{p+1}}{a_p} = 1$,

(iv) $\displaystyle\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = 1$.

If (iii) or (iv) holds, then $f_{\boldsymbol{a}} \sim f$, that is, $\lim_{x\to\infty} f_{\boldsymbol{a}}(x)/f(x) = 1$. Moreover, if f is continuous then (2.13) implies (2.11) and (2.12) and $f_{\boldsymbol{a}} \sim f$ implies (iii) and (iv).

Proof. (i) $\Rightarrow$ (ii) For $x \geq N$, since f is nondecreasing, we have that
$$f(x+1) \leq f(\lfloor x\rfloor + 2) = a_{\lfloor x\rfloor+2} \leq C^2 a_{\lfloor x\rfloor} = C^2 f(\lfloor x\rfloor) \leq C^2 f(x).$$

(ii) $\Rightarrow$ (i) For $p < N$ (i) holds by the definition of $\boldsymbol{a}$ and for $p \geq N$ it is immediate from (ii).

(iii) $\Rightarrow$ (iv) For $x \geq N$, since f is nondecreasing, we see that
$$1 \leq \frac{f(x+1)}{f(x)} \leq \frac{f(\lfloor x\rfloor+2)}{f(\lfloor x\rfloor+1)} \frac{f(\lfloor x\rfloor+1)}{f(\lfloor x\rfloor)},$$

and (iv) follows from (iii).
(iv) $\Rightarrow$ (iii) Immediate.

Furthermore, assuming (i) or, equivalently, (ii), since f is nondecreasing for $x \geq N$ we have that
$$f_{\boldsymbol{a}}(x) = a_{\lfloor x \rfloor} = f(\lfloor x \rfloor) \leq f(x) \leq f(\lfloor x \rfloor + 1) = a_{\lfloor x \rfloor + 1} \leq C a_{\lfloor x \rfloor} = C f_{\boldsymbol{a}}(x).$$
Similarly, for $x \geq N$ we observe that
$$1 \leq \frac{f(x)}{f(\lfloor x \rfloor)} \leq \frac{f(\lfloor x \rfloor + 1)}{f(\lfloor x \rfloor)},$$
so, assuming (iii) or (iv), $f_{\boldsymbol{a}} \sim f$.

In addition, if f is continuous and satisfies (2.13), for $p \geq N$, by the continuity in $x = p+1$, we have that for $\varepsilon = f(N) > 0$ there exists $\delta_{p,\varepsilon} \in (0,1)$ such that $0 \leq f(p+1) - f(p+1-\delta) < f(N)$. Consequently,
$$a_{p+1} = f(p+1) \leq 2f(p+1-\delta) \leq 2C f_{\boldsymbol{a}}(p+1-\delta) = 2C a_p, \qquad p \geq N,$$
and (i) is satisfied. Analogously, for f continuous we see that $f_{\boldsymbol{a}} \sim f$ implies (iii) and (iv). □

From the last result, we can establish an inverse version of the embedding theorems, Theorems 1.2.37 and 1.2.44, for nondecreasing functions.

Corollary 2.1.35. Let $f : [N, +\infty) \to (0, +\infty)$, with $N \in \mathbb{N}$, be a nondecreasing function. Then
$$f \quad \text{is regularly varying} \quad \text{if and only if} \quad \boldsymbol{a} \quad \text{also is.}$$
$$f \quad \text{is O-regularly varying} \quad \text{if and only if} \quad \boldsymbol{a} \quad \text{also is.}$$

Proof. If f is regularly varying, then $\lim_{x\to\infty} f(\lfloor x \rfloor)/f(x) = 1$ and we have that
$$\lim_{p\to\infty} \frac{a_{\lfloor \lambda p \rfloor}}{a_p} = \lim_{p\to\infty} \frac{f(\lfloor \lambda p \rfloor)}{f(\lambda p)} \frac{f(\lambda p)}{f(p)} = \lambda^\rho, \quad \lambda \in (0, \infty),$$
i.e., $\boldsymbol{a}$ also is. Conversely, if $\boldsymbol{a}$ is regularly varying, by Lemma 1.2.40 $\lim a_{p+1}/a_p = 1$, using the previous lemma we deduce that $f \sim f_{\boldsymbol{a}}$. Applying Theorem 1.2.37, we know that $f_{\boldsymbol{a}}$ is regularly varying and by Remark 1.2.6 we conclude that f also is.

Assuming that f is O-regularly varying, we see that
$$\limsup_{p\to\infty} \frac{a_{\lfloor \lambda p \rfloor}}{a_p} = \limsup_{p\to\infty} \frac{f(\lfloor \lambda p \rfloor)}{f(p)} \leq \limsup_{p\to\infty} \frac{f(\lambda p)}{f(p)} < \infty, \quad \lambda \in (0, \infty),$$
then $\boldsymbol{a}$ also is. Reciprocally, if $\boldsymbol{a}$ is O-regularly varying, then
$$\limsup_{p\to\infty} \frac{a_{p+1}}{a_p} \leq \limsup_{p\to\infty} \frac{a_{2p}}{a_p} < \infty,$$
so (2.11) holds. Consequently, there exists a constant $C \geq 1$ such that
$$C^{-1} f_{\boldsymbol{a}}(x) \leq f(x) \leq C f_{\boldsymbol{a}}(x) \qquad x \geq N,$$
and, applying Remark 1.2.31, we deduce that f is O-regularly varying because, by Theorem 1.2.44, $f_{\boldsymbol{a}} \in ORV$. □

In particular, if $f(2x) = O(f(x))$, or, equivalently, $\alpha(f) < \infty$ (see the proof of Proposition 2.1.28.(ii)), then it is plain to check that (2.12) holds, so we can represent our function f by the sequence $\boldsymbol{a}$.

Monotonicity hypothesis on f might be replaced by some weaker or different condition but, in the context of weight sequences and weight functions, this requirement is enough.

Remark 2.1.36. Condition (2.11) also appears in different contexts (see [80, (2.10)]). We know that if $\mathbb{M}$ is (lc) we have the following implications:

$$(\text{mg}) \Rightarrow (2.11) \Rightarrow (\text{dc}),$$

meaning that if $\mathbb{M}$ has (mg) then $\mathbf{m}$ satisfies (2.11) because, by Proposition 1.1.9, for all $p \in \mathbb{N}$ we see that

$$m_{p+1} \le m_{2p} \le C m_p,$$

and, by a simple computation, if $\mathbf{m}$ satisfies (2.11), then $\mathbb{M}$ satisfies (dc) since

$$M_{p+1} = m_0 m_1 \cdots m_p \le m_0 C^p m_0 m_1 \cdots m_{p-1} = m_0 C^p M_p, \quad p \in \mathbb{N}_0.$$

The converse implications fail in general. For the first one, we have the $q-$Gevrey sequences $\mathbb{M} = (q^{p^2})_{p\in\mathbb{N}_0}$, $q > 1$, do not satisfy (mg) but nevertheless (2.11) holds true. For the second one, we consider

$$m_p := \exp(2^j), \qquad 2^j \le p < 2^{j+1}, \quad j \in \mathbb{N}_0.$$

We observe that $\mathbf{m}$ is nondecreasing, $m_p = \exp(2^j) \le \exp(p)$ for $p \in \mathbb{N}$, then $\mathbb{M}$ satisfies (dc) and the quotient $m_{2^{j+1}}/m_{2^{j+1}-1} = \exp(2^j)$ is not bounded, then (2.11) is violated.

Remark 2.1.37. This is not the first approach to go from weight functions to weight sequences. In [17], where the connection between weight functions and regular variation was mentioned, from a function $g : [0,\infty) \to [1,\infty)$ they construct a sequence $\mathbb{M} = (((p+1)g(p))^p)_{p\in\mathbb{N}_0}$. Under suitable assumptions for g the authors show that the corresponding sequence $\mathbb{M}$ is strongly regular, that is, $\mathbb{M}$ is (lc), has (mg) and is (snq). In between these conditions, for a continuously differentiable increasing function g, three are connected to O-regular variation:

(I) $g(2x) = O(g(x))$ [17, Lemma 22.(1)].

(II) $\limsup_{x\to\infty}(g(x+1)/g(x))^x < \infty$ [17, Lemma 22.(2)].

(III) $\sup_{x\ge 0} xg'(x)/g(x) < \infty$ [17, Lemma 24.(i)].

For the construction of Example 25 [17] the function g was directly proved to satisfy (III) that implies the other two conditions (I) and (II).

It is plain to check that condition (I) is equivalent to $\alpha(g) < \infty$ (as in the proof of Proposition 2.1.28.(ii)). Regarding condition (III), one may observe that for $g : [0,\infty) \to [1,\infty)$ continuously differentiable and increasing with $g(0) = 1$, we can write

$$g(x) = 1 + \int_0^x t g'(t) \frac{dt}{t}, \qquad x \ge 0,$$

which leads to Theorem 1.2.34, since $h(t) = tg'(t)$ is a positive locally integrable function. Consequently, from the point of view of O-regular variation, one may alternatively assume that $\alpha(g') < \infty$ then, by using the almost decreasing characterization, $\alpha(h) < \infty$ and, by Theorem 1.2.34.(i), (III) is satisfied. Hence also (I) and (II) hold and the corresponding sequence $\widehat{\mathbb{M}}$ has also the expected properties.

Theorem 2.1.30 suggests that one may also find relations for the Matuszewska indices of $\mathbf{m}$, $\nu_{\mathbf{m}}$ and $\omega_{\mathbb{M}}$. This connection has been partially studied for functions in a recent work of D. Djurčić, R. Nikolić and A. Torgašev [26], where they analyze the O-regularly varying duality between a positive nondecreasing unbounded function $f : [X, \infty) \to (0, \infty)$ and the function

$$f^{\leftarrow}(x) := \inf\{y \geq X; f(y) > x\} = \sup\{y \geq X; f(y) \leq x\}.$$

for $x \geq f(X)$. Even if some information can be inferred from their proofs, there is not a explicit correspondence between the indices. In our situation, we will show the duality between a weight sequence and its counting function.

Proposition 2.1.38. Let $\mathbb{M}$ be a weight sequence. Then

$$\beta(\mathbf{m}) = \frac{1}{\alpha(\nu_{\mathbf{m}})}, \qquad \alpha(\mathbf{m}) = \frac{1}{\beta(\nu_{\mathbf{m}})},$$

(with the typical conventions for 0 and ∞). Consequently, we recover the classical equivalence for the growth conditions of $\mathbb{M}$ and $\nu_{\mathbf{m}}$:

(i) $\mathbb{M}$ has (mg) if and only if there exists $H \geq 1$ such that $\liminf_{t\to\infty} \nu_{\mathbf{m}}(Ht)/\nu_{\mathbf{m}}(t) > 1$ (see [80, Lemma 2.2]).

(ii) $\mathbb{M}$ satisfies (snq) if and only if $\nu_{\mathbf{m}}(2t) = O(\nu_{\mathbf{m}}(t))$ (see [17, Lemma 12] for one of the implications).

Proof. First we assume that $0 < \gamma < \beta(\mathbf{m})$, so, by Proposition 2.1.10, $(p^{-\gamma} m_p)_{p\in\mathbb{N}}$ is almost increasing, then $(p^{-1} {m_p}^{1/\gamma})_{p\in\mathbb{N}}$ is almost increasing with constant $D \geq 1$. We have that for every $t \geq s \geq m_0$, there exist $p, q \in \mathbb{N}_0$ such that $q \geq p$, $s \in [m_p, m_{p+1})$ and $t \in [m_q, m_{q+1})$. If $q = p$, we see that

$$\frac{\nu_{\mathbf{m}}(s)}{s^{1/\gamma}} = \frac{\nu_{\mathbf{m}}(t)}{s^{1/\gamma}} \geq \frac{\nu_{\mathbf{m}}(t)}{t^{1/\gamma}},$$

and if $q \geq p + 1 \geq 1$, we get

$$\frac{\nu_{\mathbf{m}}(s)}{s^{1/\gamma}} \geq \frac{p+1}{(m_{p+1})^{1/\gamma}} \geq \frac{q}{D(m_q)^{1/\gamma}} \geq \frac{q}{q+1}\frac{q+1}{Dt^{1/\gamma}} \geq \frac{\nu_{\mathbf{m}}(t)}{2Dt^{1/\gamma}},$$

that is, $\nu_{\mathbf{m}}(t)/t^{1/\gamma}$ is almost decreasing, then $1/\gamma \geq \alpha(\nu_{\mathbf{m}})$ and $1/\beta(\mathbf{m}) \geq \alpha(\nu_{\mathbf{m}})$.

Correspondingly, if $\gamma > \alpha(\mathbf{m})$, then $((p+1)^{-1} {m_p}^{1/\gamma})_{p\in\mathbb{N}_0}$ is almost decreasing with constant $d \in (0, 1)$. For $t \geq s \geq m_0$ taking $p, q \in \mathbb{N}_0$ as before we see that

$$\frac{\nu_{\mathbf{m}}(s)}{s^{1/\gamma}} \leq \frac{p+1}{(m_p)^{1/\gamma}} \leq \frac{q+2}{d(m_{q+1})^{1/\gamma}} \leq d^{-1}\frac{q+2}{q+1}\frac{q+1}{(t)^{1/\gamma}} \leq 2d^{-1}\frac{\nu_{\mathbf{m}}(t)}{t^{1/\gamma}},$$

then $1/\gamma \leq \beta(\nu_{\mathbf{m}})$ and $1/\alpha(\mathbf{m}) \leq \beta(\nu_{\mathbf{m}})$.

Reciprocally, if $\gamma > \alpha(\nu_{\mathbf{m}})$, there exists $\varepsilon > 0$ such that $\gamma - \varepsilon > \alpha(\nu_{\mathbf{m}})$, so $\gamma - \varepsilon > 0$, since $\alpha(\nu_{\mathbf{m}}) \geq 0$ by Proposition 2.1.28.(i). Then, $\nu_{\mathbf{m}}(t)/t^{\gamma-\varepsilon}$ is almost decreasing which implies that there exists $d \in (0, 1)$ such that for every $\lambda \geq 1$ and all $t \geq m_0$ we have that

$$\nu_{\mathbf{m}}(t) \geq d\nu_{\mathbf{m}}(\lambda t)/\lambda^{\gamma-\varepsilon}.$$

We fix $Q \in \mathbb{N}$, large enough, such that $Q^{(\varepsilon/2)/(\gamma-\varepsilon/2)} d \geq 1$ and taking $\lambda = Q^{1/(\gamma-\varepsilon/2)}$ we see that

$$\nu_{\mathbf{m}}(t)Q \geq \nu_{\mathbf{m}}(Q^{1/(\gamma-\varepsilon/2)}t), \qquad t \geq m_0. \tag{2.14}$$

Using (2.14), for $p \in \mathbb{N}$, we observe that

$$\begin{aligned} m_p =& \sup\{t \geq m_0;\ \nu_{\mathbf{m}}(t) \leq p\} \leq \sup\{t \geq m_0;\ \nu_{\mathbf{m}}(Q^{1/(\gamma-\varepsilon/2)}t) \leq Qp\} \\ =& Q^{-1/(\gamma-\varepsilon/2)} \sup\{s \geq Q^{1/(\gamma-\varepsilon/2)}m_0;\ \nu_{\mathbf{m}}(s) \leq Qp\} \leq \frac{m_{Qp}}{Q^{1/(\gamma-\varepsilon/2)}}. \end{aligned}$$

Hence we have shown that there exist $Q \in \mathbb{N}$, $Q \geq 2$ and $\delta > 0$ such that

$$\liminf_{p\to\infty} \frac{m_{Qp}}{Q^{1/\gamma} m_p} \geq Q^{\delta} > 1.$$

By Proposition 2.1.22.(viii), we obtain that $1/\gamma \leq \beta(\mathbf{m})$ and $1/\alpha(\nu_{\mathbf{m}}) \leq \beta(\mathbf{m})$. Analogously, if $0 < \gamma < \beta(\nu_{\mathbf{m}})$, there exists $\varepsilon > 0$ with $\gamma + \varepsilon < \beta(\nu_{\mathbf{m}})$ and $Q \in \mathbb{N}$, large enough, such that

$$\nu_{\mathbf{m}}(t)Q \leq \nu_{\mathbf{m}}(Q^{1/(\gamma+\varepsilon/2)}t), \qquad t \geq m_0.$$

For $p \in \mathbb{N}$, large enough, we observe that

$$\begin{aligned} m_p =& \sup\{t \geq m_0;\ \nu_{\mathbf{m}}(t) \leq p\} \geq \sup\{t \geq m_0;\ \nu_{\mathbf{m}}(Q^{1/(\gamma+\varepsilon/2)}t) \leq Qp\} \\ =& Q^{-1/(\gamma+\varepsilon/2)} \sup\{s \geq Q^{1/(\gamma+\varepsilon/2)}m_0;\ \nu_{\mathbf{m}}(s) \leq Qp\} = \frac{m_{Qp}}{Q^{1/(\gamma+\varepsilon/2)}}. \end{aligned}$$

By Proposition 2.1.22.(iv), we obtain that $1/\gamma \geq \alpha(\mathbf{m})$ and $1/\beta(\nu_{\mathbf{m}}) \geq \alpha(\mathbf{m})$.

Finally, by Proposition 2.1.18, $\mathbb{M}$ has (mg) if and only if $\alpha(\mathbf{m}) < \infty$ if and only if $\beta(\nu_{\mathbf{m}}) > 0$ which, by Proposition 2.1.28, is equivalent to the existence of $H \geq 1$, such that

$$\liminf_{t\to\infty} \nu_{\mathbf{m}}(Ht)/\nu_{\mathbf{m}}(t) > 1.$$

Similarly, we see that $\mathbb{M}$ satisfies (snq) if and only if $\nu_{\mathbf{m}}(2t) = O(\nu_{\mathbf{m}}(t))$. □

Remark 2.1.39. In particular, for a weight sequence $\mathbb{M}$, by using the previous result and Proposition 2.1.29, we can increase the list of alternative definitions of strong regularity in Corollary 2.1.6 and Remark 2.1.19, that is, the following are equivalent:

(i) $\mathbb{M}$ is strongly regular,

(v) $0 < \liminf_{t\to\infty} \dfrac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \leq \limsup_{t\to\infty} \dfrac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} < \infty$,

(vi) $\alpha(\nu_{\mathbf{m}}) < \infty$ and $\beta(\nu_{\mathbf{m}}) > 0$,

(vii) $\alpha(\omega_{\mathbb{M}}) < \infty$ and $\beta(\omega_{\mathbb{M}}) > 0$.

In this case, we also have that $\alpha(\omega_{\mathbb{M}}) = \alpha(\nu_{\mathbf{m}}) < \infty$ and $\beta(\omega_{\mathbb{M}}) = \beta(\nu_{\mathbf{m}}) > 0$ and $\rho(\omega_{\mathbb{M}})$, $\mu(\omega_{\mathbb{M}})$, $\rho(\nu_{\mathbf{m}})$, $\mu(\nu_{\mathbf{m}}) \in (0, \infty)$.

This suggests a possible explanation, which was also pointed out in [57], of some of the facts described at the end of Remark 2.1.23, one may assert that $\nu_{\mathbf{m}}$ is a dual function in terms of O-regularly varying behavior of the sequence $\mathbf{m}$, but if we have strongly regularity we can replace $\nu_{\mathbf{m}}$ by $\omega_{\mathbb{M}}$. More concretely, for a weight sequence $\mathbb{M}$, we have that

(1) $\alpha(\mathbf{m}) < \infty$ if and only if $\beta(\omega_{\mathbb{M}}) > 0$ (see [80, Lemma 2.2]).

(2) $\beta(\mathbf{m}) > 0$ implies $\alpha(\omega_{\mathbb{M}}) < \infty$ (see [17, Lemma 12]).

These conditions can be read in terms of growth properties (see Proposition 2.1.18 and Proposition 2.1.28). The fact that the implication in (2) can not be reversed (see our example in [47] where $\beta(m) = 0$ and $\alpha(\omega_{\mathbb{M}}) = 0$) might clarify why the weight function approach (see Remark 2.1.33) is more general than the weight sequence case, meaning that, even if $\alpha(\mathbf{m}) = \infty$, so $\beta(\omega_{\mathbb{M}}) = 0$, and $\beta(\mathbf{m}) = 0$ we may have $\alpha(\omega_{\mathbb{M}}) < \infty$. In other words $\omega_{\mathbb{M}}$ might be of O-regular variation even though $\mathbf{m}$ is quite pathological. Whereas if $\alpha(\omega_{\mathbb{M}}) = \infty$ and $\beta(\omega_{\mathbb{M}}) = 0$, then automatically $\alpha(\mathbf{m}) = \infty$ and $\beta(\mathbf{m}) = 0$.

Accordingly, the following notion of the dual sequence is proposed.

Definition 2.1.40. Let $\mathbb{M}$ be a weight sequence. For $p \in \mathbb{N}_0$, we define its *dual sequence* $\mathbb{D}$ using the sequence of quotients by

$$d_p := \nu_{\mathbf{m}}(p), \quad p \geq m_0; \qquad d_p := 1, \quad p < m_0.$$

Correspondingly, $D_0 := 1$ and $D_p := d_0 d_1 \cdots d_{p-1}$ for $p \in \mathbb{N}$.

It is plain to check that $\mathbb{D}$ is also a weight sequence and we can consider the bidual sequence.

Definition 2.1.41. Let $\mathbb{M}$ be a weight sequence and $\mathbb{D}$ its dual sequence. For $p \in \mathbb{N}_0$, we define its *bidual sequence* $\mathbb{E}$, as the dual sequence of $\mathbb{D}$, that is,

$$e_p := \nu_{\boldsymbol{d}}(p), \quad p \geq 1 = d_0; \qquad e_0 := 1,$$

and $E_0 := 1$ and $E_p := e_0 e_1 \cdots e_{p-1}$ for $p \in \mathbb{N}$.

Since $\nu_{\mathbf{m}}, \nu_{\boldsymbol{d}} : [0, \infty) \to \mathbb{N}_0$, $\boldsymbol{d}, \boldsymbol{e}$ are sequences of natural numbers and we can establish the expected connection between $\mathbb{M}$ and $\mathbb{E}$.

Theorem 2.1.42. Let $\mathbb{M}$ be a weight sequence and $\mathbb{E}$ its bidual. Then, $\mathbf{m} \simeq \boldsymbol{e}$, i.e., there exists $c > 1$ such that

$$c^{-1} e_p \leq m_p \leq c\, e_p, \qquad \text{for all} \qquad p \in \mathbb{N}_0.$$

In fact, $\mathbf{m} \sim \boldsymbol{e}$, that is, $\lim_{p\to\infty} e_p/m_p = 1$.

Proof. For $p > \max(1, \nu_{\mathbf{m}}(\lfloor m_0 \rfloor + 1))$, we can ensure that

$$\begin{aligned} e_p = \nu_{\boldsymbol{d}}(p) &= \max\{j \in \mathbb{N}; d_{j-1} \leq p\} = \max\{j \in \mathbb{N}; \nu_{\mathbf{m}}(j-1) \leq p\} \\ &= \max\{j \in \mathbb{N}; \max\{k \in \mathbb{N}; m_{k-1} \leq j-1\} \leq p\}. \end{aligned}$$

If $j - 1 \geq m_p$, then $\max\{k \in \mathbb{N}; m_{k-1} \leq j-1\} \geq p+1$ and $j > e_p$. Conversely, if $j - 1 < m_p$, then $\max\{k \in \mathbb{N}; m_{k-1} \leq j-1\} < p+1$, then $j \leq e_p$. For all $p > \max(1, \nu_{\mathbf{m}}(\lfloor m_0 \rfloor + 1))$ we deduce that

$$e_p = \max\{j \in \mathbb{N}; j - 1 < m_p\}.$$

Consequently, for every $p > \max(1, \nu_{\mathbf{m}}(\lfloor m_0 \rfloor + 1))$ we have shown that

$$e_p - 1 < m_p \leq e_p.$$

and we conclude that

$$m_p \leq e_p < m_p + 1 \leq (1 + m_0^{-1}) m_p.$$

By suitably choosing a constant $c \geq (1 + m_0^{-1})$, we can extend these inequalities for $p \leq \max(1, \nu_{\mathbf{m}}(\lfloor m_0 \rfloor + 1))$ and we see that $\mathbf{m} \simeq \boldsymbol{e}$. Moreover, since $\lim_{p\to\infty} m_p = \infty$, we conclude that $\mathbf{m} \sim \boldsymbol{e}$. □

We want to make use of Proposition 2.1.38 to study the O-regularly varying behavior of $\mathbb{D}$ and $\mathbb{E}$, in terms of $\nu_{\mathbf{m}}$ and $\nu_{\boldsymbol{d}}$, respectively. Hence, we need to use Lemma 2.1.34 and it seems reasonable to assume that $\nu_{\mathbf{m}}$ and $\nu_{\boldsymbol{d}}$ satisfy (2.12). However, since $\mathbf{m} \simeq \boldsymbol{e}$, $\mathbf{m}$ satisfies (2.11) if and only if $\boldsymbol{e}$ also does, or equivalently, by Lemma 2.1.34, if $\nu_{\boldsymbol{d}}$ satisfies (2.12). Let us see that this condition is enough for our purposes.

Theorem 2.1.43. Let $\mathbb{M}$ be a weight sequence such that $\mathbf{m}$ satisfies (2.11). The following relation for orders and Matuszewska indices holds

$$\beta(\mathbf{m}) = \frac{1}{\alpha(\boldsymbol{d})}, \quad \mu(\mathbf{m}) = \frac{1}{\rho(\boldsymbol{d})}, \quad \rho(\mathbf{m}) = \frac{1}{\mu(\boldsymbol{d})}, \quad \alpha(\mathbf{m}) = \frac{1}{\beta(\boldsymbol{d})}.$$

Consequently, we have that some of the growth properties for $\mathbb{M}$ and $\mathbb{D}$ are reflected:

(i) $\mathbb{M}$ has (mg) if and only if $\mathbb{D}$ satisfies (snq),

(ii) $\mathbb{M}$ satisfies (snq) if and only if $\mathbb{D}$ has (mg).

In particular, if $\mathbb{M}$ is strongly regular, all the previous indices are positive real numbers and $\mathbb{D}$ is also strongly regular.

Proof. Since $\mathbf{m}$ satisfies (2.11) and $\mathbf{m} \simeq \boldsymbol{e}$, then $\boldsymbol{e}$ also does. Since $e_p = \nu_{\boldsymbol{d}}(p)$ for $p \in \mathbb{N}$, by Lemma 2.1.34 and Remark 1.2.31, this means that

$$\beta(\boldsymbol{e}) = \beta(f_{\boldsymbol{e}}) = \beta(\nu_{\boldsymbol{d}}), \quad \rho(\boldsymbol{e}) = \rho(f_{\boldsymbol{e}}) = \rho(\nu_{\boldsymbol{d}}), \quad \alpha(\boldsymbol{e}) = \alpha(f_{\boldsymbol{e}}) = \alpha(\nu_{\boldsymbol{d}}).$$

By Proposition 2.1.38 applied to $\mathbb{D}$, we get

$$\beta(\boldsymbol{e}) = \beta(\nu_{\boldsymbol{d}}) = \frac{1}{\alpha(\boldsymbol{d})}, \qquad \alpha(\boldsymbol{e}) = \alpha(\nu_{\boldsymbol{d}}) = \frac{1}{\beta(\boldsymbol{d})}.$$

Since $\boldsymbol{e} \simeq \mathbf{m}$, we conclude that $\beta(\mathbf{m}) = 1/\alpha(\boldsymbol{d})$ and $\alpha(\mathbf{m}) = 1/\beta(\boldsymbol{d})$. Moreover, using Theorem 2.1.30 for $\mathbb{D}$, we know that

$$\rho(\mathbf{m}) = \rho(\boldsymbol{e}) = \rho(f_{\boldsymbol{e}}) = \rho(\nu_{\boldsymbol{d}}) = 1/\mu(\boldsymbol{d}).$$

Finally, for $t > \max(2, m_0 + 1)$ we observe that

$$\frac{\log(t-1)}{\log(\lfloor t \rfloor)} \frac{\log(\nu_{\mathbf{m}}(t-1))}{\log(t-1)} \leq \frac{\log(\nu_{\mathbf{m}}(\lfloor t \rfloor))}{\log(\lfloor t \rfloor)} \leq \frac{\log(\nu_{\mathbf{m}}(t))}{\log(t)} \frac{\log(t)}{\log(\lfloor t \rfloor)}.$$

then $\rho(\nu_{\mathbf{m}}) = \rho(\boldsymbol{d})$. Applying Theorem 2.1.30 for $\mathbb{M}$, we deduce that $\rho(\boldsymbol{d}) = 1/\mu(\mathbf{m})$. □

Condition (2.11) characterizes (2.13) for continuous nondecreasing functions, but since $\nu_{\mathbf{m}}$ and $\nu_{\boldsymbol{d}}$ are only nondecreasing there is some hope that this condition can be skipped by going directly from the indices of $\mathbf{m}$ to the indices of $\boldsymbol{d}$. However, as commented in Remark 2.1.36, assuming (2.11) it is not a big restriction and it is enough to illustrate the reflection between $\mathbb{D}$ and $\mathbb{M}$.

Remark 2.1.44. According to Remark 2.1.39, if $\mathbb{M}$ is strongly regular then $\nu_{\mathbf{m}}$ is O-regularly varying and

$$0 < \liminf_{t\to\infty} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \leq \limsup_{t\to\infty} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} < \infty.$$

Moreover, there exists $p_0 \in \mathbb{N}$ such that $\omega_{\mathbb{M}}(t) \geq 1$ for $t \geq p_0$, we can consider the sequence:

$$t_p := \omega_{\mathbb{M}}(p), \quad p \geq p_0; \qquad t_p := 1, \quad p < p_0.$$

Hence $\boldsymbol{d} \simeq \boldsymbol{t}$ and

$$\beta(\mathbf{m}) = \frac{1}{\alpha(\boldsymbol{t})}, \quad \mu(\mathbf{m}) = \frac{1}{\rho(\boldsymbol{t})}, \quad \rho(\mathbf{m}) = \frac{1}{\mu(\boldsymbol{t})}, \quad \alpha(\mathbf{m}) = \frac{1}{\beta(\boldsymbol{t})}.$$

Example 2.1.45. Thanks to the connection with the associated function given in the previous remark and the computations in Example 1.1.22 or also by a direct calculation, we can show that the dual sequence of the Gevrey sequence $\mathbb{M}_{\alpha,0} = (p!^{\alpha})_{p\in\mathbb{N}_0}$ is equivalent to $\mathbb{M}_{1/\alpha,0} = (p!^{1/\alpha})_{p\in\mathbb{N}_0}$. However, since the Gevrey sequences and the sequences $\mathbb{M}_{\alpha,\beta} = (n!^{\alpha}\Pi_{m=0}^{n}\log^{\beta}(e+m))_{n\in\mathbb{N}_0}$ with $\alpha > 0$ and $\beta \in \mathbb{R}$ are regularly varying, all the indices are equal (see Example 2.1.20), the duality is hidden.

For the $q-$Gevrey sequences $\mathbb{M} = (q^{p^2})_{p\in\mathbb{N}_0}$, with $q > 1$, we can consider the sequences $\mathbb{D}$ and $\mathbb{T}$ defined from $\nu_{\mathbf{m}}(t)$ and $\omega_{\mathbb{M}}(t)$, respectively. By Example 2.1.20, we know that $\alpha(\mathbf{m}) = \beta(\mathbf{m}) = \infty$ and since $\mathbb{M}$ satisfies (2.11), we obtain that $\alpha(\boldsymbol{d}) = \beta(\boldsymbol{d}) = 0$, so the duality is also concealed. By a simple computation, we can notice the following

$$\boldsymbol{d} \simeq (\log(p+e))_{p\in\mathbb{N}}, \qquad \boldsymbol{t} \simeq ((\log(p+e))^2)_{p\in\mathbb{N}},$$

in this case, $\boldsymbol{d} \not\simeq \boldsymbol{t}$ as expected by the characterization of strong regularity in Remark 2.1.39.

Remark 2.1.46. It is worthy to mention that the duality is not preserved for $\approx$ or $\simeq$, this means that in general some information is lost when passing to the dual sequence. For example we consider the sequences $\mathbb{M}$ and $\mathbb{L}$ defined for all $k \in \mathbb{N}$ in terms of their sequences of quotients:

$$m_0 = m_1 = m_2 = m_3 := 1, \quad m_p := k \qquad \text{for every} \quad 2^{(2k)^k} \le p < 2^{(2(k+1))^{k+1}},$$

$$\ell_0 = \ell_1 = \ell_2 = \ell_3 := 1, \quad \ell_p := k+1 \qquad \text{for every} \quad 2^{(2k)^k} \le p < 2^{(2(k+1))^{k+1}}.$$

Evidently, $\mathbb{M}$ and $\mathbb{L}$ are weight sequences of moderate growth with $\mathbf{m} \simeq \boldsymbol{\ell}$. We can compute their duals, for $k \ge 2$ we have that

$$d_k^{\mathbb{M}} = \nu_{\mathbf{m}}(k) = \max\{j \in \mathbb{N};\, m_{j-1} \le k\} = 2^{(2(k+1))^{k+1}},$$

$$d_k^{\mathbb{L}} = \max\{j \in \mathbb{N};\, \ell_{j-1} \le k\} = 2^{(2k)^k}.$$

For all $k \ge 2$, we observe that

$$\left(\frac{D_k^{\mathbb{M}}}{D_k^{\mathbb{L}}}\right)^{1/k} \ge \frac{(d_{k-1}^{\mathbb{M}})^{1/k}}{d_{k-1}^{\mathbb{L}}} = \frac{2^{2^k k^{k-1}}}{2^{2^{k-1}(k-1)^{k-1}}} = \exp\left(\log(2)2^{k-1}(k-1)^{k-1}\left(2\left(\frac{k}{k-1}\right)^{k-1} - 1\right)\right).$$

Hence $\mathbb{D}^{\mathbb{M}} \not\approx \mathbb{D}^{\mathbb{L}}$, because the left hand side is unbounded as k tends to ∞.

However, if we add some regularity the equivalence is kept. For instance, if $\mathbb{M}$ and $\mathbb{L}$ are weight sequences with $\mathbf{m} \simeq \boldsymbol{\ell}$ and $\beta(\mathbf{m}) > 0$, so $\beta(\boldsymbol{\ell}) > 0$, there exists $Q \in \mathbb{N}$ such that for t large enough we have that

$$\nu_{\mathbf{m}}(t) \le \nu_{\boldsymbol{\ell}}(Qt), \qquad \nu_{\boldsymbol{\ell}}(t) \le \nu_{\mathbf{m}}(Qt).$$

By Proposition 2.1.38, $\alpha(\nu_{\mathbf{m}}) < \infty$ and $\alpha(\nu_{\boldsymbol{\ell}}) < \infty$, so we conclude that $\boldsymbol{d}^{\mathbb{M}} \simeq \boldsymbol{d}^{\mathbb{L}}$. In particular, this stability holds for strongly regular sequences.

2.2 Log-convex sequences, regular variation and nonzero proximate orders

The construction of nontrivial 'fine' flat functions belonging to ultraholomorphic classes and defined in sectors of optimal opening, on which is based the $\mathbb{M}-$summability theory developed in [60, 88] by A. Lastra, J. Sanz and S. Malek, depends on the possibility of associating with the weight sequence, that defines the class, a nonzero proximate order. The link between proximate orders (see Definition 1.2.7) and a weight sequences $\mathbb{M}$, is given by the function

$$d_{\mathbb{M}}(t) := \frac{\log(\omega_{\mathbb{M}}(t))}{\log(t)}, \quad \text{for } t \text{ large enough.} \tag{2.15}$$

The characterization of the sequences for which $d_{\mathbb{M}}$ is a nonzero proximate order was an open question, successfully answered in Theorem 2.2.6 below. As a byproduct of this result, a new characterization of regularly varying sequences has been obtained.

Example 2.2.22 provides a strongly regular sequence $\mathbb{M}$, equivalent for $\simeq$ to $\mathbb{L} = (p!)_{p\in\mathbb{N}_0}$, such that $d_{\mathbb{M}}(t)$ is not a proximate order. However, it will be proved in Example 2.2.8 that $d_{\mathbb{L}}$ is a nonzero proximate order (in particular, we deduce that the property of $d_{\mathbb{M}}$ being a proximate order is not stable under equivalence of sequences, neither $\approx$ nor $\simeq$). So, we may obtain a satisfactory summability theory in the Carleman ultraholomorphic class associated with $\mathbb{L}$, which coincides with that associated with $\mathbb{M}$. This shows that asking for $d_{\mathbb{M}}$ to be a nonzero proximate order is a too demanding restriction and one could ask instead for:

(f) There exists a weight sequence $\mathbb{L}$ such that $\mathbb{L} \approx \mathbb{M}$ and $d_{\mathbb{L}}(t)$ is a nonzero proximate order.

On the other hand, J. Sanz had already observed [88, Remark 4.11(iii)] that, for the construction of nontrivial flat functions in sectors of optimal opening, $d_{\mathbb{M}}$ need not be a nonzero proximate order, but it is enough that there exist nonzero proximate orders close enough to $d_{\mathbb{M}}$, in the following sense:

Definition 2.2.1. Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. We say that $\mathbb{M}$ *admits a proximate order* if there exists a proximate order $\rho(t)$ and constants A and B such that

$$A \le \log(t)(\rho(t) - d_{\mathbb{M}}(t)) \le B, \qquad \text{for } t \text{ large enough,}$$

or, equivalently, if

$$e^A \le \frac{t^{\rho(t)}}{\omega_{\mathbb{M}}(t)} \le e^B, \qquad \text{for } t \text{ large enough.}$$

In Subsection 2.2.4, we will show that the requirement (f) and the admissibility of a nonzero proximate order are equivalent for a weight sequence $\mathbb{M}$ and we provide a Representation Theorem for such sequences. In order to prove this, we need to construct well-behaved sequences from proximate orders, employing the duality presented in Subsection 2.1.5. Finally, several pathological examples of strongly regular sequences, which will be mentioned along the chapter, are provided.

The results contained in this section are the core of our work [44]. Nevertheless, here some new information is given, for instance the Representation Theorem 2.2.19, the relation between the conditions (f), (j), (k) and (ℓ) is clarified through some counterexamples (see Remark 2.2.18) and Subsection 2.2.3, corresponding to Section 4.1 in the paper, has been completely rewritten with a different and more simple approach.

2.2.1 A new characterization of regular variation

The main aim of this subsection is to provide a new characterization of the regular variation of the sequence of quotients $\mathbf{m}$ of any sequence $\mathbb{M}$ of positive real numbers with $\mathbb{M}_0 = 1$. This result, interesting in its own right, will be used in the next subsection, Theorem 2.2.6, to describe, in terms of the existence of a limit, when it is possible to construct a proximate order from a weight sequence.

For convenience, given a sequence $\mathbb{M}$ of positive real numbers and the corresponding sequence of quotients $\mathbf{m} = (m_p)_{p\in\mathbb{N}_0}$ we define the auxiliary sequences

$$\alpha_p := \log(m_p), \quad p \in \mathbb{N}_0; \qquad \beta_0 := \alpha_0, \qquad \beta_p := \log\left(\frac{m_p}{M_p^{1/p}}\right), \quad p \geq 1. \tag{2.16}$$

From the relation between $\mathbb{M}$ and $\mathbf{m}$ we deduce a elementary connection between $(\alpha_p)_{p\in\mathbb{N}_0}$ and $(\beta_p)_{p\in\mathbb{N}_0}$.

Lemma 2.2.2. Given a sequence $\mathbb{M}$ of positive real numbers with $M_0 = 1$, for all $p \in \mathbb{N}_0$ we have

$$\beta_p = \alpha_p - \frac{1}{p}\sum_{k=0}^{p-1}\alpha_k, \tag{2.17}$$

$$\alpha_p = \sum_{k=0}^{p-1}\frac{\beta_k}{k+1} + \beta_p. \tag{2.18}$$

Proof. From the definition of $(\beta_p)_{p\in\mathbb{N}}$ we have that $\beta_0 = \alpha_0$, and for all $p \in \mathbb{N}$

$$\beta_p = \log\left(\frac{m_p}{M_p^{1/p}}\right) = \log(m_p) - \frac{1}{p}\log(m_0 m_1 \cdots m_{p-1}) = \alpha_p - \frac{1}{p}\sum_{k=0}^{p-1}\alpha_k,$$

then (2.17) holds.

For the proof of (2.18) we apply induction. It immediately holds for $p = 0$, and if we admit its validity for some $p \in \mathbb{N}_0$, then

$$\alpha_{p+1} = \log(m_{p+1}) = \alpha_p + \log\big(\frac{m_{p+1}}{m_p}\big) = \sum_{k=0}^{p-1}\frac{\beta_k}{k+1} + \beta_p + \log\big(\frac{m_{p+1}}{m_p}\big)$$
$$= \sum_{k=0}^{p}\frac{\beta_k}{k+1} + \frac{p}{p+1}\beta_p + \log\big(\frac{m_{p+1}}{m_p}\big).$$

So, we are done if it holds that

$$\frac{p}{p+1}\beta_p + \log\big(\frac{m_{p+1}}{m_p}\big) = \beta_{p+1},$$

but this equality can be easily checked by direct manipulation. □

As it has happened in the previous section when dealing with the sequence of quotients $\mathbf{m}$ of $\mathbb{M}$, that is defined for $p \in \mathbb{N}_0$, an index shift inconvenience arises. It can be solved thanks to Lemma 1.2.40 and we can state the following proposition.

Proposition 2.2.3. Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive real numbers. The following are equivalent:

(i) There exists $\lim_{p\to\infty}\log\big(m_p/M_p^{1/p}\big) \in \mathbb{R}$,

(ii) $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ is regularly varying.

In case any of these statements holds, the value of the limit in (i) and the index of regular variation of $\mathbf{m}$ are the same.

Proof. (i) $\Rightarrow$ (ii) We call ω the value of the limit in (i). If we consider the sequences $(\alpha_p)_{p\in\mathbb{N}_0}$ and $(\beta_p)_{p\in\mathbb{N}_0}$ defined in (2.16), we will show that

$$\lim_{p\to\infty}(\alpha_{\lfloor\lambda p\rfloor}-\alpha_p)=\omega\log(\lambda),\qquad \lambda>0,$$

which, by definition, shows that the shifted sequence $\boldsymbol{s}_{\mathbf{m}}=(m_p)_{p\in\mathbb{N}}$ is regularly varying and, by Lemma 1.2.40, this is equivalent to condition (ii). For $\lambda=1$ the result is immediate. Assume $\lambda>1$, using (2.18), for all $p\in\mathbb{N}_0$, $p\geq(\lambda-1)^{-1}$ we see that $\lfloor\lambda p\rfloor>p$ and we have that

$$\alpha_{\lfloor\lambda p\rfloor}-\alpha_p=\sum_{k=p}^{\lfloor\lambda p\rfloor-1}\frac{\beta_k}{k+1}+\beta_{\lfloor\lambda p\rfloor}-\beta_p.$$

Condition (i) can be written as $\lim_{p\to\infty}\beta_p=\omega$, so it is sufficient to prove that

$$\lim_{p\to\infty}\sum_{k=p}^{\lfloor\lambda p\rfloor-1}\frac{\beta_k}{k+1}=\omega\log(\lambda).$$

If we take $\varepsilon>0$, we fix $\delta>0$ such that $\delta\log(\lambda)<\varepsilon/6$. There exists $p_\delta\in\mathbb{N}$ such that $|\beta_p-\omega|<\delta$ for $p\geq p_\delta$. We remember that the $p-$th partial sum $H_p=\sum_{k=1}^{p}1/k$ of the harmonic series may be given as

$$H_p=\log(p)+\gamma+\varepsilon_p,\qquad \gamma=\text{Euler's constant},\ \lim_{p\to\infty}\varepsilon_p=0.$$

Consequently, for $p\geq\max(p_\delta,(\lambda-1)^{-1})$ we have

$$\sum_{k=p}^{\lfloor\lambda p\rfloor-1}\frac{\beta_k}{k+1}\leq(\omega+\delta)(H_{\lfloor\lambda p\rfloor}-H_p)=(\omega+\delta)\left(\log\left(\frac{\lfloor\lambda p\rfloor}{\lambda p}\right)+\log(\lambda)+\varepsilon_{\lfloor\lambda p\rfloor}-\varepsilon_p\right).$$

Using that $\lim_{p\to\infty}\lfloor\lambda p\rfloor/(\lambda p)=1$ and that $\lim_{p\to\infty}\varepsilon_p=0$, we take $p_0\geq\max(p_\delta,(\lambda-1)^{-1})$ such that for every $p\geq p_0$ one has

$$\left|\omega\log\left(\frac{\lfloor\lambda p\rfloor}{\lambda p}\right)\right|<\varepsilon/12,\quad \left|\delta\log\left(\frac{\lfloor\lambda p\rfloor}{\lambda p}\right)\right|<\varepsilon/12,\quad |\omega\varepsilon_p|<\varepsilon/6,\quad |\delta\varepsilon_p|<\varepsilon/6.$$

Then for $p\geq p_0$ we see that

$$\sum_{k=p}^{\lfloor\lambda p\rfloor-1}\frac{\beta_k}{k+1}<\omega\log(\lambda)+\varepsilon.$$

Analogously, for $p\geq p_0$ we may also get that

$$\omega\log(\lambda)-\varepsilon<\sum_{k=p}^{\lfloor\lambda p\rfloor-1}\frac{\beta_k}{k+1},$$

and we are done.

For $\lambda\in(0,1)$, the proof is similar and we omit it.

(ii) $\Rightarrow$ (i) Let $\omega\in\mathbb{R}$ be the index of regular variation of $\mathbf{m}=(m_{p-1})_{p\in\mathbb{N}}$. By Theorem 1.2.38 one may write

$$m_p=\exp\left(c_{p+1}+\sum_{k=1}^{p+1}\frac{\eta_k}{k}\right),\qquad p\in\mathbb{N}_0,$$

where $(c_p)_{p\in\mathbb{N}}$ and $(\eta_p)_{p\in\mathbb{N}}$ are sequences of real numbers converging to $c\in\mathbb{R}$ and ω, respectively. Then

$$\begin{aligned}M_p=m_0m_1\cdots m_{p-1}&=\exp\left(\sum_{j=1}^{p}c_j+\sum_{j=0}^{p-1}\sum_{k=1}^{j+1}\frac{\eta_k}{k}\right)=\exp\left(\sum_{j=1}^{p}c_j+\sum_{k=1}^{p}(p-k+1)\frac{\eta_k}{k}\right)\\&=\exp\left(\sum_{j=1}^{p}c_j+(p+1)\sum_{k=1}^{p}\frac{\eta_k}{k}-\sum_{k=1}^{p}\eta_k\right).\end{aligned}$$

Since $\lim_{p\to\infty}\eta_p/p=0$, $\lim_{p\to\infty}\eta_p=\omega$ and $\lim_{p\to\infty}c_p=c$, for the corresponding arithmetic means we have that

$$\lim_{p\to\infty}p^{-1}\sum_{j=1}^{p}\eta_j/j=0,\qquad \lim_{p\to\infty}p^{-1}\sum_{j=1}^{p}\eta_j=\omega\quad\text{and}\quad\lim_{p\to\infty}p^{-1}\sum_{j=1}^{p}c_j=c,$$

and we see that

$$\lim_{p\to\infty}\log\left(\frac{m_p}{M_p^{1/p}}\right)=\lim_{p\to\infty}\left[\frac{1}{p}\left(\sum_{k=1}^{p}\eta_k-\sum_{k=1}^{p}\frac{\eta_k}{k}-\sum_{k=1}^{p}c_k\right)+c_{p+1}\right]=\omega,$$

or, equivalently, $\lim_{p\to\infty}\beta_p=\omega$. □

Remark 2.2.4. In fact, we observe that, since $\mathbb{M}$ is any sequence of positive real numbers with $M_0=1$, $\mathbf{m}$ can be substituted by any sequence $\boldsymbol{a}=(a_p)_{p\in\mathbb{N}}$ of positive real numbers. We can consider the sequence of geometric means given by

$$\overline{a}_p:=\left(\prod_{n=1}^{p}a_n\right)^{1/p},\qquad p\in\mathbb{N}.$$

By carefully skipping the index shift nuisance, with Lemma 1.2.40, the following are equivalent:

(i) there exists $\lim_{p\to\infty}a_p/\overline{a}_p\in(0,\infty)$,

(ii) $\boldsymbol{a}$ is regularly varying.

The main advantage of this equivalent definition of a regularly varying sequence is that we avoid working with the integer part and the corresponding step function $f_{\boldsymbol{a}}(x)=a_{\lfloor x\rfloor}$.

Remark 2.2.5. The reader may note that following the same reasoning a new characterization of O-regular variation for sequences in terms of the geometric means can be provided. With the same notation as in the previous remark, the following are equivalent:

(i) $0<\liminf_{p\to\infty}a_p/\overline{a}_p\le\limsup_{p\to\infty}a_p/\overline{a}_p<\infty$,

(ii) $\boldsymbol{a}$ is O-regularly varying.

The proof has been omitted because we will not employ it in the forthcoming sections.

2.2.2 Proximate order associated with a weight sequence

The principal result, gathered in this subsection, characterizes those sequences for which one can define, in a straightforward and natural way, a nonzero proximate order. In addition, we will show how these sequences interact with the notions considered in the previous sections such as strong regularity, regular variation and the associated functions.

The function $d_{\mathbb{M}}(t) = \log(\omega_{\mathbb{M}}(t))/\log(t)$, presented in the introduction of the section, is immediately shown to be continuous and piecewise continuously differentiable in its domain (meaning that it is differentiable except at a sequence of points, tending to infinity, at any of which it is continuous and has distinct finite lateral derivatives) and nonnegative for t large enough. Then $d_{\mathbb{M}}(t)$ always verifies conditions (A) and (B) of proximate orders (see Definition 1.2.7). Hence we only need to deal with conditions (C) and (D).

In the proof of the principal theorem, we will use the theorem of L. de Haan, Theorem 1.2.42, which for monotone sequences shows that regular variation can be expressed in a much more nicer form and that is the case of $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$ when $\mathbb{M}$ is (lc).

Theorem 2.2.6. Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a weight sequence. The following are equivalent:

(a) $d_{\mathbb{M}}(t)$ is a proximate order with $\lim_{t\to\infty} d_{\mathbb{M}}(t) \in (0,\infty)$,

(b) There exists $\omega > 0$ such that $\omega\nu_{\mathbf{m}}(t) \sim \omega_{\mathbb{M}}(t)$, that is, $\lim_{t\to\infty} \omega\nu_{\mathbf{m}}(t)/\omega_{\mathbb{M}}(t) = 1$,

(c) There exists $\omega > 0$ such that for every natural number $\ell \geq 2$,

$$\lim_{p\to\infty} \frac{m_{\ell p}}{m_p} = \ell^{\omega},$$

(d) $\mathbf{m}$ is regularly varying with a positive index of regular variation,

(e) There exists $\lim_{p\to\infty} \log\left(m_p/M_p^{1/p}\right) \in (0,\infty)$.

In case any of these statements holds, the value of the limit mentioned in (e), that of the index mentioned in (d), and that of the constant ω in (b) and (c) is $\omega(\mathbb{M})$ and the limit in (a) is $1/\omega(\mathbb{M})$.

Proof. (a) $\Rightarrow$ (b) According to (2.9) and (2.15), we have that

$$d'_{\mathbb{M}}(t) = \frac{\omega'_{\mathbb{M}}(t)}{\log(t)\omega_{\mathbb{M}}(t)} - \frac{d_{\mathbb{M}}(t)}{t\log(t)} = \frac{1}{t\log(t)}\left(\frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} - d_{\mathbb{M}}(t)\right),$$

for $t \neq m_p$ large enough. Observe that (D) in Definition 1.2.7 amounts then to

$$\lim_{\substack{t\to\infty \\ t\neq m_p}} \left(\frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} - d_{\mathbb{M}}(t)\right) = 0. \tag{2.19}$$

By Theorem 2.1.30 and condition (C) in Definition 1.2.7, we know that $\lim_{t\to\infty} d_{\mathbb{M}}(t) = 1/\omega(\mathbb{M})$, and so

$$\lim_{\substack{t\to\infty \\ t\neq m_p}} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} = \frac{1}{\omega(\mathbb{M})}.$$

By Lemma 1.2.12, we know that $\omega_{\mathbb{M}} \in R_{1/\omega(\mathbb{M})}$, so $\lim_{t\to\infty} \omega_{\mathbb{M}}(t+1)/\omega_{\mathbb{M}}(t) = 1$. For every $p \in \mathbb{N}_0$ large enough, we take $\varepsilon_p \in (0,1)$ such that $m_p+\varepsilon_p$, $m_p-\varepsilon_p$, $m_p+\varepsilon_p-1$ and $m_p-\varepsilon_p+1$ are not elements of the sequence $\mathbf{m}$. By the monotonicity of $\nu_{\mathbf{m}}$ and $\omega_{\mathbb{M}}$, we have that

$$\frac{\omega_{\mathbb{M}}(m_p-\varepsilon_p)}{\omega_{\mathbb{M}}(m_p-\varepsilon_p)}\frac{\nu_{\mathbf{m}}(m_p-\varepsilon_p)}{\omega_{\mathbb{M}}(m_p-\varepsilon_p+1)} \leq \frac{\nu_{\mathbf{m}}(m_p)}{\omega_{\mathbb{M}}(m_p)} \leq \frac{\nu_{\mathbf{m}}(m_p+\varepsilon_p)}{\omega_{\mathbb{M}}(m_p+\varepsilon_p-1)}\frac{\omega_{\mathbb{M}}(m_p+\varepsilon_p)}{\omega_{\mathbb{M}}(m_p+\varepsilon_p)},$$

for all p large enough and we conclude that $\lim_{p\to\infty} \nu_{\mathbf{m}}(m_p)/\omega_{\mathbb{M}}(m_p) = 1/\omega(\mathbb{M})$.

(b) $\Rightarrow$ (c) We redefine $\omega'_{\mathbb{M}}(m_p) := \omega(m_p)/(m_p\omega)$ for all $p \in \mathbb{N}_0$. Since $\lim_{t\to\infty} \omega_{\mathbb{M}}(t) = \infty$, there exists $N \in \mathbb{N}$ such that $\omega_{\mathbb{M}}(t) > 0$ for $t \geq m_N$, we can write

$$\omega_{\mathbb{M}}(t) = \omega_{\mathbb{M}}(m_N) \exp\left(\int_{m_N}^{t} \frac{u\omega'_{\mathbb{M}}(u)}{\omega_{\mathbb{M}}(u)} \frac{du}{u}\right), \qquad t \geq m_N.$$

By (b) and the definition of $\omega'_{\mathbb{M}}(m_p)$, we have that

$$\lim_{t\to\infty} \frac{t\omega'_{\mathbb{M}}(t)}{\omega_{\mathbb{M}}(t)} = \lim_{t\to\infty} \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} = \frac{1}{\omega}.$$

Then by the Representation Theorem 1.2.4, we have that $\omega_{\mathbb{M}}(t)$ is regularly varying of index $1/\omega$ and, again by (b) and Remark 1.2.6, we have that $\nu_{\mathbf{m}}(t)$ is regularly varying of index $1/\omega$.

For any $\ell \in \mathbb{N}$ with $\ell \geq 2$, for every $\varepsilon \in (0,1)$ we take $\lambda_1(\ell,\varepsilon) := (\ell/(1+\varepsilon))^{\omega}$ and $\lambda_2(\ell,\varepsilon) := (\ell/(1-\varepsilon))^{\omega}$ then, using the regular variation of $\nu_{\mathbf{m}}(t)$, there exists $t_0 > 0$ such that

$$\lambda_j^{1/\omega}(1-\varepsilon) \leq \frac{\nu_{\mathbf{m}}(\lambda_j t)}{\nu_{\mathbf{m}}(t)} \leq \lambda_j^{1/\omega}(1+\varepsilon), \qquad \text{for all} \quad t \geq t_0,$$

for $j = 1,2$. Since $\lim_{p\to\infty} m_p = \infty$, there exists $p_0 \in \mathbb{N}$ such that $m_{p_0-1} > \max(\lambda_1 t_0, \lambda_2 t_0, t_0)$, then $\nu_{\mathbf{m}}(t_0) < p_0$. So for $p \geq p_0$ we have that

$$m_p = \sup\{t \geq m_0; \nu_{\mathbf{m}}(t) \leq p\} = \sup\{t \geq t_0; \nu_{\mathbf{m}}(t) \leq p\}.$$

Consequently, since $\nu_{\mathbf{m}}(\lambda_1 t) \leq \ell\nu_{\mathbf{m}}(t) \leq \nu_{\mathbf{m}}(\lambda_2 t)$ for every $t \geq t_0$, for every $p \geq p_0$ we see that

$$\sup\{t \geq t_0; \nu_{\mathbf{m}}(\lambda_2 t) \leq \ell p\} \leq m_p \leq \sup\{t \geq t_0; \nu_{\mathbf{m}}(\lambda_1 t) \leq \ell p\}.$$

Since $\ell p \geq p \geq p_0$, for $j = 1,2$ we have that $\nu_{\mathbf{m}}(\lambda_j t_0) < p_0$ and for all $p \geq p_0$ we observe that

$$\sup\{t \geq t_0; \nu_{\mathbf{m}}(\lambda_j t) \leq \ell p\} = (\lambda_j)^{-1} \sup\{s \geq \lambda_j t_0; \nu_{\mathbf{m}}(s) \leq \ell p\} = (\lambda_j)^{-1} m_{\ell p}.$$

Finally, for every $p \geq p_0$ we conclude that

$$\frac{\ell^{\omega}}{(1+\varepsilon)^{\omega}} = \lambda_1 \leq \frac{m_{\ell p}}{m_p} \leq \lambda_2 = \frac{\ell^{\omega}}{(1-\varepsilon)^{\omega}}.$$

(c) $\Rightarrow$ (d) Since $\boldsymbol{s}_{\mathbf{m}} = (m_p)_{p\in\mathbb{N}}$ is nondecreasing, by Theorem 1.2.42 we see that $\boldsymbol{s}_{\mathbf{m}}$ is regularly varying of index $\omega > 0$. Then it suffices to apply Lemma 1.2.40 to ensure that $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}_0}$ is regularly varying of index $\omega > 0$.

(d) $\Leftrightarrow$ (e) Apply Proposition 2.2.3.

(e) $\Rightarrow$ (a) According to (2.16), condition (e) can be written as

$$\lim_{p\to\infty} \beta_p = \omega \in (0,\infty). \tag{2.20}$$

By using (2.18), we see that

$$\lim_{p\to\infty} \frac{(\alpha_{p+1} - \beta_{p+1}) - (\alpha_p - \beta_p)}{\log(p+1) - \log(p)} = \lim_{p\to\infty} \frac{\beta_p/(p+1)}{1/p} = \omega,$$

and then we deduce by Stolz's criterion that

$$\lim_{p\to\infty} \frac{\alpha_p - \beta_p}{\log(p)} = \omega.$$

Since $\beta_p = O(1)$ (and $\alpha_p = \log(m_p)$), we get

$$\lim_{p\to\infty} \frac{\log(m_p)}{\log(p)} = \omega. \tag{2.21}$$

On the other hand from (e), there exist $a, A > 0$ and $p_0 \in \mathbb{N}$ such that

$$a < \log\left(m_p/M_p^{1/p}\right) < A, \qquad p \geq p_0,$$

what, by (1.6) and taking logarithms, amounts to

$$\log(a) + \log(p) < \log(\omega_{\mathbb{M}}(m_p)) < \log(p) + \log(A), \qquad p \geq p_0.$$

Subsequently, we see that

$$\lim_{p\to\infty} \frac{\log(\omega_{\mathbb{M}}(m_p))}{\log(p)} = 1. \tag{2.22}$$

Observe that $\omega_{\mathbb{M}}(t)$ is nondecreasing, so for every $t \in [m_{p-1}, m_p)$ we have

$$\frac{p}{\omega_{\mathbb{M}}(m_p)} \leq \frac{\nu_{\mathbf{m}}(t)}{\omega_{\mathbb{M}}(t)} \leq \frac{p}{\omega_{\mathbb{M}}(m_{p-1})},$$
$$\frac{\log(\omega_{\mathbb{M}}(m_{p-1}))}{\log(m_p)} \leq \frac{\log(\omega_{\mathbb{M}}(t))}{\log(t)} \leq \frac{\log(\omega_{\mathbb{M}}(m_p))}{\log(m_{p-1})}.$$

By (1.6) we know that $\omega_{\mathbb{M}}(m_p) = p\beta_p$ for every $p \in \mathbb{N}$, so from (2.20) and the first inequalities we see that $\lim_{t\to\infty} \nu_{\mathbf{m}}(t)/\omega_{\mathbb{M}}(t) = 1/\omega$. Now, using (2.21) and (2.22) we conclude from the second inequalities that $\lim_{t\to\infty} d_{\mathbb{M}}(t) = 1/\omega$, and also that (2.19) is satisfied. So, (C) and (D) in Definition 1.2.7 are valid and $d_{\mathbb{M}}$ is a proximate order. Moreover, by Theorem 2.1.30 we deduce that $\omega = \omega(\mathbb{M})$.

The value of the different limits or indices involved in the statements is deduced in the course of the proof. □

Remark 2.2.7. We can easily deduce some necessary conditions for $d_{\mathbb{M}}$ being a nonzero proximate order. By Remark 2.1.14 and Remark 2.1.19, if $\mathbb{M}$ is a weight sequence such that $\mathbf{m}$ is regularly varying of positive index, the following holds:

(j) $\mathbf{m}$ is nondecreasing and we have that

$$\beta(\mathbf{m}) = \gamma(\mathbb{M}) = \mu(\mathbf{m}) = \omega(\mathbb{M}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) \in (0, \infty),$$

(k) $\mathbb{M}$ is strongly regular and

$$\lim_{p\to\infty} \frac{\log(m_p)}{\log(p)} = \omega(\mathbb{M}), \tag{2.23}$$

(ℓ) $\mathbb{M}$ is strongly regular.

Condition (k) trivially implies (ℓ) and, by Remark 2.1.19 we deduce that, if (j) is valid, $\mathbb{M}$ is strongly regular and, using Proposition 2.1.10, we see that (2.23) holds, then (j) implies (k). The converse implications do not hold, see Example 2.2.24 for (j) and (k) and Example 2.2.26 for (k) and (ℓ).

It is plain to check that (j), (k) and (ℓ) are stable for $\simeq$. Moreover, if $\mathbb{M}$ is a weight sequence satisfying any of these conditions and $\mathbb{L}$ is another weight sequence with $\mathbb{L} \approx \mathbb{M}$, by Proposition 1.1.20, we have that $\boldsymbol{\ell} \simeq \mathbf{m}$, consequently, $\mathbb{L}$ also satisfies the same condition.

In some sense, Theorem 2.2.6 can be seen as the analogous version of the equivalent definitions for strongly regular sequences in Corollary 2.1.6, Remarks 2.1.19 and 2.1.39 for the case of regularly varying sequences.

Example 2.2.8. For the sequences in the Example 1.1.4 we have shown that $\omega(\mathbb{M}_{\alpha,\beta}) = \alpha$ and $\omega(\mathbb{M}_q) = \infty$ for the considered values of α, β and q (see Example 2.1.20). So, Theorem 2.2.6 shows that for the sequences $\mathbb{M}_{0,\beta}$ and $\mathbb{M}_q$ the function $d_{\mathbb{M}}$ is not a nonzero proximate order. On the contrary, one may easily check that (c) or (e) in that theorem hold for $\mathbb{M}_{\alpha,\beta}$ whenever $\alpha > 0$ and, consequently, $d_{\mathbb{M}_{\alpha,\beta}}$ is indeed a nonzero proximate order, although its handling will be difficult in general (in this sense, see Remark 2.2.16).

Among the examples at the end of the chapter, only the first one Example 2.2.21 satisfies the theorem. For the second one, Example 2.2.22, even if $d_{\mathbb{M}}$ is not a nonzero proximate order, we are still able to apply the results from the generalized summability theory, presented in Section 4.1, whereas the others are examples of pathological strongly regular sequences for which this theory is not available, but nevertheless the properties of the asymptotic Borel map can be analyzed, as it will be done in the next chapter.

2.2.3 Regularly varying sequences defined from proximate orders

In the previous subsection it has been shown how to go from weight sequences to nonzero proximate orders. Now, departing from a nonzero proximate order, and for every element V in the class $MF(\gamma, \rho(t))$ given by L.S. Maergoiz [65] (see Theorem 1.2.16 and Definition 1.2.17), we will construct a well-behaved sequence $\mathbb{V}$. This procedure is closely related to the one described in Subsection 2.1.5.

Definition 2.2.9. Let $\rho(t)$ be a nonzero proximate order, $\gamma > 0$ and $V \in MF(\gamma, \rho(t))$. We define its *associated sequence* by

$$v_p := V(p), \quad p \in \mathbb{N}, \qquad v_0 := V(1).$$

Then $V_p := v_0 v_1 \cdots v_{p-1}$ for all $p \in \mathbb{N}$ and $V_0 = 1$.

Using Remark 1.2.8, Theorem 1.2.16.(I), (III) and (VI), we see that $\mathbb{V}$ is a weight sequence and that $\boldsymbol{v}$ is regularly varying of positive index $\rho := \lim_{t\to\infty} \rho(t)$. In particular, by Theorem 2.2.6, $d_{\mathbb{V}}$ is a nonzero proximate order and, by Remark 2.2.7, $\mathbb{V}$ is strongly regular.

We have the following relation between $\mathbb{V}$ and the dual sequence $\mathbb{D}^{\mathbb{M}}$ (see Definition 2.1.40) of a weight sequence $\mathbb{M}$ admitting $\rho(t)$ as a nonzero proximate order (see Definition 2.2.1).

Lemma 2.2.10. Let $\mathbb{M}$ be a weight sequence admitting $\rho(t)$ as a nonzero proximate order. Then for any $\gamma > 0$ and every $V \in MF(\gamma, \rho(t))$ we have that $\boldsymbol{v} \simeq \boldsymbol{d}^{\mathbb{M}}$.

Proof. Since $\mathbb{M}$ admits $\rho(t)$ as a nonzero proximate order, there exist $A, B > 0$ such that

$$B \le \frac{t^{\rho(t)}}{\omega_{\mathbb{M}}(t)} \le A, \quad \text{for } t \text{ large enough.}$$

By Theorem 1.2.16.(VI), there exist $C, D > 0$ such that

$$D \leq \frac{V(t)}{\omega_{\mathbb{M}}(t)} \leq C, \quad \text{for } t \text{ large enough.}$$

Then, by Theorem 1.2.16.(I) and Remark 1.2.31, $\omega_{\mathbb{M}} \in ORV$ and $\alpha(\omega_{\mathbb{M}}) = \beta(\omega_{\mathbb{M}}) \in (0,\infty)$. Applying Theorem 2.1.29, we have that there exist $E, F > 0$ such that

$$F \leq \frac{V(t)}{\nu_{\mathbf{m}}(t)} \leq E, \quad \text{for } t \text{ large enough.}$$

then, evaluating in the natural numbers, $\boldsymbol{v} \simeq \boldsymbol{d}^{\mathbb{M}}$. □

Using the notion of conjugate proximate order, defined in Subsection 1.2.2, we can associate with a nonzero proximate order a conjugate sequence.

Definition 2.2.11. Let $\rho(t)$ be a nonzero proximate order, $\gamma > 0$ and $V \in MF(\gamma, \rho(t))$. We define its *associated conjugate sequence* by

$$u_p := U(p), \quad p \in \mathbb{N}, \qquad u_0 := U(1),$$

where $U(s)$ is the inverse of the function $V(t)$ (see Remark 1.2.18) and $U_p := u_0 u_1 \cdots u_{p-1}$ for every $p \in \mathbb{N}$ and $U_0 = 1$.

By Theorem 1.2.19, $\mathbb{U}$ is a weight sequence and $\boldsymbol{u}$ is regularly varying of positive index $1/\rho$, where $\rho = \lim_{t\to\infty} \rho(t)$. Naturally arises the question of the relation between $\mathbb{U}$ and the dual sequence $\mathbb{D}^{\mathbb{V}}$ of $\mathbb{V}$ defined by

$$d_p^V := \nu_{\boldsymbol{v}}(p), \quad p \geq m_0, \qquad d_p^V := 1, \quad p < m_0$$

and $D_0^V := 1$ and $D_p^V := d_0^V d_1^V \cdots d_{p-1}^V$ for all $p \in \mathbb{N}$.

Lemma 2.2.12. Let $\rho(t)$ be a nonzero proximate order, $\gamma > 0$ and $V \in MF(\gamma, \rho(t))$. Then we have that

$$\nu_{\boldsymbol{v}}(s) = \lfloor U(s) \rfloor + 1, \qquad s \geq V(1),$$

where $U(s)$ is the inverse of the function $V(t)$, and $\boldsymbol{d}^{\mathbb{V}} \simeq \boldsymbol{u}$.

Proof. For $s \geq V(1)$ we have that

$$\nu_{\boldsymbol{v}}(s) = \max\{j \in \mathbb{N};\, V(j-1) \leq s\} = \max\{j \in \mathbb{N}; j-1 \leq U(s)\} = \lfloor U(s) \rfloor + 1.$$

We see that for s large enough $U(s) \geq 1$ and

$$1 \leq \frac{\nu_{\boldsymbol{v}}(s)}{U(s)} = \frac{\lfloor U(s) \rfloor + 1}{U(s)} \leq \frac{U(s)+1}{U(s)} \leq 2.$$

Hence we conclude that $\boldsymbol{d}^{\mathbb{V}} \simeq \boldsymbol{u}$. □

Remark 2.2.13. By carefully combining the results in Subsections 2.1.5 and 2.1.4, for t large enough it is also possible to show that there exist positive constants A and B such that

$$A \leq \frac{\omega_{\mathbb{U}}(t)}{V(t)} \leq B,$$

confirming what was expected for the conjugate sequence $\mathbb{U}$.

Finally, Lemmas 2.2.10 and 2.2.12 suggest the next connection between the sequence $\mathbb{U}$, a weight sequence $\mathbb{M}$ admitting $\rho(t)$ as a nonzero proximate order and its bidual sequence $\mathbb{E}^{\mathbb{M}}$ (see Definition 2.1.41).

Theorem 2.2.14. Let $\mathbb{M}$ be a weight function admitting $\rho(t)$ as a nonzero proximate order. Then $\boldsymbol{u} \simeq \boldsymbol{e}^{\mathbb{M}}$ and $\boldsymbol{u} \simeq \mathbf{m}$.

Proof. By Lemma 2.2.10, $\boldsymbol{v} \simeq \boldsymbol{d}^{\mathbb{M}}$ then it exists $a > 1$ such that

$$\nu_{\boldsymbol{v}}(a^{-1}t) \leq \nu_{\boldsymbol{d}^{\mathbb{M}}}(t) \leq \nu_{\boldsymbol{v}}(at), \qquad t > 0.$$

Consequently, as in the proof of Lemma 2.2.12, for t large enough we have that

$$U(a^{-1}t) \leq \lfloor U(a^{-1}t) \rfloor + 1 \leq \nu_{\boldsymbol{d}}(t) \leq \lfloor U(at) \rfloor + 1 \leq 2U(at).$$

Using the regular variation of $U(t)$, we observe that

$$\lim_{t\to\infty} \frac{U(at)}{U(t)} = a^{1/\rho}, \qquad \lim_{t\to\infty} \frac{U(a^{-1}t)}{U(t)} = a^{-1/\rho},$$

then we see that there exist $A, B > 0$ such that $U(at) \leq AU(t)$ and $U(a^{-1}t) \geq BU(t)$ for t large enough. Hence

$$BU(t) \leq \nu_{\boldsymbol{d}}(t) \leq 2AU(t) \qquad \text{for } t \text{ large enough.}$$

We deduce that for $p \in \mathbb{N}$ large enough

$$Bu_p \leq e_p^{\mathbb{M}} = \nu_{\boldsymbol{d}^{\mathbb{M}}}(p) \leq 2Au_p,$$

which implies $\boldsymbol{e}^{\mathbb{M}} \simeq \boldsymbol{u}$ and we deduce that $\boldsymbol{u} \simeq \mathbf{m}$ by Theorem 2.1.42. □

Remark 2.2.15. Other procedures for going from functions to sequences in this context have been considered, as the one by J. Bonet, R. Meise and S.N. Melikhov [17] decribed in Remark 2.1.37.

In our work [44, Sect. 4.1], inspired by the argument by S. Mandelbrojt [72] and H. Komatsu [52] to recover a sequence from its associated function $\omega_{\mathbb{M}}(t)$, given a nonzero proximate order $\rho(t)$, $\gamma > 0$ and $V \in MF(\gamma, \rho(t))$ we define the sequence

$$M_p^V := \sup_{t>0} \frac{t^p}{e^{V(t)}}, \qquad p \in \mathbb{N}_0.$$

We show that $\mathbb{M}^V$ is strongly regular, making use of the Young conjugate we see that $\omega_{\mathbb{M}^V}(t) \sim V(t)$, finally, we prove that $\mathbb{M}^V \approx \mathbb{U}$. Since the previous proofs are simpler than the ones in the paper, this new equivalent approach has been included in the dissertation.

2.2.4 Sequences admitting a nonzero proximate order

Before proving that the weaker conditions (f) and (g) in Theorem 2.2.17 introduced at the beginning of the section, which are sufficient for the construction of nontrivial 'fine' flat functions in sectors of optimal opening, are indeed the same, some worthy remarks regarding the admissibility condition (see Definition 2.2.1) are presented.

First, note that the notion of equivalent proximate orders (see Definition 1.2.9) is more demanding (apart from the fact that here $d_{\mathbb{M}}$ need not be a proximate order). If $\mathbb{M}$ admits a proximate order, $d_{\mathbb{M}}$ verifies all the properties of proximate orders except possibly (D), since from the definition of admissibility $\lim_{t\to\infty} d_{\mathbb{M}}(t) = \lim_{t\to\infty} \rho(t)$ exists and, by Theorem 2.1.30, equals $1/\omega(\mathbb{M})$. Moreover, from the relation between regular variation, O-regular variation and proximate orders (see Lemma 1.2.12 and Remark 1.2.31), we deduce that if $\mathbb{M}$ admits a proximate order, then $\omega_{\mathbb{M}}(t) \in ORV$.

Remark 2.2.16. The admissibility condition is interesting even if $d_{\mathbb{M}}$ is a proximate order. For instance, if we consider the sequences $\mathbb{M}_{\alpha,\beta}$ in Example 1.1.4, with $\alpha > 0$, for large t we have

$$c_2 t^{1/\alpha} \log^{-\beta/\alpha}(t) \le \omega_{\mathbb{M}_{\alpha,\beta}}(t) \le c_1 t^{1/\alpha} \log^{-\beta/\alpha}(t)$$

for suitable constants $c_1, c_2 > 0$ (see [98, Example 1.2.2]), then

$$\log(c_2) \le \log(t)(d_{\mathbb{M}_{\alpha,\beta}}(t) - \rho_{\alpha,\beta}(t)) \le \log(c_1) \quad \text{for } t \text{ large enough}$$

(see Example 1.2.11 for the definition of $\rho_{\alpha,\beta}$). This shows that the proximate order $\rho_{\alpha,\beta}(t)$ is admissible for $\mathbb{M}_{\alpha,\beta}$, and therefore, for our purposes, it may substitute $d_{\mathbb{M}_{\alpha,\beta}}(t)$ whenever it is convenient. In particular, when working with Gevrey ultraholomorphic classes one may consider the constant order $\rho_{\alpha,0}(t) \equiv 1/\alpha$, as expected.

As a consequence of the results in the previous subsection, we can show that the weaker conditions are equivalent.

Theorem 2.2.17. Let $\mathbb{M}$ be a weight sequence, then the following conditions are equivalent:

(f) There exists a weight sequence $\mathbb{L}$ such that $\mathbb{L} \approx \mathbb{M}$ and $d_{\mathbb{L}}(t)$ is a nonzero proximate order,

(g) $\mathbb{M}$ admits a nonzero proximate order.

Proof. (f) $\Rightarrow$ (g) Since $\mathbb{L} \approx \mathbb{M}$, there exist positive constants A and B such that for every $t \in (0,\infty)$ one has

$$\omega_{\mathbb{L}}(At) \le \omega_{\mathbb{M}}(t) \le \omega_{\mathbb{L}}(Bt).$$

Since $d_{\mathbb{L}}(t)$ is a nonzero proximate order, $\omega_{\mathbb{L}}(t) = t^{d_{\mathbb{L}}(t)}$ is regularly varying by Lemma 1.2.12, and we deduce that there exist positive constants C and D such that

$$C \le \frac{\omega_{\mathbb{M}}(t)}{\omega_{\mathbb{L}}(t)} \le D \qquad \text{for } t \text{ large enough.}$$

Finally, taking logarithms, we conclude that $\mathbb{M}$ admits $d_{\mathbb{L}}$ as a nonzero proximate order.
(g) $\Rightarrow$ (f) Let $\rho(t)$ be the nonzero proximate order that $\mathbb{M}$ admits. By Theorem 2.2.14, for any $\gamma > 0$ and every $V \in MF(\gamma, \rho(t))$, we have that $\mathbf{m} \simeq \boldsymbol{u}$ where $\boldsymbol{u}$ is the regularly varying (of positive index) nondecreasing sequence defined in terms of the inverse function $U(s)$ of $V(t)$ (see Definition 2.2.11).

Applying Theorem 2.2.6, we know that $d_{\mathbb{U}}$ is a nonzero proximate order and, by Proposition 1.1.15, we deduce that $\mathbb{U} \approx \mathbb{M}$. ☐

Remark 2.2.18. The implication (a) $\Rightarrow$ (f) (see Theorems 2.2.6 and 2.2.17) is obvious, while Example 2.2.22 shows that the converse fails.

It is also immediate that (f) $\Rightarrow$ (j) in Remark 2.2.7 because, with the notation of (f), if $d_{\mathbb{L}}$ is a proximate order, then $\mathbb{L}$ satisfies (j) and so $\mathbb{M}$ also satisfies (j) since it is stable for $\approx$. Again, the converse implication (j) $\Rightarrow$ (f) fails, as Example 2.2.23 illustrates. Consequently, the sequences $\mathbb{M}_q$ and $\mathbb{M}_{0,\beta}$ do not admit a nonzero proximate order, since they are not strongly regular. Among the strongly regular ones, for those appearing in applications (f) and even (a) are valid, but extremely pathological examples (see Examples 2.2.23, 2.2.24 and 2.2.26) of strongly regular sequences not satisfying (f) will be constructed below.

Adding the information in Remark 2.2.7, (a) $\Rightarrow$ (f) $\Rightarrow$ (j) $\Rightarrow$ (k) $\Rightarrow$ (ℓ), and the arrows can not be reversed.

The next representation result, analogous to Theorems 1.2.38 and 1.2.46, provides a characterization of the weight sequences that satisfy (f), only in terms of their structure, i.e., not depending on the existence of another weight sequence $\mathbb{L}$ or a nonzero proximate order $\rho(t)$, closing an open question set in [44, Remark 4.15].

Theorem 2.2.19. Let $\mathbb{M}$ be a weight sequence, then the following conditions are equivalent:

(f) There exists a weight sequence $\mathbb{L}$ such that $\mathbb{L} \approx \mathbb{M}$ and $d_{\mathbb{L}}(t)$ is a nonzero proximate order,

(g) $\mathbb{M}$ admits a nonzero proximate order,

(h) There exist $\omega \in (0,\infty)$ and bounded sequences of real numbers $(b_p)_{p\in\mathbb{N}}$, $(\eta_p)_{p\in\mathbb{N}}$ such that $(\eta_p)_{p\in\mathbb{N}}$ converges to ω and we can write

$$m_p = \exp\left(b_{p+1} + \sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

In case the previous holds, $\lim_{t\to\infty} d_{\mathbb{L}}(t) = 1/\omega = 1/\omega(\mathbb{M})$.

Proof. (f) $\Leftrightarrow$ (g) Theorem 2.2.17.

(f) $\Rightarrow$ (h) Applying Theorem 2.2.6, we know that $\boldsymbol{\ell} = (\boldsymbol{\ell}_{p-1})_{p\in\mathbb{N}}$ is regularly varying of index $\omega = \omega(\mathbb{L}) = \omega(\mathbb{M})$. Then by the Representation Theorem 1.2.38, there exist sequences of real numbers $(c_p)_{p\in\mathbb{N}}$ and $(\eta_p)_{p\in\mathbb{N}}$, converging to $c \in \mathbb{R}$ and ω, respectively, such that

$$\ell_p = \exp\left(c_{p+1} + \sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

By Remark 2.2.7, $\mathbb{L}$ satisfies (j), then it has (mg) and, by Proposition 1.1.20, we deduce that $\mathbf{m} \simeq \boldsymbol{\ell}$. This means that there exists a bounded sequence $(h_p)_{p\in\mathbb{N}}$ such that

$$m_p = \exp(h_{p+1})\ell_p = \exp\left(h_{p+1} + c_{p+1} + \sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

Writing $b_p := h_p + c_p$ for $p \in \mathbb{N}$, we conclude that (h) holds.

(h) $\Rightarrow$ (f) We define the sequence

$$t_p := \exp\left(\sum_{j=1}^{p+1} \frac{\eta_j}{j}\right), \quad p \in \mathbb{N}_0.$$

Since $\lim_{p\to\infty} \eta_p = \omega \in (0,\infty)$, we fix $\varepsilon \in (0,\omega)$ and we get $p_0 \in \mathbb{N}$ such that $\eta_p > \omega - \varepsilon > 0$ for $p \geq p_0$, this implies that

$$\frac{t_{p+1}}{t_p} = \exp(\eta_{p+2}/(p+2)) \geq 1.$$

We consider

$$\ell_p := t_p \quad p \geq p_0, \qquad \ell_p = t_{p_0}, \quad p < p_0.$$

Then by the Representation Theorem 1.2.38, we know that $\boldsymbol{t}$ is regularly varying of index ω. Hence, by Lemma 1.2.40, then ℓ also is. Moreover, by construction, $\boldsymbol{\ell}$ is nondecreasing and, from the regular variation of positive index, one can easily deduce that $\lim_{p\to\infty} \ell_p = \infty$, so $\mathbb{L}$ is a weight sequence and, by Theorem 2.2.6, $d_{\mathbb{L}}$ is a proximate order of index $1/\omega$.

Finally, we observe that

$$\frac{m_p}{\ell_p} = \frac{m_p}{t_p} = \exp(b_{p+1}), \qquad p \geq p_0,$$

and we conclude that $\mathbf{m} \simeq \boldsymbol{\ell}$ because $(b_p)_{p\in\mathbb{N}}$ is bounded, then $\mathbb{L} \approx \mathbb{M}$ by Proposition 1.1.15. □

Remark 2.2.20. If $\mathbb{M}$ is a weight sequence satisfying (h), we can obtain more information from the representation formula, concretely, since $\mathbf{m}$ is nondecreasing for every $p \in \mathbb{N}_0$ we have that $b_{p+1} - b_p + \eta_{p+1}/(p+1) \geq 0$ should hold. Moreover, as we already know by Remark 2.2.18, but now directly from Theorem 1.2.46, we have that $\mathbf{m}$ is O-regularly varying and

$$\omega = \beta(\mathbf{m}) = \gamma(\mathbb{M}) = \mu(\mathbf{m}) = \omega(\mathbb{M}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) \in (0, \infty).$$

2.2.5 Examples

In this subsection five examples of pathological regularly varying and O-regularly varying sequences are provided. Hence, by the connections to weight sequences described in this chapter, we can infer some properties for $\mathbb{M}$ when these sequences are assumed to be the corresponding sequence of quotients $\mathbf{m} = (m_{p-1})_{p\in\mathbb{N}}$, clarifying several open questions. We have presented some of these examples in our papers [43, 44], where most of the computations, included here, were skipped.

The examples below, ordered attending to their regularity, are constructed by means of different techniques. Specially relevant is the one employed in Examples 2.2.21, 2.2.23 and 2.2.26 inspired by the Representation Theorems 1.2.38 and 1.2.46. If $\mathbf{m}$ has the appropriate structure provided by these theorems several conditions can be automatically checked, systematically producing new examples (see Remark 2.2.27).

In these representations, the partial sums of the harmonic series play a fundamental role, one may write the $p-$th partial sum by $H_p := \sum_{k=1}^{p} 1/k$, and we know that

$$H_p = \log(p) + \gamma + \varepsilon_p, \qquad \gamma = \text{Euler's constant}, \ \lim_{p\to\infty} \varepsilon_p = 0. \tag{2.24}$$

Before knowing Theorem 2.2.6, J. Sanz suggests in [88, Corollary 4.10] that the existence of

$$\lim_{p\to\infty} p \log\left(\frac{m_{p+1}}{m_p}\right), \tag{2.25}$$

which implies Theorem 2.2.6.(e), could be equivalent to $d_{\mathbb{M}}$ being a nonzero proximate order. This first example shows this sufficient condition is not necessary.

Example 2.2.21. We consider the sequence $\mathbb{M}$ defined by the sequence of its quotients as $m_0 = 1$, $m_1 = e$ and for all $p \in \mathbb{N}$

$$m_{2p} = e^{1/p} m_{2p-1}, \qquad m_{2p+1} = e^{1/(2p+1)} m_{2p}.$$

The following are valid:

(i) $\mathbb{M}$ is a strongly regular sequence,

(ii) $\mathbf{m}$ is regularly varying of index $\omega = 3/2$ and, by Theorem 2.2.6, $d_{\mathbb{M}}$ is a proximate order,

(iii) $\mathbf{m}$ is not 'smooth' in the sense of Theorem 1.2.41, that is, it does not satisfy

$$\frac{m_{p+1}}{m_p} = 1 + \frac{3/2}{p} + o\left(\frac{1}{p}\right) \qquad \text{as } p \to \infty,$$

(iv) $\mathbf{m}$ does not satisfy (2.25).

Proof. The sequence $(m_p)_{p\in\mathbb{N}_0}$ is nondecreasing, then $\mathbb{M}$ is (lc). We can write

$$\log(m_{2p}) = \frac{1}{2}H_p + H_{2p}, \qquad \log(m_{2p+1}) = \frac{1}{2p+1} + \frac{1}{2}H_p + H_{2p}, \qquad p \in \mathbb{N}.$$

Thanks to the well-known behavior of the partial sums of the harmonic series, we show that

$$\lim_{p\to\infty} \frac{m_{\ell p}}{m_p} = \ell^{3/2},$$

for every $\ell \in \mathbb{N}$ and $\ell \geq 2$, which, by Theorem 2.2.6, implies that $\mathbf{m}$ is regularly varying and by Corollary 2.1.6, that $\mathbb{M}$ is strongly regular. However, we observe that

$$\frac{m_{2p+1}}{m_{2p}} = e^{1/2p+1} = 1 + \frac{1}{2p+1} + o\left(\frac{1}{p}\right), \qquad \text{as } p \to \infty,$$
$$\frac{m_{2p}}{m_{2p-1}} = e^{1/p} = 1 + \frac{2}{2p} + o\left(\frac{1}{p}\right), \qquad \text{as } p \to \infty,$$

then (iii) is true. With a similar computation, we see that

$$\lim_{p\to\infty} 2p\log\left(\frac{m_{2p+1}}{m_{2p}}\right) = \lim_{p\to\infty} 2p\log\left(e^{1/2p+1}\right) = 1,$$
$$\lim_{p\to\infty} (2p-1)\log\left(\frac{m_{2p}}{m_{2p-1}}\right) = \lim_{p\to\infty} (2p-1)\log\left(e^{1/p}\right) = 2,$$

and (2.25) does not hold. □

The second example shows that the equivalent conditions in Theorem 2.2.6 are stronger than the ones in Theorem 2.2.17 (see Remark 2.2.18), that is, (f) does not imply (a) in general. The idea is to construct a nondecreasing O-regularly varying sequence that is not regularly varying but that is equivalent for $\simeq$ to $(p!)_{p\in\mathbb{N}_0}$.

Example 2.2.22. Let $\mathbb{M}$ be defined using the sequence of quotients $\mathbf{m}$. We put $m_0 = m_1 = 1$, $m_2 = m_3 = 2$ and $m_4 = m_5 = m_6 = m_7 = 6$; for every $k \in \mathbb{N}$ and $2^{2^k+1} \leq p < 2^{2^{k+1}+1}$ we define m_p as follows:

$$m_p = 2^{2^k} 3 \left(\frac{2^{2^k}}{3}\right)^{\frac{j-1}{2^k-1}}, \qquad 2^{2^k+j} \leq p \leq 2^{2^k+j+1} - 1, \quad j = 1, 2, \ldots, 2^k.$$

The following are valid:

(i) $\mathbb{M}$ is a weight sequence,

(ii) $\mathbf{m}$ is not regularly varying and $d_{\mathbb{M}}$ is not a nonzero proximate order,

(iii) There exists $\boldsymbol{\ell}$ nondecreasing and regularly varying of index $\omega = 1$ such that $\boldsymbol{\ell} \simeq \mathbf{m}$. Consequently, from Theorem 2.2.6, $d_{\mathbb{L}}$ is a nonzero proximate order and, by Theorem 2.2.17, $\mathbb{M}$ admits $d_{\mathbb{L}}$ as a nonzero proximate order,

(iv) $\mathbf{m}$ is O-regularly varying, with

$$\beta(\mathbf{m}) = \gamma(\mathbb{M}) = \mu(\mathbf{m}) = \omega(\mathbb{M}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = 1,$$

so it is also strongly regular.

Proof. (i) Since $m_8 = 12$, in order to obtain the property (lc) it is enough to show that $(m_p)_{p\geq 8}$ is nondecreasing. For every $k \in \mathbb{N}$ there are three possibilities:

1. If $2^{2^k+j} \leq p \leq p+1 \leq 2^{2^k+j+1} - 1$ for $j = 1, \dots, 2^k$, we have that $m_{p+1}/m_p = 1$.

2. If $p = 2^{2^k+j+1} - 1$ for $j = 1, \dots, 2^k - 1$, we have that $m_{p+1}/m_p = (2^{2^k}/3)^{1/(2^k-1)}$, which is greater than 1 since $k \in \mathbb{N}$.

3. If $p = 2^{2^{k+1}+1} - 1$, we have that $m_{p+1}/m_p = 2^{2^{k+1}}3/2^{2^{k+1}} = 3$.

Moreover, we deduce that for $2^{2^k+1} \leq p < 2^{2^{k+1}+1}$, $m_p \geq m_{2^{2^k+1}} = 2^{2^k}3$, so $\lim_{p\to\infty} m_p = \infty$.

(ii) Next we analyze the quotients m_{2p}/m_p. By definition, for any $2^{2^k+j} \leq p \leq 2^{2^k+j+1} - 1$ we have that $2^{2^k+j+1} \leq 2p \leq 2^{2^k+j+2} - 1$. We distinguish two cases:

1. If $2^{2^k} \leq p \leq 2^{2^k+1} - 1$, we have that $m_{2p}/m_p = 3$.

2. If $2^{2^k+j} \leq p \leq 2^{2^k+j+1} - 1$ for $j = 1, \dots, 2^k - 1$, we have that $m_{2p}/m_p = (2^{2^k}/3)^{1/(2^k-1)}$.

We observe that

$$\lim_{k\to\infty} \frac{2^{2^k/(2^k-1)}}{3^{1/(2^k-1)}} = 2.$$

From both cases, we have that

$$1 < 2 = \liminf_{p\to\infty} \frac{m_{2p}}{m_p} \leq \limsup_{p\to\infty} \frac{m_{2p}}{m_p} = 3 < \infty.$$

Since the limit $\lim_{p\to\infty} m_{2p}/m_p$ does not exist, condition (c) in Theorem 2.2.6 is violated, so $\mathbf{m}$ is not regularly varying and $d_{\mathbb{M}}(t)$ is not a nonzero proximate order.

(iii) Next, we are going to see that $\mathbf{m} \simeq \boldsymbol{l}$, where the sequence $\boldsymbol{l} = (\ell_p)_{p\in\mathbb{N}_0}$, with $\ell_p = p+1$ for every $p \in \mathbb{N}_0$, corresponds to the sequence of quotients of the Gevrey sequence of order 1, i.e., $(p!)_{p\in\mathbb{N}_0}$, which is nondecreasing and regularly varying. For every $p \geq 8$ and there exist $k \in \mathbb{N}$ and $j \in \{1, 2, \dots, 2^k\}$ such that $2^{2^k+j} \leq p \leq 2^{2^k+j+1} - 1$ and we have that

$$\frac{2^{2^k}3\left(2^{2^k}/3\right)^{\frac{j-1}{2^k-1}}}{2^{2^k+j+1}} \leq \frac{m_p}{p} \leq \frac{2^{2^k}3\left(2^{2^k}/3\right)^{\frac{j-1}{2^k-1}}}{2^{2^k+j}}.$$

Then

$$3^{\frac{2^k-j}{2^k-1}}2^{\frac{j-2^k}{2^k-1}-1} \leq \frac{m_p}{p} \leq 3^{\frac{2^k-j}{2^k-1}}2^{\frac{j-2^k}{2^k-1}}.$$

Since $j \in \{1, 2, \dots, 2^k\}$, we see that
$$2^{-2} \leq \frac{m_p}{p} \leq 3,$$
from where we conclude that $\mathbf{m} \simeq \boldsymbol{\ell}$.

(iv) This follows immediately by the stability of orders, Matuszewska indices, O-regular variation and strong regularity for $\simeq$.

□

The third example shows that condition (j) in Remark 2.2.7 is weaker than the equivalent conditions (f), (g) and (h) in Theorem 2.2.19. The construction is based on Example 1.2.33 from which it is possible to build a nondecreasing O-regularly varying function with its Matuszewska indices equal to 1 that is not equivalent in the sense of Remark 1.2.31 to any regularly varying function.

Example 2.2.23. We consider the function $f : [1, \infty) \to (0, \infty)$ in Example 1.2.33, we define $g(x) := xf(x)$. From the properties of f, we observe that g satisfies the analogous ones:

(i) g is nondecreasing and continuous,

(ii) $\lambda \leq g_{\text{low}}(\lambda) \leq g^{\text{up}}(\lambda) = \lambda \exp((\log(\lambda))^{1/2})$ for every $\lambda > 1$,

(iii) $g \in ORV$ and $\beta(g) = \mu(g) = \rho(g) = \alpha(g) = 1$,

(iv) There do not exist $A \geq a$ and measurable and bounded functions $d, \xi : [A, \infty) \to \mathbb{R}$ with $\lim_{x\to\infty} \xi(x) = 1$ such that
$$g(x) = \exp\left(d(x) + \int_A^x \xi(u)\frac{du}{u}\right), \qquad x \geq A,$$
what implies, by Theorem 1.2.4, that g is not regularly varying. Furthermore, it does not exist $h \in RV$ and $C \geq 1$ such that
$$C^{-1}h(x) \leq g(x) \leq Ch(x), \qquad x \geq 1.$$

We can consider the sequence $m_p := g(p)$ for $p \in \mathbb{N}$ and $m_0 = g(1)$. Then

(i) $\mathbf{m}$ is nondecreasing, so $\mathbb{M}$ is (lc),

(ii) $\mathbf{m}$ is O-regularly varying and $\beta(\mathbf{m}) = \mu(\mathbf{m}) = \rho(\mathbf{m}) = \alpha(\mathbf{m}) = 1$, so $\mathbb{M}$ satisfies (j),

(iii) It does not exist $\boldsymbol{\ell}$ with $\boldsymbol{\ell} \simeq \mathbf{m}$ such that $\boldsymbol{\ell}$ is regularly varying, which, by Theorem 2.2.17 and Proposition 1.1.20, implies that $\mathbb{M}$ does not satisfy (f).

Proof. (i) Immediate.

(ii) Since g is nondecreasing and $g \in ORV$, by Lemma 2.1.34, we see that there exists $C \geq 1$ such that for all $x \geq 1$, $C^{-1}g(\lfloor x \rfloor) \leq g(x) \leq Cg(\lfloor x \rfloor)$. Using that $g \in ORV$, $\beta(g) = \mu(g) = \rho(g) = \alpha(g) = 1$, Remark 1.2.31, Theorem 1.2.44 and the definition of indices and orders for sequences, we conclude that (ii) is valid.

(iii) Assuming that the contrary is true, by Theorem 1.2.37 and Lemma 2.1.12, this would mean that $f_{\boldsymbol{\ell}}(x) = \ell_{\lfloor x \rfloor}$ is regularly varying of index $\rho = 1$ and that there exists $D \geq 1$ with
$$(DC)^{-1} f_{\boldsymbol{\ell}}(x) \leq C^{-1} f_{\mathbf{m}}(x) \leq g(x) \leq C f_{\mathbf{m}}(x) \leq CD f_{\boldsymbol{\ell}}(x), \qquad x \geq 1,$$
which contradicts property (iv) of g.

□

In the fourth place, we present an example of a sequence that is strongly regular for which the lower Matuszewska index and the upper and lower orders coincide (then (k) holds) but the upper Matuszewska index takes a different value ((j) is violated).

Example 2.2.24. Let $\mathbb{M}$ be defined using the sequence of quotients $\mathbf{m}$. The construction is similar to the one in Example 2.2.22. We set $m_0 = m_1 = 1$ and $m_2 = 2$. For each $k \in \mathbb{N}_0$, we consider the intervals $A^k := (2^{2^k}, 2^{2^{k+1}}]$ which we divide in 2^k subintervals. We put $I_j^k := (2^{2^k+j}, 2^{2^k+j+1}] \cap \mathbb{N}$ for $0 \leq j \leq 2^k - 1$. For all $0 \leq j \leq k-1$ we define m_p as follows:

$$m_p := 4m_{2^{2^k+j}} = 4^{j+1}2^{2^k}, \qquad p \in I_j^k;$$

for every $k \leq j \leq 2^k - 1$ we write $\tau_k = (2^k - 2k)/(2^k - k) \geq 0$ and we set

$$m_p := 2^{\tau_k} m_{2^{2^k+j}} = 2^{(j-k+1)\tau_k} 2^{2^k+k}, \qquad p \in I_j^k.$$

For all $k \in \mathbb{N}$, we observe that

$$m_{2^{2^k}} = 2^{2^k} \qquad \text{and} \qquad m_{2^{2^k+k}} = 2^{2^k+2k}.$$

In some sense, one may say that the sequence is oscillating between $\mathbb{M}_{1,0}$ and $\mathbb{M}_{1,1}$. We can show that

(i) $\mathbb{M}$ is strongly regular,

(ii) $\mu(\mathbf{m}) = \rho(\mathbf{m}) = 1$, so (k) holds,

(iii) $\beta(\mathbf{m}) \in (0, 1]$ and $\alpha(\mathbf{m}) \geq 2$, then (j) is violated.

Proof. (i) The sequence $\mathbb{M}$ is a weight sequence, since $\mathbf{m}$ is nondecreasing, and $\lim_{p\to\infty} m_p = \infty$. By definition, for any $p \in I_j^k$ we have that $2p$ belongs to the adjacent interval. We distinguish two cases:

1. If $p \in I_j^k$ for $0 \leq j \leq k-2$ or $j = 2^k - 1$, we have that $m_{2p}/m_p = 4$.

2. If $p \in I_j^k$ for $k-1 \leq j \leq 2^k - 2$, we have that $m_{2p}/m_p = 2^{\tau_k}$.

We observe that $\lim_{k\to\infty} \tau_k = 1$. From both cases, we have that

$$1 < 2 = \liminf_{p\to\infty} \frac{m_{2p}}{m_p} \leq \limsup_{p\to\infty} \frac{m_{2p}}{m_p} = 4 < \infty.$$

Applying Corollary 2.1.6, we see that $\mathbb{M}$ is (mg) and (snq).
(ii) We are going to show that

$$\lim_{p\to\infty} \frac{\log(m_p)}{\log(p)} = 1,$$

which implies that $\mu(\mathbf{m}) = \rho(\mathbf{m}) = 1$. For all $0 \leq j \leq k-1$, if $p \in I_j^k$ we get

$$\frac{(j+1)\log(4) + 2^k \log(2)}{(2^k + j + 1)\log(2)} \leq \frac{\log(m_p)}{\log(p)} \leq \frac{(j+1)\log(4) + 2^k \log(2)}{(2^k + j)\log(2)}.$$

Since $0 \leq j \leq k-1$ we have that

$$\frac{2 + 2^k}{2^k + k} \leq \frac{\log(m_p)}{\log(p)} \leq \frac{2k + 2^k}{2^k}. \tag{2.26}$$

For every $0 \leq j \leq 2^k - k - 1$, if $p \in I^k_{k+j}$ then

$$\frac{(j+1)\cdot\tau_k\cdot\log(2)+(2^k+2k)\log(2)}{(2^k+k+j+1)\log(2)} \leq \frac{\log(m_p)}{\log(p)} \leq \frac{(j+1)\cdot\tau_k\cdot\log(2)+(2^k+2k)\log(2)}{(2^k+k+j)\log(2)},$$

or, equivalently, we see that

$$1+\frac{k(2^k-k-j-1)}{(2^k+k+j+1)(2^k-k)} \leq \frac{\log(m_p)}{\log(p)} \leq 1+\frac{k(2^k-k-j-2)+2^k}{(2^k+k+j)(2^k-k)}.$$

Since $0 \leq j \leq 2^k - k - 1$, we deduce that

$$1 \leq \frac{\log(m_p)}{\log(p)} \leq 1+\frac{k(2^k-k-2)+2^k}{2^{2k}-k^2}. \tag{2.27}$$

By (2.26) and (2.27), we conclude that $\lim_{p\to\infty}\log(m_p)/\log(p)=1$.
(iii) By Remark 2.1.9 and Proposition 2.1.18, we can show that

$$0<\beta(\mathbf{m})\leq 1=\mu(\mathbf{m})=\rho(\mathbf{m})\leq\alpha(\mathbf{m})<\infty,$$

hence only $\alpha(\mathbf{m})\geq 2$ needs to be verified. If $\alpha<2$, we take $p=2^{2^k}$ and $q=2^{2^k+k}$:

$$\frac{m_{2^{2^k}}}{m_{2^{2^k+k}}}\frac{(2^{2^k+k})^\alpha}{(2^{2^k})^\alpha}=\frac{2^{2^k}}{2^{2^k+2k}}\frac{(2^{2^k+k})^\alpha}{(2^{2^k})^\alpha}=2^{(\alpha-2)k}.$$

Therefore, since $\lim_{k\to\infty}(\alpha-2)k=-\infty$, we deduce that m_p/p^α is not almost decreasing for all $\alpha<2$ and we conclude, by using Proposition 2.1.10 that $\alpha(\mathbf{m})\geq 2$.

Moreover, a tedious but simple computation, using the almost increasing and almost decreasing characterization of $\beta(\mathbf{m})$ and $\alpha(\mathbf{m})$, leads to $\beta(\mathbf{m})=1$ and $\alpha(\mathbf{m})=2$.

□

Our last example is a strongly regular sequence for which the values $\beta(\mathbf{m})=\gamma(\mathbb{M})$, $\mu(\mathbf{m})=\omega(\mathbb{M})$, $\rho(\mathbf{m})$, $\alpha(\mathbf{m})$ are mutually distinct. Regarding the implications for the corresponding ultraholomorphic class presented in the next chapter, this will mean that the asymptotic Borel map is neither surjective nor injective for sectors whose opening is $\pi\gamma$ with $\gamma\in(\gamma(\mathbb{M}),\omega(\mathbb{M}))$. The Representation Theorem for O-regularly varying sequence plays a key role and, related to it, appears the notion of Riesz summability.

Definition 2.2.25. [18, Sect. 3.2] A numerical sequence $(s_k)_{k\in\mathbb{N}}$ of complex numbers is said to be *logarithmic summable or Riesz summable of order* 1, if there exists some $A\in\mathbb{C}$ such that

$$\lim_{p\to\infty}\frac{1}{H_p}\sum_{k=1}^{p}\frac{s_k}{k}=A, \qquad \text{where} \qquad H_p=\sum_{k=1}^{p}\frac{1}{k}. \tag{2.28}$$

This method is regular, that is, if the ordinary limit exists, then the limit in (2.28) also exists and with the same value. The base of the example is the construction of a sequence of positive real numbers bounded away from 0 and ∞ that is not Riesz summable of order 1. We were not able to find such an example in the classical literature and the reader is referred to the book of J. Boos [18] for further details regarding the summability methods. In Example 2.2.21, it is hidden the use of a sequence that is 2 on the even positions and 1 in the odd ones, which was proved to be Riesz summable with sum 3/2. Hence the next example requires more elaboration to make the logarithmic means divergent.

Example 2.2.26. We define $\mathbb{M}$ by the sequence of its quotients,

$$m_0 := 1, \qquad m_p := e^{\xi_p/p} m_{p-1} = \exp\left(\sum_{k=1}^{p} \frac{\xi_k}{k}\right), \qquad p \in \mathbb{N}.$$

We consider the sequences of subindices

$$k_n := 2^{3^n} < q_n := k_n^2 = 2^{3^n 2} < k_{n+1} = 2^{3^{n+1}}, \quad n \in \mathbb{N}_0,$$

and we choose the sequence $(\xi_k)_{k=1}^{\infty}$ as follows:

$$\begin{aligned} &\xi_1 = \xi_2 = 2, \\ &\xi_k = 3, \quad \text{if} \quad k \in \{k_n + 1, \dots, q_n\}, n \in \mathbb{N}_0, \\ &\xi_k = 2, \quad \text{if} \quad k \in \{q_n + 1, \dots, k_{n+1}\}, n \in \mathbb{N}_0. \end{aligned}$$

The following hold:

(i) $\mathbb{M}$ is strongly regular, that is, (ℓ) is valid.

(ii) $\mathbb{M}$ does not satisfy (k), i.e., the limit in (2.23),

$$\lim_{p\to\infty} \log(m_p)/\log(p),$$

does not exist. Consequently, neither (j), nor (f), nor (a) holds.

(iii) $\beta(\mathbf{m}) = \gamma(\mathbb{M}) = 2$, $\mu(\mathbf{m}) = \omega(\mathbb{M}) = 5/2$, $\rho(\mathbf{m}) = 11/4$ and $\alpha(\mathbf{m}) = 3$.

Proof. (i) From the definition we deduce immediately that $m_{p+1} > m_p$ for $p \in \mathbb{N}_0$, then $\mathbb{M}$ is (lc). For all $p \in \mathbb{N}$ we have that

$$\exp\left(2\sum_{k=p+1}^{2p} \frac{1}{k}\right) \le \frac{m_{2p}}{m_p} = \exp\left(\sum_{k=p+1}^{2p} \frac{\xi_k}{k}\right) \le \exp\left(3\sum_{k=p+1}^{2p} \frac{1}{k}\right).$$

Using the asymptotic expression (2.24) for the partial sums of the harmonic series, for every $p \in \mathbb{N}$ we have that

$$\exp\left(2\log(2) + 2\varepsilon_{2p} - 2\varepsilon_p\right) \le \frac{m_{2p}}{m_p} \le \exp\left(3\log(2) + 3\varepsilon_{2p} - 3\varepsilon_p\right).$$

From these inequalities and using Corollary 2.1.6, we deduce that $\mathbb{M}$ satisfies (mg) and (snq), therefore $\mathbb{M}$ is strongly regular, which can be also alternatively shown from Theorem 1.2.46 and Remark 2.1.19.

(ii) Observe that $\mathbb{M}$ verifies (2.23) if and only if the sequence

$$t_p := \frac{1}{\log(p)} \sum_{k=1}^{p} \frac{\xi_k}{k}, \qquad p \in \mathbb{N},$$

is convergent (in other words, precisely when the sequence $(\xi_k)_{k=1}^{\infty}$ is Riesz summable, see Definition 2.2.25). We will see that $(\xi_k)_{k=1}^{\infty}$ is not Riesz summable, more precisely we will show that

$$\lim_{n\to\infty} t_{k_n} = \frac{5}{2}, \qquad \lim_{n\to\infty} t_{q_n} = \frac{11}{4}. \tag{2.29}$$

We have the following relations:

$$t_{q_n} = \frac{\log(k_n)}{\log(q_n)}t_{k_n} + 3\frac{H_{q_n} - H_{k_n}}{\log(q_n)} = \frac{t_{k_n}}{2} + \frac{3}{2} + 3\frac{\varepsilon_{q_n} - \varepsilon_{k_n}}{\log(q_n)}, \tag{2.30}$$

$$t_{k_{n+1}} = \frac{2t_{q_n}}{3} + \frac{2}{3} + 2\frac{\varepsilon_{k_{n+1}} - \varepsilon_{q_n}}{\log(k_{n+1})}. \tag{2.31}$$

Using (2.30) and (2.31), we see that

$$t_{k_{n+1}} = \frac{t_{k_n}}{3} + 1 + 2\frac{\varepsilon_{q_n} - \varepsilon_{k_n}}{\log(q_n)} + \frac{2}{3} + 2\frac{\varepsilon_{k_{n+1}} - \varepsilon_{q_n}}{\log(k_{n+1})} = \frac{t_{k_n}}{3} + \frac{5}{3} + \widetilde{\varepsilon}_n, \tag{2.32}$$

where $\widetilde{\varepsilon}_n := (2\varepsilon_{k_{n+1}} + \varepsilon_{q_n} - 3\varepsilon_{k_n})/\log(k_{n+1})$ noting that $\lim_{n\to\infty} \widetilde{\varepsilon}_n = 0$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|\widetilde{\varepsilon}_n| < \varepsilon$ for every $n \geq n_0$. We consider the sequences $(s_n)_{n\geq n_0}$ and $(u_n)_{n\geq n_0}$ recursively defined by

$$\begin{cases} s_{n+1} = s_n/3 + 5/3 + \varepsilon \\ s_{n_0} = t_{k_{n_0}} \end{cases} \qquad \begin{cases} u_{n+1} = u_n/3 + 5/3 - \varepsilon \\ u_{n_0} = t_{k_{n_0}} \end{cases}$$

If $t_{k_{n_0}} \leq 5/2 + 3\varepsilon/2$ (resp. $t_{k_{n_0}} \geq 5/2 + 3\varepsilon/2$) using induction, for every $n \geq n_0$, we deduce that $s_n \leq 5/2 + 3\varepsilon/2$ (resp. $s_n \geq 5/2 + 3\varepsilon/2$). Then, $(s_n)_{n\geq n_0}$ is nondecreasing (resp. nonincreasing) and in both cases we prove that $\lim_{n\to\infty} s_n = 5/2 + 3\varepsilon/2$.

Analogously, we see that $\lim_{n\to\infty} u_n = 5/2 - 3\varepsilon/2$. Since $s_{n_0} = u_{n_0} = t_{k_{n_0}}$, using that $|\widetilde{\varepsilon}_n| < \varepsilon$ for every $n \geq n_0$ we prove by induction employing (2.32) that

$$u_n \leq t_{k_n} \leq s_n, \qquad n \geq n_0.$$

Taking limits, we get

$$\frac{5}{2} - \frac{3\varepsilon}{2} \leq \liminf_{n\to\infty} t_{k_n} \leq \limsup_{n\to\infty} t_{k_n} \leq \frac{5}{2} + \frac{3\varepsilon}{2}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} t_{k_n} = 5/2$. In a similar way, we also can show that $\lim_{n\to\infty} t_{q_n} = 11/4$. Using (2.30) and (2.31), we see that

$$t_{q_{n+1}} = \frac{t_{q_n}}{3} + \frac{1}{3} + \frac{\varepsilon_{k_{n+1}} - \varepsilon_{q_n}}{\log(k_{n+1})} + \frac{3}{2} + 3\frac{\varepsilon_{q_{n+1}} - \varepsilon_{k_{n+1}}}{\log(q_{n+1})} = \frac{t_{q_n}}{3} + \frac{11}{6} + \widetilde{\widetilde{\varepsilon}}_n,$$

where $\widetilde{\widetilde{\varepsilon}}_n := (3\varepsilon_{q_{n+1}} - \varepsilon_{k_{n+1}} - 2\varepsilon_{q_n})/\log(q_{n+1})$ and we conclude, reasoning as before.

(iii) Studying the monotonicity of the sequence $(t_p)_{p\geq 1}$, we will show that

$$\omega(\mathbb{M}) = \mu(\mathbf{m}) = \liminf_{p\to\infty} t_p = \lim_{n\to\infty} t_{k_n} = \frac{5}{2}, \qquad \rho(\mathbf{m}) = \limsup_{p\to\infty} t_p = \lim_{n\to\infty} t_{q_n} = \frac{11}{4}. \tag{2.33}$$

If we show, for n large enough, that $t_p \leq t_{p+1}$ for $p \in \{k_n, \dots, q_n - 1\}$ and $t_p \geq t_{p+1}$ for $p \in \{q_n, \dots, k_{n+1} - 1\}$, using (2.29), we obtain (2.33) . We observe that

$$t_{p+1} = \frac{\log(p)}{\log(p+1)}t_p + \frac{\xi_{p+1}}{(p+1)\log(p+1)} = \frac{\xi_{p+1} + (p+1)\log(p)t_p}{\log(p+1)(p+1)}. \tag{2.34}$$

Then

$$t_{p+1} - t_p = \frac{1}{\log(p+1)(p+1)}\left(\xi_{p+1} - (p+1)\log(1+1/p)t_p\right). \tag{2.35}$$

First we will show that there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, we have that

$$t_p < 3 - \frac{3}{2p} \qquad \text{for all} \quad p \in \{k_n, \dots, q_n - 1\}. \tag{2.36}$$

Since $\lim_{n\to\infty} t_{k_n} = 5/2$, there is $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ we have $t_{k_n} < 5/2 + 5/16 = 45/16$. Since $k_1 = 8$, for every $n \geq n_1$ we have that

$$t_{k_n} < 45/16 = 3 - 3/(2k_1) < 3 - 3/(2k_n).$$

Reasoning by induction if we assume that $t_p < 3 - 3/(2p)$ for a certain $n \geq n_1$ and a certain $p \in \{k_n, \dots, q_n - 1\}$, so $\xi_{p+1} = 3$, and using (2.34) we see that

$$\log(p+1)(p+1)t_{p+1} = 3 + (p+1)\log(p)t_p < 3 + (p+1)\log(p)\left(3 - \frac{3}{2p}\right).$$

Hence it suffices to prove that

$$3 + (p+1)\log(p)\left(3 - \frac{3}{2p}\right) \leq \log(p+1)(p+1)\left(3 - \frac{3}{2(p+1)}\right).$$

This happens if and only if

$$2p + (p+1)\log(p)(2p-1) \leq p\log(p+1)\left(2p+1\right),$$

or, equivalently, if

$$2p \leq (2p^2 + p)\log(1 + 1/p) + \log(p).$$

Since $x/(1+x) \leq \log(1+x)$ for every $x > 0$, we see that the last inequality holds for every p large enough which proves (2.36). Using (2.35), we see that $t_p < t_{p+1}$ if and only if

$$\xi_{p+1} > (p+1)\log(1+1/p)t_p.$$

We observe that

$$(p+1)\log(1+1/p) = 1 + \frac{1}{2p} - \frac{1}{6p^2} + o(1/p^2).$$

Consequently, there exists $n_2 \in \mathbb{N}$ such that for every $p \geq k_{n_2}$, we have that

$$(p+1)\log(1+1/p) \leq 1 + \frac{1}{2p} - \frac{1}{12p^2}.$$

By (2.36) for every $n \geq n_0 := \max(n_1, n_2)$ and $p \in \{k_n, \dots, q_n - 1\}$ we have that

$$t_p(p+1)\log\left(1 + \frac{1}{p}\right) < \left(3 - \frac{3}{2p}\right)\left(1 + \frac{1}{2p} - \frac{1}{12p^2}\right) = 3 - \frac{1}{p^2} + \frac{1}{8p^3} < 3 = \xi_{p+1}.$$

Consequently, $t_p \leq t_{p+1}$ for every $n \geq n_0$ and $p \in \{k_n, \dots, q_n - 1\}$. Analogously, we will show that for n large enough, and $p \in \{q_n, \dots, k_{n+1} - 1\}$ we have that

$$t_p > 2 + \frac{1}{p}. \tag{2.37}$$

Since $\lim_{n\to\infty} t_{q_n} = 11/4$ there exists $n_3 \in \mathbb{N}$ such that $t_{q_n} > 2 + 1/64 = 2 + 1/q_1 > 2 + 1/q_n$ for every $n \geq n_3$. Assume $t_p > 2 + 1/p$ for a certain $n \geq n_3$ and a certain $p \in \{q_n, \dots, k_{n+1} - 1\}$ by (2.34) we have that

$$\log(p+1)(p+1)t_{p+1} = 2 + (p+1)\log(p)t_p > 2 + (2 + 1/p)(p+1)\log(p).$$

Then it is enough to prove that

$$2+(2+1/p)(p+1)\log(p)\geq \log(p+1)(p+1)\left(2+\frac{1}{p+1}\right),$$

or, equivalently,

$$2p\geq (2p^2+3p)\log(1+1/p)-\log(p).$$

Since $\log(1+1/p)\leq 1/p$, the last inequality is always true for p large enough, which proves (2.37). We observe that for $n\geq n_4\geq n_3$ and $p\in\{q_n,\dots,k_{n+1}-1\}$ we have that

$$t_p(p+1)\log(p+1)>(2+1/p)(p+1)\log(p+1)\geq (2+1/p)(1+1/(3p))>2=\xi_{p+1}.$$

We conclude that $t_p\geq t_{p+1}$ for n large enough and $p\in\{q_n,\dots,k_{n+1}-1\}$. Using these monotonicity properties of the sequence $(t_p)_{p\geq 1}$ we conclude that

$$\mu(\mathbf{m})=\frac{5}{2}=\lim_{n\to\infty}t_{k_n}=\liminf_{p\to\infty}t_p\leq\limsup_{p\to\infty}t_p=\lim_{n\to\infty}t_{q_n}=\frac{11}{4}=\rho(\mathbf{m}).$$

Finally, let us see that we have $\beta(\mathbf{m})=\gamma(\mathbb{M})=2$. We will show that m_p/p^α is almost increasing if and only if $\alpha\in(0,2]$. We have that m_p/p^α is almost increasing if and only if there exists $M\geq 1$ such that for every $p\in\mathbb{N}$ and all $\ell\geq p$ we have that

$$\frac{m_p}{m_\ell}\frac{\ell^\alpha}{p^\alpha}\leq M,$$

or, equivalently, using (2.24),

$$\frac{m_p}{m_\ell}\exp(\alpha(H_\ell-H_p)-\alpha(\varepsilon_\ell-\varepsilon_p))\leq M.$$

Since $\lim_{p\to\infty}\varepsilon_p=0$, using the definition of m_p, we have that m_p/p^α is almost increasing if and only if, there exists $M\geq 1$ such that for every $p\in\mathbb{N}$ and all $\ell\geq p$ we have that

$$\exp\left(\sum_{k=p+1}^{\ell}\frac{\alpha-\xi_k}{k}\right)\leq M.$$

If $\alpha\in(0,2]$, we observe that, for every $k\in\mathbb{N}$, $\alpha-\xi_k\leq 0$. Then for every $p\in\mathbb{N}$ and all $\ell\geq p$ we see that

$$\exp\left(\sum_{k=p+1}^{\ell}\frac{\alpha-\xi_k}{k}\right)\leq 1=:M.$$

If $\alpha>2$, we see that for every $n\in\mathbb{N}$ and every $k\in\{q_n+1,\dots,k_{n+1}\}$ we have that $\alpha-\xi_k=\alpha-2>0$. Then taking $p=q_n=2^{3^n2}$ and $\ell=k_{n+1}=2^{3^{n+1}}$ and using again (2.24) we see that

$$\exp\left(\sum_{k=q_n+1}^{k_{n+1}}\frac{\alpha-\xi_k}{k}\right)=\exp((\alpha-2)(3^n\log(2)-\varepsilon_{k_{n+1}}-\varepsilon_{q_n})),$$

but the right hand side is unbounded as $n\to\infty$, then m_p/p^α is not almost increasing. Consequently, m_p/p^α is almost increasing if and only if $\alpha\in(0,2]$ and, by Proposition 2.1.10, it means that $\beta(\mathbf{m})=2$. Similarly, we see that $\alpha(\mathbf{m})=3$.

□

Remark 2.2.27. In a more general framework, given $\xi : [1,\infty) \to (0,\infty)$ locally integrable we can consider the function

$$\omega(x) = \exp\left(\int_1^x \xi(u)\frac{du}{u}\right), \qquad t > 1,$$

that is a nondecreasing function.

If ξ is bounded, then ω is automatically O-regularly varying, by Theorem 1.2.25. In particular, given four mutually distinct positive values $0 < \beta < \mu < \rho < \alpha < \infty$, for all $a > b > 1$ we define

$$\xi(t) := \begin{cases} \alpha, & \text{for} \quad t \in [2^{a^n}, 2^{ba^n}). \\ \beta, & \text{for} \quad t \in [2^{ba^n}, 2^{a^{n+1}}). \end{cases}$$

With a technique similar to the one employed in the last example, in which $\alpha = a = 3$ and $\beta = b = 2$, one can prove that

$$\mu(\mathbf{m}) = \frac{(b-1)\alpha + (a-b)\beta}{a-1}, \qquad \rho(\mathbf{m}) = \frac{a(b-1)\alpha + (a-b)\beta}{b(a-1)},$$

then taking

$$b := \frac{\alpha - \mu}{\alpha - \rho}, \qquad a := b\,\frac{\rho - \beta}{\mu - \beta} = \frac{\alpha - \mu}{\alpha - \rho}\,\frac{\rho - \beta}{\mu - \beta}$$

we obtain

$$\beta(\omega) = \beta, \quad \mu(\omega) = \mu, \quad \rho(\omega) = \rho, \quad \alpha(\omega) = \alpha.$$

Finally, for the corresponding nondecreasing sequence, that is, $m_p := \omega(p)$ for p large enough, since $\omega \in ORV$, by Lemma 2.1.34, we know that

$$\beta(\mathbf{m}) = \beta, \quad \mu(\mathbf{m}) = \mu, \quad \rho(\mathbf{m}) = \rho, \quad \alpha(\mathbf{m}) = \alpha.$$

In a similar way, one can construct sequences for which $\gamma(\mathbb{M}) = \beta(\mathbf{m})$, $\omega(\mathbb{M}) = \mu(\mathbf{m}) \in (0,\infty)$, but that $\alpha(\mathbf{m}) = \infty$. For instance, for all $t \geq 4$ and $n \geq 2$ we can define

$$\xi(t) := \begin{cases} 2, & \text{for} \quad t \in [2^{2((n-1)!)^2}, 2^{(n!)^2}). \\ n, & \text{for} \quad t \in [2^{(n!)^2}, 2^{2(n!)^2}). \end{cases}$$

The corresponding sequence $\mathbb{M}$ is a (dc) weight sequence with $\gamma(\mathbb{M}) = 2$, $\omega(\mathbb{M}) \in (0,\infty)$ and $\alpha(\mathbf{m}) = \infty$.

Such an example can also be constructed indirectly by considering the dual sequence of the sequence $\mathbb{M}$ in [57, Example 3.3] introduced before in Example 2.1.20. For this sequence, we have that $\beta(\mathbf{m}) = 0$, $\mu(\mathbf{m}) = 2$, $\rho(\mathbf{m}) = 3$ and $\alpha(\mathbf{m}) = 4$, so since $\mathbf{m}$ is O-regularly varying by Theorem 2.1.43 for the dual sequence $\mathbb{D}^{\mathbb{M}}$ we have that $\beta(\boldsymbol{d}) = 1/4$, $\mu(\boldsymbol{d}) = 1/3$, $\rho(\boldsymbol{d}) = 1/2$ and $\alpha(\boldsymbol{d}) = \infty$.

These sequences are not strongly regular (see Remark 2.1.19), but nevertheless the values $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ are meaningful, as it will be explained in the next chapter. Unfortunately, as it will be mentioned in the following chapter, not much information is available for the surjectivity of the asymptotic Borel map in these kind of situations.

Chapter 3

Injectivity and surjectivity of the asymptotic Borel map

From the sequences of positive real numbers considered in the previous chapters we will define classes of holomorphic functions in unbounded sectors of the Riemann surface of the logarithm. In this context, the study of properties of the asymptotic Borel map, sending a function f into its asymptotic expansion, appears as a natural problem. The injectivity and surjectivity of the Borel map will be examined in three instances: in Roumieu-Carleman ultraholomorphic classes and in classes of functions admitting (uniform or nonuniform) asymptotic expansion at the origin. It will be shown that the solution depends on the opening of the sector, injectivity is possible if the sector is wide enough whereas surjectivity is only attainable in narrow regions. This issue is closely related with the summability theory presented in the next chapter: injectivity provides uniqueness and surjectivity existence of the sum of a formal power series.

Injectivity had been solved in two cases by S. Mandelbrojt and B. Rodríguez-Salinas in the 1950's, respectively, and we completely solve the third one by means of the theory of proximate orders (see Theorem 3.2.15). Sanz's growth index $\omega(\mathbb{M})$ turns out to put apart the values of the opening of the sector for which injectivity holds or not. The first section ends with Theorem 3.2.16 in which it is proved that the Borel map is never bijective as an outcome of the injectivity theorems.

In the case of surjectivity, only some partial results were available by J. Schmets and M. Valdivia and by V. Thilliez at the very beginning of this century, resting on results from the ultradifferentiable setting and disregarding questions about the optimality of the opening of the sector, that was only established for the Gevrey case $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$. This last author introduced the growth index $\gamma(\mathbb{M})$ for this problem. We considerably extend here their results, proving that $\gamma(\mathbb{M})$ is indeed optimal in some standard situations, for instance for strongly regular sequences, putting now apart the values of the opening of the sector for which surjectivity holds or not.

From the information in Section 2.1, we know that for strongly regular sequences the value of theses indices $\omega(\mathbb{M})$ and $\gamma(\mathbb{M})$ is generally different. However they coincide for a large class, the ones admitting a nonzero proximate order (see Section 2.2), which contains most of the sequences appearing in the applications.

The results gathered in this chapter can be found in [48].

3.1 Asymptotic expansions and ultraholomorphic classes

In this section, we introduce three different classes of ultaholomorphic functions. We will study their relations and their elementary properties. Next the asymptotic Borel map will be defined from these classes into the algebra of formal power series. Finally, some helpful notation regarding the injectivity and surjectivity of the Borel map will be considered.

3.1.1 Basic definitions

The functions appearing below are defined in regions of the Riemann surface of the logarithm $\mathcal{R}$, some of them already introduced in the first chapter. We consider bounded *sectors*

$$S(d,\gamma,r) := \{z \in \mathcal{R} : |\arg(z) - d| < \frac{\gamma\,\pi}{2},\ |z| < r\},$$

respectively unbounded sectors

$$S(d,\gamma) := \{z \in \mathcal{R} : |\arg(z) - d| < \frac{\gamma\,\pi}{2}\},$$

with *bisecting direction* $d \in \mathbb{R}$, *opening* $\gamma\,\pi$ ($\gamma > 0$) and (in the first case) *radius* $r \in (0,\infty)$. For unbounded sectors of opening $\gamma\,\pi$ bisected by direction 0, we write $S_\gamma := S(0,\gamma)$.

In some cases, it will also be convenient to considered more general domains. A *sectorial region* $G(d,\gamma)$ with bisecting direction $d \in \mathbb{R}$ and opening $\gamma\,\pi$ will be an open connected set in $\mathcal{R}$ such that $G(d,\gamma) \subseteq S(d,\gamma)$, and for every $\beta \in (0,\gamma)$ there exists $\rho = \rho(\beta) > 0$ with $S(d,\beta,\rho) \subseteq G(d,\gamma)$. In particular, sectors are sectorial regions. If $d = 0$ we just write G_γ.

A bounded (respectively, unbounded) sector T is said to be a *proper subsector* of a sectorial region G (resp. of an unbounded sector S) , and we write $T \ll G$ (resp. $T \prec\!\!\prec S$), if $\overline{T} \subset G$ (where the closure of T is taken in $\mathcal{R}$, and so the vertex of the sector is not under consideration).

For an open set $U \subset \mathcal{R}$, the set of all holomorphic functions in U will be denoted by $\mathcal{H}(U)$. Finally, $\mathbb{C}[[z]]$ stands for the set of formal power series in z with complex coefficients.

As in the previous chapters, $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ is a sequence of positive real numbers with $M_0 = 1$. We consider the following three classes of functions defined for arbitrary sectorial regions, so also for sectors.

Definition 3.1.1. Given a sectorial region G, we say $f \in \mathcal{H}(G)$ admits the formal power series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ as its $\mathbb{M}-$*asymptotic expansion* in G (when the variable tends to 0) if for every $T \ll G$ there exist $C_T, A_T > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$\Big|f(z) - \sum_{n=0}^{p-1} a_n z^n\Big| \le C_T A_T^p M_p |z|^p, \qquad z \in T.$$

We will write $f \sim_{\mathbb{M}} \hat{f}$ in G, and $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ will stand for the space of functions admitting $\mathbb{M}-$asymptotic expansion in G.

Definition 3.1.2. Given a sectorial region G, we say $f \in \mathcal{H}(G)$ admits $\hat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ as its *uniform* $\mathbb{M}-$*asymptotic expansion in* G *(of type* $1/A$ *for some* $A > 0$) if there exists $C > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$\Big|f(z) - \sum_{n=0}^{p-1} a_n z^n\Big| \le C A^p M_p |z|^p, \qquad z \in G.$$

We will write $f \sim_{\mathbb{M}}^{u} \hat{f}$ in G, and $\tilde{\mathcal{A}}^{u}_{\mathbb{M}}(G)$ stands for the space of functions admitting uniform $\mathbb{M}-$asymptotic expansion in G (of some type).

Definition 3.1.3. Given a constant $A>0$ and a sectorial region G, we define

$$\mathcal{A}_{\mathbb{M},A}(G)=\{f\in\mathcal{H}(G):\|f\|_{\mathbb{M},A}:=\sup_{z\in G,p\in\mathbb{N}_0}\frac{|f^{(p)}(z)|}{A^p p! M_p}<\infty\}.$$

$(\mathcal{A}_{\mathbb{M},A}(G),\|\ \|_{\mathbb{M},A})$ is a Banach space, and $\mathcal{A}_{\mathbb{M}}(G):=\cup_{A>0}\mathcal{A}_{\mathbb{M},A}(G)$ is called a *Carleman ultraholomorphic class of of Roumieu type* in the sectorial region G.

The functions considered in this chapter are complex-valued, but most of the following results are also valid if they take values in a complex Banach algebra. From the conditions introduced in Chapter 1 for the sequence $\mathbb{M}$ we obtain some elementary properties of the classes.

Remark 3.1.4. For any sequence $\mathbb{M}$, the classes $\mathcal{A}_{\mathbb{M}}(G)$, $\tilde{\mathcal{A}}^u_{\mathbb{M}}(G)$ and $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ are complex vector spaces. If $\mathbb{M}$ is (lc), they are algebras and if $\mathbb{M}$ is (dc), they are stable under taking derivatives. Moreover, if $\mathbb{M}\approx\mathbb{L}$ the corresponding classes coincide.

In our results we will mainly consider sectors but some of them can be extended to sectorial regions, specially when dealing with the class $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$. For a sector S, bounded or not, since the derivatives of $f\in\mathcal{A}_{\mathbb{M},A}(S)$ are Lipschitzian, for every $n\in\mathbb{N}_0$ one may define

$$f^{(p)}(0):=\lim_{z\in S,z\to 0}f^{(p)}(z)\in\mathbb{C}. \tag{3.1}$$

We recall now the relation between this Roumieu-Carleman ultraholomorphic class and the concept of asymptotic expansion. that is obtained as a consequence of Taylor's formula and Cauchy's integral formula for the derivatives (see [7, 31] for a proof in the Gevrey case, which may be easily adapted to this more general situation).

Proposition 3.1.5. Let $\mathbb{M}$ be a sequence, S a sector and G a sectorial region. Then,

(i) if $f\in\mathcal{A}_{\mathbb{M},A}(S)$ then f admits $\hat{f}:=\sum_{p\in\mathbb{N}_0}\frac{1}{p!}f^{(p)}(0)z^p$ as its uniform $\mathbb{M}-$asymptotic expansion in S of type $1/A$ where $(f^{(p)}(0))_{p\in\mathbb{N}_0}$ is given by (3.1). Consequently, we have that

$$\mathcal{A}_{\mathbb{M}}(S)\subseteq\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)\subseteq\tilde{\mathcal{A}}_{\mathbb{M}}(S).$$

(ii) $f\in\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ if and only if for every $T\ll G$ there exists $A_T>0$ such that $f|_T\in\mathcal{A}_{\mathbb{M},A_T}(T)$. In case any of the previous holds and $f\sim_{\mathbb{M}}\sum_{p=0}^{\infty}a_pz^p$, it is plain to check that for every bounded proper subsector T of G and every $p\in\mathbb{N}_0$ one has

$$a_p=\lim_{\substack{z\to 0\\ z\in T}}\frac{f^{(p)}(z)}{p!},$$

and we can set $f^{(p)}(0):=p!a_p$.

(iii) if S is unbounded and $T\prec\!\!\prec S$, then there exists a constant $c=c(T,S)>0$ such that the restriction to T, f_T, of functions f defined on S and admitting uniform $\mathbb{M}-$asymptotic expansion in S of type $1/A>0$, belongs to $\mathcal{A}_{\mathbb{M},cA}(T)$.

(iv) if $f\in\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ its $\mathbb{M}-$asymptotic expansion $\hat{f}$ is unique.

3.1.2 The asymptotic Borel map

In this context, it is natural to consider the following map.

Definition 3.1.6. Given a sectorial region G and $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$, we define the *asymptotic Borel map* as the map sending a function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ into its $\mathbb{M}-$asymptotic expansion $\hat{f}$ and we write $\tilde{\mathcal{B}}(f) := \hat{f}$.

One may accordingly define classes of formal power series

$$\mathbb{C}[[z]]_{\mathbb{M},A} = \Big\{ \hat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]] : |\,\boldsymbol{a}\,|_{\mathbb{M},A} := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{A^p M_p} < \infty \Big\}.$$

$(\mathbb{C}[[z]]_{\mathbb{M},A}, |\ |_{\mathbb{M},A})$ is a Banach space and we put $\mathbb{C}[[z]]_{\mathbb{M}} := \cup_{A>0}\mathbb{C}[[z]]_{\mathbb{M},A}$.

Remark 3.1.7. Given $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ it is straightforward that $\tilde{\mathcal{B}}(f) \in \mathbb{C}[[z]]_{\mathbb{M}}$, so

$$\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(G) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}.$$

If $G = S$ is a sector, using Proposition 3.1.5.(i) we see that the asymptotic Borel map is also well defined on $\mathcal{A}_{\mathbb{M}}(S)$ and $\tilde{\mathcal{A}}^{u}_{\mathbb{M}}(S)$.
If $\mathbb{M}$ is (lc), $\tilde{\mathcal{B}}$ is a homomorphism of algebras; if $\mathbb{M}$ is (dc), $\tilde{\mathcal{B}}$ is a homomorphism of differential algebras. Finally, note that if $\mathbb{M} \approx \mathbb{L}$, then $\mathbb{C}[[z]]_{\mathbb{M}} = \mathbb{C}[[z]]_{\mathbb{L}}$.

A fundamental role in the discussion about the injectivity and surjectivity of the asymptotic Borel map gathered in this chapter will be played by the flat functions.

Definition 3.1.8. A function f in any of the previous classes is said to be *flat* if $\tilde{\mathcal{B}}(f)$ is the null power series, in other words, $f \sim_{\mathbb{M}} \hat{0}$.

One may express flatness in $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$ by means of the associated functions defined in Section 1.1.3.

Proposition 3.1.9 ([97], Proposition 4)**.** Given a sequence $\mathbb{M}$, a sectorial region G and $f \in \mathcal{H}(G)$, the following are equivalent:

(i) $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ and f is flat,

(ii) For every bounded proper subsector T of G there exist $c_1, c_2 > 0$ with

$$|f(z)| \le c_1 e^{-\omega_{\mathbb{M}}(1/(c_2|z|))} = c_1 h_{\mathbb{M}}(c_2|z|), \qquad z \in T.$$

In the Gevrey case of order α, thanks to the estimates in Example 1.1.22, we recover the classical result that characterizes flatness in terms of exponential decrease bounds of order $1/\alpha$, that is, in terms of $e^{-|z|^{-1/\alpha}}$.

Remark 3.1.10. In the results gathered in the next sections we will only deal with weight sequences, that is, (lc) such that $\lim_{p\to\infty} m_p = \infty$. The requirement of (lc) condition is justified in Remarks 3.1.4 and 3.1.7. Moreover, A. Gorny and H. Cartan proved that this is not a restriction in the ultradifferentiable setting (see [53, p. 104]).

For a (lc) sequence $\mathbb{M}$, since $\mathbf{m}$ is not decreasing, if $\lim_{p\to\infty} m_p \neq \infty$, then $\lim_{p\to\infty} m_p < \infty$ and also $\lim_{p\to\infty}(M_p)^{1/p} < \infty$ (see Lemma 1.1.7). Then there exists $A > 0$ such that the function $h_{\mathbb{M}}(t) = 0$ for all $t \in [0, A]$. Hence by Proposition 3.1.9, if $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ and f is flat, we have that $f(t) = 0$ for every $t \in (0, A]$ which, by the identity principle, implies that $f(z)$ identically vanishes in G. Consequently, the Borel map is always injective.

On the other hand, in the same situation, the Borel map is never surjective: we consider a holomorphic function at the origin $L(z)$ whose Taylor expansion at 0 is given by a lacunary series $\hat{L} \in \mathbb{C}\{z\} \subseteq \mathbb{C}[[z]]_{\mathbb{M}}$, whose domain of convergence is a disc of radius R where $R < |z|$ for some $z \in G$. We have that $L \sim_{\mathbb{M}} \hat{L}$ on a region $G' \subseteq G$, so by the injectivity of the Borel map it cannot exist another function $E \in \tilde{\mathcal{A}}_{\mathbb{M}}(G) \subseteq \tilde{\mathcal{A}}_{\mathbb{M}}(G')$ with $E \sim_{\mathbb{M}} \hat{L}$. Since L cannot be analytically continued to G, the Borel map is not surjective, which justifies the consideration of the limit condition for $\mathbf{m}$.

By using a simple rotation, we see that the injectivity and the surjectivity of the Borel map in any of the previously considered classes do not depend on the bisecting direction d of the sectorial region G, so we limit ourselves to the case $d = 0$. Moreover, in this dissertation we will restrict our study to the unbounded sectors S_γ, and include comments on what can be said, to our knowledge, for more general sectorial regions. So, we define

$$\begin{aligned} I_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is injective}\}, \\ \tilde{I}^u_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is injective}\}, \\ \tilde{I}_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is injective}\}. \end{aligned}$$

Whenever $\gamma > 0$ belongs to any of these sets, we say that the corresponding class is *quasianalytic*. Hence, nonquasianalyticity amounts to the existence of nontrivial flat functions in the class.

We easily observe that, by restriction and the identity principle, if $\gamma > 0$ is in any of those sets then every $\gamma' > \gamma$ also is. Consequently, $I_{\mathbb{M}}$, $\tilde{I}^u_{\mathbb{M}}$ and $\tilde{I}_{\mathbb{M}}$ are either empty or unbounded intervals contained in $(0, \infty)$, which we call *quasianalyticity or injectivity intervals*.
Similarly, we define

$$\begin{aligned} S_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \tilde{S}^u_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \tilde{S}_{\mathbb{M}} :=& \{\gamma > 0; \quad \tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}. \end{aligned}$$

It is also plain to check that if $\gamma > 0$ is in any of those sets then every $0 < \gamma' < \gamma$ also is, so $S_{\mathbb{M}}$, $\tilde{S}^u_{\mathbb{M}}$ and $\tilde{S}_{\mathbb{M}}$ are either empty or left-open intervals having 0 as endpoint, called *surjectivity intervals*. Using Proposition 3.1.5.(i), we easily see that

$$I_{\mathbb{M}} \supseteq \tilde{I}^u_{\mathbb{M}} \supseteq \tilde{I}_{\mathbb{M}}, \tag{3.2}$$

$$S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}}. \tag{3.3}$$

Remark 3.1.11. In the literature, a set of conditions different from those presented in the previous chapters appears when dealing with ultraholomorphic or ultradifferentiable classes of functions, specially if they are given in terms of bounds for the derivatives. In particular, some authors (H. Komatsu [52], H.-J. Petzsche [77], J. Bonet, R. Meise and S.N. Melikhov [17] and others) define the classes replacing the sequence $\mathbb{M}$ by $\widehat{\mathbb{M}} := (p!M_p)_{p\in\mathbb{N}_0}$ in the estimates of Definitions 3.1.1, 3.1.2 and 3.1.3, that is,

$$\sup_{z\in G, p\in\mathbb{N}_0} \frac{|f^{(p)}(z)|}{A^p \widehat{M}_p} < \infty, \qquad \text{or} \qquad \Big| f(z) - \sum_{n=0}^{p-1} a_n z^n \Big| \le C A^p \frac{\widehat{M}_p}{p!} |z|^p,$$

for z in the corresponding region. In this situation, the following conditions, with the notation of H. Komatsu, are considered for $\widehat{\mathbb{M}}$:

(M.1) $L_p^2 \leq L_{p-1}L_p$ for every $p \in \mathbb{N}$,

(M.2) There exists $A > 0$ such that

$$L_{p+k} \leq A^{p+k} L_p L_k, \qquad k, p \in \mathbb{N}_0,$$

(M.3) There exists $B > 0$ such that

$$\sum_{k=p}^{\infty} \frac{L_k}{L_{k+1}} \leq Bp \frac{L_p}{L_{p+1}}, \qquad p \in \mathbb{N}.$$

Note that $\mathbb{L}$ satisfies (M.3) if and only if $\boldsymbol{\ell}$ has (γ_1) of H. -J. Petzsche (see Remark 2.1.23). Let us clarify the relation between these two approaches: If $\mathbb{M}$ is (lc), then $\widehat{\mathbb{M}}$ satisfies (M.1), if $\mathbb{M}$ has (mg), then $\widehat{\mathbb{M}}$ satisfies (M.2) and if $\mathbb{M}$ is (snq), then $\widehat{\mathbb{M}}$ satisfies (M.3). On the other hand, if $\mathbb{M}$ satisfies (M.2) or (M.3), then the sequence $\check{\mathbb{M}} := (M_p/p!)_{p\in\mathbb{N}_0}$ has (mg) or (snq), respectively. However, if $\mathbb{M}$ is (M.1), $\check{\mathbb{M}}$ is not necessarily (lc). If this is the case, we say that $\mathbb{M}$ is *strongly logarithmically convex*, for short (slc).

In this same context, instead of considering the class of formal power series $\mathbb{C}[[z]]_{\mathbb{M}}$, some authors considered the classes of sequences

$$\Lambda_{\mathbb{M},A} = \Big\{ \boldsymbol{\mu} = (\mu_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : |\boldsymbol{\mu}|_{\mathbb{M},A} := \sup_{n\in\mathbb{N}_0} \frac{|\mu_n|}{A^n n! M_n} < \infty \Big\}.$$

$(\Lambda_{\mathbb{M},A}, |\ |_{\mathbb{M},A})$ is again a Banach space, and we put $\Lambda_{\mathbb{M}} := \cup_{A>0}\Lambda_{\mathbb{M},A}$. There is a bijection between them sending a formal power series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ into the sequence of derivatives at the origin, $(a_p p!)_{p\in\mathbb{N}_0}$ (see Proposition 3.1.5.(ii)). This map is an isomorphism of Banach algebras if $\mathbb{M}$ is (lc). Due to the fact that the product considered in these algebras is the classical for formal power series, we have found reasonable the choice of the notation $\mathbb{C}[[z]]_{\mathbb{M}}$.

Even if the relation between the sequences $\mathbb{M}$, $\widehat{\mathbb{M}}$ and $\check{\mathbb{M}}$ is known, we need to impose certain conditions in order to establish the connection between the associated functions $\omega_{\mathbb{M}}(t)$, $\omega_{\widehat{\mathbb{M}}}(t)$ and $\omega_{\check{\mathbb{M}}}(t)$. This is one of the major concerns when mixing the results from the two different approaches.

3.2 Injectivity of the asymptotic Borel map. Impossibility of bijectivity

In 1912, G.N. Watson [105] determined the injectivity interval $\tilde{I}^u_{\mathbb{M}_\alpha}$ for the Gevrey sequence $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$ by proving that if a function f has global exponential decrease bounds of order $1/\alpha$ on a sector S_α, the function f is too flat and must be 0. The modern proof, contained in most of the books, is a smart but easy consequence of Phragmén-Lindelöf principle. Extensions of these results were obtained by S. Mandelbrojt and B. Rodríguez-Salinas (see Theorems 3.2.1 and 3.2.3).

In this section, these classical results which deal with the classes $\mathcal{A}_{\mathbb{M}}$ and $\tilde{\mathcal{A}}^u_{\mathbb{M}}$ will be presented and we will reformulate them in terms of the growth index $\omega(\mathbb{M})$ thanks to its relation with the exponent of convergence of $\mathbf{m}$. Finally, the problem for the class $\tilde{\mathcal{A}}_{\mathbb{M}}$ will be solved by constructing, via proximate orders, flat functions in sectors of optimal opening. As a consequence, we will obtain the first surjectivity result stating that the Borel map is never bijective.

3.2.1 Classical injectivity results

The quasianalyticity intervals $I_{\mathbb{M}}$ and $\tilde{I}^u_{\mathbb{M}}$ were determined in the literature in the 1950's. We will rephrase the corresponding results by means of Sanz's growth index $\omega(\mathbb{M})$ (see Subection 1.1.4). This index was proved to coincide with the lower order of the sequence $\mu(\mathbf{m})$ which is closely related to the exponent of convergence of $\mathbf{m}$ (see Theorem 2.1.16 and Proposition 2.1.26). Hence, if $\mathbb{M}$ is (lc), then $\mathbf{m}$ and $\widehat{\mathbf{m}}=((p+1)m_p)_{p\in\mathbb{N}_0}$ are nondecreasing and, by the results we have just mentioned, we see that

$$\omega(\mathbb{M})=\mu(\mathbf{m})=\sup\{\mu>0;\quad \sum_{\ell=0}^{\infty}\frac{1}{(m_\ell)^{1/\mu}}<\infty\},$$

$$\omega(\mathbb{M})=\mu(\mathbf{m})=\sup\{\mu>0;\quad \sum_{\ell=0}^{\infty}\frac{1}{((\ell+1)m_\ell)^{1/(\mu+1)}}<\infty\}.$$

In order to simplify some statements and to avoid trivial situations (see Remark 3.1.10), we will frequently assume that $\mathbb{M}$ is a weight sequence, i.e., is (lc) with $\lim_{p\to\infty}m_p=\infty$. The first result we recall is due to S. Mandelbrojt in 1952.

Theorem 3.2.1 ([72], Section 2.4.III)**.** Let $\mathbb{M}$ be a weight sequence, $c\geq 0$, $H(c):=\{z\in\mathbb{C}:\mathrm{Re}(z)>c\}$ and $\gamma>0$. The following statements are equivalent:

(i) $\displaystyle\sum_{p=0}^{\infty}\left(\frac{1}{m_p}\right)^{1/\gamma}$ diverges,

(ii) If $f\in\mathcal{H}(H(c))$ and there exist $A,C>0$ such that

$$|f(z)|\leq\frac{CA^pM_p}{|z|^{\gamma p}},\qquad z\in H(c),\quad p\in\mathbb{N}_0, \tag{3.4}$$

then f identically vanishes.

Observe that a function f is holomorphic in $H:=H(0)$ and verifies the estimates (3.4) if and only if the function g given by $g(z):=f(1/z^{1/\gamma})$ belongs to $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ and is flat. Hence, the interval $I_{\mathbb{M}}$ is determined by the following equivalence $(i)\Leftrightarrow(ii)$, as an easy consequence of Theorem 3.2.1, and (iii) is a consequence of the above mentioned properties of $\omega(\mathbb{M})$.

Theorem 3.2.2 ([72])**.** Let $\mathbb{M}$ be a weight sequence and $\gamma>0$. The following statements are equivalent:

(i) $\tilde{\mathcal{B}}:\tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ is injective,

(ii) $\sum_{p=0}^{\infty}(m_p)^{-1/\gamma}=\infty$,

(iii) Either $\gamma>\omega(\mathbb{M})$, or $\gamma=\omega(\mathbb{M})$ and $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}=\infty$.

Quasianalyticity for the classes of functions with uniformly bounded derivatives in an unbounded sector was characterized in 1955 using Theorem 3.2.1 by Rodríguez-Salinas [87], although it is frequently attributed to B. I. Korenbljum [54].

Theorem 3.2.3 ([87], Th. 12)**.** Let $\mathbb{M}$ be a weight sequence and $\gamma>0$ be given. The following statements are equivalent:

(i) The class $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is quasianalytic,

(ii) $\displaystyle\sum_{p=0}^{\infty}\Big(\frac{1}{(p+1)m_p}\Big)^{1/(\gamma+1)}$ diverges.

Similarly, by Proposition 2.1.26, we deduce that Theorem 3.2.3 may be stated as follows.

Theorem 3.2.4. Let $\mathbb{M}$ be a weight sequence and $\gamma > 0$ be given. The following statements are equivalent:

(i) The class $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is quasianalytic,

(ii) $\sum_{p=0}^{\infty}((p+1)m_p)^{-1/(\gamma+1)} = \infty$,

(ii) $\gamma > \omega(\mathbb{M})$, or $\gamma = \omega(\mathbb{M})$ and $\displaystyle\sum_{p=0}^{\infty}\Big((p+1)m_p\Big)^{-1/(\omega(\mathbb{M})+1)}$ diverges.

Finally, regarding $\tilde{\mathcal{A}}_{\mathbb{M}}$ a partial version of Watson's Lemma can be easily obtained as a consequence of Theorem 3.2.2.

Theorem 3.2.5. Let $\mathbb{M}$ be a weight sequence, $\gamma > 0$ be given and G_γ a sectorial region. The following statements hold:

(i) If $\gamma > \omega(\mathbb{M})$, then $\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ is quasianalytic.

(ii) If $\gamma < \omega(\mathbb{M})$, then $\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ is nonquasianalytic.

Proof. (i) Assume that $\gamma > \omega(\mathbb{M})$. We take $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ with $f \sim_{\mathbb{M}} \hat{0}$, and consider a proper bounded sector $T \ll G$, where $T = S(0,\beta,r)$ with $\gamma > \beta > \omega(\mathbb{M})$. By the definition of $\mathbb{M}-$asymptotic expansion, there exist $C_T, A_T > 0$ such that

$$|f(z)| \le C_T A_T^p M_p |z|^p, \qquad z \in T, \quad p \in \mathbb{N}_0.$$

We consider the transformation $z(w) = 1/(w + (1/r)^{1/\beta})^\beta$, which maps the right half-plane $H = H(0)$ into a region $D \subseteq T$, and the holomorphic function $g : H \to \mathbb{C}$ defined by $g(w) := f(z(w))$. As for every $w \in H$ we have $|w + (1/r)^{1/\beta}| > |w|$, we deduce that

$$|g(w)| = |f(z(w))| \le \frac{C_T A_T^p M_p}{|(w+(1/r)^{1/\beta})^\beta|^p} \le \frac{C_T A_T^p M_p}{|w|^{\beta p}}, \qquad w \in H, \quad p \in \mathbb{N}_0.$$

Hence g verifies the bounds in Theorem 3.2.1.(ii). Since $\beta > \omega(\mathbb{M})$, by the properties of the growth index

$$\sum_{p=0}^{\infty}\left(\frac{1}{m_p}\right)^{1/\beta} = \infty.$$

Consequently, by Theorem 3.2.1 we obtain that $g \equiv 0$ and by the identity principle $f \equiv 0$, so $\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ is quasianalytic.
(ii) Assume that $\gamma < \omega(\mathbb{M})$, what implies, by the properties of the growth index, that

$$\sum_{p=0}^{\infty}\left(\frac{1}{m_p}\right)^{1/\gamma} < \infty.$$

Then, by Theorem 3.2.1, there exist $f \in \mathcal{H}(H)$, not identically zero and constants $A, C > 0$ with

$$|f(z)| \le \frac{CA^p M_p}{|z|^{\gamma p}}, \qquad z \in H, \quad p \in \mathbb{N}_0.$$

The function $w \to w^{-1/\gamma}$ maps the sector S_γ into H. We consider the function $g(w) = f(w^{-1/\gamma})$, holomorphic in S_γ, and we have that

$$|g(w)| = |f(w^{-1/\gamma})| \leq CA^p M_p |w|^p, \qquad w \in S_\gamma, \quad p \in \mathbb{N}_0.$$

Consequently, $g \not\equiv 0$ and $g \sim_{\mathbb{M}} \hat{0}$ in S_γ. We observe that the restriction of g to $G_\gamma \subseteq S_\gamma$ is a nontrivial flat function, and we deduce that $\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ is nonquasianalytic. □

Remark 3.2.6. From Theorem 3.2.2 or 3.2.4, we deduce that if $\mathbb{M}$ and $\mathbb{L}$ are weight sequences with $\mathbb{M} \approx \mathbb{L}$ then $\omega(\mathbb{M}) = \omega(\mathbb{L})$, what we have already proved directly (see Remark 2.1.32).

Remark 3.2.7. For any weight sequence $\mathbb{M}$, the information from the previous results can be summarized as follows:

(i) If $\omega(\mathbb{M}) = \infty$, by Theorem 3.2.4, we see that $I_{\mathbb{M}} = \emptyset$ and (3.2) implies $I_{\mathbb{M}} = \tilde{I}^u_{\mathbb{M}} = \tilde{I}_{\mathbb{M}} = \emptyset$.

(ii) If $\omega(\mathbb{M}) = 0$, by Theorem 3.2.5 we observe that $\tilde{I}_{\mathbb{M}} = (0, \infty)$ and, by (3.2), we have that $I_{\mathbb{M}} = \tilde{I}^u_{\mathbb{M}} = \tilde{I}_{\mathbb{M}} = (0, \infty)$.

(iii) If $\omega(\mathbb{M}) \in (0, \infty)$, we have the situation described in Table 3.1, where $\sum_{p=0}^\infty \sigma_p$ denotes the series $\sum_{p=0}^\infty ((p+1)m_p)^{-1/(\omega(\mathbb{M})+1)}$ and $\sum_{p=0}^\infty (m_p)^{-1/\omega(\mathbb{M})}$ is abbreviated to $\sum_{p=0}^\infty \mu_p$ (note that $\sum_{p=0}^\infty \sigma_p < \infty$ implies $\sum_{p=0}^\infty \mu_p < \infty$ by applying Theorems 3.2.2 and 3.2.4 and using that $\mathcal{A}_{\mathbb{M}}(S_\gamma) \subseteq \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$).

	$\sum_{p=0}^\infty \sigma_p = \infty$ $\sum_{p=0}^\infty \mu_p = \infty$	$\sum_{p=0}^\infty \sigma_p = \infty$ $\sum_{p=0}^\infty \mu_p < \infty$	$\sum_{p=0}^\infty \sigma_p < \infty$ $\sum_{p=0}^\infty \mu_p < \infty$
$I_{\mathbb{M}}$	$[\omega(\mathbb{M}), \infty)$	$[\omega(\mathbb{M}), \infty)$	$(\omega(\mathbb{M}), \infty)$
$\tilde{I}^u_{\mathbb{M}}$	$[\omega(\mathbb{M}), \infty)$	$(\omega(\mathbb{M}), \infty)$	$(\omega(\mathbb{M}), \infty)$
$\tilde{I}_{\mathbb{M}}$	$(\omega(\mathbb{M}), \infty)$ or $[\omega(\mathbb{M}), \infty)$?	$(\omega(\mathbb{M}), \infty)$	$(\omega(\mathbb{M}), \infty)$

Table 3.1: Injectivity intervals for a weight sequence with $\omega(\mathbb{M}) \in (0, \infty)$.

In conclusion, we see that the only injectivity interval not determined by the previous results is $\tilde{I}_{\mathbb{M}}$, and only when $\omega(\mathbb{M}) \in (0, \infty)$ and $\sum_{p=0}^\infty (m_p)^{-1/\omega(\mathbb{M})} = \infty$. Indeed, it only rests to decide whether $\omega(\mathbb{M}) \in \tilde{I}_{\mathbb{M}}$ or not. In the next subsection, we will show the existence of nontrivial flat functions in the class $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$, and so one always has $\omega(\mathbb{M}) \notin \tilde{I}_{\mathbb{M}}$ and $\tilde{I}_{\mathbb{M}} = (\omega(\mathbb{M}), \infty)$.

Example 3.2.8. We consider the sequence $\mathbb{M}_{\alpha,\beta} = \big(p!^\alpha \prod_{m=0}^p \log^\beta(e+m)\big)_{p\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$, we have that $\omega(\mathbb{M}_{\alpha,\beta}) = \alpha$ (See Example 2.1.20). Hence, Table 3.2 contains all the information about the injectivity intervals deduced from the classical results for the sequences $\mathbb{M}_{\alpha,\beta}$.

Please note that even if the Gevrey case $\mathbb{M}_\alpha = \big(p!^\alpha\big)_{p\in\mathbb{N}_0}$ belongs to the first column of Table 3.2, all the information is known because the function $f(z) := \exp(-1/z^{1/\alpha}) \sim_{\mathbb{M}_\alpha} \hat{0}$ and $f \in \tilde{\mathcal{A}}_{\mathbb{M}_\alpha}(G_\alpha)$, so $\tilde{I}_{\mathbb{M}_\alpha} = (\alpha, \infty)$. As mentioned before, we will find such functions for any sequence $\mathbb{M}$ using proximate orders.

Watson's Lemma is proved below for the class $\tilde{\mathcal{A}}_{\mathbb{M}}$ for arbitrary sectorial regions, regarding the other two classes the following information is available.

	$\beta \leq \alpha$	$\alpha < \beta \leq \alpha + 1$	$\beta > \alpha + 1$
$I_{\mathbb{M}_{\alpha,\beta}}$	$[\alpha, \infty)$	$[\alpha, \infty)$	(α, ∞)
$\tilde{I}^u_{\mathbb{M}_{\alpha,\beta}}$	$[\alpha, \infty)$	(α, ∞)	(α, ∞)
$\tilde{I}_{\mathbb{M}_{\alpha,\beta}}$	(α, ∞) or $[\alpha, \infty)$?	(α, ∞)	(α, ∞)

Table 3.2: Injectivity intervals for the sequence $\mathbb{M}_{\alpha,\beta}$ with $\alpha > 0$, $\beta \in \mathbb{R}$.

Remark 3.2.9. Theorem 3.2.2 holds true for bounded sectors $S(0, \gamma, r)$ with similar arguments. If $\sum_{p=0}^{\infty} (m_p)^{-1/\gamma} < \infty$ the restriction to $S(0, \gamma, r)$ of the nontrivial flat function defined in S_γ given by Theorem 3.2.2 solves the problem. Hence, we only need to modify the proof of (ii)⇒(i). We proceed as in Theorem 3.2.5, by considering the transformation $z(w) = 1/(w + (1/r)^{1/\gamma})^\gamma$, which maps H into a region D contained in $S(0, \gamma, r)$: given a flat function $g \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S(0, \gamma, r))$, the function $f(w) := g(z(w))$ is defined in H and, by Mandelbrojt's theorem, it identically vanishes.

For more general regions, including sectorial regions, the solution was also given by Mandelbrojt [72, Sect. 2.4.I] and the answer depends on the way the boundary of the region approaches the origin.

Remark 3.2.10. The problem of quasianalyticity for classes of functions with uniformly bounded derivatives in bounded regions has also been treated. In the works of K. V. Trunov and R. S. Yulmukhametov [101, 108] a characterization is given, for a convex bounded region containing 0 in its boundary, in terms of the sequence $\mathbb{M}$ and also of the way the boundary approaches 0. In particular, for bounded sectors, if $\gamma \leq 1$, $d \in \mathbb{R}$ and $r > 0$, it turns out that the class $\mathcal{A}_{\mathbb{M}}(S(d, \gamma, r))$ is quasianalytic precisely when condition (ii) in Theorem 3.2.4 is satisfied.

3.2.2 New injectivity results

In [88], J. Sanz shows how one can construct nontrivial flat functions in the class $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ for strongly regular sequences $\mathbb{M}$ (see Definition 1.1.1) such that $d_{\mathbb{M}}(t)$ is a proximate order (see (2.15)) using a function $V \in MF(\gamma, d_{\mathbb{M}}(t))$ (see Definition 1.2.17). Under these conditions for $\mathbb{M}$, by Theorem 2.1.30 and Remark 2.1.19, we know that $d_{\mathbb{M}}(t)$ is a nonzero proximate order. Moreover, by Remark 2.2.7, we know that one may equivalently assume that $\mathbb{M}$ is a weight sequence and $d_{\mathbb{M}}(t)$ is a nonzero proximate order

Theorem 3.2.11 (Watson's Lemma, Coro. 4.12 [88])**.** Suppose $\mathbb{M}$ is a strongly regular such that $d_{\mathbb{M}}(t)$ is a nonzero proximate order, and let $\gamma > 0$ be given. The following statements are equivalent:

(i) $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ is quasianalytic, i.e., it does not contain nontrivial flat functions (in other words, the Borel map is injective in this class).

(ii) $\gamma > \omega(\mathbb{M})$.

At it was pointed out in [88, Remark 4.11], it is enough to ask that $\mathbb{M}$ admits a nonzero proximate order and, by Remark 2.2.18, it suffices to verify that $\mathbb{M}$, originally strongly regular, is a weight sequence admitting a nonzero proximate order.

A better understanding of the connection between proximate orders and sequences has been achieved (see Section 2.2) allowing us to extend this last result for arbitrary weight sequence. In fact, the admissibility of a proximate order $\rho(t)$ guarantees that the associated function $\omega_{\mathbb{M}}$

is bounded above and below by a constant times the function $t^{\rho(t)}$ (see Definition 2.2.1). These bounds are needed for most of the results in [60, 88], but by suitably using the notion of regular variation we will see that the upper bounds are enough for the construction of flat functions. The existence of a proximate order such that we have the upper bounds is guaranteed for each nonnegative, nondecreasing continuous function of finite upper order (see Definition 1.2.30) by the following classical result.

Theorem 3.2.12 ([32], Ch. 2, Th. 2.1)**.** Let $\omega : (a,\infty) \to (0,\infty)$ be a nonnegative, nondecreasing continuous function with $\rho(\omega) = \limsup_{t\to\infty} \log(\omega(t))/\log(t) < \infty$. Then, it exists a proximate order $\rho(t)$ with $\lim_{t\to\infty} \rho(t) = \rho(\omega)$ such that

$$\limsup_{t\to\infty} \frac{\omega(t)}{t^{\rho(t)}} \in (0,\infty). \tag{3.5}$$

It was a particular version of this last result which motivates the introduction of the notion of proximate order for the study of the growth of entire functions. In that version, $\omega(t) = \max(0, \log(\max_{|z|=t} |f(z)|))$ and the converse is also available, that is, given a proximate order it is possible to construct an entire function such that (3.5) is valid for the previous choice of ω (see [63, Th. 25, Th. 26]).

The next property, easily deduced from the regular variation of the functions in the class $MF(\gamma, d_{\mathbb{M}}(t))$, will be employed.

Proposition 3.2.13 ([65], Property 2.9)**.** Let $\rho(t)$ be a proximate order with $\lim_{t\to\infty} \rho(t) = \rho > 0$, $\gamma \geq 2/\rho$ and $V \in MF(\gamma, \rho(t))$. Then, for every $\alpha \in (0, 1/\rho)$ there exist constants $b > 0$ and $R_0 > 0$ such that

$$\mathrm{Re}(V(z)) \geq bV(|z|), \quad z \in S_\alpha,\ |z| \geq R_0,$$

where Re stands for the real part.

We have all the ingredients for the main result in this section.

Theorem 3.2.14. Suppose $\mathbb{M}$ is a weight sequence with $\omega(\mathbb{M}) \in (0,\infty)$. Then, $\omega(\mathbb{M})$ does not belong to $\tilde{I}_{\mathbb{M}}$.

Proof. For brevity, put $\omega := \omega(\mathbb{M})$. By Theorem 2.1.30, the associated function $\omega_{\mathbb{M}}(t)$ is of finite order $\rho := \rho(\omega_{\mathbb{M}}) = 1/\omega > 0$, and by Theorem 3.2.12, there exist a nonzero proximate order $\rho(t)$ with $\lim_{t\to\infty} \rho(t) = \rho = 1/\omega > 0$, $A_1 > 0$ and $t_1 > 0$ such that

$$\omega_{\mathbb{M}}(t) \leq A_1 t^{\rho(t)}, \quad t \geq t_1. \tag{3.6}$$

Take now a function $V \in MF(2\omega, \rho(t))$. The proof will be complete if we show that $G(z) := \exp(-V(1/z))$, which is well defined and holomorphic in the sector S_ω, belongs to $\tilde{\mathcal{A}}_{\mathbb{M}}(S_\omega)$ and it is flat, for what we will apply Proposition 3.1.9. It is enough to work in subsectors $S(0,\beta,r_0) \ll S_\omega$, where $0 < \beta < \omega$ and $r_0 > 0$. If $z \in S(0,\beta,r_0)$, we have $1/z \in S_\beta$. On the one hand, according to (VI) in Theorem 1.2.16, combined with (3.6), there exist $A_2 > 0$ and $t_2 > 0$ such that

$$\omega_{\mathbb{M}}(t) \leq A_2 V(t), \quad t \geq t_2. \tag{3.7}$$

On the other hand, Proposition 3.2.13 provides us with constants $b > 0$ and $R_0 > 0$ such that

$$\mathrm{Re}(V(\zeta)) \geq bV(|\zeta|), \qquad \zeta \in S_\beta,\ |\zeta| \geq R_0. \tag{3.8}$$

Choose a positive constant c such that $c > (A_2/b)^\omega$. By property (I) in Theorem 1.2.16, we have

$$\lim_{t\to\infty} \frac{V(t/c)}{V(t)} = \left(\frac{1}{c}\right)^{1/\omega} < \frac{b}{A_2},$$

so there exists $R_1 > 0$ such that

$$bV(t) > A_2V(t/c), \quad t \geq R_1. \tag{3.9}$$

Let $R_2 := \max(R_0, R_1, ct_2)$ and $r := R_2^{-1}$. Then, using (3.8), (3.9) and (3.7), for $z \in S(0,\beta,r)$ we have

$$-\operatorname{Re}(V(1/z)) \leq -bV(1/|z|) < -A_2V(1/(c|z|)) \leq -\omega_{\mathbb{M}}(1/(c|z|)),$$

and so

$$|G(z)| = e^{-\operatorname{Re}(V(1/z))} \leq e^{-\omega_{\mathbb{M}}(1/(c|z|))}.$$

We are done whenever $r \geq r_0$. If $r \leq r_0$, by compactness there exists $K > 0$ such that the inequality

$$|G(z)| \leq Ke^{-\omega_{\mathbb{M}}(1/(c|z|))}$$

is valid throughout $S(0,\beta,r_0)$. □

Combining this with the partial version of Watson's Lemma, Theorem 3.2.5, obtained before, we have the following final statement in this respect.

Theorem 3.2.15 (Watson's Lemma)**.** Let $\mathbb{M}$ be a weight sequence, $\gamma > 0$ and G_γ a sectorial region. The following statements are equivalent:

(i) $\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ is quasianalytic.

(ii) $\gamma > \omega(\mathbb{M})$.

Hence, the question mark in Table 3.1 can be deleted and the answer for that cell is $\tilde{I}_{\mathbb{M}} = (\omega(\mathbb{M}),\infty)$, what completes the study of injectivity for unbounded sectors.

3.2.3 Impossibility of bijectivity

Our first surjectivity result is a consequence of the injectivity Theorems 3.2.2, 3.2.4 and 3.2.5. It answers in the negative whether bijectivity is possible.

Theorem 3.2.16. Let $\mathbb{M}$ be a weight sequence. Then,

$$S_{\mathbb{M}} \cap I_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} \cap \tilde{I}^u_{\mathbb{M}} = \tilde{S}_{\mathbb{M}} \cap \tilde{I}_{\mathbb{M}} = \emptyset.$$

In other words, the Borel map is never bijective.

Proof. In all three cases we will show that surjectivity for any $\gamma > 0$ implies noninjectivity.

(i) Let us see that $\tilde{S}_{\mathbb{M}} \cap \tilde{I}_{\mathbb{M}} = \emptyset$. Suppose $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective. Since the geometric series $\sum_{n=0}^\infty z^n \in \mathbb{C}[[z]]_{\mathbb{M}}$, then there exists $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ such that $f(z) \sim_{\mathbb{M}} \sum_{n=0}^\infty z^n$. The function $g(z) := f(z) - \sum_{n=0}^\infty z^n = f(z) - 1/(1-z)$ is holomorphic in $S_\gamma \setminus \{1\}$ and, by the identity principle, cannot vanish identically. Moreover, $g \in \tilde{\mathcal{A}}_{\mathbb{M}}(S(0,\gamma,1/2))$ and $g(z) \sim_{\mathbb{M}} \hat{0}$, and so the Borel map is not injective in $\tilde{\mathcal{A}}_{\mathbb{M}}(S(0,\gamma,1/2))$ and, by Theorem 3.2.15, $\gamma \leq \omega(\mathbb{M})$. Again by Theorem 3.2.15, we conclude that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is not injective.

(ii) Let us see that $\tilde{S}^u_{\mathbb{M}} \cap \tilde{I}^u_{\mathbb{M}} = \emptyset$. Suppose $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective. Since $z \in \mathbb{C}[[z]]_{\mathbb{M}}$, there exists $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ such that $f(z) \sim_{\mathbb{M}} z$ uniformly in S_γ. The function

$g(z) := f(z) - z$ is holomorphic in S_γ and, since f is bounded in S_γ and z is not, g cannot vanish identically. Furthermore, $g(z) \sim_{\mathbb{M}} \hat{0}$ uniformly in $S(0,\gamma,1)$, so there exist $C, A > 0$ such that for every $z \in S(0,\gamma,1)$ one has

$$|g(z)| \leq CA^p M_p |z|^p, \qquad p \in \mathbb{N}_0.$$

Hence, the holomorphic function $\psi : \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \to \mathbb{C}$, defined by $\psi(u) = g(1/u^\gamma)$, is not identically 0 and

$$|\psi(u)| \leq \frac{CA^p M_p}{|u|^{\gamma p}}, \qquad p \in \mathbb{N}_0,\ \mathrm{Re}(u) > 1.$$

Now, we can apply Theorem 3.2.1 in $H(1)$ and we deduce that $\sum_{p=0}^{\infty} m_p^{-1/\gamma} < \infty$. By Theorem 3.2.2, we conclude that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is not injective.

(iii) Let us show that $S_{\mathbb{M}} \cap I_{\mathbb{M}} = \emptyset$. Finally, if $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective there exists $f \in \mathcal{A}_{\mathbb{M}}(S_\gamma)$ such that $f^{(p)}(0) = \delta_{1,p}$ (see (3.1)) for every $p \in \mathbb{N}_0$, where $\delta_{1,p}$ is Kronecker's delta. By definition of the class, there exist $C, A > 0$ (without loss of generality, we may assume that $C \geq 1$ and $CAM_1 \geq 1$) such that

$$|f^{(p)}(z)| \leq CA^p p! M_p, \qquad z \in S_\gamma,\ p \in \mathbb{N}_0. \tag{3.10}$$

We consider the Laplace transform of the function $f(z) - z$,

$$g(z) := \int_0^{\infty(\varphi)} e^{-zt}(f(t) - t)\, dt, \qquad z \in S_{\gamma+1}, \tag{3.11}$$

where the integration is over the half-line parametrized by $r \in (0,\infty) \mapsto re^{i\varphi}$, whose argument is a real number

$$\varphi \in \left(-\frac{\pi\gamma}{2}, \frac{\pi\gamma}{2}\right) \quad \text{such that} \quad \arg(z) + \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.12}$$

This last condition guarantees the exponential decrease at infinity of the factor e^{-zt} which, together with the linear growth of $f(t) - t$, ascertains that the function g is well defined and holomorphic in $S_{\gamma+1}$. We proceed now to estimate $|g(z)|$. Firstly, parametrizing we have that

$$\begin{aligned} |g(z)| &\leq \left| \int_0^\infty e^{-re^{i\varphi}z} f(re^{i\varphi}) e^{i\varphi}\, dr - \int_0^\infty e^{-re^{i\varphi}z} re^{i\varphi} e^{i\varphi}\, dr \right| \\ &\leq \int_0^\infty e^{-r\,\mathrm{Re}(e^{i\varphi}z)} |f(re^{i\varphi})|\, dr + \left| \int_0^\infty e^{-re^{i\varphi}z} r\, dr \right|. \end{aligned}$$

In the first integral we use (3.10) for $p = 0$ and compute the remaining integral, and in the second one we integrate by parts, and get that

$$|g(z)| \leq \frac{C}{\mathrm{Re}(e^{i\varphi}z)} + \left| \frac{1}{e^{i\varphi}z} \int_0^\infty e^{-re^{i\varphi}z}\, dr \right| \leq \frac{C}{\mathrm{Re}(e^{i\varphi}z)} + \frac{1}{|z|\,\mathrm{Re}(e^{i\varphi}z)}. \tag{3.13}$$

for every $z \in S_{\gamma+1}$. A different estimation is obtained by integration by parts in (3.11), taking into account that $f(0) = 0$:

$$g(z) = \frac{1}{z} \int_0^{\infty(\varphi)} e^{-zt}(f'(t) - 1)\, dt, \qquad z \in S_{\gamma+1}. \tag{3.14}$$

Now we parametrize and split the integral as before, and use (3.10) for $p = 1$ to obtain that

$$|g(z)| \leq \frac{CAM_1}{|z|\,\mathrm{Re}(e^{i\varphi}z)} + \frac{1}{|z|\,\mathrm{Re}(e^{i\varphi}z)} \leq \frac{2CAM_1}{|z|\,\mathrm{Re}(e^{i\varphi}z)}. \tag{3.15}$$

Finally, if we iterate the integration by parts in (3.14) and use that $f^{(p)}(0) = \delta_{1,p}$, we get for every $p \geq 2$ the identity

$$g(z) = \frac{1}{z^p} \int_0^{\infty(\varphi)} e^{-zt} f^{(p)}(t)\, dt, \qquad z \in S_{\gamma+1}.$$

Using again (3.10) for $p \geq 2$, we deduce that

$$|g(z)| \leq \frac{CA^p p! M_p}{|z|^p \operatorname{Re}(e^{i\varphi} z)}. \tag{3.16}$$

Our aim is to apply Theorem 3.2.1 to the function h given by $h(w) = g(w^{\gamma+1})$, $w \in S_1$, when restricted to the half-plane $\{w : \operatorname{Re}(w) > 1\}$. Note that the estimates in (3.13) imply for $\operatorname{Re}(w) > 1$ (and so $|w| > 1$) that

$$|h(w)| \leq \frac{C}{\operatorname{Re}(e^{i\varphi} w^{\gamma+1})} + \frac{1}{|w^{\gamma+1}| \operatorname{Re}(e^{i\varphi} w^{\gamma+1})} \leq \frac{2C}{\operatorname{Re}(e^{i\varphi} w^{\gamma+1})}.$$

These last estimates and the ones in (3.15) and (3.16) can now be summed up for h as

$$|h(w)| \leq \frac{2CA^p p! M_p}{|w|^{p(\gamma+1)} \operatorname{Re}(e^{i\varphi} w^{\gamma+1})}, \quad \operatorname{Re}(w) > 1,\ p \in \mathbb{N}_0.$$

Now we choose φ in order to minimize the value $\operatorname{Re}(e^{i\varphi} w^{\gamma+1})$. We study two cases:

(1) If $|\arg(w)| < \gamma\pi/(2(\gamma+1))$, then $|\arg(w^{\gamma+1})| < \gamma\pi/2$ and, according to (3.12), we may choose $\varphi = -\arg(w^{\gamma+1})$, and we deduce that $\operatorname{Re}(e^{i\varphi} w^{\gamma+1}) = |w|^{\gamma+1} > 1$. So, for such w we get

$$|h(w)| \leq \frac{2CA^p p! M_p}{|w|^{p(\gamma+1)}}, \quad p \in \mathbb{N}_0. \tag{3.17}$$

(2) If $|\arg(w)| \in [\gamma\pi/(2(\gamma+1)), \pi/2)$, the previous choice is not possible, and we choose

$$\varphi_\varepsilon = \begin{cases} -\frac{\gamma\pi}{2} + \varepsilon & \text{if } \arg(w) \in \left(-\frac{\pi}{2}, -\frac{\pi\gamma}{2(\gamma+1)}\right], \\ \frac{\gamma\pi}{2} - \varepsilon & \text{if } \arg(w) \in \left[\frac{\pi\gamma}{2(\gamma+1)}, \frac{\pi}{2}\right), \end{cases}$$

for any $\varepsilon \in (0, \gamma\pi/2)$. Hence, $\operatorname{Re}(e^{i\varphi_\varepsilon} w^{\gamma+1}) = |w|^{\gamma+1} \cos((\gamma+1)|\arg(w)| - \gamma\pi/2 + \varepsilon)$, and making ε tend to 0 we obtain that

$$|h(w)| \leq \frac{2CA^p p! M_p}{|w|^{p(\gamma+1)} |w|^{\gamma+1} \cos((\gamma+1)|\arg(w)| - \gamma\pi/2)}, \quad p \in \mathbb{N}_0. \tag{3.18}$$

We observe that in this case

$$0 < \frac{\pi}{2} - |\arg(w)| \leq (\gamma+1)(\frac{\pi}{2} - |\arg(w)|) \leq \frac{\pi}{2},$$

and so

$$\begin{aligned} |w| \cos\left((\gamma+1)|\arg(w)| - \frac{\gamma\pi}{2}\right) &= |w| \sin\left((\gamma+1)\left(\frac{\pi}{2} - |\arg(w)|\right)\right) \\ &\geq |w| \sin\left(\frac{\pi}{2} - |\arg(w)|\right) = |w| \cos(\arg(w)) = \operatorname{Re}(w) > 1. \end{aligned}$$

Since we also have $|w|^\gamma > 1$, from (3.18) we obtain the same estimates (3.17) given in the first case.

Since h is not identically 0, by Theorem 3.2.1 we deduce that the series $\sum_{p=0}^\infty ((p+1)m_p)^{-1/(\gamma+1)}$ converges, and Theorem 3.2.4 implies that $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is not injective. □

Remark 3.2.17. As an easy consequence we have that if $\omega(\mathbb{M}) < \infty$, then

$$S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}} \subseteq (0, \omega(\mathbb{M})].$$

3.3 Surjectivity of the asymptotic Borel map

In 1895, É. Borel showed that $\tilde{\mathcal{B}} : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathbb{C}[[z]]$ sending a smooth function f in $\mathbb{R}$ into the formal power series $\hat{f} = \sum_{p=0}^{\infty}(f^p(0)/p!)z^p$ is surjective and in 1916, J. F. Ritt extended this result for $\tilde{\mathcal{A}}^u(S)$, the class of analytic functions in a sector S with uniform asymptotic expansion at the origin, i.e., $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u(S) \longrightarrow \mathbb{C}[[z]]$ is also surjective. Several proofs of these results are known, in [71, Th. 1.1.4.1, Coro. 1.1.4.2] the reader can find a simple one given by B. Malgrange. In between their generalizations, it is worthy to mention that for an arbitrary closed set $F \subseteq \mathbb{R}^N$ H. Whitney showed that we can construct a function $f \in \mathcal{C}^\infty(\mathbb{R}^N)$, real analytic in $\mathbb{R}^N \setminus F$, such that its value at F is determined by a given jet (see [106, 107]).

As it was pointed out before, if $f \sim_{\mathbb{M}} \hat{f}$, then $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{M}}$ and it is quite natural to restrict ourselves to study the surjectivity for the case the coefficients of the series have a prescribed growth in terms of $\mathbb{M}$. Compared to injectivity, very little is known about this situation for the classes $\mathcal{A}_{\mathbb{M}}(S)$, $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ and $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$. The first result in this direction is the Borel-Ritt-Gevrey theorem where J. P. Ramis, using the technique of the truncated Laplace transform for the Gevrey sequence $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, proved the following:

Theorem 3.3.1 ([81, 82])**.** Let $\alpha > 0$. Then $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}_\alpha}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}_\alpha}$ is surjective if and only if $\gamma \leq \alpha$.

In 1995, V. Thilliez [93, (1.3)] gave a linear and continuous extension from $\mathbb{C}[[z]]_{\mathbb{M}_\alpha,A}$ to $\mathcal{A}_{\mathbb{M}_\alpha,dA}(S_\gamma)$ for $\gamma < \alpha$ for every $A > 0$, where $d > 0$ depends only on α and γ, employing results of continuous extension from the ultradifferentiable setting. The first results for arbitrary weight sequences satisfying (dc) (see Definition 1.1.5) were given by J. Schmets and M. Valdivia in [91], their approach is based on the consideration of some nonclassical ultradifferentiable classes, $\mathcal{E}_{r,\mathbb{M}}$, $\mathcal{N}_{r,\mathbb{M}}$ and $\mathcal{L}_{r,\mathbb{M}}$ defined below for $r \in \mathbb{N}$, in which the interpolation is done only for a subsequence $(f^{(pr)}(0))_{p\in\mathbb{N}_0}$ of their derivatives at 0. They obtain results for the Roumieu and the Beurling case, this last one will be not be considered below. In that paper, although surjectivity is studied, the main focus is on the existence of linear and continuous global extension between the corresponding (LB)-spaces, which is much more demanding, and their main theorem, in the Roumieu case, is only for sequences with $\gamma(\mathbb{M}) = \infty$. In 2003, V. Thilliez proved the following:

Theorem 3.3.2 ([95], Theorem 3.2.1)**.** Let $\mathbb{M}$ be a strongly regular sequence and $0 < \gamma < \gamma(\mathbb{M})$. Then there exists $d \geq 1$ such that for every $A > 0$ there is a linear continuous operator

$$T_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},dA}(S_\gamma)$$

such that $\tilde{\mathcal{B}} \circ T_{\mathbb{M},A,\gamma} = Id_{\mathbb{C}[[z]]_{\mathbb{M},A}}$, the identity map in $\mathbb{C}[[z]]_{\mathbb{M},A}$. Hence, $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective.

This theorem was reproved by A. Lastra, S. Malek, J. Sanz [58] using the technique of the truncated Laplace transform for a suitable kernel. Finally, in [88, Theorem 6.1] J. Sanz generalized the Borel–Ritt–Gevrey theorem for strongly regular sequences such that the function $d_{\mathbb{M}}$ (see (2.15)) is a proximate order (see also Theorem 3.3.21). As it has been shown in the preceding chapter, although this condition is satisfied for most of the sequences appearing in the applications, it might be too restrictive (see Remark 2.2.18). These works are the departing point of this subsection whose main objective, partially but satisfactorily accomplished in the strongly regular case (see Table 3.4), is providing necessary and sufficient conditions for the surjectivity of the Borel map.

In the previous section, it has been shown that the appropriate value for characterizing the injectivity of the Borel map is the index $\omega(\mathbb{M})$ which equals the lower order of the sequence of quotients $\mathbf{m}$. We will show that for the surjectivity problem the suitable one is Thilliez's growth index $\gamma(\mathbb{M})$ which has been proved to coincide with the lower Matuszewska index of $\mathbf{m}$, $\beta(\mathbf{m})$ (see Theorem 2.1.16). Please note that we always have $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$ (see Proposition 1.1.28). For a strongly regular sequence such that the function $d_{\mathbb{M}}$ is a proximate order we have that $\gamma(\mathbb{M}) = \omega(\mathbb{M})$ (see Remark 2.2.7), then this difference was hidden. In Subsection 2.2.5, we have constructed strongly regular sequences for which $\gamma(\mathbb{M}) < \omega(\mathbb{M})$ (see Remark 3.3.20 below for the implications of this fact concerning $\tilde{\mathcal{B}}$).

We recall some convenient properties obtained in Section 2.1 for $\gamma(\mathbb{M})$. For any sequence $\mathbb{M}$ and every $s > 0$ one has

$$\gamma((p!^s M_p)_{p\in\mathbb{N}_0}) = \gamma(\mathbb{M}) + s, \qquad \gamma((M_p^s)_{p\in\mathbb{N}_0}) = s\gamma(\mathbb{M}).$$

(see Proposition 2.1.11). We also recall some of the information given in Proposition 2.1.22 and Remark 2.1.23, for any $\beta > 0$ we say that $\mathbf{m}$ satisfies (γ_β) if there exists $A > 0$ such that

$$(\gamma_\beta) \qquad \sum_{\ell=p}^{\infty} \frac{1}{(m_\ell)^{1/\beta}} \leq \frac{A(p+1)}{(m_p)^{1/\beta}}, \qquad p \in \mathbb{N}_0.$$

If $\mathbb{M}$ is (lc), $\gamma(\mathbb{M}) \geq 0$ and using this condition we can state an alternative definition of the index:

$$\gamma(\mathbb{M}) = \sup\{\beta > 0;\ \mathbf{m} \text{ satisfies } (\gamma_\beta)\}.$$

In addition, the next equivalences will be used several times if: $\widehat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ is (lc) and $\beta > 0$ we have that

(i) $\gamma(\mathbb{M}) > 0$ if and only if $\mathbb{M}$ is (snq) (see Proposition 2.1.18).

(ii) $\gamma(\widehat{\mathbb{M}}) > 1$ if and only if $\widehat{\mathbf{m}}$ satisfies (γ_1) (see Remark 2.1.23).

(iii) $\gamma(\widehat{\mathbb{M}}) > \beta$ if and only if $\widehat{\mathbf{m}}$ satisfies (γ_β) (using (ii) and Proposition 2.1.11).

We will start with an arbitrary weight sequence $\mathbb{M}$ for which we will obtain some necessary conditions for the surjectivity of the Borel map. Subsequently, imposing (dc) (see Definition 1.1.5) we will get some improvements on that conditions for the classes $\mathcal{A}_{\mathbb{M}}$ and $\tilde{\mathcal{A}}^u_{\mathbb{M}}$ (see Table 3.3). For strongly regular sequences, after applying the sufficient condition provided by Thilliez and some ramification arguments, we will prove that the surjectivity intervals are either $(0, \gamma(\mathbb{M}))$ or $(0, \gamma(\mathbb{M})]$ (see Table 3.4). Finally, in case $\gamma(\mathbb{M}) = \omega(\mathbb{M})$ or if, furthermore, $\mathbb{M}$ admits a nonzero proximate order we will apply the theorems from the previous section to analyze if the value $\gamma(\mathbb{M})$ belongs to these intervals or not (see Table 3.5).

3.3.1 Weight sequences

Our first result is based on a theorem by H.-J. Petzsche in the ultradifferentiable setting and we need to consider the following space.

Definition 3.3.3. We say that $f \in \mathcal{E}_{\mathbb{M}}([-1,1])$ if $f \in \mathcal{C}^\infty([-1,1])$ and there exists a constant $A > 0$ for which

$$\sup_{p\in\mathbb{N}_0,\, x\in[-1,1]} \frac{|f^{(p)}(x)|}{A^p p! M_p} < \infty.$$

Correspondingly, we consider the Borel map $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([-1,1])\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ sending f into the formal power series $\sum_{p=0}^{\infty}(f^{(p)}(0)/p!)z^p$ (we warn the reader our notations differ from those in [77], see Remark 3.1.11).

All over the paper [77], H.-J. Petszche assumes that $\widehat{\mathbb{M}}$ is a weight sequence and that $\mathbb{M}$ satisfies (nq). However, condition (nq) can be suppressed in the statement of the following theorem, since, if $\widehat{\mathbf{m}}=((p+1)m_p)_{p\in\mathbb{N}_0}$ satisfies (γ_1) then $\mathbb{M}$ satisfies (snq) and, consequently, (nq), and there is only one direction that needs to be checked. This can be done by carefully inspecting his proof.

Theorem 3.3.4 ([77], Th. 3.5)**.** Let $\mathbb{M}$ be a sequence such that $\widehat{\mathbb{M}}$ is weight sequence. Then, the Borel map $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([-1,1])\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ is surjective if and only if $\widehat{\mathbf{m}}$ satisfies (γ_1).

We are ready to give the first connection between the growth index $\gamma(\mathbb{M})$ with the surjectivity intervals which holds for arbitrary weight sequences.

Lemma 3.3.5. Let $\mathbb{M}$ be a weight sequence. If $\tilde{S}_{\mathbb{M}}\neq\emptyset$, then $\mathbb{M}$ has (snq) or, equivalently, $\gamma(\mathbb{M})>0$.

Proof. Let $\widehat{f}=\sum_{p=0}^{\infty}a_pz^p\in\mathbb{C}[[z]]_{\mathbb{M}}$. Since there exists $\gamma>0$ such that $\tilde{\mathcal{B}}:\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ is surjective, we may take a function $f_1\in\tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ such that $\tilde{\mathcal{B}}(f_1)=\widehat{f}$. A suitable rotation shows that also $\tilde{\mathcal{B}}:\tilde{\mathcal{A}}_{\mathbb{M}}(S(\pi,\gamma))\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ is surjective and so there exists a function $f_2\in\tilde{\mathcal{A}}_{\mathbb{M}}(S(\pi,\gamma))$ such that $\tilde{\mathcal{B}}(f_2)=\widehat{f}$. It is plain to check (by a recursive application of the Mean Value Theorem) that the function

$$h(x)=f_1(x),\quad x\in(0,1];\qquad h(x)=f_2(x),\quad x\in[-1,0);\qquad h(0)=a_0,$$

belongs to $\mathcal{C}^\infty([-1,1])$ and $h^{(p)}(0)=p!a_p$ for every $p\in\mathbb{N}$ (see Proposition 3.1.5). Moreover, considering suitable subsectors of S_γ (respectively, $S(\pi,\gamma)$) containing $(0,1]$ (resp., $[-1,0)$), and again by a double application of Proposition 3.1.5.(ii), one obtains a constant $A>0$ such that

$$\sup_{p\in\mathbb{N}_0,\,x\in[-1,1]}\frac{|h^{(p)}(x)|}{A^pp!M_p}<\infty.$$

Hence, we deduce that the Borel map $\mathcal{B}:\mathcal{E}_{\mathbb{M}}([-1,1])\longrightarrow\mathbb{C}[[z]]_{\mathbb{M}}$ is also surjective. Since $\mathbb{M}$ is a weight sequence, $\widehat{\mathbb{M}}$ also is, so by Theorem 3.3.4 this surjectivity amounts to the fact that the sequence of quotients of $\widehat{\mathbb{M}}=(p!M_p)_{p\in\mathbb{N}_0}$, namely $\widehat{\mathbf{m}}$, satisfies the condition (γ_1), which is precisely condition (snq) for $\mathbb{M}$ (see Remark 2.1.23). □

No other result concerning the surjectivity of the Borel map is present in the literature without adding some additional condition on the weight sequence $\mathbb{M}$ in this ultraholomorphic setting.

Our next results, Theorem 3.3.10 and Theorem 3.3.14, are inspired by statements of J. Schmets and M. Valdivia [91, Section 4] in the Beurling case. Although we do not treat this case here, some of their proofs can be adapted to, or suitably modified for, our Roumieu-like spaces.

While the aforementioned authors impose condition (dc) on the sequence $\mathbb{M}$, i.e., there exists $A>0$ such that $M_{p+1}\le A^pM_p$ for every $p\in\mathbb{N}_0$, we will show that, in some cases, one can obtain some information without it.

In the course of our arguments we will need to introduce suitable ultradifferentiable classes (the notations again differ from those in [91]):

For a natural number $r\in\mathbb{N}$ and a sequence $\mathbb{M}$, we consider the space $\mathcal{N}_{r,\mathbb{M}}([0,\infty))$ of functions $f\in\mathcal{C}^\infty([0,\infty))$ such that

(a) $f^{(pr+j)}(0)=0$ for every $p\in\mathbb{N}_0$ and $j\in\{1,\dots,r-1\}$ (this condition is empty when $r=1$),

(b) there exists a constant $A>0$ for which

$$\sup_{p\in\mathbb{N}_0,\,x\in[0,\infty)}\frac{|f^{(pr)}(x)|}{A^p p! M_p}<\infty.$$

The subspace of $\mathcal{N}_{r,\mathbb{M}}([0,\infty))$ consisting of those functions with support contained in $[0,1]$ will be denoted by $\mathcal{L}_{r,\mathbb{M}}([0,\infty))$. Similarly, we introduce the space $\mathcal{E}_{r,\mathbb{M}}([0,1])$ of functions $f\in\mathcal{C}^\infty([0,1])$ such that

(a) $f^{(pr+j)}(0)=0$ for every $p\in\mathbb{N}_0$ and $j\in\{1,\dots,r-1\}$ (this condition is empty when $r=1$),

(b) there exists a constant $A>0$ for which

$$\sup_{p\in\mathbb{N}_0,\,x\in[0,1]}\frac{|f^{(pr)}(x)|}{A^p p! M_p}<\infty.$$

Note that these spaces coincide with the classical ones for $r=1$. In this context, it is natural to consider the next auxiliary sequence.

Definition 3.3.6. Given a sequence $\mathbb{M}$ and $r\in\mathbb{N}$, its $r-$*interpolating sequence* $\mathbb{P}_{r,\mathbb{M}}=\mathbb{P}=(P_n)_{n\in\mathbb{N}_0}$ is defined by

$$P_{kr+j}=\left(M_k^{r-j}M_{k+1}^{j}\right)^{1/r},\quad k\in\mathbb{N}_0,\ j\in\{0,\dots,r\}.$$

Note that with $j=r$ for k and $j=0$ for $k+1$ we obtain the same value. As it was pointed out in [91], a simple computation leads to

(i) $\mathbb{P}_{1,\mathbb{M}}=\mathbb{M}$,

(ii) $P_{kr}=M_k$ for every $k\in\mathbb{N}_0$,

(iii) $p_{kr+j}=(m_k)^{1/r}$ for all $k\in\mathbb{N}_0$ and $j\in\{0,\dots,r-1\}$,

(iv) If $\mathbb{M}$ is a weight sequence, then $\mathbb{P}$ also is.

We also deduce the following relation for their injectivity indices.

Lemma 3.3.7. Let $\mathbb{M}$ be a sequence and $r\in\mathbb{N}$. Then

$$\omega(\mathbb{M})=r\omega(\mathbb{P}).$$

Proof. Fix $j\in\{0,\dots,r-1\}$, the lemma is deduce from the next calculation

$$\begin{aligned}\omega(\mathbb{M})&=\liminf_{k\to\infty}\frac{\log m_k}{\log k}=r\liminf_{k\to\infty}\frac{\log(m_k)^{1/r}}{\log k}=r\liminf_{k\to\infty}\frac{\log p_{kr+j}}{\log(kr+j)}\frac{\log(kr+j)}{\log(k)}\\&=r\liminf_{k\to\infty}\frac{\log p_{kr+j}}{\log(kr+j)}.\end{aligned}$$

□

The introduction of this $r-$interpolating sequence is motivated by the following estimates, independently obtained by A. Gorny and H. Cartan (see [72, Sect. 6.4.IV]).

Lemma 3.3.8. If $f \in \mathcal{C}^r([-1,1])$ for some $r \in \mathbb{N}$ and

$$Q_0 := \sup_{x\in[-1,1]} |f(x)|, \quad \text{and} \quad Q_r := \sup_{x\in[-1,1]} |f^{(r)}(x)|,$$

then

$$\sup_{x\in[-1,1]} |f^{(j)}(x)| \le (8er/j)^j \max(Q_0^{1-j/r}, Q_r^{j/r}, (r/2)^j Q_0).$$

for every $j \in \{1,\dots,r-1\}$.

We will employ the integral representation for the reciprocal Gamma function, usually referred to as *Hankel's formula* (see [7, p. 228]):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\gamma_\phi} w^{-z}e^{w}dw$$

for all $z \in \mathbb{C}$ where γ_ϕ is a path consisting of a half-line in direction $-\phi\pi/2$ (for any $\phi \in (1,2)$) with end point w_0 on the ray $\arg(w) = -\phi\pi/2$ then the circular arc $|w| = |w_0|$ from w_0 to the point w_1 on the ray $\arg(w) = \phi\pi/2$ (traversed anticlockwise), and finally the half-line starting at w_1 in direction $\phi\pi/2$. Now, for every $\beta \in (1,3/2)$ and any $t \in S_{(\beta-1)/2}$, we define

$$\phi_{\beta,t} := \beta + 2\arg(t)/\pi \in ((\beta+1)/2, (3\beta-1)/2) \subseteq (1,7/4).$$

Hence, the change of variables $u = t/w$ maps $\gamma_{\phi_{\beta,t}}$ into δ_β which is a path consisting of a segment from the origin to a point u_0 with $\arg(u_0) = \beta\pi/2$, then the circular arc $|u| = |u_0|$ from u_0 to the point u_1 on the ray $\arg(u) = -\beta\pi/2$ (traversed clockwise), and finally the segment from u_1 to the origin. Therefore, for every $z \in \mathbb{C}$ and all $t \in S_{(\beta-1)/2}$ we have that

$$\frac{t^{z-1}}{\Gamma(z)} = \frac{-1}{2\pi i}\int_{\delta_\beta} u^{z-1}e^{t/u}\frac{du}{u}. \tag{3.19}$$

Our first result is obtained as a consequence of the next proposition and the proof is inspired by Theorem 4.6 in [91].

Proposition 3.3.9 ([91], Prop. 5.1)**.** Let $\mathbb{M}$ be a sequence such that $\widehat{\mathbb{M}}$ is a weight sequence and $r \in \mathbb{N}$. If the restriction map

$$\mathcal{B}_r : \mathcal{L}_{r,\mathbb{M}}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$$

sending f to the formal power series $\sum_{p=0}^{\infty}(f^{(pr)}(0)/p!)z^p$ is surjective, then $\widehat{\mathbf{m}}$ satisfies (γ_r).

Theorem 3.3.10. Let $\mathbb{M}$ be a weight sequence.

(i) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, be such that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\alpha) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective. Then, $\gamma(\mathbb{M}) > \lfloor\alpha\rfloor$.

(ii) If we have that $\tilde{S}_{\mathbb{M}} = (0,\infty)$, then $\gamma(\mathbb{M}) = \infty$.

Proof. (i) Consider first the case $\alpha \in (0,1)$. Then, it suffices to apply Lemma 3.3.5 to obtain that $\mathbb{M}$ has (snq), or equivalently $\gamma(\mathbb{M}) > 0 = \lfloor\alpha\rfloor$, as desired.

Suppose now that $\alpha > 1$ and put $r = \lfloor\alpha\rfloor$, a positive natural number. Firstly, for $\breve{\mathbb{M}} = (M_p/p!)_{p\in\mathbb{N}_0}$ we will prove that the restriction map $\mathcal{B}_r : \mathcal{E}_{r,\breve{\mathbb{M}}}([0,1]) \longrightarrow \mathbb{C}[[z]]_{\breve{\mathbb{M}}}$ is surjective. Since $\alpha \notin \mathbb{N}$, we may choose two numbers β_1, β_2 with

$$1 < \beta_1 < \beta_2 < \min\{\frac{\alpha}{r}, \frac{3}{2}\}.$$

Given $\hat{g}=\sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{\widetilde{\mathbb{M}}}$, we write $b_p := a_p p!$ for all $p\in\mathbb{N}_0$, and there exist $C_0, A_0>0$ such that

$$|b_p| \le C_0 A_0^p p! \widetilde{M}_p = C_0 A_0^p M_p, \quad p\in\mathbb{N}_0.$$

Hence, the formal Laplace transform of $\hat{g}$, defined by $\hat{f} := \hat{L}\hat{g} = \sum_{p=0}^{\infty} b_p z^p$ belongs to $\mathbb{C}[[z]]_{\mathbb{M}}$. By hypothesis, there exists $\psi \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\alpha)$ such that $\tilde{\mathcal{B}}(\psi)=\hat{f}$. Hence, given β_2 and $R>1$, there exist $C, A>0$ such that for every $p\in\mathbb{N}_0$ one has

$$\Big|\psi(z)-\sum_{k=0}^{p-1} b_k z^k\Big| \le CA^p M_p |z|^p, \qquad z\in S(0, r\beta_2, R^r). \tag{3.20}$$

The function $\varphi : S_{\alpha/r}\to\mathbb{C}$ given by $\varphi(u)=\psi(u^r)$, is well defined and holomorphic in $S_{\alpha/r}$, which contains S_{β_2} as a proper unbounded subsector. Moreover, according to (3.20) for $p=0$, for every $w\in S(0,\beta_2,R)$ one has

$$|\varphi(u)| = |\psi(u^r)| \le CM_0. \tag{3.21}$$

We consider now a path δ_{β_1} in $S(0,\beta_2,R)$ like the ones used in the classical Borel transform, made up of a segment δ_1 from the origin to a point u_0 with $|u_0|=R_0<R$ and $\arg(u_0)=\pi\beta_1/2$, then the circular arc δ_2, traversed clockwise on the circumference $|u|=R_0$ and going from u_0 to the point u_1 on the ray $\arg(u_1)=-\pi\beta_1/2$, and finally the segment δ_3 from u_1 to the origin.

Define the function $f : S_{(\beta_1-1)/2}\to\mathbb{C}$ given by

$$f(t) = \frac{-1}{2\pi i}\int_{\delta_{\beta_1}} e^{t/u}\varphi(u)\frac{du}{u}.$$

Observe that $\varphi(u)$ is holomorphic and bounded at 0 in $S(0,\beta_2,R)$, and for every $t\in S_{(\beta_1-1)/2}$ one may easily check that t/u runs over a half-line in the open left half-plane and tends to infinity as u runs over any of the segments δ_1 or δ_3 and tends to 0. Hence, f is holomorphic in the sector $S_{(\beta_1-1)/2}$. We note that, by virtue of Cauchy's theorem, the value assigned to R_0 in the definition of δ_{β_1} is irrelevant for the value of f.

Let us fix in the following estimations some $t\in S(0,(\beta_1-1)/2,R)$ and some natural number $p\in\mathbb{N}$. Hankel's formula (3.19) for $z=kr+1$ allows us to write

$$\begin{aligned} f(t)-\sum_{k=0}^{p-1} b_k\frac{t^{kr}}{(kr)!} &= -\frac{1}{2\pi i}\int_{\delta_{\beta_1}} e^{t/u}\left(\varphi(u)-\sum_{k=0}^{p-1} b_k u^{kr}\right)\frac{du}{u} \\ &= -\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\delta_j} e^{t/u}\left(\varphi(u)-\sum_{k=0}^{p-1} b_k u^{kr}\right)\frac{du}{u}. \end{aligned} \tag{3.22}$$

Taking into account (3.20), for every $u\in S(0,\beta_2,R)$ we have

$$\left|\varphi(u)-\sum_{k=0}^{p-1} b_k u^{kr}\right| = \left|\psi(u^r)-\sum_{k=0}^{p-1} b_k (u^r)^k\right| \le CA^p M_p |u|^{pr}. \tag{3.23}$$

So, if we choose $R_0=|t|/p<R$, we may apply (3.23) and see that

$$\left|\int_{\delta_2} e^{t/u}\left(\varphi(u)-\sum_{k=0}^{p-1} b_k u^{kr}\right)\frac{du}{u}\right| \le \pi\beta_1 e^p CA^p M_p\left(\frac{|t|}{p}\right)^{pr}. \tag{3.24}$$

On the other hand, by the same estimates (3.23) and by the choice made for R_0, for $j = 1, 3$ we have

$$\left| \int_{\delta_j} e^{t/u} \left(\varphi(u) - \sum_{k=0}^{p-1} b_k u^{kr} \right) \frac{du}{u} \right| \leq CA^p M_p \int_0^{|t|/p} s^{pr} |e^{t/(se^{\pm i\pi\beta_1/2})}| \frac{ds}{s} \leq CC_1 A^p M_p \left(\frac{|t|}{p} \right)^{pr}, \tag{3.25}$$

where C_1 is a constant, independent of both t and p, given by

$$\begin{aligned} C_1 &= \sup_{t \in S(0,(\beta_1-1)/2,R),\ p\in\mathbb{N}} \int_0^{|t|/p} |e^{t/(se^{\pm i\pi\beta_1/2})}| \frac{ds}{s} \\ &= \sup_{t \in S(0,(\beta_1-1)/2,R),\ p\in\mathbb{N}} \int_0^{|t|/p} e^{|t|\cos(\arg(t)\mp\pi\beta_1/2)/s} \frac{ds}{s} \\ &\leq \sup_{|t|<R,\ p\in\mathbb{N}} \int_0^{|t|/p} e^{-|t|\cos(\pi(\beta_1-1)/4)/s} \frac{ds}{s} = \sup_{p\in\mathbb{N}} \int_0^{1/p} e^{-\cos(\pi(\beta_1-1)/4)/u} \frac{du}{u} \\ &\leq \int_0^1 e^{-\cos(\pi(\beta_1-1)/4)/u} \frac{du}{u} < \infty. \end{aligned}$$

According to (3.22), (3.24) and (3.25), and using Stirling's formula, we find that there exist constants $C_2, A_2 > 0$ such that for every $p \in \mathbb{N}$ and $t \in S(0, (\beta_1 - 1)/2, R)$ one has

$$\left| f(t) - \sum_{k=0}^{p-1} b_k \frac{t^{kr}}{(kr)!} \right| \leq C_2 A_2^p \frac{M_p}{(pr)!} |t|^{pr}. \tag{3.26}$$

This last estimation also holds for $p = 0$, in a similar way, taking $R_0 = |t|$ and using the definition of f and (3.21). Hence one can show that f admits the series $\sum_{p=0}^{\infty} b_p t^{pr}/(pr)!$ as its asymptotic expansion as t tends to 0 in the sector (if $r \geq 2$ observe that for $(p-1)r + 1 \leq n < pr$ we have $|t|^{pr} \leq |t|^n$ whenever $|t| \leq 1$). It is then a standard fact that for every $m \in \mathbb{N}_0$ and every proper subsector T of $S(0, (\beta_1 - 1)/2, R)$ there exists

$$\lim_{t\to 0,\ t\in T} f^{(m)}(t) = \begin{cases} b_p & \text{if } m = pr \text{ for some natural number } p \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

Finally, we define the function $F\colon [0, 1] \to \mathbb{C}$ given by $F(t) = f(t)$ for $t \in (0, 1]$, $F(0) = b_0$. Since f is holomorphic in $S(0, (\beta_1 - 1)/2, R)$ and we have (3.27), we immediately deduce that F belongs to $\mathcal{C}^\infty([0, 1])$ and

$$F^{(m)}(0) = \begin{cases} b_p & \text{if } m = pr \text{ for some } p \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we may take $\varepsilon > 0$ such that for every $t \in (0, 1]$ the disk $D(t, \varepsilon t)$ is contained in $S(0, (\beta_1 - 1)/2, R)$. Then, Cauchy's integral formula together with (3.26) allow us to deduce that for every $p \in \mathbb{N}_0$,

$$|F^{(pr)}(t)| = \left| \left(f(t) - \sum_{k=1}^{p-1} b_k \frac{t^{kr}}{(kr)!} \right)^{(pr)} \right| \leq (pr)! \left(\frac{1+\varepsilon}{\varepsilon} \right)^{pr} \frac{C_2 A_2^p M_p}{(pr)!} = C_3 A_3^p M_p.$$

In conclusion, $F \in \mathcal{E}_{r,\breve{\mathbb{M}}}([0,1])$ and $\mathcal{B}_r(F) = \hat{g}$. So, S is surjective.

Secondly, according to Theorem 3.2.16 the map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\alpha) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is not injective, this means by Theorem 3.2.15 that $\alpha \leq \omega(\mathbb{M})$, then $r = \lfloor \alpha \rfloor < \omega(\mathbb{M})$ because $\alpha \notin \mathbb{N}$. By Lemma 3.3.7 and Proposition 2.1.11, if $\mathbb{P} = \mathbb{P}_{r,\mathbb{M}}$ we have that

$$\omega(\breve{\mathbb{P}}) = \omega(\mathbb{P}_{r,\mathbb{M}}) - 1 = \omega(\mathbb{M})/r - 1 > 0.$$

Hence, since $\mathbb{P}$ is (lc), by Proposition 2.1.26.(ii), $\breve{\mathbb{P}}$ has (nq) (see Definition 1.1.5), so by the Denjoy-Carleman theorem (see [38, Ch. 1]) there exists a $\mathcal{C}^\infty$ nonnegative function φ in $\mathbb{R}$ with support contained in $[-1,1]$ and which takes the value 1 in a neighborhood of 0, such that there exists $A > 0$ with

$$\sup_{t\in\mathbb{R},\ n\in\mathbb{N}_0} \frac{|\varphi^{(n)}(t)|}{A^n P_n} < \infty.$$

Applying the Gorny-Cartan estimates of Lemma 3.3.8, for every $h \in \mathcal{E}_{r,\breve{\mathbb{M}}}([0,1])$ one can check that the product φh belongs to $\mathcal{L}_{r,\breve{\mathbb{M}}}([0,\infty))$ and, moreover, $(\varphi h)^{(p)}(0) = h^{(p)}(0)$ for every $p \in \mathbb{N}_0$.

Since $\mathcal{B}_r : \mathcal{E}_{r,\breve{\mathbb{M}}}([0,1]) \to \mathbb{C}[[z]]_{\breve{\mathbb{M}}}$ is surjective, we deduce that $\mathcal{B}_r : \mathcal{L}_{r,\breve{\mathbb{M}}}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{\breve{\mathbb{M}}}$ also is. By Proposition 3.3.9, we conclude that $\mathbf{m}$ satisfies (γ_r), what amounts to $\gamma(\mathbb{M}) > r = \lfloor \alpha \rfloor$.

(ii) It is an immediate consequence of (i). □

Corollary 3.3.11. Whenever $\mathbb{M}$ is a weight sequence, if $\gamma(\mathbb{M}) < \infty$ one always has

$$\tilde{S}_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1].$$

In case $\gamma(\mathbb{M}) \in \mathbb{N}$, then $\tilde{S}_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}) + 1)$. Note that if $\gamma(\mathbb{M}) = \infty$, the previous theorem does not provide any relevant information.

Proof. The case $\tilde{S}_{\mathbb{M}} = \emptyset$ is trivial. So, we treat the case in which the surjectivity interval is not empty, what according to Lemma 3.3.5 implies $\gamma(\mathbb{M}) > 0$.

Let $\alpha \in \tilde{S}_{\mathbb{M}}$. On one hand, if $\alpha \notin \mathbb{N}$, by Theorem 3.3.10 we have $\lfloor \alpha \rfloor < \gamma(\mathbb{M})$, and so $\alpha - 1 < \lfloor \alpha \rfloor \leq \lfloor \gamma(\mathbb{M}) \rfloor$, from where $\alpha < \lfloor \gamma(\mathbb{M}) \rfloor + 1$. On the other hand, if $\alpha \in \mathbb{N}$ then we can apply Theorem 3.3.10 for any $\beta \in (\alpha - 1, \alpha)$ (since $\beta \in \tilde{S}_{\mathbb{M}}$ too) and deduce that $\alpha - 1 = \lfloor \beta \rfloor < \gamma(\mathbb{M})$, hence $\alpha < \gamma(\mathbb{M}) + 1$. We deduce that $\alpha \leq \lfloor \gamma(\mathbb{M}) + 1 \rfloor = \lfloor \gamma(\mathbb{M}) \rfloor + 1$, except in case $\gamma(\mathbb{M}) \in \mathbb{N}$, where moreover α cannot coincide with $\gamma(\mathbb{M}) + 1$. The conclusion easily follows. □

Remark 3.3.12. Summing up, for a weight sequence $\mathbb{M}$ and taking into account (3.3) and Theorem 3.2.16 we see that:

(i) if $\gamma(\mathbb{M}) = 0$ (equivalently, if $\mathbb{M}$ has not (snq)) then $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = \tilde{S}_{\mathbb{M}} = \emptyset$.

(ii) if $\gamma(\mathbb{M}) \in (0, \infty)$ and

   (a) $\gamma(\mathbb{M}) \notin \mathbb{N}$, then $S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1] \cap (0, \omega(\mathbb{M})]$,

   (b) $\gamma(\mathbb{M}) \in \mathbb{N}$, then $S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}) + 1) \cap (0, \omega(\mathbb{M})]$.

If $\omega(\mathbb{M}) = \infty$, the second interval in these intersections should be taken as $(0, \infty)$.

3.3.2 Weight sequences satisfying derivation closedness condition

As it has been pointed out in Remark 3.3.12, Corollary 3.3.11 provides also information about $\tilde{S}^u_{\mathbb{M}}$. In order to slightly improve it, one needs to impose (dc), which is a natural condition on the sequence $\mathbb{M}$, in the sense that it guarantees that the ultraholomorphic classes under consideration, consisting of holomorphic functions, are closed with respect to taking derivatives (see Remarks 3.1.4 and 3.1.7). We will also need the next result.

Proposition 3.3.13 ([91], Prop. 5.2)**.** Let $r \in \mathbb{N}$ and $\mathbb{M}$ be a sequence such that $\widehat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ is a weight sequence. If the map $\mathcal{B}_r : \mathcal{N}_{r,\mathbb{M}}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ sending f to the formal power series $\sum_{p=0}^{\infty}(f^{(pr)}(0)/p!)z^p$ is surjective, then the sequence $\widehat{\mathbf{m}} = ((p+1)m_p)_{p\in\mathbb{N}_0}$ satisfies the condition (γ_r).

Following the ideas in the proof of Proposition 4.6 in [91], we will be able to deal also with the case $\alpha \in \mathbb{N}$ whenever $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\alpha) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective.

Theorem 3.3.14. Let $\mathbb{M}$ be a weight sequence satisfying (dc).

(i) Let $\alpha > 0$ be such that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\alpha) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective. Then, $\gamma(\mathbb{M}) > \lfloor\alpha\rfloor$.

(ii) If we have that $\tilde{S}^u_{\mathbb{M}} = (0,\infty)$, then $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = \tilde{S}_{\mathbb{M}} = (0,\infty)$ and $\gamma(\mathbb{M}) = \infty$.

Proof. (i) Consider first the case $\alpha \in (0,1)$, then $\alpha \in \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}}$ and $\alpha \notin \mathbb{N}$, so by Theorem 3.3.10 we conclude that $\gamma(\mathbb{M}) > 0$. Note that in this case no use has been made of (dc).

Suppose now that $\alpha \geq 1$ and put $r = \lfloor\alpha\rfloor$, a positive natural number (note that, by Theorem 3.3.10, we only would need to consider the case $\alpha = r \in \mathbb{N}$ but the proof works anyway). Our aim is to show that $\mathcal{B}_r : \mathcal{N}_{r,\widecheck{\mathbb{M}}}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{\widecheck{\mathbb{M}}}$ is surjective.

Given $\hat{g} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{\widecheck{\mathbb{M}}}$, we write $b_p := a_p p!$ for all $p \in \mathbb{N}_0$ and we see that there exist $C_0, A_0 > 0$ such that

$$|b_p| \leq C_0 A_0^p p! \widecheck{M}_p = C_0 A_0^p M_p, \quad p \in \mathbb{N}_0. \tag{3.28}$$

Consider the formal power series $\widehat{f} = \sum_{p=0}^{\infty}(-1)^{pr} b_p z^p \in \mathbb{C}[[z]]_{\mathbb{M}}$. By hypothesis, there exists $\psi \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\alpha)$ such that $\tilde{\mathcal{B}}(\psi) = \widehat{f}$, and so there exist $C, A > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$\Big|\psi(z) - \sum_{k=0}^{p-1}(-1)^{kr} b_k z^k\Big| \leq CA^p M_p |z|^p, \qquad z \in S_\alpha. \tag{3.29}$$

The function $\varphi : S_{\alpha/r} \to \mathbb{C}$ given by $\varphi(w) = \psi(w^{-r}) - b_0$, is well defined and holomorphic in $S_{\alpha/r} \supseteq S_1$. Moreover, according to (3.29) for $p = 1$, for every $w \in S_1$ one has

$$\left|\frac{\varphi(w)}{w}\right| = \frac{1}{|w|}|\psi(w^{-r}) - b_0| \leq \frac{CAM_1}{|w|^{r+1}}. \tag{3.30}$$

So, the function $f : \mathbb{R} \to \mathbb{C}$ given by

$$f(t) = \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{tu}\frac{\varphi(u)}{u}\,du$$

is well defined and continuous on $\mathbb{R}$. By the classical Hankel formula for the reciprocal Gamma function 3.19, for every natural number $p \geq 2$ and every $t \in \mathbb{R}$ we may write

$$f(t) - \sum_{k=1}^{p-1}(-1)^{kr} b_k \frac{t^{kr}}{(kr)!} = \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{tu}\left(\frac{\varphi(u)}{u} - \sum_{k=1}^{p-1}\frac{(-1)^{kr}b_k}{u^{kr+1}}\right) du. \tag{3.31}$$

Since, again by (3.29), we have

$$\left|\frac{\varphi(u)}{u} - \sum_{k=1}^{p-1}(-1)^{kr} b_k \frac{1}{u^{kr+1}}\right| = \frac{1}{|u|}\left|\psi(u^{-r}) - \sum_{k=0}^{p-1}(-1)^{kr} b_k (u^{-r})^k\right| \leq \frac{CA^p M_p}{|u|^{pr+1}} \tag{3.32}$$

for every $u \in S_1$, we can apply Leibniz's theorem for parametric integrals and deduce that the function

$$f(t) - \sum_{k=1}^{p-1}(-1)^{kr} b_k \frac{t^{kr}}{(kr)!}$$

belongs to $\mathcal{C}^{pr-1}(\mathbb{R})$. Moreover, all of its derivatives of order $m \leq pr-1$ at $t=0$ vanish. This fact can be checked by differentiating the right-hand side of (3.31) m times under the integral sign, evaluating at $t=0$, and then computing the integral by means of Cauchy's theorem. For that, consider the paths Γ_s, $s>0$, consisting of the arc of circumference centered at 1, joining $1+si$ and $1-si$ and passing through $1+s$, and the segment $[1-si, 1+si]$. It is plain to check that $\int_{\Gamma_s} u^{m-1}(\varphi(u) - \sum_{k=1}^{p-1}(-1)^{kr} b_k u^{-kr})du = 0$, and applying (3.32) a limiting process when $s \to \infty$ leads to the conclusion.

As p is arbitrary, we have that $f \in \mathcal{C}^\infty(\mathbb{R})$ and, moreover,

$$f^{(m)}(0) = \begin{cases} (-1)^{pr} b_p & \text{if } m = pr \text{ for some } p \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define the function

$$F(t) = b_0 + f(-t), \quad t \geq 0.$$

Obviously, $F \in \mathcal{C}^\infty([0,\infty))$ and $F^{(pr)}(0) = b_p$, $p \in \mathbb{N}_0$; $F^{(m)}(0) = 0$ otherwise. In order to conclude, we estimate the derivatives of F of order pr for some $p \in \mathbb{N}_0$. For $p=0$ and $t \geq 0$, we take into account (3.28) and (3.30) in order to obtain that

$$|F^{(0)}(t)| \leq |b_0| + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t}\frac{CAM_1}{|1+yi|^{r+1}}\,dy \leq C_0 + \frac{CAM_1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^{(r+1)/2}}\,dy, \tag{3.33}$$

and so F is bounded. For $p \geq 1$ we may write formula (3.31) evaluated at $-t$ as

$$f(-t) - \sum_{k=1}^{p} b_k \frac{t^{kr}}{(kr)!} = \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{-tz}\left(\frac{\varphi(z)}{z} - \sum_{k=1}^{p}\frac{(-1)^{kr} b_k}{z^{kr+1}}\right)dz.$$

Then,

$$\begin{aligned} F^{(pr)}(t) &= b_p + \left(f(-t) - \sum_{k=1}^{p} b_k \frac{t^{kr}}{(kr)!}\right)^{(pr)}(t) \\ &= b_p + \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{-tz}(-z)^{pr}\left(\frac{\varphi(z)}{z} - \sum_{k=1}^{p}\frac{(-1)^{kr} b_k}{z^{kr+1}}\right)dz, \end{aligned}$$

and we may apply (3.28), and (3.32) in order to obtain

$$|F^{(pr)}(t)| \leq C_0 A_0^p M_p + \frac{CA^{p+1}M_{p+1}}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^{(r+1)/2}}\,dy. \tag{3.34}$$

From (3.33) and (3.34), and since $\mathbb{M}$ satisfies (dc), we deduce that there exist $C_1, A_1 > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$|F^{(pr)}(t)| \leq C_1 A_1^p M_p = C_1 A_1^p p! \widecheck{M}_p, \quad t \geq 0,$$

and so $F \in \mathcal{N}_{r,\widecheck{\mathbb{M}}}([0,\infty))$ and $\mathcal{B}_r(F) = \hat{g}$. In conclusion, $\mathcal{B}_r$ is surjective as desired, and by Proposition 3.3.13 we deduce that $\mathbf{m}$ satisfies (γ_r), what amounts to $\gamma(\mathbb{M}) > r = \lfloor \alpha \rfloor$.

(ii) The fact that all the intervals of surjectivity are $(0,\infty)$ is an easy consequence of (3.3) and Proposition 3.1.5.(iii), while $\gamma(\mathbb{M}) = \infty$ stems from (i). □

Corollary 3.3.15. Whenever $\mathbb{M}$ is a weight sequence satisfying (dc), one has

$$S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1).$$

If moreover $\gamma(\mathbb{M}) \in \mathbb{N}$, then $S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}))$.

Proof. The arguments are similar to those in the proof of Corollary 3.3.11. The case $\tilde{S}^u_{\mathbb{M}} = \emptyset$ is trivial. Otherwise, $\tilde{S}_{\mathbb{M}} \neq \emptyset$ and, by Lemma 3.3.5, $\gamma(\mathbb{M}) > 0$.

Let $\alpha \in \tilde{S}^u_{\mathbb{M}}$. By Theorem 3.3.14 we have $\lfloor \alpha \rfloor < \gamma(\mathbb{M})$, and so $\alpha < \lfloor \alpha \rfloor + 1 \leq \lfloor \gamma(\mathbb{M}) \rfloor + 1$, which is the first statement. In case $\gamma(\mathbb{M}) \in \mathbb{N}$, the condition $\lfloor \gamma(\mathbb{M}) \rfloor < \gamma(\mathbb{M})$ does not hold, and so $\gamma(\mathbb{M}) \notin \tilde{S}^u_{\mathbb{M}}$ and the interval $\tilde{S}^u_{\mathbb{M}}$ has to be contained in $(0, \gamma(\mathbb{M}))$. □

Recall that if $\mathbb{M}$ has not (snq) the problem is solved (see Remark 3.3.12). Let $\mathbb{M}$ be (lc), (snq) and (dc) (by Lemma 1.1.7, the first two conditions imply that $\mathbb{M}$ is a weight sequence). Then $\gamma(\mathbb{M}) \in (0, \infty]$, and we have the situation described in Table 3.3, with the corresponding conventions if $\gamma(\mathbb{M}) = \infty$ or $\omega(\mathbb{M}) = \infty$. With the same assumptions, one might be able to at show least that $\tilde{S}_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}))$ and $\tilde{S}_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M})]$ but it seems that a technique that only employs the properties of the spaces $\mathcal{E}_{r,\mathbb{M}}$, $\mathcal{N}_{r,\mathbb{M}}$ and $\mathcal{L}_{r,\mathbb{M}}$ is not sufficient.

As it was mentioned in Remark 2.2.27, there exist sequences that are not strongly regular such that $\gamma(\mathbb{M}), \omega(\mathbb{M}) \in (0, \infty)$, so these values refer to some concrete openings in the injectivity and surjectivity problems.

$\gamma(\mathbb{M}) \in \mathbb{N}$	$\gamma(\mathbb{M}) \in \mathbb{R}\backslash\mathbb{N}$
$S_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}))$	$S_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1) \cap (0, \omega(\mathbb{M})]$
$\tilde{S}^u_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}))$	$\tilde{S}^u_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1) \cap (0, \omega(\mathbb{M})]$
$\tilde{S}_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}) + 1) \cap (0, \omega(\mathbb{M})]$	$\tilde{S}_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1] \cap (0, \omega(\mathbb{M})]$

Table 3.3: Surjectivity intervals when $\mathbb{M}$ is (lc), (snq) and (dc).

3.3.3 Strongly regular sequences

We need to impose more conditions on the sequence $\mathbb{M}$ in order to get extra information about surjectivity. We recall that $\mathbb{M}$ is said to be strongly regular if is (lc), (snq) and (mg). As commented before, the first two conditions are natural in this context and moderate growth condition (mg), which is stronger than (dc), is our additional assumption. A quite complete study of strong regularity has been presented in Section 2.1, we just remember that for these sequences $0 < \gamma(\mathbb{M}) \leq \omega(\mathbb{M}) < \infty$ (see Remark 2.1.19).

The main known result regarding surjectivity for strongly regular sequences was provided by V. Thilliez (see Theorem 3.3.2). Except in the classical Gevrey classes, no information about the optimality of $\gamma(\mathbb{M})$ was provided. Our next attempt will be to obtain as much information as possible in this direction. The following result rests on Theorem 3.3.14 and a ramification argument. As usual, $\mathbb{Q}$ denotes the rational numbers and $\mathbb{I}$ the irrationals.

Theorem 3.3.16. Let $\mathbb{M}$ be a strongly regular sequence, and let $r \in \mathbb{Q}$, $r > 0$ be given. The following assertions are equivalent:

(i) $r < \gamma(\mathbb{M})$,

(ii) there exists $d \geq 1$ such that for every $A > 0$ there is a linear continuous operator

$$T_{\mathbb{M},A,r} : \mathbb{C}[[z]]_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},dA}(S_r)$$

such that $\tilde{\mathcal{B}} \circ T_{\mathbb{M},A,\gamma} = Id_{\mathbb{C}[[z]]_{\mathbb{M},A}}$ the identity map in $\mathbb{C}[[z]]_{\mathbb{M},A}$,

(iii) the Borel map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_r) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(iv) the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_r) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) This is Theorem 3.3.2.

(iii) $\Longrightarrow$ (iv) Trivial by contention.

(iv) $\Longrightarrow$ (i) In case $r \in \mathbb{N}$, we use Theorem 3.3.14.(i) and we conclude.

Otherwise, we write $r = p/q$ with $p, q \in \mathbb{N}$ relatively prime, $q \geq 2$. Consider the sequence $\mathbb{M}^q = (M_n^q)_{n\in\mathbb{N}_0}$, which also turns out to be strongly regular (see [95, Lemma 1.3.4] or, alternatively, Proposition 2.1.11 and Remark 2.1.19). We will prove that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}^q}(S_p) \to \mathbb{C}[[z]]_{\mathbb{M}^q}$ is surjective, so, again by Theorem 3.3.14.(i), we see that $p < \gamma(\mathbb{M}^q)$. Hence, we get that $r = p/q < \gamma(\mathbb{M})$, as desired.

Let us prove the aforementioned surjectivity. Given $\hat{f} = \sum_{j=0}^{\infty} a_j z^j \in \mathbb{C}[[z]]_{\mathbb{M}^q}$, there exist $C, A > 0$ such that $|a_j| \leq CA^j M_j^q$ for every $j \in \mathbb{N}_0$. Let us define a new formal power series $\hat{g} = \sum_{j=0}^{\infty} b_j z^j$ with coefficients

$$b_{qj} = a_j,\ j \in \mathbb{N}_0; \quad b_m = 0 \text{ otherwise.}$$

The log-convexity of $\mathbb{M}$ implies that $M_j^q \leq M_{qj}$ for every j, so we have that

$$|b_{qj}| \leq CA^j M_j^q \leq C(A^{1/q})^{qj} M_{qj},$$

and consequently, $\hat{g} \in \mathbb{C}[[z]]_{\mathbb{M}}$. By hypothesis, there exists a function $g \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_r)$ such that $\tilde{\mathcal{B}}(g) = \hat{g}$, and so there exist $C_1, A_1 > 0$ such that for every $z \in S_r$ and $n \in \mathbb{N}_0$ one has

$$\left| g(z) - \sum_{j=0}^{n-1} b_j z^j \right| \leq C_1 A_1^n M_n |z|^n. \tag{3.35}$$

Consequently, the function $f\colon S_p \to \mathbb{C}$ given by $f(w) = g(w^{1/q})$ is well-defined and holomorphic in S_p. Moreover, for every $w \in S_p$ and $n \in \mathbb{N}_0$ one deduces from (3.35) that

$$\begin{aligned} \left| f(w) - \sum_{j=0}^{n-1} a_j w^j \right| &= \left| g(w^{1/q}) - \sum_{j=0}^{n-1} b_{qj} (w^{1/q})^{qj} \right| = \left| g(w^{1/q}) - \sum_{k=0}^{qn-1} b_k (w^{1/q})^k \right| \\ &\leq C_1 A_1^{qn} M_{qn} |w^{1/q}|^{qn}. \end{aligned} \tag{3.36}$$

We apply now the property (mg) of $\mathbb{M}$: by Remark 1.1.10, there exists $A_0 > 0$ such that for all $n \in \mathbb{N}_0$ we have $M_{qn} \leq A_0^n M_n^q$. We may use this fact in (3.36) and obtain that

$$\left| f(w) - \sum_{j=0}^{n-1} a_j w^j \right| \leq C_1 (A_0 A_1^q)^n M_n^q |w|^n.$$

So, $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}^q}(S_p)$ and $\tilde{\mathcal{B}}(f) = \hat{f}$, what shows the surjectivity as intended. □

This result has several important consequences.

Corollary 3.3.17. Let $\mathbb{M}$ be a strongly regular sequence with $\gamma(\mathbb{M}) \in \mathbb{Q}$. Then, $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = (0, \gamma(\mathbb{M}))$.

Proof. By Theorem 3.3.2 and (3.3), we have $(0, \gamma(\mathbb{M})) \subseteq S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}}$, while (iii) $\Longrightarrow$ (i) in Theorem 3.3.16 ensures that, $\gamma(\mathbb{M})$ being rational, it cannot be the case that $\gamma(\mathbb{M}) \in \tilde{S}^u_{\mathbb{M}}$, and so $\tilde{S}^u_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M}))$. □

Corollary 3.3.18. Let $\mathbb{M}$ be a strongly regular sequence, and let $t \in \mathbb{R}$, $t > 0$ be given. Each assertion implies the following one:

(i) $t < \gamma(\mathbb{M})$,

(ii) the Borel map $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(iii) the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(iv) the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_t) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(v) for every $\xi \in \mathbb{I}$ with $\xi < t$, the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\xi) \to \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective,

(vi) $t \leq \gamma(\mathbb{M})$.

Hence, $(0, \gamma(\mathbb{M})) \subseteq S_{\mathbb{M}} \subseteq \tilde{S}^u_{\mathbb{M}} \subseteq \tilde{S}_{\mathbb{M}} \subseteq (0, \gamma(\mathbb{M})]$.

Proof. Only (v) $\Longrightarrow$ (vi) needs a short proof. For every $q \in \mathbb{N}$ we have that $\zeta = \xi q \notin \mathbb{N}$, we will show that $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}^q}(S_\zeta) \to \mathbb{C}[[z]]_{\mathbb{M}^q}$ is surjective so, by Theorem 3.3.10.(i), we see that $\lfloor \zeta \rfloor < \gamma(\mathbb{M}^q)$. Then $\gamma(\mathbb{M}) > \lfloor \xi q \rfloor / q > \xi - 1/q$. Since q is arbitrary, making q tend to ∞ we deduce that $\xi \leq \gamma(\mathbb{M})$ for every irrational $\xi < t$, so $t \leq \gamma(\mathbb{M})$.

The proof of the surjectivity follows the same ramification argument used in (iv) $\Longrightarrow$ (i) of Theorem 3.3.16, where the asymptotic relations obtained for bounded subsectors of S_ξ are transformed into the analogous ones for the corresponding bounded subsectors of S_ζ. □

Remark 3.3.19. The situation for strongly regular sequences is summed up in Table 3.4. The conjecture is that, at least for strongly regular sequences, one always has $\tilde{S}_{\mathbb{M}} = (0, \gamma(\mathbb{M})]$ and $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = (0, \gamma(\mathbb{M}))$. The main difference with the injectivity problem, in which the belonging of the value $\omega(\mathbb{M})$ to the injectivity interval depends on the convergence of a series, might lie in the fact that the value of $\gamma(\mathbb{M})$ completely characterized (snq) condition, that is, $\gamma(\mathbb{M}) > 0$ if and only if $\mathbb{M}$ has (snq), whereas for $\omega(\mathbb{M})$ we remember that if $\omega(\mathbb{M}) > 0$ then $\mathbb{M}$ is (nq), but if $\mathbb{M}$ is (nq) then only $\omega(\mathbb{M}) \geq 0$ is known.

	$\gamma(\mathbb{M})\in\mathbb{Q}$	$\gamma(\mathbb{M})\in\mathbb{I}$
$S_{\mathbb{M}}$	$(0,\gamma(\mathbb{M}))$	$(0,\gamma(\mathbb{M}))$ or $(0,\gamma(\mathbb{M})]$
$\tilde{S}^u_{\mathbb{M}}$	$(0,\gamma(\mathbb{M}))$	$(0,\gamma(\mathbb{M}))$ or $(0,\gamma(\mathbb{M})]$
$\tilde{S}_{\mathbb{M}}$	$(0,\gamma(\mathbb{M}))$ or $(0,\gamma(\mathbb{M})]$	

Table 3.4: Surjectivity intervals for strongly regular sequences

Remark 3.3.20. A question which was open for some time is: Are $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ always equal for strongly regular sequences? After some trial and error, a strongly regular sequence has been constructed with $\gamma(\mathbb{M})=2<\omega(\mathbb{M})=5/2$ (see Example 2.2.26). In fact, given any pair of values $0<\gamma<\omega<\infty$ we are able to provide a strongly regular sequence $\mathbb{M}$ such that $\gamma(\mathbb{M})=\gamma$ and $\omega(\mathbb{M})=\omega$ (see Remark 2.2.27). This means that for opening $\alpha\pi$ with α in the interval (γ,ω), the Borel map is neither injective nor surjective and the corresponding injectivity and surjectivity intervals for this sequence are either $[\omega,\infty)$ or (ω,∞) and $(0,\gamma)$ or $(0,\gamma]$, respectively.

3.3.4 Sequences admitting a nonzero proximate order

In this final subsection, taking into account that the Borel map is never bijective, Theorem 3.2.16, we will deduce more information regarding the surjectivity intervals. In order to be able to infer from that result whether or not $\gamma(\mathbb{M})$ belongs to $S_{\mathbb{M}}$ and $\tilde{S}^u_{\mathbb{M}}$, strongly regularity is not enough and we need to assume $\gamma(\mathbb{M})=\omega(\mathbb{M})$. Then,

(i) If $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}=\infty$, we know that $\tilde{I}^u_{\mathbb{M}}=I_{\mathbb{M}}=[\omega(\mathbb{M}),\infty)=[\gamma(\mathbb{M}),\infty)$, and then

$$S_{\mathbb{M}}=\tilde{S}^u_{\mathbb{M}}=(0,\gamma(\mathbb{M})),\qquad (0,\gamma(\mathbb{M}))\subseteq\tilde{S}_{\mathbb{M}}\subseteq(0,\gamma(\mathbb{M})].$$

(ii) If $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}<\infty$ and $\sum_{p=0}^{\infty}((p+1)m_p)^{-1/(\omega(\mathbb{M})+1)}=\infty$, we know that $I_{\mathbb{M}}=[\gamma(\mathbb{M}),\infty)$ and $\tilde{I}^u_{\mathbb{M}}=(\gamma(\mathbb{M}),\infty)$, and so

$$S_{\mathbb{M}}=(0,\gamma(\mathbb{M})),\qquad (0,\gamma(\mathbb{M}))\subseteq\tilde{S}^u_{\mathbb{M}}\subseteq\tilde{S}_{\mathbb{M}}\subseteq(0,\gamma(\mathbb{M})].$$

Hence, the information we have for strongly regular sequences with $\gamma(\mathbb{M})=\omega(\mathbb{M})$ is summarized in the first two rows of Table 3.5. Note that for nonuniform asymptotics this assumption does not produce any improvements and we will need to go one step further.

Our final result was given by J. Sanz, Theorem 6.1 in [88] for strongly regular sequences $\mathbb{M}$ such that $d_{\mathbb{M}}$ is a proximate order. For nonuniform asymptotics, he proved that $\tilde{S}_{\mathbb{M}}=(0,\gamma(\mathbb{M})]$ employing the truncated Laplace transform technique where the classical exponential kernel was replaced by a function e_V (see Remark 4.1.4) which is constructed using proximate orders and Maerogiz's functions. As it is deduced from [88, Remark 4.11.(iii)] and Remark 2.2.18, this construction is also available whenever $\mathbb{M}$ is a weight sequence admitting a nonzero proximate order. We recall that if $\mathbb{M}$ admits a nonzero proximate order then it is strongly regular and $\gamma(\mathbb{M})=\omega(\mathbb{M})\in(0,\infty)$ but, as explained in the previous chapter, the converse does not hold, so this is the most regular situation we will consider.

Theorem 3.3.21 (Generalized Borel–Ritt–Gevrey theorem)**.** *Let $\mathbb{M}$ be a weight sequence admitting a nonzero proximate order and $\gamma>0$ be given. The following statements are equivalent:*

(i) $\gamma\le\omega(\mathbb{M})=\gamma(\mathbb{M})$,

(ii) For every $\hat{f} = \sum_{p\in\mathbb{N}_0} a_p z^p \in \mathbb{C}[[z]]_{\mathbb{M}}$ there exists a function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$ such that

$$f \sim_{\mathbb{M}} \hat{f},$$

i.e., $\tilde{\mathcal{B}}(f) = \hat{f}$. In other words, the Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective.

Hence, $\tilde{S}_{\mathbb{M}} = (0, \gamma(\mathbb{M})] = (0, \omega(\mathbb{M})]$.

Table 3.5 gathers the information about surjectivity in case $\mathbb{M}$ admits a nonzero proximate order. For the sequence $\mathbb{M}_{\alpha,\beta} = \big(p!^\alpha \prod_{m=0}^{p} \log^\beta(e+m)\big)_{p\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$, the information is summarized in Table 3.6, note that the Gevrey case always belongs to the first column.

	$\gamma(\mathbb{M}) \in \mathbb{Q}$	$\gamma(\mathbb{M}) \in \mathbb{I}$		
		$\sum_{p=0}^{\infty} \left(\frac{1}{m_p}\right)^{\frac{1}{\omega(\mathbb{M})}} = \infty$	$\sum_{p=0}^{\infty} \left(\frac{1}{(p+1)m_p}\right)^{\frac{1}{\omega(\mathbb{M})+1}} = \infty$	$\sum_{p=0}^{\infty} \left(\frac{1}{(p+1)m_p}\right)^{\frac{1}{\omega(\mathbb{M})+1}} < \infty$
$S_{\mathbb{M}}$	$(0, \gamma(\mathbb{M}))$			
$\tilde{S}^u_{\mathbb{M}}$			$(0, \gamma(\mathbb{M}))$ or $(0, \gamma(\mathbb{M})]$	
$\tilde{S}_{\mathbb{M}}$	$(0, \gamma(\mathbb{M})]$			

Table 3.5: Surjectivity intervals for weight sequences admitting a nonzero proximate order.

	$\beta \le \alpha$	$\alpha < \beta \le \alpha + 1$	$\beta > \alpha + 1$
$S_{\mathbb{M}_{\alpha,\beta}}$	$(0, \alpha)$	$(0, \alpha)$	$(0, \alpha)$ or $(0, \alpha]$
$\tilde{S}^u_{\mathbb{M}_{\alpha,\beta}}$	$(0, \alpha)$	$(0, \alpha)$ or $(0, \alpha]$	$(0, \alpha)$ or $(0, \alpha]$
$\tilde{S}_{\mathbb{M}_{\alpha,\beta}}$	$(0, \alpha]$	$(0, \alpha]$	$(0, \alpha]$

Table 3.6: Surjectivity intervals for the sequences $\mathbb{M}_{\alpha,\beta}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Chapter 4

Multisummability via proximate orders

The method of summation of formal power series by means of Borel and Laplace operators was introduced by É. Borel in the 'simplest' case of level 1, and the extension of the method to any level k is quite straightforward but technical. In 1981 multisummability notion was created in a somewhat different and more general form by J. Écalle [27], using what he called acceleration operators. In 1992, W. Balser [5] reformulated this method for the Gevrey case by means of the iterated Laplace integrals. This iteration of a finite number of $k-$summability procedures, which has been proved to be stronger than any of them, is based on the fact that a convenient Borel transform of the series is itself summable. Six different approaches to multisummability can be found in the book of M. Loday [64].

The technique of multisummability (in Balser's sense) has been successfully applied to the study of formal power series solutions at a singular point of linear and nonlinear (systems of) meromorphic ordinary differential equations in the complex domain (see, to cite but a few, the works [6, 7, 9, 19, 73, 85]), of partial differential equations (for example, [8, 10, 36, 69, 76]), as well as of singular perturbation problems (see [11, 23, 59], among others).

Nevertheless, it is known that nonGevrey formal power series solutions may appear for different kinds of equations. For example, V. Thilliez has proven some results on solutions within these general classes for algebraic equations in [97]. Also, G. K. Immink in [40, 41] has obtained some results on summability for solutions of difference equations whose coefficients grow at an intermediate rate between Gevrey classes, called of 1^+ level, that is governed by a strongly regular sequence. More recently, S. Malek [70] has studied some singularly perturbed small step size difference-differential nonlinear equations whose formal solutions with respect to the perturbation parameter can be decomposed as sums of two formal series, one with Gevrey order 1, the other of 1^+ level, a phenomenon already observed for difference equations [20].

All these results invite one to try to generate summability tools so they are able to deal with formal power series whose coefficients' growth is controlled by a general strongly regular sequence, so including Gevrey, 1^+ level and other interesting examples. These generalized summability methods have been developed by A. Lastra, S. Malek, J. Sanz in [60, 88, 89] and will be briefly presented in the first section.

The aim of this chapter is to put forward the corresponding multisummability theory, in Balser's sense, in this context by suitably combining the methods created from different sequences $\mathbb{M}_1, \mathbb{M}_2, \dots, \mathbb{M}_n$ instead of different Gevrey levels $k_1, k_2, \dots, k_n$. In the second section, we will analyze the main difficulties of this general approach, such as the comparability of different sequences, the properties of the product and quotient sequences and the extension of the classical tauberian theorems. The definition of a meaningful multisummability notion depends on the existence of these tauberian results that will be valid if the growth indices $\omega(\mathbb{M}_j)$ of the sequences are mutually distinct. In this situation, supplementary summability kernels will be constructed from some given kernel e_j for $\mathbb{M}_j-$summability in the third section. They will provide Laplace and Borel-like transforms for the quotient and the product of two moment sequences m_{e_j}, m_{e_k} allowing us to recover the multisum of a formal power series.

4.1 $\mathbb{M}-$summability

For a given sectorial region G of wide opening and a weight sequence $\mathbb{M}$, i.e., (lc) with quotients tending to infinity, Watson's Lemma in $\tilde{\mathcal{A}}_{\mathbb{M}}(G)$, Theorem 3.2.15, ensures that every function f in such a class is determined by its asymptotic expansion $\hat{f}$. This fact motivates the concept, developed by A. Lastra, S. Malek, J. Sanz in [60, 88, 89], of summability of formal (i.e. divergent in general) power series with controlled growth in their coefficients in the framework of general Carleman ultraholomorphic classes in sectors, described in the preceding chapter, so generalizing the by-now classical and powerful tool of $k-$summability of formal Gevrey power series, introduced by J.-P. Ramis [81, 82].

Definition 4.1.1. Let $d \in \mathbb{R}$ and $\mathbb{M}$ be a weight sequence. We say $\hat{f} = \sum_{p\geq 0} a_p z^p \in \mathbb{C}[[z]]$ is $\mathbb{M}-$*summable in direction* d if there exist a sectorial region $G = G(d,\gamma)$, with $\gamma > \omega(\mathbb{M})$, and a function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ such that $f \sim_{\mathbb{M}} \hat{f}$.

By Remark 3.1.7, we have that $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{M}}$, according to Theorem 3.2.15, f is unique with the property stated, and will be denoted $f = \mathcal{S}_{\mathbb{M},d}\hat{f}$, the $\mathbb{M}-$*sum of* $\hat{f}$ *in direction* d.

The aim of this section is to briefly recall the suitable tools developed in [60, 88, 89], where the ideas in the theory of general moment summability methods put forward W. Balser in [7] were followed, in order to recover f from $\hat{f}$ by means of formal and analytic transforms, in the same vein as in the classical theory for Gevrey case, so-called $k-$summability.

4.1.1 $\mathbb{M}-$summability kernels

Balser's moment summability methods, equivalent in a sense to $k-$summability, rely on the determination of a pair of kernel functions e and E with suitable asymptotic and growth properties, in terms of which to define formal and analytic Laplace- and Borel-like transforms. The definition of $k-$summability kernels in [7, Section 5.5] is extended for strongly regular sequences as follows, where the case $\omega(\mathbb{M}) < 2$ is mainly treated and indications will be given below on how to work in the opposite situation.

Definition 4.1.2. Let $\mathbb{M}$ be a strongly regular sequence with $\omega(\mathbb{M}) < 2$. A pair of complex functions e, E are said to be *kernel functions for* $\mathbb{M}-$*summability* if:

(I) e is holomorphic in $S_{\omega(\mathbb{M})}$.

(II) $z^{-1}e(z)$ is locally uniformly integrable at the origin, i.e., there exists $t_0 > 0$, and for every $z_0 \in S_{\omega(\mathbb{M})}$ there exists a neighborhood U of z_0, $U \subseteq S_{\omega(\mathbb{M})}$, such that the integral $\int_0^{t_0} t^{-1}\sup_{z\in U}|e(t/z)|dt$ is finite.

(III) For every $\varepsilon > 0$ there exist $c, k > 0$ such that

$$|e(z)| \leq c h_{\mathbb{M}}\left(\frac{k}{|z|}\right) = c\, e^{-\omega_{\mathbb{M}}(|z|/k)}, \qquad z \in S_{\omega(\mathbb{M})-\varepsilon}, \tag{4.1}$$

where $h_{\mathbb{M}}$ and $\omega_{\mathbb{M}}$ are the functions associated with $\mathbb{M}$ defined Subsection 1.1.3.

(IV) For $x \in \mathbb{R}$, $x > 0$, the values of $e(x)$ are positive real.

(V) If we define the *moment function* associated with e,

$$m_e(\lambda) := \int_0^\infty t^{\lambda-1} e(t)dt, \qquad \mathrm{Re}(\lambda) \geq 0,$$

from $(I) - (IV)$ we see that m_e is continuous in $\{\mathrm{Re}(\lambda) \geq 0\}$, holomorphic in $\{\mathrm{Re}(\lambda) > 0\}$, and $m_e(x) > 0$ for every $x \geq 0$. Then, the function E given by

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_e(n)}, \qquad z \in \mathbb{C},$$

is entire, and there exist $C, K > 0$ such that

$$|E(z)| \leq \frac{C}{h_{\mathbb{M}}(K/|z|)} = Ce^{\omega_{\mathbb{M}}(|z|/K)}, \quad z \in \mathbb{C}. \tag{4.2}$$

(VI) $z^{-1}E(1/z)$ is locally uniformly integrable at the origin in the sector $S(\pi, 2 - \omega(\mathbb{M}))$, in the sense that there exists $t_0 > 0$, and for every $z_0 \in S(\pi, 2 - \omega(\mathbb{M}))$ there exist a neighborhood U of z_0, $U \subseteq S(\pi, 2 - \omega(\mathbb{M}))$, such that the integral $\int_0^{t_0} t^{-1} \sup_{z \in U} |E(z/t)|dt$ is finite.

We recall that if $\mathbb{M}$ is strongly regular $\omega(\mathbb{M}) \in (0, \infty)$ (see Remark 2.1.19) and the sectors in the above definition are meaningful.

Remark 4.1.3. (i) According to Definition 4.1.2(V), the knowledge of e is enough to determine the pair of kernel functions. So, in the sequel we will frequently omit the function E in our statements.

(ii) In case $\omega(\mathbb{M}) \geq 2$, condition (VI) in Definition 4.1.2 does not make sense. However, we note that for a positive real number $s > 0$ the sequence of $1/s$−powers $\mathbb{M}^{(1/s)} := (M_p^{1/s})_{p \in \mathbb{N}_0}$ is also a strongly regular, $\omega(\mathbb{M}^{(1/s)}) = \omega(\mathbb{M})/s$ (see Proposition 2.1.11 and Theorem 2.1.16 and Remark 2.1.19) and, as it is easy to check,

$$h_{\mathbb{M}^{(1/s)}}(t) = \big(h_{\mathbb{M}}(t^s)\big)^{1/s}, \quad t \geq 0.$$

So, following the ideas of Section 5.6 in [7, p.90], we will say that a complex function e is a kernel of $\mathbb{M}$−summability if there exist $s > 0$ with $\omega(\mathbb{M})/s < 2$, and a kernel $\tilde{e} : S_{\omega(\mathbb{M})/s} \to \mathbb{C}$ for $\mathbb{M}^{(1/s)}$−summability such that

$$e(z) := \tilde{e}(z^{1/s})/s, \qquad z \in S_{\omega(\mathbb{M})}.$$

If one defines the moment function m_e as before, it is plain to see that $m_e(\lambda) = m_{\tilde{e}}(s\lambda)$, $\mathrm{Re}(\lambda) \geq 0$. The properties verified by $\tilde{e}$ and $m_{\tilde{e}}$ are easily translated into similar ones for e, but in this case the function

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_e(n)} = \sum_{n=0}^{\infty} \frac{z^n}{m_{\tilde{e}}(sn)}$$

does not have the same properties as before, and one rather pays attention to the kernel associated with $\tilde{e}$,

$$\tilde{E}(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_{\tilde{e}}(n)} = \sum_{n=0}^{\infty} \frac{z^n}{m_e(n/s)}, \tag{4.3}$$

which will behave as indicated in (V) and (VI) of Definition 4.1.2 for such a kernel of $\mathbb{M}^{(1/s)}-$summability.

It is worth remarking that, once such s and $\tilde{e}$ as in the definition exist, one easily checks that for any real number $t > \omega(\mathbb{M})/2$ a kernel $\overline{e}$ for $\mathbb{M}^{(1/t)}-$summability exists with $e(z) = \overline{e}(z^{1/t})/t$.

Remark 4.1.4. (i) Note that Definition 4.1.2 can be given for arbitrary weight sequences with $\omega(\mathbb{M}) \in (0, 2)$. If, moreover, $\mathbb{M}$ is (dc) and there exists an $\mathbb{M}-$summability kernel, one can show, following the ideas in [58, 88], that $\mathbb{M}$ satisfies (snq), so this condition is obtained automatically and, furthermore, $\gamma(\mathbb{M}) = \omega(\mathbb{M})$.

However, (mg) seems not to be deduced from the definition of the kernels but it turns out that for the proofs below (mg) is essential. In any case, since the existence of such kernels is only guaranteed for a subfamily of strongly regular sequences, the ones admitting a nonzero proximate order (see (ii) in this remark), the definition in [60] has been kept.

(ii) The existence of such kernels have been proved in [60], by taking into account the construction of nontrivial flat functions in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ accomplished by J. Sanz in [88], whenever the function $d_{\mathbb{M}}(t)$ is a nonzero proximate order (see (2.15) and Theorem 2.2.6) which can be extended whenever the sequence $\mathbb{M}$ admits a nonzero proximate order. In Remark 2.2.18, we have explained that this condition holds for the strongly regular sequences appearing in the applications but it is strictly stronger. We recall that a weight sequence $\mathbb{M}$ admits a nonzero proximate order if there exists a nonzero proximate order $\rho(t)$ and constants A and B such that

$$A \leq \log(t)(\rho(t) - d_{\mathbb{M}}(t)) \leq B, \qquad \text{for } t \text{ large enough},$$

or, equivalently, if

$$e^A \leq \frac{t^{\rho(t)}}{\omega_{\mathbb{M}}(t)} \leq e^B, \qquad \text{for } t \text{ large enough}.$$

In this situation we know that $\lim_{t\to\infty} d_{\mathbb{M}}(t) = \lim_{t\to\infty} \rho(t) = 1/\omega(\mathbb{M})$ (see Subsection 2.2.4). Then, for such a sequence and for every $V \in MF(2\omega(\mathbb{M}), \rho(t))$ they consider the function e_V defined in $S_{\omega(\mathbb{M})}$ by

$$e_V(z) = z\,\exp(-V(z)).$$

Since V is holomorphic in $S_{2\omega(\mathbb{M})}$ and real in $(0, \infty)$, the same is true for e_V, so (I) and (IV) in Definition 4.1.2 hold. Property (II) in that Definition has been obtained, as in [88, Lemma 5.3], as a consequence of Proposition 3.2.13 which ensures that for every $\varepsilon \in (0, \omega(\mathbb{M}))$ there exists $b > 0$ such that for any $z \in S_{\omega(\mathbb{M})-\varepsilon}$,

$$|e_V(z)| \leq |z|\exp(-\operatorname{Re}(V(z))) \leq |z|\exp(-bV(|z|)) \leq |z|\exp(-Ab\omega_{\mathbb{M}}(|z|)) \leq |z|, \qquad (4.4)$$

because $\omega_{\mathbb{M}}(|z|) \geq 0$. Similarly for (III), using in addition Lemma 1.1.24, also as in [88, Lemma 5.3], we see that there exist constants $c, k > 0$ such that

$$|e_V(z)| \leq |z|\exp(-Ab\omega_{\mathbb{M}}(|z|)) \leq c\exp(-\omega_{\mathbb{M}}(|z|/k))$$

for every $z \in S_{\omega(\mathbb{M})-\varepsilon}$. Then, the moment function associated with e_V,

$$m_V(\lambda) := \int_0^\infty t^{\lambda-1} e_V(t)dt = \int_0^\infty t^\lambda e^{-V(t)}dt,$$

is well defined in $\{\operatorname{Re}(\lambda) \geq 0\}$, continuous in its domain, holomorphic in $\{\operatorname{Re}(\lambda) > 0\}$ and $m_V(x) > 0$ for every $x \geq 0$. Moreover, we have the following result of L.S. Maergoiz.

Proposition 4.1.5 ([65], Th. 3.3)**.** The function

$$E_V(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_V(n)}, \qquad z \in \mathbb{C},$$

is entire and there exist constants $C_1, K_1 > 0$ such that for every $z \in \mathbb{C}$ one has

$$|E_V(z)| \leq C_1 \exp(K_1 V(|z|)).$$

Consequently, by the admissibility condition, for every $z \in \mathbb{C}$ and suitably large constants $\tilde{C}, \tilde{K} > 0$, we have that

$$|E_V(z)| \leq \tilde{C} \exp(\tilde{K} \omega_{\mathbb{M}}(|z|))$$

and so condition (V) in Definition 4.1.2 is satisfied. Finally, we take into account the following.

Proposition 4.1.6 ([65], (3.25))**.** Let $\rho(t)$ be a proximate order with $\rho > 1/2$, $\gamma \geq 2/\rho$ and $V \in MF(\gamma, \rho(t))$. Then, for every $\varepsilon > 0$ such that $\varepsilon < \pi/2(2-1/\rho)$ we have, uniformly as $|z| \to \infty$, that (in Landau's notation)

$$E_V(z) = O\left(\frac{1}{|z|}\right), \qquad \frac{\pi}{2\rho} + \varepsilon \leq |\arg z| \leq \pi. \tag{4.5}$$

In this case $\rho = 1/\omega(\mathbb{M})$ and this information easily implies that also condition (VI) in Definition 4.1.2 is fulfilled, so e_V is a kernel of $\mathbb{M}-$summability.

(iii) In Balser's theory for Gevrey sequences $\mathbb{M}_{1/k} = (p!^{1/k})_{p\in\mathbb{N}_0}$ (see [7, Sect. 5.5]), the classical example of kernels is given by $e_k(z) = kz^k \exp(-z^k)$, the moment function is $\Gamma(1+\lambda/k)$ and the Borel kernel E is a Mittag-Leffler function.

Next, we recall the following key result by H. Komatsu that characterizes the growth of a entire function in terms of its Taylor coefficients and it was useful in the proof of Proposition 4.1.8 and will be employed afterwards.

Proposition 4.1.7 ([52], Prop. 4.5)**.** Let $\omega_{\mathbb{M}}(t)$ be the function associated with a weight sequence $\mathbb{M}$. Given an entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, the following statements are equivalent:

(i) There exist $C, K > 0$ such that $|F(z)| \leq Ce^{\omega_{\mathbb{M}}(K|z|)}$, $z \in \mathbb{C}$.

(ii) There exist $c, k > 0$ such that for every $n \in \mathbb{N}_0$, $|a_n| \leq ck^n/M_n$.

The following result is key for the development of a satisfactory summability theory because it ensures that the classes of functions and formal power series defined from $\mathbb{M}$ and $\mathfrak{m}_e$ coincide. In the proof, the estimates, for the kernels e and E appearing in (4.1) and (4.2), respectively, are crucial

Proposition 4.1.8 ([88], Prop. 5.7)**.** Let e be a kernel function for $\mathbb{M}-$summability, and $\mathfrak{m}_e = (m_e(p))_{p\in\mathbb{N}_0}$ the sequence of moments associated with e. Then $\mathbb{M} \approx \mathfrak{m}_e$.

Remark 4.1.9. (i) As mentioned in Remark 4.1.4, in the Gevrey case of order $\alpha > 0$, $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, it is usual to choose the kernel

$$e_{1/\alpha}(z) = \frac{1}{\alpha} z^{1/\alpha} \exp(-z^{1/\alpha}), \qquad z \in S_\alpha.$$

Then we obtain that $m_{e_\alpha}(\lambda) = \Gamma(1+\alpha\lambda)$ for $\mathrm{Re}(\lambda) \geq 0$. Of course, the sequences $\mathbb{M}_\alpha$ and $\mathfrak{m}_{e_\alpha} = (m_\alpha(p))_{p\in\mathbb{N}_0}$ are equivalent.

(ii) Indeed, for any kernel e for $\mathbb{M}-$summability, up to multiplication by a constant scaling factor, one may always suppose that $m_e(0) = 1$. One may also prove that the sequence of moments $\mathfrak{m}_e = (m_e(p))_{p\in\mathbb{N}_0}$ is also (lc), which is a consequence of Hölder's inequality. Then the strong regularity is deduced from the equivalence between $\mathbb{M}$ and $\mathfrak{m}_e$ (see Proposition 1.1.20).

Bearing this fact in mind, in Definition 4.1.2 one could depart not from a weight sequence $\mathbb{M}$, but from a kernel e, initially defined and positive in direction $d = 0$, whose moment function $m_e(\lambda)$ is supposed to be well-defined for $\lambda \geq 0$, and such that the sequence $\mathfrak{m}_e$ is strongly regular. With this approach, $\omega(\mathbb{M})$ will be replaced by $\omega(\mathfrak{m}_e)$ and the function $\omega_{\mathbb{M}}(t)$ by $\omega_{\mathfrak{m}_e}(t)$ in the definition of the kernels.

(iii) In a more general framework (see O. Blasco [15]), departing from a continuous and piecewise continuously differentiable nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\log(t) = o(\phi(t))$ as $t \to \infty$, we can construct two sequences:

(i) the moment sequence $M_p(\phi) = \displaystyle\int_0^\infty t^p e^{-\phi(t)}dt$, so $e(t) = te^{-\phi(t)}$ for all $p \in \mathbb{N}_0$.

(ii) the Legendre sequence: $L_p(\phi) = \sup_{t>0} t^p e^{-\phi(t)}$ for all $p \in \mathbb{N}_0$ (compare with the log-convex minorant in Proposition 1.1.23).

We have that both sequences are (lc) and we have that

$$\frac{L_{p+1}}{p+1} \leq M_p \leq \frac{\sqrt{L_1}}{p}\sqrt{L_{2p+1}}, \qquad p \in \mathbb{N}.$$

Hence, if the lower Matuszewska index of ϕ is positive, $\beta(\phi) > 0$, we deduce that it exists $H \geq 1$ such that $\phi(t) \leq \phi(Ht) + H$ and, as in the proof of Lemma 1.1.24, we see that $\mathbb{L}$ has (mg) and we conclude that $\mathbb{M} \approx \mathbb{L}$.

4.1.2 Generalized Laplace and Borel transforms

In this subsection, it will be shown how Laplace- and Borel-like transforms are defined from the $\mathbb{M}-$summability kernels, summarizing their main properties. The first definition resembles that of functions of exponential growth of order $1/k$, playing a fundamental role when dealing with Laplace and Borel transforms in $k-$summability for Gevrey classes. For convenience, we will say a holomorphic function f in a sector S is *continuous at the origin* if $\lim_{z\to 0,\ z\in T} f(z)$ exists for every $T \ll S$.

Definition 4.1.10. Let $\mathbb{M}$ be a weight sequence, and consider an unbounded sector S in $\mathcal{R}$. The set $\mathcal{O}^{\mathbb{M}}(S)$ consists of the holomorphic functions f in S, continuous at the origin and having $\mathbb{M}-$growth in S, i.e. such that for every unbounded proper subsector T of S, we write $T \prec\!\!\prec S$, there exist $r, c, k > 0$ such that for every $z \in T$ with $|z| \geq r$ one has

$$|f(z)| \leq \frac{c}{h_{\mathbb{M}}(k/|z|)} = ce^{\omega_{\mathbb{M}}(|z|/k)}. \tag{4.6}$$

Remark 4.1.11. Since continuity at 0 has been asked for, $f \in \mathcal{O}^{\mathbb{M}}(S)$ implies that for every $T \prec\!\!\prec S$ there exist $c, k > 0$ such that for every $z \in T$ with $|z| < r$ one has (4.6).

We are ready for the introduction of the $e-$Laplace transform. Given a sector $S = S(d,\alpha)$, a kernel e for $\mathbb{M}-$summability and $f \in \mathcal{O}^{\mathbb{M}}(S)$, for any direction τ in S we define the operator

$T_{e,\tau}$ sending f to its $e-$*Laplace transform in direction* τ, defined as

$$(T_{e,\tau}f)(z) := \int_0^{\infty(\tau)} e(u/z)f(u)\frac{du}{u}, \quad |\arg(z)-\tau| < \omega(\mathbb{M})\pi/2, \ |z| \text{ small enough}, \tag{4.7}$$

where the integral is taken along the half-line parametrized by $t \in (0,\infty) \mapsto te^{i\tau}$. We have the following result.

Proposition 4.1.12 ([60], Prop. 3.11)**.** For a sector $S = S(d,\alpha)$ and $f \in \mathcal{O}^{\mathbb{M}}(S)$, the family $\{T_{e,\tau}f\}_{\tau \,\text{in}\, S}$ defines a holomorphic function $T_e f$ in a sectorial region $G(d, \alpha + \omega(\mathbb{M}))$.

We now define the generalized Borel transforms.

Definition 4.1.13. Suppose $\omega(\mathbb{M}) < 2$, and let $G = G(d,\alpha)$ be a sectorial region with $\alpha > \omega(\mathbb{M})$, and $f : G \to \mathbb{C}$ be holomorphic in G and continuous at 0. For $\tau \in \mathbb{R}$ such that $|\tau - d| < (\alpha - \omega(\mathbb{M}))\pi/2$ we may consider a path $\delta_{\omega(\mathbb{M})}(\tau)$ in G like the ones used in the classical Borel transform, consisting of a segment from the origin to a point z_0 with $\arg(z_0) = \tau + \omega(\mathbb{M})(\pi+\varepsilon)/2$ (for some suitably small $\varepsilon \in (0,\pi)$), then the circular arc $|z| = |z_0|$ from z_0 to the point z_1 on the ray $\arg(z) = \tau - \omega(\mathbb{M})(\pi+\varepsilon)/2$ (traversed clockwise), and finally the segment from z_1 to the origin.

Given kernels e, E for $\mathbb{M}-$summability, we define the operator $T^-_{e,\tau}$ sending f to its $e-$*Borel transform in direction* τ, defined as

$$(T^-_{e,\tau}f)(u) := \frac{-1}{2\pi i}\int_{\delta_{\omega(\mathbb{M})}(\tau)} E(u/z)f(z)\frac{dz}{z}, \quad u \in S(\tau,\varepsilon_0), \quad \varepsilon_0 \text{ small enough.}$$

Proposition 4.1.14 ([60], Prop. 3.12)**.** For $G = G(d,\alpha)$ and $f : G \to \mathbb{C}$ as above, the family

$$\{T^-_{e,\tau}f\}_\tau,$$

where τ is a real number such that $|\tau - d| < (\alpha - \omega(\mathbb{M}))\pi/2$, defines a holomorphic function $T^-_e f$ in the sector $S = S(d, \alpha - \omega(\mathbb{M}))$. Moreover, $T^-_e f$ is of $\mathbb{M}-$growth in S.

Remark 4.1.15. In case $\omega(\mathbb{M}) \geq 2$, choose $s > 0$ and a kernel $\tilde{e}$ for $\mathbb{M}^{(1/s)}-$summability as in Remark 4.1.3.(ii), and let $T^-_{\tilde{e},\tau}$ be defined as before, where the kernel under the integral sign is the function $\tilde{E}$ given in (4.3). Then, if ϕ_s is the operator sending a function f to the function $f(z^s)$, we define $T^-_{e,\tau}$ by the identity

$$\phi_s \circ T^-_{e,\tau} = T^-_{\tilde{e},\tau} \circ \phi_s,$$

in the same way as in [7, p. 90].

One can compute the $e-$transforms of a monomial.

Proposition 4.1.16 ([60], p. 1187)**.** Given $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$, the function $f_\lambda(z) = z^\lambda$ belongs to the space $\mathcal{O}^{\mathbb{M}}(S)$ and we have that

$$T_e f_\lambda(z) = \int_0^\infty e(t) z^{\lambda-1} t^{\lambda-1} z dt = m_e(\lambda) z^\lambda,$$

$$T^-_e f_\lambda(u) = \frac{u^\lambda}{m_e(\lambda)}.$$

The last proposition justifies the forthcoming definition of formal Laplace and Borel transforms.

Definition 4.1.17. Given a sequence $\mathbb{M}$ and a kernel of $\mathbb{M}-$summability e, the formal $e-$Laplace transform $\hat{T}_e : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ is given by

$$\hat{T}_e\big(\sum_{p=0}^{\infty} a_p z^p\big) := \sum_{p=0}^{\infty} m_e(p) a_p z^p.$$

Accordingly, we define the formal $e-$Borel transform $\hat{T}_e^- : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ by

$$\hat{T}_e^-\big(\sum_{p=0}^{\infty} a_p z^p\big) := \sum_{p=0}^{\infty} \frac{a_p}{m_e(p)} z^p.$$

The operators $\hat{T}_e$ and $\hat{T}_e^-$ are inverse to each other.

The next result lets us know how these analytic and formal transforms interact with general asymptotic expansions. Given two sequences of positive real numbers $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ and $\mathbb{M}' = (M'_p)_{p\in\mathbb{N}_0}$, we consider the sequences $\mathbb{M}\cdot\mathbb{M}' = (M_p M'_p)_{n\in\mathbb{N}_0}$ and $\mathbb{M}'/\mathbb{M} = (M'_p/M_p)_{p\in\mathbb{N}_0}$.

Theorem 4.1.18 ([60], Th. 3.16)**.** Suppose $\mathbb{M}$ is a sequence and e is a kernel of $\mathbb{M}-$summability. For any sequence $\mathbb{M}'$ of positive real numbers the following hold:

(i) If $f \in \mathcal{O}^{\mathbb{M}}(S(d,\alpha))$ and $f \sim_{\mathbb{M}'} \hat{f}$, then $T_e f \sim_{\mathbb{M}\cdot\mathbb{M}'} \hat{T}_e \hat{f}$ in a sectorial region $G(d, \alpha + \omega(\mathbb{M}))$.

(ii) If $f \sim_{\mathbb{M}'} \hat{f}$ in a sectorial region $G(d,\alpha)$ with $\alpha > \omega(\mathbb{M})$, then $T_e^- f \sim_{\mathbb{M}'/\mathbb{M}} \hat{T}_e^- \hat{f}$ in the sector $S(d, \alpha - \omega(\mathbb{M}))$.

Note that if both sequences are weight sequences, $\mathbb{M}\cdot\mathbb{M}'$ is again a weight sequence, but $\mathbb{M}'/\mathbb{M}$ might not be. This last theorem motivates the study of the quotient and the product sequence achieved in Subsection 4.2.2.

4.1.3 $\mathbb{M}-$summability and $e-$summability

With the tools presented in the previous subsections, we are ready to give a definition of summability in a direction with respect to a kernel e of $\mathbb{M}-$summability. We recall that $\mathfrak{m}_e$ is also strongly regular and equivalent to $\mathbb{M}$ (see Proposition 4.1.8 and Remark 4.1.9), so, on one hand, $\mathbb{C}[[z]]_{\mathbb{M}} = \mathbb{C}[[z]]_{\mathfrak{m}_e}$ and, on the other hand, it makes sense to consider the space $\mathcal{O}^{\mathfrak{m}_e}(S)$ for any unbounded sector S and, moreover, $\mathcal{O}^{\mathfrak{m}_e}(S) = \mathcal{O}^{\mathbb{M}}(S)$ (see (1.7)).

Definition 4.1.19. Let e be a kernel of $\mathbb{M}-$summability. We say $\hat{f} = \sum_{p\ge 0} a_p z^p$ is *$e-$summable in direction $d \in \mathbb{R}$* if:

(i) $\hat{f} \in \mathbb{C}[[z]]_{\mathfrak{m}_e}$, so $g := \hat{T}_e^- \hat{f} = \displaystyle\sum_{p\ge 0} \frac{a_p}{m_e(p)} z^p$ converges in a disc and

(ii) g admits analytic continuation in a sector $S = S(d,\varepsilon)$ for some $\varepsilon > 0$, and $g \in \mathcal{O}^{\mathfrak{m}_e}(S)$.

The next result states the equivalence between $\mathbb{M}-$summability and $e-$summability in a direction, and provides a way to recover the $\mathbb{M}-$sum in a direction of a summable power series by means of the formal and analytic transforms previously introduced.

Theorem 4.1.20 ([60], Th. 3.18)**.** Given a weight sequence $\mathbb{M}$, a direction $d \in \mathbb{R}$ and a formal power series $\hat{f} = \sum_{p\ge 0} a_p z^p$, the following are equivalent:

(i) $\hat{f}$ is $\mathbb{M}$−summable in direction d.

(ii) For every kernel e of $\mathbb{M}$−summability, $\hat{f}$ is e−summable in direction d.

(iii) For some kernel e of $\mathbb{M}$−summability, $\hat{f}$ is e−summable in direction d.

In case any of the previous holds, we deduce from the Watson's Lemma that we have (after analytic continuation)

$$\mathcal{S}_{\mathbb{M},d}\hat{f} = T_e(\hat{T}_e^- \hat{f}) \tag{4.8}$$

for any kernel e of $\mathbb{M}$−summability.

Remark 4.1.21. In case $\mathbb{M} = \mathbb{M}_{1/k}$, the summability methods described are just the classical k−summability and e−summability (in a direction) for kernels e of order $k > 0$, as defined by W. Balser.

Finally, we will present some basic properties of $\mathbb{M}$−summable series. As a consequence of Watson's Lemma and the basic properties of asymptotics, we have that:

Lemma 4.1.22. Let $\mathbb{M}$ be a weight sequence, the following holds:

(i) Let $\hat{f}$ be convergent. Then for every d, the series $\hat{f}$ is $\mathbb{M}$−summable in direction d and $\mathcal{S}_{\mathbb{M},d}\hat{f}(z) = \mathcal{S}\hat{f}(z)$ for every z where both sides are defined, where $\mathcal{S}$ maps each convergent power series to its natural sum.

(ii) If $\hat{f}$ is $\mathbb{M}$−summable in direction d for every $d \in (\alpha, \beta)$ with $\alpha < \beta$, then

$$\mathcal{S}_{\mathbb{M},d_1}\hat{f}(z) = \mathcal{S}_{\mathbb{M},d_2}\hat{f}(z), \qquad d_1, d_2 \in (\alpha, \beta)$$

where both functions are defined.

(iii) Let $\hat{f}$ be $\mathbb{M}$−summable in direction d, there exists $\varepsilon > 0$ such that $\hat{f}$ is $\mathbb{M}$−summable in all directions $\tilde{d}$ with $|\tilde{d} - d| < \varepsilon$.

(iv) For $\tilde{d} = d + 2\pi$, the $\mathbb{M}$−summability of $\hat{f}$ in direction d is equivalent to the $\mathbb{M}$−summability of $\hat{f}$ in direction $\tilde{d}$. Moreover, we have that

$$\mathcal{S}_{\mathbb{M},\tilde{d}}\hat{f}(z) = \mathcal{S}_{\mathbb{M},d}\hat{f}(ze^{-2\pi i}),$$

where both functions are defined.

Remark 4.1.23. In particular (i) in the last Lemma says that our summability method is *regular*, that is, if the ordinary sum exists, then the sum in (4.8) also exists and with the same value.

From Lemma 4.1.22.(iv), we know that we can identify directions d that differ by integer multiples of 2π. By Lemma 4.1.22.(iii), we know that the set of directions for which the formal power series is not $\mathbb{M}$−summable is closed, specially interesting is the case in which this set is finite (mod 2π).

Definition 4.1.24. Let $\mathbb{C}\{z\}_{\mathbb{M},d}$ be the set of formal power series $\hat{f}$ which are $\mathbb{M}$−summable in direction d. Let $\mathbb{C}\{z\}_{\mathbb{M}}$ be the set of $\mathbb{M}$−summable formal power series $\hat{f}$ which are $\mathbb{M}$−summable in every direction except for a finite set of directions (mod 2π), denoted by $\operatorname{sing}(\hat{f}) = \{d_1, \ldots, d_m\}$.

Please note that $\mathbb{C}\{z\} \subseteq \mathbb{C}\{z\}_{\mathbb{M},d} \subseteq \mathbb{C}[[z]]_{\mathbb{M}}$. We have the following properties of these sets. Only the proof of (i) is indicated due to its importance in the proof of the tauberian results.

Proposition 4.1.25. Let $\mathbb{M}$ be a weight sequence. Then,

(i) If $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}}$ and $\text{sing}(\hat{f}) = \emptyset$, then $\hat{f}$ is convergent.

(ii) $\mathbb{C}\{z\}_{\mathbb{M},d}$, $\mathbb{C}\{z\}_{\mathbb{M}}$ are algebras. Moreover, if $\mathbb{M}$ is (dc) they are differential algebras.

Proof. (i) Using Lemma 4.1.22.(ii), we know that $f(z) := \mathcal{S}_{\mathbb{M},d}\hat{f}(z)$ is well-defined (independent from direction d). By Lemma 4.1.22.(iv) we know that f is single-valued, then f can be expanded into a convergent power series about the origin. By the uniqueness of the asymptotic expansion, we deduce that $\hat{f}$ coincides with this convergent power series.

□

4.2 Tauberian theorems

Classical Tauberian theorems ([71, Th. 2.2.4.2] and [7, Th. 37]) serve to compare the processes of $k-$summability for various k; these theorems are strongly related to multisummability. In this section, we will see what can be said about the connection between the algebras $\mathbb{C}\{z\}_{\mathbb{M}}$ and $\mathbb{C}\{z\}_{\mathbb{L}}$ for two given sequences $\mathbb{M}$ and $\mathbb{L}$. For this purpose in the first subsection we will thoroughly examine the comparability notion introduce at the beginning of the dissertation. Secondly, we will analyze the properties of the quotient and the product sequences of $\mathbb{M}$ and $\mathbb{L}$. Finally, with all these tools, we will formulate our main results, generalizing the Gevrey case if $\omega(\mathbb{M}) < \omega(\mathbb{L})$, and if $\omega(\mathbb{M}) = \omega(\mathbb{L})$ showing that such a generalization is not possible for our definition of summability. The results in this section are stated for a couple of sequences but can be easily extended for a finite set of sequences $\mathbb{M}_1, \mathbb{M}_2, \dots, \mathbb{M}_k$.

4.2.1 Comparison of sequences

In this subsection, we want to see whether or not it is possible to compare two different sequences in such a way that the Tauberian theorems are available, so a multisummability notion, that generalizes the Gevrey situation, can be formulated. The example at the end of the subsection will show that, in general, we can not always determine which of two given weight sequences, i.e., (lc) with quotients tending to infinity, $\mathbb{M}$ and $\mathbb{L}$ is greater, not even for sequences whose quotients are regularly varying which implies the admissibility of a nonzero proximate order (see Remark 2.2.18). Hence, it will be natural to impose some comparability condition between $\mathbb{M}$ and $\mathbb{L}$ in the forthcoming results.

Let $\mathbb{M}$ and $\mathbb{L}$ be sequences, we recall (see Definition 1.1.12) that $\mathbb{M} \precsim \mathbb{L}$ if there exists some positive constant $A > 0$ such that

$$M_p \leq A^p L_p, \qquad \text{for all } \ p \in \mathbb{N}_0.$$

We say that $\mathbb{M}$ and $\mathbb{L}$ are comparable if $\mathbb{M} \precsim \mathbb{L}$ or $\mathbb{L} \precsim \mathbb{M}$ holds. If both conditions hold, we say that $\mathbb{M}$ is equivalent to $\mathbb{L}$, and we write $\mathbb{M} \approx \mathbb{L}$. We also remind that if $\mathbb{M} \approx \mathbb{L}$ then the corresponding classes of functions are the same (see Remark 3.1.4) and $\omega(\mathbb{M}) = \omega(\mathbb{L})$ (see Remark 2.1.32). Since equivalent sequences define the same classes, we are particularly interested in comparable but not equivalent sequences, i.e., $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$, which is true if and only if

$$\inf_{p\in\mathbb{N}} \left(\frac{L_p}{M_p}\right)^{1/p} = 0, \qquad \text{and} \quad \sup_{p\in\mathbb{N}} \left(\frac{L_p}{M_p}\right)^{1/p} < \infty,$$

or, equivalently, if

$$\liminf_{p\to\infty}\left(\frac{L_p}{M_p}\right)^{1/p}=0,\qquad\text{and}\qquad \limsup_{p\to\infty}\left(\frac{L_p}{M_p}\right)^{1/p}<\infty.$$

In other words, we want to avoid noncomparable sequences, that is,

$$\liminf_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p}=0,\qquad\text{and}\qquad \limsup_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p}=\infty. \tag{4.9}$$

For the construction of our example of noncomparable sequences, we need to characterize (4.9) in terms of the corresponding associated functions (see (1.4)). For any weight sequence $\mathbb{M}$ admitting a nonzero proximate order, by Lemma 1.2.12 and Remark 1.2.31, we know that the associated function $\omega_{\mathbb{M}}(t)$ is O-regularly varying with all the Matuszewska indices and the orders coinciding with $1/\omega(\mathbb{M})\in(0,\infty)$ and we deduce the next Lemma.

Lemma 4.2.1. Let $\mathbb{M}$ be a weight sequence admitting a nonzero proximate order. Then for any $A>0$ there exist $t_A,E,F>0$ such that

$$\omega_{\mathbb{M}}(Et)<A\omega_{\mathbb{M}}(t)<\omega_{\mathbb{M}}(Ft),\qquad t>t_A,$$

and for any $B>0$ there exist constants $t_B,G,H>0$ such that

$$G\omega_{\mathbb{M}}(t)<\omega_{\mathbb{M}}(Bt)<H\omega_{\mathbb{M}}(t),\qquad t>t_B.$$

The characterization of comparability in terms of the associated function is stated below in the most regular case, i.e., for a sequence $\mathbb{M}$ admitting a nonzero proximate order, because these are the ones used to develop the summability theory, but Lemma 4.2.1 is still valid if weaker conditions are satisfied by $\mathbb{M}$ (see [90, Lemmma 3.18] by G. Schindl). Hence, the next proposition also holds for simpler but less regular sequences, for instance strongly regular, and examples of noncomparability can be constructed in a similar way.

Proposition 4.2.2. Let $\mathbb{M}$ and $\mathbb{L}$ be two weight sequences such that $\mathbb{M}$ admits a nonzero proximate order. We have that

$$\liminf_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p}=0\qquad\text{if and only if}\qquad \liminf_{t\to\infty}\frac{\omega_{\mathbb{L}}(t)}{\omega_{\mathbb{M}}(t)}=0,$$

and

$$\limsup_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p}=\infty\qquad\text{if and only if}\qquad \limsup_{t\to\infty}\frac{\omega_{\mathbb{L}}(t)}{\omega_{\mathbb{M}}(t)}=\infty.$$

Proof. If we suppose that $\liminf_{t\to\infty}\omega_{\mathbb{L}}(t)/\omega_{\mathbb{M}}(t)>0$, then there exists $A>0$ such that $\omega_{\mathbb{L}}(t)\geq A\omega_{\mathbb{M}}(t)$ for every $t\geq t_0$. By Proposition 1.1.23, we have that

$$\liminf_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p}=\liminf_{p\to\infty}\left(\frac{M_p}{\sup_{t>0}(t^p/e^{\omega_{\mathbb{L}}(t)})}\right)^{1/p}\geq\liminf_{p\to\infty}\left(\frac{M_p}{\sup_{t>0}(t^p/e^{A\omega_{\mathbb{M}}(t)})}\right)^{1/p}.$$

By Lemma 4.2.1, there exists $E>0$ such that $A\omega_{\mathbb{M}}(t)>\omega_{\mathbb{M}}(Et)$ for t large enough. It is easy to check that the supremum for $t>0$ of the function $f_{p,\mathbb{M}}(t)=t^pe^{-A\omega_{\mathbb{M}}(t)}$ is attained in $[m_{\lfloor p/A\rfloor},\infty)$, so for p large enough we have that

$$\sup_{t>0}(t^p/e^{A\omega_{\mathbb{M}}(t)})<\sup_{t>0}(t^p/e^{\omega_{\mathbb{M}}(Et)}),$$

and we deduce that

$$\liminf_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p} > E > 0.$$

Now, if we suppose that $\liminf_{p\to\infty}(M_p/L_p)^{1/p} > 0$, then there exists $B > 0$ such that $M_p \geq B^p L_p$ for every $p \in \mathbb{N}$. Consequently, $\omega_{\mathbb{L}}(t) \geq \omega_{\mathbb{M}}(Bt)$ for every $t > m_0$ and, using Lemma 4.2.1, we have that

$$\liminf_{t\to\infty}\frac{\omega_{\mathbb{L}}(t)}{\omega_{\mathbb{M}}(t)} \geq \liminf_{t\to\infty}\frac{\omega_{\mathbb{M}}(Bt)}{\omega_{\mathbb{M}}(t)} \geq G > 0.$$

The same arguments lead to the other equivalence. □

We will use the construction of sequences from proximate orders, described in Subsection 2.2.3, in order to build noncomparable sequences from noncomparable proximate orders. The main advantage of this procedure is that it may be more suitable to work with proximate orders rather than directly with sequences.

Definition 4.2.3. Two proximate orders $\rho_1(t)$ and $\rho_2(t)$ are said to be *noncomparable* if the functions $V_1(t) = t^{\rho_1(t)}$ and $V_2(t) = t^{\rho_2(t)}$ satisfy that

$$\liminf_{t\to\infty}\frac{V_1(t)}{V_2(t)} = 0 \qquad \text{and} \qquad \limsup_{p\to\infty}\frac{V_1(t)}{V_2(t)} = \infty.$$

If $\rho_1, \rho_2 > 0$ are the corresponding values of their limit at infinity and $\rho_1 \neq \rho_2$, using the property (C) of proximate orders (see Remark 1.2.8), one can show that the proximate orders are comparable.

Example 4.2.4. We consider the following functions

$$\rho_1(t) = 1, \quad t \in (0,\infty),$$

$$\rho_2(t) = 1 + \frac{\sin(\log_2(t))}{\log_2(t)}, \quad t \in (e,\infty), \quad \log_2(t) := \log(\log(t)).$$

The function $\rho_1(t)$ is evidently a nonzero proximate order and it is easy to check that $\rho_2(t)$ verifies conditions (A), (B) and (C). Since

$$\rho_2'(t) = \frac{\cos(\log_2(t))\log_2(t) - \sin(\log_2(t))}{t\log(t)(\log_2(t))^2},$$

we see that

$$\lim_{t\to\infty}\rho_2'(t)t\log(t) = \lim_{t\to\infty}\left(\frac{\cos(\log_2(t))}{\log_2(t)} - \frac{\sin(\log_2(t))}{(\log_2(t))^2}\right) = 0,$$

so $\rho_2(t)$ is a nonzero proximate order. We consider the sequences

$$r_n = \exp(\exp(\pi/2 + 2\pi n)), \qquad s_n = \exp(\exp(3\pi/2 + 2\pi n)), \qquad n \in \mathbb{N}.$$

We write $V_1(t) = t^{\rho_1(t)}$ and $V_2(t) = t^{\rho_2(t)}$ and we observe that

$$\lim_{n\to\infty}\frac{V_2(s_n)}{V_1(s_n)} = 0, \qquad \lim_{n\to\infty}\frac{V_2(r_n)}{V_1(r_n)} = \infty.$$

Hence, the proximate orders $\rho_1(t)$ and $\rho_2(t)$ are noncomparable.

We fix $\gamma > 0$, $\tilde{V}_1 \in MF(\gamma, \rho_1(t))$ and $\tilde{V}_2 \in MF(\gamma, \rho_2(t))$ and we consider the sequences $\mathbb{U}_1$ and $\mathbb{U}_2$ (see Definition 2.2.11) defined from the corresponding inverse functions $\tilde{U}_1(t)$ and $\tilde{U}_2(t)$. These sequences $\mathbb{U}_1$ and $\mathbb{U}_2$ are regularly varying of index 1, so the functions $d_{\mathbb{U}_j}(t)$ are nonzero proximate orders. According to Theorem 1.2.16.(VI) and Remark 2.2.13 for $j = 1, 2$ we see that

$$0 < \liminf_{t\to\infty} \frac{\omega_{\mathbb{U}_j}(t)}{V_j(t)} \leq \limsup_{t\to\infty} \frac{\omega_{\mathbb{U}_j}(t)}{V_j(t)} < \infty.$$

Since $\rho_1(t)$ and $\rho_2(t)$ are noncomparable, we deduce that

$$\liminf_{t\to\infty} \frac{\omega_{\mathbb{U}_2}(t)}{\omega_{\mathbb{U}_1}(t)} = 0, \qquad \text{and} \qquad \limsup_{p\to\infty} \frac{\omega_{\mathbb{U}_2}(t)}{\omega_{\mathbb{U}_1}(t)} = \infty.$$

Finally, by Proposition 4.2.2, we conclude that $\mathbb{U}_1$ and $\mathbb{U}_2$ are noncomparable.

4.2.2 Product and quotient of sequences

In the study of multisummability in this general context there naturally appear the product sequence $\mathbb{M}\cdot\mathbb{L} := (M_pL_p)_{p\in\mathbb{N}_0}$ and the quotient sequence $\mathbb{M}/\mathbb{L} := (M_p/L_p)_{p\in\mathbb{N}_0}$ of two sequences $\mathbb{M}$ and $\mathbb{L}$. In this subsection, some elementary properties of these sequences will be obtained and the connection with the comparability notion in the last subsection will be established. Since many of these properties are stated in terms of the sequence of quotients, note that the corresponding ones for $\mathbb{M}\cdot\mathbb{L}$ and $\mathbb{M}/\mathbb{L}$ are $\mathbf{m}\cdot\boldsymbol{\ell} = (m_p\ell_p)_{p\in\mathbb{N}_0}$ and $\mathbf{m}/\boldsymbol{\ell} = (m_p/\ell_p)_{p\in\mathbb{N}_0}$, respectively.

Proposition 4.2.5. Suppose given two weight sequences $\mathbb{M}$ and $\mathbb{L}$, each one admitting a nonzero proximate order. Then, $\mathbb{M}\cdot\mathbb{L}$ is a weight sequence and it admits a nonzero proximate order. In this situation, we have that $\omega(\mathbb{M}\cdot\mathbb{L}) = \omega(\mathbb{M}) + \omega(\mathbb{L})$.

Proof. This is immediate using Theorems 2.2.6 and 2.2.17 and the stability of (lc), regular variation and the index of regular variation for the product. □

Recall that, for sequences admitting a nonzero proximate order, the orders and the Matuszewska indices are all equal to $\omega(\mathbb{M}) \in (0,\infty)$ (see Remark 2.2.18).

Remark 4.2.6. We observe that the product sequence of two sequences also preserves some weaker properties. In particular, if $\mathbb{M}$ and $\mathbb{L}$ are strongly regular sequences then $\mathbb{M}\cdot\mathbb{L}$ is strongly regular and $\omega(\mathbb{M})+\omega(\mathbb{L}) \leq \omega(\mathbb{M}\cdot\mathbb{L})$. However, the equality $\omega(\mathbb{M}\cdot\mathbb{L}) = \omega(\mathbb{M})+\omega(\mathbb{L})$ is not always valid, such an example can be constructed with the techniques described in Remark 2.2.27.

Remark 4.2.7. If there exists $a > 0$ such that the sequences of quotients associated with M_p and L_p satisfy

$$a^{-1}\ell_p \leq m_p \leq a\ell_p, \qquad p \in \mathbb{N}_0,$$

then $\mathbb{M} \approx \mathbb{L}$. Consequently, if $(\ell_p)_{p\in\mathbb{N}_0}$ and $(m_p)_{p\in\mathbb{N}_0}$ are equivalent in the classical sense, that is, $\lim_{p\to\infty} \ell_p/m_p = 1$, then we also have that $\mathbb{M} \approx \mathbb{L}$.

The main difficulty when dealing with the quotient sequence is to ensure that it satisfies (lc). Applying Theorem 1.2.41 of R. Bojanic and E. Seneta, we will solve this problem, if the sequences considered are regular enough, by switching $\mathbb{M}/\mathbb{L}$ for an equivalent sequence.

Proposition 4.2.8. Given two weight sequences $\mathbb{M}$ and $\mathbb{L}$, each one admitting a nonzero proximate order, assume that $\omega(\mathbb{L}) < \omega(\mathbb{M})$. Then it exists a weight sequence $\mathbb{A}$ equivalent to $\mathbb{M}/\mathbb{L}$ whose sequence of quotients is regularly varying with index $\omega(\mathbb{M}) - \omega(\mathbb{L})$.

Proof. By Theorem 2.2.6 and 2.2.17, we know that there exist weight sequences $\mathbb{L}'$ and $\mathbb{M}'$ equivalent to $\mathbb{L}$ and $\mathbb{M}$, respectively, whose sequences of quotients $(\ell'_p)_{p\in\mathbb{N}_0}$ and $(m'_p)_{p\in\mathbb{N}_0}$ are regularly varying with indices $\omega(\mathbb{L})$ and $\omega(\mathbb{M})$.

Applying Theorem 1.2.41 there exist sequences of positive real numbers $(\ell''_p)_{p\in\mathbb{N}_0}$ and $(m''_p)_{p\in\mathbb{N}_0}$ equivalent in the classical sense to $(\ell'_p)_{p\in\mathbb{N}_0}$ and $(m'_p)_{p\in\mathbb{N}_0}$, respectively, i.e, $\lim_{p\to\infty}\ell'_p/\ell''_p=1$ and $\lim_{p\to\infty}m'_p/m''_p=1$, and satisfying (1.15). It is plain to check that the sequence $\boldsymbol{b}=(b_p:=m''_p/\ell''_p)_{p\in\mathbb{N}_0}$ is regularly varying of index $\omega(\mathbb{M})-\omega(\mathbb{L})$. By (1.15), we observe that

$$\frac{b_{p+1}}{b_p}=\frac{m''_{p+1}}{m''_p}\frac{\ell''_p}{\ell''_{p+1}}=\left(1+\frac{\omega(\mathbb{M})}{p}+o\left(\frac{1}{p}\right)\right)\frac{1}{1+\omega(\mathbb{L})/p+o(1/p)}.$$

We take $\alpha\in(\omega(\mathbb{L}),\omega(\mathbb{M}))$, then there exists $p_0\in\mathbb{N}$ such that

$$\frac{b_{p+1}}{b_p}>\left(1+\frac{\alpha}{p}\right)\frac{1}{(1+\alpha/p)}=1,\qquad p\geq p_0.$$

We define the sequence $a_p:=b_{p_0}$ for $p<p_0$ and $a_p:=b_p$ for $p\geq p_0$. The sequence $\boldsymbol{a}$ is nondecreasing and regularly varying of index $\omega(\mathbb{M})-\omega(\mathbb{L})$ and $\boldsymbol{a}\simeq\boldsymbol{b}$.

Consequently, the corresponding sequence $\mathbb{A}$ is a weight sequence with $\mathbb{A}\approx\mathbb{B}=\mathbb{M}''/\mathbb{L}''$ (see Proposition 1.1.15). Since $\mathbf{m}'/\boldsymbol{\ell}'\sim\mathbf{m}''/\boldsymbol{\ell}''$, we have that $\mathbf{m}'/\boldsymbol{\ell}'\simeq\mathbf{m}''/\boldsymbol{\ell}''$, so $\mathbb{A}\approx\mathbb{M}'/\mathbb{L}'$. Finally, using that $\mathbb{L}'$ and $\mathbb{M}'$ are equivalent to $\mathbb{L}$ and $\mathbb{M}$, we conclude that the proposition holds. □

Remark 4.2.9. If $\omega(\mathbb{L})<\omega(\mathbb{M})$, using the above proposition, we change $\mathbb{M}$ for the equivalent sequence $\mathbb{A}\cdot\mathbb{L}$ which is (lc) and admits a nonzero proximate order. Without loss of generality, we can always assume that $\mathbb{M}/\mathbb{L}$ is (lc) and that its sequence of quotients is regularly varying of positive index.

Some information about the behavior of these sequences can be obtained even if the conditions on $\mathbb{M}$ and $\mathbb{L}$ are relaxed. In particular, we observe that the indices $\omega(\mathbb{M}\cdot\mathbb{L})$ and $\omega(\mathbb{M}/\mathbb{L})$ can be computed from $\omega(\mathbb{M})$ and $\omega(\mathbb{L})$.

Remark 4.2.10. Assume that $\mathbb{M}$ and $\mathbb{L}$ are weight sequences, such that $\mathbb{L}$ satisfies (2.23), which is guaranteed in case $\mathbb{L}$ admits a nonzero proximate order (see Remark 2.2.18), then

$$\omega(\mathbb{M}\cdot\mathbb{L})=\liminf_{p\to\infty}\frac{\log(m_p\ell_p)}{\log(p)}=\liminf_{p\to\infty}\frac{\log(m_p)}{\log(p)}+\lim_{p\to\infty}\frac{\log(\ell_p)}{\log(p)}=\omega(\mathbb{M})+\omega(\mathbb{L})\in[0,\infty],$$

$$\omega(\mathbb{M}/\mathbb{L})=\liminf_{p\to\infty}\frac{\log(m_p/\ell_p)}{\log(p)}=\liminf_{p\to\infty}\frac{\log(m_p)}{\log(p)}+\lim_{p\to\infty}-\frac{\log(\ell_p)}{\log(p)}=\omega(\mathbb{M})-\omega(\mathbb{L})\in\mathbb{R}.$$

Finally, the following proposition shows that $\mathbb{L}\precsim\mathbb{M}$ and $\mathbb{L}\not\approx\mathbb{M}$ (comparability but not equivalence) can be characterized in terms of the quotient sequence.

Proposition 4.2.11. Given two sequences $\mathbb{M}$ and $\mathbb{L}$, we have that

(i) if the sequence of quotients of $\mathbb{M}/\mathbb{L}$ tends to infinity, then $\lim_{p\to\infty}(M_p/L_p)^{1/p}=\infty$ and $\mathbb{L}\precsim\mathbb{M}$ and $\mathbb{L}\not\approx\mathbb{M}$.

(ii) Assume that $\mathbb{M}/\mathbb{L}$ is (lc). Then

(ii.a) $\mathbb{L}\precsim\mathbb{M}$ and $\mathbb{L}\not\approx\mathbb{M}$ if and only if $\lim_{p\to\infty}(M_p/L_p)^{1/p}=\infty$.

(ii.b) $\mathbb{L}\precsim\mathbb{M}$ and $\mathbb{L}\not\approx\mathbb{M}$ if and only if the sequence of quotients of $\mathbb{M}/\mathbb{L}$ tends to infinity.

(iii) Assume that $\mathbb{L}$ is a weight sequence satisfying (2.23), i.e., $\omega(\mathbb{L}) = \mu(\boldsymbol{\ell}) = \rho(\boldsymbol{\ell})$ and that $\omega(\mathbb{L}) < \omega(\mathbb{M})$. Then the sequence of quotients of $\mathbb{M}/\mathbb{L}$ tends to infinity, $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$ and $\lim_{p\to\infty}(M_p/L_p)^{1/p} = \infty$.

Proof. (i) If $\mathbf{m}/\boldsymbol{\ell}$ tends to infinity, then for any $K > 0$ it exists $p_0 \in \mathbb{N}$ such that $m_p/\ell_p \geq K$ for every $p \geq p_0$. Hence, we deduce that

$$\left(\frac{M_p}{L_p}\right)^{1/p} \geq \left(\frac{K^{p-p_0} M_{p_0}}{L_{p_0}}\right)^{1/p}, \qquad p \geq p_0.$$

Taking limit inferior in both sides we see that $\liminf_{p\to\infty}(M_p/L_p)^{1/p} \geq K$. Since this is true for any $K > 0$ we conclude that

$$\lim_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p} = \liminf_{p\to\infty}\left(\frac{M_p}{L_p}\right)^{1/p} = \infty$$

which implies that $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$.

(ii.a) If $\lim_{p\to\infty}(M_p/L_p)^{1/p} = \infty$, we immediately get that $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$.

Conversely, using Lemma 1.1.7.(vi), if $\mathbb{M}/\mathbb{L}$ is (lc), we obtain that $((M_p/L_p)^{1/p})_{p\in\mathbb{N}}$ is nondecreasing. Hence, from the fact that $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$ we show that $((M_p/L_p)^{1/p})_{p\in\mathbb{N}}$ tends ∞.

(ii.b) Since $\mathbb{M}/\mathbb{L}$ is (lc), it is immediate from (ii.a) and Lemma 1.1.7.(vii).

(iii) We have that

$$\liminf_{p\to\infty}\frac{\log(m_p)}{\log(p)} = \omega(\mathbb{M}), \qquad \lim_{p\to\infty}\frac{\log(\ell_p)}{\log(p)} = \omega(\mathbb{L}).$$

Since $\omega(\mathbb{L}) < \omega(\mathbb{M})$ we can fix $0 < \varepsilon < (\omega(\mathbb{M}) - \omega(\mathbb{L}))/2$ and we observe that it exists $p_0 \in \mathbb{N}$ such that for every $p \geq p_0$ we get that

$$\frac{m_p}{\ell_p} \geq \frac{p^{\omega(\mathbb{M})-\varepsilon}}{p^{\omega(\mathbb{L})+\varepsilon}} = p^{\omega(\mathbb{M})-\omega(\mathbb{L})-2\varepsilon},$$

then $\lim_{p\to\infty} m_p/\ell_p = \infty$. From (i), we see that $\lim_{p\to\infty}(M_p/L_p)^{1/p} = \infty$, $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$.

□

Remark 4.2.12. By the previous proposition, if $\mathbb{L}$ satisfies (2.23), which is valid whenever $\mathbb{L}$ admits a nonzero proximate order, we have seen that if $\omega(\mathbb{L}) < \omega(\mathbb{M})$, $\mathbb{M}$ and $\mathbb{L}$ are comparable but not equivalent.

Consequently, in our framework, comparability conditions need only to be assumed when $\omega(\mathbb{M}) = \omega(\mathbb{L})$. In this situation, according to the last result, it is natural to assume that $\mathbb{M}/\mathbb{L}$ is a weight sequence or, equivalently, that $\mathbb{M}/\mathbb{L}$ is (lc) and $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$, which can not be deduced from the regularity of $\mathbb{M}$ and $\mathbb{L}$ (see Example 4.2.4). Note that this is not a weird condition.

4.2.3 Tauberian theorems

Assuming a basic comparability hypothesis, justified in the previous subsections, and that one of the sequences is regular enough in order to employ summability techniques, we are ready to clarify the relation between $\mathbb{C}\{z\}_{\mathbb{M}}$ and $\mathbb{C}\{z\}_{\mathbb{L}}$. First, we observe that if a formal power series is summable for two different sequences, then its sums agree, extending what was proved for $k-$summability by J.-P. Ramis and J. Martinet [83, Ch. 2, Prop. 4.3] (see also [7, Lemma 8] and [64, Coro. 5.3.15]).

Proposition 4.2.13. Let $\mathbb{L}$ and $\mathbb{M}$ be weight sequences such that $\mathbb{L}$ admits a nonzero proximate order and $\mathbb{M}/\mathbb{L}$ is (lc). If $\omega(\mathbb{L}) < \omega(\mathbb{M})$, or if $\omega(\mathbb{L}) = \omega(\mathbb{M})$ assuming in addition that $\mathbb{L} \precsim \mathbb{M}$ and $\mathbb{L} \not\approx \mathbb{M}$, then for every $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}[[z]]_{\mathbb{L}}$, we have that

(i) $\operatorname{sing}_{\mathbb{L}}(\hat{f}) \subseteq \operatorname{sing}_{\mathbb{M}}(\hat{f})$ and $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}\{z\}_{\mathbb{L}}$.

(ii) For every $d \notin \operatorname{sing}_{\mathbb{M}}(\hat{f})$, $(\mathcal{S}_{\mathbb{L},d}\hat{f})(z) \sim_{\mathbb{L}} \hat{f}$ on $G(d,\alpha)$ with $\alpha > \omega(\mathbb{M})$,

(iii) $(\mathcal{S}_{\mathbb{L},d}\hat{f})(z) = (\mathcal{S}_{\mathbb{M},d}\hat{f})(z)$ for every z where both functions are defined.

Proof. By Proposition 4.2.11, from the hypothesis in both situations $\omega(\mathbb{L}) < \omega(\mathbb{M})$ or $\omega(\mathbb{L}) = \omega(\mathbb{M})$, we deduce that $\mathbb{M}/\mathbb{L}$ is a weight sequence. Then, Watson's Lemma is available.

(i) Since $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}}$, we have that $\operatorname{sing}_{\mathbb{M}}(\hat{f})$ is finite. Let d be a nonsingular direction of $\mathbb{M}-$summability. We write $f(z) := \mathcal{S}_{\mathbb{M},d}\hat{f}(z)$. We choose a kernel of $\mathbb{L}-$summability, that exists by Remark 4.1.4, and we consider $g := T^-_{\mathbb{L}} f$. Since f is defined on a sectorial region $G(d,\alpha)$ with $\alpha > \omega(\mathbb{M}) \geq \omega(\mathbb{L})$, by Proposition 4.1.14, g is holomorphic in a sector $S(d, \alpha - \omega(\mathbb{L}))$ with $\mathbb{L}-$growth on this sector.

On the other hand, $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{L}}$, hence we have that $\hat{g} := \hat{T}^-_{\mathbb{L}} \hat{f} \in \mathbb{C}\{z\}$. By Theorem 4.1.18.(ii), we have that $g \sim_{\mathbb{M}/\mathbb{L}} \hat{g}$ on $S(d, \alpha - \omega(\mathbb{L}))$ and, since $\hat{g} \in \mathbb{C}\{z\}$, $\mathcal{S}\hat{g} \sim_{\mathbb{M}/\mathbb{L}} \hat{g}$ in a disc, and then in a sectorial region of opening $\pi(\alpha - \omega(\mathbb{L}))$. By the Watson's Lemma, Theorem 3.2.15, we have that

$$\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}/\mathbb{L}}(G_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}/\mathbb{L}}$$

is injective for every $\gamma > \omega(\mathbb{M}/\mathbb{L}) = \omega(\mathbb{M}) - \omega(\mathbb{L})$ (see Remark 4.2.10). Since g is holomorphic in a sector of opening $\pi(\alpha - \omega(\mathbb{L})) > \pi(\omega(\mathbb{M}) - \omega(\mathbb{L}))$, g is unique and, consequently, it is an analytic extension of $\hat{g} := \hat{T}^-_{\mathbb{L}} \hat{f}$ with $\mathbb{L}-$growth in the sector $S(d, \alpha - \omega(\mathbb{L}))$. Using Theorem 4.1.20, we see that $\hat{f}$ is $\mathbb{L}-$summable in direction d. We conclude that $\operatorname{sing}_{\mathbb{L}}(\hat{f}) \subseteq \operatorname{sing}_{\mathbb{M}}(\hat{f})$, then $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{L}}$.

(ii) From (i), we know that $\hat{g} = \hat{T}^-_{\mathbb{L}} \hat{f}$ converges in a disc and admits analytic continuation g in a sector $S = S(d, \alpha - \omega(\mathbb{L}))$, $g \in \mathcal{O}^{\mathbb{L}}(S)$ and we have that $\mathcal{S}\hat{g} \sim_{\mathbb{M}'} \hat{T}^-_{\mathbb{L}} \hat{f}$ in S with $\mathbb{M}' = (1)_{n\in\mathbb{N}_0}$. Then, in Theorem 4.1.18.(i), we obtain that the function $f := \mathcal{S}_{\mathbb{L},d}\hat{f} = T_{\mathbb{L}} g$ is holomorphic in a sectorial region $G(d,\alpha)$ and $f \sim_{\mathbb{L}} \hat{f}$ there.

(iii) With the notation in (i), we have that

$$(\mathcal{S}_{\mathbb{L},d}\hat{f})(z) = (T_{\mathbb{L}} g)(z) = (T_{\mathbb{L}} T^-_{\mathbb{L}} f)(z) = f(z) = (\mathcal{S}_{\mathbb{M},d}\hat{f})(z)$$

for every z where these functions are defined.

□

As a consequence of the last proposition, a generalization of the classical Tauberian result for $k-$summability of J.-P. Ramis (see [84, Th. 3.8.1], [7, Th. 37] and [64, Coro. 5.3.16]) can be stated when the indices $\omega(\mathbb{L})$ and $\omega(\mathbb{M})$ are distinct.

Theorem 4.2.14. Let $\mathbb{L}$ and $\mathbb{M}$ be weight sequences such that $\mathbb{L}$ admits a nonzero proximate order, $\mathbb{M}/\mathbb{L}$ is (lc) and $\omega(\mathbb{L}) < \omega(\mathbb{M})$. If $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}[[z]]_{\mathbb{L}}$, then $\hat{f}$ is convergent.

Proof. Using Proposition 4.2.13, we know that $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}\{z\}_{\mathbb{L}}$ and

$$(\mathcal{S}_{\mathbb{L},d}\hat{f})(z) = (\mathcal{S}_{\mathbb{M},d}\hat{f})(z),$$

for every z where both functions are defined. Given $\theta \in \text{sing}_{\mathbb{M}}(\hat{f}) = \{\theta_1, \theta_2, \dots, \theta_m\}$, we can take $d \notin \text{sing}_{\mathbb{M}}(\hat{f})$ such that $|d-\theta| < \delta := \pi(\omega(\mathbb{M}) - \omega(\mathbb{L}))$. Then $(\mathcal{S}_{\mathbb{M},d}\hat{f})(z) = (\mathcal{S}_{\mathbb{L},d}\hat{f})(z)$ is defined in a sectorial region G of opening $\pi\alpha > \pi\omega(\mathbb{M})$ bisected by direction d. We observe that $f = (\mathcal{S}_{\mathbb{L},d}\hat{f})(z)$ is a holomorphic function defined in a sectorial region $\tilde{G}$ contained in G, bisected by direction θ, and of opening

$$\alpha\pi - |d-\theta| > \alpha\pi - \delta = \alpha\pi - \pi\omega(\mathbb{M}) + \pi\omega(\mathbb{L}) > \pi\omega(\mathbb{L}).$$

By Proposition 4.2.13.(ii), we have that $f \sim_{\mathbb{L}} \hat{f}$ in this region, then $\hat{f} \in \mathbb{C}\{z\}_{\mathbb{L},\theta}$. Since $\text{sing}_{\mathbb{L}}(\hat{f}) \subseteq \text{sing}_{\mathbb{M}}(\hat{f})$, we deduce that $\text{sing}_{\mathbb{L}}(\hat{f}) = \emptyset$ and, by Proposition 4.1.25, we conclude that $\hat{f} \in \mathbb{C}\{z\}$. □

Remark 4.2.15. Regarding the last two results, in the case $\omega(\mathbb{L}) < \omega(\mathbb{M})$, if $\mathbb{M}$ also admits a nonzero proximate order, the logarithmic convexity of the sequence $\mathbb{M}/\mathbb{L}$ does not need to be assumed since it is automatically guaranteed (see Remark 4.2.9).

Finally, we will show that this theorem is not valid when the indices coincide.

Theorem 4.2.16. Let $\mathbb{L}$ and $\mathbb{M}$ be weight sequences with $\mathbb{L} \precsim \mathbb{M}$ and $\omega(\mathbb{L}) = \omega(\mathbb{M})$. Then

$$\mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}\{z\}_{\mathbb{L}} = \mathbb{C}\{z\}_{\mathbb{L}}.$$

Moreover, if $\mathbb{L}$ admits a nonzero proximate order, $\mathbb{L} \not\approx \mathbb{M}$ and $\mathbb{M}/\mathbb{L}$ is (lc) we have that

$$\mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}[[z]]_{\mathbb{L}} = \mathbb{C}\{z\}_{\mathbb{M}} \cap \mathbb{C}\{z\}_{\mathbb{L}} = \mathbb{C}\{z\}_{\mathbb{L}}.$$

Proof. If $\hat{f}$ is $\mathbb{L}-$summable in direction d, then it exists $\alpha > \omega(\mathbb{L}) = \omega(\mathbb{M})$ and f holomorphic in $G = G(d,\alpha)$ such that $f \sim_{\mathbb{L}} \hat{f}$ in G. Since $\mathbb{L} \precsim \mathbb{M}$, we deduce that $f \sim_{\mathbb{M}} \hat{f}$ in G, and we conclude that $\hat{f}$ is $\mathbb{M}-$summable in direction d. Then, $\mathbb{C}\{z\}_{\mathbb{L},d} \subseteq \mathbb{C}\{z\}_{\mathbb{M},d}$ and, consequently, $\mathbb{C}\{z\}_{\mathbb{L}} \subseteq \mathbb{C}\{z\}_{\mathbb{M}}$. The last statement is obtained immediately using Proposition 4.2.13. □

Example 4.2.17. With the notation and computations in Examples 1.1.4 and 1.1.30, we deduce that $\mathbb{C}\{z\}_{\mathbb{M}_{\alpha,\beta}} \cap \mathbb{C}[[z]]_{\mathbb{M}_{\alpha,\beta'}} = \mathbb{C}\{z\}_{\mathbb{M}_{\alpha,\beta'}}$, but $\mathbb{C}\{z\}_{\mathbb{M}_{\alpha,\beta}} \cap \mathbb{C}[[z]]_{\mathbb{M}_{\alpha',\beta}} = \mathbb{C}\{z\}$ for any $\alpha > \alpha' > 0$ and every $\beta > \beta'$.

Remark 4.2.18. In order to generalize this result and to put forward a satisfactory multisummability theory when the indices coincide we would need to redefine the notion of $\mathbb{M}-$summability according to the classical theorem of S. Mandelbrojt [72, Sect. 2.4.I]. In this new tentative definition, as it will be specified in the conclusions of the dissertation, the sectorial regions are replaced by regions of uniqueness whose boundary is tangent to the boundary of the sector $S_{\omega(\mathbb{M})}$ near 0.

4.3 Multisummability

Whenever the Tauberian Theorem 4.2.14 is available it makes sense to give a definition of multisummability in this context. In the recent book of M. Loday-Richaud [64, Ch. 7], several equivalent definitions of multisummability are provided together with a careful study of their peculiarities. In this dissertation, since the $\mathbb{M}-$summability tools are defined using moment summability methods, the approach of W. Balser [7, Ch. 10] has been chosen, that is, the decomposition into sums. Due to a ramification inconvenience, this splitting definition is only compatible with the others if the corresponding indices $\omega(\mathbb{M}_j)$ are all smaller than 2.

Definition 4.3.1. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be weight sequences, i.e., (lc) with quotients tending to infinity, admitting a nonzero proximate order (see Definition 2.2.1) such that $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < 2$ and $d_1, d_2 \in \mathbb{R}$ such that $|d_1 - d_2| < \pi(\omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))/2$. A formal power series $\hat{f}(z) = \sum_{p\geq 0} a_p z^p \in \mathbb{C}[[z]]$ is said to be $(\mathbb{M}_1, \mathbb{M}_2)-$*summable in the multidirection* (d_1, d_2), if there exist a formal series $\hat{f}_1(z)$ which is $\mathbb{M}_1-$summable in d_1 with $\mathbb{M}_1-$sum f_1 and a formal series $\hat{f}_2(z)$ which is $\mathbb{M}_2-$summable in d_2 with $\mathbb{M}_2-$sum f_2 such that

$$\hat{f} = \hat{f}_1 + \hat{f}_2.$$

Furthermore, the holomorphic function $f(z) = f_1(z) + f_2(z)$ defined on $G(d_1, \alpha_1)$ for some $\alpha_1 > \omega(\mathbb{M}_1)$ is called the $(\mathbb{M}_1, \mathbb{M}_2)-$*sum of* $\hat{f}$ *in the multidirection* (d_1, d_2) and we write $f(z) = (\mathcal{S}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}\hat{f})(z)$ and $\hat{f} \in \mathbb{C}\{z\}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}$.

Remark 4.3.2. In the conditions of the previous definition, there always exists a sectorial region $G = G(d_1, \alpha_1)$ with $\alpha_1 > \omega(\mathbb{M}_1)$ such that

$$f \sim_{\mathbb{M}_2} \hat{f} \qquad \text{on} \quad G. \tag{4.10}$$

However, since the region is not wide enough, f is not the sole function in between the ones satisfying (4.10). Hence, this condition is weaker than the multisummability notion, because the next proposition shows that the multisum is unique and the splittings are essentially unique.

Proposition 4.3.3. In the conditions of Definition 4.3.1, assume that there exist two pairs of formal power series $\hat{f}_1, \hat{f}_2$ and $\hat{g}_1, \hat{g}_2$ such that

$$\hat{f} = \hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2.$$

Then there exist $\alpha_2 > \omega(\mathbb{M}_2)$ and $\hat{u}_1 \in \mathbb{C}[[z]]$ such that u_1 is $\mathbb{M}_1-$summable on a sectorial region $G(d_2, \alpha_2)$ and

$$\hat{g}_1 = \hat{f}_1 - \hat{u}_1, \qquad \text{and} \qquad \hat{g}_2 = \hat{u}_1 + \hat{f}_2.$$

Moreover, we have that the $(\mathbb{M}_1, \mathbb{M}_2)-$sum of $\hat{f}$ is unique, that is,

$$f_1(z) + f_2(z) = f(z) = g_1(z) + g_2(z),$$

in a sectorial region $G(d_1, \alpha_1)$ with $\alpha_1 > \omega(\mathbb{M}_1)$.

Proof. We define $\hat{u}_1 := \hat{f}_1 - \hat{g}_1$, so $\hat{u}_1 \in \mathbb{C}\{z\}_{\mathbb{M}_1,d_1}$, in particular $\hat{u}_1 \in \mathbb{C}[[z]]_{\mathbb{M}_1}$. We observe that $\hat{g}_2 - \hat{f}_2 = \hat{u}_1$, then $\hat{u}_1 \in \mathbb{C}\{z\}_{\mathbb{M}_2,d_2}$. By Proposition 4.2.13.(ii), there exists $\alpha_2 > \omega(\mathbb{M}_2)$ such that $\hat{u}_1$ is $\mathbb{M}_1-$summable in $G(d_2, \alpha_2)$.

Furthermore, by Lemma 4.1.22.(ii), $(\mathcal{S}_{\mathbb{M}_1,d_1}\hat{u}_1)(z) = (\mathcal{S}_{\mathbb{M}_1,d_2}\hat{u}_1)(z)$ on a sectorial region $G = G(d_1, \alpha_1)$ with $\alpha_1 > \omega(\mathbb{M}_1)$ and, using Proposition 4.2.13.(iii), for all $z \in G$ we conclude that

$$\begin{aligned} f_1(z) - g_1(z) &= (\mathcal{S}_{\mathbb{M}_1,d_1}(\hat{f}_1 - \hat{g}_1))(z) = (\mathcal{S}_{\mathbb{M}_1,d_2}\hat{u}_1)(z) \\ &= (\mathcal{S}_{\mathbb{M}_2,d_2}\hat{u}_1)(z) = (\mathcal{S}_{\mathbb{M}_2,d_2}(\hat{g}_2 - \hat{f}_2))(z) = g_2(z) - f_2(z). \end{aligned}$$

□

Remark 4.3.4. Similarly to the classical situation, this definition can be recursively extended for a finite set of sequences $\mathbb{M}_1, \mathbb{M}_2, \dots, \mathbb{M}_k$ with $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < \cdots < \omega(\mathbb{M}_k) < 2$ (see [7, Ch. 10] and [64, Ch. 7]).

The rest of this section is devoted to recover the multisum by means of some suitable integral transform.

4.3.1 Moment-kernel duality

The main aim of this subsection is to prove that a kernel e of $\mathbb{M}-$summability is uniquely determined by its sequence of moments $\mathfrak{m}_e$, similarly to the result of W. Balser for the moment summability methods [7, Sect. 5.8].

For bounded functions on sectors, the following auxiliary lemma shows that the domain of holomorphy of their $e-$Laplace transform is not an arbitrary sectorial region, as it is shown in Proposition 4.1.12, but an unbounded sector.

Lemma 4.3.5. Let a kernel function $e(z)$ of $\mathbb{M}-$summability with corresponding operator $T = T_e$ be given. Given f a function defined in a sector $S = S(d, \alpha)$, assume that for every $0 < \beta < \alpha$, there exists a constant $C_\beta > 0$ such that we have that

$$|f(u)| \le C_\beta, \qquad u \in S(d, \beta).$$

Then $g = Tf$ is holomorphic in $S(d, \alpha + \omega(\mathbb{M}))$.

Proof. We have that $f \in \mathcal{O}^{\mathbb{M}}(S)$ with $S = S(d, \alpha)$. Let $\tau \in \mathbb{R}$ be a direction in S, i.e., such that $|\tau - d| < \pi\alpha/2$. For every $u, z \in \mathcal{R}$ with $\arg(u) = \tau$ and $|\tau - \arg(z)| < \omega(\mathbb{M})\pi/2$ we have that $u/z \in S_{\omega(\mathbb{M})}$, so the expression under the integral sign in (4.7) makes sense. We fix $a > 0$, and write

$$g(z) = \int_0^{\infty(\tau)} e(u/z) f(u) \frac{du}{u} = \int_0^{ae^{i\tau}} e(u/z) f(u) \frac{du}{u} + \int_{ae^{i\tau}}^{\infty(\tau)} e(u/z) f(u) \frac{du}{u}.$$

Since f is bounded at the origin following direction τ and by Definition 4.1.2.(II), it is straightforward to apply Leibniz's rule for parametric integrals and deduce that the first integral in the right-hand side defines a holomorphic function in $S(\tau, \omega(\mathbb{M}))$. Regarding the second integral, we take $\beta < \alpha$ such that $|\tau - d| < \beta\pi/2$ and we fix $0 < \gamma < \omega(\mathbb{M})$. We have that $u/z \in S_\gamma$, for $\arg(u) = \tau$ and z such that $|\tau - \arg(z)| < \gamma\pi/2$. The property in Definition 4.1.2.(III) provides us with constants $c, k > 0$ such that

$$|e(u/z)| \le c h_{\mathbb{M}}(k|z|/|u|), \qquad \arg(u) = \tau, \quad z \in S(\tau, \gamma),$$

then

$$\Big|\frac{1}{u} e(u/z) f(u)\Big| \le \frac{cC_\beta}{|u|} h_{\mathbb{M}}(k|z|/|u|), \qquad \arg(u) = \tau, \quad z \in S(\tau, \gamma).$$

For any $z_0 \in S(\tau, \gamma)$ we fix a bounded neighborhood U of z_0 contained in $S(\tau, \gamma)$. We have that $|z| < r$ for every $z \in U$, and from the monotonicity of $h_{\mathbb{M}}$ we deduce that

$$\Big|\frac{1}{u} e(u/z) f(u)\Big| \le \frac{cC_\beta}{|u|} h_{\mathbb{M}}(kr/|u|).$$

By the definition of $h_{\mathbb{M}}$, we have that $h_{\mathbb{M}}(kr/|u|) \le M_1 kr/|u|$, so the right-hand side of the last inequality is an integrable function of $|u|$ in (a, ∞), and again Leibniz's rule allows us to conclude the desired analyticity for the second integral.

Consequently, g is holomorphic in $S(\tau, \gamma)$ for every $|\tau - d| < \pi\alpha$ and every $0 < \gamma < \omega(\mathbb{M})$. As in the proof of Proposition 4.1.12, we see that the value of $g(z)$ is the same in the intersection of these regions and we have that g is holomorphic in $S(d, \alpha + \omega(\mathbb{M}))$. $\square$

Remark 4.3.6. Moreover, if $f(z) \sim_{\mathbb{M}'} \sum_{n=0}^{\infty} a_n z^n$, by Theorem 4.1.18.(i), we deduce that $g = T_e f \sim_{\mathbb{M}\mathbb{M}'} \sum_{n=0}^{\infty} a_n m_e(n) z^n$ on a sectorial region $G(d, \alpha+\omega(\mathbb{M}))$. Since the notion of asymptotic expansion only depends on the behavior of the function on bounded subsectors, we can say that $g \sim_{\mathbb{M}\mathbb{M}'} \sum_{n=0}^{\infty} a_n m_e(n) z^n$ on $S(d, \alpha + \omega(\mathbb{M}))$, whenever g is holomorphic in $S(d, \alpha + \omega(\mathbb{M}))$.

As it happens in the Gevrey case, since the moment function $m_e(\lambda)$ is the Mellin transform of $e(z)$ (see [100, Sect. 1.29]), there is a duality between $m_e(\lambda)$ and e and the next lemma shows how one can recover $e(z)$ from its moment sequence $\mathfrak{m}_e$, thanks to the inversion formula. However, observe that, as it was mentioned in [7], we shall not be concerned with the harder question of how to characterize such $\mathfrak{m}$ to which a kernel $e(z)$ exists. The following lemma generalizes Lemma 7 in [7].

Lemma 4.3.7. Let a kernel function $e(z)$ of $\mathbb{M}-$summability with corresponding operator $T = T_e$ be given. For $f(u) := (1-u)^{-1}$, we have that $g = Tf$ is holomorphic in $S(\pi, 2+\omega(\mathbb{M}))$, is $\mathbb{M}-$asymptotic to $\hat{g}(z) = \sum_0^{\infty} m(n) z^n$ there and $g(z) \to 0$ as $|z| \to \infty$ uniformly for $z \in S(\pi, 2+\gamma)$ for every $\gamma < \omega(\mathbb{M})$. Moreover,

$$g(z) - g(ze^{2\pi i}) = 2\pi i e(1/z), \qquad z \in S_{\omega(\mathbb{M})}. \tag{4.11}$$

Proof. The function $f(u)$ is defined in the sector $S(\pi, 2)$ and continuous at the origin. For every $1 < \beta < 2$, we have that

$$|f(u)| \leq \frac{1}{|1-u|} \leq \frac{1}{\sin(\pi - \beta\pi/2)}, \qquad u \in S(\pi, \beta).$$

Hence, by Lemma 4.3.5, we have that g is holomorphic in $S(\pi, 2+\omega(\mathbb{M}))$. Since f is convergent at 0, we have that $f(z) \sim_{\mathbb{M}'} \sum_{n=0}^{\infty} z^n$ with $\mathbb{M}' = (1)_{n\in\mathbb{N}_0}$ and, by Remark 4.3.6, we deduce that $g \sim_{\mathbb{M}} \sum_{n=0}^{\infty} m(n) z^n$ on $S(\pi, 2+\omega(\mathbb{M}))$.

The behavior at infinity can be again read off from the integral representation as follows. We fix a direction $\tau \in (0, 2\pi)$, $\tau \neq \pi$, and we consider a direction $\theta \in (-\pi\omega(\mathbb{M})/2, 2\pi + \pi\omega(\mathbb{M})/2)$ such that $|\tau - \theta| < \pi\gamma/2 < \pi\omega(\mathbb{M})/2$. For every $z \in S(\tau, \gamma)$ with $\arg(z) = \theta$, we have that

$$g(z) = \int_0^{\infty} e\left(\frac{r}{|z|} e^{i(\tau-\theta)}\right) \frac{dr}{r(1-re^{i\tau})} = \int_0^{\infty} e(se^{i(\tau-\theta)}) \frac{ds}{s(1-s|z|e^{i\tau})}.$$

We split the integral into two parts and we see that

$$\left| \int_0^{1/|z|^{1/2}} e(se^{i(\tau-\theta)}) \frac{ds}{s(1-s|z|e^{i\tau})} \right| \leq \frac{1}{\inf_{0<s<\infty}\{|1-s|z|e^{i\tau}|\}} \int_0^{1/|z|^{1/2}} \frac{|e(se^{i(\tau-\theta)})|}{s} ds.$$

If $\tau \in (0, \pi/2) \cup (3\pi/2, 2\pi)$, we have that

$$\inf_{0<s<\infty} \{|1-s|z|e^{i\tau}|\} = |\sin(\tau)| \neq 0,$$

and if $\tau \in [\pi/2, 3\pi/2]$ $(\tau \neq \pi)$, we observe that

$$\inf_{0<s<\infty} \{|1-s|z|e^{i\tau}|\} = 1 \geq |\sin(\tau)| \neq 0.$$

Consequently, we have shown that

$$\left| \int_0^{1/|z|^{1/2}} e(se^{i(\tau-\theta)}) \frac{ds}{s(1-s|z|e^{i\tau})} \right| \leq \frac{1}{|\sin(\tau)|} \int_0^{1/|z|^{1/2}} \sup_{|\phi|<\pi\gamma/2} |e(se^{i\phi})| \frac{ds}{s}. \tag{4.12}$$

Using Definition 4.1.2.(ii), we see that the integral in the right hand side is convergent. Subsequently, it tends to 0, uniformly as $|z|$ goes to infinity in $S(\tau,\gamma)$.

On the other hand, since the function e is uniformly bounded in S_γ for every $|\tau-\theta|<\pi\gamma/2$, there exists $c>0$ such that

$$\left|\int_{1/|z|^{1/2}}^{\infty} e(se^{i(\tau-\theta)})\frac{ds}{s(1-s|z|e^{i\tau})}\right| \leq c\int_{1/|z|^{1/2}}^{\infty}\frac{ds}{s|1-s|z|e^{i\tau}|}.$$

We have that $|1-s|z|e^{i\tau}|\geq |s|z|-1| = |s|z|^{1/2}|z|^{1/2}-1|$. Since always $s|z|^{1/2}\geq 1$, if $|z|^{1/2}\geq 1$ we see that $|1-s|z|e^{i\tau}|\geq s|z|-1$. We observe that if $|z|>4$, we have that $1/|z|^{1/2}>2/|z|$, then $s>2/|z|$, and consequently, $s|z|-1\geq s|z|/2$. Hence, for every $z\in S(\tau,\gamma)$ with $|z|>4$ we see that

$$\left|\int_{1/|z|^{1/2}}^{\infty} e(se^{i(\tau-\theta)})\frac{ds}{s(1-s|z|e^{i\tau})}\right| \leq \frac{2c}{|z|}\int_{1/|z|^{1/2}}^{\infty}\frac{ds}{s^2} = \frac{2c}{|z|^{1/2}}. \tag{4.13}$$

The right hand side of this inequality tends to 0 as $|z|$ goes to infinity. By (4.12) and (4.13), we see that $g(z)\to 0$ as $|z|\to\infty$ uniformly for $z\in S(\tau,\gamma)$. By a compactness argument, we see that $g(z)\to 0$ as $|z|\to\infty$ whenever $z\in S(\pi,2+\omega(\mathbb{M}))$ uniformly for $z\in S(\pi,2+\gamma)$.

Let $\theta\in\mathbb{R}$ be a direction such that $|\theta|<\pi\omega(\mathbb{M})/2$ and $z\in\mathcal{R}$ with $\arg(z)=\theta$. There exists $\varepsilon\in(0,\pi\omega(\mathbb{M})/2)$ such that:

1. For every $u\in\mathcal{R}$ with $\arg(u)=\varepsilon$ we have that $u/z\in S_{\omega(\mathbb{M})}$.
2. For every $u\in\mathcal{R}$ with $\arg(u)=2\pi-\varepsilon$ we have that $u/ze^{2\pi i}\in S_{\omega(\mathbb{M})}$.

Then, since f is single-valued, we have that

$$\begin{aligned} g(z)-g(ze^{2\pi i}) &= \int_0^{\infty(\varepsilon)}\frac{e(u/z)du}{(1-u)u} - \int_0^{\infty(2\pi-\varepsilon)}\frac{e(u/(ze^{2\pi i}))du}{(1-u)u} \\ &= \int_0^{\infty(\varepsilon)}\frac{e(u/z)du}{(1-u)u} - \int_0^{\infty(-\varepsilon)}\frac{e(w/z)dw}{(1-w)w}. \end{aligned}$$

We denote by γ_r the arc of radius $r>1$ from $re^{i\varepsilon}$ to $re^{-i\varepsilon}$ clockwise. We observe that

$$\left|\int_{\gamma_r}\frac{e(u/z)du}{(1-u)u}\right| = \left|\int_{-\varepsilon}^{\varepsilon}\frac{e(re^{i\theta}/z)id\theta}{(1-re^{i\theta})}\right| \leq \int_{-\varepsilon}^{\varepsilon}\frac{|e(re^{i\theta}/z)|d\theta}{|r-1|} \leq 2\varepsilon c\frac{h_{\mathbb{M}}(k|z|/r)}{r-1} \leq \frac{2\varepsilon c}{r-1}.$$

Hence, we deduce that

$$\lim_{r\to\infty}\left|\int_{\gamma_r}\frac{e(u/z)du}{(1-u)u}\right| = 0.$$

If we compute the residue of $h_z(u)=e(u/z)/(u(1-u))$ at $u=1$, we see that

$$\lim_{u\to 1}\frac{e(u/z)}{(1-u)u}(u-1) = -e(1/z).$$

According to the Residue theorem, we conclude that

$$g(z)-g(ze^{2\pi i}) = 2\pi i e(1/z).$$

□

Remark 4.3.8. Let e and $\overline{e}$ be two $\mathbb{M}-$summability kernels whose moment sequence $\mathfrak{m} = (m(n))_{n\in\mathbb{N}_0}$ is the same. By the above lemma, for $f(z) = 1/(1-z)$, we have that $T_e f \sim_{\mathbb{M}} \sum_{n=0}^{\infty} m(n)z^n$ and $T_{\overline{e}} f \sim_{\mathbb{M}} \sum_{n=0}^{\infty} m(n)z^n$ on $S(\pi, 2+\omega(\mathbb{M}))$. Observe that, according to Watson's Lemma, Theorem 3.2.15, $T_e f = T_{\overline{e}} f$. By (4.11), we deduce that $e(z) = \overline{e}(z)$.

Remark 4.3.9. Since $g(z) \to 0$ as $|z| \to \infty$ uniformly for $z \in S(\pi, 2+\gamma)$ for every $\gamma < \omega(\mathbb{M})$, by (4.11) we also deduce that $e(z) \to 0$, $|z| \to 0$, uniformly for $z \in S_\gamma$ for every $\gamma < \omega(\mathbb{M})$, which does not follow immediately from Definition 4.1.2.

4.3.2 Strong kernels of $\mathbb{M}-$summability

In order to recover the multisum of a formal power series, we need to combine a kernel e_1 of $\mathbb{M}_1-$summability with a kernel e_2 of $\mathbb{M}_2-$summability. The idea is to define new kernels $e_1 * e_2$ and $e_1 \triangleleft e_2$ whose sequences of moments are $\mathfrak{m}_{e_1} \cdot \mathfrak{m}_{e_2}$ and $\mathfrak{m}_{e_2}/\mathfrak{m}_{e_1}$, respectively. This construction is based on the one given in the Gevrey case by W. Balser [7, Sect. 5.8]. Nevertheless, in this general situation, a stronger notion of summability kernel should be considered which will not suppose a significant restriction (see Remark 4.3.13).

Definition 4.3.10. Let $\mathbb{M}$ be a strongly regular sequence with $\omega(\mathbb{M}) < 2$. A pair of complex functions e, E are said to be *strong kernel functions for* $\mathbb{M}-$*summability* if:

(I) e is holomorphic in $S_{\omega(\mathbb{M})}$.

(II.B) It exists $\alpha > 0$ such that for every $\tau \in (0, \omega(\mathbb{M}))$, there exist $C_\tau, \varepsilon_\tau > 0$ such that

$$|e(z)| \le C_\tau |z|^\alpha, \qquad \text{for all } z \in S_\tau, \qquad \text{with } |z| \le \varepsilon_\tau.$$

(III) For every $\varepsilon > 0$ there exist $c, k > 0$ such that

$$|e(z)| \le c h_{\mathbb{M}}\left(\frac{k}{|z|}\right) = c\, e^{-\omega_{\mathbb{M}}(|z|/k)}, \qquad \text{for all } z \in S_{\omega(\mathbb{M})-\varepsilon},$$

where $h_{\mathbb{M}}$ and $\omega_{\mathbb{M}}$ are the functions associated with $\mathbb{M}$ defined by (1.4).

(IV) For $x \in \mathbb{R}$, $x > 0$, the values of $e(x)$ are positive real.

(V) If we define the *moment function* associated with e,

$$m_e(\lambda) := \int_0^\infty t^{\lambda-1} e(t)dt, \qquad \mathrm{Re}(\lambda) \ge 0,$$

from $(I)-(IV)$ we see that $m_e(\lambda)$ is continuous in $\{\mathrm{Re}(\lambda) \ge 0\}$, holomorphic in $\{\mathrm{Re}(\lambda) > 0\}$, and $m_e(x) > 0$ for every $x \ge 0$. Then, the function E given by

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_e(n)}, \qquad z \in \mathbb{C},$$

is entire, and there exist $C, K > 0$ such that

$$|E(z)| \le \frac{C}{h_{\mathbb{M}}(K/|z|)} = Ce^{M(|z|/K)}, \quad \text{for all } z \in \mathbb{C}.$$

(VI.B) It exists $\beta > 0$ such that for every $\tau \in (0, 2-\omega(\mathbb{M}))$, there exist $K_\tau, M_\tau > 0$ such that

$$|E(z)| \leq \frac{K_\tau}{|z|^\beta}, \qquad \text{for all } z \in S(\pi,\tau), \qquad \text{with } |z| \geq M_\tau.$$

Remark 4.3.11. In case $\omega(\mathbb{M}) \geq 2$, condition (VI.B) makes no sense and, in the same way as in [7, p. 90] and in Remark 4.1.3, suitable adaptations should be made. For simplicity, we will omit this situation from now on.

Lemma 4.3.12. Let $\mathbb{M}$ be a strongly regular sequence with $\omega(\mathbb{M}) < 2$. Let e and E be a pair of complex function satisfying Definition 4.3.10, then they fulfill the conditions in Definition 4.1.2.

Proof. We only have to check that e and E satisfy conditions (II) and (VI) in Definition 4.1.2, respectively. We take $z_0 \in S_{\omega(\mathbb{M})}$, we fix $r_0 > 0$ and $\tau_0 \in (0, \omega(\mathbb{M}))$ such that $U := B(z_0, r_0) \subseteq S_{\tau_0}$. By condition (II.B), we have that

$$|e(z)| \leq C_0|z|^\alpha, \qquad z \in S_{\tau_0}, \qquad |z| \leq \varepsilon_0.$$

For $t \in (0, \varepsilon_0(|z_0| - r_0))$ and for every $z \in U$ we observe that $t/z \in S_{\tau_0}$ and $|t/z| \leq \varepsilon_0$. Then

$$\int_0^{\varepsilon_0(|z_0|-r_0)} \sup_{z\in U} |e(t/z)| \frac{dt}{t} \leq \int_0^{\varepsilon_0(|z_0|-r_0)} \frac{C_0 t^{\alpha-1} dt}{(|z_0|-r_0)^\alpha} \leq \frac{C_0 \varepsilon_0^\alpha}{\alpha}.$$

We fix $T > 0$, if $\varepsilon_0(|z_0| - r_0) \geq T$ condition (II) is immediately satisfied. If $\varepsilon_0(|z_0| - r_0) < T$ we define $D_0 := \{t/z;\ z \in U,\ t \in [\varepsilon_0(|z_0| - r_0), T]\} \subseteq S_{\tau_0}$, by condition (I), e is continuous on S_{τ_0} and, since D_0 is contained on a compact subset of S_{τ_0} , we have that $\sup_{w\in D_0} |e(w)| = K_0 < \infty$. Then

$$\int_0^T \sup_{z\in U} |e(t/z)| \frac{dt}{t} \leq \frac{C_0 \varepsilon_0^\alpha}{\alpha} + \frac{T K_0}{\varepsilon_0(|z_0| - r_0)} < \infty.$$

Analogously, we will verify condition (VI). We take $z_0 \in S(\pi, 2 - \omega(\mathbb{M}))$, we fix $r_0 > 0$ and $\tau_0 \in (0, 2 - \omega(\mathbb{M}))$ such that $U := B(z_0, r_0) \subseteq S(\pi, \tau_0)$. By condition (VI.B), we have that

$$|E(z)| \leq \frac{K_0}{|z|^\beta}, \qquad z \in S(\pi, \tau_0), \qquad |z| \geq M_0.$$

For $0 < t \leq (|z_0| - r_0)/M_0$ and for every $z \in U$ we observe that $z/t \in S(\pi, \tau_0)$ and $|z/t| \geq M_0$. Then

$$\int_0^{(|z_0|-r_0)/M_0} \sup_{z\in U} |E(z/t)| \frac{dt}{t} \leq \int_0^{(|z_0|-r_0)/M_0} \frac{K_0 dt}{(|z_0|-r_0)^\beta t^{1-\beta}} \leq \frac{K_0}{M_0^\beta \beta},$$

and, since E is entire, we conclude as before. □

Remark 4.3.13. In general, thanks to Remark 4.3.9 one can only guarantee that e tends to 0 in the regions considered in (II.B) but it seems not possible to ensure that it has power-like growth, likewise for E.

However, either the classical kernels in the Gevrey theory $e_k(z) = kz^k \exp(-z^k)$ (see [7]), or the new ones $e_V(z) = z\exp(-V(z))$, constructed for sequences admitting a nonzero proximate order (see Remark 4.1.4), using the functions V defined in [65] (see [60, Th. 4.8, Prop. 4.11]), satisfy conditions (II.B) and (VI.B) (see (4.4), (4.5)).

Moreover, if we want to proof the integrability conditions (II) or (VI) in some concrete example, we end up showing estimates similar to those appearing in (II.B) and (VI.B).

Remark 4.3.14. This stronger notion need to be considered to assure that the convolution and the acceleration kernels, defined in the forthcoming subsections from two given kernels e_1 and e_2, also satisfy adequate integrability properties which seem not to be preserved in the standard situation.

Remark 4.3.15. We observe that once condition (II.B) or (VI.B) is satisfied for some values α and β, it is possible to replace α for any $0 < \alpha' < \alpha$ and β for any $0 < \beta' < \beta$ and the corresponding conditions hold.

4.3.3 Convolution kernels

In this subsection, we consider two strong kernels e_1 and e_2 satisfying the properties in Definition 4.3.10 for two sequences $\mathbb{M}_1$ and $\mathbb{M}_2$ with corresponding operators $T_{e_j}, T^-_{e_j}$ and moment functions $m_{e_j}(\lambda)$ for $j = 1, 2$. We will find a pair of operators T, T^- such that T coincides with $T_{e_1} \circ T_{e_2}$ for a suitable class of functions containing the monomials. Hence, we will deduce that the moment function $m(\lambda)$ associated with T equals $m_{e_1}(\lambda)m_{e_2}(\lambda)$. The kernel that defines the operator T will be obtained as a Mellin convolution of the kernels e_1 and e_2, which justifies its name.

First, we prove an auxiliary lemma that connects the associated function of two weight sequences $\mathbb{M}_1$ and $\mathbb{M}_2$ with the associated function of their product sequence $\mathbb{M}_1 \cdot \mathbb{M}_2$. This will be essential when dealing with functions in the classes $\mathcal{O}^{\mathbb{M}_1}$, $\mathcal{O}^{\mathbb{M}_2}$ and $\mathcal{O}^{\mathbb{M}_1 \cdot \mathbb{M}_2}$.

Lemma 4.3.16. Let $\mathbb{M}_j$, $j = 1, 2$, be weight sequences, for every $s, r > 0$ we have that

$$e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(r)} \leq e^{\omega_{\mathbb{M}_1}(s)} e^{\omega_{\mathbb{M}_2}(r/s)}. \tag{4.14}$$

Proof. We write $\mathbb{M}_1 = (M_{1,p})_{p \in \mathbb{N}_0}$ and $\mathbb{M}_2 = (M_{2,p})_{p \in \mathbb{N}_0}$. For every $s, r > 0$, we observe that

$$e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(r)} = \sup_{p \in \mathbb{N}_0} \frac{r^p}{M_{1,p} M_{2,p}} = \sup_{p \in \mathbb{N}_0} \frac{s^p}{M_{1,p}} \frac{(r/s)^p}{M_{2,p}} \leq \sup_{p \in \mathbb{N}_0} \frac{s^p}{M_{1,p}} \sup_{p \in \mathbb{N}_0} \frac{(r/s)^p}{M_{2,p}} = e^{\omega_{\mathbb{M}_1}(s)} e^{\omega_{\mathbb{M}_2}(r/s)}.$$

□

Remark 4.3.17. We say that a weight sequence $\mathbb{M}$ is *normalized* if $m_0 = M_1 = 1$, which by log-convexity implies that $m_p \geq 1$, $M_p \leq M_{p+1}$ and $M_p \geq 1$ for all $p \in \mathbb{N}_0$.

Given two normalized weight sequences $\mathbb{L}$ and $\mathbb{M}$, then

$$\min(\omega_{\mathbb{L}}(t), \omega_{\mathbb{M}}(t)) \geq \omega_{\mathbb{L} \cdot \mathbb{M}}(t) \qquad \text{for } t > 0. \tag{4.15}$$

which follows directly from the definition of the associated functions since $L_p \leq L_p M_p$ and $M_p \leq L_p M_p$ for all $p \in \mathbb{N}_0$.

For arbitrary weight sequences, (4.15) is satisfied for t large enough. However, normalization is not a significant restriction since $m_p \geq 1$ for p large and we can modify the first terms of a sequence according to Remark 1.1.19 getting a normalized weight sequence $\mathbb{M}'$ with $\mathbf{m}' \simeq \mathbf{m}$. This assumption simplifies in a considerable way the proofs of the forthcoming results.

Remark 4.3.18. The results until the end of the chapter might be valid for normalized strongly regular sequences such that we can associate with them a strong kernel. Nevertheless, the existence of such kernels has only been proved for sequences admitting a nonzero proximate order which, as it was pointed out in Remark 2.2.18, are the ones appearing in the applications.

In the second place, the following proposition shows the convergence of the double integral (4.16) that will ensure that the operators T and $T_{e_1} \circ T_{e_2}$ coincide.

Proposition 4.3.19. Let $\mathbb{M}_j$, $j = 1, 2$, be normalized weight sequences admitting a nonzero proximate order. We consider strong kernels e_j for $\mathbb{M}_j-$summability, its moment function m_{e_j} and T_{e_j} the corresponding Laplace-like operators. If $f \in \mathcal{O}^{\mathbb{M}_1 \cdot \mathbb{M}_2}(S(d, \gamma))$, then it exists a sectorial region $G(d, \gamma + \omega(\mathbb{M}_1) + \omega(\mathbb{M}_2))$ such that for every $z_0 \in G(d, \gamma + \omega(\mathbb{M}_1) + \omega(\mathbb{M}_2))$ there exist a neighborhood $U_0 \subseteq G(d, \gamma + \omega(\mathbb{M}_1) + \omega(\mathbb{M}_2))$ of z_0 and directions θ and ϕ (depending on z_0) such that we have

$$\int_0^{\infty(\theta+\phi)} \int_0^{\infty(\theta)} \left| e_1\left(\frac{v}{z}\right) e_2\left(\frac{u}{v}\right) f(u) \frac{du}{u}\frac{dv}{v} \right| < \infty, \tag{4.16}$$

for every $z \in U_0$. Consequently, $T_{e_1} \circ T_{e_2}(f)(z)$ is holomorphic in the sectorial region $G(d, \gamma + \omega(\mathbb{M}_1) + \omega(\mathbb{M}_2))$ and we observe that

$$T_{e_1} \circ T_{e_2}(f)(z) = \int_0^{\infty(\theta)} f(u) \left(\int_0^{\infty(\phi)} e_1(wu/z) e_2(1/w) \frac{dw}{w} \right) \frac{du}{u}.$$

Proof. We write $\mathbb{M}_1 = (M_{1,p})_{p \in \mathbb{N}_0}$ and $\mathbb{M}_2 = (M_{2,p})_{p \in \mathbb{N}_0}$ and, for simplicity, $\omega_1 = \omega(\mathbb{M}_1)$ and $\omega_2 = \omega(\mathbb{M}_2)$.

We fix $\psi_0 \in (d - (\omega_1 + \omega_2 + \gamma)\pi/2, d + (\gamma + \omega_1 + \omega_2)\pi/2)$, we choose directions $\tau_1 \in (0, \omega_1)$, $\tau_2 \in (0, \omega_2)$, $\tau_3 \in (0, \gamma)$, θ and ϕ with $|\theta - d| < \tau_3\pi/2$ and $|\phi| \leq \pi\tau_2/2$ such that

$$|\theta + \phi - \psi_0| < \pi\tau_1/2. \tag{4.17}$$

Then, it exists $\varepsilon > 0$, such that $[\psi_0 - \varepsilon, \psi_0 + \varepsilon] \subseteq (d - (\omega_1 + \omega_2 + \gamma)\pi/2, d + (\gamma + \omega_1 + \omega_2)\pi/2)$ and (4.17) remains true if we replace ψ_0 by ψ for every $\psi \in (\psi_0 - \varepsilon, \psi_0 + \varepsilon)$. By Definition 4.3.10 (II.B), for e_1 and e_2 we know that there exist $\alpha_1, \alpha_2 > 0$ (not depending on τ_1 and τ_2), and constants $C_1, C_2 > 0$, $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that

$$|e_1(w)| \leq C_1 |w|^{\alpha_1}, \qquad w \in S_{\tau_1}, \qquad |w| \leq \varepsilon_1, \tag{4.18}$$
$$|e_2(w)| \leq C_2 |w|^{\alpha_2}, \qquad w \in S_{\tau_2}, \qquad |w| \leq \varepsilon_2. \tag{4.19}$$

Using condition (III), for e_1 and e_2, there exist $d_1, d_2, k_1, k_2 > 0$ such that

$$|e_1(w)| \leq d_1 \, e^{-\omega_{\mathbb{M}_1}(k_1|w|)}, \qquad w \in S_{\tau_1}, \tag{4.20}$$
$$|e_2(w)| \leq d_2 \, e^{-\omega_{\mathbb{M}_2}(k_2|w|)}, \qquad u \in S_{\tau_2}. \tag{4.21}$$

Since $f \in \mathcal{O}^{\mathbb{M}_1 \cdot \mathbb{M}_2}(S(d, \gamma))$, we see that there exist $d_3, k_3 > 0$ such that

$$|f(w)| \leq d_3 \, e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3|w|)}, \qquad w \in S(d, \tau_3). \tag{4.22}$$

Now, we define $k_4 := \max(\varepsilon_2, A_2/k_2)$ where A_2 is the constant appearing in (1.8) for $\mathbb{M}_2$ and $s = 2$. We fix $z_0 \in S(d, \gamma + \omega_1 + \omega_2)$, with $\arg(z_0) \in (\psi_0 - \varepsilon, \psi_0 + \varepsilon)$ and $|z_0| < k_1/(k_3 k_4 A_1)$, where A_1 is the constant appearing in (1.8) for $\mathbb{M}_1$ and $s = 1$. We consider $U_0 := B(z_0, \rho_0)$ centered in z_0 such that $\overline{U_0} \subseteq S(\psi_0, \varepsilon, k_1/(k_3 k_4 A_1))$.

In order to prove (4.16), parametrizing the integral and using Tonelli's Theorem, it is enough to show that

$$\int_0^{\infty} \left| e_1\left(\frac{se^{i(\theta+\phi)}}{z}\right) \right| \left(\int_0^{\infty} \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r} \right) \frac{ds}{s} < \infty,$$

for every $z \in U_0$. We fix $\alpha < \min(\alpha_1, 1)$ and

$$s_0 = \min(1, k_2/(k_3 A_{1,2}), \varepsilon_1(|z_0| - \rho_0)),$$

where $A_{1,2}$ is the constant appearing in (1.8) for $\mathbb{M}_1 \cdot \mathbb{M}_2$ and $s = 2$. For all $s < s_0$, we observe that

$$I(s) := \int_0^\infty \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r} = \left(\int_0^{\varepsilon_2 s} + \int_{\varepsilon_2 s}^{\varepsilon_2 s^\alpha} + \int_{\varepsilon_2 s^\alpha}^\infty \right) \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r}.$$

We split the integral into three parts $I_j(s)$ for $j = 1, 2, 3$ defined below. Since $r/(se^{i\phi}) \in S_{\tau_2}$ and $|r/(se^{i\phi})| \leq \varepsilon_2$ for all $r \in (0, \varepsilon_2 s)$, by (4.19) and (4.22), we have that

$$I_1(s) := \int_0^{\varepsilon_2 s} \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r} \leq \frac{C_2 d_3}{s^{\alpha_2}} \int_0^{\varepsilon_2 s} r^{\alpha_2} e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Using that $\omega_{\mathbb{M}_1 \mathbb{M}_2}(k_3 r)$ is nondecreasing we see that

$$I_1(s) \leq \frac{C_2 d_3 \varepsilon_2^{\alpha_2}}{\alpha_2} \exp(\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 \varepsilon_2 s)). \tag{4.23}$$

By (4.21) and (4.22), we see that

$$I_2(s) := \int_{\varepsilon_2 s}^{\varepsilon_2 s^\alpha} \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r} \leq d_2 d_3 \int_{\varepsilon_2 s}^{\varepsilon_2 s^\alpha} e^{-\omega_{\mathbb{M}_2}(k_2 r/s)} e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Using again that $\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)$ is nondecreasing and that

$$\exp(-\omega_{\mathbb{M}_2}(k_2 r/s)) = h_{\mathbb{M}_2}(s/(k_2 r)) = \inf_{p \in \mathbb{N}_0} M_{2,p} \frac{s^p}{(k_2 r)^p} \leq M_{2,1} \frac{s}{k_2 r}, \tag{4.24}$$

we get that

$$I_2(s) \leq \frac{d_2 d_3 M_{2,1}}{k_2} s \exp(\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 \varepsilon_2 s^\alpha)) \int_{\varepsilon_2 s}^{\varepsilon_2 s^\alpha} \frac{dr}{r^2} \leq \frac{d_2 d_3 M_{2,1}}{k_2 \varepsilon_2} \exp(\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 \varepsilon_2 s^\alpha)). \tag{4.25}$$

By (4.21) and (4.22) again, we can see that

$$I_3(s) := \int_{\varepsilon_2 s^\alpha}^\infty \left| e_2\left(\frac{r}{se^{i\phi}}\right) \right| |f(re^{i\theta})| \frac{dr}{r} \leq d_2 d_3 \int_{\varepsilon_2 s^\alpha}^\infty e^{-\omega_{\mathbb{M}_2}(k_2 r/s)} e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Using (4.15) and Lemma 1.1.24 for $\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}$, we see that

$$I_3(s) \leq d_2 d_3 \int_{\varepsilon_2 s^\alpha}^\infty e^{-\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_2 r/s)} e^{\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r} \leq d_2 d_3 \int_{\varepsilon_2 s^\alpha}^\infty e^{-2\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}((k_2 r)/(s A_{1,2})) + \omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Since $\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(t)$ is nondecreasing and $s < s_0 \leq k_2/(k_3 A_{1,2})$, we have that

$$I_3(s) \leq d_2 d_3 \int_{\varepsilon_2 s^\alpha}^\infty e^{-\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Finally, by the definition of $\omega_{\mathbb{M}_1 \cdot \mathbb{M}_2}(t)$ we obtain

$$I_3(s) \leq d_2 d_3 \int_{\varepsilon_2 s^\alpha}^\infty \frac{M_{1,1} M_{2,1}}{k_3 r^2} dr = \frac{d_2 d_3 M_{1,1} M_{2,1}}{k_3 \varepsilon_2 s^\alpha}. \tag{4.26}$$

Consequently, by (4.23), (4.25) and (4.26), for all $s < s_0$ we see that

$$I(s) \leq \frac{C_2 d_3 \varepsilon_2^{\alpha_2}}{\alpha_2} \exp(\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(k_3\varepsilon_2 s_0)) + \frac{d_2 d_3 M_{2,1}}{k_2\varepsilon_2} \exp(\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(k_3\varepsilon_2 s_0^{\alpha})) + \frac{d_2 d_3 M_{1,1} M_{2,1}}{k_3 \varepsilon_2 s^{\alpha}}$$
$$= a_1 + \frac{a_2}{s^{\alpha}}. \tag{4.27}$$

Now, for every $s \geq s_0$ we split the integral into two parts $\tilde{I}_j(s)$ for $j = 1, 2$. Since $r/(se^{i\phi}) \in S_{\tau_2}$ and $|r/(se^{i\phi})| \leq \varepsilon_2$ for all $r \in (0, \varepsilon_2 s)$, by (4.19) and (4.22), as before we get that

$$\tilde{I}_1(s) := \int_0^{\varepsilon_2 s} \left| e_2\left(\frac{r}{se^{i\phi}}\right)\right| |f(re^{i\theta})| \frac{dr}{r} \leq \frac{C_2 d_3}{s^{\alpha_2}} \int_0^{\varepsilon_2 s} r^{\alpha_2} e^{\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Using (4.15) and the monotonicity of $\omega_{\mathbb{M}_1}(t)$, we see that

$$\tilde{I}_1(s) \leq \frac{C_2 d_3 \varepsilon_2^{\alpha_2}}{\alpha_2} \exp(\omega_{\mathbb{M}_1}(k_3\varepsilon_2 s)). \tag{4.28}$$

By (4.21) and (4.22) again, we can see that

$$\tilde{I}_2(s) := \int_{\varepsilon_2 s}^{\infty} \left| e_2\left(\frac{r}{se^{i\phi}}\right)\right| |f(re^{i\theta})| \frac{dr}{r} \leq d_2 d_3 \int_{\varepsilon_2 s}^{\infty} e^{-\omega_{\mathbb{M}_2}(k_2 r/s)} e^{\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(k_3 r)} \frac{dr}{r}.$$

Applying Lemma 1.1.24 to $\omega_{\mathbb{M}_2}$ and by (4.14) we can show that

$$\tilde{I}_2(s) \leq d_2 d_3 \int_{\varepsilon_2 s}^{\infty} e^{-2\omega_{\mathbb{M}_2}(k_2 r/(sA_2))} e^{\omega_{\mathbb{M}_1}(k_3 A_2 s/k_2)} e^{\omega_{\mathbb{M}_2}(k_2 r/(sA_2))} \frac{dr}{r}.$$

Using the definition of $\omega_{\mathbb{M}_2}(t)$, as in (4.24), we deduce that

$$\tilde{I}_2(s) \leq d_2 d_3 e^{\omega_{\mathbb{M}_1}(k_3 A_2 s/k_2)} \int_{\varepsilon_2 s}^{\infty} e^{-\omega_{\mathbb{M}_2}(k_2 r/(sA_2))} \frac{dr}{r} \leq \frac{d_2 d_3 A_2 M_{2,1}}{k_2 \varepsilon_2} e^{\omega_{\mathbb{M}_1}(k_3 A_2 s/k_2)}.$$

Together with (4.28), for all $s \geq s_0$ we get that

$$I(s) \leq \frac{C_2 d_3 \varepsilon_2^{\alpha_2}}{\alpha_2} \exp(\omega_{\mathbb{M}_1}(k_3\varepsilon_2 s)) + \frac{d_2 d_3 A_2 M_{2,1}}{k_2\varepsilon_2} \exp(\omega_{\mathbb{M}_1}(k_3 A_2 s/k_2)).$$

Since $k_4 = \max(\varepsilon_2, A_2/k_2)$ and $\omega_{\mathbb{M}_1}(t)$ is nondecreasing, for every $s \geq s_0$ we have shown that

$$I(s) \leq b_1 \exp(\omega_{\mathbb{M}_1}(k_3 k_4 s)). \tag{4.29}$$

Consequently, for any $z \in U_0$ and any $s \in (0, s_0)$ we have that $se^{i(\theta+\phi)}/z \in S_{\tau_1}$ and $s/|z| \leq s_0/|z| \leq \varepsilon_1$ and, by (4.18) and (4.27), we deduce that

$$J_1(z) := \int_0^{s_0} \left| e_1\left(\frac{se^{i(\theta+\phi)}}{z}\right)\right| I(s)\frac{ds}{s} \leq \frac{C_1}{|z|^{\alpha_1}} \int_0^{s_0} (a_1 s^{\alpha_1} + a_2 s^{\alpha_1-\alpha}) \frac{ds}{s}.$$

Since $\alpha < \alpha_1$, for every $z \in U_0$ we see that

$$\int_0^{s_0} \left| e_1\left(\frac{se^{i(\theta+\phi)}}{z}\right)\right| I(s)\frac{ds}{s} \leq \frac{C_1 a_1 s_0^{\alpha_1}}{\alpha_1(|z_0| - \rho_0)^{\alpha_1}} + \frac{C_1 a_2 s_0^{\alpha_1-\alpha}}{(\alpha_1 - \alpha)(|z_0| - \rho_0)^{\alpha_1}}. \tag{4.30}$$

In the same way, applying (4.20) and (4.29), we get that

$$J_2(z) := \int_{s_0}^{\infty} \left| e_1\left(\frac{se^{i(\theta+\phi)}}{z}\right)\right| I(s)\frac{ds}{s} \leq d_1 b_1 \int_{s_0}^{\infty} \exp(-\omega_{\mathbb{M}_1}(sk_1/|z|) + \omega_{\mathbb{M}_1}(k_3k_4s))\frac{ds}{s}.$$

By Lemma 1.1.24 for $\omega_{\mathbb{M}_1}$ and since $|z| < k_1/(k_3k_4A_1)$ for every $z \in U_0$, we deduce that

$$J_2(z) \leq d_1b_1 \int_{s_0}^{\infty} \exp(-2\omega_{\mathbb{M}_1}(sk_1/(|z|A_1)) + \omega_{\mathbb{M}_1}(k_3k_4s))\frac{ds}{s} \leq d_1b_1 \int_{s_0}^{\infty} \exp(-\omega_{\mathbb{M}_1}(k_3k_4s))\frac{ds}{s},$$

Using the definition of $\omega_{\mathbb{M}_1}(t)$, as in (4.24), we get that

$$J_2(z) \leq \frac{M_{1,1}}{k_3k_4s_0}. \tag{4.31}$$

From (4.30) and (4.31), we see that (4.16) holds.

Hence, by the Leibniz's rule we deduce that the double integral

$$\int_0^{\infty(\theta+\phi)} \int_0^{\infty(\theta)} e_1\left(\frac{v}{z}\right) e_2\left(\frac{u}{v}\right) f(u)\frac{du}{u}\frac{dv}{v} \tag{4.32}$$

defines a holomorphic function in U_0. One can easily show, as in the proof of Proposition 4.1.12, that the value of such function does not depend on the directions ϕ and θ, so it defines a holomorphic function in a sectorial region $G(d,\gamma+\omega_1+\omega_2)$ and we observe that

$$T_{e_1} \circ T_{e_2}(f)(z) = \int_0^{\infty(\theta+\phi)} e_1(v/z)(T_{e_2}f)(v)\frac{dv}{v} = \int_0^{\infty(\theta+\phi)} \int_0^{\infty(\theta)} e_1\left(\frac{v}{z}\right) e_2\left(\frac{u}{v}\right) f(u)\frac{du}{u}\frac{dv}{v},$$

where θ and ϕ (depending on z) are chosen as in the beginning of the proof. Finally, since the double integral in (4.32) converges for any $z \in G(d,\gamma+\omega_1+\omega_2)$, applying Fubini's Theorem and making the change of variables $v = wu$, we see that

$$T_{e_1} \circ T_{e_2}(f)(z) = \int_0^{\infty(\theta)} f(u) \left(\int_0^{\infty(\phi)} e_1(wu/z)e_2(1/w)\frac{dw}{w}\right)\frac{du}{u}.$$

□

Remark 4.3.20. In the proof of the last proposition, we have shown that for any $f \in \mathcal{O}^{\mathbb{M}_1\cdot\mathbb{M}_2}(S)$ we have $T_{e_2}(f) \in \mathcal{O}^{\mathbb{M}_1}(S(d,\gamma+\omega(\mathbb{M}_2)))$ (see (4.27)+(4.29)), which extends the classical Gevrey result.

Finally, we construct the convolution kernel of two strong kernels and we prove that it is also a strong kernel of $\mathbb{M}_1\cdot\mathbb{M}_2-$summability.

Proposition 4.3.21. Let $\mathbb{M}_j$, $j = 1,2$, be normalized weight sequences admitting a nonzero proximate order. Assume $\omega(\mathbb{M}_1)+\omega(\mathbb{M}_2) < 2$ and consider strong kernels e_j of $\mathbb{M}_j-$summability, its moment function m_{e_j} and $T_{e_j}, T^-_{e_j}$ the corresponding Laplace or Borel operators.

1. We define *the convolution of* e_1 *and* e_2, denoted $e_1 * e_2$, by

$$e_1 * e_2(z) := T_{e_1}(e_2(1/u))(1/z).$$

Then, $e_1 * e_2$ is a strong kernel of $\mathbb{M}_1\cdot\mathbb{M}_2-$summability whose moment function is $m(\lambda) = m_{e_1}(\lambda)m_{e_2}(\lambda)$. Moreover if E_1 and E are the kernels associated by Definition 4.3.10.(v) with e_1 and $e_1 * e_2$, respectively, we have that

$$E(z) = T^-_{e_2}E_1(z).$$

2. The function $e_1 * e_2$ is the unique moment summability kernel with moment sequence $(m(p) = m_{e_1}(p)m_{e_2}(p))_{p\in\mathbb{N}_0}$.

3. Let $T_{e_1} * T_{e_2}$ denote the Laplace-like integral operator associated with $e_1 * e_2$. If S is an unbounded sector and $f \in \mathcal{O}^{\mathbb{M}_1\cdot\mathbb{M}_2}(S)$, then

$$(T_{e_1} * T_{e_2})f = T_{e_1} \circ T_{e_2}(f).$$

4. We consider $f(u) = (1-u)^{-1}$. We define $g(z) := ((T_{e_1} \circ T_{e_2})f)(z)$ and

$$e(1/z) := \frac{g(z) - g(ze^{2\pi i})}{2\pi i}.$$

Then, e is well defined in $S_{\omega(\mathbb{M}_1)+\omega(\mathbb{M}_2)}$ and $e(z) = e_1 * e_2(z)$.

Proof. For simplicity we write $\omega_1 = \omega(\mathbb{M}_1)$ and $\omega_2 = \omega(\mathbb{M}_2)$. We observe that $\omega(\mathbb{M}_1 \cdot \mathbb{M}_2) = \omega_1 + \omega_2$ (see Remark 4.2.10).

1. Let us show that the function $e_1 * e_2(z) = T_{e_1}(e_2(1/u))(1/z)$ has the necessary properties for a strong kernel function of $\mathbb{M}_1\cdot\mathbb{M}_2-$summability, as listed in Definition 4.3.10. Since the function $e_2(1/u)$ is holomorphic in S_{ω_2} and bounded on every sector S_β with $0 < \beta < \omega_2$, by Lemma 4.3.5, we have that $T_{e_1}(e_2(1/u))(z)$ is holomorphic in $S_{\omega_1+\omega_2}$ which proves (I).

   Regarding the integrability condition (II.B), we fix $\tau \in (0, \omega_1+\omega_2)$ and we take $\tau_1 \in (0, \omega_1)$ and $\tau_2 \in (0, \omega_2)$ such that $\tau < \tau_1 + \tau_2$. By Definition 4.3.10 (II.B) for e_1 and e_2 we know that there exist $\alpha_1, \alpha_2 > 0$ (not depending on τ_1 and τ_2), and constants $C_1, C_2 > 0$, $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that

$$|e_1(z)| \le C_1|z|^{\alpha_1}, \qquad z \in S_{\tau_1}, \qquad |z| \le \varepsilon_1, \tag{4.33}$$
$$|e_2(z)| \le C_2|z|^{\alpha_2}, \qquad z \in S_{\tau_2}, \qquad |z| \le \varepsilon_2. \tag{4.34}$$

   We fix $z \in S_\tau$ with $|z| \le (\varepsilon_1\varepsilon_2)^2$ and we choose $|\theta| < \pi\tau_2/2$, such that $ze^{i\theta} \in S_{\tau_1}$. We have that

$$\begin{aligned}|e_1 * e_2(z)| &= \left|\int_0^{\infty(\theta)} e_1(uz)e_2\left(\frac{1}{u}\right)\frac{du}{u}\right| \\ &\le \int_0^{\varepsilon_1/|z|^{1/2}} \left|e_1(re^{i\theta}z)e_2\left(\frac{1}{re^{i\theta}}\right)\right|\frac{dr}{r} + \int_{\varepsilon_1/|z|^{1/2}}^{\infty} \left|e_1(re^{i\theta}z)e_2\left(\frac{1}{re^{i\theta}}\right)\right|\frac{dr}{r}.\end{aligned}$$

   If $r \le \varepsilon_1/|z|^{1/2}$, we have that $|re^{i\theta}z| \le \varepsilon_1|z|^{1/2} \le \varepsilon_1^2\varepsilon_2 \le \varepsilon_1$, and if $r \ge \varepsilon_1/|z|^{1/2}$, we see that $|1/(re^{i\theta})| \le |z|^{1/2}/\varepsilon_1 \le \varepsilon_2$. Applying (4.33) and (4.34), we obtain

$$|e_1 * e_2(z)| \le C_1|z|^{\alpha_1}\int_0^{\varepsilon_1/|z|^{1/2}} \left|e_2\left(\frac{1}{re^{i\theta}}\right)\right|\frac{dr}{r^{1-\alpha_1}} + C_2\int_{\varepsilon_1/|z|^{1/2}}^{\infty}\left|e_1(re^{i\theta}z)\right|\frac{dr}{r^{1+\alpha_2}}.$$

   By condition (III) for e_1 and e_2, we know that there exist constants D_1, D_2 such that $|e_1(w)| \le D_1$ for every $w \in S_{\tau_1}$ and $|e_2(w)| \le D_2$ for every $w \in S_{\tau_2}$. We deduce that

$$|e_1 * e_2(z)| \le \frac{C_1D_2\varepsilon_1^{\alpha_1}}{\alpha_1}|z|^{\alpha_1/2} + \frac{C_2D_1}{\varepsilon_1^{\alpha_2}\alpha_2}|z|^{\alpha_2/2}.$$

   Consequently, condition (II.B) is satisfied with $\alpha = \min(\alpha_1/2, \alpha_2/2)$.

By condition (III) for e_2, for every $\varepsilon > 0$, there exist $c, k > 0$ such that

$$|e_2(1/u)| \le c\, e^{-\omega_{\mathbb{M}_2}(k/|u|)}, \qquad u \in S_{\omega_2-\varepsilon}.$$

Then, by Proposition 3.1.9, we have that $e_2(1/u) \sim_{\mathbb{M}_2} \hat{0}$ in S_{ω_2}. By Remark 4.3.6, we see that $e_1 * e_2(1/z) = T_{e_1}(e_2(1/u))(z) \sim_{\mathbb{M}_1\mathbb{M}_2} \hat{0}$ in $S_{\omega_1+\omega_2}$ which implies, again by Proposition 3.1.9, that for every $\varepsilon > 0$ there exist $c, k, r > 0$ such that

$$|e_1 * e_2(z)| \le c\, e^{-\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(|z|/k)}, \qquad z \in S_{\omega_1+\omega_2-\varepsilon}, \qquad |z| > r. \tag{4.35}$$

By condition (II.B) for $e_1 * e_2$, we know that $|e_1 * e_2(z)| \le C$ for $z \in S_{\omega_1+\omega_2-\varepsilon}$ with $|z| \le \delta$. Since $e_1 * e_2(z)$ is continuous, (4.35) holds for every $z \in S_{\omega_1+\omega_2-\varepsilon}$ and we conclude that condition (III) is satisfied.

Condition (IV) holds immediately because for $x > 0$ we have that

$$e_1 * e_2(x) = \int_0^\infty e_1(xy) e_2(1/y) \frac{dy}{y}.$$

Since e_1 and e_2 are positive real over the positive real axis, we deduce $e_1 * e_2(x)$ also is. Let us show that

$$m_{e_1*e_2}(\lambda) = m_{e_1}(\lambda) m_{e_2}(\lambda). \tag{4.36}$$

We have that

$$m_{e_1*e_2}(\lambda) = \int_0^\infty x^{\lambda-1}(e_1 * e_2)(x)dx = \int_0^\infty \int_0^\infty x^{\lambda-1} e_1(xy)e_2(1/y)\frac{dy}{y}dx.$$

We make the change of variables $t = xy$ and $s = 1/y$ and we get

$$m_{e_1*e_2}(\lambda) = \int_0^\infty \int_0^\infty (st)^{\lambda-1} e_1(t)e_2(s)dtds = m_{e_1}(\lambda) m_{e_2}(\lambda).$$

Consequently, using property (V) of e_1 and e_2, we deduce that $m_{e_1*e_2}(\lambda)$ is continuous in $\{\mathrm{Re}(\lambda) \ge 0\}$, holomorphic in $\{\mathrm{Re}(\lambda) > 0\}$ and $m_e(x) > 0$ for every $x \ge 0$.

We define the function E by

$$E(z) := \sum_{n=0}^\infty \frac{z^n}{m_{e_1*e_2}(n)}, \qquad z \in \mathbb{C}.$$

If we compute the radius of convergence of this series, using (4.36), we see that

$$r = \liminf_{n\to\infty} \sqrt[n]{m_{e_1*e_2}(n)} = \liminf_{n\to\infty} \sqrt[n]{m_{e_1}(n) m_{e_2}(n)} = \infty$$

hence E is entire. We see that $(m_{e_1*e_2}(p))_{p\in\mathbb{N}_0}$, again by (4.36), is equivalent to the sequence $\mathbb{M}_1 \cdot \mathbb{M}_2$, then, by Proposition 4.1.7, we see that there exist $C, K > 0$ such that $|E(z)| \le Ce^{\omega_{\mathbb{M}_1\cdot\mathbb{M}_2}(K|z|)}$, $z \in \mathbb{C}$, then (V) holds.

Finally, regarding condition (VI.B), we need first to show that

$$E(u) = (T^-_{e_2} E_1)(u), \qquad u \in \mathbb{C}^*. \tag{4.37}$$

We fix $u \in \mathbb{C}^*$, write $\tau = \arg(u)$ and consider a path $\delta_{\omega_2}(\tau)$ (see Definition 4.1.13). Since E_1 is entire and $\delta_{\omega_2}(\tau)$ compact, we have that

$$|\sum_{n=0}^N z^n/m_{e_1}(n)| \le E_1(|z|) \le C_\tau, \qquad z \in \delta_{\omega_2}(\tau), \qquad N \in \mathbb{N}_0,$$

and by condition (VI.B) for E_2, $|E_2(u/z)z^{-1}|$ is integrable on $\delta_{\omega_2}(\tau)$, so we can apply Dominated Convergence Theorem. Therefore, we can exchange integral and sum and, by Proposition 4.1.16, we see that

$$(T_{e_2}^{-}E_1)(u)=\sum_{n=0}^{\infty}(T_{e_2}^{-}(z^n/m_{e_1}(n)))(u)=\sum_{n=0}^{\infty}\frac{u^n}{m_{e_1*e_2}(n)}=E(u).$$

We fix $\tau\in(0,2-(\omega_1+\omega_2))$, we take $\tau_1\in(0,2-\omega_1)$ and $\tau_2\in(0,2-\omega_2)$ such that $\tau+2\in(0,\tau_1+\tau_2-(2-\omega_2-\tau_2))$ and we can choose $\varepsilon\in(2-\omega_2-\tau_2,\tau_1+\tau_2-2-\tau)$. By (VI.B) for E_1 and E_2, we know that there exist $\beta_1,\beta_2>0$ (not depending on τ_1 and τ_2), and constants $K_1,K_2>0$, $M_1,M_2\geq 1$ such that

$$|E_1(z)|\leq\frac{K_1}{|z|^{\beta_1}},\qquad z\in S(\pi,\tau_1),\qquad |z|\geq M_1, \tag{4.38}$$

$$|E_2(z)|\leq\frac{K_2}{|z|^{\beta_2}},\qquad z\in S(\pi,\tau_2),\qquad |z|\geq M_2. \tag{4.39}$$

We fix $u\in S(\pi,\tau)$ with $|u|\geq M_1M_2$. We write $\phi=\arg(u)\in(0,2\pi)$ and we may consider a path $\delta_{\omega_2}(\phi)$ (see Definition 4.1.13). We can write $\delta_{\omega_2}(\phi)=\delta_1+\delta_2+\delta_3$ (δ_1 and δ_3 are segments in directions $\theta_1=\phi+(\pi/2)(\omega_2+\varepsilon)$ and $\theta_3=\phi-(\pi/2)(\omega_2+\varepsilon)$, respectively, and δ_2 is a circular arc with radius $R=|u|/M_2$, that can be chosen in this way because E_1 is entire). Then, using (4.37), we see that

$$E(u)=\frac{-1}{2\pi i}\int_{\delta_{\omega_2}(\phi)}E_2\left(\frac{u}{z}\right)E_1\left(z\right)\frac{dz}{z}.$$

We have that

$$|E(u)|\leq\frac{1}{2\pi}\left(\int_0^{|u|/M_2}\left|E_2\left(\frac{u}{re^{i\theta_1}}\right)E_1\left(re^{i\theta_1}\right)\right|\frac{dr}{r}+\int_{\theta_3}^{\theta_1}\left|E_2\left(\frac{uM_2}{|u|e^{i\theta}}\right)E_1\left(\frac{|u|e^{i\theta}}{M_2}\right)\right|d\theta\right.$$
$$\left.+\int_0^{|u|/M_2}\left|E_2\left(\frac{u}{re^{i\theta_3}}\right)E_1\left(re^{i\theta_3}\right)\right|\frac{dr}{r}\right). \tag{4.40}$$

First, we study the second integral in (4.40). By condition (V) for E_2, we know that there exists a constant H_2 such that $|E_2(z)|\leq H_2$ for every $z\in D(0,M_2+1)$ and we deduce that

$$\int_{\theta_3}^{\theta_1}\left|E_2\left(M_2e^{i(\phi-\theta)}\right)E_1\left(\frac{|u|e^{i\theta}}{M_2}\right)\right|d\theta\leq H_2\int_{\theta_3}^{\theta_1}\left|E_1\left(\frac{|u|e^{i\theta}}{M_2}\right)\right|d\theta.$$

Using the upper bounds of ε we see that $2-\tau-\omega_2-\varepsilon>2-\tau_1$ and $2+\tau+\omega_2+\varepsilon<2+\tau_1$, then $[\theta_3,\theta_1]\subseteq((\pi/2)(2-\tau-\omega_2-\varepsilon),(\pi/2)(2+\tau+\omega_2+\varepsilon))$ and we get that

$$|u|e^{i\theta}/M_2\in S(\pi,\tau_1),\qquad \text{if } \theta\in[\theta_3,\theta_1]. \tag{4.41}$$

Since $|u|\geq M_1M_2$ we have that $|ue^{i\theta}|/M_2\geq M_1$ and by (4.38), we deduce that

$$H_2\int_{\theta_3}^{\theta_1}\left|E_1\left(\frac{|u|e^{i\theta}}{M_2}\right)\right|d\theta\leq\frac{H_2K_1M_2^{\beta_1}}{|u|^{\beta_1}}\pi\tau_1,\qquad |u|\geq M_1M_2. \tag{4.42}$$

We study now the first and the last integral in (4.40). If $r\in(0,|u|/M_2)$, we observe that $|u/re^{i\theta_j}|\geq M_2$. Since $\omega_2+\tau_2<2$ and $\tau_1-2-\tau<-\omega_1-\tau<0$, using the bounds for ε, we have that

$$\omega_2+\varepsilon\in(2-\tau_2,\tau_2+\omega_2+\tau_1-2-\tau)\subseteq(2-\tau_2,2)$$

and we deduce that $\arg(u/re^{i\theta_3}) = (\pi/2)(\omega_2+\varepsilon) \in ((\pi/2)(2-\tau_2),\pi)$ and $\arg(u/re^{i\theta_1}) \in (-\pi,-(\pi/2)(2-\tau_2))$. Then for $j=1,3$ we see that $u/re^{i\theta_j} \in S(\pi,\tau_2)$ and, by (4.39), we show that

$$\int_0^{|u|/M_2} \left|E_2\left(\frac{u}{re^{i\theta_j}}\right)E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r} \le \frac{K_2}{|u|^{\beta_2}}\int_0^{|u|/M_2} r^{\beta_2}\left|E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r}.$$

Since $|u|\ge M_1M_2$, we can write $(0,|u|/M_2)$ as the disjoint union of the intervals $(0,M_1)$ and $[M_1,|u|/M_2)$. By condition (V) for E_1, we know that there exists a constant H_1 such that $|E_1(z)|\le H_1$ for every $z\in D(0,M_1+1)$, then

$$\frac{K_2}{|u|^{\beta_2}}\int_0^{|u|/M_2} r^{\beta_2}\left|E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r} \le \frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_2}{|u|^{\beta_2}}\int_{M_1}^{|u|/M_2} r^{\beta_2}\left|E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r}.$$

If $r\ge M_1$, by (4.41), we can use (4.38) and we obtain

$$\frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_2}{|u|^{\beta_2}}\int_{M_1}^{|u|/M_2} r^{\beta_2}\left|E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r} \le \frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_1K_2}{|u|^{\beta_2}}\int_{M_1}^{|u|/M_2} r^{\beta_2-\beta_1}\frac{dr}{r}.$$

According to Remark 4.3.15, we may assume that $\beta_1<\beta_2$, and we conclude that

$$\frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_1K_2}{|u|^{\beta_2}}\int_{M_1}^{|u|/M_2} r^{\beta_2-\beta_1}\frac{dr}{r} \le \frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_1K_2}{|u|^{\beta_1}(\beta_2-\beta_1)M_2^{\beta_2-\beta_1}}.$$

Then for $j=1,3$ and for every $u\in S(\pi,\tau)$ with $|u|\ge M_1M_2$ we have that

$$\int_0^{|u|/M_2} \left|E_2\left(\frac{u}{re^{i\theta_j}}\right)E_1\left(re^{i\theta_j}\right)\right|\frac{dr}{r} \le \frac{K_2H_1M_1^{\beta_2}}{|u|^{\beta_2}\beta_2}+\frac{K_1K_2}{|u|^{\beta_1}(\beta_2-\beta_1)M_2^{\beta_2-\beta_1}}. \tag{4.43}$$

Consequently, by (4.42) and (4.43), condition (VI.B) is satisfied with $\beta=\min(\beta_1,\beta_2)$, for every $u\in S(\pi,\tau)$ with $|u|\ge M_1M_2$.

2. Uniqueness follows from Remark 4.3.8.

3. We take $f\in\mathcal{O}^{\mathbb{M}_1\cdot\mathbb{M}_2}(S(d,\alpha))$. Since e_1*e_2 is a kernel of $\mathbb{M}_1\dot{\mathbb{M}}_2-$summability, by Proposition 4.1.12 we know that $T_{e_1}*T_{e_2}(f)$ is holomorphic in a sectorial region $G(d,\alpha+\omega_1+\omega_2)$. We fix $z\in G(d,\alpha+\omega_1+\omega_2)$, there exists θ with $|\theta-d|<\pi\alpha/2$ such that if $\arg(u)=\theta$, then $u/z\in S_{\omega_1+\omega_2}$ and there exists ϕ with $|\phi|<\pi\omega_2/2$ such that if $\arg(w)=\phi$, then $wu/z\in S_{\omega_1}$. We have that

$$\begin{aligned} T_{e_1}*T_{e_2}(f)(z) &= \int_0^{\infty(\theta)} e_1*e_2(u/z)f(u)\frac{du}{u}\\ &= \int_0^{\infty(\theta)} f(u)\left(\int_0^{\infty(\phi)} e_1(wu/z)e_2(1/w)\frac{dw}{w}\right)\frac{du}{u}.\end{aligned}$$

We conclude, using Proposition 4.3.19, that this last expression is equal to $T_{e_1}\circ T_{e_2}(f)(z)$.

4. We consider $f(u)=(1-u)^{-1}$ and we define $g(z):=((T_{e_1}\circ T_{e_2})f)(z)$. By Lemma 4.3.7, we know that $T_{e_2}f$ is holomorphic in $S(\pi,2+\omega_2)$ and $T_{e_2}f(z)\to 0$ as $z\to\infty$ uniformly on every sector $S(\pi,2+\gamma)$ with $\gamma<\omega_2$. Moreover, $T_{e_2}f\sim_{\mathbb{M}_2}\sum_{n=0}^{\infty}m_2(n)z^n$ on $S(\pi,2+\omega_2)$. Then we deduce that $T_{e_2}f$ is bounded on every sector $S(\pi,2+\gamma)$ with $\gamma<\omega_2$. We can apply

the T_{e_1} transform to $T_{e_2}f$ and, by Lemma 4.3.5, we have that $g = T_{e_1}T_{e_2}f$ is holomorphic in $S = S(\pi, 2+\omega_1+\omega_2)$. Then, the function

$$e(1/z) := \frac{g(z) - g(ze^{2\pi i})}{2\pi i},$$

is holomorphic in $S_{\omega_1+\omega_2}$. Since $f \in \mathcal{O}^{\mathbb{M}_1\cdot\mathbb{M}_2}(S_{\omega_1+\omega_2})$, then, by statement 3, $(T_{e_1} * T_{e_2})(f) = (T_{e_1} \circ T_{e_2})(f)$ and, by Lemmma 4.3.7, we deduce that

$$e(1/z) = \frac{g(z) - g(ze^{2\pi i})}{2\pi i} = \frac{(T_{e_1} * T_{e_2})(f)(z) - ((T_{e_1} * T_{e_2})f)(ze^{2\pi i})}{2\pi i} = e_1 * e_2(1/z).$$

□

Remark 4.3.22. We know that $\mathbb{M}_1 \cdot \mathbb{M}_2$ is a weight sequence with quotients tending to infinity admitting nonzero proximate order (see Proposition 4.2.5). Consequently, we can construct a kernel e of $\mathbb{M}_1 \cdot \mathbb{M}_2-$summability (see Remark 4.1.4). However, we do not have any control on the corresponding moment sequence of e apart from being equivalent to $\mathbb{M}_1 \cdot \mathbb{M}_2$. Last proposition guarantees that the moment sequence associated with $e_1 * e_2$ is $(m_{e_1}(n)m_{e_2}(n))_{n\in\mathbb{N}_0}$, which is important because it ensures good behavior of formal and analytic Borel-Laplace operators for asymptotics.

Remark 4.3.23. Note that $\mathcal{O}^{\mathbb{M}_1\cdot\mathbb{M}_2}(S) \subset \mathcal{O}^{\mathbb{M}_2}(S)$. Consequently, $T_{e_1} \circ T_{e_2}$ extends the operator $T_{e_1} * T_{e_2}$. The opposite situation occurs for the acceleration operator that will be presented in the next subsection, whose main advantage is that it extends the composition operator.

4.3.4 Acceleration kernels

In the same conditions as in the previous subsection, assuming in addition that $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2)$, we will construct a pair of operators T, T^- such that T extends $T_{e_1}^- \circ T_{e_2}$. This new operator will be called the acceleration operator from e_2 to e_1 because it will send a function $f \in \mathcal{O}^{\mathbb{M}_2}(S)$ into a function with greater growth $Tf \in \mathcal{O}^{\mathbb{M}_1}(\tilde{S})$. According to this property, the kernel associated with T will be called acceleration kernel and its corresponding sequence of moments will be $m_{e_2}(\lambda)/m_{e_1}(\lambda)$.

Our first result is the analogous version of Proposition 4.3.19 that guarantees that the operators T and $T_{e_1}^- \circ T_{e_2}$ coincide for a large enough class of functions.

Proposition 4.3.24. Let $\mathbb{M}_j$, $j = 1, 2$, be weight sequence admitting a nonzero proximate order. We consider strong kernels e_j of $\mathbb{M}_j-$summability. Let T_{e_j} $T_{e_j}^{-1}$ be the corresponding integral operators. Assume that $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < 2$.

If $f \in \mathcal{O}^{\mathbb{M}_2}(S(d,\gamma))$, then for every $z_0 \in S(d, \gamma + \omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))$ there exist a neighborhood $U_0 \subseteq S(d, \gamma + \omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))$ of z_0 and a direction ϕ with $|d - \phi| < \pi\gamma/2$ (depending on z_0) such that we have

$$\int_{\delta_{\omega_1}(\arg(z_0))} \int_0^{\infty(\phi)} \left| E_1(z/v)e_2(u/v)f(u)\frac{du}{u}\frac{dv}{v} \right| < \infty \tag{4.44}$$

for every $z \in U_0$, where $\delta_{\omega_1}(\arg(z_0))$ is a path as considered in Definition 4.1.13. Moreover, the function

$$F(z) := \int_{\delta_{\omega_1}(\arg(z))} \int_0^{\infty(\phi)} E_1(z/v)e_2(u/v)f(u)\frac{du}{u}\frac{dv}{v}$$

is holomorphic in $S(d,\alpha+\omega(\mathbb{M}_2)-\omega(\mathbb{M}_1))$ and

$$T_{e_1}^{-}\circ T_{e_2}(f)(z)=\int_0^{\infty(\phi)} f(u)\left(\frac{-1}{2\pi i}\int_{\delta_{\omega_1}(\arg(z)-\phi)} E_1(z/wu)e_2\left(1/w\right)\frac{dw}{w}\right)\frac{du}{u}.$$

Proof. For simplicity we write $\omega_1=\omega(\mathbb{M}_1)$ and $\omega_2=\omega(\mathbb{M}_2)$, $\mathbb{M}_1=(M_{1,p})_{p\in\mathbb{N}_0}$ and $\mathbb{M}_2=(M_{2,p})_{p\in\mathbb{N}_0}$. We observe that $\omega(\mathbb{M}_2/\mathbb{M}_1)=\omega_2-\omega_1$ (see Remark 4.2.10).

We fix $z_0\in S(d,\alpha+\omega_2-\omega_1)$, since $\omega_2-\omega_1>0$ we can consider directions $\tau_1\in(0,2-\omega_1)$, $\tau_2\in(0,\omega_2)$, $\tau_3\in(0,\gamma)$ with $2-\tau_1-\omega_1<\tau_2+\tau_1-2+\tau_3-\gamma$. Then, we can choose

$$\varepsilon\in(2-\tau_1-\omega_1,\tau_2+\tau_1-2+\tau_3-\gamma)\subseteq(0,\omega_2-\omega_1),$$

and $\phi>0$ with $|\phi-d|<\tau_3\pi/2$ such that

$$|\arg(z_0)-\phi|<(\tau_2+\tau_1-2-\varepsilon+\tau_3-\gamma)\pi/2. \tag{4.45}$$

We observe that we can take $\rho_0>0$ small enough such that $\overline{B(z_0,\rho_0)}\subseteq S(d,\gamma+\omega_2-\omega_1)$ and the inequality (4.45) remains valid if we replace $\arg(z_0)$ by $\arg(z)$ for every $z\in B(z_0,\rho_0)$. We write $\theta_1=\arg(z_0)+(\omega_1+\varepsilon)\pi/2$ and $\theta_3=\arg(z_0)-(\omega_1+\varepsilon)\pi/2$, since $\tau_2<2$ we observe that the value of ε guarantees that $2-\omega_1-\varepsilon>0$ and $\omega_1+\tau_1-2+\varepsilon>0$. Then, by suitably reducing the radius ρ_0, we also have that

$$|\arg(z)-\arg(z_0)|<\delta:=\min((2-\omega_1-\varepsilon)\pi/2,(\omega_1+\tau_1-2+\varepsilon)\pi/2)/2, \tag{4.46}$$

for every $z\in U_0=B(z_0,\rho_1)$ with $\rho_1\le\rho_0$, which implies

$$-\pi<\arg(z)-\theta_1<(\tau_1-2)\pi/2,\qquad (2-\tau_1)\pi/2<\arg(z)-\theta_3<\pi. \tag{4.47}$$

Moreover, from (4.45) we deduce that

$$|\theta-\phi|\le|\theta-\arg(z_0)|+|\arg(z_0)-\phi|<(\omega_1+\tau_1-2+\tau_2+\tau_3-\gamma)\pi/2\le\tau_2\pi/2 \tag{4.48}$$

for every $\theta\in[\theta_3,\theta_1]$.

By Definition 4.3.10.(VI.B) for E_1 and (II.B) for e_2 we know that there exist $\beta_1,\alpha_2>0$ (not depending on τ_1 and τ_2), and constants $K_1,C_2>0$, $M\ge1$, $\varepsilon_2\in(0,1)$ such that

$$|E_1(z)|\le\frac{K_1}{|z|^{\beta_1}},\qquad z\in S(\pi,\tau_1),\qquad |z|\ge M, \tag{4.49}$$

$$|e_2(z)|\le C_2|z|^{\alpha_2},\qquad z\in S_{\tau_2},\qquad |z|\le\varepsilon_2. \tag{4.50}$$

By condition (III) for e_2, there exist $d_2,k_2>0$ such that

$$|e_2(w)|\le d_2\,e^{-\omega_{\mathbb{M}_2}(k_2|w|)},\qquad u\in S_{\tau_2},$$

and since $f\in\mathcal{O}^{\mathbb{M}_2}(S(d,\gamma))$, we see that there exist $d_3,k_3>0$ such that

$$|f(w)|\le d_3\,e^{\omega_{\mathbb{M}_2}(k_3|w|)},\qquad w\in S(d,\tau_3). \tag{4.51}$$

We fix $s_0=\min(1,k_2/(k_3A_2),(|z_0-\rho_1|/M))$ and $\beta<\min(\beta_1,1)$, where A_2 is the constant appearing in (1.8) for $\mathbb{M}_2$ and $s=2$. We consider a path $\delta_{\omega_1}(\arg(z_0))$ (see Definition 4.1.13). We can write $\delta_{\omega_1}(\arg(z_0))=\delta_1+\delta_2+\delta_3$ (δ_1 and δ_3 are segments in directions $\theta_1=\arg(z_0)+(\omega_1+\varepsilon)\pi/2$ and $\theta_3=\arg(z_0)-(\omega_1+\varepsilon)\pi/2$, respectively, and δ_2 is a circular arc with radius $R=s_0$).

In order to prove (4.44), parametrizing the integral and using Tonelli's Theorem, it is enough to show that

$$J_i(z)=\int_0^{s_0}|E_1(z/se^{i\theta_i})|\int_0^{\infty}|e_2(re^{i\phi-\theta_i}/s)f(re^{i\phi})|\frac{dr}{r}\frac{ds}{s}<\infty,\qquad i=1,3,$$

$$J_2(z)=\int_{\theta_3}^{\theta_1}|E_1(z/s_0e^{i\theta})|\int_0^{\infty}|e_2(re^{i\phi-\theta}/s_0)f(re^{i\phi})|\frac{dr}{r}d\theta<\infty,$$

for every $z\in U_0$. For $s\le s_0$ and $\theta\in[\theta_3,\theta_1]$ we consider

$$I(s,\theta):=\int_0^{\infty}|e_2(re^{i\phi-\theta}/s)f(re^{i\phi})|\frac{dr}{r}.$$

By (4.48) we show that $re^{i\phi-i\theta}/s\in S_{\tau_2}$ for all $\theta\in[\theta_3,\theta_1]$ and every $s\le s_0$. Then, splitting the interval into three parts $(0,\varepsilon_2 s)$, $(\varepsilon_2 s,\varepsilon_2 s^{\beta})$, $(\varepsilon_2 s^{\beta},\infty)$ as in the proof of Proposition 4.3.19 and using (4.50), (4.51), Lemma 1.1.24 and that $\omega_{\mathbb{M}_2}(t)$ is nondecreasing, we get that

$$I(s,\theta)\le\frac{C_2d_3\varepsilon_2^{\alpha_2}}{\alpha_2}\exp(\omega_{\mathbb{M}_2}(k_3\varepsilon_2 s_0))+\frac{d_2d_3M_{2,1}}{k_2\varepsilon_2}\exp(\omega_{\mathbb{M}_2}(k_3\varepsilon_2 s_0^{\beta}))+\frac{d_2d_3M_{2,1}}{k_3\varepsilon_2 s^{\beta}}=:a_1+\frac{a_2}{s^{\beta}}.\quad(4.52)$$

Applying (4.47), we see that $z/se^{i\theta_i}\in S(\pi,\tau_1)$ for $i=1,3$ and every $z\in U_0$ and since $|z/s|\ge|z|/s_0\ge M$, we can apply (4.49) and (4.52) and we see that

$$J_i(z)\le K_1\int_0^{s_0}\frac{s^{\beta_1}}{|z|^{\beta_1}}\left(a_1+\frac{a_2}{s^{\beta}}\right)\frac{ds}{s}\le\frac{K_1a_1s_0^{\beta_1}}{\beta_1(|z_0|-\rho_1)^{\beta_1}}+\frac{K_1a_2s_0^{\beta_1-\beta}}{(\beta_1-\beta)(|z_0|-\rho_1)^{\beta_1}},\qquad(4.53)$$

for $i=1,3$ and every $z\in U_0$. Using that E_1 is entire we have that $|E_1(w)|\le H_1$ for every $w\in B(0,(|z_0|+\rho_1+1)/s_0)$, and (4.52) shows that

$$J_2(z)\le\left(a_1+\frac{a_2}{s_0^{\beta}}\right)(\theta_1-\theta_3)H_1\qquad(4.54)$$

for every $z\in U_0$. Using (4.53) and (4.54) we see that (4.44) holds. Hence, by the Leibniz's rule the double integral

$$\int_{\delta_{\omega_1}(\arg(z_0))}\int_0^{\infty(\phi)}E_1(z/v)e_2(u/v)f(u)\frac{du}{u}\frac{dv}{v}$$

is a holomorphic function in the neighborhood U_0 of z_0 for every $z_0\in S(d,\alpha+\omega_2-\omega_1)$. If $z\in U_0\cap U_1$, with U_1 the corresponding neighborhood of z_1, the choice of $\delta>0$ in (4.46) guarantees that $\lim_{s\to0}|E_1(z/se^{i\theta})|I(s,\theta)=0$, uniformly for θ between θ_i and $\theta_i'=\arg(z_1)\pm(\omega_1+\varepsilon)\pi/2$ for $i=1,3$. This fact ensures that we can apply Cauchy's theorem to deform the path of integration from $\delta_{\omega_1}(\arg(z_0))$ to $\delta_{\omega_1}(\arg(z_1))$, and we deduce that

$$\int_{\delta_{\omega_1}(\arg(z))}\int_0^{\infty(\phi)}E_1(z/v)e_2(u/v)f(u)\frac{du}{u}\frac{dv}{v}\qquad(4.55)$$

defines a holomorphic in the sector $S(d,\alpha+\omega_2-\omega_1)$. We observe that

$$\begin{aligned}T_{e_1}^{-}\circ T_{e_2}(f)(z)=&\frac{-1}{2\pi i}\int_{\delta_{\omega_1}(\arg(z))}E_1(z/v)(T_{e_2}f)(v)\frac{dv}{v}\\=&\frac{-1}{2\pi i}\int_{\delta_{\omega_1}(\arg(z))}\int_0^{\infty(\phi)}E_1(z/v)e_2\left(\frac{u}{v}\right)f(u)\frac{du}{u}\frac{dv}{v},\end{aligned}$$

where ϕ and $\delta_{\omega_1}(\arg(z))$ are chosen as in the beginning of the proof. We write $\eta = \arg(z) - \phi$ and we make the change of variables $v = wu$. Then the path $\delta_{\omega_1}(\arg(z))$ is transformed into the path $\delta_{\omega_1}(\eta)$. We can write $\delta_{\omega_1}(\eta) = \gamma_1 + \gamma_2 + \gamma_3$ (γ_1 and γ_3 are segments in directions $\theta_1'' = \eta + (\pi/2)(\omega_1 + \varepsilon)$ and $\theta_3'' = \eta - (\pi/2)(\omega_1 + \varepsilon)$, respectively, and γ_2 is a circular arc with radius $R = s_0/|u|$, the path $\delta_{\omega_1}(\eta)$ stays inside S_{ω_2}. We have that

$$T_{e_1}^{-} \circ T_{e_2}(f)(z) = \frac{-1}{2\pi i} \int_{\delta_{\omega_1}(\eta)} \int_0^{\infty(\phi)} E_1(z/wu) e_2\left(1/w\right) f(u) \frac{du}{u} \frac{dw}{w}.$$

Finally, since the double integral in (4.55) converges for any $z \in S(d, \alpha + \omega_2 - \omega_1)$, we can apply Fubini's theorem and we can interchange the integration order, then we see that

$$T_{e_1}^{-} \circ T_{e_2}(f)(z) = \int_0^{\infty(\phi)} f(u) \left(\frac{-1}{2\pi i} \int_{\gamma_{\omega_1}(\arg(z)-\phi)} E_1(z/wu) e_2\left(1/w\right) \frac{dw}{w} \right) \frac{du}{u}.$$

□

We are ready to prove the main result, essential for the construction of the multisum.

Proposition 4.3.25. Let $\mathbb{M}_j$, e_j, m_{e_j}, and $T_{e_j}, T_{e_j}^{-}$, $j = 1, 2$, be as in Proposition 4.3.21. Assume that $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < 2$.

1. We define *the acceleration from e_2 to e_1*, denoted $e_1 \triangleleft e_2$, by

$$(e_1 \triangleleft e_2)(z) = T_{e_1}^{-}(e_2(1/u))(1/z).$$

   Then, $e_1 \triangleleft e_2$ is a strong kernel of $\mathbb{M}_2/\mathbb{M}_1-$ summability whose moment function is $m(\lambda) = m_{e_2}(\lambda)/m_{e_1}(\lambda)$. Moreover, if E_2 and E are the functions associated by Definition 4.3.10.(v) with e_2 and $e_1 \triangleleft e_2$, respectively, we have that

$$E(u) = T_{e_1}(E_2(u)).$$

2. The function $e_1 \triangleleft e_2$ is the unique moment summability kernel with moment sequence $(m(p) = m_{e_2}(p)/m_{e_1}(p))_{p \in \mathbb{N}_0}$.

3. Let A_{e_1,e_2} denote the Laplace-like integral operator associated with $e_1 \triangleleft e_2$. If S is an unbounded sector and $f \in \mathcal{O}^{\mathbb{M}_2}(S)$, then

$$A_{e_1,e_2} f = T_{e_1}^{-} \circ T_{e_2}(f).$$

4. We define $g(z) := ((T_1^{-} \circ T_{e_2})f)(z)$ with $f(u) = (1-u)^{-1}$ and

$$e(1/z) := \frac{g(z) - g(ze^{2\pi i})}{2\pi i}.$$

   Then, e is well defined in $S_{\omega(\mathbb{M}_2)-\omega(\mathbb{M}_1)}$ and $e(z) = e_1 \triangleleft e_2(z)$.

Proof. For simplicity we write $\omega_1 = \omega(\mathbb{M}_1)$ and $\omega_2 = \omega(\mathbb{M}_2)$. We observe that $\omega(\mathbb{M}_2/\mathbb{M}_1) = \omega_2 - \omega_1$ (see Remark 4.2.10).

1. Let us show that the function $(e_1 \triangleleft e_2)(z) = T_{e_1}^{-}(e_2(1/u))(1/z)$ has the necessary properties for a strong kernel function of $\mathbb{M}_2/\mathbb{M}_1-$summability, as listed in Definition 4.3.10.

   Since the function $e_2(1/u)$ is holomorphic in S_{ω_2}, continuous at the origin and $\omega_2 > \omega_1$, by Proposition 4.1.14 we have that $T_{e_1}^{-}(e_2(1/u))(z)$ is holomorphic in $S_{\omega_2-\omega_1}$ which proves requirement (I).

   Regarding the integrability condition (II.B), we fix $\tau \in (0, \omega_2 - \omega_1)$ and we take $\tau_1 \in (0, 2-\omega_1)$ and $\tau_2 \in (0, \omega_2)$ such that $2+\tau \in (0, \tau_1+\tau_2-(2-\omega_1-\tau_1))$ and we can choose $\varepsilon \in (2-\omega_1-\tau_1, \tau_1+\tau_2-2-\tau)$. By Definition 4.3.10.(VI.B) for E_1 and (II.B) for e_2 we know that there exist $\beta_1, \alpha_2 > 0$ (not depending on τ_1 and τ_2), and contants $K_1, C_2 > 0$, $M_1 \geq 1$, $\varepsilon_2 \in (0,1)$ such that

$$|E_1(z)| \leq \frac{K_1}{|z|^{\beta_1}}, \qquad z \in S(\pi, \tau_1), \qquad |z| \geq M_1, \tag{4.56}$$

$$|e_2(z)| \leq C_2|z|^{\alpha_2}, \qquad z \in S_{\tau_2}, \qquad |z| \leq \varepsilon_2. \tag{4.57}$$

   We fix $z \in S_\tau$ with $|z| \leq \varepsilon_2^2/M_1^2$, then $|z| \leq 1$. We write $\phi = \arg(z) \in (-\pi\tau/2, \pi\tau/2)$ and we may consider a path $\delta_{\omega_1}(-\phi)$ (see Definition 4.1.13). We can write $\delta_{\omega_1}(-\phi) = \delta_1+\delta_2+\delta_3$ (δ_1 and δ_3 are segments in directions $\theta_1 = -\phi+(\pi/2)(\omega_1+\varepsilon)$ and $\theta_3 = -\phi-(\pi/2)(\omega_1+\varepsilon)$, respectively, and δ_2 is a circular arc with radius $R = 1/(|z|M_1)$, that can be chosen in this way because e_2 is holomorphic in S_{ω_2}, that is unbounded, and the path $\delta_{\omega_1}(-\phi)$ stays inside S_{τ_2}). Then, by definition, we see that

$$e_1 \triangleleft e_2(z) = \frac{-1}{2\pi i}\int_{\delta_{\omega_1}(-\phi)} E_1\left(\frac{1}{uz}\right) e_2\left(1/u\right)\frac{du}{u}.$$

   We have that

$$\begin{aligned} |e_1 \triangleleft e_2(z)| \leq & \frac{1}{2\pi}\left(\int_0^{1/(|z|M_1)} \left|E_1\left(\frac{1}{zre^{i\theta_1}}\right) e_2\left(\frac{1}{re^{i\theta_1}}\right)\right| \frac{dr}{r}\right. \\ & + \int_{\theta_3}^{\theta_1} \left|E_1\left(\frac{|z|M_1}{ze^{i\theta}}\right) e_2\left(\frac{|z|M_1}{e^{i\theta}}\right)\right| d\theta \\ & \left. + \int_0^{1/(|z|M_1)} \left|E_1\left(\frac{1}{zre^{i\theta_3}}\right) e_2\left(\frac{1}{re^{i\theta_3}}\right)\right| \frac{dr}{r}\right). \end{aligned} \tag{4.58}$$

   First, we study the second integral in (4.58). By condition (V) for E_1, we know that there exists a constant H_1 such that $|E_1(w)| \leq H_1$ for every $w \in D(0, M_1+1)$. Using the bounds for ϕ, we see that

$$[\theta_3, \theta_1] \subseteq ((\pi/2)(-\tau-\omega_1-\varepsilon), (\pi/2)(\tau+\omega_1+\varepsilon)).$$

   Employing the upper bound for ε, we obtain that $\tau+\omega_1+\varepsilon < \tau_2-(2-\omega_1-\tau_1)$, then we deduce that

$$|z|M_1e^{-i\theta} \in S_{\tau_2}, \qquad \theta \in [\theta_3, \theta_1]. \tag{4.59}$$

   Since $|z| \leq |z|^{1/2} \leq \varepsilon_2/M_1$, we have that $|ze^{-i\theta}M_1| \leq \varepsilon_2$ and, by (4.57), we conclude that

$$\int_{\theta_3}^{\theta_1} \left|E_1\left(\frac{|z|M_1}{ze^{i\theta}}\right) e_2\left(\frac{|z|M_1}{e^{i\theta}}\right)\right| d\theta \leq H_1C_2|z|^{\alpha_2}M_1^{\alpha_2}\pi\tau_2. \tag{4.60}$$

We study now the first and the last integral in (4.58). If $r \in (0, 1/(|z|M_1))$, we observe that $|1/(zre^{i\theta_j})| \geq M_1$. Since $\omega_1 + \tau_1 < 2$ and $\tau_2 < 2$, using the bounds for ε, we have that

$$\omega_1 + \varepsilon \in (2 - \tau_1, \tau_2 + \omega_1 + \tau_1 - 2 - \tau) \subseteq (2 - \tau_1, 2)$$

and we deduce that $\arg(1/(zre^{i\theta_3})) = (\pi/2)(\omega_1 + \varepsilon) \in ((\pi/2)(2 - \tau_1), \pi)$ and also that $\arg(1/(zre^{i\theta_1})) \in (-\pi, -(\pi/2)(2 - \tau_1))$. Then for $j = 1, 3$ we see that $1/(zre^{i\theta_j}) \in S(\pi, \tau_1)$ and, by (4.56), we show that

$$\int_0^{(|z|M_1)^{-1}} \left| E_1\left(\frac{1}{zre^{i\theta_j}}\right) e_2\left(\frac{1}{re^{i\theta_j}}\right) \right| \frac{dr}{r} \leq K_1|z|^{\beta_1} \int_0^{(|z|M_1)^{-1}} r^{\beta_1} \left| e_2\left(\frac{1}{re^{i\theta_j}}\right) \right| \frac{dr}{r}.$$

Since $|z| \leq 1$, we can split $(0, 1/(|z|M_1))$ as the union of the intervals $(0, 1/(|z|^{1/2}M_1))$ and $[1/(|z|^{1/2}M_1), 1/(|z|M_1))$. By condition (III) for e_2, we know that there exists a constant D_2 such that $|e_2(w)| \leq D_2$ for every $w \in S_{\tau_2}$. If $r \geq 1/(|z|^{1/2}M_1)$, then $1/r \leq \varepsilon_2$, by (4.59), we can apply (4.57) and we obtain

$$\int_0^{(|z|M_1)^{-1}} \left| E_1\left(\frac{1}{zre^{i\theta_j}}\right) e_2\left(\frac{1}{re^{i\theta_j}}\right) \right| \frac{dr}{r} \leq \frac{K_1|z|^{\beta_1/2}D_2}{M_1^{\beta_1}\beta_1} + K_1C_2|z|^{\beta_1} \int_{(|z|^{1/2}M_1)^{-1}}^{(|z|M_1)^{-1}} r^{\beta_1 - \alpha_2} \frac{dr}{r}.$$

According to Remark 4.3.15, we may assume that $\alpha_2 < \beta_1$. Then for $j = 1, 3$ and for every $u \in S_\tau$ with $|z| \leq \varepsilon_2^2/M_1^2$ we have that

$$\int_0^{(|z|M_1)^{-1}} \left| E_1\left(\frac{1}{zre^{i\theta_j}}\right) e_2\left(\frac{1}{re^{i\theta_j}}\right) \right| \frac{dr}{r} \leq \frac{K_1D_2|z|^{\beta_1/2}}{M_1^{\beta_1}\beta_1} + \frac{K_1C_2|z|^{\alpha_2}}{(\beta_1 - \alpha_2)M_1^{\beta_1 - \alpha_2}}. \tag{4.61}$$

Consequently, by (4.60) and (4.61), condition (II.B) is satisfied with $\alpha = \min(\beta_1/2, \alpha_2)$.

By condition (III) for e_2, for every $\varepsilon > 0$ there exist $c, k > 0$ such that

$$|e_2(1/u)| \leq c\, e^{-\omega_{\mathbb{M}_2}(k/|u|)}, \qquad u \in S_{\omega_2 - \varepsilon}.$$

Then, by Proposition 3.1.9, we have that $e_2(1/u) \sim_{\mathbb{M}_2} \hat{0}$ in S_{ω_2}.

By Theorem 4.1.18, we see that $e_1 \triangleleft e_2(1/z) = T_{e_1}^{-}(e_2(1/u))(z) \sim_{\mathbb{M}_2/\mathbb{M}_1} \hat{0}$ in $S_{\omega_2 - \omega_1}$ which implies, again by Proposition 3.1.9, that for every $\varepsilon > 0$ there exist $c, k, r > 0$ such that

$$|e_1 \triangleleft e_2(z)| \leq c\, e^{-\omega_{\mathbb{M}_2/\mathbb{M}_1}(|z|/k)}, \qquad z \in S_{\omega_2 - \omega_1 - \varepsilon}, \qquad |z| > r. \tag{4.62}$$

By condition (II.B) for $e_1 \triangleleft e_2$, we know that $|e_1 \triangleleft e_2(z)| \leq C$ for $z \in S_{\omega_2 - \omega_1 - \varepsilon}$ with $|z| \leq \delta$. Since $e_1 \triangleleft e_2(z)$ is continuous, (4.62) holds for every $z \in S_{\omega_2 - \omega_1 - \varepsilon}$ and we conclude that condition (III) is satisfied.

For $x > 0$ we have that

$$\begin{aligned}\overline{e_1 \triangleleft e_2(x)} = &\frac{1}{2\pi i}\left(\int_0^R \overline{E_1\left(\frac{1}{xre^{i\theta_1}}\right) e_2\left(\frac{1}{re^{i\theta_1}}\right)} \frac{dr}{r}\right. \\ &\left. - \int_{\theta_1}^{\theta_3} \overline{E_1\left(\frac{1}{xRe^{i\theta}}\right) e_2\left(\frac{1}{Re^{i\theta}}\right) i d\theta} + \int_R^0 \overline{E_1\left(\frac{1}{xre^{i\theta_3}}\right) e_2\left(\frac{1}{re^{i\theta_3}}\right)} \frac{dr}{r}\right)\end{aligned}$$

with $\theta_1 = (\pi/2)(\omega_1 + \varepsilon)$ and $\theta_3 = -(\pi/2)(\omega_1 + \varepsilon)$. Since E_1 and e_2 are positive real over the positive real axis and holomorphic in S_{ω_2}, we deduce that

$$\begin{aligned}\overline{e_1 \triangleleft e_2(x)} =& \frac{1}{2\pi i}\left(\int_0^R E_1\left(\frac{1}{xre^{-i\theta_1}}\right) e_2\left(\frac{1}{re^{-i\theta_1}}\right)\frac{dr}{r}\right.\\ &\left. - \int_{\theta_1}^{\theta_3} E_1\left(\frac{1}{xRe^{-i\theta}}\right) e_2\left(\frac{1}{Re^{-i\theta}}\right) id\theta + \int_R^0 E_1\left(\frac{1}{xre^{-i\theta_3}}\right) e_2\left(\frac{1}{re^{-i\theta_3}}\right)\frac{dr}{r}\right).\end{aligned}$$

Since $\theta_1 = -\theta_3$, we observe that $\overline{e_1 \triangleleft e_2(x)} = e_1 \triangleleft e_2(x)$, then (IV) holds.

Let us show that

$$m_{e_1 \triangleleft e_2}(\lambda) = m_{e_2}(\lambda)/m_{e_1}(\lambda) \tag{4.63}$$

for $\mathrm{Re}(\lambda) \geq 0$. We have that

$$\begin{aligned} m_{e_1 \triangleleft e_2}(\lambda) &= \int_0^\infty x^{\lambda-1}(e_1 \triangleleft e_2)(x)dx \\ &= \frac{-1}{2\pi i}\int_0^\infty x^{\lambda-1}\int_{\delta_{\omega_1}(0)} E_1\left(\frac{1}{ux}\right) e_2(1/u)\frac{du}{u}dx.\end{aligned}$$

We make the change of variables $t = 1/(xu)$. Then the path $\delta_{\omega_1}(0)$ (with radius $R = 1/x$, that can be chosen in this way because e_2 is holomorphic in S_{ω_2}) stays inside S_{ω_2} and it is transformed into $\Delta_{\omega_1}(0)$, with $\Delta_{\omega_1}(0) = \Delta_3 + \Delta_2 + \Delta_1$ (Δ_3 is the line in direction $\theta_3 = -(\pi/2)(\omega_1 + \varepsilon)$ from infinity to $e^{i\theta_3}$, Δ_2 is a circular arc with radius $R = 1$, and Δ_1 is the line from $e^{i\theta_1}$ in direction $\theta_1 = (\pi/2)(\omega_1 + \varepsilon)$ to infinity) and we get

$$m_{e_1 \triangleleft e_2}(\lambda) = \frac{1}{2\pi i}\int_0^\infty x^\lambda \int_{\Delta_{\omega_1}(0)} E_1(t)\, e_2(xt)\frac{dt}{t}\frac{dx}{x}.$$

We make the change of variables $xt = s$, we see that

$$m_{e_1 \triangleleft e_2}(\lambda) = \frac{1}{2\pi i}\int_0^\infty s^\lambda e_2(s)\int_{\Delta_{\omega_1}(0)} E_1(t)\frac{dt}{t^{\lambda+1}}\frac{ds}{s}.$$

Using condition (VI.B) for E_1 and Cauchy's theorem to deform the path of integration and replace $\Delta_{\omega_1}(0)$ by the disc $D(0,1)$, we obtain

$$m_{e_1 \triangleleft e_2}(\lambda) = \int_0^\infty s^\lambda e_2(s)\left(\frac{1}{2\pi i}\int_{D(0,1)} E_1(t)\frac{dt}{t^{\lambda+1}}\right)\frac{ds}{s}.$$

By Cauchy's formula for E_1, we see that

$$m_{e_1 \triangleleft e_2}(\lambda) = \int_0^\infty s^\lambda e_2(s)\frac{1}{m_{e_1}(\lambda)}\frac{ds}{s}.$$

Finally, by condition (V) for e_2 we show that (4.63) is satisfied.

We deduce that $m_{e_1 \triangleleft e_2}(\lambda)$ is continuous in $\{\mathrm{Re}(\lambda) \geq 0\}$, holomorphic in $\{\mathrm{Re}(\lambda) > 0\}$ and $m_{e_1 \triangleleft e_2}(x) > 0$ for every $x \geq 0$. We define the function $E_\triangleleft$ by

$$E_\triangleleft(z) := \sum_{p=0}^\infty \frac{z^n}{m_{e_1 \triangleleft e_2}(p)}, \qquad z \in \mathbb{C}.$$

If we compute the radius of convergence of this series, using (4.63) and that $\omega_1 < \omega_2$ (see Proposition 4.2.11.(iii)) we show that

$$r = \liminf_{p\to\infty} \sqrt[n]{m_{e_1\triangleleft e_2}(p)} = \liminf_{p\to\infty} \sqrt[p]{\frac{m_{e_2}(p)}{m_{e_1}(p)}} = \infty,$$

hence $E_\triangleleft$ is entire. We see that $(m_{e_1\triangleleft e_2}(p))_{p\in\mathbb{N}_0}$, again by (4.63), is equivalent to the sequence $\mathbb{M}_2/\mathbb{M}_1$. Then, by Proposition 4.1.7, we see that there exist $C, K > 0$ such that $|E_\triangleleft(z)| \le C\exp(\omega_{\mathbb{M}_2/\mathbb{M}_1}(K|z|))$, for all $z\in\mathbb{C}$, then (v) holds.

Finally, regarding condition (vi.b), we need first to show that

$$E_\triangleleft(u) = (T_{e_1}E_2)(u), \qquad u\in\mathbb{C}^*. \tag{4.64}$$

We prove this equality for $u\in(0,\infty)$, and we conclude using the identity principle since $E_\triangleleft(u)$ is entire and, by Proposition 4.1.12, $(T_{e_1}E_2)(u)$ is holomorphic in a sectorial region $G(0, 2+\omega_1)$.

We fix $u\in(0,\infty)$, we have that

$$(T_{e_1}E_2)(u) = \int_0^\infty e_1\left(\frac{z}{u}\right)E_2(z)\frac{dz}{z}.$$

Since e_1 and E_2 are positive over $(0,\infty)$, then we can exchange integral and sum and applying (4.63) we see that

$$\begin{aligned}(T_{e_1}E_2)(u) &= \int_0^\infty e_1\left(\frac{z}{u}\right)E_2(z)\frac{dz}{z} = \int_0^\infty e_1\left(\frac{z}{u}\right)\sum_{n=0}^\infty \frac{z^n}{m_{e_2}(n)}\frac{dz}{z}\\ &= \sum_{n=0}^\infty (T_{e_1}(z^n/m_{e_2}(n)))(u) = \sum_{n=0}^\infty \frac{u^n}{m_{e_1\triangleleft e_2}(n)} = E_\triangleleft(u).\end{aligned}$$

Now, we fix $\tau\in(0, 2-\omega_2+\omega_1)$, and we take $\tau_1\in(0,\omega_1)$ and $\tau_2\in(0,2-\omega_2)$ such that $\tau_2<\tau<\tau_1+\tau_2$. By Definition 4.3.10.(ii.b) for e_1 and (vi.b) for E_2 we know that there exist $\alpha_1,\beta_2>0$ (not depending on τ_1 and τ_2), and constants $C_1, K_2 > 0$, $\varepsilon_1\in(0,1)$, $M_2\ge 1$ such that

$$|e_1(z)| \le C_1|z|^{\alpha_1}, \qquad z\in S_{\tau_1}, \qquad |z|\le\varepsilon_1, \tag{4.65}$$

$$|E_2(z)| \le \frac{K_2}{|z|^{\beta_2}}, \qquad z\in S(\pi,\tau_2), \qquad |z|\ge M_2. \tag{4.66}$$

We fix $u\in S(\pi,\tau)$ with $|u|\ge M_2/\varepsilon_1$. If $u\in S(\pi,\tau_2)$, we define $\theta_u := \arg(u)$ so we have that

$$\frac{e^{i\theta_u}}{u}\in S_{\tau_1}, \qquad e^{i\theta_u}\in S(\pi,\tau_2). \tag{4.67}$$

If $\arg(u)\in((\pi/2)(2-\tau), (\pi/2)(2-\tau_2)]$, we define $\theta_u := \arg(u)+\varepsilon_u$ with

$$\varepsilon_u\in((\pi/2)(2-\tau_2)-\arg(u), \min((\pi/2)(2+\tau_2)-\arg(u), (\pi/2)\tau_1)).$$

This interval is not empty since $\arg(u)(2/\pi) > 2-\tau > 2-\tau_2-\tau_1$, then $(\pi/2)(2-\tau_2)-\arg(u) < \tau_1\pi/2$. We observe that $\arg(e^{i\theta_u}/u) = \varepsilon_u\in[0,(\pi/2)\tau_1)$ and we also have that $e^{i\theta_u}\in S(\pi,\tau_2)$ and we deduce (4.67).

Analogously, if $\arg(u) \in [(\pi/2)(2+\tau_2), (\pi/2)(2+\tau))$ we choose $\theta_u := \arg(u) - \varepsilon_u$ with

$$\varepsilon_u \in (-(\pi/2)(2+\tau_2) + \arg(u), \min(-(\pi/2)(2-\tau_2) + \arg(u), (\pi/2)\tau_1)),$$

and we also obtain (4.67) for this choice of θ_u. By (4.64), since $|u| \geq M_2/\varepsilon_1$ we have that

$$\begin{aligned}|E_{\triangleleft}(u)| = &\left|\int_0^{\infty(\theta_u)} e_1\left(\frac{z}{u}\right) E_2(z)\frac{dz}{z}\right| \leq \int_0^{M_2} \left|e_1\left(\frac{re^{i\theta_u}}{u}\right) E_2(re^{i\theta_u})\right| \frac{dr}{r}\\ &+ \int_{M_2}^{|u|\varepsilon_1} \left|e_1\left(\frac{re^{i\theta_u}}{u}\right) E_2(re^{i\theta_u})\right| \frac{dr}{r} + \int_{|u|\varepsilon_1}^{\infty} \left|e_1\left(\frac{re^{i\theta_u}}{u}\right) E_2(re^{i\theta_u})\right| \frac{dr}{r}.\end{aligned}$$

If $r \leq |u|\varepsilon_1$, we have that $|re^{i\theta_u}/u| \leq \varepsilon_1$, by (4.67) we can apply (4.65) and we obtain

$$\begin{aligned}|E_{\triangleleft}(u)| \leq &\frac{C_1}{|u|^{\alpha_1}}\left(\int_0^{M_2} r^{\alpha_1}\left|E_2(re^{i\theta_u})\right|\frac{dr}{r} + \int_{M_2}^{|u|\varepsilon_1} r^{\alpha_1}\left|E_2(re^{i\theta_u})\right|\frac{dr}{r}\right)\\ &+ \int_{|u|\varepsilon_1}^{\infty} \left|e_1\left(\frac{re^{i\theta_u}}{u}\right) E_2(re^{i\theta_u})\right| \frac{dr}{r}.\end{aligned}$$

By condition (III) for e_1, and (V) for E_2 we know that there exist constants D_1, H_2 such that $|e_1(w)| \leq D_1$ for every $w \in S_{\tau_1}$ and $|E_2(w)| \leq H_2$ for every $w \in D(0, M_2+1)$. We deduce that

$$|E_{\triangleleft}(u)| \leq \frac{C_1 H_2 M_2^{\alpha_1}}{\alpha_1 |u|^{\alpha_1}} + \frac{C_1}{|u|^{\alpha_1}}\int_{M_2}^{|u|\varepsilon_1} r^{\alpha_1}\left|E_2(re^{i\theta_u})\right|\frac{dr}{r} + D_1\int_{|u|\varepsilon_1}^{\infty}\left|E_2(re^{i\theta_u})\right|\frac{dr}{r}.$$

If $r \geq M_2$, by (4.67) we can apply (4.66) and we have

$$|E_{\triangleleft}(u)| \leq \frac{C_1 H_2 M_2^{\alpha_1}}{\alpha_1 |u|^{\alpha_1}} + \frac{C_1 K_2}{|u|^{\alpha_1}}\int_{M_2}^{|u|\varepsilon_1} r^{\alpha_1-\beta_2}\frac{dr}{r} + \frac{D_1 K_2}{\beta_2 |u|^{\beta_2}\varepsilon_1^{\beta_2}}.$$

According to Remark 4.3.15, we may assume that $\alpha_1 > \beta_2$, then

$$|E_{\triangleleft}(u)| \leq \frac{C_1 H_2 M_2^{\alpha_1}}{\alpha_1 |u|^{\alpha_1}} + \frac{C_1 K_2 \varepsilon_1^{\alpha_1-\beta_2}}{(\alpha_1-\beta_2)|u|^{\beta_2}} + \frac{D_1 K_2}{\beta_2 |u|^{\beta_2}\varepsilon_1^{\beta_2}}.$$

Consequently, condition (VI.B) is satisfied with $\beta = \min(\alpha_1, \beta_2)$.

2. Uniqueness follows from Remark 4.3.8.

3. We take $f \in \mathcal{O}^{\mathbb{M}_2}(S(d,\alpha)) \subseteq \mathcal{O}^{\mathbb{M}_2/\mathbb{M}_1}(S(d,\alpha))$. Since $e_1 \triangleleft e_2$ is a kernel of $\mathbb{M}_2/\mathbb{M}_1-$summability, by Proposition 4.1.12 we know that $A_{e_1,e_2}(f)$ is holomorphic in a sectorial region $G(d, \alpha+\omega_2-\omega_1)$. We fix $z \in G(d, \alpha+\omega_2-\omega_1)$, there exists ϕ with $|d-\phi| < \pi\alpha/2$, such that if $\arg(u) = \phi$, then $u/z \in S_{\omega_2-\omega_1}$. We write $\eta = \arg(z/u) = \arg(z) - \phi$ and we consider a path $\delta_{\omega_1}(\eta)$ chosen as in Proposition 4.3.24, what is possible because e_2 is holomorphic in S_{ω_2} and the path $\delta_{\omega_1}(\eta)$ stays inside S_{ω_2}. We have that

$$A_{e_1,e_2}(f)(z) = \int_0^{\infty(\phi)} f(u)\left(\frac{-1}{2\pi i}\int_{\delta_{\omega_1}(\eta)} E_1\left(\frac{z}{wu}\right) e_2(1/w)\frac{dw}{w}\right)\frac{du}{u}.$$

We conclude, using Proposition 4.3.24, that this last expression equals $T_{e_1}^{-} \circ T_{e_2}(f)(z)$.

4. We consider $f(u) = (1-u)^{-1}$ and we define $g(z) := ((T^-_{e_1} \circ T_{e_2})f)(z)$. By Lemma 4.3.7, we know that $T_{e_2} f$ is holomorphic in $S(\pi, 2+\omega_2)$. Moreover, $T_{e_2} f \sim_{\mathbb{M}_2} \sum_{n=0}^{\infty} m_2(n) z^n$ on $S(\pi, 2+\omega_2)$, then it is continuous at the origin. We can apply the $T^-_{e_1}$ transform to $T_{e_2} f$ and, by Proposition 4.1.14, we have that g is holomorphic in $S = S(\pi, 2+\omega_2-\omega_1)$. Then, the function
$$e(1/z) := \frac{g(z) - g(ze^{2\pi i})}{2\pi i},$$
is holomorphic in $S_{\omega_2-\omega_1}$. Since $f \in \mathcal{O}^{\mathbb{M}_2}(S_{\omega_2-\omega_1})$, by statement 3 we have $A_{e_1,e_2}(f) = (T^-_{e_1} \circ T_{e_2})(f)$ and, by Lemma 4.3.7, we deduce that
$$e(1/z) = \frac{g(z) - g(ze^{2\pi i})}{2\pi i} = \frac{A_{e_1,e_2}(f)(z) - A_{e_1,e_2}(f)(ze^{2\pi i})}{2\pi i} = e_1 \triangleleft e_2(1/z).$$

□

Remark 4.3.26. Note that $\mathcal{O}^{\mathbb{M}_2}(S) \subseteq \mathcal{O}^{\mathbb{M}_2/\mathbb{M}_1}(S)$. Consequently, A_{e_1,e_2} extends the operator $T^{-1}_{e_1} \circ T_{e_2}$. Moreover, by the Proposition 4.3.25.3 and Proposition 4.1.14, for every $f \in \mathcal{O}^{\mathbb{M}_2}(S(d,\gamma))$ we deduce $A_{e_1,e_2} f \in \mathcal{O}^{\mathbb{M}_1}(S(d, \gamma + \omega(\mathbb{M}_2) - \omega(\mathbb{M}_1)))$ which justifies the name of the operator.

Remark 4.3.27. We know that $\mathbb{M}_2/\mathbb{M}_1$ is equivalent to a weight sequence admitting nonzero proximate order (see Proposition 4.2.8). As indicated in Remark 4.3.22, a strong kernel of $\mathbb{M}_2/\mathbb{M}_1-$summability, according to Remark 4.1.4, can be constructed but it may not behave well for the asymptotic relations.

Remark 4.3.28. From the uniqueness of the convolution and the acceleration kernels we deduce some basic properties:
$$e_1 * e_2 = e_2 * e_1, \qquad e_1 * (e_1 \triangleleft e_2) = e_2, \qquad e_1 \triangleleft (e_1 * e_2) = e_2, \qquad e_2 \triangleleft (e_1 * e_2) = e_1.$$

4.3.5 Multisummability through acceleration

In order to describe the procedure to recover the multisum of a formal power series presented below, we need to analyze the behavior of asymptotics under the operator A_{e_1,e_2} defined in Proposition 4.3.25 and to extend what was known for the Gevrey case (see [7, Th. 55 and 56]).

Theorem 4.3.29. Let $\mathbb{M}_j$, e_j, m_{e_j}, and $T_{e_j}, T^-_{e_j}$, $j = 1, 2$, be as in Proposition 4.3.21. Assume $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < 2$. Let A_{e_1,e_2} denote the Laplace-like integral operator associated with $e_1 \triangleleft e_2$ (see Proposition 4.3.25) and $\mathbb{M}'$ be any sequence of positive real numbers. Then,

(i) If $f \in \mathcal{O}^{\mathbb{M}_2/\mathbb{M}_1}(S(d,\alpha))$ and $f \sim_{\mathbb{M}'} \hat{f}$, then $A_{e_1,e_2} f \sim_{\mathbb{M}' \cdot (\mathbb{M}_2/\mathbb{M}_1)} \hat{A}_{e_1,e_2} \hat{f}$ in a sectorial region $G(d, \alpha + \omega(\mathbb{M}_2/\mathbb{M}_1))$, where
$$\hat{A}_{e_1,e_2}\left(\sum_{p=0}^{\infty} a_p z^p\right) := \sum_{p=0}^{\infty} \frac{a_p m_{e_2}(p)}{m_{e_1}(p)} z^p.$$

(ii) If, moreover, $f \in \mathcal{O}^{\mathbb{M}_2}(S(d,\alpha))$, then $A_{e_1,e_2} f \in \mathcal{O}^{\mathbb{M}_1}(S(d, \alpha + \omega(\mathbb{M}_2/\mathbb{M}_1)))$ and
$$T_{e_1}(A_{e_1,e_2} f) = T_{e_2} f.$$

Proof. (i) By Proposition 4.3.25, $e_1 \triangleleft e_2$ is a strong kernel of $\mathbb{M}_2/\mathbb{M}_1-$ summability. Then, the conclusion follows applying Theorem 4.1.18.

(ii) By Proposition 4.3.25.3, we know that

$$A_{e_1,e_2}f = (T^-_{e_1} \circ T_{e_2})f.$$

By Proposition 4.1.12, $T_{e_2}f$ is holomorphic in a sectorial region $G(d, \alpha + \omega(\mathbb{M}_2))$. Since $\omega(\mathbb{M}_2) > \omega(\mathbb{M}_1)$, by Proposition 4.1.14 $(T^-_{e_1} \circ T_{e_2})f$ is holomorphic in the unbounded sector $S = S(d, \alpha + \omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))$ and it is of $\mathbb{M}_1-$growth in S. We observe that $\omega(\mathbb{M}_2/\mathbb{M}_1) = \omega(\mathbb{M}_2) - \omega(\mathbb{M}_1)$ (see Remark 4.2.10), then $A_{e_1,e_2}f = (T^-_{e_1} \circ T_{e_2})f \in \mathcal{O}^{\mathbb{M}_1}(S)$. We can apply T_{e_1} to $A_{e_1,e_2}f$, and we get

$$T_{e_1}(A_{e_1,e_2}f) = T_{e_1}T^-_{e_1}T_{e_2}f = T_{e_2}f.$$

□

In a natural way, we define $\hat{A}^-_{e_1,e_2}\left(\sum_{p=0}^{\infty} a_p z^p\right) := \sum_{p=0}^{\infty}(a_p m_{e_1}(p)/m_{e_2}(p))z^p$. With the tools presented in the previous subsections and in the conditions of Proposition 4.3.25, we are ready for giving a definition of multisummability in a multidirection with respect to the multikernel (e_1, e_2).

Definition 4.3.30. In the conditions of Proposition 4.3.25, we say that $\hat{f} = \sum_{p\geq 0} a_p z^p$ is $(e_1, e_2)-$*summable in the multidirection* (d_1, d_2) with $|d_1 - d_2| < \pi(\omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))/2$ and $d_1, d_2 \in \mathbb{R}$ if:

(i) $\hat{g} := \hat{T}^-_{e_1}\hat{f} = \displaystyle\sum_{p\geq 0} \frac{a_p}{m_{e_1}(p)} z^p$ is $\mathbb{M}_2/\mathbb{M}_1-$summable in direction d_2.

(ii) The sum $\mathcal{S}_{\mathbb{M}_2/\mathbb{M}_1,d_2}\hat{g}$ admits analytic continuation g in a sector $S = S(d_1, \varepsilon)$ for some $\varepsilon > 0$, and $g \in \mathcal{O}^{\mathbb{M}_1}(S)$.

In this situation we can define the corresponding multisum by:

$$\mathcal{S}_{(e_1,e_2),(d_1,d_2)}\hat{f} := T_{e_1} \circ A_{e_1,e_2} \circ \hat{A}^-_{e_1,e_2} \circ \hat{T}^-_{e_1}\hat{f}.$$

The next result states the equivalence between $(\mathbb{M}_1, \mathbb{M}_2)-$multisummability and $(e_1, e_2)-$multisummability in a multidirection, and provides a way to recover the multisum by means of the formal and analytic acceleration operators previously introduced (see [7, Ch. 10] for the Gevrey case).

Theorem 4.3.31. Given $\mathbb{M}_1, \mathbb{M}_2$ weight sequences admitting a nonzero proximate order with $\omega(\mathbb{M}_1) < \omega(\mathbb{M}_2) < 2$, directions $d_1, d_2 \in \mathbb{R}$ with $|d_1 - d_2| < \pi(\omega(\mathbb{M}_2) - \omega(\mathbb{M}_1))/2$ and a formal power series $\hat{f}$, the following are equivalent:

(i) $\hat{f} \in \mathbb{C}\{z\}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}$.

(ii) For every pair of strong kernels, e_1 of $\mathbb{M}_1-$summability and e_2 of $\mathbb{M}_2-$summability, $\hat{f}$ is $(e_1, e_2)-$multisummable in multidirection (d_1, d_2).

(iii) For some pair of strong kernels, e_1 of $\mathbb{M}_1-$summability and e_2 of $\mathbb{M}_2-$summability, $\hat{f}$ is $(e_1, e_2)-$multisummable in multidirection (d_1, d_2).

In case any of the previous holds, we deduce that the $(\mathbb{M}_1, \mathbb{M}_2)-$sum of $\hat{f}$ on the multidirection (d_1, d_2) is given by

$$\mathcal{S}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}\hat{f} = T_{e_1} \circ A_{e_1,e_2} \circ \hat{T}^-_{e_2}\hat{f}.$$

for any pair of kernels e_1, e_2.

Proof. For simplicity we write $\omega_1 = \omega(\mathbb{M}_1)$ and $\omega_2 = \omega(\mathbb{M}_2)$. We observe that $\omega(\mathbb{M}_2/\mathbb{M}_1) = \omega_2 - \omega_1$ (see Remark 4.2.10).

(i) $\Longrightarrow$ (ii) With the notation in Definition 4.3.1, we write $\hat{f} = \hat{f}_1 + \hat{f}_2$. We put $\hat{g} := \hat{T}^-_{e_1}\hat{f}$ and we observe that $\hat{A}^-_{e_1,e_2}\hat{g} = \hat{A}^-_{e_1,e_2}\hat{T}^-_{e_1}\hat{f} = \hat{T}^-_{e_2}\hat{f}$. By Theorem 4.1.20, we know $\hat{h}_2 := \hat{T}^-_{e_2}\hat{f}_2$ converges in a disc, admits analytic continuation h_2 in a sector $S_2 = S(d_2, \varepsilon_2)$ for some $\varepsilon_2 > 0$, and $h_2 \in \mathcal{O}^{\mathbb{M}_2}(S_2)$. Since $\hat{f}_1 \in \mathbb{C}[[z]]_{\mathbb{M}_1}$, we see that $\hat{h}_1 = \hat{T}^-_{e_2}\hat{f}_1$ defines an entire function h_1 and, by Proposition 4.1.7, we have that h_1 is of $\mathbb{M}_2/\mathbb{M}_1-$growth on S_2.

Hence, $\hat{h} := \hat{T}^-_{e_2}\hat{f}$ converges in a disc, admits analytic continuation h in the sector S_2 where h is of $\mathbb{M}_2/\mathbb{M}_1-$growth there because $\mathcal{O}^{\mathbb{M}_2}(S_2) \subseteq \mathcal{O}^{\mathbb{M}_2/\mathbb{M}_1}(S_2)$. By Theorem 4.1.20, this means that the formal power series $\hat{A}_{e_1,e_2}\hat{T}^-_{e_2}\hat{f} = \hat{T}^-_{e_1}\hat{f} = \hat{g}$ is $\mathbb{M}_2/\mathbb{M}_1-$summable in direction d_2, so Definition 4.3.30.(i) is valid.

On the other hand, we observe that

$$\mathcal{S}_{\mathbb{M}_2/\mathbb{M}_1,d_2}\hat{g} = A_{e_1,e_2}\hat{A}^-_{e_1,e_2}\hat{g} = A_{e_1,e_2}\hat{A}^-_{e_1,e_2}(\hat{g}_1 + \hat{g}_2)$$

where $\hat{g}_1 := \hat{T}^-_{e_1}\hat{f}_1$ and $\hat{g}_2 := \hat{T}^-_{e_1}\hat{f}_2$. Since $\hat{f}_1 \in \mathbb{C}\{z\}_{\mathbb{M}_1,d_1}$, we have that $\hat{g_1} = \hat{T}^-_{e_1}\hat{f}_1$ converges in a disc, admits analytic continuation g_1 in the sector $S_1 = (d_1, \varepsilon_1)$ for some $\varepsilon_1 \in (0, \varepsilon_2)$ and $g_1 \in \mathcal{O}^{\mathbb{M}_1}(S_1)$. Moreover, thanks to the convergence, $A_{e_1,e_2}\hat{A}^-_{e_1,e_2}\hat{g}_1 = \mathcal{S}\hat{g}_1$. Regarding $\hat{g}_2$, we observe that

$$A_{e_1,e_2}\hat{A}^-_{e_1,e_2}\hat{g}_2 = A_{e_1,e_2}\hat{A}^-_{e_1,e_2}\hat{T}^-_{e_1}\hat{f}_2 = A_{e_1,e_2}\hat{T}^-_{e_2}\hat{f}_2.$$

Since $\hat{h}_2 = \hat{T}^-_{e_2}\hat{f}_2$ converges in a disc and admits analytic continuation $h_2 \in \mathcal{O}^{\mathbb{M}_2}(S_2)$, by Theorem 4.3.29, we see that $A_{e_1,e_2}h_2 = T^-_{e_1}T_{e_2}h_2$. Furthermore, $T^-_{e_1}T_{e_2}h_2$ is holomorphic in the sector $S(d_2, \omega_2 - \omega_1 + \varepsilon_2)$, which contains the sector S_1 because $|d_1 - d_2| < \pi(\omega_2 - \omega_1)/2$, and $T^-_{e_1}T_{e_2}h_2 \in \mathcal{O}^{\mathbb{M}_1}(S_1)$, so $\mathcal{S}_{\mathbb{M}_2/\mathbb{M}_1,d_2}\hat{g}$ can be written as the sum of two functions $A_{e_1,e_2}\hat{h}_2$ and $\mathcal{S}\hat{g}_1$ whose analytic continuations in S_1, $T^-_{e_1}T_{e_2}h_2$ and g_1, have $\mathbb{M}_1-$growth there, that is, Definition 4.3.30.(ii) holds.

(ii) $\Longrightarrow$ (iii) Trivial.

(iii) $\Longrightarrow$ (i) By Definition 4.3.30.(i), $g = \mathcal{S}_{\mathbb{M}_2/\mathbb{M}_1,d_2}\hat{g}$ is holomorphic in a sectorial region $G = G(d_2, \alpha)$ with $\alpha > \omega_2 - \omega_1$ and $g \sim_{\mathbb{M}_2/\mathbb{M}_1} \hat{g}$ in G. Let T be a subsector of G, bisected by direction d_2 and of opening $\pi\beta$ with $\beta \in (\omega_2 - \omega_1, 2)$, such that $\overline{T} \subseteq G$ and let γ denote the positively oriented boundary of $\overline{T}$. Decomposing $\gamma = \gamma_1 + \gamma_2$ where γ_1 is the circular part and γ_2 is the radial part, we define

$$g_j(z) := \frac{1}{2\pi i}\int_{\gamma_j}\frac{g(w)}{w - z}dw, \qquad \text{for all } z \in T, \quad j = 1, 2.$$

Since g is continuous at the origin, by Cauchy's Formula, we can write $g = g_1 + g_2$. By Leibniz's rule we see that g_1 is holomorphic at the origin. Hence, $g_2 = g - g_1 \sim_{\mathbb{M}_2/\mathbb{M}_1} \hat{g}_2$ where $\hat{g}_2 := \hat{g} - \hat{g_1}$ and $\hat{g}_1$ is the formal power series of g_1 at the origin.

We define $\hat{f}_1 := \hat{T}_{e_1}\hat{g}_1$ and we immediately observe that $\hat{f}_1 \in \mathbb{C}[[z]]_{\mathbb{M}_1}$. By (ii) in Definition 4.3.30, g admits analytic continuation in a sector $S_1 = S(d_1, \varepsilon)$ for some $\varepsilon > 0$, and this analytic continuation has $\mathbb{M}_1-$growth there. Again by the Leibniz's rule, we can see that g_2 is holomorphic in $S(d_2, \beta)$ and we can prove that tends to 0 as $|z| \to \infty$ therein, so $g_2 \in \mathcal{O}^{\mathbb{M}_1}(S(d_2, \beta))$. Since $|d_1 - d_2| < \pi(\omega_2 - \omega_1)/2$, we may assume, by suitably reducing ε, that $S_1 \subseteq S(d_2, \beta)$. Hence, $g_1 = g - g_2$ has an analytic continuation to S_1 and has $\mathbb{M}_1-$growth there, this means by Theorem 4.1.20 that $\hat{f}_1$ is $\mathbb{M}_1-$summable in direction d_1.

Now, we consider $\hat{f}_2 := \hat{T}_{e_1}\hat{g}_2$, we can apply Theorem 4.1.18.(i) to g_2 and we deduce that $T_{e_1}g_2 \sim_{\mathbb{M}_2} \hat{f}_2$ on a sectorial region $G(d_2, \beta + \omega_1)$. Since $\beta + \omega_1 > \omega_2$, this means that $\hat{f}_2$

is $\mathbb{M}_2$−summable in direction d_2. Consequently, we can write $\hat{f} = \hat{T}_{e_1}\hat{g} = \hat{f}_1 + \hat{f}_2$, so $\hat{f} \in \mathbb{C}\{z\}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}$.
In case any of the previous equivalent conditions holds, we have seen that by Theorem 4.3.29,

$$f_2 = T_{e_2}\hat{T}^{-}_{e_2}\hat{f}_2 = T_{e_1}A_{e_1,e_2}\hat{T}^{-}_{e_2}\hat{f}_2,$$

and, thanks to the convergence of $\hat{T}^{-}_{e_1}\hat{f}_1$, we have shown that

$$f_1 = T_{e_1}\hat{T}^{-}_{e_1}\hat{f}_1 = T_{e_1}A_{e_1,e_2}\hat{A}^{-}_{e_1,e_2}\hat{T}^{-}_{e_1}\hat{f}_1 = T_{e_1}A_{e_1,e_2}\hat{T}^{-}_{e_2}\hat{f}_1.$$

Hence, we conclude that $\mathcal{S}_{(\mathbb{M}_1,\mathbb{M}_2),(d_1,d_2)}\hat{f} = \mathcal{S}_{(e_1,e_2),(d_1,d_2)}\hat{f}$.

□

Remark 4.3.32. Classical multisummability theory can be also stated in a cohomological form. In this general context, this approach is also possible and one can provide a version of the relative Watson's Lemma (see [64, Th. 7.2.1] for the Gevrey case), which is the cohomological equivalent of the Tauberian Theorem. Apart from the Watson's Lemma and the Borel-Ritt Theorem, a Ramis-Sibuya-like result given by A. Lastra, S. Malek and J. Sanz in [61, Lemma 3] is necessary for the proof (for a reference on the classical version of Ramis–Sibuya theorem, the reader may consult to [39, Theorem XI-2-3]).

The main reason why the analytical point of view have been chosen is that an explicit construction of the acceleration kernels and operators can be given with the corresponding explicit expression for the multisums. This cohomological version and the results of this chapter are included in our work [42].

Chapter 5

A Phragmén-Lindelöf theorem via proximate orders and the propagation of asymptotics

In 1999, A. Fruchard and C. Zhang [29] proved that for a holomorphic function in a sector S which is bounded in every proper subsector of S, the existence of an asymptotic expansion following just one direction implies global (nonuniform) asymptotic expansion in the whole of S. Moreover, a Gevrey version of this result is provided with a control on the type, namely:

Theorem A ([29], Theorem 11)**.** Let f be a function analytic and bounded in an open sector $S = S(d, \gamma, r)$ of bisecting direction $d \in \mathbb{R}$, opening $\pi\gamma$ and radius r, with $\gamma, r > 0$. Suppose f has asymptotic expansion $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ of Gevrey order $1/k$ ($k > 0$) and type (at least) $R(\theta_0) > 0$ in some direction θ_0 with $|\theta_0 - d| < \pi\gamma/2$, i.e., for every $\delta > 0$ there exists $C = C(\delta) > 0$ such that for every $z \in S$ with $\arg(z) = \theta_0$ and every nonnegative integer p we have that

$$|f(z) - \sum_{n=0}^{p-1} a_n z^n| \leq C \left(\frac{1}{R(\theta_0)} + \delta \right)^p \Gamma(1 + \frac{p}{k})|z|^p. \tag{5.1}$$

Then, in every direction θ of S, f admits $\hat{f}$ as its asymptotic expansion of Gevrey order $1/k$ and type $R(\theta)$ given as follows:

$$R(\theta) = \begin{cases} R(\theta_0) \left(\frac{\sin(k(\theta-\alpha))}{\sin(k(\alpha'-\alpha))} \right)^{1/k} & \text{if} \quad \theta \in (\alpha, \alpha'], \\ R(\theta_0) & \text{if} \quad \theta \in [\alpha', \beta'], \\ R(\theta_0) \left(\frac{\sin(k(\theta-\beta)}{\sin(k(\beta'-\beta))} \right)^{1/k} & \text{if} \quad \theta \in [\beta', \beta). \end{cases}$$

Here, $\alpha = d - \pi\gamma/2$ and $\beta = d + \pi\gamma/2$ are the directions of the radial boundary of S, $\alpha' = \min(\theta_0, \alpha + \frac{\pi}{2k}) \in (\alpha, \theta_0]$, and $\beta' = \max(\theta_0, \beta - \frac{\pi}{2k}) \in [\theta_0, \beta)$.

We warn the reader that there is no agreement about the terminology in this respect: while most authors adhere, as we will do, to the convention that the asymptotics in (5.1) is Gevrey of order $1/k$, others (for example, Fruchard and Zhang or W. Balser in [6]) say this is of order k. Moreover, the notion of type is not standard, compare to the definition by M. Canalis-Durand [22] for whom the type in case one has (5.1) is $(1/R+\delta)^k$. It should also be mentioned that the factor

$\Gamma(1+p/k)$ could be changed into $(p!)^{1/k}$ without changing the asymptotics, but this would affect the base of the geometric factor providing the type (by Stirling's formula, see [22, pp. 3-4]) in any case. As it will be explained below, our interest in the type will be limited, and so we will choose a simple approach in this respect, see Definitions 5.1.1 and 5.1.2.

The proof of this result is based on the classical Phragmén-Lindelöf theorem and on the so-called Borel-Ritt-Gevrey theorem. This last statement provides the surjectivity, as long as the opening of the sector is at most π/k, of the Borel map sending a function with Gevrey asymptotic expansion of order $1/k$ in a sector to its series of asymptotic expansion, whose coefficients will necessarily satisfy Gevrey-like estimates (see Section 3.3). Also, the injectivity of the Borel map in sectors of opening greater than π/k (known as Watson's lemma) plays an important role when specifying conditions that guarantee the uniqueness of a function with a prescribed Gevrey asymptotic expansion of order $1/k$ in a direction (see Section 3.2).

The main aim of this chapter is to extend these results for other types of asymptotic expansions available in the literature. This possibility was already mentioned in [62], where A. Lastra, J. Mozo-Fernández and J. Sanz generalized the results of Fruchard and Zhang for holomorphic functions of several variables in a polysector (cartesian product of sectors) admitting strong asymptotic expansion in the sense of H. Majima [67, 68], considering also the Gevrey case as introduced by Y. Haraoka in [34].

The asymptotics we will consider are those associated with a general ultraholomorphic class of functions, as the ones studied in the previous chapters, defined by constraining the growth of the sequence of their successive derivatives in a sector in terms of a sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive numbers ($\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$), which will play the role of $(\Gamma(1 + p/k))_{p\in\mathbb{N}_0}$ in (5.1). The possibility of extending to this more general framework the results on the injectivity or surjectivity of the Borel map, gathered in Chapter 3, and a Phragmén-Lindelöf-like statement, obtained below applying the relation between weight sequences, proximate orders and the property of regular variation established in Chapter 2, are the keys to our success.

As in the Gevrey case, the study of the type as the direction moves in the sector is possible, although some information is lost in general (see Remark 5.1.7). This is due to the fact that the classical exponential kernel appearing in the finite Laplace transform providing the solution of the Borel-Ritt-Gevrey theorem in the Gevrey case is now replaced by $e_V(z)$ (see Remark 4.1.4) whose behavior at infinity is only given by some asymptotic relations, which is not enough for an accurate handling of the resulting type.

This chapter, whose contents may be found in our work [46], is organized as follows. In Section 5.1, several lemmas of a Phragmén-Lindelöf flavor are obtained. A paradigm is Lemma 5.1.6, where exponential decrease is extended from just one direction to a whole small (in the sense of its opening) sector adjacent to it. Section 5.2 contains several versions of Watson's lemma on the uniqueness of a function admitting a given asymptotic expansion in a direction, and in the final Section 5.3 we characterize the functions with an asymptotic expansion in a sectorial region as those asymptotically bounded and admitting such expansion in just one direction in the region.

5.1 $\mathbb{M}-$flatness extension

As in the previous chapters $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ always stands for a sequence of positive real numbers, and we always assume that $M_0 = 1$. In most of the statements, $\mathbb{M}$ will be also assumed to be a weight sequence, that is, (lc) with $\lim_{p\to\infty} m_p = \infty$. The reader is referred to Chapters 1 and 2 for the information involving sequences, proximate orders and regular and O-regular variation and to Chapter 3 for the notation and results concerning the ultraholomorphic classes of functions defined in sectors and sectorial regions.

We are going to consider two definitions useful for our purposes regarding the asymptotic expansions. First, we recall the notion of type for the functions in $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ previously considered.

Definition 5.1.1. Given a sector S, we say $f \in \mathcal{H}(S)$ admits $\hat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ as its *uniform* $\mathbb{M}-$*asymptotic expansion in* S *(of type* $1/A$ *for some* $A > 0$) if there exists $C > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$\Big|f(z) - \sum_{n=0}^{p-1} a_n z^n\Big| \le CA^p M_p |z|^p, \qquad z \in S.$$

We will write $f \sim^u_{\mathbb{M}} \hat{f}$ in S, and $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ stands for the space of functions admitting uniform $\mathbb{M}-$asymptotic expansion in S (of some type).

Secondly, we need to settle on the concept of asymptotic expansion in a direction.

Definition 5.1.2. Let f be a function defined in a sectorial region $G = G(d,\gamma)$, and θ be a direction in G, i.e. $|\theta - d| < \pi\gamma/2$. We say f has $\mathbb{M}-$*asymptotic expansion* $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ *in direction* θ if there exist $r_\theta, C_\theta, A_\theta > 0$ such that the segment $(0, r_\theta e^{i\theta}]$ is contained in G, and for every $z \in (0, r_\theta e^{i\theta}]$ and every $p \in \mathbb{N}_0$ one has

$$\Big|f(z) - \sum_{n=0}^{p-1} a_n z^n\Big| \le C_\theta A_\theta^p M_p |z|^p.$$

In this case, we say the type is $1/A_\theta$. Of course, the definition makes sense as long as the function is defined only in direction θ near the origin, i.e. in a segment $(0, re^{i\theta}]$ for suitable $r > 0$.

Remark 5.1.3. In the conditions of Definition 5.1.2, if $\hat{f}$ is the null series we say that f is $\mathbb{M}-$*flat in direction* θ. As in Proposition 3.1.9, this can be characterized in terms of $\omega_{\mathbb{M}}$ and it amounts to the existence of $r_\theta, C_\theta, A_\theta > 0$ such that the segment $(0, r_\theta e^{i\theta}]$ is contained in G, and for every $z \in (0, r_\theta e^{i\theta}]$ one has

$$|f(z)| \le C_\theta e^{-\omega_{\mathbb{M}}(1/(A_\theta |z|))}.$$

Suppose moreover that f is bounded throughout the (bounded or not) sectorial region G. Since the function $e^{-\omega_{\mathbb{M}}(t)}$ is nonincreasing in $[0,\infty)$, it is obvious that f is $\mathbb{M}-$flat in direction θ if and only if there exist $\tilde{C}_\theta > 0$ and the same constant $A_\theta > 0$ as before, such that for every $z \in G$ with $\arg(z) = \theta$ one has

$$|f(z)| \le \tilde{C}_\theta e^{-\omega_{\mathbb{M}}(1/(A_\theta |z|))}. \tag{5.2}$$

This fact will be used later on.

In getting the Phragmén-Lindelöf-like results contained in this section, similarly to the $\mathbb{M}-$summability theory presented in Section 4.1, a fundamental role is played by the functions $V \in MF(\gamma, \rho(t))$ for a given $\gamma > 0$ and a given nonzero proximate order $\rho(t)$. Using the characterizations established in Section 2.2, we know that the possibility of associating a function V with a weight sequence $\mathbb{M}$ in a suitable way (see Remark 5.1.4) depends on the regularity of the function $d_{\mathbb{M}}(t) = \log(\omega_{\mathbb{M}}(t))/\log(t)$ defined for large t, where $\omega_{\mathbb{M}}(t)$ is the associated function introduced in Section 1.1.3. If $d_{\mathbb{M}}(t)$ is a nonzero proximate order or, less restrictive, if it can be approximated by one in the sense of Definition 2.2.1, that is, $\mathbb{M}$ admits a nonzero proximate order, this possibility is available.

Remark 5.1.4. If $\mathbb{M}$ admits a nonzero proximate order $\rho(t)$, for every $\gamma > 0$, thanks to (VI) in Theorem 1.2.16, we know that there exist $V \in MF(\gamma, \rho(t))$ and positive constants A, B, t_0 such that

$$AV(t) \le t^{d_{\mathbb{M}}(t)} = \omega_{\mathbb{M}}(t) \le BV(t), \qquad t > t_0. \tag{5.3}$$

It is worth recalling that weight sequences admitting a proximate order are strongly regular and all the Matuszewska indices and orders are positive real numbers and coincide (See Remarks 2.2.7 and 2.2.18). Then it is a matter of convention how we name this value, $\gamma(\mathbb{M})$, $\omega(\mathbb{M})$, $\beta(\mathbf{m})$, $\mu(\mathbf{m})$, $\rho(\mathbf{m})$ or $\alpha(\mathbf{m})$. Since we mainly deal with quasianalyticity results, the notation $\omega(\mathbb{M})$ is the choice. Furthermore, from this point on we will assume that

$\mathbb{M}$ is a given weight sequence admitting a nonzero proximate order.

In Subsection 2.2.4, it has been shown that this is not a strong assumption for strongly regular sequences, since it is satisfied by every such sequence appearing in applications. However, as it has also been proved, there are strongly regular sequences which do not satisfy it.

We are ready for proving an important lemma about the extension of $\mathbb{M}-$flatness from a boundary direction into a whole small sector for functions bounded there and admitting a continuous extension to the boundary (considered in $\mathcal{R}$, i.e., disregarding the origin). First, we recall a classical version of Phragmén-Lindelöf theorem needed in the proof.

Theorem 5.1.5 (Phragmén-Lindelöf theorem, [100], p. 177)**.** Let f be a function holomorphic in a sector $S = S(d, \gamma, \rho)$, continuous and bounded by C in the boundary ∂S. Suppose there exist $K, L > 0$ and $\omega > \gamma$ such that

$$|f(z)| < Ke^{L|z|^{-1/\omega}}$$

for every $z \in S$. Then f is bounded by C in the sector S.

Now we obtain an analogue of Phragmén-Lindelöf theorem for $\mathbb{M}-$flat functions in a sector.

Lemma 5.1.6. Let $0 < \gamma < \omega(\mathbb{M})$ be given. Suppose f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is $\mathbb{M}-$flat in direction $d = \pi\gamma/2$. Then for every $0 < \delta < \pi\gamma$, there exist constants $k_1(\delta), k_2(\delta) > 0$ with

$$|f(z)| \le k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2].$$

Proof. For simplicity, we denote $\omega := \omega(\mathbb{M})$. We fix $0 < \delta < \pi\gamma$. Since $\gamma < \omega$, we have that

$$\frac{\pi}{2} < \beta = \beta(\delta) := \frac{1}{\omega}(\frac{\pi}{2}\omega + \frac{\delta}{2}) < \pi, \quad -\frac{\pi}{2} + \frac{\delta}{2\omega} < \alpha = \alpha(\delta) := \frac{1}{\omega}(\frac{\pi}{2}\omega - \pi\gamma + \frac{\delta}{2}) < \frac{\pi}{2}.$$

Then we take $\varepsilon, \eta > 0$ (depending on δ) such that

$$\cos(\beta) + \varepsilon \le -\eta < 0.$$

Since $\mathbb{M}$ admits a nonzero proximate order $\rho(t)$, by Remark 5.1.4 there exist a function $V \in MF(2\omega, \rho(t))$ and positive constants A, B, t_0 such that (5.3) holds. According to Remark 5.1.3, and specifically to (5.2), there exist $c_1, c_2 > 0$ with

$$|f(z)| \le c_1 e^{-\omega_{\mathbb{M}}(1/(c_2|z|))}, \qquad \text{if } \arg(z) = \pi\gamma/2. \tag{5.4}$$

Choose $d_2 > 0$ such that $c_2^{-1/\omega} > d_2$, and take $a \in \mathcal{R}$ with

$$\arg(a) = \frac{\omega\pi}{2} - \frac{\pi\gamma}{2} + \frac{\delta}{2}, \qquad 0 < |a| < \left(\frac{Ad_2}{2}\right)^{\omega}.$$

We see that $\varepsilon < 1$, so we have that

$$\cos\left(\frac{\arg(a) - \arg(z)}{\omega}\right) + \varepsilon \le 2 \tag{5.5}$$

for every $z \in \overline{S_\gamma}$ (where the closure is taken in $\mathcal{R}$, and so the vertex of the sector is not under consideration).

We observe that $\arg(a/z) \in [\omega\alpha, \omega\beta] \subseteq (-\pi\omega/2, \pi\omega)$ for every $z \in \overline{S_\gamma}$. Taking into account the comments at the beginning of Subsection 2.2.4 and using property (I) of the functions in $MF(2\omega, \rho(t))$, we see that

$$\lim_{|z|\to 0} \frac{V(a/z)}{|a|^{1/\omega} V(1/|z|)} = e^{i(\arg(a)-\arg(z))/\omega}$$

uniformly for $\arg(z) \in [-\pi\gamma/2, \pi\gamma/2]$. Consequently,

$$\lim_{|z|\to 0} \mathrm{Re}\left(\frac{V(a/z)}{|a|^{1/\omega} V(1/|z|)}\right) = \cos((\arg(a) - \arg(z))/\omega)$$

uniformly for $\arg(z) \in [-\pi\gamma/2, \pi\gamma/2]$, and we deduce that

$$|a|^{1/\omega} V\left(\frac{1}{|z|}\right) (\cos((\arg(a) - \arg(z))/\omega) - \varepsilon)) \leq \mathrm{Re}\left(V\left(\frac{a}{z}\right)\right), \tag{5.6}$$

$$|a|^{1/\omega} V\left(\frac{1}{|z|}\right) (\cos((\arg(a) - \arg(z))/\omega) + \varepsilon) \geq \mathrm{Re}\left(V\left(\frac{a}{z}\right)\right), \tag{5.7}$$

for $|z| < s_1$ small enough and $\arg(z) \in [-\pi\gamma/2, \pi\gamma/2]$. For convenience, we choose $s_1 < 1/(t_0 c_2)$. Consider the function

$$F(z) := f(z) e^{V(a/z)}.$$

The function $V(a/z)$ is holomorphic in $S(\arg(a), 2\omega) \supset \overline{S_\gamma}$, so $F(z)$ is holomorphic in S_γ and continuous up to ∂S_γ. Our aim is to apply the Phragmén-Lindelöf theorem 5.1.5 to this function in a suitable bounded sector $S(0, \gamma, s_3)$.

If $\arg(z) = -\pi\gamma/2$, we have that $\arg(a) - \arg(z) = \beta\omega$. Then, since f is bounded in $\overline{S_\gamma}$ by a constant $K > 0$, by using (5.7) we see that for $|z| < s_1$,

$$|F(z)| \leq K e^{\mathrm{Re}(V(a/z))} \leq K e^{(\cos(\beta)+\varepsilon)|a|^{1/\omega} V(1/|z|)} \leq K e^{-\eta |a|^{1/\omega} V(1/|z|)}.$$

Now, observe that $V(1/|z|) > 0$ (property (III)), so we deduce that $|F(z)| \leq K$ for every z with $|z| < s_1$ and $\arg(z) = -\pi\gamma/2$.

If $\arg(z) = \pi\gamma/2$, we have that $\arg(a) - \arg(z) = \alpha\omega$. Then, from (5.4), (5.3), (5.5) and (5.7) we see that, if $|z| < s_1$,

$$|F(z)| \leq c_1 e^{-\omega_{\mathbb{M}}(1/(c_2|z|))} e^{(\cos(\alpha)+\varepsilon)|a|^{1/\omega} V(1/|z|)} \leq c_1 e^{-AV(1/(c_2|z|)) + 2|a|^{1/\omega} V(1/|z|)}.$$

Using property (I) of the functions in $MF(2\omega, \rho(t))$ we have that

$$\lim_{|z|\to 0} \frac{V(1/(c_2|z|))}{V(1/|z|)} = c_2^{-1/\omega}.$$

Then, for $|z| < s_2 \leq s_1$ small enough we have that $V(1/(c_2|z|)) \geq d_2 V(1/|z|)$, and we conclude that

$$|F(z)| \leq c_1 e^{(-Ad_2 + 2|a|^{1/\omega}) V(1/|z|)}, \qquad \text{for } |z| < s_2, \quad \arg(z) = \pi\gamma/2.$$

Since $|a|$ has been chosen small enough in order that $-Ad_2 + 2|a|^{1/\omega} < 0$, we deduce that $|F(z)| \leq c_1$ for every $|z| < s_2$ and $\arg(z) = \pi\gamma/2$.

For $z \in S_\gamma$ with $|z| < s_1$, by using (5.5) and (5.7) we have that

$$\operatorname{Re}\left(V\left(\frac{a}{z}\right)\right) \le 2|a|^{1/\omega} V\left(\frac{1}{|z|}\right).$$

As $\gamma < \omega$, there exists $\mu > 0$ such that $\gamma < \mu < \omega$. By property (VI), we know that $\log(V(t))/\log(t)$ is a proximate order equivalent to $\rho(t)$, hence tending to $1/\omega$ at infinity. Then, we can apply Remark 1.2.8: there exists $0 < s_3 \le s_2$ small enough such that for every $z \in S_\gamma$, $|z| \le s_3$,

$$\operatorname{Re}\left(V\left(\frac{a}{z}\right)\right) \le 2|a|^{1/\omega} \left(\frac{1}{|z|}\right)^{1/\mu}.$$

Since $f(z)$ is bounded in S_γ, we have that

$$|F(z)| \le K\exp(2|a|^{1/\omega}|z|^{-1/\mu}), \qquad \text{for all } z \in S_\gamma, \quad \text{with } |z| \le s_3,$$

and, in particular,

$$|F(z)| \le K\exp(2|a|^{1/\omega}s_3^{-1/\mu}), \qquad \text{for every } z \in S_\gamma, \quad \text{with } |z| = s_3.$$

By applying Phragmén-Lindelöf theorem 5.1.5 to the function $F(z)$ in $S(0,\gamma,s_3)$, we obtain that

$$|F(z)| \le K_0 := \max(K, c_1, K\exp(2|a|^{1/\omega}s_3^{-1/\mu}))$$

for $|z| \le s_3$ and $\arg(z) \in [-\pi\gamma/2, \pi\gamma/2]$.

Consequently, using (5.6), if $|z| \le s_3$ and $\arg(z) \in [-\pi\gamma/2, \pi\gamma/2]$ we have that

$$|f(z)| \le K_0 e^{\operatorname{Re}(-V(a/z))} \le K_0 e^{-(\cos((\arg(a)-\arg(z))/\omega)-\varepsilon)|a|^{1/\omega}V(1/|z|)}.$$

Assuming that $\arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2]$, we deduce that

$$\cos((\arg(a) - \arg(z))/\omega) \ge \cos\left(\frac{\pi}{2} - \frac{\delta}{2\omega}\right) = -\cos(\beta) \ge \eta + \varepsilon > 0.$$

Then, for $r_2 := \eta|a|^{1/\omega} > 0$ we find that for every z with $\arg(z) \in [-\pi\gamma/2+\delta, \pi\gamma/2]$ and $|z| < s_3$ we have that

$$|f(z)| \le K_0 e^{-r_2 V(1/|z|)}.$$

Choose $k_2 > 0$ such that $(1/k_2)^{1/\omega} < r_2/B$. Property (I) of the functions in $MF(2\omega, \rho(t))$ implies that, for z with $|z| < s_4 < \min(s_3, 1/(t_0 k_2))$, small enough, and $\arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2]$, we have

$$|f(z)| \le K_0 e^{-BV(1/(k_2|z|))} \le K_0 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}.$$

We take $k_1 := K_0 e^{\omega_{\mathbb{M}}(1/(k_2 s_4))} \ge K_0$. Then, since $\omega_{\mathbb{M}}(t)$ is nondecreasing, if $|z| \ge s_4$ and $\arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2]$ we have

$$|f(z)| \le K \le K_0 = k_1 e^{-\omega_{\mathbb{M}}(1/(k_2 s_4))} \le k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))},$$

which concludes the proof. □

Remark 5.1.7. Some comments are in order concerning the statement or proof of the previous result.

By a simple rotation, one may easily check that the validity of Lemma 5.1.6 and of the subsequent results in this chapter does not depend on the bisecting direction of the sector where the function f is defined. Moreover, one could slightly weaken the hypotheses by considering a function f holomorphic in S_γ that admits a continuous extension to the direction $d = \pi\gamma/2$, in which it is $\mathbb{M}-$flat, and that is bounded in every (half-open) sector

$$\{z \in \mathcal{R} : \arg(z) \in (-\frac{\pi\gamma}{2} + \delta, \frac{\pi\gamma}{2}]\}, \quad \delta > 0.$$

Indeed, we may give a more precise information about the type. Following the previous proof, one notes that

$$k_2 = k_2(\delta) > \Big(\frac{B}{r_2}\Big)^\omega = \Big(\frac{B}{\eta|a|^{1/\omega}}\Big)^\omega \geq \Big(\frac{2B}{Ad_2\cos(\frac{\pi}{2} - \frac{\delta}{2\omega})}\Big)^\omega \geq \Big(\frac{2B}{A}\Big)^\omega \Big(\frac{1}{\sin(\frac{\delta}{2\omega})}\Big)^\omega c_2,$$

and k_2 may be made arbitrarily close to the last expression at the price of enlarging the constant $k_1 = k_1(\delta)$. So, the original type c_2 is basically affected by a precise factor when moving to a direction $\theta = -\pi\gamma/2+\delta$ with $0 < \delta < \pi\gamma$. It is obvious that $k_2(\delta)$ explodes at least like $1/\sin^\omega(\delta)$ as $\delta \to 0$. This means that the type of the null asymptotic expansion tends to 0 as the direction in the sector approaches the boundary $d = -\pi\gamma/2$, in the same way as in the Gevrey case (see Theorem A).

Moreover, the constant 2 in $\delta/(2\omega)$ could be any number greater than 1 and, by suitably choosing the value ε in the proof, the constant $2B/A$ appearing before can be made as close to B/A as desired, so the only indeterminacy in the previous factor is caused by the values A, B involved in (5.3). In the common situation that the function $d_{\mathbb{M}}(t)$ is indeed a proximate order, the constants A and B can also be taken as near to 1 as wanted, what makes the expression even more explicit.

Finally, note that, by using Theorem 2.2.17 one may change $\mathbb{M}$ by an equivalent sequence $\mathbb{L}$ such that $d_{\mathbb{L}}$ is a proximate order. However, this fact does not improve the proof, since again Theorem 1.2.16 will be applied to obtain a function $V \in MF(2\omega, d_{\mathbb{L}}(t))$ and we will work with the same type of estimate that we have in (5.3).

The following lemma shows that imposing $\gamma < \omega(\mathbb{M})$ is only a technical condition in order to apply Phragmén-Lindelöf theorem 5.1.5.

Lemma 5.1.8. Let $\gamma > 0$ be given. Suppose f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is $\mathbb{M}-$flat in direction $d = \pi\gamma/2$. Then for every $0 < \delta < \pi\gamma$, there exist constants $k_1(\delta), k_2(\delta) > 0$ with

$$|f(z)| \leq k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2].$$

Proof. For simplicity we write $\omega = \omega(\mathbb{M})$, and put $\theta_0 := \pi\gamma/2$. We can obviously choose a suitable natural number m and directions $\theta_j \in (-\pi\gamma/2, \pi\gamma/2)$, $j = 1, 2, \ldots, m$, such that

$$\theta_j := \theta_{j-1} - \pi\omega/2, \quad \theta_j \geq -\pi\gamma/2 + \delta, \quad j = 1, \ldots, m-1,$$
$$\theta_m \in (-\pi\gamma/2, -\pi\gamma/2 + \delta), \quad \theta_{m-1} - \theta_m < \pi\omega/2.$$

We fix $0 < \varepsilon < \pi\omega/4$. Since $\theta_0 - \theta_1 + \varepsilon < 3\pi\omega/4 < \pi\omega$, we can apply Lemma 5.1.6 to the function f restricted to the sector $S_1 = \{z \in \mathcal{R} : \arg(z) \in [\theta_1 - \varepsilon, \theta_0]\}$. We deduce that there exist constants $k_{1,1}, k_{2,1} > 0$ with

$$|f(z)| \leq k_{1,1} e^{-\omega_{\mathbb{M}}(1/(k_{2,1}|z|))}, \qquad \text{if } \arg(z) \in [\theta_1, \theta_0].$$

By recursively reasoning in the sectors

$$S_j = \{z \in \mathcal{R} : \arg(z) \in [\theta_j - \varepsilon, \theta_{j-1}]\}, \quad j = 2, 3, \dots, m-1,$$

and finally in the sector

$$S_m = \{z \in \mathcal{R} : \arg(z) \in [\theta_m, \theta_{m-1}]\},$$

we obtain constants $k_{1,j}, k_{2,j} > 0$ such that

$$|f(z)| \leq k_{1,j} e^{-\omega_{\mathbb{M}}(1/(k_{2,j}|z|))}, \qquad \text{if } \arg(z) \in [\theta_j, \theta_{j-1}].$$

Then for $k_1 := \max_j k_{1,j}$ and $k_2 := \max_j k_{2,j}$ we have that

$$|f(z)| \leq k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2].$$

□

In the next result we impose $\mathbb{M}-$flatness in both boundary directions of the sector, and conclude uniform $\mathbb{M}-$flatness throughout the sector.

Lemma 5.1.9. Let $\gamma > 0$ be given. Suppose f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is $\mathbb{M}-$flat in directions $d = \pi\gamma/2$ and $-d$. Then there exist constants $k_1, k_2 > 0$ with

$$|f(z)| \leq k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2, \pi\gamma/2]. \tag{5.8}$$

Proof. By Lemma 5.1.8, there exist constants $k_{1,1}, k_{2,1}, k_{1,2}, k_{2,2} > 0$ such that

$$|f(z)| \leq k_{1,1} e^{-\omega_{\mathbb{M}}(1/(k_{2,1}|z|))}, \qquad \text{if } \arg(z) \in [0, \pi\gamma/2]$$

and

$$|f(z)| \leq k_{1,2} e^{-\omega_{\mathbb{M}}(1/(k_{2,2}|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2, 0].$$

We conclude taking $k_1 := \max\{k_{1,1}, k_{1,2}\}$ and $k_2 := \max\{k_{2,1}, k_{2,2}\}$. □

Remark 5.1.10. By carefully inspecting its proof, we see that Lemma 5.1.6 holds true in any bounded sector $S(d, \gamma, r)$ and, consequently, Lemma 5.1.8 and Lemma 5.1.9 are also valid in bounded sectors.

We show next that, as Remark 5.1.10 suggests, it is also possible to work in sectorial regions.

Proposition 5.1.11. Let $\gamma > 0$ be given. Suppose f is holomorphic in a sectorial region G_γ, bounded in every $T \ll G$, and $\mathbb{M}-$flat in a direction θ in G_γ. Then, for every $T \ll G_\gamma$ there exist constants $k_1(T), k_2(T) > 0$ with

$$|f(z)| \leq k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{for all } z \in T. \tag{5.9}$$

Proof. By suitably enlarging the opening of the subsector, we can assume that θ is one of the directions in T. There exist $R, c_1, c_2 > 0$ with

$$|f(z)| \leq c_1 e^{-\omega_{\mathbb{M}}(1/(c_2|z|))}, \qquad \text{if } \arg(z) = \theta \quad \text{and } |z| \leq R. \tag{5.10}$$

If $\theta_1 < \theta_2$ are the (radial) boundary directions of T, we consider $\delta > 0$ such that $-\pi\gamma/2 < \theta_1 - \delta$ and $\theta_2 + \delta < \pi\gamma/2$. There exists $0 < r < R$ such that the sectors $S_1 = \{z \in \mathcal{R} : |z| \leq r, \arg(z) \in [\theta_1 - \delta, \theta]\}$ and $S_2 = \{z \in \mathcal{R} : |z| \leq r, \arg(z) \in [\theta, \theta_2 + \delta]\}$ are contained in G_γ. Taking into account (5.10) and Remark 5.1.10, we can apply Lemma 5.1.8 to the restriction of f to each sector and we conclude that f is $\mathbb{M}-$flat uniformly for $\arg(z) \in [\theta_1, \theta_2]$ and $|z| \leq r$. Since $\omega_{\mathbb{M}}(t)$ is nondecreasing, by suitably enlarging the constant k_1 we obtain (5.9). □

Example 5.1.12. Boundedness of the considered function is necessary in any of the previous results in this section. The next example shows that having an $\mathbb{M}-$asymptotic expansion in a direction d does not guarantee its validity in any sector containing that direction. Our inspiration comes from a similar example in W. Wasow's book [104, p. 38], which concerned the function $f(z) = \sin(e^{1/z})e^{-1/z}$.

By Remark 5.1.4, for every $\gamma > 0$ there exists $V \in MF(\gamma, \rho(t))$ such that we have (5.3). We consider the function

$$f(z) = \sin(e^{V(1/z)})e^{-V(1/z)}, \qquad z \in S_\gamma.$$

Since $\sin(e^{V(1/z)})$ is bounded for real $z > 0$, we see that f is $\mathbb{M}-$flat in direction 0. If we compute the derivative of f in S_γ we see that

$$\begin{aligned} f'(z) &= \frac{V'(1/z)}{z^2}\left(\sin(e^{V(1/z)})e^{-V(1/z)} - \cos(e^{V(1/z)})\right) \\ &= \frac{V'(1/z)}{zV(1/z)}\frac{V(1/z)}{z}\left(\sin(e^{V(1/z)})e^{-V(1/z)} - \cos(e^{V(1/z)})\right). \end{aligned}$$

Since for $z > 0$ we have $\lim_{z\to 0}(1/z)V'(1/z)/V(1/z) = 1/\omega(\mathbb{M})$ (by property (VI), see [65, Prop. 1.2]) and $\lim_{z\to 0} V(1/z)/z = \infty$ (property (III)), we deduce that $\lim_{z\to 0} f'(z)$ does not exist. By Proposition 3.1.5.(ii), f can not have $\mathbb{M}-$asymptotic expansion in any sectorial region containing direction 0. Consequently, f is not $\mathbb{M}-$flat in any such sectorial region. We note that, in particular, the example of Wasow corresponds to the Gevrey case of order 1, i.e., to the sequence $\mathbb{M} = (p!)_{p\in\mathbb{N}_0}$.

Remark 5.1.13. At this point it is worth saying a few words about a situation which, although not usually considered in the theory of asymptotic expansions, plays an important role in the general framework of ultradifferentiable or ultraholomorphic classes, namely that of the so-called Carleman classes of Beurling type. We will not give full details here, but let us say that a function f, holomorphic in a sectorial region G, has *Beurling* $\mathbb{M}-$*asymptotic expansion* $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ *in a direction* θ in G if there exists $r_\theta > 0$ such that the segment $(0, r_\theta e^{i\theta}]$ is contained in G, and for every $A_\theta > 0$ (small) there exists $C_\theta > 0$ (large) such that for every $z \in (0, r_\theta e^{i\theta}]$ and every $p \in \mathbb{N}_0$ one has

$$\left| f(z) - \sum_{n=0}^{p-1} a_n z^n \right| \le C_\theta A_\theta^p M_p |z|^p.$$

Following the idea in Remark 5.1.3, one can prove that f, bounded throughout G, is Beurling $\mathbb{M}-$flat in direction θ if and only if for every $c_2 > 0$ (small) there exist $c_1 > 0$ (large) such that for every $z \in G$ with $\arg(z) = \theta$ one has

$$|f(z)| \le c_1 e^{-\omega_{\mathbb{M}}(1/(c_2|z|))}. \tag{5.11}$$

Then, the following analogue of Lemma 5.1.6 is valid: Given $0 < \gamma < \omega(\mathbb{M})$, suppose f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is Beurling $\mathbb{M}-$flat in direction $d = \pi\gamma/2$. Then for every $0 < \delta < \pi\gamma$ and every $k_2 > 0$, there exists a constant $k_1 = k_1(\delta, k_2) > 0$ such that

$$|f(z)| \le k_1 e^{\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-\pi\gamma/2 + \delta, \pi\gamma/2].$$

The proof of this statement follows the same lines as that of the original lemma, by carefully tracing the dependence of the different constants involved in the estimates. Indeed, the constants $A, B, \alpha, \beta, \varepsilon, \eta$ are determined in the same way. Choose $r_2 > 0$ such that $r_2/B > k_2^{-1/\omega}$, and a point a with the specified argument and modulus $(r_2/\eta)^\omega$. Take a positive d_2 such that $d_2 > 2|a|^{1/\omega}/A$, and then $c_2 > 0$ such that $c_2 < d_2^{-\omega}$. By definition of Beurling $\mathbb{M}-$flatness in direction $\gamma\pi/2$, there exists $c_1 > 0$ such that (5.11) holds for $\arg(z) = \gamma\pi/2$. Then, the desired estimates hold for the same $k_1 > 0$ obtained in the proof of that lemma.

Note that also Lemma 5.1.8, Lemma 5.1.9 and Proposition 5.1.11 will be valid in this Beurling setting.

5.2 Watson's Lemmas

Given a weight sequence $\mathbb{M}$ admitting a nonzero proximate order, we will now obtain several quasianalyticity results by combining those in Subsections 3.2.1 and 3.2.2 with the results on the propagation of null asymptotics in Section 5.1.

Remark 5.2.1. In a similar way as in the proof of Proposition 3.1.9 (see [97]), it is easy to deduce that, given a bounded holomorphic function f in a sector S_γ that admits a continuous extension to the boundary ∂S_γ, the fact that $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ and f is $\mathbb{M}-$flat amounts to the existence of constants $k_1, k_2 > 0$ such that (5.8) holds.

In the first version, an immediate consequence of previous information, we assume the function is flat at both boundary directions.

Lemma 5.2.2. Let $\gamma > 0$ be given, such that either $\gamma > \omega(\mathbb{M})$, or $\gamma = \omega(\mathbb{M})$ and the series $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}$ diverges. Suppose f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is $\mathbb{M}-$flat in directions $d = \pi\gamma/2$ and $-d$. Then $f \equiv 0$.

Proof. By Lemma 5.1.9, we know that (5.8) holds for suitable $k_1, k_2 > 0$. The previous remark implies that $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ and $f \sim_{\mathbb{M}} \hat{0}$, and by Theorem 3.2.2 we deduce that $f \equiv 0$. ☐

In the second, improved version we assume only that the function is flat in one of the boundary directions.

Lemma 5.2.3. Assume the same hypotheses as in Lemma 5.2.2, except that now f is $\mathbb{M}-$flat only in direction $d = \pi\gamma/2$. Then $f \equiv 0$.

Proof. For simplicity we write $\omega = \omega(\mathbb{M})$. The argument is simple if $\gamma > \omega$: We fix $\omega < \mu < \gamma$ and $\delta = (\gamma - \mu)\pi > 0$. By Lemma 5.1.8 we know that there exist constants $k_1(\delta), k_2(\delta) > 0$ with

$$|f(z)| \le k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [\pi\gamma/2 - \mu\pi, \pi\gamma/2].$$

Then, Remark 5.2.1 implies that $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$, with $S = \{z \in \mathcal{R} : \arg(z) \in (\pi\gamma/2 - \mu\pi, \pi\gamma/2)\}$ and $f \sim_{\mathbb{M}} \hat{0}$. Since $\mu > \omega$, we can apply Theorem 3.2.2 to the function f in S, using a suitable rotation, and we deduce that $f \equiv 0$.

If $\gamma = \omega$ we fix $\delta = \pi\omega/8 > 0$. Lemma 5.1.8 ensures there exist $k_1(\delta), k_2(\delta) > 0$ with

$$|f(z)| \le k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))}, \qquad \text{if } \arg(z) \in [-3\pi\omega/8, \pi\omega/2]. \tag{5.12}$$

As in the proof of Lemma 5.1.6, since $\mathbb{M}$ admits a nonzero proximate order $\rho(t)$, there exist $V \in MF(2\omega, \rho(t))$ and positive constants A, B, t_0 such that we have (5.3). Choose $q_2 > 0$ such that $k_2^{-1/\omega} > q_2$, and take $a \in \mathcal{R}$ such that

$$\arg(a) = \frac{\omega\pi}{4}, \qquad 0 < |a| < \left(\frac{Aq_2}{2}\right)^\omega.$$

We observe that for every z with $\arg(z) \in [-\pi\omega/2, \pi\omega/2]$ one has

$$\arg(a/z) \in [-\pi\omega/4, 3\pi\omega/4] \subseteq (-\pi\omega/2, \pi\omega).$$

Using property (I) of the functions in $MF(2\omega, \rho(t))$, we see that

$$\lim_{|z|\to 0} \mathrm{Re}\left(\frac{V(a/z)}{|a|^{1/\omega}V(1/|z|)}\right) = \cos((\arg(a) - \arg(z))/\omega)$$

uniformly for $\arg(z) \in [-\pi\omega/2, \pi\omega/2]$. We fix $0 < \varepsilon < 1$ such that

$$\cos(3\pi/4) + \varepsilon \leq \cos(5\pi/8) + \varepsilon \leq -1/3 < 0.$$

We deduce that we have (5.6) and (5.7) for $\arg(z) \in [-\pi\omega/2, \pi\omega/2]$ and $|z| < s_1$, small enough and subject to the restriction $s_1 < 1/(t_0 k_2)$. Consider the function

$$F(z) := f(z)e^{V(a/z)}, \qquad \text{for } \arg(z) \in [-\pi\omega/2, \pi\omega/2].$$

Then we see that $F(z)$ is holomorphic in S_ω and continuous in $\overline{S_\omega}$.

If $\arg(z) \in [-\pi\omega/2, -3\pi\omega/8]$, we have that $\arg(a/z) \in [5\pi\omega/8, 3\pi\omega/4]$. Then, since $f(z)$ is bounded by $K > 0$ in $\overline{S_\omega}$ and using (5.7) for $|z| < s_1$, one has

$$|F(z)| \leq Ke^{\mathrm{Re}(V(a/z))} \leq Ke^{(\cos(5\pi/8)+\varepsilon)|a|^{1/\omega}V(1/|z|)} \leq Ke^{-|a|^{1/\omega}V(1/|z|)/3}.$$

Using property (I) of the functions in $MF(2\omega, \rho(t))$ we see that

$$\lim_{|z|\to 0} \frac{V((|a|/(3B)^\omega)(1/2|z|))}{(|a|^{1/\omega}/(3B))V(1/|z|)} = (1/2)^{1/\omega} < 1.$$

We define $b_2 := (|a|/(3B)^\omega)/2$. Then for $|z| < s_2 < \min(s_1, b_2/t_0)$, small enough, we have that

$$|F(z)| \leq Ke^{-BV(b_2/|z|)}, \qquad \text{if } |z| < s_2, \qquad \arg(z) \in [-\pi\omega/2, -3\pi\omega/8].$$

Using (5.3), we see that

$$|F(z)| \leq Ke^{-\omega_{\mathbb{M}}(b_2/|z|)}, \qquad \text{if } |z| < s_2, \qquad \arg(z) \in [-\pi\omega/2, -3\pi\omega/8]. \tag{5.13}$$

We define $C = \max\{\mathrm{Re}(V(a/z)) : |z| \geq s_2, -\pi\omega/2 \leq \arg(z) \leq -3\pi\omega/8\}$ and we take

$$c_1 := K\max\{\exp(C), 1\} < \infty.$$

Then, since $\omega_{\mathbb{M}}(t) \geq 0$ we have that

$$|F(z)| \leq c_1 \leq c_1 e^{\omega_{\mathbb{M}}(b_2/|z|)} \qquad \text{if } |z| \geq s_2, \qquad \arg(z) \in [-\pi\omega/2, -3\pi\omega/8]. \tag{5.14}$$

Since $c_1 \geq K$, from (5.13) and (5.14) we deduce that F is $\mathbb{M}$−flat uniformly for $\arg(z) \in [-\pi\omega/2, -3\pi\omega/8]$.

If $\arg(z) \in [-3\pi\omega/8, \pi\omega/2]$, we have that $\arg(a/z) \in [-\pi\omega/4, 5\pi\omega/8]$. Using (5.3), (5.7) and (5.12), for $|z| < s_1$ we see that

$$|F(z)| \leq k_1 e^{-\omega_{\mathbb{M}}(1/(k_2|z|))} e^{(\cos(\arg(a/z)/\omega)+\varepsilon)|a|^{1/\omega}V(1/|z|)} \leq k_1 e^{-AV(1/k_2|z|)+2|a|^{1/\omega}V(1/|z|)}.$$

Now, property (I) of the functions in $MF(2\omega, \rho(t))$ lets us write

$$\lim_{|z|\to 0} \frac{V(1/k_2|z|)}{V(1/|z|)} = k_2^{-1/\omega},$$

so, for $|z| < s_3 \leq s_2$ small enough, we have that $V(1/k_2|z|) \geq q_2 V(1/|z|)$. We conclude that

$$|F(z)| \leq k_1 e^{(-Aq_2+2|a|^{1/\omega})V(1/|z|)}, \qquad \text{if } |z| < s_3, \quad \arg(z) \in [-3\pi\omega/8, \pi\omega/2].$$

Since $|a|$ has been chosen small enough in order that $-Aq_2 + 2|a|^{1/\omega} < 0$, proceeding as before, we find that $F(z)$ is $\mathbb{M}-$flat uniformly for $\arg(z) \in [-3\pi\omega/8, \pi\omega/2]$.

Consequently, F verifies estimates of the type (5.8) in $\overline{S_\omega}$ and, by Remark 5.2.1, $F \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\omega)$ and $F \sim_{\mathbb{M}} \hat{0}$. Since $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}$ is assumed to be divergent, we can apply Theorem 3.2.2 to the function $F(z)$ in S_ω, and deduce that $F(z) \equiv 0$ and $f \equiv 0$. □

In the proof of Lemma 5.2.3 we need to distinguish two situations: in case $\gamma > \omega(\mathbb{M})$ we have been given an $\mathbb{M}-$flat function f in a wide enough sector (what entails uniqueness), while in case $\gamma = \omega(\mathbb{M})$ an $\mathbb{M}-$flat function F in a sector of opening $\pi\omega(\mathbb{M})$ has to be constructed in order to apply Theorem 3.2.2, what is possible thanks to the additional assumption on the series $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}$.

It is interesting to note that in the Gevrey case the aforementioned series diverges, so the previous result extends Lemma 5 in [29]. Indeed, in that instance the very divergence of the series allows one to treat the case $\gamma > \omega(\mathbb{M})$ by restricting the function to a sector with $\gamma = \omega(\mathbb{M})$, an argument which is not available in our general situation.

Remark 5.2.4. In most situations we can obtain converse statements to Lemma 5.2.2 and Lemma 5.2.3. Observe that if $\gamma < \omega(\mathbb{M})$ and we take $\gamma < \mu < \omega(\mathbb{M})$, by Theorem 3.2.2 we know there exists a nontrivial flat function $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\mu)$. Then (the restriction of) f is a bounded holomorphic function in S_γ that admits a continuous extension to the boundary ∂S_γ, and that is $\mathbb{M}-$flat in directions $d = \pi\gamma/2$ and $-d$.

Analogously, if $\gamma = \omega$ and $\sum_{p=0}^{\infty}((p+1)m_p)^{-1/(\omega(\mathbb{M})+1)}$ converges, we deduce that the series $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}$ converges too. Hence, by Theorem 3.2.4 there exists a nontrivial flat function $f \in \mathcal{A}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$. Since the derivatives of f are Lipschitzian, one may continuously extend f to the boundary of $S_{\omega(\mathbb{M})}$ preserving the estimates, and again obtain that f is $\mathbb{M}-$flat in directions $\pi\omega(\mathbb{M})/2$ and $-\pi\omega(\mathbb{M})/2$.

However, the converse of Lemma 5.2.2 and Lemma 5.2.3 fails in case $\gamma = \omega(\mathbb{M})$, the series $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})}$ converges and $\sum_{p=0}^{\infty}((p+1)m_p)^{-1/(\omega(\mathbb{M})+1)}$ diverges (for instance, this is the situation for the sequence $\mathbb{M}_{1,3/2}$, see Example 1.1.4(i)). Although nontrivial flat functions in $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ exist in this situation, there is no warranty that they can be continuously extended to the boundary of the sector.

Finally, we provide a version of Watson's Lemma for functions in sectorial regions which are $\mathbb{M}-$flat in a direction.

Proposition 5.2.5. Let $\gamma > 0$ be given with $\gamma > \omega(\mathbb{M})$. Suppose f is holomorphic in a sectorial region G_γ, bounded in every $T \ll G$, and $\mathbb{M}-$flat in a direction θ in G_γ. Then $f \equiv 0$.

Proof. Using Proposition 5.1.11 we know that for every $T \ll G_\gamma$ we have (5.9) for suitable $k_1, k_2 > 0$ depending on T and for every $z \in T$. Then, Proposition 3.1.9 implies that $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ and $f \sim_{\mathbb{M}} \hat{0}$, and Theorem 3.2.15 leads to the conclusion. □

Remark 5.2.6. By Theorem 3.2.15, if $\gamma \leq \omega(\mathbb{M})$ we can find a nontrivial function $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ such that $f \sim_{\mathbb{M}} \hat{0}$, so it is bounded on every proper bounded subsector T of G_γ and $\mathbb{M}-$flat in any direction $\theta_0 \in (-\pi\gamma/2, \pi\gamma/2)$. Consequently, in this situation we have a complete version of Watson's Lemma.

5.3 Asymptotic expansion extension

From the generalized version of the Borel–Ritt–Gevrey, Theorem 3.3.21, for a weight sequence $\mathbb{M}$ admitting a nonzero proximate order we may generalize Theorem 1 in [29].

Theorem 5.3.1. Given $\gamma > 0$, suppose f is holomorphic in a sectorial region G_γ, it is bounded in every $T \ll G_\gamma$, and it admits $\hat{f} \in \mathbb{C}[[z]]$ as its $\mathbb{M}-$asymptotic expansion in a direction $\theta \in (-\pi\gamma/2, \pi\gamma/2)$. Then, $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ and $f \sim_{\mathbb{M}} \hat{f}$ in G_γ.

Proof. We distinguish two cases:

1. Sectorial regions of small opening: If $\gamma < \omega$, we take $\gamma < \mu < \omega$. By the Borel-Ritt-Gevrey Theorem 3.3.21 we know that there exists a function $f_0 \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_\mu)$ such that $f_0 \sim_{\mathbb{M}} \hat{f}$ in S_μ. Then the function $g := f - f_0$ is holomorphic in G_γ, bounded in every proper bounded subsector of G_γ and it is $\mathbb{M}-$flat in direction θ. Using Proposition 5.1.11, we see that g is $\mathbb{M}-$flat in G_γ.

   Then, for every proper bounded subsector T of G_γ, there exist positive constants $A(T)$, $B(T)$, $C(T)$, $D(T) > 0$ such that

$$|f(z) - \sum_{n=0}^{p-1} a_n z^n| \leq |g(z)| + |f_0(z) - \sum_{n=0}^{p-1} a_n z^n|$$
$$\leq AC^p M_p |z|^p + BD^p M_p |z|^p \leq 2\max(A,B)\max(C^p, D^p) M_p |z|^p,$$

   for every $z \in T$ and every $p \in \mathbb{N}_0$. Consequently, $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ and $f \sim_{\mathbb{M}} \hat{f}$ in G_γ.

2. Sectorial regions of large opening: If $\gamma \geq \omega$, we may choose natural numbers ℓ and m, and for all $j = -\ell, \dots, -1, 0, 1, \dots, m$ we may consider directions $\theta_j \in (-\pi\gamma/2, \pi\gamma/2)$ such that

$$\theta_0 := \theta, \qquad \theta_j := \theta_{j-1} + \pi\omega/8, \qquad j = 1, \dots, m, \qquad \pi\gamma/2 - \theta_m < \pi\omega/8;$$
$$\theta_j := \theta_{j+1} - \pi\omega/8, \qquad j = -1, \dots, -l, \qquad -\pi\gamma/2 + \theta_{-l} > -\pi\omega/8.$$

   There exists $\rho_0 > 0$ such that $S_0 = S(\theta_0, \pi\omega/4, \rho_0) \subseteq G_\gamma$. We apply the first part in the sector S_0 and we see that $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S_0)$ and $f \sim_{\mathbb{M}} \hat{f}$ in S_0. In particular, f admits $\hat{f}$ as its $\mathbb{M}-$asymptotic expansion in directions θ_1 and θ_{-1} for $|z| < \rho_0$. Repeating the process we see that $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma)$ and $f \sim_{\mathbb{M}} \hat{f}$ in G_γ.

□

The proof of our last statement is now straightforward.

Corollary 5.3.2. Given $\gamma > 0$ and $\theta \in (-\pi\gamma/2, \pi\gamma/2)$, we have that

$$\tilde{\mathcal{A}}_{\mathbb{M}}(G_\gamma) = \{f \in \mathcal{H}(G_\gamma) : f \text{ is bounded in every proper bounded subsector } T \text{ of } G_\gamma \text{ and } f \text{ admits } \mathbb{M}-\text{asymptotic expansion in direction } \theta\}.$$

Conclusiones y trabajo futuro

El objetivo de esta tesis era explorar ciertas propiedades de las clases ultraholomorfas de funciones y su aplicación a los desarrollos asintóticos y la teoría de multisumabilidad, analizando qué resultados del caso Gevrey pueden ser extendidos a este marco general. Una primera meta alcanzada con éxito en el Capítulo 2 es la descripción de las relaciones entre diversas propiedades de las sucesiones peso y las nociones de orden aproximado, variación regular y O-variación regular. De ese capítulo, el Teorema 2.2.19 merece mención especial porque caracteriza la forma de las sucesiones para las que la teoría de sumabilidad desarrollada en [60, 88, 89] esté disponible.

El tercer capítulo está dedicado al estudio de la inyectividad y sobreyectividad de la aplicación de Borel asintótica. El resultado principal de este estudio, completado para la inyectividad y sucesiones peso generales y casi finalizado para la sobreyectividad y sucesiones fuertemente regulares, es la existencia de dos índices $\omega(\mathbb{M})$ y $\gamma(\mathbb{M})$, uno para cada problema y en general distintos (ver el Ejemplo 2.2.26), que miden la apertura límite de los sectores para los que la aplicación de Borel es inyectiva o, respectivamente, sobreyectiva. Puesto que para todos los ejemplos en las aplicaciones el valor de estos índices coincide, esta división ha resultado difícil de detectar. Finalmente, en los Capítulos 4 y 5 se acentúa el significado de estos resultados relativos a los desarrollos asintóticos. En esta dirección, el Teorema Tauberiano 4.2.14 clarifica cuándo tiene sentido la herramienta de multisumabilidad en este contexto. En ese caso, se ha proporcionado una construcción explícita y detallada de los núcleos de aceleración, lo que nos permite recuperar la multisuma de una serie de potencias formal. Por último, se ha probado que, como sucede en el caso Gevrey, se puede extender un $\mathbb{M}-$desarrollo asintótico de una función holomorfa en el origen en una dirección a la región donde dicha función está acotada.

Esta tesis representa un primer paso hacia una mejor comprensión de las condiciones que se asumen frecuentemente para las funciones y sucesiones peso mediante la noción de O-variación regular, expresando propiedades cualitativas en términos de ciertos valores cuantitativos. Estos hallazgos y técnicas presentes podrían ayudar a resolver otros asuntos en el contexto ultraholomorfo y ultradiferenciable. Ambos están estrechamente relacionados, como se ha resaltado mediante el estudio de la aplicación de Borel, lo que ha potenciado nuestro entendimiento sobre su conexión. Una consecuencia adicional que emerge de este trabajo es la aplicación potencial al estudio de ciertas ecuaciones, particularmente ecuaciones en diferencias, del método de multisumabilidad, que proporcione un tratamiento unificado del problema.

Este análisis de clases ultraholomorfas de funciones se ha ocupado principalmente de aquellas definidas por medio de una sucesión peso, de tipo Roumieu, y en el caso de una variable. Sin embargo, algunos de los resultados podrían ser válidos también para clases ultraholomorfas definidas mediante una función peso, o incluso una matriz peso, como ha sido recientemente considerado por A. Rainer y G. Schindl. Además, las clases de tipo Beurling son adecuadas para el estudio de problemas similares, o se podrían considerar clases de funciones de varias variables complejas definidas en polisectores (productos cartesianos de sectores) o en regiones más generales. Al mismo tiempo, el estudio presente ha utilizado solo parcialmente la información disponible proveniente de la teoría de variación regular o de O-variación regular, por lo tanto se podría profundizar en algunas de las conclusiones del segundo capítulo. Finalmente, merece la pena mencionar que, aunque los métodos de $\mathbb{M}-$sumabilidad se han aplicado en [60, 61], su desarrollo permanece aún en un plano bastante teórico.

Antes de terminar, y a la luz de los resultados obtenidos en esta tesis, sus implicaciones y sus limitaciones, se listan a continuación algunas líneas de investigación futura.

- En primer lugar, parece que todavía se puede explotar la conexión entre sucesiones peso y la O-variación regular, establecida en la Sección 2.1, para dilucidar el significado de otras condiciones que aparecen frecuentemente cuando se consideran clases ultraholomorfas o ultradiferenciables de funciones. En esta misma dirección, se podría analizar la noción, también clásica, de E-variación regular (véase [13, Sect. 2.1]), que se encuentra entre la variación regular y la O-variación regular. Adjunto a este concepto interviene un par adicional de índices, los de Karamata, distintos en general de los órdenes y de los índices de Matuszewska. Por lo tanto, es natural el análisis de si estos también describen propiedades significativas de las sucesiones peso. Una línea diferente, mencionada en la Observación 2.1.33, tiene que ver con la consideración de clases ultraholomorfas y ultradiferenciables definidas por medio de funciones peso. Hemos mostrado en [45, 47] que los requerimientos impuestos sobre estas funciones tienen una interpretación en términos de propiedades o índices de O-variación regular. Sin embargo, como ocurre en el caso de sucesiones peso, parece que no hemos explotado todavía toda la información que se puede obtener a partir de estas potentes herramientas.

- La sucesión dual introducida en la Subsección 2.1.5 sugiere la existencia de cierta dualidad entre los espacios correspondientes. La dualidad clásica en espacios de Orlicz, construidos a partir de una función monótona y de O-variación regular cuya inversa 'por la derecha' determina el espacio dual (ver [86]), apoya esta conjetura. Puesto que la función de conteo $\nu_{\mathbf{m}}$ puede verse como una suerte de inversa de la función escalonada $f_{\mathbf{m}}(x) = m_{\lfloor x \rfloor}$, puede esperarse dicha dualidad.

- Uno de los objetivos más evidentes que alcanzar es el estudio completo de la sobreyectividad de la aplicación de Borel. Como se señaló en la Observación 3.3.19, se espera que $\tilde{S}_{\mathbb{M}} = (0, \gamma(\mathbb{M})]$ y $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = (0, \gamma(\mathbb{M}))$, al menos para sucesiones fuertemente regulares. En este caso, solo queda determinar si el valor $\gamma(\mathbb{M})$ pertenece o no a alguno de estos intervalos.

  Cuando $\mathbb{M}$ no es fuertemente regular, se debe observar que solo se tiene información acerca de la extensión máxima de los intervalos de sobreyectividad pero, hasta donde sabemos, perfectamente podrían ser vacíos. Por lo tanto, se debería dedicar algún esfuerzo a la construcción de operadores de extensión bajo la condición necesaria (snq) junto con, posiblemente, alguna otra condición más débil que (mg).

- La existencia de núcleos de $\mathbb{M}-$sumabilidad ha sido probada únicamente para sucesiones peso que admiten un orden aproximado no nulo (ver la Observación 4.1.4). Es un problema abierto el determinar si tales núcleos existen para sucesiones fuertemente regulares arbitrarias. Estrechamente relacionado con este, se podría también intentar caracterizar las sucesiones que pueden ser escritas como momentos de un núcleo de $\mathbb{M}-$sumabilidad; observemos que la solución de este problema no se conoce ni siquiera en el caso Gevrey, como ha señalado W. Balser [7, p. 94]. Se comentó en la Observación 4.1.3 que, para una sucesión peso $\mathbb{M}$ que satisfaga (dc), con $0 < \omega(\mathbb{M}) < 2$ y para la que exista un núcleo de $\mathbb{M}-$sumabilidad, se puede deducir que no solo $\mathbb{M}$ es (snq), sino que de hecho $\gamma(\mathbb{M}) = \omega(\mathbb{M})$. Un asunto interesante sería el estudio de las posibles implicaciones o equivalencias entre diferentes propiedades de este tipo, como la existencia de núcleos de $\mathbb{M}-$sumabilidad, la existencia de funciones planas 'buenas' en sectores óptimos, y el hecho de que $\mathbb{M}$ satisfaga algún conjunto específico de propiedades entre (snq), (dc), (mg), $0 < \gamma(\mathbb{M}) = \omega(\mathbb{M}) < \infty$, etc.

- El lector puede haber advertido que, incluso habiéndose establecido un método de sumabilidad para la sucesión $\mathbb{M}_{\alpha,\beta}$, paradójicamente no se sabe si la serie de potencias formal $\sum_{p=0}^{\infty}\big(p!^{\alpha}\prod_{m=0}^{p}\log^{\beta}(e+m)\big)z^p$ es $\mathbb{M}_{\alpha,\beta}-$sumable. Este problema fundamental surge del hecho de que en el método de $\mathbb{M}_{\alpha,\beta}-$sumabilidad la sucesión es reemplazada por la sucesión de momentos $\mathfrak{m}_{e_V}$ construida a partir de una función de Maergoiz V. Este fenómeno, que es especialmente molesto cuando se consideran métodos de sumabilidad de momentos, es en general una debilidad del método de sumabilidad de Borel, puesto que no hay ningún procedimiento sistemático para encontrar información acerca del comportamiento global de una función analítica, tal como su prolongación analítica o la posición de sus singularidades, en términos de sus coeficientes de Taylor en un punto. Por ejemplo, en un trabajo reciente de O. Costin y X. Xia [24] se ha probado la $1-$sumabilidad de la serie aparentemente ingenua $\sum_{p=0}^{\infty}p^{p+1}z^p$, y la prueba requiere de herramientas y conceptos sofisticados como la analizabilidad de sus coeficientes y el uso de trans-series. Valdría la pena clarificar si esto es posible en el caso anterior, y si la variación regular puede arrojar algo de luz en este asunto.

- Es relevante verificar si es posible redefinir la noción de $\mathbb{M}-$sumabilidad de modo que el Teorema Tauberiano 4.2.14 esté disponible para dos sucesiones peso $\mathbb{M}$ and $\mathbb{L}$, comparables pero no equivalentes y con $\omega(\mathbb{M}) = \omega(\mathbb{L})$. Puesto que el Lema de Watson 3.2.15 para desarrollos asintóticos no uniformes se verifica para regiones sectoriales arbitrarias, es posible que se hayan de considerar desarrollos asintóticos uniformes. En esta situación, gracias a un teorema de S. Mandelbrojt [72, Sect. 2.4.I], podemos determinar regiones de casianaliticidad para la sucesión más pequeña, digamos $\mathbb{L}$, que no tienen esta propiedad para la más grande, $\mathbb{M}$, por lo tanto el Teorema Tauberiano debería ser cierto en este contexto. Sin embargo, dado que el índice de crecimiento de la sucesión cociente es $\omega(\mathbb{M}/\mathbb{L}) = 0$, para recuperar la multisuma se debería proporcionar una nueva definición de núcleo de sumabilidad, definido solamente en el eje real positivo. Además, la función asociada $\omega_{\mathbb{M}/\mathbb{L}}$ puede ser de variación rápida, por lo que las funciones de Maergoiz no son útiles y la construcción de núcleos requiere considerar un procedimiento distinto.

- Las técnicas de multisumabilidad desarrolladas en el Capítulo 4 nos permitirán trabajar al mismo tiempo con $\mathbb{M}-$ y con $k-$sumabilidad. Esto lleva a considerar las propiedades de sumabilidad de las soluciones en serie de potencias formal de diferentes tipos de ecuaciones. En el estudio de ecuaciones en diferencias aparece el llamado nivel 1^+ (véanse los trabajos de G. I. Immink [40, 41]), que corresponde a la sucesión $\mathbb{M}_{1,-1}$. Siempre que los otros niveles de la solución formal, aparte del 1^+, sean distintos de 1, podremos aplicar nuestro método de multisumabilidad. No obstante, existe una situación interesante en la que se necesita aplicar un proceso de aceleración para la sucesión Gevrey de orden 1 y la sucesión $\mathbb{M}_{1,-1}$, y para el que es imprescindible resolver previamente el asunto planteado en el apartado anterior.

Conclusions and future work

This thesis had the aim of exploring certain properties of ultraholomorphic classes of functions and their application to asymptotic expansions and multisummability theory, analyzing which results from the Gevrey case can be extended to this general framework. A first objective successfully accomplished in Chapter 2 is the description of the relations between several properties of the weight sequences and the notions of proximate order, regular and O-regular variation. From that chapter, Theorem 2.2.19 deserves a special mention because it characterizes the shape of the sequences for which the summability theory developed in [60, 88, 89] is available.

The third chapter is devoted to the study of the injectivity and surjectivity of the asymptotic Borel map. The main outcome of this study, completed for injectivity for general weight sequences and nearly finished for surjectivity and strongly regular sequences, is the existence of two indices $\omega(\mathbb{M})$ and $\gamma(\mathbb{M})$, one for each problem and generally different (see Example 2.2.26), measuring the limit opening of the sectors for which the Borel map is injective or, respectively, surjective. Since for all the examples in the applications the value of these indices coincides, this division has been hard to detect. Finally, in Chapters 4 and 5 the significance of these results regarding asymptotics is stressed. In this direction, Tauberian Theorem 4.2.14 clarifies when the multisummability tool makes sense in this context. In that situation, an explicit and detailed construction of the acceleration kernels has been provided, allowing us to recover the multisum of a formal power series. Finally, it has been shown that, as it happens in the Gevrey case, one may extend the $\mathbb{M}-$asymptotic expansion at the origin in a direction of a holomorphic function to the region where it is bounded.

This dissertation represents the first step towards a better insight into the conditions frequently assumed for weight functions and weight sequences through the notion of O-regular variation, expressing qualitative properties in terms of some quantitative values. These present findings and techniques might help to solve other issues in the ultraholomorphic and ultradifferentiable settings. Both settings are closely related, as it has highlighted the study of the Borel map, enhancing our understanding of their connection. An additional implication that emerges from this work is the potential application to certain equations, particularly difference equations, of the multisummability method, so providing a unified treatment of the problem.

This analysis of ultraholomorphic classes of functions has been primarily concerned with those defined by means of a weight sequence, of Roumieu type, and in the one-variable situation. However, some of the outcomes might be also valid in ultraholomorphic classes defined by means of a weight function or even a weight matrix, as it has been recently considered by A. Rainer and G. Schindl. Also, the Beurling type classes are suitable for the study of similar problems, or one could consider classes of functions of several complex variables defined in polysectors (cartesian products of sectors) or more general regions. At the same time, the present study has only partially employed the information available from the regular and O-regular variation theory, so some of the conclusions in the second chapter might be sharpened. Eventually, it is worth mentioning that, although $\mathbb{M}-$summability methods have been applied in [60, 61], their development remains still on a quite theoretical plane.

Before concluding, and in the light of the results obtained in this dissertation, their implications and limitations, some possible future lines of research are listed below.

- In the first place, it seems that one may still take advantage of the connection between weight sequences and O-regular variation, established in Section 2.1, in order to elucidate the meaning of other conditions frequently appearing when dealing with ultraholomorphic and ultradifferentiable classes of functions. In this same direction, one might be guided to consider the, also classical, notion of E-regular variation (see [13, Sect. 2.1]) which is in between regular and O-regular variation. Attached to this concept, an extra pair of indices, Karamata indices, generally different from orders and Matuszewska indices, come into play. Hence, one is tempted to analyze whether or not they are also describing significant properties of weight sequences. A different line, mentioned in Remark 2.1.33, has to do with the consideration of ultraholomorphic and ultradifferentiable classes defined by means of weight functions. We have shown in [45, 47] that the requirements imposed on these functions have an interpretation in terms of properties or indices of O-regular variation. However, as for the weight sequence case, it seems that we have not put to work yet all the information that could be obtained from this powerful machinery.

- The dual sequence introduced in Subsection 2.1.5 suggests the existence of some duality between the corresponding spaces. Supporting this conjecture is the classical duality of Orlicz spaces which are constructed from a O-regulary varying monotone function f whose right inverse function g determines the dual space (see [86]). Since the counting function $\nu_{\mathbf{m}}$ can be seen as a sort of inverse of the step function $f_{\mathbf{m}}(x) = m_{\lfloor x \rfloor}$, such a duality might be expected.

- One of the most evident objectives to be achieved is the complete study of the surjectivity of the Borel map. As it was pointed out in Remark 3.3.19, it is expected that $\tilde{S}_{\mathbb{M}} = (0, \gamma(\mathbb{M})]$ and $S_{\mathbb{M}} = \tilde{S}^u_{\mathbb{M}} = (0, \gamma(\mathbb{M}))$ at least for strongly regular sequences. In this case, it only rests to determine whether the value $\gamma(\mathbb{M})$ belongs to some of these intervals or not.

  In case $\mathbb{M}$ is not strongly regular, one should note that we only have information about the maximal extent of the surjectivity intervals, but as far as we know they could perfectly be empty. So, some efforts should be concentrated in the construction of extension operators under the necessary condition (snq) plus possibly some other condition weaker than (mg).

- The existence of $\mathbb{M}-$summability kernels has only been proved for weight sequences admitting a nonzero proximate order (see Remark 4.1.4). It is an open question to determine whether or not they exist for an arbitrary strongly regular sequence. Closely related to it, one might also try to characterize the sequences that can be written as the moments of an $\mathbb{M}-$summability kernel; we note that the solution of this problem is not even known in the Gevrey case, as W. Balser [7, p. 94] pointed out. It was commented in Remark 4.1.3 that, for a weight sequence $\mathbb{M}$ satisfying (dc), $0 < \omega(\mathbb{M}) < 2$ and for which a kernel of $\mathbb{M}-$summability exists, one may deduce that not only $\mathbb{M}$ is (snq), but indeed $\gamma(\mathbb{M}) = \omega(\mathbb{M})$. An interesting issue would be the study of the possible implications or equivalences among different properties of this kind, such as the existence of kernels of $\mathbb{M}-$summability, the existence of 'fine' flat functions in optimal sectors, and the fact that $\mathbb{M}$ satisfies some specific set of properties among (snq), (dc), (mg), $0 < \gamma(\mathbb{M}) = \omega(\mathbb{M}) < \infty$, etc.

- The reader may have noticed that even if a summability method for the sequence $\mathbb{M}_{\alpha,\beta}$ has been developed, it is paradoxically unknown if the formal power series $\sum_{p=0}^{\infty} \big(p!^{\alpha} \prod_{m=0}^{p} \log^{\beta}(e + m)\big) z^p$ is $\mathbb{M}_{\alpha,\beta}-$summable. This fundamental problem arises from the fact that in the method of $\mathbb{M}_{\alpha,\beta}-$summability the sequence is replaced by the moment sequence $\mathfrak{m}_{e_V}$ constructed from a Maergoiz function V. This phenomenon, which is specially disturbing when dealing with moment summability methods, is in general a weakness of the Borel summability method, since there is not a systematic procedure to find information about the global behavior of an analytic function, such as the analytic continuation or the position of its singularities, in terms of its Taylor coefficients. For instance, in a recent work by O. Costin and X. Xia [24] the $1-$summability of the apparently naive series $\sum_{p=0}^{\infty} p^{p+1} z^p$ has been shown, and their proof requires sophisticated tools and concepts such as the analyzability of its coefficients and the use of transseries. It would be worthy to clarify if this could be done in our situation and if regular variation could bring some light to this issue.

- It is relevant to verify if it is possible to redefine the $\mathbb{M}-$summability notion in such a way that the Tauberian Theorem 4.2.14 is available for two comparable but not equivalent weight sequences $\mathbb{M}$ and $\mathbb{L}$ with $\omega(\mathbb{M}) = \omega(\mathbb{L})$. Since Watson's Lemma 3.2.15 for nonuniform asymptotics holds for arbitrary sectorial regions, one might need to consider uniform asymptotics. In this situation, thanks to a theorem of S. Mandelbrojt [72, Sect. 2.4.I] we can determine regions of quasianalyticity for the smaller sequence, say $\mathbb{L}$, that do not have this property for the bigger one $\mathbb{M}$, so the Tauberian theorem should be true in this context. However, since the growth index of the quotient sequence is $\omega(\mathbb{M}/\mathbb{L}) = 0$, in order to recover the multisum a new concept of summability kernel, defined just over the positive real axis, should be given. Moreover, the associated function $\omega_{\mathbb{M}/\mathbb{L}}$ may be rapidly varying, so Maergoiz functions are useless and a different procedure has to be considered for the construction of the kernels.

- The multisummability techniques developed in Chapter 4 will allow to us work at the same time with $\mathbb{M}-$ and $k-$summability. This leads to the consideration of the summability properties of the formal power series solutions of different types of equations. In the study of difference equations, the so-called level 1^+ (see the works of G. I. Immink [40, 41]), which corresponds to the sequence $\mathbb{M}_{1,-1}$, appears. Whenever the other levels, apart from the 1^+, of the formal solution are distinct from 1 we might apply our multisummability method. Nevertheless, there is an interesting situation in which one needs to apply an acceleration process for the Gevrey sequence of order 1 and the sequence $\mathbb{M}_{1,-1}$, and for which the issue presented in the last paragraph needs to be previously solved.

Notation

List of Symbols

$\boldsymbol{g}_r$	$\boldsymbol{g}_r = (p^r)_{p\in\mathbb{N}}$ for $r \in \mathbb{R}\setminus\{0\}$	62
$\gamma(\mathbb{M})$	Thilliez's growth index for $\mathbb{M}$	38
$G(d,\gamma)$	sectorial region in $\mathcal{R}$ bisected by the direction $d \in \mathbb{R}$ of opening $\pi\gamma$	108
G_γ	sectorial region in $\mathcal{R}$ bisected by the direction $d = 0$ of opening $\pi\gamma$	108
$\mathcal{H}(\mathbb{C})$	entire functions space.	
$\mathcal{H}(U)$	holomorphic functions in an open set $U \subset \mathcal{R}$.	
H_p	$p-$th partial sum of the harmonic series	96
H	$H(0) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, the open right half-plane of $\mathbb{C}$	113
$H(c)$	the open half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > c\}$	113
$I_{\mathbb{M}}, \tilde{I}^u_{\mathbb{M}}, \tilde{I}_{\mathbb{M}}$	injectivity intervals	111
$\lambda(\boldsymbol{a})$	exponent of convergence of a nondecreasing sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$	71
(lc)	logarithmic convexity condition	29
$\mathbb{M}^{(lc)}$	logarithmically convex minorant sequence of $\mathbb{M}$	37
$\mu(\boldsymbol{a})$	lower order of the sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$	61
$\mu(f)$	lower order of a positive function f	46
$\check{\mathbb{M}}$	$\check{\mathbb{M}} = (M_p/p!)_{p\in\mathbb{N}_0}$ for any sequence $\mathbb{M}$	111
$\widehat{\mathbb{M}}$	$\widehat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ for any sequence $\mathbb{M}$	111
(mg)	moderate growth condition	29
$MF(\gamma,\rho(t))$	Maergoiz's class of analytic functions in S_γ associated with $\rho(t)$	43
$\mathbb{N}$	natural numbers $\{1, 2, \dots\}$.	
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.	
$\nu_{\mathbf{m}}(t)$	counting function for the sequence of quotients $\mathbf{m}$ of a sequence $\mathbb{M}$	72
(nq)	nonquasianalyticity condition	30
$\mathcal{O}^{\mathbb{M}}(S)$	set of holomorphic functions of $\mathbb{M}-$growth on S	142
ORV	class of O-regularly varying functions	44
(P_γ)	property used for the definition of $\gamma(\mathbb{M})$	38
$\mathcal{R}$	Riemann surface of the logarithm.	
$\rho(\boldsymbol{a})$	upper order of the sequence $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}}$	61
$\rho(f)$	upper order of a positive function f	46
RV, R_ρ	class of regularly varying functions and regularly varying functions of index ρ	40

Glossary

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