Superintegrable systems on 3-dimensional curved spaces: Eisenhart formalism and separability

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Abstract

The Eisenhart geometric formalism, which transforms an Euclidean natural Hamiltonian $H = T + V$ into a geodesic Hamiltonian $\mathcal{T}$ with one additional degree of freedom, is applied to the four families of quadratically superintegrable systems with multiple separability in the Euclidean plane. Firstly, the separability and superintegrability of such four geodesic Hamiltonians $\mathcal{T}_r$ $(r = a, b, c, d)$ in a three-dimensional curved space are studied and then these four systems are modified with the addition of a potential $\mathcal{U}_r$ leading to $\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r$. Secondly, we study the superintegrability of the four Hamiltonians $\widetilde{\mathcal{H}}_r = \mathcal{H}_r/\mu_r$, where μ_r is a certain position-dependent mass, that enjoys the same separability as the original system $\mathcal{H}_r$. All the Hamiltonians here studied describe superintegrable systems on non-Euclidean three-dimensional manifolds with a broken spherically symmetry.

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1 Introduction

It is well known that the harmonic (isotropic) oscillator and the Kepler–Coulomb (KC) problem are integrable systems admitting additional constants of motion (Demkov–Fradkin tensor [1, 2] and Laplace–Runge–Lenz vector, respectively). Systems endowed with this property are called superintegrable. It is also known that if a system is separable (Hamilton–Jacobi (HJ) separable in the classical case or Schrödinger separable in the quantum case), then it is integrable with integrals of motion of at most second-order in momenta. Thus, if a system admits multiseparability (separability in several different systems of coordinates) then it is endowed with 'quadratic superintegrability' (superintegrability with linear or quadratic integrals of motion).

Fris *et al.* studied in [3] the two-dimensional (2D) Euclidean systems admitting separability in more than one coordinate system and they obtained four families of potentials V_r, $r = a, b, c, d$, possessing three functionally independent integrals of motion (they were mainly interested in the quantum 2D Schrödinger equation but their results also hold at the classical level). Then other authors studied similar problems on higher-dimensional Euclidean spaces [4]–[6], on 2D spaces with a pseudo-Euclidean metric (Drach potentials) [7]–[10], and on curved spaces [11]–[21] (see [22] for a recent review on superintegrability that includes a long list of references).

The superintegrability property is related with different formalisms and it can be studied by making use of different approaches, that is, proving that all bounded classical trajectories are closed, HJ separability, action-angle variables formalism, exact solvability, degenerate quantum energy levels, complex functions whose Poisson bracket with the Hamiltonian are proportional to themselves, etc. In this paper, we relate superintegrability with a geometric formalism introduced many years ago by Eisenhart [23].

The theory of general relativity states that the motion of a particle under the action of gravitational forces is described by a geodesic in the 4D Riemannian spacetime. The Eisenhart formalism (also known as Eisenhart lift) associates to a system governed by a natural Hamiltonian $H = T + V$ (a kinetic term plus a potential) a new geodesic Hamiltonian $\mathcal{T}$ (so without any potential) with an additional degree of freedom (it is in fact an extended formalism). The important point is that the solutions of the equations of motion for such a Hamiltonian H come from geodesics of $\mathcal{T}$ in an enlarged curved space. That is, it is a geometric formalism introduced with the idea of relating classical nonrelativistic Lagrangian or Hamiltonian mechanics with relativistic gravitation [24]–[35]. Our idea is that this formalism can also be applied for the study of superintegrable systems on non-Euclidean spaces.

One important point is that although the number of superintegrable systems can be considered as rather limited, they are not, however, isolated ones but, on the contrary, they frequently appear grouped into families; for example, each of the above mentioned 2D potentials V_r ($r = a, b, c, d$), has the structure of a 3D vector space. In this paper we prove that the 2D Euclidean potentials V_r are related, via the Eisenhart formalism, with some superintegrable geodesic Hamiltonian systems $\mathcal{T}_r$ on 3D curved spaces, generally of nonconstant curvature and with a broken spherically symmetry. Furthermore, natural 3D Hamiltonians, $\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r$, can then be constructed by preserving the same superintegrability and separability properties.

On the other hand, in these last years the interest for the study of systems with a position-dependent mass (PDM) has become a matter of great interest and has attracted a lot of attention of many authors [36]–[50]. It seems therefore natural to enlarge the study of superintegrability and separability to include systems with a PDM by following the same constructive approach. Consequently, as a new step in this procedure, we also prove that $\mathcal{T}_r$ and $\mathcal{H}_r$ admit deformations, say $\widetilde{\mathcal{T}}_r = \mathcal{T}_r/\mu_r$ and $\widetilde{\mathcal{H}}_r = \mathcal{H}_r/\mu_r$, with a PDM $\mu_r(\lambda)$ depending of a real parameter λ, in such a way that the latter are superintegrable for all the values of λ (in the domain of the parameter) and that for $\lambda = 0$ they reduce to the previously studied superintegrable Hamiltonians.

We must mention that there exists a certain relationship between the approach presented in this paper and some previous studies on curved oscillators and KC potentials related to the so-called Bertrand spacetimes (spherically symmetric and static Lorentzian spacetimes), firstly introduced by Perlick in [51] and further studied in [52, 53], where generalisations of superintegrable Hamiltonians fulfilling Bertrand's theorem [54] on conformally flat spaces have been achieved. We stress that one of the main differences (in addition to the use of the Eisenhart formalism) is that, in this paper the potentials are not necessary central, that is, our results mainly concern systems defined on non-conformally flat spaces.

The structure of the paper is as follows. In the next section we establish the main characteristics of the Eisenhart formalism (a rigorous geometrical description can be found in the Appendix). In Section 3 we briefly review the classification of the four families of quadratic in the momenta superintegrable Hamiltonians on the Euclidean plane $H_r = T + V_r$ $(r = a, b, c, d)$. In Section 4, the Eisenhart approach is applied in order to construct superintegrable 3D geodesic Hamiltonians $\mathcal{T}_r$ from the previous V_r. The addition of a potential $\mathcal{U}_r$ to $\mathcal{T}_r$ leading to superintegrable/separable Hamiltonians $\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r$ is addressed in Section 5. Next in Section 6, a PDM μ_r is introduced in the 3D geodesic Hamiltonians $\mathcal{T}_r$, by preserving separability, so giving rise to new superintegrable geodesic Hamiltonians $\widetilde{\mathcal{T}}_r = \mathcal{T}_r/\mu_r$. In Section 7, a separable potential $\widetilde{\mathcal{U}}_r$ is added to $\widetilde{\mathcal{T}}_r$ providing new superintegrable Hamiltonians $\widetilde{\mathcal{H}}_r$ which constitute the main result of this paper. We conclude in the last section with some remarks and open problems.

2 Eisenhart formalism

Let us first recall some basic properties relating Riemannian geometry with Lagrangian dynamics for natural systems.

Suppose a nD manifold M endowed with a Riemannian metric g. If we denote by $\{q^i\,; i = 1, \dots, n\}$, a set of coordinates on M and by $g_{ij}(q)$ the components of g, the expressions of g and $\mathrm{d}s^2$ are given by

$$g = g_{ij}(q)\, \mathrm{d}q^i \otimes \mathrm{d}q^j , \qquad \mathrm{d}s^2 = g_{ij}(q) \mathrm{d}q^i \mathrm{d}q^j \,.$$

Then the corresponding equation of the geodesics on M,

$$\ddot{q}^i + \Gamma^i_{jk}\, \dot{q}^j\, \dot{q}^k = 0\,, \qquad \Gamma^i_{jk} = \frac{1}{2}\, g^{il} \left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right), \quad i, j, k = 1, \dots, n,$$

can be obtained as the Euler–Lagrange equations from a Lagrangian L with only a quadratic kinetic term T_g and without any potential

$$L = T_g = \frac{1}{2}\, g_{ij}(q) v^i v^j \,.$$

Conversely, the Lagrangian formalism establishes that the trajectories of free motion of a particle in a configuration space Q are (i) the solutions of the equations determined by a pure kinetic Lagrangian (quadratic kinetic term without potential) and that (ii) these trajectories are just the geodesics on the space Q. Hence the Lagrangians describing the free motion are also known as geodesic Lagrangians.

As it is well known, the relativistic theory of gravitation introduced by Einstein in 1915 establishes that the trajectory of a particle under external gravitational forces can be described as a geodesic on the 4D spacetime. This, in turn, means that the spatial paths of particles in the 3D Euclidean space can alternatively be considered as geodesics in a higher-dimensional non-Euclidean space by introducing a new metric. This was the idea introduced later on by Eisenhart in 1928/29 in nonrelativistic Lagrangian and Hamiltonian dynamics [23]. Thus the equations of motion of a particle under the action of a potential force in a nD configuration space Q can be reformulated as the equations of geodesics in a $(n+1)$D new configuration space $\widetilde{Q}$ with a new (pseudo-)Riemannian metric constructed by combining the original metric with the potential defined on Q.

More explicitly, assume that we are given a natural Lagrangian (quadratic kinetic term minus a potential)

$$L(q,v) = T(q,v) - V(q)\,, \qquad T(q,v) = \frac{1}{2}\,g_{ij}(q)v^i v^j\,,$$

where the coefficients $g_{ij}(q)$ are symmetric functions of the coordinates and $V(q)$ is a potential. We can then consider the configuration space Q of the system as a Riemannian space with a metric determined by the coefficients of the kinetic term

$$\mathrm{d}s^2 = g_{ij}(q)\mathrm{d}q^i\mathrm{d}q^j\,.$$

Since the matrix $[g_{ij}(q)]$ is invertible the Legendre transformation, $p_i = g_{ij}(q)\,v^j$, leads to the Hamiltonian function H given by

$$H(q,p) = T(q,p) + V(q)\,, \qquad T = \frac{1}{2}\,g^{ij}(q)p_i p_j\,,$$

where $g_{ij}\,g^{jk} = \delta_i^k$.

The Eisenhart formalism (also known as Eisenhart lift) is an extended formalism. The main idea is introducing a new degree of freedom with a new coordinate, say z, i.e. Q is replaced by $\widetilde{Q} = \mathbb{R}\times Q$ and its corresponding momentum p_z, in such a way that the new metric $\mathrm{d}\sigma^2$ and the new Hamiltonian $\mathcal{T}\in C^\infty(T^*\widetilde{Q})$ are given by

$$\mathrm{d}\sigma^2 = g_{ij}(q)\mathrm{d}q^i\mathrm{d}q^j + \frac{\mathrm{d}z^2}{V(q)}\,, \qquad \mathcal{T} = \frac{1}{2}\,g^{ij}(q)p_i p_j + \frac{1}{2}\,V(q)\,p_z^2\,, \tag{2.1}$$

so that $\mathcal{T}$ is homogeneous of degree two in the momenta, and this defines a geodesic Hamiltonian. As the variable z is cyclic p_z is a constant of motion and fixing its value $p_z = 1$ the parameter of the integral curves coincides with the arc-length. In this way the motion of a particle under external forces arising from a potential V is described as a geodesic motion in an extended configuration space determined by $\mathcal{T}$. Although the origin of this formalism is related with properties of relativistic mechanics, this procedure has been studied by making use of different approaches (see [24]–[35] and references therein). A more detailed geometric study of the Eisenhart lift is presented in the Appendix.

We must mention that Eisenhart also considered another more extended formalism that introduces not just one but two additional degrees of freedom; that is, two new variables (Q is replaced by $\widetilde{Q} = \mathbb{R}^2\times Q$) and two conjugated momenta [23], [32]. This more genera Eisenhart lift is related with the study of time-dependent systems and with problems with external gauge fields (Lagrangians with terms linear in the velocities). Nevertheless in what follows we study time-independent natural Hamiltonians without gauge fields and, therefore (see Section 6 of [23]), we will make use the Eisenhart formalism with only one extra degree of freedom.

3 Quadratic superintegrability in the Euclidean plane

Let us denote by V_r, $r = a, b, c, d$, the four 2D potentials with separability in two different coordinate systems in the Euclidean plane [3]–[6]. Each resulting potential V_r is, in fact, a superposition of three potentials

$$V_r = k_1 V_1 + k_2 V_2 + k_3 V_3 \,,$$

where, hereafter, k_1, k_2, k_3 are three arbitrary real constants. We remark that, from a mathematical/physical viewpoint, the k_1-term will be the 'principal' potential so that each family will be 'shortly' named according to it.

For our purposes we write these four families in terms of Cartesian coordinates (x, y) with conjugate momenta (p_x, p_y), in such a manner that the Hamiltonian H_r reads

$$H_r = T + V_r = \frac{1}{2}\,(p_x^2 + p_y^2) + V_r(x, y)\,, \qquad r = a, b, c, d\,. \tag{3.1}$$

These four types of Hamiltonians determine quadratically superintegrable systems as they are endowed with *three* functionally independent constants of motion which are quadratic in the momenta. Notice that for $n = 2$, the superintegrability property is, in fact, maximal since $2n - 1 = 3$ is the maximum number of independent integrals.

The two first potentials, V_a and V_b, represent nonlinear oscillators (harmonic oscillators with additional terms), meanwhile the two remaining potentials, V_c and V_d, correspond to the superposition of the KC problem with two other terms.

3.1 Family a: Isotropic oscillator

This corresponds to the potential

$$V_a = \frac{1}{2}\,k_1(x^2 + y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2}\,, \tag{3.2}$$

which is separable in (i) Cartesian coordinates and (ii) polar ones. The k_1-potential is just the isotropic oscillator with frequency ω whenever $k_1 = \omega^2 > 0$, meanwhile the two remaining potentials are Rosochatius or Winternitz terms (which provide centrifugal barriers when $k_2 > 0$ and $k_3 > 0$). We recall that the Hamiltonian H_a is just the 2D version of the so-called Smorodinsky–Winternitz system [3] which has been widely studied (see, e.g., [5, 6, 15, 55, 56, 57] and references therein).

Three functionally independent constants of motion are the two 1D energies, I_{a1} and I_{a2}, along with a third integral I_{a3} related to the angular momentum; namely,

$$I_{a1} = \frac{1}{2}\,p_x^2 + \frac{1}{2}\,k_1 x^2 + \frac{k_2}{x^2}\,, \qquad I_{a2} = \frac{1}{2}\,p_y^2 + \frac{1}{2}\,k_1 y^2 + \frac{k_3}{y^2}\,,$$

$$I_{a3} = (x p_y - y p_x)^2 + 2k_2\left(\frac{y}{x}\right)^2 + 2k_3\left(\frac{x}{y}\right)^2 .$$

3.2 Family b: Anisotropic oscillator

The following potential

$$V_b = \frac{1}{2}\,k_1(4x^2 + y^2) + \frac{k_2}{y^2} + k_3 x \tag{3.3}$$

is separable in (i) Cartesian coordinates and (ii) parabolic ones. The k_1-potential is just the anisotropic $2:1$ oscillator provided that $k_1 = \omega^2 > 0$ (so with frequencies $\omega_x = 2\omega$ and $\omega_y = \omega$), the k_2-potential is a Rosochatius–Winternitz term, and the (trivial) k_3-potential simply corresponds to a translation along the x-axis.

Three constants of motion are the two 1D energies, I_{b1}, I_{b2}, and a third integral I_{b3}, related to one component of the 2D Laplace–Runge–Lenz vector, which are given by

$$
\begin{aligned}
I_{b1} &= \frac{1}{2}\,p_x^2 + 2k_1x^2 + k_3x\,, \qquad I_{b2} = \frac{1}{2}\,p_y^2 + \frac{1}{2}\,k_1y^2 + \frac{k_2}{y^2}\,,\\
I_{b3} &= (xp_y - yp_x)p_y - k_1xy^2 + \frac{2k_2x}{y^2} - \frac{k_3y^2}{2}\,.
\end{aligned}
$$

3.3 Family c: Kepler–Coulomb I

The potential given by

$$
V_c = \frac{k_1}{\sqrt{x^2+y^2}} + \frac{k_2}{y^2} + \frac{k_3x}{y^2\sqrt{x^2+y^2}} \tag{3.4}
$$

is separable in (i) polar coordinates and (ii) parabolic ones. In this case, the k_1-term is the KC potential and the k_2-term is a Rosochatius–Winternitz potential.

One constant of motion is the Hamiltonian itself, that is $I_{c1} = H_c$, and two other integrals, I_{c2}, and I_{c3}, read

$$
\begin{aligned}
I_{c2} &= (xp_y - yp_x)^2 + \frac{2k_2x^2}{y^2} + \frac{2k_3x\sqrt{x^2+y^2}}{y^2}\,,\\
I_{c3} &= (xp_y - yp_x)p_y + \frac{k_1x}{\sqrt{x^2+y^2}} + \frac{2k_2x}{y^2} + \frac{k_3(2x^2+y^2)}{y^2\sqrt{x^2+y^2}}\,.
\end{aligned}
$$

Hence I_{c2} comes from the angular momentum, while I_{c3} is provided by a component of the Laplace–Runge–Lenz vector.

3.4 Family d: Kepler–Coulomb II

Finally, the fourth potential is given by

$$
V_d = \frac{k_1}{\sqrt{x^2+y^2}} + k_2\,\frac{\left[\sqrt{x^2+y^2}+x\right]^{1/2}}{\sqrt{x^2+y^2}} + k_3\,\frac{\left[\sqrt{x^2+y^2}-x\right]^{1/2}}{\sqrt{x^2+y^2}}\,, \tag{3.5}
$$

which is separable in (i) parabolic coordinates (τ, σ) and (ii) a second system of parabolic coordinates (α, β) obtained from (τ, σ) by a rotation. Thus, we recall that the KC potential (k_1-term) can be superposed with two other potentials which are different from the previous ones (3.4) keeping superintegrability.

One constant of motion is again the Hamiltonian itself, $I_{d1} = H_d$, meanwhile two other integrals, I_{d2}, and I_{d3}, turn out to be

$$
\begin{aligned}
I_{d2} &= (xp_y - yp_x)p_y + \frac{k_1x}{\sqrt{x^2+y^2}} - \frac{k_2y\left[\sqrt{x^2+y^2}-x\right]^{1/2}}{\sqrt{x^2+y^2}} + \frac{k_3y\left[\sqrt{x^2+y^2}+x\right]^{1/2}}{\sqrt{x^2+y^2}}\,,\\
I_{d3} &= (xp_y - yp_x)p_x - \frac{k_1y}{\sqrt{x^2+y^2}} - \frac{k_2x\left[\sqrt{x^2+y^2}-x\right]^{1/2}}{\sqrt{x^2+y^2}} + \frac{k_3x\left[\sqrt{x^2+y^2}+x\right]^{1/2}}{\sqrt{x^2+y^2}}\,.
\end{aligned}
$$

These are related to both components of the 2D Laplace–Runge–Lenz vector.

Obviously, when $k_2 = k_3 = 0$ both KC I and II families reduce to the common KC k_1-potential. Nevertheless, throughout the paper we shall deal with the three generic k_i-terms so describing two essential different families of superintegrable Hamiltonians.

4 Geodesic Hamiltonians $\mathcal{T}$ endowed with multiple separability on 3D curved spaces

Let us consider the 2D Euclidean Hamiltonian H_r (3.1) with one of the superintegrable potentials V_r given in the above section. By applying the Eisenhart lift (2.1) with $g_{ij} = \delta_{ij}$, $q_1 = x, q_2 = y$, we obtain a new 3D Riemannian metric and associated free Hamiltonian defined by

$$\mathrm{d}\sigma_r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{\mathrm{d}z^2}{V_r(x,y)}\,, \qquad \mathcal{T}_r = \frac{1}{2}\left(p_x^2 + p_y^2 + V_r(x,y)\,p_z^2\right)\,, \qquad r = a, b, c, d\,, \tag{4.1}$$

where (x, y, z) are Cartesian coordinates and (p_x, p_y, p_z) their conjugate momenta.

In what follows we study the separability of the corresponding HJ equation for each of the four types of geodesic Hamiltonians $\mathcal{T}_r$. We stress that the separability of $\mathcal{T}_r$ is, in fact, provided by the separability of H_r, that is, if H_r is separable in the coordinates (q_1, q_2), we assume that $\mathcal{T}_r$ is separable in the coordinates (q_1, q_2, z). We advance that we shall obtain *four* independent integrals for each $\mathcal{T}_r$ in an explicit form (three of them being mutually in involution). Consequently, $\mathcal{T}_r$ will determine a superintegrable system but not a maximal superintegrable one, since an additional *fifth* constant of motion would be necessary to get the maximum number of $2n-1 = 5$ integrals (corresponding to $n = 3$ degrees of freedom). In this sense, $\mathcal{T}_r$ can be regarded as either a minimally superintegrable Hamiltonian [4] or a quasi-maximally superintegrable one [57].

At this point we mention that the idea of obtaining a new superintegrable $(n+1)$D Hamiltonian starting with a simpler and previously known superintegrable Hamiltonian with n degrees of freedom is a matter that has been analyzed by some authors (see e.g. [58, 59, 60]) but making use of other approaches different to the Eisenhart formalism presented in this paper.

4.1 Geodesic Hamiltonian $\mathcal{T}_a$ from isotropic oscillator

We construct the Hamiltonian $\mathcal{T}_a$ (4.1) with the potential V_a (3.2). Since the initial 2D Hamiltonian H_a is separable in Cartesian (x, y) and polar variables (r, ϕ), we now analyse the separability of the new 3D system $\mathcal{T}_a$ in Cartesian (x, y, z) and cylindrical (r, ϕ, z) coordinates.

4.1.1 Cartesian separability

The HJ equation takes the form

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + V_a(x,y)\left(\frac{\partial W}{\partial z}\right)^2 = 2E\,,$$

so that if we assume that W can be written as $W = W_x(x) + W_y(y) + W_z(z)$, then we can perform a separation of variables which leads to the following one-variable expressions

$$(W_z')^2 = -\,\alpha\,, \quad (W_x')^2 - \alpha\left(\frac{1}{2}\,k_1 x^2 + \frac{k_2}{x^2}\right) = \beta + E\,, \quad (W_y')^2 - \alpha\left(\frac{1}{2}\,k_1 y^2 + \frac{k_3}{y^2}\right) = -\,\beta + E\,,$$

where α and β denote two constants associated with separability. Each one of these expressions determines a constant of motion; so the following functions

$$K_{a1}=p_z\,,\qquad K_{a2}=p_x^2+\left(\frac{1}{2}\,k_1x^2+\frac{k_2}{x^2}\right)p_z^2\,,\qquad K_{a3}=p_y^2+\left(\frac{1}{2}\,k_1y^2+\frac{k_3}{y^2}\right)p_z^2\,,\qquad(4.2)$$

are three functionally independent constants of motion,

$$\mathrm{d}K_{a1}\,\wedge\,\mathrm{d}K_{a2}\,\wedge\,\mathrm{d}K_{a3}\neq 0\,,$$

satisfying the following properties

$$\{K_{a1},K_{a2}\}=0\,,\qquad\{K_{a1},K_{a3}\}=0\,,\qquad\{K_{a2},K_{a3}\}=0\,,\qquad\mathcal{T}_a=\frac{1}{2}\big(K_{a2}+K_{a3}\big)\,.$$

4.1.2 Cylindrical separability

We introduce the usual polar coordinates, $x=r\cos\phi$ and $y=r\sin\phi$, finding that the free Hamiltonian T_a reads

$$\mathcal{T}_a=\frac{1}{2}\left(p_r^2+\frac{p_\phi^2}{r^2}+V_a\,p_z^2\right)\,,\qquad V_a=\frac{1}{2}\,k_1r^2+\frac{k_2}{r^2\cos^2\phi}+\frac{k_3}{r^2\sin^2\phi}\,.$$

The HJ equation turns out to be

$$\left(\frac{\partial W}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial W}{\partial\phi}\right)^2+V_a(r,\phi)\left(\frac{\partial W}{\partial z}\right)^2=2E\,.$$

If we suppose that $W=W_r(r)+W_\phi(\phi)+W_z(z)$, then we can perform a separation of variables; we first obtain $(W_z')^2=-\,\gamma$ and next

$$r^2(W_r')^2-2r^2E-\frac{1}{2}\,k_1r^4\gamma=-\,(W_\phi')^2+\gamma\left(\frac{k_2}{\cos^2\phi}+\frac{k_3}{\sin^2\phi}\right)=\delta,$$

where γ and δ are two constants. Hence we obtain the following constants of motion

$$J_{a1}=p_z\equiv K_{a1}\,,\qquad J_{a2}=p_\phi^2+\left(\frac{k_2}{\cos^2\phi}+\frac{k_3}{\sin^2\phi}\right)p_z^2\,,\qquad J_{a3}=r^2p_r^2+\frac{1}{2}\,k_1r^4p_z^2-2r^2\mathcal{T}_a\,,\ (4.3)$$

such that $\{J_{a1}\,,J_{a2}\}=0$ and $J_{a2}+J_{a3}=0$.

We summarize the above results in the following statement.

Proposition 1. *The 3D geodesic Hamiltonian*

$$\mathcal{T}_a=\frac{1}{2}\left(p_x^2+p_y^2+V_a\,p_z^2\right)\,,\qquad V_a=\frac{1}{2}\,k_1(x^2+y^2)+\frac{k_2}{x^2}+\frac{k_3}{y^2}\,,\qquad(4.4)$$

is HJ separable in Cartesian (x,y,z) and cylindrical (r,ϕ,z) coordinates. This determines a superintegrable system endowed with four independent constants of motion given by K_{a1},K_{a2},K_{a3} (4.2) and J_{a2} (4.3). Furthermore, K_{a1},K_{a2},K_{a3} are mutually in involution and $\mathcal{T}_a=\frac{1}{2}\big(K_{a2}+K_{a3}\big)$.

4.2 Geodesic Hamiltonian $\mathcal{T}_b$ from anisotropic oscillator

Let $\mathcal{T}_b$ be the 3D free Hamiltonian (4.1) with V_b (3.3). Since V_b is separable in Cartesian and parabolic (a, b) coordinates, we study the separability of $\mathcal{T}_b$ in Cartesian (x, y, z) and parabolic-cylindrical (a, b, z) coordinates.

4.2.1 Cartesian separability

The HJ equation yields

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + V_b(x,y)\left(\frac{\partial W}{\partial z}\right)^2 = 2E\,,$$

so that if we assume that W can be written as $W = W_x(x)+W_y(y)+W_z(z)$ then we can perform a separation of variables obtaining the one-variable expressions

$$(W_z')^2 = -\,\alpha\,,\quad (W_x')^2 - \alpha\left(2k_1x^2 + k_3x\right) = \beta + E\,,\quad (W_y')^2 - \alpha\left(\frac{1}{2}\,k_1y^2 + \frac{k_2}{y^2}\right) = -\beta + E\,,$$

where α and β are two constants. Each one of these expressions determines a constant of motion, namely,

$$K_{b1} = p_z\,,\qquad K_{b2} = p_x^2 + \left(2k_1x^2 + k_3x\right)p_z^2\,,\qquad K_{b3} = p_y^2 + \left(\frac{1}{2}\,k_1y^2 + \frac{k_2}{y^2}\right)p_z^2\,, \tag{4.5}$$

which, moreover, are functionally independent

$$\mathrm{d}K_{b1}\wedge\mathrm{d}K_{b2}\wedge\mathrm{d}K_{b3}\neq 0\,,$$

and they satisfy the following properties

$$\{K_{b1},K_{b2}\} = 0\,,\qquad \{K_{b1},K_{b3}\} = 0\,,\qquad \{K_{b2},K_{b3}\} = 0\,,\qquad \mathcal{T}_b = \frac{1}{2}\big(K_{b2}+K_{b3}\big)\,.$$

4.2.2 Parabolic-cylindrical separability

If we introduce the parabolic coordinates defined by

$$x = \frac{1}{2}\left(\tau^2-\sigma^2\right)\,,\qquad y = \tau\sigma\,, \tag{4.6}$$

the Hamiltonian $\mathcal{T}_b$ (4.1) and the potential V_b (3.3) become

$$\mathcal{T}_b = \frac{1}{2}\left(\frac{p_\tau^2+p_\sigma^2}{\tau^2+\sigma^2} + V_b\,p_z^2\right)\,,\quad V_b = \frac{1}{\tau^2+\sigma^2}\left[\frac{k_1}{2}(\tau^6+\sigma^6) + k_2\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right) + \frac{k_3}{2}(\tau^4-\sigma^4)\right]\,,$$

and the HJ equation adopts the form

$$\frac{1}{\tau^2+\sigma^2}\left[\left(\frac{\partial W}{\partial \tau}\right)^2 + \left(\frac{\partial W}{\partial \sigma}\right)^2\right] + V_b(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2 = 2E\,,$$

so that if we assume that W is of the form $W = W_\tau(\tau) + W_\sigma(\sigma) + W_z(z)$ we can perform a separation of variables obtaining first $(W_z')^2 = -\,\gamma$ and then

$$\big((W_\tau')^2 - 2\tau^2E\big) + \big((W_\sigma')^2 - 2\sigma^2E\big) = \gamma\left(\frac{k_1}{2}\,\tau^6 + \frac{k_2}{\tau^2} + \frac{k_3}{2}\,\tau^4\right) + \gamma\left(\frac{k_1}{2}\,\sigma^6 + \frac{k_2}{\sigma^2} - \frac{k_3}{2}\,\sigma^4\right)\,,$$

providing three integrals

$$
\begin{aligned}
&J_{b1} = p_z \equiv K_{b1}\,, \qquad J_{b2} = p_\tau^2 + \left(\frac{k_1}{2}\,\tau^6 + \frac{k_2}{\tau^2} + \frac{k_3}{2}\,\tau^4 \right) p_z^2 - 2\tau^2 \mathcal{T}_b\,, \\
&J_{b3} = p_\sigma^2 + \left(\frac{k_1}{2}\,\sigma^6 + \frac{k_2}{\sigma^2} - \frac{k_3}{2}\,\sigma^4 \right) p_z^2 - 2\sigma^2 \mathcal{T}_b\,,
\end{aligned} \tag{4.7}
$$

such that $\{J_{b1}\,, J_{b2}\} = 0$ and $J_{b2} + J_{b3} = 0$.

The following proposition summarizes these results.

Proposition 2. *The 3D geodesic Hamiltonian given by*

$$
\mathcal{T}_b = \frac{1}{2}\left(p_x^2 + p_y^2 + V_b\, p_z^2\right) \qquad V_b = \frac{1}{2}\,k_1(4x^2 + y^2) + \frac{k_2}{y^2} + k_3 x\,, \tag{4.8}
$$

is HJ separable in Cartesian (x, y, z) and parabolic-cylindrical (τ, σ, z) coordinates. It represents a superintegrable system endowed with four independent constants of motion corresponding to K_{b1}, K_{b2}, K_{b3} (4.5) and J_{b2} (4.7). Moreover, K_{b1}, K_{b2}, K_{b3} are mutually in involution and $\mathcal{T}_b = \frac{1}{2}\big(K_{b2} + K_{b3}\big)$.

4.3 Geodesic Hamiltonian $\mathcal{T}_c$ from Kepler–Coulomb I

Now we consider $\mathcal{T}_c$ (4.1) with V_c (3.4). Recall that V_c is separable in polar and parabolic coordinates, so that we analyse the separability of $\mathcal{T}_c$ in cylindrical and parabolic-cylindrical coordinates.

4.3.1 Cylindrical separability

In the variables (r, ϕ, z), the geodesic Hamiltonian $\mathcal{T}_c$ is expressed as

$$
\mathcal{T}_c = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2} + V_c\, p_z^2 \right)\,, \qquad V_c = \frac{k_1}{r} + \frac{k_2}{r^2 \sin^2\phi} + \frac{k_3 \cos\phi}{r^2 \sin^2\phi}\,.
$$

The HJ equation reads

$$
\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2 + V_c(r,\phi)\left(\frac{\partial W}{\partial z}\right)^2 = 2E\,.
$$

Hence if we assume that W is of the form $W = W_r(r) + W_\phi(\phi) + W_z(z)$ we can perform a separation of variables, finding first that $(W_z')^2 = -\,\gamma$ and then

$$
r^2 (W_r')^2 - 2r^2 E - k_1 \gamma\, r = -\,(W_\phi')^2 + \gamma\left(\frac{k_2}{\sin^2\phi} + \frac{k_3\cos\phi}{\sin^2\phi} \right) = \delta\,,
$$

so that the following functions

$$
\begin{aligned}
&K_{c1} = p_z\,, \qquad K_{c2} = p_\phi^2 + \Big(\frac{k_2}{\sin^2\phi} + \frac{k_3\cos\phi}{\sin^2\phi} \Big)\, p_z^2\,, \\
&K_{c3} = r^2 p_r^2 + k_1 r\, p_z^2 - 2\, r^2\, \mathcal{T}_c\,,
\end{aligned} \tag{4.9}
$$

are constants of motion such that $\{K_{c1}\,, K_{c2}\} = 0$ and $K_{c2} + K_{c3} = 0$.

4.3.2 Parabolic-cylindrical separability

By introducing the parabolic coordinates (4.6) we find that the Hamiltonian $\mathcal{T}_c$ (4.1) is given by

$$\mathcal{T}_c = \frac{1}{2}\left(\frac{p_\tau^2 + p_\sigma^2}{\tau^2+\sigma^2} + V_c\, p_z^2\right) , \qquad V_c = \frac{1}{\tau^2+\sigma^2}\left[2k_1 + k_2\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right) + k_3\left(\frac{1}{\sigma^2}-\frac{1}{\tau^2}\right)\right] ,$$

so that the HJ equation becomes

$$\frac{1}{\tau^2+\sigma^2}\left[\left(\frac{\partial W}{\partial \tau}\right)^2 + \left(\frac{\partial W}{\partial \sigma}\right)^2\right] + V_c(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2 = 2E\,.$$

By writing $W = W_\tau(\tau) + W_\sigma(\sigma) + W_z(z)$, separability leads to

$$\left((W_\tau')^2 - 2\tau^2 E\right) + \left((W_\sigma')^2 - 2\sigma^2 E\right) = \gamma\left(k_1 + \frac{k_2-k_3}{\tau^2}\right) + \gamma\left(k_1 + \frac{k_2+k_3}{\sigma^2}\right) .$$

Therefore the following functions

$$\begin{aligned} &J_{c1} = p_z \equiv K_{c1}\,, \qquad J_{c2} = p_\tau^2 + \left(k_1 + \frac{k_2-k_3}{\tau^2}\right) p_z^2 - 2\tau^2\, \mathcal{T}_c\,, \\ &J_{c3} = p_\sigma^2 + \left(k_1 + \frac{k_2+k_3}{\sigma^2}\right) p_z^2 - 2\sigma^2\, \mathcal{T}_c\,, \end{aligned} \tag{4.10}$$

are three constants of motion fulfilling $\{J_{c1}\,, J_{c2}\} = 0$ and $J_{c2} + J_{c3} = 0$.

We conclude with the following statement.

Proposition 3. *The 3D geodesic Hamiltonian*

$$\mathcal{T}_c = \frac{1}{2}\left(p_x^2 + p_y^2 + V_c\, p_z^2\right) , \qquad V_c = \frac{k_1}{\sqrt{x^2+y^2}} + \frac{k_2}{y^2} + \frac{k_3 x}{y^2\sqrt{x^2+y^2}} , \tag{4.11}$$

is HJ separable in cylindrical (r,ϕ,z) and parabolic-cylindrical (τ,σ,z) coordinates. This is endowed with four independent constants of motion: the Hamiltonian itself, $\mathcal{T}_c$, along with K_{c1}, K_{c2} (4.9) and J_{c2} (4.10). The three integrals $\mathcal{T}_c, K_{c1}, K_{c2}$ are mutually in involution.

4.4 Geodesic Hamiltonian $\mathcal{T}_d$ from Kepler–Coulomb II

Finally, we consider the four family $\mathcal{T}_d$ (4.1) with V_d (3.5). Since V_d is separable in two types of parabolic coordinates, (τ,σ) and (α,β), we study the separability of $\mathcal{T}_d$ in the corresponding two types of parabolic-cylindrical coordinates.

4.4.1 Parabolic-cylindrical separability I

We introduce the parabolic coordinates (τ,σ) (4.6) in the Hamiltonian $\mathcal{T}_d$ yielding

$$\mathcal{T}_d = \frac{1}{2}\left(\frac{p_\tau^2 + p_\sigma^2}{\tau^2+\sigma^2} + V_d\, p_z^2\right) , \qquad V_d = 2\,\frac{k_1 + k_2\tau + k_3\sigma}{\tau^2+\sigma^2}\,.$$

Hence the corresponding HJ equation

$$\frac{1}{\tau^2+\sigma^2}\left[\left(\frac{\partial W}{\partial \tau}\right)^2 + \left(\frac{\partial W}{\partial \sigma}\right)^2\right] + V_d(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2 = 2E$$

admits separation of variables and it leads to $(W_z')^2 = -\gamma$ and

$$\left[(W_\tau')^2 - 2\tau^2 E - (k_1 + 2k_2\tau)\,\gamma\right] + \left[(W_\sigma')^2 - 2\sigma^2 E - (k_1 + 2k_3\sigma)\,\gamma\right] = 0\,.$$

Therefore, the following functions

$$\begin{aligned} &K_{d1} = p_z\,, \qquad K_{d2} = p_\tau^2 + (k_1 + 2k_2\tau)\,p_z^2 - 2\tau^2\mathcal{T}_d\,, \\ &K_{d3} = p_\sigma^2 + (k_1 + 2k_3\sigma)\,p_z^2 - 2\sigma^2\mathcal{T}_d\,, \end{aligned} \tag{4.12}$$

are constants of motion satisfying $\{K_{d1}\,, K_{d2}\} = 0$ and $K_{d2} + K_{d3} = 0$.

4.4.2 Parabolic-cylindrical separability II

We consider a second system of parabolic coordinates (α, β) by rotating the original one (τ, σ) (4.6) in the form

$$\tau = \frac{1}{\sqrt{2}}(\alpha + \beta)\,, \qquad \sigma = \frac{1}{\sqrt{2}}(\alpha - \beta)\,, \tag{4.13}$$

in such a way that the Hamiltonian $\mathcal{T}_d$ is now given by

$$\mathcal{T}_d = \frac{1}{2}\left(\frac{p_\alpha^2 + p_\beta^2}{\alpha^2 + \beta^2} + V_d\,p_z^2\right)\,, \qquad V_d = \frac{2k_1 + k_2\sqrt{2}(\alpha + \beta) + k_3\sqrt{2}(\alpha - \beta)}{\alpha^2 + \beta^2}\,.$$

Thus $\mathcal{T}_d$ determines the HJ equation

$$\frac{1}{\alpha^2 + \beta^2}\left[\left(\frac{\partial W}{\partial \alpha}\right)^2 + \left(\frac{\partial W}{\partial \beta}\right)^2\right] + V_d(\alpha, \beta)\left(\frac{\partial W}{\partial z}\right)^2 = 2E$$

that also admits separability leading to three integrals

$$\begin{aligned} &J_{d1} = p_z \equiv K_{d1}\,, \qquad J_{d2} = p_\alpha^2 + \Big(k_1 + \sqrt{2}(k_2 + k_3)\alpha\Big)\,p_z^2 - 2\alpha^2\mathcal{T}_d\,, \\ &J_{d3} = p_\beta^2 + \Big(k_1 + \sqrt{2}(k_2 - k_3)\beta\Big)\,p_z^2 - 2\beta^2\mathcal{T}_d\,, \end{aligned} \tag{4.14}$$

such that $\{J_{d1}\,, J_{d2}\} = 0$ and $J_{d2} + J_{d3} = 0$.

These results are summarized as follows.

Proposition 4. *The 3D geodesic Hamiltonian given by*

$$\mathcal{T}_d = \frac{1}{2}\left(p_x^2 + p_y^2 + V_d\,p_z^2\right)\,, \qquad V_d = \frac{k_1}{r} + k_2\frac{\sqrt{r + x}}{r} + k_3\frac{\sqrt{r - x}}{r}\,, \qquad r^2 = x^2 + y^2\,, \tag{4.15}$$

is HJ separable in two sets of parabolic-cylindrical coordinates (τ, σ, z) and (α, β, z) which are related by a rotation. This system is endowed with four independent constants of motion: the Hamiltonian $\mathcal{T}_d$, together with the K_{d1}, K_{d2} (4.12) and J_{d2} (4.14). The three integrals $\mathcal{T}_d, K_{d1}, K_{d2}$ are mutually in involution.

Notice that K_{d2} and J_{d2} can be written in the first set of parabolic-cylindrical coordinates (τ, σ, z) as

$$\begin{aligned} K_{d2} &= \frac{\sigma^2 p_\tau^2 - \tau^2 p_\sigma^2}{\tau^2 + \sigma^2} + \left(\frac{k_1(\sigma^2 - \tau^2) + 2k_2\tau\sigma^2 - 2k_3\tau^2 b}{\tau^2 + \sigma^2}\right)p_z^2\,, \\ J_{d2} &= \frac{(\tau p_\sigma - \sigma p_\tau)(\tau p_\tau - \sigma p_\sigma)}{\tau^2 + \sigma^2} - \left(\frac{2k_1\tau\sigma + k_2(\tau^2 - \sigma^2)\sigma - k_3(\tau^2 - \sigma^2)\tau}{\tau^2 + \sigma^2}\right)p_z^2\,, \end{aligned} \tag{4.16}$$

and they can be interpreted as generalised versions of the two Laplace–Runge–Lenz constants of motion of the 2D KC problem.

5 Hamiltonians $\mathcal{H}$ endowed with multiple separability on 3D curved spaces

The next step in our approach is to add a 3D potential $\mathcal{U}_r$ to each superintegrable geodesic Hamiltonian $\mathcal{T}_r$ constructed in the previous section, thus leading to a natural Hamiltonian $\mathcal{H}_r$ in the form

$$\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r = \frac{1}{2}\left(p_x^2 + p_y^2 + V_r(x,y)\, p_z^2\right) + \mathcal{U}_r(x,y,z)\,, \qquad r = a,b,c,d\,. \tag{5.1}$$

We now require the complete Hamiltonian $\mathcal{H}_r$ to be HJ separable in the same two sets of coordinates as its kinetic component. Therefore, due to the structure of $\mathcal{T}_r$, we assume that the 3D potential is given by

$$\mathcal{U}_r(x,y,z) = U_r(x,y) + V_r(x,y)Z(z)\,, \qquad r = a,b,c,d\,, \tag{5.2}$$

where $U_r(x,y)$ is a function to be determined for each family, $V_r(x,y)$ is just the known 2D potential, and $Z(z)$ is always an arbitrary smooth function for the four families. Hence the generic initial Hamiltonian reads

$$\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r = \frac{1}{2}\left(p_x^2 + p_y^2 + V_r(x,y)\, p_z^2\right) + U_r(x,y) + V_r(x,y)Z(z)\,, \qquad r = a,b,c,d\,. \tag{5.3}$$

Then multiseparability will give rise to the compatible explicit form for $U_r(x,y)$ together with four independent constants of motion. We remark that, for the four families, U_r will be formally similar to V_r but with different coefficients t_i instead of k_i ($i = 1,2,3$). Consequently, the resulting Hamiltonian $\mathcal{H}_r$ will always determine a superintegrable system but, in general, not a maximally superintegrable one.

Since computations are quite similar to the previous ones, we shall omit most technical details and only provide the main results.

5.1 Hamiltonian $\mathcal{H}_a$ from isotropic oscillator

Let us consider $\mathcal{H}_a$ (5.3) with $\mathcal{T}_a$ given in (4.4) and impose that such a system preserve the same multiple separability studied in Section 4.1. In this case, separability in Cartesian and cylindrical coordinates implies that

$$U_a(x,y) = A(x) + B(y)\,, \qquad U_a(r,\phi) = F(r) + \frac{G(\phi)}{r^2}\,.$$

These two restrictions determine the form of the potential U_a through the above functions that turn out to be

$$A(x) = \frac{1}{2}\,t_1 x^2 + \frac{t_2}{x^2}\,, \qquad B(y) = \frac{1}{2}\,t_1 y^2 + \frac{t_3}{y^2}\,,$$
$$F(r) = \frac{1}{2}\,t_1 r^2\,, \qquad G(\phi) = \frac{t_2}{\cos^2\phi} + \frac{t_3}{\sin^2\phi}\,,$$

where, from now on, t_1, t_2, t_3 denote three arbitrary real constants. Thus U_a is a function formally similar to V_a but with different coefficients.

The final results are summarized in the following statement which generalises Proposition 1.

Proposition 5. *The 3D Hamiltonian $\mathcal{H}_a$ (5.3) with V_a (4.4) and similar U_a with coefficients t_i is HJ separable in Cartesian (x, y, z) and cylindrical (r, ϕ, z) coordinates and it is endowed with four independent quadratic constants of motion given by*

$$
\begin{aligned}
\mathcal{K}_{a1} &= p_z^2 + 2Z(z)\,, \\
\mathcal{K}_{a2} &= p_x^2 + \left(\frac{1}{2}\,k_1 x^2 + \frac{k_2}{x^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 x^2 + \frac{2t_2}{x^2}, \\
\mathcal{K}_{a3} &= p_y^2 + \left(\frac{1}{2}\,k_1 y^2 + \frac{k_3}{y^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 y^2 + \frac{2t_3}{y^2}, \\
\mathcal{J}_{a2} &= p_\phi^2 + \left(\frac{k_2}{\cos^2\phi} + \frac{k_3}{\sin^2\phi}\right)\big(p_z^2 + 2Z(z)\big) + \frac{2t_2}{\cos^2\phi} + \frac{2t_3}{\sin^2\phi}.
\end{aligned}
$$

The three integrals $\mathcal{K}_{a1}, \mathcal{K}_{a2}, \mathcal{K}_{a3}$ are mutually in involution and $\mathcal{H}_a = \frac{1}{2}\big(\mathcal{K}_{a2} + \mathcal{K}_{a3}\big)$.

5.2 Hamiltonian $\mathcal{H}_b$ from anisotropic oscillator

Let $\mathcal{H}_b$ be the Hamiltonian (5.3) with kinetic term $\mathcal{T}_b$ given in (4.8). According to Section 4.2, we impose separability in both Cartesian and parabolic-cylindrical coordinates which means that

$$
U_b(x, y) = A(x) + B(y)\,, \qquad U_b(\tau, \sigma) = \frac{C(\tau) + D(\sigma)}{\tau^2 + \sigma^2}\,,
$$

leading to

$$
\begin{aligned}
A(x) &= 2t_1 x^2 + t_3 x\,, \qquad B(y) = \frac{1}{2}\,t_1 y^2 + \frac{t_2}{y^2}\,, \\
C(\tau) &= \frac{t_1}{2}\,\tau^6 + \frac{t_2}{\tau^2} + \frac{t_3}{2}\,\tau^4\,, \qquad D(\sigma) = \frac{t_1}{2}\,\sigma^6 + \frac{t_2}{\sigma^2} - \frac{t_3}{2}\,\sigma^4\,.
\end{aligned}
$$

Hence U_b is again a function formally similar to V_b.

The final results generalise those achieved in Proposition 2 as follows.

Proposition 6. *The 3D Hamiltonian $\mathcal{H}_b$ (5.3) with V_b (4.8) and similar U_b with coefficients t_i is HJ separable in Cartesian (x, y, z) and parabolic-cylindrical (τ, σ, z) coordinates. This is endowed with the following four independent quadratic constants of motion*

$$
\begin{aligned}
\mathcal{K}_{b1} &= p_z^2 + 2Z(z)\,, \\
\mathcal{K}_{b2} &= p_x^2 + \big(2k_1 x^2 + k_3 x\big)\big(p_z^2 + 2Z(z)\big) + 4t_1 x^2 + 2t_3 x\,, \\
\mathcal{K}_{b3} &= p_y^2 + \left(\frac{1}{2}\,k_1 y^2 + \frac{k_2}{y^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 y^2 + \frac{2t_2}{y^2}\,, \\
J_{b2} &= p_\tau^2 + \left(\frac{k_1}{2}\,\tau^6 + \frac{k_2}{\tau^2} + \frac{k_3}{2}\,\tau^4\right)\big(p_z^2 + 2Z(z)\big) + t_1\tau^6 + \frac{2t_2}{\tau^2} + t_3\tau^4 - 2\tau^2\mathcal{H}_b\,,
\end{aligned}
$$

such that $\mathcal{K}_{b1}, \mathcal{K}_{b2}, \mathcal{K}_{b3}$ are mutually in involution and $\mathcal{H}_b = \frac{1}{2}\big(\mathcal{K}_{b2} + \mathcal{K}_{b3}\big)$.

5.3 Hamiltonian $\mathcal{H}_c$ from Kepler–Coulomb I

Now we consider $\mathcal{H}_c$ (5.3) with $\mathcal{T}_c$ (4.11) and impose the separability in cylindrical and parabolic-cylindrical coordinates as in Section 4.3. This means that U_c must admit the following expressions

$$
U_c(r, \phi) = F(r) + \frac{G(\phi)}{r^2}\,, \qquad U_c(\tau, \sigma) = \frac{C(\tau) + D(\sigma)}{\tau^2 + \sigma^2}\,.
$$

Compatibility among these two separabilities leads to

$$
\begin{aligned}
F(r) &= \frac{t_1}{r}\,, \qquad G(\phi) = \frac{t_2}{\sin^2\phi} + \frac{t_3\cos\phi}{\sin^2\phi}\,, \\
C(\tau) &= t_1 + \frac{t_2 - t_3}{\tau^2}\,, \qquad D(\sigma) = t_1 + \frac{t_2 + t_3}{\sigma^2}\,.
\end{aligned}
$$

Notice that U_c is formally similar to V_c. Then we conclude with the following statement (to be compared with Proposition 3).

Proposition 7. *The 3D Hamiltonian $\mathcal{H}_c$ (5.3) with V_c (4.11) and similar U_c with coefficients t_i is HJ separable in cylindrical (r, ϕ, z) and parabolic-cylindrical (τ, σ, z) coordinates. This system is endowed with four independent integrals, which are the Hamiltonian itself, $\mathcal{H}_c$, along with*

$$
\begin{aligned}
\mathcal{K}_{c1} &= p_z^2 + 2Z(z)\,, \\
\mathcal{K}_{c2} &= p_\phi^2 + \left(\frac{k_2}{\sin^2\phi} + \frac{k_3\cos\phi}{\sin^2\phi}\right)\big(p_z^2 + 2Z(z)\big) + \frac{2t_2}{\sin^2\phi} + \frac{2t_3\cos\phi}{\sin^2\phi}\,, \\
\mathcal{J}_{c2} &= p_\tau^2 + \left(k_1 + \frac{k_2 - k_3}{\tau^2}\right)\big(p_z^2 + 2Z(z)\big) + 2t_1 + \frac{2(t_2 - t_3)}{\tau^2} - 2\tau^2\mathcal{H}_c\,,
\end{aligned}
$$

such that the three constants of motion $\mathcal{H}_c, \mathcal{K}_{c1}, \mathcal{K}_{c2}$ are mutually in involution.

5.4 Hamiltonian $\mathcal{H}_d$ from Kepler–Coulomb II

Finally, we consider the fourth family $\mathcal{H}_d$ (5.3) with $\mathcal{T}_d$ (4.15) and require to preserve the separability in parabolic-cylindrical coordinates of type I (4.6) and type II (4.13) similarly to Section 4.4. Hence multiple separability means that U_d must admit the following expressions

$$
U_d(\tau, \sigma) = \frac{C(\tau) + D(\sigma)}{\tau^2 + \sigma^2}\,, \qquad U_d(\alpha, \beta) = \frac{L(\alpha) + M(\beta)}{\alpha^2 + \beta^2}\,,
$$

which yields

$$
\begin{aligned}
C(\tau) &= t_1 + 2t_2\tau\,, \qquad D(\sigma) = t_1 + 2t_3\sigma\,, \\
L(\alpha) &= t_1 + \sqrt{2}(t_2 + t_3)\alpha\,, \qquad M(\beta) = t_1 + \sqrt{2}(t_2 - t_3)\beta\,,
\end{aligned}
$$

providing a potential U_d formally similar to V_d. The final results, that generalise those given in Proposition 4, are summarized as follows.

Proposition 8. *The 3D Hamiltonian $\mathcal{H}_d$ (5.3) with V_d (4.15) and similar U_d with coefficients t_i is HJ separable in two types of parabolic-cylindrical coordinates: (τ, σ, z) and (α, β, z). This is endowed with four independent constants of motion: the Hamiltonian itself $\mathcal{H}_d$ together with the following three functions*

$$
\begin{aligned}
\mathcal{K}_{d1} &= p_z^2 + 2Z(z)\,, \\
\mathcal{K}_{d2} &= p_\tau^2 + (k_1 + 2k_2\tau)\big(p_z^2 + 2Z(z)\big) + 2t_1 + 4t_2\tau - 2\tau^2\mathcal{H}_d\,, \\
\mathcal{J}_{d2} &= p_\alpha^2 + \left(k_1 + \sqrt{2}(k_2 + k_3)\alpha\right)\big(p_z^2 + 2Z(z)\big) + 2t_1 + 2\sqrt{2}(t_2 + t_3)\alpha - 2\alpha^2\mathcal{H}_d\,.
\end{aligned}
$$

The three integrals $\mathcal{H}_d, \mathcal{K}_{d1}, \mathcal{K}_{d2}$ are mutually in involution.

In parabolic-cylindrical coordinates (τ, σ, z), the integrals $\mathcal{K}_{d2}$ and $\mathcal{J}_{d2}$ can be rewritten as

$$
\begin{aligned}
\mathcal{K}_{d2} &= \frac{\sigma^2 p_\tau^2 - \tau^2 p_\sigma^2}{\tau^2+\sigma^2} + \left(\frac{k_1(\sigma^2-\tau^2) + 2k_2\tau\sigma^2 - 2k_3\tau^2\sigma}{\tau^2+\sigma^2} \right) \left(p_z^2 + 2Z(z) \right) \\
&\qquad + \frac{2t_1(\sigma^2-\tau^2) + 4t_2\tau\sigma^2 - 4t_3\tau^2\sigma}{\tau^2+\sigma^2}\,, \\
\mathcal{J}_{d2} &= \frac{(\tau p_\sigma - \sigma p_\tau)(\tau p_\tau - \sigma p_\sigma)}{\tau^2+\sigma^2} - \left(\frac{2k_1\tau\sigma + k_2(\tau^2-\sigma^2)\sigma - k_3(\tau^2-\sigma^2)\tau}{\tau^2+\sigma^2} \right) \left(p_z^2 + 2Z(z) \right) \\
&\qquad - \frac{4t_1\tau\sigma + 2t_2(\tau^2-\sigma^2)\sigma - 2t_3(\tau^2-\sigma^2)\tau}{\tau^2+\sigma^2}\,,
\end{aligned}
\tag{5.4}
$$

which, as we shall see later on, can be regarded as the generalised counterpart of the 2D Laplace–Runge–Lenz vector corresponding to the 2D KC system.

5.5 Comments

So far, by applying the Eisenhart formalism, we have achieved the extension sequence:

$$
H_r = T + V_r \;\longrightarrow\; \mathcal{T}_r \;\longrightarrow\; \mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r\,, \qquad r = a, b, c, d\,. \tag{5.5}
$$

Obviously, the reverse process, firstly, corresponds to set $\mathcal{U}_r \equiv 0$, that is, $t_i = 0$ $(i = 1, 2, 3)$ and $Z(z) \equiv 0$, so that $\mathcal{H}_r \to \mathcal{T}_r$ and the integrals $\mathcal{K}_{ri} \to K_{ri}$, $\mathcal{J}_{ri} \to J_{ri}$; thus Propositions 5–8 reduce to Propositions 1–4, respectively. And secondly, to set p_z constant, say $p_z = \sqrt{2}$, gives $\mathcal{T}_r \to H_r = T + V_r$ in the form described in Section 3.

Now we comment on some characteristics of these new four families of 3D superintegrable Hamiltonians endowed with four independent constants of motion.

The four geodesic Hamiltonians $\mathcal{T}_r$ are endowed with an exact Noether symmetry since they are invariant under translation along the z-axis and the Noether theorem states the conservation of the momentum p_z, which is just their common integral K_{r1}. The remaining three constants of motion are quadratic and homogeneous in the momenta. The coefficients of such integrals can be considered as the components of Killing tensors of the underlying 3D Riemannian metric (2.1). In this respect, let us recall the relation of Killing tensors with Hamiltonian dynamics.

A Killing tensor $\mathbf{K}$ of valence p defined in a Riemannian manifold (M, g) is a symmetric $(p, 0)$ tensor satisfying the Killing tensor equation [61]–[65]

$$
[\mathbf{K}, g]_S = 0\,, \tag{5.6}
$$

where $[\cdot,\cdot]_S$ denotes the Schouten bracket (bilinear operator representing the natural generalisation of the Lie bracket of vector fields). When $p = 1$ the Killing tensor reduces to a Killing vector $X \in \mathfrak{X}(M)$ (generator of isometries), the bracket becomes a Lie derivative and the Killing equation reduces to $\mathcal{L}_X(g) = 0$. When $p = 2$ the metric tensor g is itself a trivial Killing tensor. The set $\mathcal{K}^p(M)$ of all the Killing tensors of valence p on M is a vector space. If M is a space of constant curvature then the dimension d of $\mathcal{K}^p(M)$ is giving by the Delong–Takeuchi–Thompson formula [63] (that generalises the expression $\frac{1}{2}n(n+1)$ for Killing vectors)

$$
d = \dim \mathcal{K}^p(M) = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}\,, \quad p \geq 1\,.
$$

In the more general case of a space of nonconstant curvature, the dimension of the vector space is a value lower than d.

We are now interested in the particular case with $p=2$. In this case, the Killing tensor $\mathbf{K}$ determines a homogeneous quadratic function $F_K = K^{ij}p_ip_j$ and then the Killing equation can be rewritten as the vanishing of the Poisson bracket of two functions

$$\{K^{ij}p_ip_j\,,\,g^{ij}p_ip_j\} = 0\,.$$

This means that the function F_K, associated to the tensor $\mathbf{K}$, is a first integral of the geodesic flow determined by the Hamiltonian $\mathcal{T} = \frac{1}{2}g^{ij}p_ip_j$.

The four families of 3D geodesic Hamiltonian systems, $\mathcal{T}_r$ $(r=a,b,c,d)$, are determined by metric tensors g_r given by

$$g_r^{ij} = \mathrm{diag}(1,1,V_r)\,, \quad r=a,b,c,d\,,$$

and the result is that the four configuration spaces $(\mathbb{R}^3, g_r)$ are endowed with a Killing vector $X = \partial/\partial z$ (determining the linear constant p_z) and three $p=2$ Killing tensors determining the three quadratic integrals of motion.

We must mention that there exist systems with higher-order constants of motion that, in differential geometric terms, are related with the existence of $p>2$ Killing tensors. For example, a cubic integral of motion [66]–[70] means that the configuration space admits a nontrivial symmetric $(3,0)$ tensor satisfying the Killing tensor equation (5.6) and determining a function $F_K = K^{ijk}p_ip_jp_k$ satisfying

$$\{K^{ijk}p_ip_jp_k\,,\,g^{ij}p_ip_j\} = 0\,,$$

and representing a first-integral for the geodesic motion (the existence of higher-order Killing tensors for systems in external gauge fields is analyzed in [71, 72]). Nevertheless we restrict our study to the two above mentioned cases: Killing vector fields and $p=2$ Killing tensors.

Another interesting property deserving to be mentioned is the close relation of the two integrals $(\mathcal{K}_{d2}, \mathcal{J}_{d2})$ (5.4) of $\mathcal{H}_d$ (and also (K_{d2}, J_{d2}) (4.16) for $\mathcal{T}_d$) with the Laplace–Runge–Lenz 2-vector since their first term in parabolic (τ,σ) and Cartesian (x,y) coordinates reads as

$$\begin{aligned}
\mathcal{K}_{d2} \;&:\; \frac{\sigma^2 p_\tau^2 - \tau^2 p_\sigma^2}{\tau^2+\sigma^2} = -2(xp_y - yp_x)p_y\,,\\
\mathcal{J}_{d2} \;&:\; \frac{(\tau p_\sigma - \sigma p_\tau)(\tau p_\tau - \sigma p_\sigma)}{\tau^2+\sigma^2} = 2(xp_y - yp_x)p_x\,.
\end{aligned}$$

As it is well known the existence of this conserved vector is one of the main characteristics of the KC problem and the importance of this fact have led to the study of systems admitting generalisations of the Laplace–Runge–Lenz vector [54], [73]–[83]. In particular, the Hamiltonian $\mathcal{H}_d$, defined on a 3D curved space of nonconstant curvature, could be considered as a new cornerstone in order to generalised the KC problem and consequently $\mathcal{K}_{d2}$ and $\mathcal{J}_{d2}$ could be considered as new ways of representing the Laplace–Runge–Lenz vector; clearly, this property also holds for the geodesic Hamiltonian $\mathcal{T}_d$.

6 3D geodesic Hamiltonians $\widetilde{\mathcal{T}}$ with a position-dependent mass

The sequence (5.5) can further be enlarged through the introduction of an 'appropriate' position-dependent mass (PDM) [36]–[50], in such a manner that new (generalised) Hamiltonians can be obtained by requiring once again to preserve separability/superintegrabilty.

More specifically, let us consider the Euclidean plane with metric $\mathrm{d}s^2$, free Lagrangian L and geodesic Hamiltonian T in Cartesian coordinates. The introduction of a PDM, $\mu(x,y)$, determines a metric $\mathrm{d}s_\mu^2$ on a 2D Riemannian space (generally, of nonconstant curvature) with associated free Lagrangian L_μ and geodesic Hamiltonian T_μ given by

$$\mathrm{d}s_\mu^2 = \mu\,\mathrm{d}s^2 = \mu\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)\,, \qquad L_\mu = \mu\,L = \frac{1}{2}\,\mu\big(v_x^2 + v_y^2\big)\,, \qquad T_\mu = \frac{1}{\mu}\,T = \frac{1}{2\mu}\,\big(p_x^2 + p_y^2\big)\,,$$

i.e. the new metric is conformally Euclidean.

Therefore in the Eisenhart formalism we are considering, which starting from the Euclidean plane with a potential $V_r(x,y)$ leads to a geodesic motion in a 3D Riemannian configuration space, we can express the new metric $\mathrm{d}\widetilde{\sigma}_r^2$ and the new geodesic Hamiltonian $\widetilde{\mathcal{T}}_r$ as (see (2.1) and (4.1))

$$\begin{aligned}
\mathrm{d}\widetilde{\sigma}_r^2 &= \mu_r\,\mathrm{d}\sigma_r^2 = \mu_r(x,y)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \frac{\mathrm{d}z^2}{V_r(x,y)}\right), \\
\widetilde{\mathcal{T}}_r &= \frac{1}{\mu_r}\,\mathcal{T}_r = \frac{1}{2\mu_r(x,y)}\,\big(p_x^2 + p_y^2 + V_r(x,y)\,p_z^2\big)\,, \qquad r = a,b,c,d\,.
\end{aligned} \tag{6.1}$$

The new geodesic dynamics determined by $\widetilde{\mathcal{T}}_r$ must be a deformation of the initial one provided by $\mathcal{T}_r$ [84] in the sense that the PDM μ_r, and so $\widetilde{\mathcal{T}}_r$, will depend on a real parameter λ in such a way that the following properties must be satisfied:

(i) The PDM $\mu_r(\lambda)$ must preserve the multiple separability of the original dynamics as established in Section 4.

(ii) The new geodesic Hamiltonian $\widetilde{\mathcal{T}}_r(\lambda)$ must be a continuous function of λ (in a certain domain of the parameter).

(iii) When taking the limit $\lambda \to 0$ the PDM must satisfy $\mu_r(\lambda) \to 1$, so that the dynamics of the original geodesic Hamiltonian $\mathcal{T}_r$ is recovered.

In what follows we study the separability and superintegrability of the geodesic Hamiltonians $\widetilde{\mathcal{T}}_r(\lambda)$ obtained by introducing an adequate λ-PDM $\mu_r(\lambda)$ in the four families of geodesic Hamiltonians $\mathcal{T}_r$ $(r = a,b,c,d)$ described in Section 4, fulfilling the three above requirements. We remark that the explicit form for μ_r is, in fact, determined by the 'principal' potential within each family, that is, the k_1-term.

6.1 Geodesic Hamiltonian $\widetilde{\mathcal{T}}_a$ with PDM from isotropic oscillator

Let us consider the geodesic Hamiltonian $\widetilde{\mathcal{T}}_a$ (6.1) with $\mathcal{T}_a$ given in (4.4) and PDM defined by

$$\mu_a(x,y) = 1 - \lambda\,r^2\,, \qquad r^2 = x^2 + y^2\,. \tag{6.2}$$

The parameter λ can take both positive and negative values. If $\lambda < 0$, the dynamics from $\widetilde{\mathcal{T}}_a$ is correctly defined for all the values of the variables; nevertheless, when $\lambda > 0$, the Hamiltonian (and the associated dynamics) has a singularity at $1 - \lambda r^2 = 0$, so in this case the dynamics is only defined in the interior of the circle with radius $r = 1/\sqrt{\lambda}$, that is, the region in which $\widetilde{\mathcal{T}}_a$ is positive definite.

The HJ equation in Cartesian coordinates takes the form

$$\frac{1}{1-\lambda(x^2+y^2)} \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + V_a(x,y) \left(\frac{\partial W}{\partial z}\right)^2 \right] = 2E \,,$$

and admits separability giving rise to three independent first integrals

$$\begin{aligned} \widetilde{K}_{a1} &= p_z \,, \qquad \widetilde{K}_{a2} = p_x^2 + \left(\frac{1}{2}\, k_1 x^2 + \frac{k_2}{x^2} \right) p_z^2 + 2\lambda\, x^2\, \widetilde{\mathcal{T}}_a \,, \\ \widetilde{K}_{a3} &= p_y^2 + \left(\frac{1}{2}\, k_1 y^2 + \frac{k_3}{y^2} \right) p_z^2 + 2\lambda\, y^2\, \widetilde{\mathcal{T}}_a \,. \end{aligned} \tag{6.3}$$

Meanwhile, the HJ equation in cylindrical coordinates

$$\frac{1}{1-\lambda\, r^2} \left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \phi}\right)^2 + V_a(r,\phi) \left(\frac{\partial W}{\partial z}\right)^2 \right] = 2E$$

also admits separability providing a fourth functionally independent constant of motion

$$\widetilde{J}_{a2} = p_\phi^2 + \left(\frac{k_2}{\cos^2\phi} + \frac{k_3}{\sin^2\phi} \right) p_z^2 \,. \tag{6.4}$$

Consequently, we find:

Proposition 9. *The 3D λ-dependent geodesic Hamiltonian $\widetilde{\mathcal{T}}_a(\lambda)$ (6.1), with $\mathcal{T}_a$ (4.4) and PDM (6.2),*

$$\widetilde{\mathcal{T}}_a = \frac{1}{\mu_a}\, \mathcal{T}_a = \frac{1}{2\mu_a(x,y)} \left(p_x^2 + p_y^2 + V_a(x,y)\, p_z^2 \right) , \qquad \mu_a(x,y) = 1 - \lambda\, r^2 \,,$$

is HJ separable in Cartesian (x,y,z) and cylindrical (r,ϕ,z) coordinates. This is endowed with four independent constants of motion given by $\widetilde{K}_{a1}, \widetilde{K}_{a2}, \widetilde{K}_{a3}$ (6.3) and $\widetilde{J}_{a2}$ (6.4). Moreover, $\widetilde{K}_{a1}, \widetilde{K}_{a2}, \widetilde{K}_{a3}$ are mutually in involution and $\widetilde{\mathcal{T}}_a = \frac{1}{2}\big(\widetilde{K}_{a2} + \widetilde{K}_{a3}\big)$.

Notice that the integral $\widetilde{J}_{a2}$ is λ-independent and coincides with the original one J_{a2} (4.3). The two remaining constants of motion satisfy the limits $\widetilde{K}_{a2} \to K_{a2}$ and $\widetilde{K}_{a3} \to K_{a3}$ when $\lambda \to 0$, so recovering (4.2) and Proposition 1.

6.2 Geodesic Hamiltonian $\widetilde{\mathcal{T}}_b$ with PDM from anisotropic oscillator

For the second family $\widetilde{\mathcal{T}}_b$ (6.1) with $\mathcal{T}_b$ given in (4.8) we define the PDM by

$$\mu_b(x,y) = 1 - \lambda\, x \,. \tag{6.5}$$

Hence the region in which $\mu_b(x,y)$ is positive definite is given by $x < 1/\lambda$.

Next, the HJ equation written in Cartesian coordinates reads

$$\frac{1}{1-\lambda x}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+V_b(x,y)\left(\frac{\partial W}{\partial z}\right)^2\right]=2E\,,$$

so it admits separability and leads to the following three independent constants of motion

$$\begin{aligned}&\widetilde{K}_{b1}=p_z\,,\qquad \widetilde{K}_{b2}=p_x^2+\left(2k_1x^2+k_3x\right)p_z^2+2\lambda x\widetilde{\mathcal{T}}_b\,,\\&\widetilde{K}_{b3}=p_y^2+\left(\frac{1}{2}\,k_1y^2+\frac{k_2}{y^2}\right)p_z^2\,.\end{aligned}\tag{6.6}$$

The HJ equation when written in parabolic coordinates (τ,σ,z) (4.6),

$$\frac{1}{1-\frac{1}{2}\lambda(\tau^2-\sigma^2)}\left\{\frac{1}{\tau^2+\sigma^2}\left[\left(\frac{\partial W}{\partial \tau}\right)^2+\left(\frac{\partial W}{\partial \sigma}\right)^2\right]+V_b(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2\right\}=2E\,,$$

also admits separability, yielding to another functionally independent constant of motion, namely

$$\widetilde{J}_{b2}=p_\tau^2+\left(\frac{k_1}{2}\,\tau^6+\frac{k_2}{\tau^2}+\frac{k_3}{2}\,\tau^4\right)p_z^2-\tau^2\left(2-\lambda\tau^2\right)\widetilde{\mathcal{T}}_b\,.\tag{6.7}$$

Thus we conclude with the following statement.

Proposition 10. *The 3D λ-dependent geodesic Hamiltonian $\widetilde{\mathcal{T}}_b(\lambda)$ (6.1) with $\mathcal{T}_b$ (4.8) and PDM (6.5)*

$$\widetilde{\mathcal{T}}_b=\frac{1}{\mu_b}\,\mathcal{T}_b=\frac{1}{2\mu_b(x,y)}\left(p_x^2+p_y^2+V_b(x,y)\,p_z^2\right),\qquad \mu_b(x,y)=1-\lambda\,x\,,$$

is HJ separable in Cartesian (x,y,z) and parabolic-cylindrical (τ,σ,z) coordinates. This is endowed with four independent constants of motion corresponding to $\widetilde{K}_{b1},\widetilde{K}_{b2},\widetilde{K}_{b3}$ (6.6) and $\widetilde{J}_{b2}$ (6.7). Furthermore, $\widetilde{K}_{b1},\widetilde{K}_{b2},\widetilde{K}_{b3}$ are mutually in involution and $\widetilde{\mathcal{T}}_b=\frac{1}{2}\big(\widetilde{K}_{b2}+\widetilde{K}_{b3}\big)$.

Notice that the results given in Proposition 2 are straightforwardly recovered under taking the limit $\lambda\to 0$.

6.3 Geodesic Hamiltonian $\widetilde{\mathcal{T}}_c$ with PDM from Kepler–Coulomb I

Now we consider the geodesic Hamiltonian $\widetilde{\mathcal{T}}_c$ (6.1) with $\mathcal{T}_c$ given in (4.11) and with a PDM defined by

$$\mu_c(x,y)=1-\frac{\lambda}{r}\,,\qquad r^2=x^2+y^2\,.\tag{6.8}$$

Hence, if $\lambda<0$ the dynamics is correctly defined for all the values of r, but when $\lambda>0$, the Hamiltonian $\widetilde{\mathcal{T}}_c$ has a singularity at $r=\lambda$. Note that μ_c is only positive when $r>\lambda$, so in this case the dynamics is only defined outside this circle, that is, the region in which $\widetilde{\mathcal{T}}_c$ is positive definite. We note that this PDM shows a certain similarity with the coefficient (related to the singularity) in the Schwarzschild metric.

The separability of the HJ equation in cylindrical coordinates (r,ϕ,z)

$$\frac{r}{r-\lambda}\left[\left(\frac{\partial W}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2+V_c(r,\phi)\left(\frac{\partial W}{\partial z}\right)^2\right]=2E\,,$$

leads to the following constants of motion

$$\widetilde{K}_{c1} = p_z\,, \qquad \widetilde{K}_{c2} = p_\phi^2 + \Big(\frac{k_2}{\sin^2\phi} + \frac{k_3\cos\phi}{\sin^2\phi}\Big)\, p_z^2\,,$$
$$\widetilde{K}_{c3} = r^2 p_r^2 + k_1 r\, p_z^2 - 2r(r-\lambda)\widetilde{\mathcal{T}}_c\,. \tag{6.9}$$

Note that $\widetilde{K}_{c2} + \widetilde{K}_{c3} = 0$ and that $\{\widetilde{K}_{c1}\,, \widetilde{K}_{c2}\} = 0$. The separability of the HJ equation in parabolic-cylindrical coordinates (τ, σ, z),

$$\frac{1}{\tau^2+\sigma^2-2\lambda}\left[\left(\frac{\partial W}{\partial \tau}\right)^2 + \left(\frac{\partial W}{\partial \sigma}\right)^2 + (\tau^2+\sigma^2)V_c(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2\right] = 2E\,,$$

also leads to three constants of motion but only one is functionally independent with respect to the above ones, namely

$$\widetilde{J}_{c2} = p_\tau^2 + \left(k_1 + \frac{k_2-k_3}{\tau^2}\right) p_z^2 + 2(\lambda - \tau^2)\,\widetilde{\mathcal{T}}_c\,. \tag{6.10}$$

We summarize the results in the following proposition.

Proposition 11. *The 3D λ-dependent geodesic Hamiltonian $\widetilde{\mathcal{T}}_c(\lambda)$ (6.1) with $\mathcal{T}_c$ (4.11) and PDM (6.8)*

$$\widetilde{\mathcal{T}}_c = \frac{1}{\mu_c}\,\mathcal{T}_c = \frac{1}{2\mu_c(x,y)}\left(p_x^2 + p_y^2 + V_c(x,y)\,p_z^2\right), \qquad \mu_c(x,y) = 1 - \frac{\lambda}{r}\,,$$

is HJ separable in cylindrical (r,ϕ,z) and parabolic-cylindrical (τ,σ,z) coordinates. This is endowed with four independent constants of motion: the Hamiltonian itself, $\widetilde{\mathcal{T}}_c$, along with $\widetilde{K}_{c1}, \widetilde{K}_{c2}$ (6.9) and $\widetilde{J}_{c2}$ (6.10). The three integrals $\widetilde{\mathcal{T}}_c, \widetilde{K}_{c1}, \widetilde{K}_{c2}$ are mutually in involution.

Notice that the functions $\widetilde{K}_{c1}$ and $\widetilde{K}_{c2}$ are λ-independent, so coinciding with the original constants (4.9). Clearly, when taking the limit $\lambda \to 0$ the results given in Proposition 3 are recovered.

6.4 Geodesic Hamiltonian $\widetilde{\mathcal{T}}_d$ with PDM from Kepler–Coulomb II

As far as the last family is concerned, we consider the geodesic Hamiltonian $\widetilde{\mathcal{T}}_d$ (6.1) with $\mathcal{T}_d$ (4.15) and with the same previous PDM $\mu_d \equiv \mu_c$ (6.8).

The separability of the HJ equation in parabolic-cylindrical coordinates of type I (τ,σ,z),

$$\frac{1}{\tau^2+\sigma^2-2\lambda}\left[\left(\frac{\partial W}{\partial \tau}\right)^2 + \left(\frac{\partial W}{\partial \sigma}\right)^2 + (\tau^2+\sigma^2)V_d(\tau,\sigma)\left(\frac{\partial W}{\partial z}\right)^2\right] = 2E\,,$$

yields three first integrals

$$\widetilde{K}_{d1} = p_z\,, \qquad \widetilde{K}_{d2} = p_\tau^2 + (k_1 + 2k_2\tau)\,p_z^2 + 2\left(\lambda - \tau^2\right)\widetilde{\mathcal{T}}_d\,,$$
$$\widetilde{K}_{d3} = p_\sigma^2 + (k_1 + 2k_3\sigma)\,p_z^2 + 2\left(\lambda - \sigma^2\right)\widetilde{\mathcal{T}}_d\,, \tag{6.11}$$

but note that $\widetilde{K}_{d2} + \widetilde{K}_{d3} = 0$. Moreover, $\widetilde{K}_{d1}$ and $\widetilde{K}_{d2}$ are in involution, i.e. $\{\widetilde{K}_{d1}\,, \widetilde{K}_{d2}\} = 0$. The separability of the HJ equation in parabolic-cylindrical coordinates of type II (α,β,z),

$$\frac{1}{\alpha^2+\beta^2-2\lambda}\left[\left(\frac{\partial W}{\partial \alpha}\right)^2 + \left(\frac{\partial W}{\partial \beta}\right)^2 + (\alpha^2+\beta^2)V_d(\alpha,\beta)\left(\frac{\partial W}{\partial z}\right)^2\right] = 2E\,,$$

also gives rise to three constants of motion, but only the following one is actually functionally independent,

$$\widetilde{J}_{d2} = p_\alpha^2 + \left(k_1 + \sqrt{2}(k_2 + k_3)\alpha\right) p_z^2 + 2\left(\lambda - \alpha^2\right) \widetilde{\mathcal{T}}_d \,. \tag{6.12}$$

In this way, we obtain the λ-generalisation of Proposition 4 as follows.

Proposition 12. *The 3D λ-dependent geodesic Hamiltonian $\widetilde{\mathcal{T}}_d(\lambda)$ (6.1) with $\mathcal{T}_d$ (4.15) and PDM (6.8)*

$$\widetilde{\mathcal{T}}_d = \frac{1}{\mu_d}\,\mathcal{T}_d = \frac{1}{2\mu_d(x,y)}\left(p_x^2 + p_y^2 + V_d(x,y)\,p_z^2\right), \qquad \mu_d(x,y) = 1 - \frac{\lambda}{r}\,,$$

is HJ separable in two sets of parabolic-cylindrical coordinates (τ,σ,z) and (α,β,z) which are related by a rotation. This system is endowed with four independent constants of motion: the Hamiltonian $\widetilde{\mathcal{T}}_d$, together with $\widetilde{K}_{d1}, \widetilde{K}_{d2}$ (6.11) and $\widetilde{J}_{d2}$ (6.12). The three first integrals $\widetilde{\mathcal{T}}_d, \widetilde{K}_{d1}, \widetilde{K}_{d2}$ are mutually in involution.

The two constants of motion $\widetilde{K}_{d2}$ and $\widetilde{J}_{d2}$ are the λ-dependent version of the generalised Laplace–Runge–Lenz 2-vector (4.16), which in the coordinates (τ,σ,z) read as

$$\begin{aligned}
\widetilde{K}_{d2} &= \frac{\sigma^2 p_\tau^2 - \tau^2 p_\sigma^2 + \lambda(p_\sigma^2 - p_\tau^2)}{\tau^2 + \sigma^2 - 2\lambda} + \left(\frac{k_1(\sigma^2 - \tau^2) + 2k_2(\sigma^2 - \lambda)\tau - 2k_3(\tau^2 - \lambda)\sigma}{\tau^2 + \sigma^2 - 2\lambda}\right) p_z^2\,,\\
\widetilde{J}_{d2} &= \frac{(\tau p_\sigma - \sigma p_\tau)(\tau p_\tau - \sigma p_\sigma) - 2\lambda\, p_\tau p_\sigma}{\tau^2 + \sigma^2 - 2\lambda}\\
&\qquad - \left(\frac{2k_1\tau\sigma + k_2(\tau^2 - \sigma^2 + 2\lambda)\sigma - k_3(\tau^2 - \sigma^2 - 2\lambda)\tau}{\tau^2 + \sigma^2 - 2\lambda}\right) p_z^2\,.
\end{aligned} \tag{6.13}$$

So far, we have constructed 3D systems from 2D ones by following the Eisenhart prescription. Next, a generalisation of such a procedure has been achieved by considering a PDM in the 2D system, so that we have obtained PDM functions depending only on the two first coordinates. In this respect, we remark that one could also consider an alternative construction inserting a PDM term in the 3D system (4.1) obtained once the Eisenhart lift has been applied. This would allow a further generalisation such that the new metric $\mathrm{d}\widetilde{\sigma}_r^2$ and the new geodesic Hamiltonian $\widetilde{\mathcal{T}}_r$ would be (see (2.1) and (4.1))

$$\begin{aligned}
\mathrm{d}\widetilde{\sigma}_r^2 &= \mu_r\, \mathrm{d}\sigma_r^2 = \mu_r(x,y,z)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \frac{\mathrm{d}z^2}{V_r(x,y)}\right),\\
\widetilde{\mathcal{T}}_r &= \frac{1}{\mu_r}\,\mathcal{T}_r = \frac{1}{2\mu_r(x,y,z)}\left(p_x^2 + p_y^2 + V_r(x,y)\,p_z^2\right), \qquad r = a,b,c,d\,.
\end{aligned}$$

However this point requires a deeper analysis as well as other related question when looking for alternative expressions to those here studied in (6.2), (6.5) and (6.8).

7 3D Hamiltonians $\widetilde{\mathcal{H}}$ with a position-dependent mass

The last step in our approach is to add a potential $\widetilde{\mathcal{U}}_r$ to the geodesic Hamiltonian with a PDM $\widetilde{\mathcal{T}}_r$ (6.1). We impose that the resulting Hamiltonian, $\widetilde{\mathcal{H}}_r = \widetilde{\mathcal{T}}_r + \widetilde{\mathcal{U}}_r$, be once again separable, so superintegrable, in the same two sets of coordinate systems corresponding to $\widetilde{\mathcal{T}}_r$ and fulfilling the same three requirements assumed at the beginning of Section 6. Therefore, we introduce the PDM μ_r in the Hamiltonian (5.1), $\widetilde{\mathcal{H}}_r = \mathcal{H}_r/\mu_r$, and by taking into account (5.2) and (5.3), we are led to consider the following family of Hamiltonians ($r = a, b, c, d$)

$$
\begin{aligned}
\widetilde{\mathcal{H}}_r &= \widetilde{\mathcal{T}}_r + \widetilde{\mathcal{U}}_r = \frac{1}{\mu_r}\,\mathcal{T}_r + \frac{1}{\mu_r}\,\mathcal{U}_r \\
&= \frac{1}{2\mu_r(x,y)}\left(p_x^2 + p_y^2 + V_r(x,y)\,p_z^2\right) + \frac{1}{\mu_r(x,y)}\left(U_r(x,y) + V_r(x,y)Z(z)\right), \qquad (7.1)
\end{aligned}
$$

where $U_r(x,y)$ is the function to be determined for each family while $Z(z)$ is an arbitrary smooth function. Then, as in Section 5, it is found that U_r keeps the same formally form as the 2D Euclidean potential V_r but with different coefficients t_i instead of k_i ($i = 1,2,3$).

We display the resulting 3D superintegrable Hamiltonians (7.1) with a PDM along with their independent constants of motion, $\widetilde{\mathcal{K}}_{ri}$ and $\widetilde{\mathcal{J}}_{ri}$, in Table 1. We stress that, in fact, this comprises the main results of the paper that generalise all the previous ones.

Now some remarks are in order.

- Notice that if we set $t_i = 0$ and $Z(z) \equiv 0$ in Table 1, then $\widetilde{\mathcal{U}}_r \equiv 0$ and $\widetilde{\mathcal{H}}_r \to \widetilde{\mathcal{T}}_r$, $\widetilde{\mathcal{K}}_{ri} \to \widetilde{K}_{ri}$, $\widetilde{\mathcal{J}}_{ri} \to \widetilde{J}_{ri}$ recovering Propositions 9–12 of Section 6 for the geodesic Hamiltonian with a PDM $\widetilde{\mathcal{T}}_r$.

- The limit $\lambda \to 0$, so $\mu_r \to 1$, in Table 1 gives rise to the limits $\widetilde{\mathcal{H}}_r \to \mathcal{H}_r$, $\widetilde{\mathcal{K}}_{ri} \to \mathcal{K}_{ri}$, $\widetilde{\mathcal{J}}_{ri} \to \mathcal{J}_{ri}$, thus reproducing the results presented in Propositions 5–8 of Section 5 for the superintegrable Hamiltonian $\mathcal{H}_r$.

- We remark that the PDM $\mu_a = 1 - \lambda r^2$ is just the same conformal factor used in the construction of the so called Darboux III oscillator [45, 85]. Recall that this is an exactly solvable model which can be regarded as a generalisation of the isotropic oscillator to a conformally flat space of nonconstant curvature, being a particular Bertrand space [51, 52, 54]. Explicitly, if we consider $\widetilde{\mathcal{H}}_a$ in Table 1 and set $k_2 = k_3 = t_i = 0$ ($i = 1,2,3$), $Z(z) \equiv 0$, $k_1 = \omega^2$ and $p_z = \sqrt{2}$, we obtain the following 2D Hamiltonian

$$
\widetilde{\mathcal{H}}_a \equiv \widetilde{\mathcal{T}}_a = \frac{1}{2(1-\lambda r^2)}\left(p_x^2 + p_y^2 + \omega^2 r^2\right) = \frac{p_x^2 + p_y^2}{2(1-\lambda r^2)} + \frac{\omega^2 r^2}{2(1-\lambda r^2)}\,,
$$

  which is the 2D counterpart of the Darboux III oscillator (change $\lambda \to -\lambda$ in [45]). Consequently, the Hamiltonian $\widetilde{\mathcal{H}}_a$ in Table 1 turns out to be a 3D superintegrable generalisation of that system by breaking spherical symmetry.

- The two KC families have the same PDM $\mu_c = \mu_d = 1 - \lambda/r$. We stress that this is just the conformal factor considered in the construction of an exactly solvable deformation of the KC problem [86]; such a system is related to a reduction [87] of the geodesic motion on the Taub–NUT space which turns out to be another Bertrand space [51, 52, 54]. In particular, if we set $k_2 = k_3 = t_i = 0$ ($i = 1,2,3$), $Z(z) \equiv 0$, $k_1 = -k$ and $p_z = \sqrt{2}$ in either $\widetilde{\mathcal{H}}_c$ or $\widetilde{\mathcal{H}}_d$ in Table 1, we find that

$$
\widetilde{\mathcal{H}}_c \equiv \widetilde{\mathcal{H}}_d \equiv \widetilde{\mathcal{T}}_c \equiv \widetilde{\mathcal{T}}_d = \frac{r}{2(r-\lambda)}\left(p_x^2 + p_y^2 - \frac{2k}{r}\right) = \frac{r(p_x^2 + p_y^2)}{2(r-\lambda)} - \frac{k}{r-\lambda}\,,
$$

  which is the 2D version of the deformed KC problem (change $\eta \to -\lambda$ in [86]) Therefore, the Hamiltonians $\widetilde{\mathcal{H}}_c$ and $\widetilde{\mathcal{H}}_d$ in Table 1 provide two different possible 3D superintegrable generalisations for the above system.

Table 1: The four families of 3D superintegrable Hamiltonians $\widetilde{\mathcal{H}}_r$ ($r = a, b, c, d$) with a PDM. For each family, we write the PDM μ_r, the 2D potential V_r (U_r is the same but with coefficients t_i), and the four independent constants of motion in the two sets of separable coordinates. All of them share a common integral $\widetilde{\mathcal{K}}_{r1} = p_z^2 + 2Z(z)$.

Generic Hamiltonian: $\displaystyle \widetilde{\mathcal{H}}_r = \frac{1}{2\mu_r}\big(p_x^2 + p_y^2 + V_r\, p_z^2\big) + \frac{1}{\mu_r}\big(U_r + V_r Z(z)\big)$

• Family a: From isotropic oscillator PDM: $\mu_a = 1 - \lambda(x^2+y^2) = 1 - \lambda r^2$

Separable in Cartesian (x, y, z) and cylindrical (r, ϕ, z) coordinates

$$V_a = \frac{1}{2}k_1(x^2+y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2} = \frac{1}{2}k_1 r^2 + \frac{k_2}{r^2\cos^2\phi} + \frac{k_3}{r^2\sin^2\phi}$$

$$\widetilde{\mathcal{K}}_{a2} = p_x^2 + \left(\frac{1}{2}k_1 x^2 + \frac{k_2}{x^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 x^2 + \frac{2t_2}{x^2} + 2\lambda x^2 \widetilde{\mathcal{H}}_a$$

$$\widetilde{\mathcal{K}}_{a3} = p_y^2 + \left(\frac{1}{2}k_1 y^2 + \frac{k_3}{y^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 y^2 + \frac{2t_3}{y^2} + 2\lambda y^2 \widetilde{\mathcal{H}}_a$$

$$\widetilde{\mathcal{J}}_{a2} = p_\phi^2 + \left(\frac{k_2}{\cos^2\phi} + \frac{k_3}{\sin^2\phi}\right)\big(p_z^2 + 2Z(z)\big) + \frac{2t_2}{\cos^2\phi} + \frac{2t_3}{\sin^2\phi}$$

Independent integrals: $\widetilde{\mathcal{K}}_{a1}, \widetilde{\mathcal{K}}_{a2}, \widetilde{\mathcal{K}}_{a3}, \widetilde{\mathcal{J}}_{a2}$ with $\widetilde{\mathcal{H}}_a = \frac{1}{2}(\widetilde{\mathcal{K}}_{a2} + \widetilde{\mathcal{K}}_{a3})$ In involution: $\widetilde{\mathcal{K}}_{a1}, \widetilde{\mathcal{K}}_{a2}, \widetilde{\mathcal{K}}_{a3}$

• Family b: From anisotropic oscillator PDM: $\mu_b = 1 - \lambda\, x = 1 - \frac{1}{2}\lambda(\tau^2 - \sigma^2)$

Separable in Cartesian (x, y, z) and parabolic-cylindrical (τ, σ, z) coordinates

$$V_b = \frac{1}{2}k_1(4x^2+y^2) + \frac{k_2}{y^2} + k_3 x = \frac{1}{\tau^2+\sigma^2}\left[\frac{k_1}{2}(\tau^6+\sigma^6) + k_2\left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right) + \frac{k_3}{2}(\tau^4 - \sigma^4)\right]$$

$$\widetilde{\mathcal{K}}_{b2} = p_x^2 + \big(2k_1 x^2 + k_3 x\big)\big(p_z^2 + 2Z(z)\big) + 4t_1 x^2 + 2t_3 x + 2\lambda x \widetilde{\mathcal{H}}_b$$

$$\widetilde{\mathcal{K}}_{b3} = p_y^2 + \left(\frac{1}{2}k_1 y^2 + \frac{k_2}{y^2}\right)\big(p_z^2 + 2Z(z)\big) + t_1 y^2 + \frac{2t_2}{y^2}$$

$$\widetilde{\mathcal{J}}_{b2} = p_\tau^2 + \left(\frac{k_1}{2}\tau^6 + \frac{k_2}{\tau^2} + \frac{k_3}{2}\tau^4\right)\big(p_z^2 + 2Z(z)\big) + t_1\tau^6 + \frac{2t_2}{\tau^2} + t_3\tau^4 - \tau^2\left(2 - \lambda\tau^2\right)\widetilde{\mathcal{H}}_b$$

Independent integrals: $\widetilde{\mathcal{K}}_{b1}, \widetilde{\mathcal{K}}_{b2}, \widetilde{\mathcal{K}}_{b3}, \widetilde{\mathcal{J}}_{b2}$ with $\widetilde{\mathcal{H}}_b = \frac{1}{2}(\widetilde{\mathcal{K}}_{b2} + \widetilde{\mathcal{K}}_{b3})$ In involution: $\widetilde{\mathcal{K}}_{b1}, \widetilde{\mathcal{K}}_{b2}, \widetilde{\mathcal{K}}_{b3}$

• Family c: From Kepler–Coulomb I PDM: $\mu_c = 1 - \lambda/r = 1 - 2\lambda/(\tau^2+\sigma^2)$

Separable in cylindrical (r, ϕ, z) and parabolic-cylindrical (τ, σ, z) coordinates

$$V_c = \frac{k_1}{r} + \frac{k_2}{r^2\sin^2\phi} + \frac{k_3\cos\phi}{r^2\sin^2\phi} = \frac{1}{\tau^2+\sigma^2}\left[2k_1 + k_2\left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right) + k_3\left(\frac{1}{\sigma^2} - \frac{1}{\tau^2}\right)\right]$$

$$\widetilde{\mathcal{K}}_{c2} = p_\phi^2 + \left(\frac{k_2}{\sin^2\phi} + \frac{k_3\cos\phi}{\sin^2\phi}\right)\big(p_z^2 + 2Z(z)\big) + \frac{2t_2}{\sin^2\phi} + \frac{2t_3\cos\phi}{\sin^2\phi}$$

$$\widetilde{\mathcal{J}}_{c2} = p_\tau^2 + \left(k_1 + \frac{k_2 - k_3}{\tau^2}\right)\big(p_z^2 + 2Z(z)\big) + 2t_1 + \frac{2(t_2 - t_3)}{\tau^2} + 2(\lambda - \tau^2)\widetilde{\mathcal{H}}_c$$

Independent integrals: $\widetilde{\mathcal{H}}_c, \widetilde{\mathcal{K}}_{c1}, \widetilde{\mathcal{K}}_{c2}, \widetilde{\mathcal{J}}_{c2}$ In involution: $\widetilde{\mathcal{H}}_c, \widetilde{\mathcal{K}}_{c1}, \widetilde{\mathcal{K}}_{c2}$

• Family d: From Kepler–Coulomb II PDM: $\mu_d = 1 - 2\lambda/(\tau^2+\sigma^2) = 1 - 2\lambda/(\alpha^2+\beta^2)$

Separable in parabolic-cylindrical (τ, σ, z) and (α, β, z) coordinates

$$V_d = 2\frac{k_1 + k_2\tau + k_3\sigma}{\tau^2+\sigma^2} = \frac{2k_1 + k_2\sqrt{2}(\alpha+\beta) + k_3\sqrt{2}(\alpha-\beta)}{\alpha^2+\beta^2}$$

$$\widetilde{\mathcal{K}}_{d2} = p_\tau^2 + (k_1 + 2k_2\tau)\big(p_z^2 + 2Z(z)\big) + 2t_1 + 4t_2\tau + 2\left(\lambda - \tau^2\right)\widetilde{\mathcal{H}}_d$$

$$\widetilde{\mathcal{J}}_{d2} = p_\alpha^2 + \big(k_1 + \sqrt{2}(k_2+k_3)\alpha\big)\big(p_z^2 + 2Z(z)\big) + 2t_1 + 2\sqrt{2}(t_2+t_3)\alpha + 2\left(\lambda - \alpha^2\right)\widetilde{\mathcal{H}}_d$$

Independent integrals: $\widetilde{\mathcal{H}}_d, \widetilde{\mathcal{K}}_{d1}, \widetilde{\mathcal{K}}_{d2}, \widetilde{\mathcal{J}}_{d2}$ In involution: $\widetilde{\mathcal{H}}_d, \widetilde{\mathcal{K}}_{d1}, \widetilde{\mathcal{K}}_{d2}$

8 Concluding remarks and outlook

As already observed in the Introduction, the first studies on superintegrability were mainly concerned with the analysis of potentials defined on (2D and 3D) Euclidean spaces. Then, a second step was the study of potentials on Riemannian spaces of constant curvature (spherical and hyperbolic geometries), and only recently the existence of superintegrable systems on more general Riemannian spaces has become a matter of study. In this last situation the problem becomes much more complicated since the superintegrability depends, not only on the potential, but also on the coefficients of the non-Euclidean metric. In particular, the existence of integrals of motion for the free particle (geodesic motion determined by the metric) must be studied; only when this question has been solved, the existence of potentials with superintegrability can be analysed.

We have here applied the Eisenhart formalism to the four families of 2D superintegrable Euclidean Hamiltonians $H_r = T + V_r$, $(r = a, b, c, d)$, and we have studied the separability of the four 3D geodesic Hamiltonians

$$\mathcal{T}_r = \frac{1}{2}\Big(p_x^2 + p_y^2 + V_r(x,y)\,p_z^2\Big)\,,$$

and then we have extended the study to the separability of Hamiltonians with the addition of a potential

$$\mathcal{H}_r = \mathcal{T}_r + \mathcal{U}_r = \frac{1}{2}\Big(p_x^2 + p_y^2 + V_r(x,y)p_z^2\Big) + \Big(U_r(x,y) + V_r(x,y)Z(z)\Big)\,.$$

Furthermore, we have also study the separability of the 3D geodesic Hamiltonians with a PDM μ_r

$$\widetilde{\mathcal{T}}_r(\lambda) = \frac{1}{\mu_r}\mathcal{T}_r = \frac{1}{2\mu_r(x,y)}\Big(p_x^2 + p_y^2 + V_r(x,y)p_z^2\Big)\,,$$

and finally, as our most general result shown in Table 1, we have obtained superintegrable Hamiltonians with both potential and PDM:

$$\widetilde{\mathcal{H}}_r(\lambda) = \frac{1}{\mu_r}\mathcal{H}_r = \frac{1}{2\mu_r(x,y)}\Big(p_x^2 + p_y^2 + V_r(x,y)p_z^2\Big) + \frac{1}{\mu_r(x,y)}\Big(U_r(x,y) + V_r(x,y)Z(z)\Big)\,.$$

We remark that the PDM μ_r depends on a parameter λ in such a way that the superintegrability is preserved for all the values of λ and that the Hamiltonian $\widetilde{\mathcal{H}}_r(\lambda)$ can be considered as continuos deformation of the previously studied Hamiltonian $\mathcal{H}_r$.

We conclude with the following open problems. First, all the constants of motion we have obtained are a straightforward consequence of the existence of symmetries (in most of cases hidden symmetries); so it would be convenient to study the properties of these symmetries from a geometric approach (that is, symplectic formalism and Lie algebra of vector fields). It would also be convenient the study of higher-order constants of motion; this means to study the existence of Killing tensors $\mathbf{K}$ of valence $p > 2$ (see Section 5.5). In this respect, the results of this paper can be regarded as a first step in this direction; in fact, the obtention of superintegrable systems with higher-order integrals and with broken spherically symmetry remains as a non-trivial open task. Second, we have applied the geometric Eisenhart formalism starting with superintegrable Hamiltonians H_r defined on the Euclidean plane; a possible generalisation should be to consider as starting point superintegrable systems defined not on the Euclidean plane but on the 2D spaces with constant curvature, that is, either on the sphere (positive curvature) or on the hyperbolic plane (negative curvature). Third, it is known that HJ separability is related to separability of the Schrödinger equation; hence it would be also convenient to study the quantum counterparts of the Hamiltonian systems studied throughout this paper. The Eisenhart lift has been related to the properties of Killing tensors defined on the Riemannian space; thus this is also a matter to be studied.

Finally, we recall the comment a the end of Section 6; the PDM μ_r, $r = a, b, c, d$, of the systems we have studied are functions of the initial variables x and y and independent of the new degree of freedom; this two-dimensional dependence of the PDMs is not a limitation of the approach but a property directly related with the particular form of the Eisenhart formalism considered in this paper. Nevertheless, a possible extension of these systems can be obtained by considering the more general case of z-dependent PDM; we have already obtained some results (introducing an additional term coupling the three coordinates and depending of a second parameter) but this point remains as a question for future work.

Appendix: A geometric approach to Eisenhart lift

Natural Lagrangians, also called Lagrangians of mechanical type, are defined by a differentiable function V in a (pseudo-)Riemannian manifold (M, g). We denote $L_{g,V}$ such Lagrangians:

$$L_{g,V}(q, v) = \frac{1}{2}\, g_q(v, v) - (\tau_M^* V)(q, v) = \frac{1}{2}\, g_q(v, v) - V(q), \tag{A.1}$$

where $\tau_M : TM \to M$ is the tangent bundle projection, i.e. the Lagrangian function is of the form $L_{g,V} = T_g - \tau_M^* V$, where the function $T_g \in C^\infty(TM)$ represents the kinetic energy. Here we follow the notation of [88] where more mathematical details can be found. In local coordinates (q^i) in an open set U of M and the associated coordinates (q^i, v^i) in its tangent bundle, the local expressions for the Riemannian metric g and kinetic energy T_g are respectively written as

$$g = g_{ij}(q)\, \mathrm{d}q^i \otimes \mathrm{d}q^j, \qquad T_g(v) = \frac{1}{2}\, g_{ij}(\tau_M(v))\, v^i v^j.$$

Nondegeneracy of the Riemann structure means that $L_{g,V}$ is a regular Lagrangian and defines a Hamiltonian dynamical system $(TM, \omega_{L_{g,V}}, E_{L_{g,V}})$ (see [88] and references therein) and the dynamical vector field $\Gamma_{L_{g,V}}$, Hamiltonian vector field defined by the energy function E_L, i.e. defined by $i(\Gamma_{L_{g,V}})\omega_{L_{g,V}} = \mathrm{d}E_{L_{g,V}}$, takes the form

$$\Gamma_{L_{g,V}}(q, v) = v^i \frac{\partial}{\partial q^i} - \left(\Gamma^i_{jk}(q) v^j v^k + g^{ij}(q) \frac{\partial V}{\partial q^j}(q)\right) \frac{\partial}{\partial v^i},$$

where Γ^i_{jk} are the Christoffel symbols of the second kind with respect to the Levi–Civita connection defined by the metric g, given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right).$$

Consequently, the curves in the manifold M whose tangent lifts are integral curves of Γ_{T_g} are such that $\nabla_{\dot\gamma}\dot\gamma + \widehat{g}^{-1}(\mathrm{d}V) = 0$, with local coordinate expression

$$g_{li}\left(\ddot q^i + \Gamma^i_{jk}\, \dot q^j\, \dot q^k\right) = -\frac{\partial V}{\partial q^l}, \qquad l = 1, \ldots, \dim M. \tag{A.2}$$

In the particular case $V = 0$ we see that such curves in M are but the geodesics of the Riemannian metric and the geodesic motion is called free motion. Recall that the arc-length of a curve γ in M between the points $\gamma(t_1)$ and $\gamma(t_2)$ is given by

$$\int_{t_1}^{t_2} \sqrt{g(\dot\gamma,\dot\gamma)}\,\mathrm{d}t\,,$$

and the extremal length curves are those of the action defined by the Lagrangian $\ell(v) = \sqrt{g(v,v)}$, even if we have to restrict ourselves to the open submanifold $T_0M = \{v \in TM \mid v \neq 0\}$ in order to preserve the differentiability. Then we can consider the Lagrangian $\ell(v) = \sqrt{2\,T_g(v)}$, which is a singular Lagrangian whose relation to the Lagrangian $T_g = L_{g,0}$ has been studied in [89].

Note that since

$$\frac{\partial \ell}{\partial \dot q^i} = \frac{g_{ij}\dot q^j}{\ell}, \qquad \frac{\partial \ell}{\partial q^i} = \frac{1}{2\ell}\left(\frac{\partial g_{jk}}{\partial q^i}\dot q^j\,\dot q^k\right),$$

the Euler–Lagrange equations of the Lagrangian ℓ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g_{ij}\dot q^j}{\ell}\right) = \frac{1}{2\ell}\left(\frac{\partial g_{jk}}{\partial q^i}\dot q^j\,\dot q^k\right).$$

If we parametrize the curves by the arc-length, and then as $\mathrm{d}s/\mathrm{d}t = \ell$, $\dot q^i = \ell\, q'^i = \ell\,\mathrm{d}q^i/\mathrm{d}s$, we find that the previous system becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{ij}\,q'^j\right) = \frac{\partial g_{jk}}{\partial q^i}\,q'^j\,q'^k,$$

which are the same equations as those obtained from T_g with q'^k instead of $\dot q^k$, and therefore in terms of such parametrization the extremal arc-length curves are geodesics of the corresponding metric.

As $L_{g,V}$ is regular, we can carry out the Legendre transformation and then the elements $(q,v) \in TQ$ correspond to elements $(q,p) \in T^*Q$ in such a way that

$$p_i = \frac{\partial L_{g,V}}{\partial v^i} = \frac{\partial T_g}{\partial v^i} = g_{ik}(q)\,v^k \iff v^i = g^{ij}(q)\,p_j\,,$$

with $g^{ik}(q)\,g_{kj}(q) = \delta^i_j$, and the Hamiltonian is nothing but the expression of the kinetic energy in terms of momenta plus the potential term

$$H = \frac{1}{2}\,g\big(\widehat g^{-1}(p), \widehat g^{-1}(p)\big) + V(q) = \frac{1}{2}\,g^{ij}\,p_i\,p_j + V(q)\,, \tag{A.3}$$

where $\widehat g : TM \to T^*M$ is the bundle map over the identity from the tangent bundle $\tau_M : TM \to M$ to the cotangent bundle $\pi_M : T^*M \to M$, defined by $\langle \widehat g(v), w\rangle = g(v,w)$.

In a seminal paper [23] Eisenhart showed the possibility of relating the dynamical trajectories of a Lagrangian system of mechanical type (A.1) with the projections on M of extremal length curves on a extended manifold $\bar M = \mathbb{R} \times M$ with a Riemann structure

$$\bar g = \mathrm{pr}_2^* g - \frac{1}{2\,V}\,\mathrm{d}u \otimes \mathrm{d}u\,, \qquad u \in \mathbb{R}\,.$$

More explicitly, if we assume that we choose g_{00} as a function A of the coordinates $q^1,\ldots,q^n$, the arc-length reads

$$\mathrm{d}s^2 = g_{ij}(q)\,\mathrm{d}q^i \otimes \mathrm{d}q^j + A(q)\,\mathrm{d}u \otimes \mathrm{d}u\,,$$

with associated free motion described by

$$T_g = \frac{1}{2}\left(g_{ij}(q)\,v^i\,v^j + A(q)\,v_u^2\right). \tag{A.4}$$

Then the equations of motion in terms of the arc-length s turn out to be

$$q''^i + \Gamma^i_{jk}(q)\,q'^j\,q'^k - \frac{1}{2}g^{ij}\frac{\partial A}{\partial q^j}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 = 0\,, \qquad i = 1,\ldots,n,$$

together with the constant of motion corresponding to the invariance under translations in the variable u:

$$A(q)\,\frac{\mathrm{d}u}{\mathrm{d}s} = a \in \mathbb{R}\,. \tag{A.5}$$

For each value of the parameter a we can use a new parameter t such that $t = a\,s$ and then the differential equations reduce respectively to

$$\ddot{q}^i + \Gamma^i_{jk}(q)\,\dot{q}^j\,\dot{q}^k - g^{ij}\frac{1}{2A^2}\frac{\partial A}{\partial q^j} = 0\,, \quad i = 1,\ldots,n, \qquad A(q)\,\frac{\mathrm{d}u}{\mathrm{d}t} = 1\,.$$

Note that when $a = 1$ the parameter t coincides with s and condition (A.5) corresponds to set $p_u = 1$.

Suppose now a natural mechanical system in which the potential function V is bounded from below and that using the ambiguity in the choice of the potential we can assume that $V(q) > 0$. Then if we choose $A = \dfrac{1}{2V}$, the preceding system of differential equations becomes equivalent to (A.2)

$$\ddot{q}^i + \Gamma^i_{jk}(q)\,\dot{q}^j\,\dot{q}^k + g^{ij}\frac{\partial V}{\partial q^j} = 0\,, \qquad i = 1,\ldots,n\,.$$

The free particle determined by the metric $\bar{g}$ is defined by the kinetic energy (A.4) and the Legendre transformation leads to the new Hamiltonian [35]

$$\bar{H}(q,u,p,p_u) = \frac{1}{2}\left(g^{ij}\,p_i p_j + V\,p_u^2\right), \tag{A.6}$$

which coincides for $p_u = \sqrt{2}$ with (A.3).

As pointed out by Benenti [90] the HJ separability of the Hamiltonian (A.3) can be studied from the integrability of the geodesic Hamiltonian (A.6)

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