The forgotten decision rules: Majority rules based on difference of votes

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Abstract

In this paper we point out some interesting properties of a class of decision rules located between simple and unanimous majorities. These majority rules are based on difference of votes: an alternative wins when the difference between the number of votes obtained by this alternative and that obtained by the other is greater than a previously fixed quantity. We also give some characterizations of these majority rules by means of two properties well known in the literature: cancellation and decisiveness.

Key words: Majority rules, Cancellation, Decisiveness, Stability, p-Pareto

JEL classification: D71.

1 Introduction

In the last decades, the Social Choice Theory has experienced a great development. The pioneer works of Black (1948), Arrow (1963) and May (1952) have originated the analysis and characterization of numerous voting schemes. The simplest situation happens when a group of individuals has to choose between two alternatives. When we consider an egalitarian treatment between the alternatives, one of the most common voting procedures is simple majority. In the same way, other decision rules such as absolute majority —and, in general, absolute special majorities such as two-thirds or three-fourths majorities— are used when it is desired that the winning alternative have a "wide" support.

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However, such as it has been emphasized by García-Lapresta and Llamazares (2001), these voting procedures have important drawbacks. For instance, under simple majority rule an alternative can be elected with very poor support if a lot of individuals are indifferent between the alternatives. Although absolute special majorities solve the problem of minimum support, it is necessary a large number of votes in order that an alternative may be elected. Therefore, they have a problem of decisiveness. Since wide support and decisiveness are conflicting concepts, it seems necessary to look for a balance between them.

On the other hand, simple and absolute majorities also have a problem of stability because there are situations where the winning alternative can change when a single "turncoat" alters their preference.

In order to avoid the previous drawbacks, García-Lapresta and Llamazares (2001) consider in the context of fuzzy preferences a class of decision rules based on difference of votes, the M_k majorities. These voting procedures had been previously cited by Fishburn (1973, p. 18) and Saari (1990, p. 122), but not analysis had been done of them. For M_k majorities, the winning alternative is the one with a number of votes exceeding that obtained by the other in a previously fixed quantity. We note that M_k majorities are found between simple majority and unanimous majority.

The aim of this paper is to characterize this class of majority rules. For this, we use two sets of properties. In the first set will be essential the cancellation property. This property was used by Young (1974), with a equivalent formulation, to characterize the Borda rule. In the second set, decisiveness (used by Ferejohn and Grether (1974)) and stability will play a fundamental role. In fact, we will prove that M_k majorities are the most decisive in a class of stable majority rules.

The paper is organized as follows. Section 2 provides the basic definitions and notation. In this section we also point out some drawbacks of well known majority rules. Section 3 contains some characterizations of M_k majorities based on cancellation and p-Pareto properties. In Sections 4 we show some characterizations of M_k majorities based on stability and decisiveness properties. We conclude in Section 5.

2 Preliminaries and motivation

Let $N = \{1, \ldots, n\}$ be the set of voters, with $n \geq 3$, and x, y two alternatives. The individual preferences between the two alternatives are described by a profile $D = (d_1, \ldots, d_n)$, where d_i is $1, -1$ or 0 depending on whether individual i prefers x to y, y to x or is indifferent between them. The set of profiles of preferences will be denoted by $\mathcal{D}$.

Given $D, D' \in \mathcal{D}$ and σ a permutation on N, we will use the following notation: $n^+(D) = \#\{i \in N \mid d_i = 1\}$, $n^-(D) = \#\{i \in N \mid d_i = -1\}$, $-D = (-d_1, \ldots, -d_n)$, $D_\sigma = (d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ and $D' \geq D$ means $d'_i \geq d_i$ for all $i \in N$. For all $j \in N \cup \{0\}$, E^j will denote the following profile of preferences:

$$e_i^j = \begin{cases} 1, \text{ if } i \leq j, \\ 0, \text{ otherwise.} \end{cases}$$

So, $E^0 = (0, \ldots, 0)$ and $E^n = (1, \ldots, 1)$. Furthermore, $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}$ will denote the sets of profiles of preferences where one individual is indifferent between the alternatives and the rest agree that x is better than y, y is better than x, and that an alternative is better than the other, respectively; i.e.,

$$\mathcal{U}^+ = \{D \in \mathcal{D} \mid n^+(D) = n - 1,\ n^-(D) = 0\},$$

$$\mathcal{U}^- = \{D \in \mathcal{D} \mid n^+(D) = 0,\ n^-(D) = n - 1\},$$

$$\mathcal{U} = \mathcal{U}^+ \cup \mathcal{U}^-.$$

The collective preference is obtained by means of social welfare functions. A *social welfare function* (SWF) is a mapping $F : \mathcal{D} \longrightarrow \{-1, 0, 1\}$. The possible values of F, $-1, 0$ and 1, have a similar interpretation that in the case of individual preferences.

In this paper we focus our attention on SWF's based on difference of votes. This family of decision rules was noted by Fishburn (1973, p. 18), Saari (1990, p. 122) and García-Lapresta and Llamazares (2001).

Definition 1 *Given $k \in \{0, 1, \ldots, n-1\}$, the M_k majority is the SWF defined by*

$$M_k(D) = \begin{cases} 1, \textit{ if } \displaystyle\sum_{i=1}^{n} d_i > k, \\ -1, \textit{ if } \displaystyle\sum_{i=1}^{n} d_i < -k, \\ 0, \textit{ otherwise.} \end{cases}$$

Since $\sum_{i=1}^{n} d_i = n^+(D) - n^-(D)$, the M_k majority can also be defined by

$$M_k(D) = \begin{cases} 1, \text{ if } \; n^+(D) > n^-(D) + k, \\ -1, \text{ if } \; n^-(D) > n^+(D) + k, \\ 0, \text{ otherwise.} \end{cases}$$

Therefore, for these SWF's, an alternative is collectively preferred to another if the number of individuals who prefer the first to the second exceeds those who prefer the second to the first in a previously fixed quantity.

It is important to emphasize that simple and unanimous majorities are particular cases of M_k majorities. Thus, when $k = 0$ we obtain simple majority, i.e.,

$$M_0(D) = \operatorname{sgn}\Big(n^+(D) - n^-(D)\Big),$$

where $\operatorname{sgn} : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ is the sign function ($\operatorname{sgn}(x)$ is 1 if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$); and when $k = n-1$ we obtain unanimous majority, i.e.,

$$M_{n-1}(D) = \begin{cases} 1, \text{ if } \; D = E^n, \\ -1, \text{ if } \; D = -E^n, \\ 0, \text{ otherwise.} \end{cases}$$

We now consider some properties of SWF's that are well known in the literature: Anonymity, neutrality, monotonicity and weak and strong Pareto principles. Anonymity, also referred to as equality and symmetry, means that collective preference depends on only the set of individual preferences, but not on which individuals have these preferences, i.e., all individuals are treated equally. Neutrality, also referred to as duality, says that if everyone reverses their preferences between x and y, then the collective preference is also reversed, i.e., the alternatives are treated equally. Monotonicity means that increased support for an alternative does not hurt this alternative. Weak and strong Pareto principles mean that when there exists a certain degree of consensus among the voters, the society holds that preference. Weak Pareto principle, also referred to as unanimity, says that if all individuals prefer one alternative over the other, then the collective preference agree with the individuals'. Strong Pareto principle is similar to weak Pareto principle but without taking into account the indifferent voters.

Definition 2 *Let F be a SWF.*

(1) F is anonymous if for all permutation σ on N and all profile $D \in \mathcal{D}$ we have $F(D_\sigma) = F(D)$.
(2) F is neutral if for all profile $D \in \mathcal{D}$ we have $F(-D) = -F(D)$.
(3) F is monotonic if for all pair of profiles $D, D' \in \mathcal{D}$ such that $D' \geq D$ we have $F(D') \geq F(D)$.
(4) F is weak Pareto if $F(E^n) = 1$ and $F(-E^n) = -1$.
(5) F is strong Pareto if for all profile $D \in \mathcal{D}$,
(a) If $n^+(D) > 0$ and $n^-(D) = 0$, then $F(D) = 1$.
(b) If $n^-(D) > 0$ and $n^+(D) = 0$, then $F(D) = -1$.

Remark 3 *If F is a neutral SWF, then:*

(1) $F(E^0) = 0$.
(2) F is characterized by the set $F^{-1}(\{1\})$, since

$$F^{-1}(\{-1\}) = \{D \in \mathcal{D} \mid -D \in F^{-1}(\{1\})\},$$

$$F^{-1}(\{0\}) = \mathcal{D} \setminus \Big(F^{-1}(\{1\}) \cup F^{-1}(\{-1\})\Big).$$

Since the M_k majority is neutral, by Remark 3 we can define M_k as the neutral SWF given by

$$M_k(D) = 1 \;\Leftrightarrow\; n^+(D) > n^-(D) + k.$$

In the same way, some of the most popular voting systems can be define using neutrality. Next, we present some of them.

(1) The *simple majority* is the neutral SWF defined by

$$F(D) = 1 \;\Leftrightarrow\; n^+(D) > n^-(D).$$

There exist numerous references in the literature about simple majority. Nevertheless, it is important to emphasize that the first axiomatic characterization of it was given by May (1952).

(2) The *absolute majority* is the neutral SWF defined by

$$F(D) = 1 \;\Leftrightarrow\; n^+(D) > \frac{n}{2}.$$

Absolute majority has been characterize by Fishburn (1973, p. 60) (he called it *weak majority*).

(3) The *unanimous majority* is the neutral SWF defined by

$$F(D) = 1 \;\Leftrightarrow\; n^+(D) = n.$$

Unanimous majority has been characterized by Woeginger (2003).

(4) The *Pareto majority* is the neutral SWF defined by

$$F(D) = 1 \iff n^+(D) > 0 \ \text{ and } \ n^-(D) = 0.$$

Pareto majority has been characterized by Sen (1970, p. 76).

(5) Given $\alpha \in [1/2, 1)$, the *absolute special majority* A_α is the neutral SWF defined by

$$A_\alpha(D) = 1 \iff n^+(D) > \alpha n.$$

These rules are found between absolute majority, for $\alpha = 1/2$, and unanimous majority, for $\alpha \geq (n-1)/n$. Absolute special majorities have been studied by Fishburn (1973, p. 67) (without neutrality assumption) and Ferejohn and Grether (1974) (they called them *α-rules*).

(6) Given $\beta \geq 1$, the *relative special majority* R_β is the neutral SWF defined by

$$R_\beta(D) = 1 \iff n^+(D) > \beta n^-(D).$$

These rules are found between simple majority, for $\beta = 1$, and Pareto majority, for $\beta \geq n-1$. Relative special majorities have been studied by Craven (1971), Fishburn (1973, p. 68) (without neutrality assumption), Ferejohn and Grether (1974) (with a equivalent formulation and they called them *extended α-rules*) and Jain (1983, 1986a) (with the same formulation than Ferejohn and Grether (1974)).

(7) Given $\delta \in [0, 1]$, the *semi-strict majority* S_δ is the neutral SWF defined by

$$\begin{aligned} S_\delta(D) = 1 &\iff n^+(D) > \frac{1}{2}\Big(\delta n + (1-\delta)(n^+(D) + n^-(D))\Big) \\ &\iff n^+(D) > \frac{\delta}{1+\delta}\, n + \frac{1-\delta}{1+\delta}\, n^-(D). \end{aligned}$$

These rules are found between simple majority, for $\delta = 0$, and absolute majority, for $\delta = 1$. Semi-strict majorities have been studied by Pattanaik (1971, p. 54) (with a equivalent formulation and he called them *M-rules*) and Jain (1986b).

We now point out some drawbacks of the previous SWF's. For a sake of clearness, we consider 101 voters. An ordered pair (n_1, n_2) will denote the result of a ballot, where n_1 and n_2 will be the number of votes for x and y, respectively. Suppose that the result of a ballot is $(1, 0)$. In this case, under simple majority voting, x wins. So, in simple majority an alternative can be elected with very poor support. Moreover, in this situation —in fact, this happens when the result is $(n_1 + 1, n_1)$ or $(n_1, n_1 + 1)$— the winning alternative can change when a single “turncoat” alters their preference.

In order to avoid the problem of minimum support, we would be able to consider using the absolute majority. In this case, the problem of minimum support disappears but we continue to have a problem of stability when the result of the ballot is $(51, 50)$ or $(50, 51)$. Furthermore, a new drawback appears because under absolute majority the winning alternative needs a lot of votes. Consequently, there is a loss of decisiveness and, in many instances, there is no winning alternative. On the other hand, if we use absolute majority so that the winning alternative has a wide support, it is paradoxical that x wins if the result of the ballot is $(51, 50)$ but not when the result is $(50, 0)$.

Pareto majority has similar drawbacks to simple and absolute majorities. Since x wins when the result of a ballot is $(1, 0)$, there are problems of minimum support and stability. Moreover, there exists a winning alternative only if the result of the ballot is $(n_1, 0)$ or $(0, n_1)$, with $n_1 \geq 1$; therefore, there is a problem of decisiveness. For its part, unanimous majority has a large problem of decisiveness: It is the least decisive neutral, monotonic and no constant SWF.

In order to reduce the previous drawbacks, we can try to consider the SWF's that are dependent on parameters. However, absolute special majorities, A_α, have a problem of decisiveness because the winning alternative needs at least $[\alpha \cdot 101] + 1$ votes although the another alternative lacks support. Moreover, this loss of decisiveness increases as α increases.

Relative special majorities, R_β, have problems of minimum support and stability when the result of the ballot is $(1, 0)$ or $(0, 1)$. In relation to semi-strict majorities, S_δ, we can vary the value of δ to get a balance between "wide" support and decisiveness. For instance, if we consider $\delta = \frac{1}{3}$, the winning alternative needs at least 26 votes when the another alternative receives 0 or 1 vote, 27 votes when the losing alternative receives 2 or 3 votes and so on. However, semi-strict majorities have a problem of stability when the result of the ballot is $(51, 50)$ or $(50, 51)$.

In the case of M_k majorities, we can note that there is also a balance between wide support and decisiveness, but these decision rules have less problems of stability. Actually, as we will prove in Section 4, the M_k majorities are the most decisive in a class of stable SWF's. For instance, if we consider $k = 25$, the winning alternative needs at least 26 votes when there is not support for the another alternative, 27 votes when the losing alternative receives 1 vote and so on. However, in this case, no coalition with less than 26 voters can change the winning alternative.

3 Characterizations of M_k majorities based on cancellation and p-Pareto properties

In this section we consider two properties, cancellation and p-Pareto, to characterize the M_k majorities. Cancellation says that if two individuals have opposed preferences, i.e., one prefers x and another prefers y, then the collective preference is the same than if both individuals are indifferent between x and y. On the other hand, we introduce p-Pareto property to distinguish among the different degrees of consensus that exist between strong and weak Pareto principles, depending on the number of voters supporting an alternative.

Definition 4 *Let F be a SWF.*

(1) F is cancellative if for all pair of profiles $D, D' \in \mathcal{D}$ such that $d_i = 1$, $d_j = -1$ and $d'_i = d'_j = 0$ for some $i, j \in N$, and $d'_l = d_l$ for all $l \in N \setminus \{i, j\}$, we have $F(D) = F(D')$.

(2) Given $p \in \{0, 1, \ldots, n-1\}$, F is p-Pareto if:
- *(a) For all profile $D \in \mathcal{D}$,*
  - *(i) If $n^+(D) > p$ and $n^-(D) = 0$, then $F(D) = 1$.*
  - *(ii) If $n^-(D) > p$ and $n^+(D) = 0$, then $F(D) = -1$.*
- *(b) If $p \in \{1, \ldots, n-1\}$, there exists $D \in \mathcal{D}$ such that it satisfies one of the following conditions:*
  - *(i) $n^+(D) = p$, $n^-(D) = 0$ and $F(D) < 1$.*
  - *(ii) $n^-(D) = p$, $n^+(D) = 0$ and $F(D) > -1$.*

In relation to the p-Pareto property, it is obvious that we obtain strong Pareto principle when $p = 0$. Moreover, for all $p \in \{0, 1, \ldots, n-1\}$, any p-Pareto SWF is weak Pareto. Reciprocally, given a weak Pareto SWF F there exists $p \in \{0, 1, \ldots, n-1\}$ such that F is p-Pareto.

Cancellation property was given by Young (1974) with a different formulation, although in the same spirit, in order to characterize the Borda rule. On the other hand, it is easy to check that if $n = 2$, then any anonymous and neutral SWF is cancellative (for this reason we consider $n \geq 3$). Moreover, cancellation property has a narrow relation with anonymity, as it states the following proposition.

Proposition 5 *If F is a cancellative SWF, then $F(D_\sigma) = F(D)$ for all profile $D \in \mathcal{D} \setminus \mathcal{U}$ and all permutation σ on N.*

PROOF. Given $D \in \{D \in \mathcal{D} \setminus \mathcal{U} \mid n^+(D) = n^-(D) + m\}$, with $m \geq 0, m \neq n-1$, we are going to prove that $F(D) = F(E^m)$ (the case $n^+(D) < n^-(D)$ can be proven in a similar way). Since $n^+(D_\sigma) = n^+(D)$ and $n^-(D_\sigma) =$

$n^-(D)$ for all permutation σ on N, we will have $F(D_\sigma) = F(E^m) = F(D)$.

Moreover, since F is cancellative there exists $D^* \in \mathcal{D} \setminus \mathcal{U}$, with $n^+(D^*) = m$ and $n^-(D^*) = 0$, such that $F(D) = F(D^*)$. Therefore, we can suppose without loss of generality that $n^+(D) = m$ and $n^-(D) = 0$.

We distinguish two cases:

(1) If $m = n$ or $m = 0$, then $D = E^m$ and the result is obvious.
(2) If $1 \leq m \leq n-2$ and $D \neq E^m$, then there exist $j \in \{1, \ldots, m\}$ and $l \in \{m+1, \ldots, n\}$ such that $d_j = 0$ and $d_l = 1$. Since $n^+(D) \leq n-2$, there exists $r \in N$, $r \neq j$ such that $d_r = 0$. Now we consider the profiles D' and D'' defined by

$$d'_i = \begin{cases} 1, \text{ if } i = j, \\ \\ -1, \text{ if } i = r, \\ \\ d_i, \text{ otherwise,} \end{cases}$$

$$d''_i = \begin{cases} 0, \text{ if } i = r, l, \\ \\ d'_i, \text{ otherwise.} \end{cases}$$

Since F is cancellative we have $F(D) = F(D') = F(D'')$. If $D'' \neq E^m$, then we can repeat the previous process and concluding that $F(D) = F(E^m)$. $\square$

From definition, it is clear that any cancellative SWF is completely determined by its values in the set of profiles where non-indifferent voters agree that an alternative is better than the another, i.e., $\{D \in \mathcal{D} \mid \min(n^+(D), n^-(D)) = 0\}$. However, Proposition 5 allows us to reduce the previous set, as we show in Corollary 6. Moreover, we can also know the set of profiles that characterize a cancellative SWF when it satisfies other properties.

Corollary 6 *If F is a cancellative SWF, then F is completely determined by its values in the set $\mathcal{U} \cup \{E^j \mid j \in N \cup \{0\}\} \cup \{-E^j \mid j \in N\}$.*

Corollary 7 *Let F be a cancellative SWF.*

(1) If F is anonymous, then F is completely determined by its values in the set

$$\{E^j \mid j \in N \cup \{0\}\} \cup \{-E^j \mid j \in N\}.$$

(2) If F is neutral, then F is completely determined by its values in the set

$$\mathcal{U}^+ \cup \{E^j \mid j \in N \cup \{0\}\}.$$

(3) If F is anonymous and neutral, then F is completely determined by its values in the set

$$\{E^j \mid j \in N \cup \{0\}\}.$$

(4) If F is p-Pareto, then F is completely determined by its values in the set
(a) $\{E^0\}$, if $p = 0$.
(b) $\{E^j \mid j \in \{0, 1, \dots, p\}\} \cup \{-E^j \mid j \in \{1, \dots, p\}\}$, if $p \in \{1, \dots, n-2\}$.
(c) $\mathcal{U} \cup \{E^j \mid j \in \{0, 1, \dots, n-2\}\} \cup \{-E^j \mid j \in \{1, \dots, n-2\}\}$, if $p = n-1$.

It is important to emphasize that cancellative and strong Pareto SWF's are characterized by their value in the profile E^0. This fact will be used in the characterization of simple majority.

Now, from the previous results it is easy to characterize the class of M_k majorities by means of independent properties.

Theorem 8 *A SWF F is a M_k majority if and only if it is anonymous, neutral, monotonic, weak Pareto and cancellative.*

PROOF. It is easy to check that any M_k majority satisfies the properties. Reciprocally, suppose that F is anonymous, neutral, monotonic, weak Pareto and cancellative. By (3) of Corollary 7 F is determined by the set $\{E^j \mid j \in N \cup \{0\}\}$. Since F is weak Pareto, $F(E^n) = 1$, and by (1) of Remark 3 we also have $F(E^0) = 0$. Moreover, $F(E^i) \geq F(E^j)$ for all $i > j$ because F is monotonic. Therefore, there exists $k \in \{1, \dots, n-1\}$ such that $F(E^{k+1}) = 1$ and $F(E^k) = 0$; i.e., F is the M_k majority. □

Remark 9

(1) The SWF F defined by

$$F(D) = \begin{cases} 1, \text{ if } D = E^n, \\ -1, \text{ if } D = -E^n, \\ d_1, \text{ if } D \in \mathcal{U}, \\ 0, \text{ otherwise,} \end{cases}$$

is neutral, monotonic, weak Pareto and cancellative, but not anonymous.

(2) The SWF F defined by

$$F(D) = \begin{cases} 1, & \text{if } D \in \mathcal{U}^+ \cup \{E^n\}, \\ -1, & \text{if } D = -E^n, \\ 0, & \text{otherwise}, \end{cases}$$

is anonymous, monotonic, weak Pareto and cancellative, but not neutral.

(3) The SWF F defined by

$$F(D) = \begin{cases} 1, & \text{if } D \in \mathcal{U}^- \cup \{E^n\}, \\ -1, & \text{if } D \in \mathcal{U}^+ \cup \{-E^n\}, \\ 0, & \text{otherwise}, \end{cases}$$

is anonymous, neutral, weak Pareto and cancellative, but not monotonic.

(4) The null SWF, i.e., $F(D) = 0$ for all $D \in \mathcal{D}$, is anonymous, neutral, monotonic and cancellative, but not weak Pareto. In fact, from the proof of Theorem 8 it is easy to check that the null SWF is the only that satisfies these conditions.

(5) The absolute majority, i.e.,

$$F(D) = \begin{cases} 1, & \text{if } n^+(D) > n/2, \\ -1, & \text{if } n^-(D) > n/2, \\ 0, & \text{otherwise}, \end{cases}$$

is anonymous, neutral, monotonic and weak Pareto, but not cancellative.

Nevertheless, the class of M_k majorities is very wide and it includes some decision procedures as different as simple and unanimous majorities. For this reason, in the following theorems we characterize each M_k majority by means of independent properties. We begin with simple majority, which is possibly the most popular voting system. It has been widely studied in Social Choice Theory and, consequently, there exist numerous characterizations in the context of SWF's that we can classify in three groups:

(1) Characterizations considering two alternatives and a fixed number of voters (a society) have been given by May (1952) and Fishburn (1973, p. 58, 1983).
(2) Characterizations considering two alternatives and a variable number of voters (a set of societies) have been given by Aşan and Sanver (2002), Woeginger (2003) and Miroiu (2004).

(3) Characterizations considering three or more alternatives and a fixed number of voters have been given by Campbell (1982, 1988), Maskin (1995), Campbell and Kelly (2000, 2003) and Yi (2005).

Our characterization is in the first group. However, we use the strong Pareto principle, which is usual in the characterizations of the second group (see Aşan and Sanver (2002) and Woeginger (2003)).

Theorem 10 *A SWF F is the simple majority if and only if it is cancellative, strong Pareto and $F(E^0) = 0$.*

PROOF. It is easy to check that simple majority satisfies the properties. Reciprocally, suppose that F is cancellative, strong Pareto and $F(E^0) = 0$. By item (4a) of Corollary 7 any cancellative and strong Pareto SWF is determined by its value in the profile E^0. Therefore, if $F(E^0) = 0$, then F is the simple majority. □

Remark 11

(1) The SWF F defined by

$$F(D) = \begin{cases} 1, \textit{ if } n^+(D) > 0 \textit{ and } n^-(D) = 0, \\ -1, \textit{ if } n^-(D) > 0 \textit{ and } n^+(D) = 0, \\ d_1, \textit{ otherwise}, \end{cases}$$

is strong Pareto and $F(E^0) = 0$ but it is not cancellative.

(2) The SWF F defined by $F(D) = -M_0(D)$ for all $D \in \mathcal{D}$ is cancellative and $F(E^0) = 0$ but it is not strong Pareto.

(3) The SWF F defined by

$$F(D) = \begin{cases} 1, \textit{ if } n^+(D) \geq n^-(D), \\ -1, \textit{ if } n^-(D) > n^+(D), \end{cases}$$

is cancellative and strong Pareto but $F(E^0) \neq 0$.

By item (1) of Remark 3 we can obtain another characterization of simple majority by replacing the condition $F(E^0) = 0$ by neutrality.

Corollary 12 *A SWF F is the simple majority if and only if it is cancellative, strong Pareto and neutral.*

The examples given in Remark 11 go to show that cancellation, strong Pareto and neutrality are also independent properties.

Next we characterize any M_k majority different from simple and unanimous majorities. In this case, the properties are similar those given in Theorem 8, but anonymity can be omitted and weak Pareto is replaced by k-Pareto.

Theorem 13 *Given $k \in \{1,\dots,n-2\}$, a SWF F is the M_k majority if and only if it is neutral, monotonic, cancellative and k-Pareto.*

PROOF. It is easy to check that the M_k majority satisfies the properties. Reciprocally, suppose that F is neutral, monotonic, cancellative and k-Pareto. By Proposition 5, $F(D_\sigma) = F(D)$ for all $D \in \mathcal{D} \setminus \mathcal{U}$ and all permutation σ on N. Moreover, $F(D) = 1$ for all $D \in \mathcal{U}$ because F is k-Pareto. Therefore, F is anonymous and by Theorem 8 we have that F is a M_j majority for some $j \in \{1,\dots,n-2\}$. Finally, F is the M_k majority because it is k-Pareto. □

Remark 14 *Let $k \in \{1,\dots,n-2\}$.*

(1) The SWF F defined by

$$F(D) = \begin{cases} 1, \text{ if } n^+(D) > n^-(D) + k, \\ -1, \text{ if } n^-(D) > n^+(D), \\ 0, \text{ otherwise}, \end{cases}$$

is monotonic, cancellative and k-Pareto, but not neutral.

(2) The SWF F defined by

$$F(D) = \begin{cases} 1, \text{ if } n^+(D) > n^-(D) + k \ \text{ or } \ n^+(D) + k \geq n^-(D) > n^+(D), \\ -1, \text{ if } n^-(D) > n^+(D) + k \ \text{ or } \ n^-(D) + k \geq n^+(D) > n^-(D), \\ 0, \text{ otherwise}, \end{cases}$$

is neutral, cancellative and k-Pareto, but not monotonic.

(3) The SWF F defined by

$$F(D) = \begin{cases} 1, \text{ if } n^+(D) > n^-(D) + k, \\ -1, \text{ if } n^-(D) > n^+(D) + k, \\ d_1, \text{ otherwise}, \end{cases}$$

is neutral, monotonic and k-Pareto, but not cancellative.

(4) The SWF given in item (1) of Remark 9 is neutral, monotonic and cancellative, but not k-Pareto.

Finally, we characterize unanimous majority. Again, the properties are similar those given in Theorem 8, but cancellation is not necessary and weak Pareto is replaced by $(n-1)$-Pareto.

Theorem 15 *A SWF F is the unanimous majority if and only if it is anonymous, neutral, monotonic and $(n-1)$-Pareto.*

PROOF. It is easy to check that unanimous majority satisfies the properties. Reciprocally, suppose that F is anonymous, neutral, monotonic and $(n-1)$-Pareto. By the last property, $F(E^n) = 1$, $F(-E^n) = -1$ and there exists $D^* \in \mathcal{U}^+$ such that $F(D^*) < 1$ or $D^* \in \mathcal{U}^-$ with $F(D^*) > -1$. Assume that $D^* \in \mathcal{U}^+$ (the case $D^* \in \mathcal{U}^-$ can be proven in a similar way). Since F is neutral, by (1) of Remark 3, $F(E^0) = 0$, and by the monotonicity of F, $F(D^*) = 0$. Because F is anonymous, $F(D) = 0$ for all $D \in \mathcal{U}^+$, and by the neutrality, $F(D)$ is also zero for all $D \in \mathcal{U}^-$. Now, given $D \in \mathcal{D} \setminus \{E^n, -E^n\}$, there exist $D' \in \mathcal{U}^-$ and $D'' \in \mathcal{U}^+$ such that $D' \leq D \leq D''$. Consequently, by the monotonicity of F, $F(D) = 0$. $\square$

In order to show that anonymity, neutrality, monotonicity and $(n-1)$-Pareto are independent properties, we can consider the SWF's given in items (1), (2), (3) and (4) of Remark 9 by changing the term weak Pareto to $(n-1)$-Pareto. In the last case, it is possible to find SWF's different from the null; for instance, we can take any M_k majority where $k \in \{0, 1, \ldots, n-2\}$.

4 Characterizations of M_k majorities based on stability and decisiveness

We have seen in Section 2 that a drawback of simple majority is that the winning alternative can change when only a voter alters their preference. This problem can also happen in absolute majority, although this voting procedure is less decisive that simple majority. To avoid this situation, it seems necessary that the majority rules used have certain stability.

When we come to defining the notion of stability, it is necessary to bear in mind that for any no constant SWF there exist profiles where an alternative stops winning when a single voter changes their preference. We only show the result for x because it can be obtained for y in a similar way.

Proposition 16 *Let F be a no constant SWF such that $F^{-1}(\{1\})$ be no empty. Then there exist $D, D' \in \mathcal{D}$ such that $\#\{i \in N \mid d_i \neq d'_i\} = 1$, $F(D) = 1$ and $F(D') < 1$.*

PROOF. Since $F^{-1}(\{1\}) \neq 0$, there exists $D^n \in \mathcal{D}$ such that $F(D^n) = 1$. Moreover, since F is not constant, there exists $D^0 \in \mathcal{D}$ such that $F(D^0) < 1$. For $j \in \{1, \ldots, n-1\}$ we consider the profile D^j defined by

$$d_i^j = \begin{cases} d_i^n, \text{ if } i \leq j, \\ d_i^0, \text{ if } i > j. \end{cases}$$

Since $F(D^0) < 1$ and $F(D^n) = 1$, there exists $j \in \{0, \ldots, n-1\}$ such that $F(D^j) < 1$ and $F(D^{j+1}) = 1$. Furthermore, $\#\{i \in N \mid d_i^j \neq d_i^{j+1}\} = 1$. $\square$

Consequently, we consider that a SWF is stable of grade q (q-stable) when given any profile where there exists a winning alternative, q voters can change their preferences without the another alternative being winner. Since the case where an alternative can never win lacks interest, we also ask in the definition of q-stability that there be profiles where the change in the opinion of $q+1$ individuals produces the switch of the winning alternative.

Definition 17 *Given $q \in \{0, 1, \ldots, n-1\}$, a SWF F is q-stable if it satisfies the following conditions:*

(1) For all $D, D' \in \mathcal{D}$ such that $\#\{i \in N \mid d_i \neq d_i'\} \leq q$,

$$F(D) = 1 \;\Rightarrow\; F(D') \geq 0; \qquad F(D) = -1 \;\Rightarrow\; F(D') \leq 0.$$

(2) There exist $D, D' \in \mathcal{D}$ such that $\#\{i \in N \mid d_i \neq d_i'\} = q+1$ satisfying $F(D) = 1$ and $F(D') = -1$.

When both alternatives can win, any SWF can be classified in relation to its stability. In the following proposition we give the grade of stability of M_k majorities.

Proposition 18 *Given $k \in \{0, 1, \ldots, n-1\}$, the M_k majority is k-stable.*

PROOF. We are going to prove that the M_k majority satisfies the conditions of Definition 17. Let $D, D' \in \mathcal{D}$ such that $\#\{i \in N \mid d_i \neq d_i'\} \leq k$. If $M_k(D) = 1$, then $n^+(D) > n^-(D) + k$. Since $n^+(D') \geq n^+(D) - k$ and $n^-(D') \leq n^-(D) + k$, we have

$$n^-(D') \leq n^-(D) + k < n^+(D) \leq n^+(D') + k,$$

i.e., $M_k(D') \geq 0$. The case $M_k(D) = -1$ can be easily proven by the neutrality of M_k.

On the other hand, in order to prove the second condition of Definition 17 it is sufficient to consider E^{k+1} and $-E^{k+1}$. $\square$

The stability, together with neutrality and monotonicity, require that the winning alternative has at least a certain support. This result will become very useful in the characterization of M_k majorities.

Proposition 19 *Let $q \in \{0, 1, \dots, n-1\}$, F be a neutral, monotonic and q-stable SWF and $D \in \mathcal{D}$. Then:*

(1) If $F(D) = 1$, then $n^+(D) > q$.
(2) If $F(D) = -1$, then $n^-(D) > q$.

PROOF.

(1) This is proven by contradiction. Suppose $n^+(D) \leq q$ and consider $D' \in \mathcal{D}$ defined by

$$d'_i = \begin{cases} -1, \text{ if } d_i = 1, \\ \\ d_i, \text{ otherwise.} \end{cases}$$

Since $F(D) = 1$, $-D \geq D'$, $\#\{i \in N \mid d_i \neq d'_i\} \leq q$ and F is neutral, monotonic and q-stable we have

$$-1 = F(-D) \geq F(D') \geq 0,$$

a contradiction.

(2) Since F is neutral, then $F(-D) = 1$, and by the previous case, $n^-(D) = n^+(-D) > q$. $\square$

From this result, it is easy to obtain a characterization of unanimous majority by means of independent properties.

Theorem 20 *A SWF F is the unanimous majority if and only if it is neutral, monotonic and $(n-1)$-stable.*

PROOF. It is easy to check that unanimous majority satisfies the properties. Reciprocally, suppose that F is neutral, monotonic and $(n-1)$-stable. By the last property, there exist $D, D' \in \mathcal{D}$ such that $F(D) = 1$ and $F(D') = -1$. By Proposition 19, $D = E^n$, $D' = -E^n$ and $F(D) = 0$ for all $D \in \mathcal{D} \setminus \{E^n, -E^n\}$, i.e., F is the unanimous majority. $\square$

Remark 21

(1) *The SWF F defined by*

$$F(D) = \begin{cases} 1, \text{ if } n^+(D) > n^-(D) + (n-2), \\ -1, \text{ if } D = -E^n, \\ 0, \text{ otherwise}, \end{cases}$$

is monotonic and $(n-1)$-stable, but not neutral.

(2) *The SWF F defined by $F(D) = -M_{n-1}(D)$ for all $D \in \mathcal{D}$ is neutral and $(n-1)$-stable, but not monotonic.*

(3) *M_0 is neutral and monotonic, but not $(n-1)$-stable.*

We can not obtain similar characterizations for the remaining M_k majorities because when $k \in \{0, 1, \dots, n-2\}$ there exist neutral, monotonic and k-stable SWF's different from M_k. Nevertheless, when we consider the class of SWF's that satisfy the previous properties together with anonymity, the M_k majority is the most decisive in the following sense: given a profile of preferences, if an alternative is elected with an anonymous, neutral, monotonic and k-stable SWF, then this alternative is also winner with M_k. This concept was called *strong* by Ferejohn and Grether (1974).

Definition 22 *A SWF F is as decisive as another SWF G if for all $D \in \mathcal{D}$ we have*

$$G(D) = 1 \;\Rightarrow\; F(D) = 1; \qquad G(D) = -1 \;\Rightarrow\; F(D) = -1.$$

Remark 23 *Obviously, if F is as decisive as G, then $F = G$ or there exists $D \in \mathcal{D}$ such that $G(D) = 0$ and $F(D) \neq 0$, i.e., F is more decisive than G.*

In the characterization of M_k majorities we will use the following remark.

Remark 24 *If F is an anonymous and monotonic SWF, then for all pair of profiles $D, D' \in \mathcal{D}$ with the same number of non-indifferent voters, i.e., $n^+(D) + n^-(D) = n^+(D') + n^-(D')$, we have*

$$n^+(D) \leq n^+(D') \;\Rightarrow\; F(D) \leq F(D').$$

Theorem 25 *Given $k \in \{0, 1, \dots, n-2\}$, the M_k majority is the most decisive anonymous, neutral, monotonic and k-stable SWF.*

PROOF. By Remark 23, it is sufficient to prove that the M_k majority is as decisive as any anonymous, neutral, monotonic and k-stable SWF F. Given

$D \in \mathcal{D}$, suppose $F(D) = 1$. By Proposition 19 we have $n^+(D) > k$. Consider $D' \in \mathcal{D}$ obtained from D by changing the first k 1's to -1, i.e.,

$$d'_i = \begin{cases} -1, \text{ if } d_i = 1 \text{ and } \#\{j \leq i \mid d_j = 1\} \leq k, \\ \\ d_i, \text{ otherwise.} \end{cases}$$

By the k-stability and neutrality of F we have $F(D') \geq 0$ and $F(-D) = -1$. Since $F(D') > F(-D)$, $n^+(D') + n^-(D') = n^+(-D) + n^-(-D)$ and F is anonymous and monotonic, by Remark 24, we have $n^+(D') > n^+(-D)$, i.e.,

$$n^+(D) - k > n^-(D),$$

and, consequently, $M_k(D) = 1$.

The case $F(D) = -1$ can be easily proven by the neutrality of F and M_k. □

Since M_k is more decisive than $M_{k'}$ when $k' > k$, we also have the following result.

Corollary 26 *Given $k \in \{0, 1, \ldots, n-2\}$ and $k' \in \{k, \ldots, n-1\}$, the M_k majority is more decisive than any anonymous, neutral, monotonic and k'-stable SWF.*

Moreover, since M_0 is more decisive than the null SWF and any neutral and no null SWF is q-stable for some $q \in \{0, 1, \ldots, n-1\}$, we can also obtain another characterization of simple majority.

Corollary 27 *Simple majority is the most decisive anonymous, neutral and monotonic SWF.*

The results given in Theorem 25 (for $k \in \{1, \ldots, n-2\}$) and Corollary 27 are only satisfied on the respective classes of SWF's. Thus, in the following remark we show that if we drop one of the conditions then the previous results are not valid.

Remark 28 *Let $k \in \{0, 1, \ldots, n-2\}$.*

(1) The SWF F where there exits an oligarchy constituted by the first $k+1$ individuals, i.e.,

$$F(D) = \begin{cases} 1, \textit{ if } \ d_i = 1 \textit{ for all } i \leq k+1, \\ \\ -1, \textit{ if } \ d_i = -1 \textit{ for all } i \leq k+1, \\ \\ 0, \textit{ otherwise,} \end{cases}$$

is neutral, monotonic and k-stable, but not anonymous. If we consider $D \in \mathcal{D}$ defined by

$$d_i = \begin{cases} 1, \text{ if } i \leq k+1, \\ -1, \text{ otherwise,} \end{cases}$$

then $F(D) = 1$ and $M_k(D) < 1$.

(2) The SWF F defined by

$$F(D) = \begin{cases} 1, \text{ if } n^+(D) > n^-(D) + (k-1), \\ -1, \text{ if } n^-(D) > n^+(D) + k, \\ 0, \text{ otherwise,} \end{cases}$$

is anonymous, monotonic and k-stable, but not neutral. In this case, we have $F(E^k) = 1$ and $M_k(E^k) = 0$.

(3) The SWF F defined by $F(D) = -M_k(D)$ for all $D \in \mathcal{D}$ is anonymous, neutral and k-stable, but not monotonic. Here, we have $F(-E^n) = 1$ and $M_k(-E^n) = -1$.

(4) If $k \in \{1, \ldots, n-2\}$, M_0 is anonymous, neutral and monotonic, but not k-stable for $k \in \{1, \ldots, n-2\}$. In this case, $M_0(E^1) = 1$ and $M_k(E^1) = 0$ for all $k \in \{1, \ldots, n-2\}$.

5 Concluding Remarks

The axiomatic characterization of SWF's is important to know their qualities. In a similar way to the work of Young (1974), we use the cancellation property to characterize the M_k majorities. Likewise, we also have proven that these majority rules are the most decisive in the class of anonymous, neutral, monotonic and k-stable SWF's. This result is similar to that obtained by Ferejohn and Grether (1974) in relation to absolute and relative special majorities. Moreover, it shows that these majority rules are the best choice when we look for a balance between stability and decisiveness.

Since simple and unanimous majorities are particular cases of M_k majorities, we also have obtained some characterizations of these majority rules by means of independent properties. To be exact, we have obtained three new characterizations of simple majority (Theorem 10 and Corollaries 12 and 27) and two new characterizations of unanimous majority (Theorems 15 and 20).

From a practical point of view, the value of k can be given through a percentage of the total number of individuals. So, we can take k as the integer part of θn, $k = [\theta n]$, where $\theta \in [0, 1)$, and we can define the M_k majorities as the neutral SWF's given by

$$M_{[\theta n]}(D) = 1 \;\Leftrightarrow\; n^+(D) > n^-(D) + \theta n.$$

In the same spirit, it is possible to define neutral SWF's where we use a percentage of the number of non-indifferent individuals instead of the total number of individuals. In this case,

$$F(D) = 1 \;\Leftrightarrow\; n^+(D) > n^-(D) + \theta(n^+(D) + n^-(D)),$$

where $\theta \in [0, 1)$. But this expression can be written as

$$F(D) = 1 \;\Leftrightarrow\; n^+(D) > \beta n^-(D),$$

where $\beta = \frac{1+\theta}{1-\theta} \geq 1$. Therefore, these neutral SWF's are in fact relative special majorities; or reciprocally, the relative special majorities admit the previous interpretation.

Acknowledgements

This work is partially supported by the Junta de Castilla y León (Consejería de Educación y Cultura, Proyecto VA040A05). The author is grateful to an anonymous referee for valuable comments.

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