An Integro-Differential Equation of the Fractional Form: Cauchy Problem and Solution

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To prof. Veronique Hussin on his 60th birthday.

Abstract We solve the Cauchy problem defined by the fractional partial differential equation $[\partial_{tt} - \kappa \mathbb{D}]u = 0$, with $\mathbb{D}$ the pseudo-differential Riesz operator of first order, and the initial conditions $u(x,0) = \mu(\sqrt{\pi} x_0)^{-1} e^{-(x/x_0)^2}$, $u_t(x,0) = 0$. The solution of the Cauchy problem resulting from the substitution of the Gaussian pulse $u(x,0)$ by the Dirac delta distribution $\varphi(x) = \mu\delta(x)$ is obtained as corollary.

Keywords Fractional partial differential equations · Fox H-functions · Dirac delta distribution · Pseudo-differential Riesz operator · Complementary equation

1 Introduction

Linear partial differential equations of second order are useful in physics to model phenomena like wave propagation, heat diffusion, and transport processes [1–3]. In analogy to conics of analytic geometry, the wave equation is hyperbolic, while the heat and transport equations are parabolic. In a recent work [4] we have reported a fractional formulation that permits the study of such equations in unified form. Additionally, we have introduced an integro-differential version of the parabolic equation $u_{tt} - \kappa u_x = 0$ (hereafter called *complementary equation*) that is solvable in analytic form. That is, in [4] we have solved the Cauchy problem for $u_{tt} - \kappa \mathbb{D} u = 0$ with zero initial velocity and the Dirac delta pulse $\varphi(x) = \mu\delta(x)$ as initial condition. The symbol $\mathbb{D}$ stands for the pseudo-differential Riesz operator [5] (for contemporary notions on the matter, see, e.g., [6]). In the present work we provide

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Ş. Kuru et al. (eds.), *Integrability, Supersymmetry and Coherent States*, CRM Series in Mathematical Physics, https://doi.org/10.1007/978-3-030-20087-9_18