Trabajo Fin de Máster:

# Dynamical Algebras of Hyperbolic Pöschl-Teller Potentials

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## Resumen

En este trabajo se resuelve algebraicamente la ecuación de Schrödinger para el potencial de Pöschl-Teller (PT) hiperbólico. Se proponen dos parejas de operadores, los cuales permiten resolver tal ecuación por el método de factorización. Se obtienen las álgebras dinámicas de una "jerarquía" de hamiltonianos de Pöschl–Teller con dos párametros. Dichos operadores actúan sobre las funciones propias de un hamiltoniano relacionándolas con las funciones propias de un segundo hamiltoniano. Este tipo de operadores se denomina "shift" porque cambia solo los párametros del potencial pero mantiene la energía. Dichos operadores cierran un álgebra de Lie $su(1,1) \oplus su(1,1)$ la cual es isomorfa al álgebra de Lie $so(2,2)$. En una segunda parte se obtiene el hamiltoniano de PT a partir del álgebra $su(1,1) \oplus su(1,1)$. Primero, construyendo las representaciones adecuadas en una pseudo esfera de dimensión tres y a continuación identificando la representación de los generadores del álgebra con los operadores calculados mediante la factorización. También se obtienen las funciones propias mediante las dos aproximaciones.

## Abstract

In this work the Schrödinger equation for hyperbolic Pöschl–Teller (PT) potential is solved algebraically. Two couples of operators are proposed, which allow us to solve the equation by means of the factorization method. The dynamical algebras of the two-parametric hyperbolic Pöschl–Teller Hamiltonian hierarchies are obtained. These operators act on the eigenfunctions of each Hamiltonian relating them to the eigenfunctions of a second Hamiltonian. This kind of operators are called "shift" because they change the parameters of the potential but keep the energy. Such operators close the Lie algebra $su(1,1) \oplus su(1,1)$ which is isomorphic to the $so(2,2)$ Lie algebra. In the second part of this work, the PT Hamiltonian is obtained starting from the Lie algebra $su(1,1) \oplus su(1,1)$. First, by building the appropriate representations on a pseudo sphere of three dimensions and afterwards we identify the realizacion of the generators of the algebra with the operators computed by the factorization method. We have also obtained the eigenfunctions by means of both approximations.

# 1 Introduction

One of the most important problems in Physics is the generation of exactly solvable models. In this context, an important class of solvable quantum systems is obtained by means of the factorization method [1,2]. The factorization method make use of a kind of operator algebras that has been applied in different ways to study the eigenstates and spectra of the Schrödinger equations for certain one-dimensional potentials [1–7]. The operators taking part in the factorization give rise to a Hamiltonian hierarchy. Futhermore, these factor operators link different Hamiltonians in such hierarchy, so sometimes they are also called "intertwining" or "shift" operators.

The Pöschl-Teller (PT) potential [8] is one of the systems amenable to this method. We will consider here the hyperbolic version of this system, which is less known than the trigonometric case [2]. The PT Hamiltonian has many applications, like the modelling of diatomic molecules [9], it allows to find analytic resonances [10], it has been applied in nuclei theory [11], or used to analysing nonlinear optical properties [12]. Moreover, the Pöschl-Teller system has also been studied from more mathematical point of views such as group theory, differential equations or special functions [2, 13–19].

Although the PT potential has been extensively revised since its introduction by G Pöschl and E Teller [8] in 1933, in this work we will focus on a new algebraic perspective. In our work the intertwinig operators give us a dynamical algebra that constitute a direct sum of Lie algebras $su(1,1) \oplus su(1,1)$ which is isomorphic to the $so(2,2)$ Lie algebra. In this construction, in principle we will restrict ourselves to the two free parameters in the hyperbolic Pöschl-Teller potential; however in the process another discrete parameter will emerge: $n = 0, 1, 2\ldots$, which is for the energy level in the discrete spectrum.

The plan of the current work is as follows. In Section 2, we discuss the standard factorization of the Pöschl-Teller potential. Two sets of "intertwining" or "shift operators" are defined. The conditions of their corresponding ground states are carefully studied. Besides, the connection between these two sets of intertwining operators and two $su(1,1)$ algebras are established. Next, in Section 3 we will discuss the conditions on a set of states such that the two kinds of operators can act simultaneously in a suitable way. We find the common ground states which allow to construct this space, the commutation relations, and their eigenfunctions which correspond to a representation of the $so(2,2)$ algebra. In this section the spectrum and eigenfunctions are computed by using the shift operators. Next, in section 4 the group approach towards the PT Hamiltonian is developed. In particular, the solutions are obtained from the Casimir of the Lie algebra. In section 5, some remarks about the relation of $so(2,2)$ and $su(1,1) \oplus su(1,1)$ are included for completeness. The work ends with some conclusions and outlook in Section 6.

# 2 Standard Factorizations of Pöschl-Teller Potential

We consider the eigenvalue problem of the Schrödinger equation

$$H_{a,b}\psi_{a,b} = E\psi_{a,b}\,, \tag{2.1}$$

for the following type of Pöschl-Teller Hamiltonians:

$$H_{a,b} = -\partial_x^2 + V(x,a,b) = -\partial_x^2 - \frac{a^2 - \frac{1}{4}}{\cosh^2 x} + \frac{b^2 - \frac{1}{4}}{\sinh^2 x}, \qquad x \in (0, +\infty) \tag{2.2}$$

where $V(x,a,b)$ is a family of potentials labeled by the real parameters $a$, $b$. We have made use of units such that $\hbar = 2m = 1$, since these constants will not play any important role in this work. Along this section we will obtain two kinds of "shift" or "intertwining" operators, which will connect pairs of Hamiltonians in this two-parametric family. They will act on the eigenfunctions of one Hamiltonian to give eigenfunctions of a second one, both of type (2.2), keeping their energy eigenvalues. The fact that these operators connect the eigenfunctions of two Hamiltonians with shifted parameters $a, b$ of the potential, but not the eigenvalue, is the reason why they are also called "shift operators". Each pair of shift operators is composed by two Hermitian conjugate operators with respect to the standard inner product.

## 2.1 The first set of "shift operators" $M^{\pm}$

The Hamiltonian (2.2) is a linear second order differential operator that can be factorized as follows,

$$H_{a,b} = M^+_{a,b} M^-_{a,b} + \mu_{a,b} = M^-_{a+1,b+1} M^+_{a+1,b+1} + \mu_{a+1,b+1}, \tag{2.3}$$

where $M^{\pm}_{a,b}$ are two first order operators of the form

$$M^{\pm}_{a,b} = \pm\partial_x - (a - 1/2)\tanh x - (b - 1/2)\coth x\,, \tag{2.4}$$

and $\mu_{a,b}$ is a function of the parameters $a, b$, called factorization energy of the Hamiltonian $H_{a,b}$, given by

$$\mu_{a,b} = -(a + b - 1)^2\,. \tag{2.5}$$

Notice that the operators $M^{\pm}_{a,b}$ above defined are explicitly Hermitian conjugate with the standard inner product: $(M^-_{a,b})^\dagger = M^+_{a,b}$. The factorization (2.3) implies the following intertwining relations between the 'consecutive' Hamiltonians $H_{a,b}$ and $H_{a-1,b-1}$

$$M^-_{a,b} H_{a,b} = H_{a-1,b-1} M^-_{a,b} \tag{2.6}$$

$$M^+_{a,b} H_{a-1,b-1} = H_{a,b} M^+_{a,b}\,. \tag{2.7}$$

The above intertwining property of the operators $M^{\pm}_{a,b}$ leads to the following action on eigenfunctions:

$$M^-_{a,b} : \psi_{a,b} \;\longrightarrow\; M^-_{a,b}\psi_{a,b} \propto \psi_{a-1,b-1} \tag{2.8}$$

$$M^+_{a,b} : \psi_{a-1,b-1} \;\longrightarrow\; M^+_{a,b}\psi_{a-1,b-1} \propto \psi_{a,b} \tag{2.9}$$

where $\psi_{a,b}$ ($\psi_{a-1,b-1}$) are eigenfunctions of $H_{a,b}$ ($H_{a-1,b-1}$). This shows that eigenfunctions of 'consecutive' Hamiltonians are connected through the operators $M^+$ and $M^-$. The eigenvalues (physical or not) of these eigenfunctions $\psi_{a,b}$ and $\psi_{a-1,b-1}$ are the same for the respective Hamiltonians:

$$H_{a,b}\psi_{a,b} = E\psi_{a,b}, \iff H_{a-1,b-1}\psi_{a-1,b-1} = E\psi_{a-1,b-1} \tag{2.10}$$

In both cases (2.8) and (2.9), we can compute the normalization coefficient of the action of $M^{\pm}$ operators on the physical eigenfunctions by means of the factorization (2.3). For instance, in the first case (2.8) let us assume that the initial state $\psi_{a,b}$ is normalized, $\langle\psi_{a,b}, \psi_{a,b}\rangle = 1$. Then we compute

$$\langle M^-_{a,b}\psi_{a,b}, M^-_{a,b}\psi_{a,b}\rangle = \langle\psi_{a,b}, M^+_{a,b}M^-_{a,b}\psi_{a,b}\rangle = \langle\psi_{a,b}, (H_{a,b} - \mu_{a,b})\psi_{a,b}\rangle = E - \mu_{a,b}\,. \tag{2.11}$$

Therefore, we can write (up to a global phase)

$$M^-_{a,b}\psi_{a,b} = \sqrt{E - \mu_{a,b}}\,\psi_{a-1,b-1} \tag{2.12}$$

where $E$ is the common eigenvalue of $\psi_{a,b}$ and $\psi_{a-1,b-1}$. In the case of the operator $M^+_{a,b}$ we obtain

$$M^+_{a,b}\psi_{a-1,b-1} = \sqrt{E - \mu_{a,b}}\,\psi_{a,b} \tag{2.13}$$

This is a consequence of the fact that the operators $M^{\pm}_{a,b}$ are adjoint of each other. Remark that in these relations we are assuming that $\psi_{a,b}$ and $\psi_{a-1,b-1}$ are physical square integrable states.

### 2.1.1 Ground states

It is easy to find the 'ground state' of $H_{a,b}$ or states annihilated by the intertwining operators. If we consider the ground states annihilated by the operator $M^-$ then we obtain

$$M^-_{a,b}\psi^-_{a,b} = 0 \implies \psi^-_{a,b} = \cosh x^{-(a-\frac{1}{2})}\sinh x^{-(b-\frac{1}{2})}, \tag{2.14}$$

whose eigenvalue, according to the factorization (2.3) is

$$E^-_{a,b} = \mu_{a,b} = -(a+b-1)^2 \tag{2.15}$$

The conditions on the parameters for the ground wave functions (2.14) to be physically acceptable are:

- Finite at the origin: $-(b-\frac{1}{2}) \geq 0 \implies b \leq \frac{1}{2}$.
- Decaying behaviour at $x \to \infty$: $-(a-\frac{1}{2}) - (b-\frac{1}{2}) < 0 \implies a > -b+1$.

If we consider the ground states annihilated by the operator $M^+$, then we get

$$M^+_{a+1,b+1}\psi^+_{a,b} = 0 \Rightarrow \psi^+_{a,b} = \cosh x^{(a+\frac{1}{2})}\sinh x^{(b+\frac{1}{2})}, \quad E^+_{a,b} = -(a+b+1)^2 \tag{2.16}$$

where in this case the conditions on the parameters are $b \geq \frac{1}{2}$ and $a < -b+1$.

From the physical ground state of the Hamiltonian $H_{a,b}$ we will get another physical state of one of the consecutive Hamiltonians $H_{a\pm1,b\pm1}$ and so on (in principle, this sequence may be finite or infinite). The knowledge of the ground states of all the Hamiltonians $H_{a\pm n,b\pm n}$ will allow us to get all the discrete spectrum and eigenfunctions of this lattice of Hamiltonians. This is at the basis of the factorization method.

### 2.1.2 Commutation relations

From the operators $M^{\pm}_{a\pm n,b\pm n}$ acting on the sequence $H_{a\pm n,b\pm n}$, we can define operators $\tilde{M}^{\pm}$ and $\tilde{M}$ in the following way

$$\tilde{M}^{-}\psi_{a,b} := \tfrac{1}{2}M^{-}_{a,b}\psi_{a,b} \propto \psi_{a-1,b-1}, \qquad \tilde{M}^{+}\psi_{a,b} := \tfrac{1}{2}M^{+}_{a+1,b+1}\psi_{a,b} \propto \psi_{a+1,b+1},$$
$$\tilde{M}\psi_{a,b} := \tfrac{1}{2}M_{a,b}\psi_{a,b} := \tfrac{1}{2}(a+b)\psi_{a,b} \tag{2.17}$$

We have introduced for completeness the diagonal operator $\tilde{M}\psi_{a,b} = \frac{1}{2}(a+b)\psi_{a,b}$. Considering the operators $\tilde{M}^{\pm}, \tilde{M}$ defined on the set of functions $\psi_{a,b}$ obtained from a ground state we can compute their commutation relations. After an straightforward calculation (see the appendix) we get

$$[\tilde{M}, \tilde{M}^{\pm}] = \pm\tilde{M}^{\pm}, \quad [\tilde{M}^{-}, \tilde{M}^{+}] = 2\tilde{M}, \tag{2.18}$$

where these commutation rules correspond to the $su(1,1)$ algebra. The Casimir operator of this algebra is given by:

$$\mathcal{C} = \tilde{M}(\tilde{M}-1) - \tilde{M}^{+}\tilde{M}^{-} \tag{2.19}$$

The value of the Casimir operator will depend on the physical ground state.

## 2.2 The second set of "shift operators" $P^{\pm}$

From the definition of the Hamiltonian (2.2) we notice that it is invariant under the discrete transformation of the parameters,

$$I_a : (a,b) \longrightarrow (-a,b), \qquad I_b : (a,b) \longrightarrow (a,-b). \tag{2.20}$$

This invariance allow us define two pairs of factorization operators; the first one is given by

$$P^{\pm}_{a,b} = I_b M^{\pm}_{a,b} I_b \,. \tag{2.21}$$

The second pair of operators is obtained by $I_a$, but in this case we find the same pair of operators in different order. Due to this property, the Hamiltonian operator admits a second factorization as follows,

$$H_{a,b} = P^{+}_{a,b}P^{-}_{a,b} + \pi_{a,b} = P^{-}_{a+1,b-1}P^{+}_{a+1,b-1} + \pi_{a+1,b-1}, \tag{2.22}$$

where the operators $P^{\pm}_{a,b}$, generated from (2.21), are given by

$$P^{\pm}_{a,b} = \pm\partial_x - (a-1/2)\tanh x + (b+1/2)\coth x\,, \tag{2.23}$$

and $\pi_{a,b}$ is a function of the labels $a, b$, called factorization energy of the Hamiltonian $H_{a,b}$, which is given by

$$\pi_{a,b} = -(a-b-1)^2 \tag{2.24}$$

In the same way as $M^{\pm}_{a,b}$, the operators $P^{\pm}_{a,b}$ are explicitly Hermitian conjugate of each other with respect to the standard inner product of wavefunctions in $(0,+\infty)$. These operators fulfill the following intertwining relations between the 'consecutive' (but in a perpendicular direction to that defined in the previous factorization) Hamiltonians $H_{a,b}$ and $H_{a+1,b-1}$

$$P^{-}_{a,b}H_{a,b} = H_{a-1,b+1}P^{-}_{a,b} \tag{2.25}$$

$$P^{+}_{a,b}H_{a-1,b+1} = H_{a,b}P^{+}_{a,b}\,. \tag{2.26}$$

This relation of the operators $P^{\pm}_{a,b}$ produces the following action on eigenfunctions:

$$P^{-}_{a,b} : \psi_{a,b} \longrightarrow \psi_{a-1,b+1} \tag{2.27}$$

$$P^{+}_{a,b} : \psi_{a-1,b+1} \longrightarrow \psi_{a,b} \tag{2.28}$$

where the function $\psi_{a,b}$ designs an eigenfunction of $H_{a,b}$. The eigenvalues (physical or not) of the eigenfunctions $\psi_{a,b}$ and $\psi_{a+1,b-1}$, linked by $P^{\pm}_{a,b}$, are the same:

$$H_{a,b}\psi_{a,b} = E\psi_{a,b}, \iff H_{a+1,b-1}\psi_{a+1,b-1} = E\psi_{a+1,b-1}. \tag{2.29}$$

In order to obtain the normalization constants of the action of $P^{\pm}_{a,b}$ on physical wavefunctions, let us assume that the initial state $\psi_{a,b}$ is normalized, $\langle\psi_{a,b},\psi_{a,b}\rangle = 1$. Then we compute

$$\langle P^{-}_{a,b}\psi_{a,b}, P^{-}_{a,b}\psi_{a,b}\rangle = \langle\psi_{a,b}, P^{+}_{a,b}P^{-}_{a,b}\psi_{a,b}\rangle = \langle\psi_{a,b}, (H_{a,b}-\pi_{a,b})\psi_{a,b}\rangle = E-\pi_{a,b}, \tag{2.30}$$

consequently, we can write

$$P^{-}_{a,b}\psi_{a,b} = \sqrt{E-\pi_{a,b}}\,\psi_{a-1,b+1}, \tag{2.31}$$

where $E$ is the common eigenvalue of $\psi_{a,b}$ and $\psi_{a-1,b+1}$. In the case of the operator $P^{+}_{a,b}$ we obtain

$$P^{+}_{a+1,b-1}\psi_{a,b} = \sqrt{E-\pi_{a+1,b-1}}\,\psi_{a+1,b-1} \tag{2.32}$$

This is consistent with the fact that the operators $P^{\pm}_{a,b}$ are adjoint of each other. Remark that $E \geq \pi_{a,b}$ and $E \geq \pi_{a+1,b-1}$. This suggest that $\pi_{a,b}$ (or $\pi_{a+1,b-1}$) might be the ground energy of $H_{a,b}$, but we will postpone this point to the next subsection.

### 2.2.1 Ground states

In order to obtain the 'ground state' of $H_{a,b}$, or states annihilated by the intertwining operators, we consider two options; the first one is when the ground state is annihilated by the operator $P^-$ then, we obtain

$$P^{-}_{a,b}\psi^{-}_{a,b} = 0 \implies \psi^{-}_{a,b} = \cosh x^{-(a-\frac{1}{2})}\sinh x^{(b+\frac{1}{2})}, \tag{2.33}$$

whose eigenvalue, according to the factorization (2.3) is

$$E^{-}_{a,b} = \pi_{a,b} = -(a-b-1)^2 \tag{2.34}$$

In this case, the conditions on the parameters for the ground wavefunctions (2.33) to be physically aceptable are are:

- Finite at the origin: $b + \frac{1}{2} \geq 0 \implies b \geq -\frac{1}{2}$.
- Decaying behaviour at $x \to \infty$: $-(a - \frac{1}{2}) + (b + \frac{1}{2}) < 0 \implies a > b + 1$.

The second case consists of the ground states annihilated by the operator $P^+$; then we obtain

$$P^+_{a+1,b-1}\psi^+_{a,b} = 0 \Rightarrow \psi^+_{a,b} = \cosh x^{(a+\frac{1}{2})} \sinh x^{-(b-\frac{1}{2})}, \quad E^+_{a,b} = -(a-b+1)^2 \tag{2.35}$$

where the conditions on the parameters are $b \leq \frac{1}{2}$ and $a < b - 1$.

Observe that a Hamiltonian $H_{a,b}$ can not have two physical ground states annihilated by $P^+$ and $P^-$, since the conditions are incompatible.

From the physical ground state of the Hamiltonian $H_{a,b}$ we will get another state of one of the consecutive $H_{a\pm1,b\pm1}$ and so on.

### 2.2.2 Commutation relations

We can define operators $\tilde{P}^\pm$ acting on any eigenfunction in terms of the operators $P^\pm_{a\pm n,b\pm n}$ as follows

$$\begin{gathered} \tilde{P}^-\psi_{a,b} := \tfrac{1}{2}\, P^-_{a,b}\psi_{a,b} \propto \psi_{a-1,b+1}, \qquad \tilde{P}^+\psi_{a,b} := \tfrac{1}{2}\, P^+_{a+1,b-1}\psi_{a,b} \propto \psi_{a+1,b-1}, \\ \tilde{P}\psi_{a,b} := \tfrac{1}{2}\, P_{a,b}\psi_{a,b} := \tfrac{1}{2}(a-b)\psi_{a,b} \end{gathered} \tag{2.36}$$

We have introduced a new diagonal operator by $\tilde{P}\psi_{a,b} = \frac{1}{2}(a-b)\psi_{a,b}$. Taking into account these definitions we can compute their commutation relations (for more details see the appendix):

$$[\tilde{P}, \tilde{P}^\pm] = \pm\tilde{P}^\pm, \quad [\tilde{P}^-, \tilde{P}^+] = 2\tilde{P}\,. \tag{2.37}$$

These commutation rules close the $su(1,1)$ algebra, i.e. the same algebra as the set $\tilde{M}^\pm, \tilde{M}$. The Casimir operator of this algebra is again:

$$\mathcal{C} = \tilde{P}(\tilde{P} - 1) - \tilde{P}^+\tilde{P}^- \tag{2.38}$$

# 3 Two-dimensional Factorizations

This section is devoted to analyse the conditions on a set of physical states such that the operators $M^\pm$, of the first factorization, and $P^\pm$, of the second one, can act such that they are are leaving invariant this set of states. Each space (sometimes we will call it "sector") will be related to a unitary representation of $so(2,2)$ characterized by the Casimir eigenvalues. This implies that we can find the commutation relations valid on the space spanned by these states, the total algebra and the common fundamental state characterizing each of these invariant spaces (or sectors).

### 3.1 Commutation relations

Applying the intertwining relations given in equations (2.6), (2.7), (2.25), (2.26) is possible to calculate the conmutation relation between the two sets of operators $\{M, M^{\pm}\}$ and $\{P, P^{\pm}\}$ when their action is well defined. The result is that they commute:

$$[P^{\pm}, M^{\pm}] = 0 \tag{3.1}$$

If we recall that from the previous commutation rules of each set, (2.18) of $\{M, M^{\pm}\}$ and (2.37) of $\{P, P^{\pm}\}$, they correspond to the $su(1,1)$ algebras, hence we conclude that the total set of operators will close a direct sum of $su(1,1)$ Lie algebras,

$$\langle M, M^{\pm}, P, P^{\pm}\rangle \;\approx\; su(1,1) \oplus su(1,1) \tag{3.2}$$

### 3.2 The common ground states

The next objective is to find a set of eigenstates $\psi_{a,b}$ of the Hamiltonians $H_{a,b}$, such that the whole set of intertwining operators $\{M, M^{\pm}, P, P^{\pm}\}$ can act on them and their action is well defined in the sense that we will always get physical square integrable wave functions. This is possible if we can find a common ground state. We will focus on the ground states $\psi^0_{a,b}$ annihilated by the operators $M^-$ and $P^-$:

$$M^-_{a,b}\psi^0_{a,b} = P^-_{a,b}\psi^0_{a,b} = 0 \quad\Longrightarrow\quad \psi^0_{a,0} = \cosh x^{(-a+\frac{1}{2})}\sinh x^{\frac{1}{2}} \tag{3.3}$$

- Therefore, a necessary condition is is that $b = 0$.
- In order this ground wavefunction $\psi^0_{a,0}$ be square integrable, the parameter $a$ must satisfy $a > 1$.

We will pay attention to common ground states for integer values of $a$, starting from the first allowed value ($a = 2, b = 0$) corresponding to $\psi^0_{2,0}$. Of course, we could start from another allowed value of $a$ and the results would be quite similar, but we will not discuss all the possibilities.

From the common ground state by means of the action of the creation operators $M^+$ and $P^+$ we generate a set of eigenstates, of a corresponding set of the respective Hamiltonians, where all the operators (3.2) can act. We will call the underlying set of eigenstates with obtained in this way with labels $(a, b)$ a "sector" or $su(1,1) \oplus su(1,1)$ representation (very similar sectors where considered in [19] for the confluent functions. In that case the Lie algebra was a direct sum of Heisenberg algebras). We describe its elements below.

**The first sector (or $\psi^0_{2,0}$-sector) of eigenfunctions**. This is composed by:

- The common ground state of the sector $\psi^0_{2,0}$, where $a = 2, b = 0$ .
- All the eigenstates $\{\psi_{a,b}\}$ obtained by the consecutive action of the operators $M^+$ and $P^+$ acting on the ground state $\psi^0_{2,0}$.

- All the Hamiltonians $\{H_{a,b}\}$ of these eigenfunctions, which are intertwined by the operators $M^+$ and $P^+$ starting from $H_{2,0}$.
- The set of points in the plane $\mathbb{R}^2$ corresponding to the labels of eigenstates and eigenfunctions:

  $$\{(a,b)\},\ \text{such that}\ (a,b) = (2,0) + m(1,1) + n(1,-1), \qquad m,n = 0,1,2\dots$$

  These parameters are obtained by consecutive application of $M^+$ and $P^+$ on the ground state labeled by $(2,0)$.
- The eigenstates $\{\psi_{a,b}\}$ of the $\psi^0_{2,0}$-sector have the same energy as the initial ground state: $E_{a,b} = E^0_{2,0} = -(1)^2$.
- In this case, the eigenvalue of the Casimir operator for any of the $su(1,1)$ Lie algebras in this representation is given by $\mathcal{C} = \frac{a}{2}(\frac{a}{2} - 1) = 0$. Usually the value of the Casimir is denoted by $k$ such that $\mathcal{C} = k(k-1)$. Using this notation, this unitary representation is characterized by the value $k = 1$.

This sector is represented in the plane in Figs. 1 and 2 where it is shown the points of the labels $(a,b)$ connected by the creation operators. Some eigenfunctions corresponding to this sector, all of them with the same energy $E = -1$, are shown in Fig. 4 ($\psi^0_{2,0}$, $\psi^0_{3,1}$ and $\psi^0_{4,2}$), in Fig. 5 ($\psi^1_{4,0}$, $\psi^1_{5,1}$ and $\psi^1_{6,2}$) and in Fig. 6 ($\psi^2_{6,0}$, $\psi^2_{7,1}$ and $\psi^2_{8,2}$). The notation $\psi^n_{a,b}$ is for the $n$-excited wavefunction of Hamiltonian $H_{a,b}$. All of them are plotted in red color.

Next we will describe the second sector based on another common ground state $\phi^0_{4,0}$. The points of the plane $(a',b')$, and the Hamiltonians, $H_{a',b'}$, are also included in the first sector, but the eigenfunctions are different, because they are obtained from a different ground state and the common energy is lower than in the first sector $E_{a',b'}$. We give a similar description.

**The second sector (or $\phi^0_{4,0}$-sector) of eigenfunctions**. This is composed by:

- The common ground state $\phi^0_{4,0}$
- All the eigenstates $\{\phi_{a',b'}\}$ obtained from $\phi^0_{4,0}$ by the action of the operators $M^+$ and $P^+$ acting on the ground state.
- All the Hamiltonians $\{H_{a',b'}\}$ with these eigenfunctions, which are intertwined by the operators $M^+$ and $P^+$ starting from $H_{4,0}$.
- The eigenfunctions of the $\phi^0_{4,0}$-sector have the same energy as the initial ground state: $E'_{a',b'} = E^0_{4,0} = -(3)^2$.
- The set of points in the plane $\mathbb{R}^2$ corresponding to the labels of eigenstates and eigenfunctions is:

  $$\{(a',b')\},\ \text{such that}\ (a',b') = (4,0) + m(1,1) + n(1,-1), \qquad m,n = 0,1,2\dots$$

  These parameters are obtained by consecutive application of $M^+$ and $P^+$ on the ground state parameters $(4,0)$.

- The Casimir operator for both $su(1,1)$ algebras is $\mathcal{C} = \frac{a}{2}(\frac{a}{2}-1) = 2$. In the $k$ notation, this representation is $k=2$ (recall that thee previous sector had $k=1$).

This sector is represented in the plane in Fig. 1 and in Fig. 2 where it is shown the points of the labels $(a', b')$ connected by the creation operators. Some eigenfunctions corresponding to this sector with the same energy $E = -3^2$ are shown in Fig. 5 ($\phi^0_{4,0}$, $\phi^0_{5,1}$ and $\phi^0_{6,2}$) and in Fig. 6 ($\phi^1_{6,0}$, $\phi^1_{7,1}$ and $\phi^1_{8,2}$), all of them with energy $E'_{a',b'} = -9$. They are plotted in green color.

The same can be said of the following sectors which are sub-sectors of all the previous ones. We see that each new sector includes one additional energy level below the ones of the previous sector. This is due to the energy of the respective ground states: $E^0_{2p,0} = -(2p-1)^2$. We have represented the first three of these sectors at different levels in Fig. 3 to show that the eigenfunctions of each sector have the same energy specified by its ground state.

By means of this process we can get the spectrum and eigenfunctions of any Hamiltonian $H_{a,b}$ in the initial sector. Let us consider the example of the Hamiltonian $H_{8,2}$ whose eigenfunctions are represented in Fig. 6.

- **Spectrum.** In order to see how is its spectrum we must find the sectors where the point $(8,2)$ is included. In this case it belongs to three sectors: a) the main sector with vertex $(2,0)$; b) the second sector with vertex $(4,0)$; and c) the third sector with vertex $(6,0)$. Therefore this Hamiltonian has three eigenvalues, one for each sector, given by the eigenvalues of the respective ground states: $E^0_{2,0} = -1$, $E^0_{4,0} = -3^2$ and $E^0_{6,0} = -5^2$.

- **Eigenfunctions.** In order to get the eigenfunction $\psi^0_{8,2}$ of the ground state of $H_{8,2}$, corresponding to the eigenvalue $E^0_{6,0} = -5^2$ we start with the common ground state $\psi^0_{6,0}$, and apply the operators $M^+$, in this case:

$$\psi^0_{8,2} \propto (M^+)^2 \psi^0_{6,0}$$

To find the eigenfunction $\psi^1_{8,2}$ of the first excited state of $H_{8,2}$, corresponding to the eigenvalue $E^0_{4,0} = -3^2$ we start with the common ground state of the previous sector $\psi^0_{4,0}$, and apply the operators $M^+$ and $P^+$ in this case:

$$\psi^1_{8,2} \propto (P^+)(M^+)^3 \psi^0_{4,0}$$

Finally, the eigenfunction $\psi^2_{8,2}$ of the second excited state of $H_{8,2}$, with eigenvalue $E^0_{2,0} = -1$ we start with the common ground state of the main sector $\psi^0_{2,0}$, and apply the operators $M^+$ and $P^+$, in this case:

$$\psi^1_{8,2} \propto (P^+)^2 (M^+)^4 \psi^0_{2,0}$$

Some plots of eigenfunctions corresponding to Hamiltonians of the first sector (labeled by $\alpha, \alpha_1, \alpha_2$ in Fig. 2) of the second sector (labeled by $\beta, \beta_1, \beta_2$ in Fig. 2) and of the third sector (labeled by $\gamma, \gamma_1, \gamma_2$ in Fig. 2) are shown in Figs. 4-5-6.

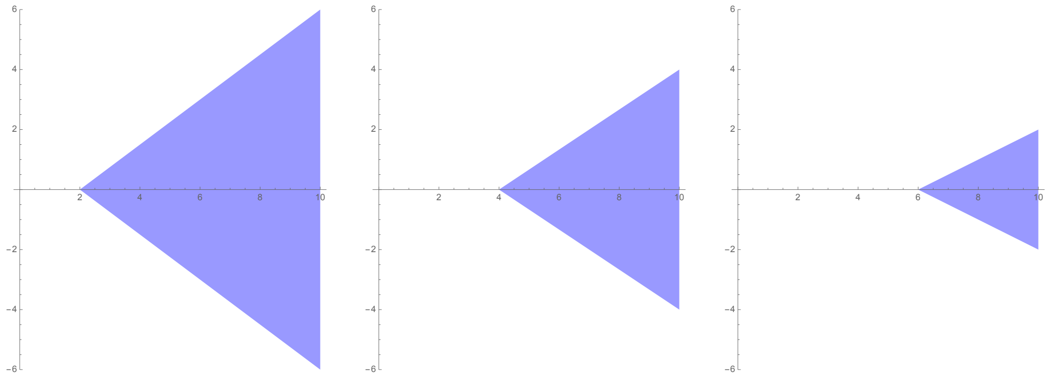


Figure 1: The three sectors corresponding to ground states with subindexes $(2, 0)$, $(4, 0)$ and $(6, 0)$. The energy levels of the corresponding functions are $E = -(1)^2$ (sector at the left), $E = -(3)^2$ (center) and $E = -(5)^2$ (right).

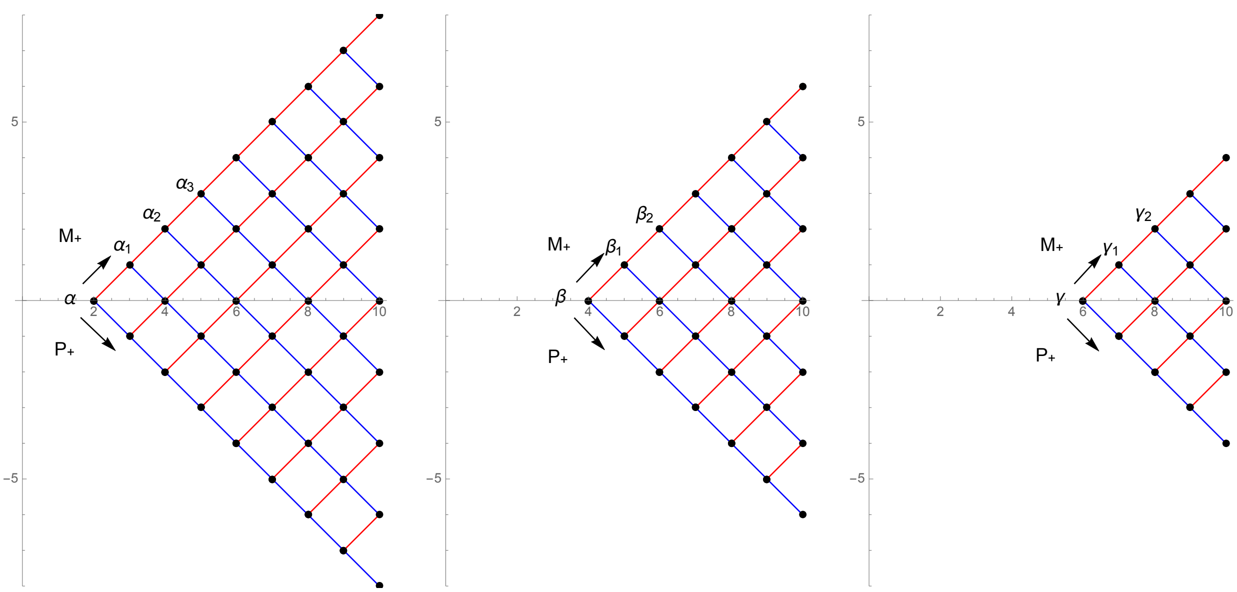


Figure 2: The points of the three sectors correspond to Hamiltonians and to states. It is possible to scroll down or up the points using the operators $P^+$ and $M^+$ respectively.

## 3.3 The eigenfunctions

One important point in this process is to know the eigenfunctions of the Hamiltonian. We can proceed in the same way as mentioned at the end of the previous section. But we would like a more automatic method which it is shown below.

One form to obtain some of the eigenfunctions is by apply subsequently the rising operator. As we saw previously the action of $M^-$ (or $P^-$) operators allow us to obtain the ground state while the action of $M^+$ (or $P^+$) operators give us the next excited states, then, we expect that

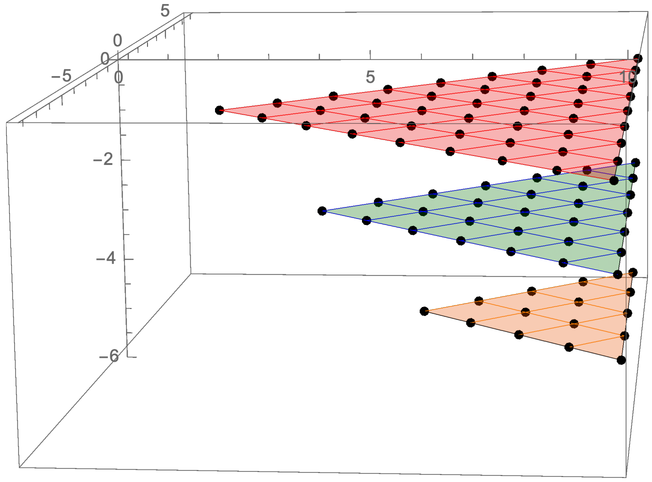


Figure 3: The highest sector shows the biggest energy states $E = -(1^2)$, in the middle sector the states with energy $E = -(3^2)$ and the third sector is for lower energy states $E = -(5^2)$. The next sectors have a similar representation.

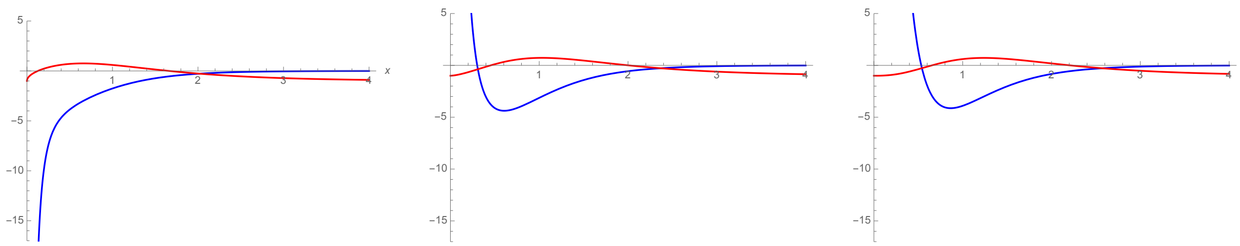


Figure 4: In the three cases the potential is the blue curve and their ground state wavefunction is the red curve. In the left hand side we have $\alpha$(2,0), in the center $\alpha_1$(3,1) and in the rhs $\alpha_2$(4,2) wavefunctions

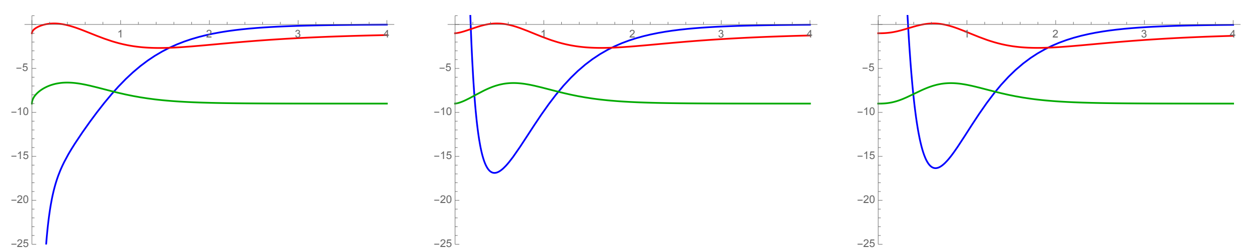


Figure 5: In the three cases the potential is the blue curve, the ground states are in green (they belong to the second sector) and the red color is for the first excited states (they belong to the first sector). The states in the second sector are in green: In the left side we have $\beta$(4,0), in the center $\beta_1$(5,1) and in the right side $\beta_2(6,2)$ wavefunctions.

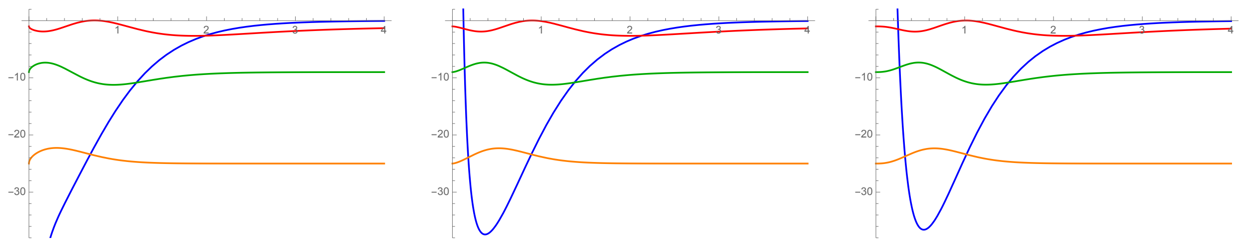


Figure 6: In the three cases the potential is the blue curve, the ground states are in orange (they belong to the third sector), the green color is the first excited states (in the second sector) and the red curves are for the second excited states (in the first sector). The states of the 3rd sector are: In the left side we have $\gamma$(6,2), in the center \$$\gamma_1$(6,0) and in the right side $\gamma_2$(8,2).

the consecutive action of $M^+$ (or $P^+$) will allow us to find a n-excited state $\psi^n_{a,b}$.

Although in previous sections a definition of $M^+$ operator was presented, we are now interested in a different but equivalent form of $M^+$ operator, this, in order to obtain $\psi^n_{a,b}$ eigenfunction. Then the $M^+$ operator also can be written as

$$M^+_{a,b} = n_1 \cosh x^{(a-\frac{1}{2})} \sinh x^{(b-\frac{1}{2})} \partial_x n_1 \cosh x^{-(a-\frac{1}{2})} \sinh x^{-(b-\frac{1}{2})}, \tag{3.4}$$

now with this definition we can apply this $M^+$ until obtain the $\psi^n_{a,b}$ as follows

Suppose that we want to get the $n$-excited state of the Hamiltonian $H_{a,b}$ by means of the $M^+$ operators if it exists. Then we start from a ground state $\psi^0_{a-n,b-n}$ and we will act $n$ times on it with the $M^+$ operator:

$$\psi^n_{a,b} = N^n_{a,b}\,(M^+)^n \psi^0_{a-n,b-n} \tag{3.5}$$

where $N$ is a normalization constant. Taking into account the definition of the action of the $M^+$ operators, this formula is given by

$$\psi^n_{a,b}(x) = N^n_{a,b} M^+_{a,b} M^+_{a-1,b-1} M^+_{a-2,b-2} \cdot\cdot\cdot ... M^+_{a-n+1,b-n+1} \psi^0_{a-n,b-n} \tag{3.6}$$

At this point let us recall the form of one of these operators as it was given in (2.4):

$$M^\pm_{a,b} = \partial_x - (a-1/2)\tanh x - (b-1/2)\coth x\,. \tag{3.7}$$

We can try to simplify it as follows:

$$M^+_{a,b} = \cosh x^{(a-\frac{1}{2})} \sinh x^{(b-\frac{1}{2})} \big(\partial_x\big) \cosh x^{-(a-\frac{1}{2})} \sinh x^{-(b-\frac{1}{2})}\,. \tag{3.8}$$

We must have also in mind the expression of the ground state given in (2.14):

$$\psi^0_{a,b} = \cosh x^{-(a-\frac{1}{2})} \sinh x^{-(b-\frac{1}{2})}\,. \tag{3.9}$$

Replacing this expressions in (3.6) we obtain

$$\psi^n_{a,b}(x) = N^n_{a,b} (\cosh x)^{\frac{a}{2}} (\sinh x)^{\frac{b}{2}} \big(\cosh x^{-1} \sinh x^{-1} \partial_x\big)^n (\cosh x)^{-(2a-2n)} (\sinh x)^{-(2b-2n)}\,. \tag{3.10}$$

Next, we can perform the natural change of variable $t = \cosh 2x$, to obtain:

$$\psi^n_{a,b}(x) = (t+1)^{-\frac{a}{2}}(t-1)^{-\frac{b}{2}}(t+1)^a(t-1)^b \frac{d^n}{dt^n}(t+1)^{-a+n}(t-1)^{-b+n}\,. \tag{3.11}$$

If $\beta = -a$ and $\alpha = -b$ then,

$$\psi^n_{a,b}(x) = (t+1)^{\frac{\beta}{2}}(t-1)^{\frac{\alpha}{2}}\left[(t+1)^{-\beta}(t-1)^{-\alpha}\frac{d^n}{dt^n}(t+1)^{\beta+n}(t-1)^{\alpha+n}\right], \tag{3.12}$$

where we can identify the last part of this expression (in square brackets) as the Rodrigues's formula of Jacobi polynomials [20]:

$$\psi^n_{a,b}(x) = (t+1)^{\frac{\beta}{2}}(t-1)^{\frac{\alpha}{2}}\, P_n^{(\alpha,\beta)}(t), \qquad t = \cosh 2x \in (1,+\infty) \tag{3.13}$$

Remark the following conditions for this formula to be sensible. The first and second are related to the function in front of the Jacobi polynomials, which correspond to a ground wave function with subindexes $(a,b)$. The last condition tell us that the Hamiltonian $H_{a,b}$ if $b<0$ belong to the sector $k = \frac{a+b}{2}$. In this case the number of bound states is $n = 0,1,\ldots,k-1$. In other words $n \le k = \frac{a+b}{2}$. Conditions on the Jacobi formula (3.13):

1. $\frac{\alpha}{2} > 0 \implies \alpha \ge 0$, or $b \le 0$.
2. $\beta + \alpha < -1 \implies a > -b+1$
3. $n \le \frac{a+b}{2} = -\frac{\alpha+\beta}{2}$

A similar formula can be obtained for the eigenfunctions with the parameter $b \ge 0$, by means of the $P^+$ operators.

# 4 Group Theory of the PT Hamiltonian

We have just seen that the PT system has two sets of operators, $\tilde{M}^\pm, \tilde{M}$ and $\tilde{P}^\pm, \tilde{P}$, that close the structure $su(1,1)\oplus su(1,1)$. In this section we want to make this way in opposite direction: by starting from this Lie algebra we will obtain the PT Hamiltonians as well as all the shift operators, which will be interpreted in terms of the representation of the generators of the Lie algebra.

## 4.1 The Lie group $SU(1,1)$

The group $SU(1,1)$ is made up of the $2\times 2$ complex matrices $A$ which act in the space $\mathbb{C}^2$ of the complex plane by matrix multiplication

$$A : \mathbf{z} \to \mathbf{z}' = A\mathbf{z}, \qquad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad z_i, \alpha, \dots \in \mathbb{C} \tag{4.1}$$

These matrices will leave invariant the Hermitian scalar product with metric $g = \mathrm{diag}(1, -1)$ defined as follows:

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1^* b_1 - a_2^* b_2 = \mathbf{a}^\dagger G\, \mathbf{b}, \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.2}$$

That is, the $SU(1,1)$ matrices will satisfy

$$\begin{aligned} &\langle A\mathbf{a}, A\mathbf{b} \rangle = \mathbf{a}^\dagger (A^\dagger G\, A)\mathbf{b} = \mathbf{a}^\dagger G\, \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle, \qquad \forall \mathbf{a}\ \mathbf{b} \in \mathbb{C}^2 \\ &\iff \quad A^\dagger G\, A = G \end{aligned} \tag{4.3}$$

Besides, such matrices must have determinant equal to one $|A| = 1$. Hence the notation $SU(1,1)$ for this special unitary group with this invariant metric. Therefore, the conditions for a matrix $A$ to belong to this group are, according to (4.3),

$$A^\dagger G A = G, \text{ and } |A| = 1\,, \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{4.4}$$

In total, from (4.4), there are five independent equations for the eight real parameters leaving three parameters free. We can write a general $SU(1,1)$ in the form

$$A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1\,. \tag{4.5}$$

A choice for the three independent parameters is given in Vilenkin's book [21]

$$A = \begin{pmatrix} e^{i(\varphi+\psi)/2} \cosh \dfrac{t}{2} & e^{i(\varphi-\psi)/2} \sinh \dfrac{t}{2} \\ e^{-i(\varphi-\psi)/2} \sinh \dfrac{t}{2} & e^{-i(\varphi+\psi)/2} \cosh \dfrac{t}{2} \end{pmatrix}, \tag{4.6}$$

$$0 \le \psi < 2\pi, \quad 0 < t < \infty, \quad -2\pi \le \psi < 2\pi$$

The elements of $SU(1,1)$, according to (4.5) are determined by two complex numbers ($\alpha = a_1 + ia_2, \beta = b_1 + ib_2$), or equivalently by the four real numbers $(a_1, a_2, b_1, b_2)$, together with the condition (4.5),

$$(a_1^2 + a_2^2) - (b_1^2 + b_2^2) = 1\,. \tag{4.7}$$

This corresponds to the points of a $(2,2)$-pseudosphere in $\mathbb{R}^4$ [22] The topological properties of $SU(1,1)$ are the same as this three dimensional surface.

The complex vector plane $\mathbb{C}^2$ where the matrices $A \in SU(1,1)$ act, can also be seen as a four dimensional real space:

$$\mathbf{z} = \begin{pmatrix} z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{pmatrix} \iff \vec{\mathbf{z}} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \qquad \mathbf{z} \in \mathbb{C}^2,\ \vec{\mathbf{z}} \in \mathbb{R}^4 \tag{4.8}$$

Under a transformation $A \in SU(1,1)$, this complex $(1,1)$-Hermitian product is conserved:

$$A : \mathbf{z} \to \mathbf{z}' = A\mathbf{z}, \qquad |z_1|^2 - |z_2|^2 = |z_1'|^2 - |z_2'|^2 \tag{4.9}$$

or in other words, the induced action of $SU(1,1)$ in $\mathbb{R}^4$ is such that:

$$A : \vec{\mathbf{z}} \to \vec{\mathbf{z}}' = A\vec{\mathbf{z}}, \qquad (x_1^2 + y_1^2) - (x_2^2 + y_2^2) = ({x_1'}^2 + {y_1'}^2) - ({x_2'}^2 + {y_2'}^2) \tag{4.10}$$

Therefore this induced transformations of $SU(1,1)$ in $\mathbb{R}^4$ leave the $(2,2)$-pseudospheres, $(x_1^2 + y_1^2) - (x_2^2 + y_2^2) = \text{const}$, invariant. The unit pseudo sphere can be parametrized in a similar way as the group elements, we will use the following one:

$$x_1 = \cosh\theta \cos\phi\,, \qquad y_1 = \cosh\theta \sin\phi\,, \qquad x_2 = \sinh\theta \cos\eta\,, \qquad y_2 = \sinh\theta \sin\eta\,. \tag{4.11}$$

This kind of $(2,2)$-pseudospheres are, by definition, invariant under the group $SO(2,2)$. We conclude that the group $SU(1,1)$ is a subgroup of $SO(2,2)$ (or at least is homomorphic to a subgroup).

## 4.2 The Lie algebra and representations of $su(1,1)$

The matrix Lie algebra of $SU(1,1)$ is given by the matrices $X$ satisfying

$$A = e^{\epsilon X}, \quad A \in SU(1,1) \tag{4.12}$$

where $\epsilon$ is a real parameter. We say that the matrix $X$ is a generator of a one-parametric subgroup of $SU(1,1)$. Next, we will characterize all these generators.

- First, having in mind that $\det A = 1 \Rightarrow \mathrm{Tr}X = 0$.
- Second, expanding the exponential up to first order in $\epsilon$, $e^{\epsilon X} \approx I + \epsilon X$, after replacing in (4.4), we have that the vector space of the generators $X$ of $SU(1,1)$ must also satisfy the following linear restriction

$$X^\dagger G + G X = 0 \tag{4.13}$$

A basis for the vector space of generators, fulfilling these two conditions, is the following:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.14}$$

Remark that $X_1$ and $X_2$ are Hermitian but $X_3$ is anti-Hermitian. This means that the exponentials $e^{\epsilon X_1}$ and $e^{\epsilon X_2}$ are not unitary but $e^{\epsilon X_3}$ gives a unitary subgroup. The commutation rules of $su(1,1)$ in this basis is

$$[X_1, X_2] = -X_3, \quad [X_3, X_1] = X_2, \quad [X_3, X_2] = -X_1 \tag{4.15}$$

In the unitary representations we use the 'Hermitian' basis $K_j := iX_j$ and their commutation rules in this case are

$$[K_1, K_2] = -iK_3, \quad [K_3, K_1] = iK_2, \quad [K_3, K_2] = -iK_1 \tag{4.16}$$

Next, we introduce the lowering-raising basis $\{K_3, K_\pm\}$, where

$$K_\pm = K_1 \pm iK_2 \tag{4.17}$$

In a unitary representation we will have $(K_-)^\dagger = K_+$, $K_3^\dagger = K_3$. The commutation rules (4.16) in the new basis become

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3\,. \tag{4.18}$$

The Casimir operator is

$$\mathcal{C} = -K_1^2 - K_2^2 + K_3^2 = K_3(K_3 - 1) - K_+K_- = K_3(K_3 + 1) - K_-K_+ \tag{4.19}$$

There are three types of unitary representations of $SU(1,1)$ which are called [21]: principal, discrete and supplementary series. We are here interested in the discrete series. The basis of the support space of this series $T^k$, labeled by the number $k = 1/2, 1, \ldots$, with Casimir eigenvalue $k(k-1)$ is: $\{|k,m\rangle\}$, $m = 0, 1, \ldots$. The action of the operators is

$$\begin{aligned}
&\mathcal{C}|k,m\rangle = k(k-1)|k,m\rangle, && K_3|k,m\rangle = (k+m)|k,m\rangle,\\
&K_+|k,m-1\rangle = \sqrt{m(2k+m-1)}|k,m\rangle && K_-|k,m\rangle = \sqrt{m(2k+m-1)}|k,m-1\rangle
\end{aligned} \tag{4.20}$$

## 4.3 The groups $SU(1,1)$ and $SO(2,1)$

Although it is not essential in our study, we will include in this subsection the relation of the groups $SU(1,1)$ and $SO(2,1)$, which is similar to that of $SU(2)$ and $SO(3)$.

For each point $\mathbf{x} = (x_1, x_2, x_3)$ of the real vector space $\mathbb{R}^3$ we form the hermitian matrix

$$h(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \tag{4.21}$$

By means of a matrix $A \in SU(1,1)$, we transform this matrix into another one having the same form:

$$A : h(\mathbf{x}) \longrightarrow Ah(\mathbf{x})A^\dagger = h(\mathbf{x}')\,, \qquad h(\mathbf{x}') = \begin{pmatrix} x_3' & x_1' + ix_2' \\ x_1' - ix_2' & x_3' \end{pmatrix} \tag{4.22}$$

such that the determinant of $h(\mathbf{x})$ and $h(\mathbf{x}')$ are equal

$$\det h(\mathbf{x}) = \det h(\mathbf{x}') \implies -x_1^2 - x_2^2 + x_3^2 = -{x_1'}^2 - {x_2'}^2 + {x_3'}^2 \tag{4.23}$$

Therefore, for each matrix $A \in SU(1,1)$ there is a transformation $g(A) \in SO(2,1)$. It can be shown that this correspondence is two to one: $SO(2,1) \approx SU(1,1)/Z_2$. Although globally these two groups are not isomorphic, from a local point of view they are isomorphic and have the same Lie algebra: $su(1,1) \approx so(2,1)$.

## 4.4 Differential representations of $su(1,1)$

Let us start by finding the representations of the generators (4.14) of the Lie algebra $su(1,1)$ on the real 4D space $\mathbb{R}^4$. This is quite reasonable, since the generators of $su(1,1)$ are $2\times 2$ complex matrices that act on the complex plane $\mathbb{C}^2$, such plane can be seen as a real vector space of 4 dimensions (4D), and the complex linear action of the elements of $su(1,1)$ become a real linear action on $\mathbb{R}^4$.

### 4.4.1 Differential representations of $su(1,1)$ on a pseudo sphere of $\mathbb{R}^{(2,2)}$

In our case the generators are defined as follows

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.24}$$

Considering the action of $X_2$ on $\mathbb{C}^2$

$$X_2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i\,z_2 \\ -i\,z_1 \end{pmatrix}$$

In other words, taking into account the correspondence (4.8), this can be seen as

$$\tilde{X}_2\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -y_2 \\ x_2 \\ y_1 \\ -x_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

In conclusion, we have the following representation of this generator:

$$X_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Longrightarrow \tilde{X}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.25}$$

In the same way we have the representations:

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow \tilde{X}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.26}$$

$$X_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \implies \tilde{X}_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.27}$$

Besides, we can define the differential representation of these generators in the same way as it is applied to the $so(3)$ differential generators

$$\hat{X}_j = \vec{\mathbf{z}}^t\, \tilde{X}_j\, \boldsymbol{\nabla} \tag{4.28}$$

where

$$\vec{\mathbf{z}}^t = (x_1, y_1, x_2, y_2)\,, \quad \boldsymbol{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \partial_{y_1} \\ \partial_{x_2} \\ \partial_{y_2} \end{pmatrix}$$

In this way, we obtain the differential generators acting on the functions defined on $\mathbb{R}^4$:

$$\begin{aligned} \hat{X}_1 &= \tfrac{1}{2}\big(x_1\partial_{x_2} + x_2\partial_{x_1} + y_1\partial_{y_2} + y_2\partial_{y_1}\big) \\ \hat{X}_2 &= \tfrac{1}{2}\big(-x_1\partial_{y_2} + x_2\partial_{y_1} + y_1\partial_{x_2} - y_2\partial_{x_1}\big) \\ \hat{X}_3 &= \tfrac{1}{2}\big(-x_1\partial_{y_1} + x_2\partial_{y_2} + y_1\partial_{x_1} - y_2\partial_{x_1}\big) \end{aligned} \tag{4.29}$$

The next step is to restrict this linear action to the surface of the pseudo sphere by changing to the "angular coordinates" $\theta, \phi, \eta$ defined in (4.11). After a lengthy but straightforward change of variables we get the following expressions for the differential generators (we will design them by the same notation, since there will be no confusion)

$$\begin{aligned} \hat{X}_1 &= \tfrac{1}{2}\big(\tanh\theta\sin(\eta-\phi)\partial_\phi - \coth\theta\sin(\eta-\phi)\partial_\eta + \cos(\eta-\phi)\partial_\theta\big) \\ \hat{X}_2 &= \tfrac{1}{2}\big(\tanh\theta\cos(\eta-\phi)\partial_\phi - \coth\theta\cos(\eta-\phi)\partial_\eta - \sin(\eta-\phi)\partial_\theta\big) \\ \hat{X}_3 &= \tfrac{1}{2}\big(-\partial_\phi + \partial_\eta\big) \end{aligned} \tag{4.30}$$

We can check that, indeed these generators fulfill the conmutator relations given in (4.15) corresponding to the $su(1,1)$ Lie algebra.

If we use the 'Hermitian' basis this are defined as $K_j := iX_j$, and the lowering-raising basis $\{K_3, K_\pm\}$, where $K_\pm = K_1 \pm iK_2$,

$$\begin{aligned} K_+ &= e^{-i(\eta-\phi)}\tfrac{1}{2}\left(-\tanh\theta\partial\phi + \coth\theta\partial\eta + i\partial\theta\right) \\ K_- &= e^{i(\eta-\phi)}\tfrac{1}{2}\left(-\tanh\theta\partial\phi - \coth\theta\partial\eta + i\partial\theta\right) \\ K_3 &= \tfrac{i}{2}\left(-\partial_\phi + \partial_\eta\right) \end{aligned} \tag{4.31}$$

They satisfy the $su(1,1)$ commutation rules (4.18).

### 4.4.2 A second differential representations of $su(1,1)$ on the pseudo sphere

We can also generate a second representation of $su(1,1)$ in the real matrices that act on $\mathbb{R}^{(2,2)}$ in the following way. Consider the complex conjugate operator $C : z \to z^*$. Then we define the operator $T$ that applies the conjugation only to the first component of the complex two-vector,

$$T = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}, \qquad T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^* \\ z_2 \end{pmatrix}, \quad T^2 = I$$

Then, we can define the representation

$$\tilde{Y}_i := T X_i T$$

We can check that the explicit form of $Y_k$ are:

$$\tilde{Y}_1 = \frac{1}{2}\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad \tilde{Y}_2 = \frac{1}{2}\begin{pmatrix} 0 & -iC \\ -iC & 0 \end{pmatrix}, \quad \tilde{Y}_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.32}$$

The corresponding real $4 \times 4$ matrix representation on $\mathbb{R}^4$ will be:

$$\tilde{Y}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tilde{Y}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tilde{Y}_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.33}$$

At this point we have two representations of $su(1,1)$ in $\mathbb{R}^4$: $\{\tilde{X}_i\}$ and $\{\tilde{Y}_i\}$, $i = 1,2,3$. We can also check that, due to their construction, they commute:

$$[\tilde{X}_i, \tilde{Y}_j] = 0, \qquad \forall i, j\,. \tag{4.34}$$

In the same way that we did for $\tilde{X}_j$, we can define the differential representation in (4.35) now we have

$$\hat{Y}_j = \vec{\mathbf{z}}^t\, \tilde{Y}_j\, \boldsymbol{\nabla} \tag{4.35}$$

Then we get the following differential operators,

$$\begin{aligned} \hat{Y}_1 &= \tfrac{1}{2}\left(x_1\partial_{x_2} + x_2\partial_{x_1} - y_1\partial_{y_2} - y_2\partial_{y_1}\right) \\ \hat{Y}_2 &= \tfrac{1}{2}\left(-x_1\partial_{y_2} - x_2\partial_{y_1} - y_1\partial_{x_2} - y_2\partial_{x_1}\right) \\ \hat{Y}_3 &= \tfrac{1}{2}\left(x_1\partial_{y_1} + x_2\partial_{y_2} - y_1\partial_{x_1} - y_2\partial_{x_2}\right) \end{aligned} \tag{4.36}$$

In the same way we did for the $X_i$ generators we change to the variables $\theta, \phi, \eta$ of the pseudo sphere defined in (4.11) to obtain the following differential representation:

$$\begin{aligned} \hat{Y}_1 &= \tfrac{1}{2}\left(-\tanh\theta\sin(\eta+\phi)\partial_\phi - \coth\theta\sin(\eta+\phi)\partial_\eta + \cos(\eta+\phi)\partial_\theta\right) \\ \hat{Y}_2 &= \tfrac{1}{2}\left(-\tanh\theta\cos(\eta+\phi)\partial_\phi - \coth\theta\cos(\eta+\phi)\partial_\eta - \sin(\eta+\phi)\partial_\theta\right) \\ \hat{Y}_3 &= \tfrac{1}{2}\left(\partial_\phi + \partial_\eta\right) \end{aligned} \tag{4.37}$$

If we use the 'Hermitian' basis this are defined as $L_j := iY_j$, and then change to the lowering-raising basis $\{L_3, L_\pm\}$, where $L_\pm = L_1 \pm iL_2$, we finally get

$$\begin{aligned} L_+ &= e^{-i(\eta+\phi)}\tfrac{1}{2}\big(\tanh\theta\partial_\phi + \coth\theta\partial_\eta + i\partial_\theta\big) \\ L_- &= e^{i(\eta+\phi)}\tfrac{1}{2}\big(-\tanh\theta\partial_\phi - \coth\theta\partial_\eta + i\partial_\theta\big) \\ L_3 &= \tfrac{i}{2}\big(\partial_\phi + \partial_\eta\big) \end{aligned} \tag{4.38}$$

In order to match these operators with $\tilde{M}^\pm, \tilde{M}$, let us redefine:

$$\tilde{L}_+ = iL_+, \quad \tilde{L}_- = -iL_+, \quad \tilde{L}_3 = L_3$$
$$\tilde{K}_+ = iK_+, \quad \tilde{K}_- = -iK_+, \quad \tilde{K}_3 = K_3$$

These redefinitions are equivalences that do not change the commutation rules. We can check that all these operators also fulfil the commutator relations given in (4.18). In this way we are ready to continue.

Next, we compute both Casimir operators for both $su(1)$ in the differential realization just obtained in (4.31) and (4.38), according to (4.19). It happens that both Casimir operators are equal:

$$\begin{aligned} -\mathcal{C} &= -\tilde{K}_3(\tilde{K}_3-1) + \tilde{K}_+\tilde{K}_- = -\tilde{L}_3(\tilde{L}_3-1) + \tilde{L}_+\tilde{L}_- \\ &= \frac{1}{4}\Big(-\partial_\theta^2 - 2\coth(2\theta)\partial_\theta + \Big(-\frac{1}{\sinh^2\theta}\partial_\phi^2 + \frac{1}{\cosh^2\theta}\partial_\eta^2\Big)\Big) \end{aligned} \tag{4.39}$$

In order to compare these operators with the factorization operators, let us apply them to functions which are at the same time eigenfunctions of the diagonal operators $K_3$ and $L_3$:

$$\psi(\theta,\eta,\phi) = \psi_{a,b}(\theta)e^{-i(a\,\eta + b\,\phi)} \tag{4.40}$$

where $a$ and $b$ are real numbers. We have the following results:

$$\begin{aligned} \tilde{K}_\pm\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} &= \tfrac{1}{2}(\pm\partial_\theta - (a)\tanh x + (b)\coth x\,,)\psi_{a,b}(\theta)e^{-i((a\pm1)\eta+(b\pm1)\phi)} \\ \tilde{K}_3\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} &= \tfrac{1}{2}(a-b)\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} \\ \tilde{L}_\pm\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} &= \tfrac{1}{2}(\pm\partial_x - (a)\tanh x - (b)\coth x\,,)\psi_{a,b}(\theta)e^{-i((a\pm1)\,\eta+(b\pm\mp1)\,\phi)} \\ \tilde{L}_3\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} &= \tfrac{1}{2}(a+b)\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} \end{aligned} \tag{4.41}$$

When the Casimir operator act on this type of functions we find

$$\begin{aligned} &-\mathcal{C}\,\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} = \\ &\frac{1}{4}\Big(-\partial_\theta^2 - 2\coth(2\theta)\partial_\theta + \Big(\frac{b^2}{\sinh^2\theta} - \frac{a^2}{\cosh^2\theta}\Big)\Big)\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} = \\ &-k(k-1)\,\psi_{a,b}(\theta)e^{-i(a\,\eta+b\,\phi)} \end{aligned} \tag{4.42}$$

From the previous expressions we see the following similitudes of operators:

$$\begin{aligned}
&\tilde{K}_\pm \sim \tilde{P}^\pm, \quad \tilde{K}_3 \sim \tilde{P} \\
&\tilde{L}_\pm \sim \tilde{M}^\pm, \quad \tilde{L}_3 \sim \tilde{M} \\
&-4\mathcal{C} \sim H_{a,b}
\end{aligned} \tag{4.43}$$

There are yet some differences. Let us see the difference of the Casimir and the Hamiltonian of the last line above. As we saw in (2.2)

$$H_{a,b} = -\partial_x^2 - \frac{a^2 - \frac{1}{4}}{\cosh^2 x} + \frac{b^2 - \frac{1}{4}}{\sinh^2 x}, \qquad x \in (0, +\infty) \tag{4.44}$$

While the action of the operator on this kind of function gives:

$$-4\mathcal{C} = -\partial_\theta^2 - 2\coth(2\theta)\partial_\theta + \Big(\frac{b^2}{\sinh^2\theta} - \frac{a^2}{\cosh^2\theta}\Big) \tag{4.45}$$

We see that the difference is in the first order derivative in the Casimir operator, and in that there are some 1/4 coefficients. The same happens with the other operators with respect to some $\pm 1/2$ coefficients. All these details are solved by means of an equivalence transformation:

$$\psi_{a,b}(\theta) = \frac{1}{\sqrt{\sinh 2\theta}}\tilde{\psi}_{a,b}(\theta) \tag{4.46}$$

By means of this equivalence all the problems are solved and we obtain total coincidence (changing the variable $x$ by $\theta$). For instance we check that

$$\sqrt{\sinh 2\theta}\,(-4\mathcal{C})\,\frac{1}{\sqrt{\sinh 2\theta}} = H_{a,b}(\theta)\,.$$

### 4.4.3 Differential equation

In this section, the eigenfunctions are obtained by means the differential equation corresponding to the eigenvalue of the Casimir operator. We use the unitary representation of $su(1,1)$ in the discrete series where the Casimir of su(1,1) is defined as

$$C = K_1^2 + K_2^2 - K_3^2 = k(k-1), \tag{4.47}$$

where $k$ is the label of the Casimir eigenvalue and $K_1$, $K_2$, $K_3$ are defined as follows

$$K_1 = \frac{i}{2}\Big(\tanh(\theta)\sin(\eta-\phi)\partial_\phi - \coth(\theta)\sin(\eta-\phi)\partial_\eta + \cos(\eta-\phi)\partial_\theta\Big), \tag{4.48}$$

$$K_2 = \frac{i}{2}\Big(\tanh(\theta)\cos(\eta-\phi)\partial_\phi - \coth(\theta)\cos(\eta-\phi)\partial_\eta + \sin(\eta-\phi)\partial_\theta\Big), \tag{4.49}$$

$$K_3 = \frac{i}{2}\Big(-\partial_\phi + \partial_\eta\Big), \tag{4.50}$$

when these $K_i$ are sustituted in equation (4.47) we have the differential equation

$$\frac{1}{4}\Big(-\partial_\theta^2 - 2\coth(2\theta)\partial_\theta + \Big(\frac{b^2}{\sinh^2\theta} - \frac{a^2}{\cosh^2\theta}\Big)\Big)\psi(\theta) = -k(k-1)\,. \tag{4.51}$$

If we consider a change of variable $\cosh(2\theta) = t$, we can simplify the above equation as follows

$$\Big(-(t^2-1)\frac{d^2}{dt^2} - 2t\frac{d}{dt} + \frac{b^2}{2(t-1)} - \frac{a^2}{2(t+1)}\Big)\psi = -k(k-1), \tag{4.52}$$

now, in order to simplify the singularities of the equation, we propose this transformation:

$$\psi = (t+1)^\mu (t-1)^\nu f(t)\,,$$

then we can calcute $\psi'$,$\psi''$ and finally we substitute in the equation (4.52 ). After reducing this expression we obtain:

$$[-(t^2-1)\partial_t^2 + (-2\mu(t-1) - 2\nu(t+1) - 2t)\partial_t + (-\mu(\mu-1)\frac{t+1-2}{t+1} - 2\mu\nu - \nu(\nu-1)\frac{t-1+2}{t-1}$$

$$-2t\frac{\mu}{t+1} - 2t\frac{\nu}{t-1} + \frac{b^2}{2(t-1)} - \frac{a^2}{2(t+1)} + k(k-1))]\psi = 0. \tag{4.53}$$

In order to compare the (4.53) with the Jacobi equation [20] it is necessary to find the conditions where $\mu$ and $\nu$ simplify the last term. This conditions are the following:

$$\mu = \pm\frac{a}{2} \qquad \nu = \pm\frac{b}{2}. \tag{4.54}$$

We chose the case $\mu = -\frac{a}{2}$ and $\nu = \frac{b}{2}$ (in order to find physical states of the upper half sector) and replace in the equation (4.53) we obtain:

$$(1-t^2)\psi'' + [a - b - (a+b+2)t]\psi' + \big[-\frac{a}{2}\Big(\frac{a}{2}+1\Big) - \frac{b}{2}\Big(\frac{b}{2}+1\Big) - \frac{ab}{2} + k(k-1)\big]\psi = 0. \tag{4.55}$$

The standard Jacobi equation [20] is given by

$$(1-x^2)y'' + [(\beta-\alpha) - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta)y = 0, \tag{4.56}$$

then, this implies that $a = \beta$, $b = \alpha$ and propose for $k = \frac{a-b}{2} - n$ such it matches exactly with the Jacobi equation (4.56). Considering this fact we can work with the Jacobi's polynomials. Then, we can propose $k_0$ as the principal quantum number defined by $k_0 = \frac{a-b}{2}$, its meaning is that it gives the 'last sector' including the wavefunction with label $(a, b)$. Then, the possible polynomial solutions come from the previous sectors, where $k = k_0 - 1, k_0 - 2, \ldots, 1$, where $k = 1$ is the minimum value of $k$. Then the polynomials of degree $n$ are such that $k_0 - k = n$

In order to be consistent we are going to show one example for particular case of the Hamiltonian $H_{8,2}$ this means that $a = 8$, $b = 2$. This fact implies that $K_3 = 3$ and the eigenfunctions are $\psi^0_{8,2}$ corresponds to the ground state, $\psi^1_{8,2}$ corresponds to the first excited state and $\psi^2_{8,2}$ corresponds to the second excited state). Then, the n-excited state can be written as follows

$$\psi^n_{8,2} = \cosh\theta^{-\frac{8}{2}+\frac{1}{2}}\sinh\theta^{-\frac{2}{2}+\frac{1}{2}}P_n^{2,-8}(\cosh 2\theta). \tag{4.57}$$

Such that the the n-excited state is expressed as

$$\psi^n_{a,b} = \cosh\theta^{-\frac{a}{2}+\frac{1}{2}}\sinh\theta^{-\frac{b}{2}+\frac{1}{2}}P_n^{b,-a}(\cosh 2\theta). \tag{4.58}$$

However, somethings are differents like the interval of the Jacobi polynomials definition which is given in this case as $[1,\infty]$, this is just an extension of orthogonal polynomials [23]. Finally, in order to guarantee the equivalence between this two methods in this subsections some examples are presented. This results was computing with *Wolfram Mathematica*

Consider the particular values $a = 8$, $b = 2$ and $n = 3$. The ground state for this parameters is defined as

$$\psi\gamma^0_{8,2}(x) = \frac{\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.59}$$

if we apply once the $M^+$ operator into the ground state we obtain the first excited state

$$\psi\gamma^1_{8,2}(x) = \frac{\cosh(2x)\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.60}$$

applying one more time the $M^+$ operator into the first excited state

$$\psi\gamma^2_{8,2}(x) = \frac{\cosh(2x) + \cosh(4x)\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.61}$$

now, if we use the second method (Jacobi polynomials) we obtain for the ground state:

$$\psi\gamma^0_{8,2}(x) = C_0\frac{\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.62}$$

for the first excited state the expression:

$$\psi\gamma^1_{8,2}(x) = C_1\frac{\cosh(2x)\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.63}$$

and finally for the second excited state:

$$\psi\gamma^2_{8,2}(x) = C_3\frac{\cosh(2x) + \cosh(4x)\sinh x^{\frac{5}{2}}}{\cosh x^{\frac{15}{2}}}, \tag{4.64}$$

as we can notice comparing the expression has te same form except maybe for a constant, this constant is given by the normalization process.[1]

[1] $\gamma$ corresponds the third section in the graphs.

## 5 The Lie Algebra $so(2,2)$ as a Direct Sum $su(1,1) \oplus su(1,1)$

In this section, which is included for completness, we want to show that the Lie algebra associated to the operators of the Pöschl–Teller system is the special orthogonal algebra $so(2,2)$. In particular, we will recall the definition of the matrix Lie algebra $so(2,2)$ and afterwards the way to express this Lie algebra as the direct sum $su(1,1) \oplus su(1,1)$. We will also comment on the kind of representation of $so(2,2)$ associated to the Pöschl–Teller quantum system.

We have seen from previous sections that each set of generators $\langle \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \rangle$ and $\langle \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \rangle$, close an $su(1,1)$ algebra, both sets commute and they have a realization as $4 \times 4$ matrices which are generators of $so(2,2)$. All this shows that, in fact, these two sets generate the $so(2,2)$ Lie algebra and that $su(1,1) \oplus su(1,1) \approx so(2,2)$. In the following, we would like to examine this question in the opposite direction, that is, if we start from the $so(2,2)$ Lie algebra, we will get its $su(1,1) \oplus su(1,1)$ structure.

In order to define the group $SO(2,2)$ let us consider the real space $\mathbb{R}^4$ with metric $G$ of signature $(2,-2)$:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t G \mathbf{y} \tag{5.1}$$

where

$$\mathbf{x}^t = (x_1, x_2, x_3, x_4)\,, \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{5.2}$$

The matrix $G$ has the metric coefficients $g = \mathrm{diag}(1,1,-1,-1)$. The Lie group $SO(2,2)$ consists in the matrices $S$ that conserve this metric,

$$(S\mathbf{x}) \cdot (S\mathbf{y}) = \mathbf{x} \cdot \mathbf{y},\ \forall \mathbf{x}, \mathbf{y} \qquad \Longleftrightarrow \qquad S^t G S = G \qquad S \in SO(2,2) \tag{5.3}$$

and have determinant one: $|S| = 1$.

The matrix generators of this Lie group $so(2,2)$ are the $4 \times 4$ real matrices $M$ such that $e^{\epsilon M} \in SO(2,2)$, for $\epsilon \in \mathbb{R}$. Replacing this expression in (5.3) we obtain the characterization of the generators of $so(2,2)$:

$$M^t G + G M = 0 \tag{5.4}$$

In order to compute explicitly a basis let us express the $4 \times 4$ matrices in blocks of $2 \times 2$ submatrices:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad G = \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \tag{5.5}$$

Then, the condition (5.4) means that

$$A^\dagger + A = 0, \quad C^\dagger - B = 0, \quad D^\dagger + D = 0 \tag{5.6}$$

Therefore $A$ and $D$ are antisymmetric, while $C$ is the transpose of $B$. In total there are six linearly independent generators, so we choose the following basis (they are given in terms of set of matrices $E_{ij}$, defined by having all the elements equal to zero except that one in position $(i,j)$ which is equal to one),

$$\begin{aligned} &M_{12} = E_{12} - E_{21}, \qquad M_{13} = -E_{13} - E_{31}, \qquad M_{23} = -E_{23} - E_{32} \\ &M_{14} = -E_{14} - E_{41}, \qquad M_{24} = -E_{24} - E_{42}, \qquad M_{34} = -E_{34} + E_{43} \end{aligned} \tag{5.7}$$

This basis satisfy the usual commutation rules of the pseudo–orthogonal groups:

$$[M_{ij}, M_{kl}] = g_{jk}M_{il} + g_{il}M_{jk} - g_{jl}M_{ik} - g_{ik}M_{jl} \tag{5.8}$$

where the metric elements $g_{ij}$ are given by $G$.

The key point to connect the $so(2,2)$ algebra with that of $su(1,1)$ is based on the following isomorphism of Lie algebras: $su(1,1) \approx so(2,1)$. Therefore, if we restrict to the first three components of $\mathbb{R}^4$, the subalgebra generated by $\langle M_{12}, M_{13}, M_{23}\rangle$ leave invariant this subspace and the restricted metric, so they will close $so(2,1)$. We will use the notation

$$M_{12} \equiv -M_3, \quad M_{13} \equiv -M_2, \quad M_{23} \equiv M_1$$

and the three generators are referred to by $\mathbf{M}$. Their commutation rules are

$$[M_3, M_1] = M_2, \quad [M_3, M_2] = -M_1, \quad [M_1, M_2] = -M_3 \tag{5.9}$$

They are the same as those of (4.15) corresponding to $su(1,1)$. In vector notation the commutators (5.9) are expressed by

$$[\mathbf{M}, \mathbf{M}] = \mathbf{M}$$

The other three generators $M_{14}, M_{24}, M_{34}$ constitute the components of an $so(2,1)$ vector and the notation for them will be

$$M_{14} \equiv N_1, \quad M_{24} \equiv N_2, \quad M_{34} \equiv N_3$$

The three components are denoted by $\mathbf{N}$. These two properties allow us to build two $su(1,1)$ commuting Lie algebras. The commutation rules (5.8) of $so(2,2)$ can be expressed in vector notation as

$$[\mathbf{M}, \mathbf{M}] = \mathbf{M}, \qquad [\mathbf{M}, \mathbf{N}] = \mathbf{N}, \qquad [\mathbf{N}, \mathbf{N}] = \mathbf{M} \tag{5.10}$$

Next, we form two $so(2,1)$ vectors as follows:

$$\mathbf{U} = \frac{1}{2}(\mathbf{M} + \mathbf{N}), \qquad \mathbf{V} = \frac{1}{2}(\mathbf{M} - \mathbf{N}) \tag{5.11}$$

Then, making use of (5.10) we can check that these two vectors generate the direct sum we were looking for:

$$so(2,2) = \langle \mathbf{U}\rangle \oplus \langle \mathbf{V}\rangle = so(2,1) \oplus so(2,1) \approx su(1,1) \oplus su(1,1) \tag{5.12}$$

This is the structure we wanted to show. The essential point was that the generators of $so(2,2)$ are divided in the $so(2,1)$ subalgebra $\langle \mathbf{M}\rangle$ and in the $so(2,1)$ vector $\mathbf{N}$.

# 6 Conclusions

In this section the conclusion are presented. In this work we find two kinds of operators that intertwine a family of hyperbolic Pöschl-Teller Hamiltonians. We have shown the action of the intertwining operators and how these operators factorize the Pöschl-Teller Hamiltonians. Each couple of these operators close a $su(1,1)$ algebra and if we consider both operators couples they close the direct sum $su(1,1) \oplus su(1,1)$. Considering this fact, in the last section was shown how this direct sum generates the Lie Algebra of $so(2,2)$. The eigenfunctions were determined in two ways. The first was based by the fact that $M^-(P^-)$ acts as an annihilator and the subsequent application of $M^+$ (or $P^+$) on the ground state of a proper element of the hierarchy gives the $n$-excited state of the Hamiltonian. The second was considering the Casimir, changes of variables and applying physical conditions. In both cases the form of the eigenfunction was determined by Jacobi polynomials.

1. We find two kinds of operators that intertwine a family or hierarchy of hyperbolic Pöschl-Teller Hamiltonians. Both belong to the kind of "shift" operators (changing only the potential parameters).

2. We have shown the action of the intertwining operators and how these operators factorize the hypergeometric Pöschl-Teller Hamiltonian. These operators act on the eigenfunctions of a Hamiltonian relating them to the eigenfunctions of another Hamiltonian.

3. Each couple of these operators close a su(1,1) Lie algebra. If we consider both couples of operators, they close a direct sum of $su(1,1) \oplus su(1,1)$ which is isomorphic to $so(2,2)$.

4. The eigenfunctions were determined in two forms. The first way, they were obtained considering that the operators $M^-(P^-)$ act as an annihilators that define ground states and the subsequent application of appropriate operators $M^+$ (or $P^+$) on this ground state give us the $n$-excited state of the Hamiltonians. The second procedure was by considering the Casimir, changes of variables and applying physical conditions. In both cases the form of the eigenfunction was determined by Jacobi polynomials.

5. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ describing the eigenstates are somewhat special, in the sense that the interval is $x \in (1, +\infty)$ and that some of the parameters $\alpha$ or $\beta$ can be negative numbers. Anyway, we have checked that the solutions are correct.

6. We obtain the exactly the same Casimir regardless the couple of operators chosen ($M^\pm$ or $P^\pm$). Therefore the representation of both $su(1,1)$ have the same label $k$ of the Casimir and the representation of $su(1,1) \oplus su(1,1)$ can be written as the tensor product $k \otimes k$ of the same representation.

7. The Schrödinger equation for hyperbolic Pöschl–Teller potential was obtained under another perspective based on the representations of the algebra $su(1,1) \oplus su(1,1)$. This representation uses a kind of coordinates $\theta, \phi, \eta$ which are not the usual spherical coordinates in the pseudo sphere of $\mathbb{R}^{2,2}$.

# 7 Appendix

## 7.1 A.1 Conmutator $[M, M^{\pm}]$

Considering a proof function we can write

$$[M, M^{+}]\psi_{a,b} = (M_{a+1,b+1}M^{+}_{a+1,b+1} - M^{+}_{a+1,b+1}M_{a,b})\psi_{a,b} \tag{7.1}$$

$$= \frac{1}{2}(a + b + 2 - a - b)M^{+}_{a+1,b+1}\psi_{a,b}$$

$$\therefore [M, M^{+}] = M^{+}$$

$$[M, M^{-}]\psi_{a,b} = (M_{a-1,b-1}M^{-}_{a-1,b-1} - M^{-}_{a-1,b-1}M_{a,b})\psi_{a,b} \tag{7.2}$$

$$= \frac{1}{2}(a + b + 2 - a - b)M^{-}_{a+1,b+1}\psi_{a,b}$$

$$\therefore [M, M^{-}] = -M^{-}$$

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