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On real analogues of the Poincaré series

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Abstract
There exist several equivalent equations for the Poincaré series of a collection of valuations on the ring of germs of functions on a complex analytic variety. We give definitions of the Poincaré series of a collection of valuations in the real setting (i.e., on the ring of germs of functions on a real analytic variety), compute them for the case of one curve valuation on the plane and discuss some of their properties.

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1 | INTRODUCTION

Let $\mathbb{K}$ be either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers, let $(V, 0)$ be a germ of an analytic variety over $\mathbb{K}$, and let $\mathcal{E}_{V,0}$ be the ring of germs of functions on V (if $\mathbb{K} = \mathbb{C}$, $\mathcal{E}_{V,0}$ is usually denoted by $\mathcal{O}_{V,0}$). A function $\nu : \mathcal{E}_{V,0} \to \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ is called an *order function* on $\mathcal{E}_{V,0}$ if

(1) $\nu(0) = +\infty$;
(2) $\nu(\lambda f) = \nu(f)$ for $f \in \mathcal{E}_{V,0}$, $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$;
(3) $\nu(f_1 + f_2) \geqslant \min\{\nu(f_1), \nu(f_2)\}$ for $f_1, f_2 \in \mathcal{E}_{V,0}$.

If besides that, one has

$$\nu(f_1 f_2) = \nu(f_1) + \nu(f_2),$$

the function ν is called a *valuation*. (Sometimes for a valuation one demand that $\nu(f) \neq +\infty$ for $f \neq 0$. In this case, a function described above is called a pre-valuation.)

Let $\{\nu_i; i = 1, 2, \dots, r\}$, be a collection of order functions on $\mathcal{E}_{V,0}$. The Poincaré series of the collection $\{\nu_i\}$ was defined in [6]. For $\underline{v} = (v_1, \dots, v_r)$ and $\underline{v}' = (v'_1, \dots, v'_r)$ from $\mathbb{Z}^r$, one says that $\underline{v} \geqslant \underline{v}'$ if $v_i \geqslant v'_i$ for all i. For $f \in \mathcal{E}_{V,0}$, let $\underline{\nu}(f) := (\nu_1(f), \dots, \nu_r(f))$. For $\underline{v} \in \mathbb{Z}^r$, let $J(\underline{v}) := \{f \in \mathcal{E}_{V,0} : \underline{\nu}(f) \geqslant \underline{v}\}$. Let

$$\mathcal{L}_{\{\nu_i\}}(\underline{t}) := \sum_{\underline{v} \in \mathbb{Z}^r} \dim \left(J(\underline{v})/J(\underline{v} + \underline{1})\right) \cdot \underline{t}^{\underline{v}},$$

where $\underline{t} = (t_1, \dots, t_r)$, $\underline{1} = (1, \dots, 1)$. (Pay attention that the sum is over all $\underline{v} \in \mathbb{Z}^r$ and therefore the series $\mathcal{L}_{\{\nu_i\}}(\underline{t})$ contains summands with negative exponents (at least for $r > 1$).) The *Poincaré series* of the collection $\{\nu_i\}$ is defined by

$$P_{\{\nu_i\}}(\underline{t}) = \frac{\mathcal{L}_{\{\nu_i\}}(\underline{t}) \cdot \prod_{i=1}^{r}(t_i - 1)}{(t_1 \cdot \ldots \cdot t_r - 1)}\,. \tag{1}$$

For one curve valuation $\nu = \nu_C$ on a complex analytic variety $(V, 0)$ ($\mathbb{K} = \mathbb{C}$, $(C, 0)$ is a curve germ on $(V, 0)$), this definition gives the following. Let $S_C \subset \mathbb{Z}_{\geqslant 0}$ be the set (a semigroup) of values of the valuation ν. Then

$$P_\nu(t) = \sum_{v \in S_C} t^v. \tag{2}$$

In [2], it was shown that, for $\mathbb{K} = \mathbb{C}$ and for a collection $\{\nu_i\}$ of curve valuations on the algebra $\mathcal{O}_{\mathbb{C}^2,0}$ of function germs in two variables defined by the irreducible components $(C_i, 0)$ of a plane curve germ $(C, 0)$, the Poincaré series $P_{\{\nu_i\}}(\underline{t})$ coincides with the Alexander polynomial (in several variables) of the link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ (for the number r of valuations $\geqslant 2$; for $r = 1$, $P_\nu(t)$ is equal to the Alexander polynomial divided by $1 - t$). In particular, for an embedded resolution $\pi : (X, D) \to (\mathbb{C}^2, 0)$ of the curve C, one has an "A'Campo type" representation

$$P_{\{\nu_i\}}(\underline{t}) = \prod_\sigma (1 - \underline{t}^{\,\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}, \tag{3}$$

where $\underline{m}_\sigma \in \mathbb{Z}^r_{>0}$ are some multiplicities defined for the components E_σ of the exceptional divisor $D = \pi^{-1}(0)$ and $\overset{\circ}{E}_\sigma$ is the "smooth part" of the irreducible component E_σ of D, that is E_σ itself minus the intersection points with all other components of the total transform $\pi^{-1}(C)$ of the curve C.

The computation of the Poincaré series in [2] was based on its relation with the so-called extended semigroup of a collection of (curve) valuations. For a collection $\{\nu_i\}$ of valuations on the algebra $\mathcal{E}_{V,0}$ its *extended semigroup* $\widehat{S}_{\{\nu_i\}}$ is the union over $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$ of the *fibers*

$$F_{\underline{v}} := \left(J(\underline{v})/J(\underline{v} + \underline{1})\right) \setminus \bigcup_{I \subset I_0,\, I \neq \emptyset} \left(J(\underline{v} + \underline{1}_I)/J(\underline{v} + \underline{1})\right),$$

where $I_0 = \{1, \dots, r\}$, the ith component of $\underline{1}_I \in \mathbb{Z}^r$ is equal to 1 if $i \in I$ and to 0 otherwise. The semigroup structure on $\widehat{S}_{\{\nu_i\}}$ is induced by the product of function germs. In [2], it was shown that (for $\mathbb{K} = \mathbb{C}$) the Poincaré series of the collection $\{\nu_i\}$ is given by the equation

$$P_{\{\nu_i\}}(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^r_{\geqslant 0}} \chi(\mathbb{P}F_{\underline{v}}) \cdot \underline{t}^{\underline{v}}, \tag{4}$$

where $\mathbb{P}F_{\underline{v}}$ is the projectivization $F_{\underline{v}}/\mathbb{C}^*$ of the fiber of the extended semigroup. (Here, we always have in mind the *additive Euler characteristic* defined as the alternating sum of the ranks of the cohomology groups with compact support. In fact, for a space being a complex quasiprojective variety, in particular for the projectivizations of the fibers of the extended semigroup in the complex case: $\mathbb{K} = \mathbb{C}$, this Euler characteristic coincides with the "usual" one.)

In [10] (also for $\mathbb{K} = \mathbb{C}$), the Poincaré series $P_{\{\nu_i\}}(\underline{t})$ was expressed in terms of the integral with respect to the Euler characteristic (appropriately defined) over the projectivization of ring $\mathcal{O}_{V,0}$:

$$P_{\{\nu_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{\nu}(f)} d\chi\,. \tag{5}$$

This equation led to a new, relatively simple computation of the Poincaré series in a number of situations.

In the complex setting, all three equations of the Poincaré series: (1), (4), and (5) (and Equation 2 for the case of one curve valuation) are equivalent to each other. For collections of curve or divisorial valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ the Poincaré series has a cyclotomic form, that is, is product/ratio of powers of binomials of the form $(1 - \underline{t}^{\underline{m}})$ like in (3). Here, we give analogues of definitions of the Poincaré series of a collection of valuations (or of order functions) in the real setting based on these equations. The definitions appear to be different. For the case of one curve valuation on the plane, we compute three of these Poincaré series. (All three of them are cyclotomic: see Theorems 4 and 8.)

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2 | ANALOGUES OF THE POINCARÉ SERIES IN THE REAL SETTING

Let $(V, 0)$ be a germ of a real analytic variety, let $(V_{\mathbb{C}}, 0)$ be its complexification, $\mathcal{E}_{V,0}$ is the ring of germs of real analytic functions on $(V, 0)$, $\mathcal{O}_{V_{\mathbb{C}},0}$ is the ring of germs of complex analytic functions on $(V_{\mathbb{C}}, 0)$. ($\mathcal{E}_{V,0}$ is a subspace of $\mathcal{O}_{V_{\mathbb{C}},0}$.) Let $\{\nu_i : i = 1, \dots, r\}$, be a collection of valuations on the ring $\mathcal{O}_{V_{\mathbb{C}},0}$. The valuations ν_i define valuations (denoted in the same way) on the ring $\mathcal{E}_{V,0}$. (Each valuation on $\mathcal{E}_{V,0}$ is the restriction of a valuation on $\mathcal{O}_{V_{\mathbb{C}},0}$: see, e.g., [11, chapter 4, Theorem 1].) Taking into account Equations (1), (4), and (5) for the Poincaré series in the complex setting, one can consider some (different) analogues of it in the described situation.

(1) Let $P_{\{\nu_i\}}(\underline{t})$ be the *classical Poincaré series* defined by Equation (1), where $J(\underline{v}) = \{f \in \mathcal{E}_{V,0} : \underline{\nu}(f) \geqslant \underline{v}\}$. The space $J(\underline{v})/J(\underline{v} + \underline{1})$ is a (real) subspace of the complex vector space $J_{\mathbb{C}}(\underline{v})/J_{\mathbb{C}}(\underline{v} + \underline{1})$, where $J_{\mathbb{C}}(\underline{v}) = \{f \in \mathcal{O}_{V_{\mathbb{C}},0} : \underline{\nu}(f) \geqslant \underline{v}\}$. Therefore $J(\underline{v})/J(\underline{v} + \underline{1})$ is the direct sum of a complex vector subspace of $J_{\mathbb{C}}(\underline{v})/J_{\mathbb{C}}(\underline{v} + \underline{1})$ and of a "purely real" one. (By a purely real subspace A of a complex vector space B we mean a (real) subspace such that $A \cap iA = \{0\}$ ($i = \sqrt{-1}$).) In the case of one curve valuation ν on $\mathcal{O}_{V_{\mathbb{C}},0}$ all the coefficients of the series $P_\nu(t)$ are equal to 0, 1, or 2 (if the subspace $J(\underline{v})/J(\underline{v} + \underline{1})$ is zero-dimensional, one-dimensional

over $\mathbb{R}$, or one-dimensional over $\mathbb{C}$, respectively). This is a consequence of the following reasoning. Let the curve valuation ν be defined by a curve germ $(C,0) \subset (V_{\mathbb{C}},0)$ which is the image of a map $\varphi : (\mathbb{C},0) \to (V_{\mathbb{C}},0)$ (an uniformization of $(C,0)$). For $f \in \mathcal{O}_{V_{\mathbb{C}},0}$ the value $\nu(f)$ is the degree of the leading term in the power series decomposition $f(\varphi(\tau)) = a(f)\tau^{\nu(f)} + \ terms\ of\ higher\ degree$, $a(f) \neq 0$. (If $f\circ\varphi \equiv 0, \nu(f) := +\infty$.) If, for some function germs f_i with $\nu(f_i) = v$, the linear combination $\sum \lambda_i a(f_i)$ with real λ_i is equal to zero, one has $\nu(\sum \lambda_i f_i)$ is greater than v. Therefore, $J(v)/J(v+1)$ can be identified with a real subspace of the complex line $\mathbb{C}$, what implies the statement.

(2) Let the *real Poincaré series*$P^{\mathbb{R}}_{\{\nu_i\}}(\underline{t})$ be defined as the integral with respect to the Euler characteristic of $\underline{t}^{\underline{v}}$ over the projectivization (i.e., the quotient by $\mathbb{R}^*$) of the extended semigroup: Equation (4). In the case of one curve valuation on $\mathcal{O}_{V_{\mathbb{C}},0}$ one has $P_\nu(t) = \sum_{v=0}^{\infty} a_v t^v$ with $a_v = 0, 1$ or 2. This gives $P^{\mathbb{R}}_\nu(t) = \sum_{v=0}^{\infty} \chi(\mathbb{RP}^{a_v-1})t^v$. For $a_v = 0,1,2$, $\chi(\mathbb{RP}^{a_v-1})$ is equal to 0,1,0, respectively. Therefore, the series $P^{\mathbb{R}}_\nu(\underline{t})$ is obtained from $P_\nu(t)$ by substituting all the monomials with the coefficient 2 by 0 (equal to the Euler characteristic of $\mathbb{RP}^1$) and all the coefficients of $P^{\mathbb{R}}_\nu(t)$ are equal to 0 or 1.

(3) One has an analogous version defined as the integral with respect to the Euler characteristic (appropriately defined) of $\underline{t}^{\underline{\nu}(f)}$ over the projectivization of the algebra $\mathcal{E}_{V,0}$: Equation (5). One can give an equation for it in the spirit of [3, Theorem 1], however the right-hand side of it seems to be not really computable. (At least even in the simplest cases the result looks very involved.) Therefore, we shall not discuss it below (in particular, for one curve valuation on $\mathcal{O}_{\mathbb{C}^2,0}$).

(4) For the case of one valuation, one has the *semigroup Poincaré series* $P^S_\nu(t)$ defined as the generating series of the semigroup S_ν of values of ν on $\mathcal{E}_{V,0}$ (an analogue of Equation 2):

$$P^S_\nu(t) = \sum_{v\in S_\nu} t^v.$$

All the coefficients of $P^S_\nu(t)$ are equal to 0 or 1. (In the case of one curve valuation on $\mathcal{O}_{V_{\mathbb{C}},0}$, it can be obtained from $P_\nu(t)$ substituting all the monomials of the form $2t^v$ by the monomials t^v.) A reasonable analogue of this Poincaré series for the case of several valuations is not clear.

Example 1. Let the plane curve $(C,0) \subset (\mathbb{C}^2,0)$ be given by the parameterization $x = t^4$, $y = \alpha t^6 + t^7$ with a generic complex α. The monomials 1, x, y, x^2, xy, and x^2y give the initial terms $1 = t^0$, t^4, αt^6, t^8, αt^{10}, and αt^{14}, respectively. Therefore $\dim(J(v)/J(v+1))$ is equal to 1 for $v = 0,4,6,8,10,14$. There are two monomials that give the initial terms of degree 12: x^3 (the initial term t^{12}) and y^2 (the initial term $\alpha^2 t^{12}$). These initial terms are linear independent over $\mathbb{R}$, therefore $\dim(J(12)/J(13)) = 2$ and $J(12)/J(13) \cong \mathbb{C}$. The same holds for all even $v \geqslant 16$: $\dim(J(v)/J(v+1)) = 2, J(v)/J(v+1) \cong \mathbb{C}$. There are three monomials that give the initial terms of degree 24: x^6 (the initial term t^{24}), x^3y^2 (the initial term $\alpha^2 t^{24}$), and y^4 (the initial term $\alpha^4 t^{24}$). There is only one (up to proportionality) nontrivial linear combination of these monomials whose term of degree 24 vanishes: $y^4 - (\alpha^2 + \overline{\alpha}^2)x^3y^2 + \alpha^2\overline{\alpha}^2 x^6$. The initial term of this linear combination is a multiple of t^{25}. Therefore, $\dim(J(25)/J(26)) = 1$. The same ($\dim(J(v)/J(v+1)) = 1$) holds for $v = 29,31,33,35,39$. The monomials that give pairs of linear independent (over $\mathbb{R}$) initial terms with $v = 12,16,18,\ldots$ multiplied by the function that represents $v = 25$ (the combination described above) give pairs of linear independent initial terms with $v = 37$ and all odd $v \geqslant 41$. This gives the data for computing the Poincaré series $P_\nu(t)$, $P^{\mathbb{R}}_\nu(t)$, and $P^S_\nu(t)$. (One can easily verify that the results are given by Theorems 4 and 8.)

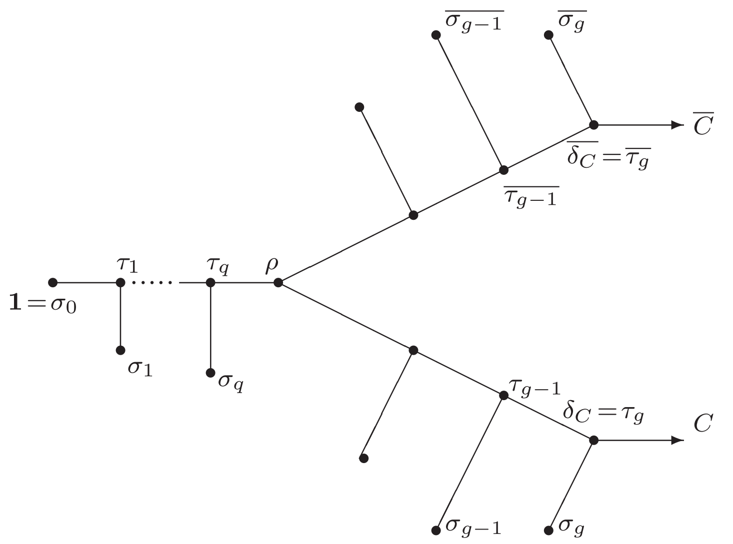

FIGURE 1 The minimal real resolution graph Γ of the valuation $\{\nu_C\}$.

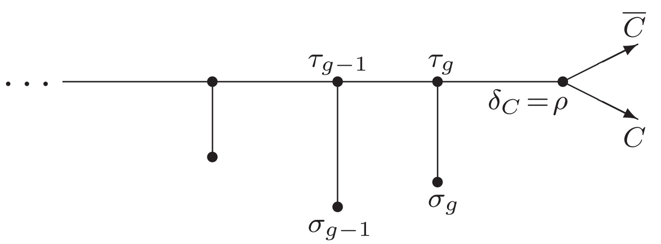

FIGURE 2 The case when the resolutions of the curves C and $\overline{C}$ split after each of the curves is resolved.

3 | THE POINCARÉ SERIES FOR ONE CURVE VALUATION

Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an irreducible plane curve germ (where we regard the complex plane $\mathbb{C}^2$ as the complexification of the real plane $\mathbb{R}^2$) and let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be the minimal *real resolution* of the curve C (or of the valuation ν_C). This means that it is the minimal resolution of the curve $C \cup \overline{C}$, where $\overline{C}$ is the complex conjugate of the curve C. This resolution can be obtained by a sequence of blow-ups either at real points or at pairs of complex conjugate points.

The (dual) resolution graph of π looks, in general, like on Figure 1. The vertex δ_C corresponds to the component intersecting the strict transform of the curve C, the vertices σ_i and $\overline{\sigma_i}$, $i = 0, 1, \ldots, g$, are *the dead ends* of the graph, the vertices τ_i and $\overline{\tau_i}$, $i = 1, 2, \ldots, g$, are *the rupture points*, the vertex ρ is *the splitting point* between the resolutions of C and of $\overline{C}$. The vertex ρ (the splitting point) may coincide with one of the rupture points τ_q, $1 \leqslant q \leqslant g$, or with the initial point σ_0. There are two options for the vertex δ_C: either it coincides with the rupture point τ_g (as on Figure 1; this happens if the resolutions of C and $\overline{C}$ split not later than each of these curves is resolved), or with the splitting point ρ if, at the moment when each of the curves is resolved, their resolutions do not split yet. In the latter case, the end of the resolution graph Γ looks like on Figure 2.

The complex conjugation acts on the resolution graph Γ (keeping fixed the part before the splitting point and exchanging the parts after it). The quotient of Γ by the complex conjugation looks

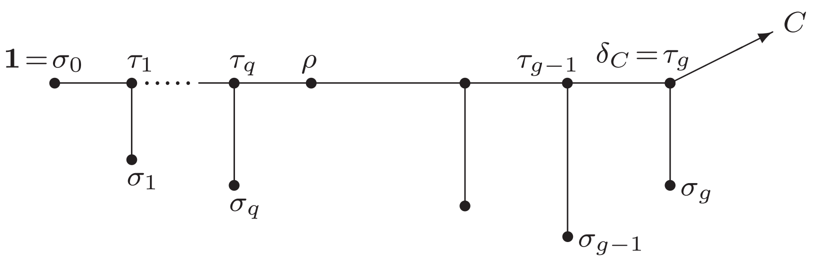

FIGURE 3 The quotient of the resolution graph Γ by the complex conjugation.

like on Figure 3 and is a resolution graph of the curve C (the minimal one if $\delta_C = \tau_g$). (If $\delta_C \neq \tau_g$, the end of this graph looks like on Figure 2.)

Let the exceptional divisor D of the resolution π be the union of its irreducible components E_σ, $\sigma \in \Gamma$. (Each E_σ is isomorphic to the complex projective line.) One has the natural involution of complex conjugation on Γ. Let $(E_\sigma \circ E_\delta)$ be the intersection matrix of the components of the exceptional divisor D (that is σ and δ run over all the vertices of the graph Γ). (The self-intersection number $E_\sigma \circ E_\sigma$ is a negative integer and, for $\delta \neq \sigma$, $E_\sigma \circ E_\delta$ is either 1 (if E_σ and E_δ intersect) or 0 otherwise.) Let $(m_{\sigma\delta}) := -(E_\sigma \circ E_\delta)^{-1}$. The entries $m_{\sigma\delta}$ have the following meaning. Let γ_σ be the germ of a smooth curve on the surface (X, D) of the resolution π intersecting the exceptional divisor D transversally at a smooth point of the component E_σ (that is not at an intersection point with another component). Let the curve $\widehat{\gamma}_\sigma := \pi(\gamma_\sigma) \subset (\mathbb{C}^2, 0)$ be given by an equation $g_\sigma = 0$ ($g_\sigma \in \mathcal{O}_{\mathbb{C}^2,0}$). (The plane curve $\widehat{\gamma}_\sigma$ or the function germ g_σ is called *a curvette* at the component E_σ.) Then $m_{\sigma\delta}$ is the multiplicity of the lifting $g_\sigma \circ \pi$ of the function g_σ along the component E_δ. In particular, $m_{\sigma\delta}$ are positive integers (see, e.g., [13]). Let, as above, the strict transform of the curve C intersect the exceptional divisor D at a point of the component E_{δ_C} and let $m_\sigma := m_{\sigma\delta_C}$.

Let $M_\sigma \in \mathbb{Z}_{>0}$ be defined in the following way. If $E_{\overline{\sigma}} = E_\sigma$, one puts $M_\sigma = m_\sigma$; if $E_{\overline{\sigma}} \neq E_\sigma$, one puts $M_\sigma := m_\sigma + m_{\overline{\sigma}}$.

Let S_C be the usual semigroup of the complex curve C, that is, $S_C = \{\nu_C(f) : f \in \mathcal{O}_{\mathbb{C}^2,0}\}$. The set of multiplicities $\{m_{\sigma_0}, \dots, m_{\sigma_g}\}$ is the minimal set of generators of S_C. Let $e_i := \gcd\{m_{\sigma_0}, \dots, m_{\sigma_i}\}$ for $i = 0, 1, \dots, g$ and let $N_i = e_{i-1}/e_i$ for $i = 1, 2, \dots, g$. (One has $m_{\tau_i} = N_i m_{\sigma_i}$ for $i = 1, 2, \dots, g$: see, e.g., [13].)

For $\delta \in \Gamma$, let $\pi_\delta : (X_\delta, D_\delta) \to (\mathbb{C}^2, 0)$ be the minimal modification of $(\mathbb{C}^2, 0)$ such that $E_\delta \subset D_\delta$. In particular, E_δ is the last exceptional component appearing in X_δ and is produced by blow-up at a point p_δ of a previous component. For $f \in \mathcal{O}_{\mathbb{C}^2,0}$, we will denote by $e_\delta(f)$ the multiplicity of the strict transform of the curve $\{f = 0\}$ at the point p_δ. Notice that $e_\delta(f)$ coincides with the intersection multiplicity of the strict transform of $\{f = 0\}$ and E_δ in the surface X_δ.

Lemma 2. *The integers $M_{\sigma_0}, M_{\sigma_1}, \dots, M_{\sigma_g}$ generate the semigroup $S_C^{\mathbb{R}}$ of values of the valuation ν_C on $\mathcal{E}_{\mathbb{R}^2,0}$. One has $(N_i - 1)M_{\sigma_i} \notin \langle M_{\sigma_0}, \dots, M_{\sigma_{i-1}} \rangle$, $N_i M_{\sigma_i} \in \langle M_{\sigma_0}, \dots, M_{\sigma_{i-1}} \rangle$ and $N_i M_{\sigma_i} < M_{\sigma_{i+1}}$. (In particular, $M_{\sigma_0}, M_{\sigma_1}, \dots, M_{\sigma_g}$ is the minimal set of generators of the semigroup $S_C^{\mathbb{R}}$.)*

Proof. Let $h \in \mathcal{E}_{\mathbb{R}^2,0}$ be irreducible, let $H = \{h = 0\}$ be the curve defined by h and let $\widetilde{H}$ denote the strict transform of H on X. Making additional blow-ups at intersection points of pairs of exceptional components one can assume that $\widetilde{H}$ intersects D at a smooth point of the component $E_\delta \subset D$. Along the proof, we will use known results relating the position of δ in the dual graph and the corresponding value $\nu_C(h)$: see, for example, [7, 8, 13].

Let q be the maximal index such that τ_q precedes (or is equal to) ρ. (If such q in between 1 and g does not exist (i.e., the resolutions of C and $\overline{C}$ split before τ_1), we put $q = 0$.) If δ precedes E_ρ, that is, $E_\delta \subset D_\rho$, then $\nu_C(h) \in \langle m_{\sigma_0}, \dots, m_{\sigma_q} \rangle$ and so $\nu_C(h) \in \langle M_{\sigma_0}, \dots, M_{\sigma_q} \rangle$. In particular, $m_\rho \in \langle M_{\sigma_0}, \dots, M_{\sigma_q} \rangle$ because $m_\rho = \nu_C(\varphi_\rho)$, where $\{\varphi_\rho = 0\}$ is a curvette at the divisor E_ρ.

Now, let us assume that δ is after ρ. In this case h is the product of two irreducible complex conjugate functions, $h = f \cdot \overline{f}$, equivalently $H = F \cup \overline{F}$ with $F = \{f = 0\}$ being a complex irreducible curve germ. As earlier, denote by E_δ the (unique) divisor on the minimal resolution of C intersecting the strict transform $\widetilde{F}$ of F, so $E_{\overline{\delta}}$ is the corresponding component for the conjugate $\overline{F}$ of F. The splitting point of $\overline{\delta}$ and C is ρ and therefore $\nu_C(\overline{f}) = e_\rho(f) m_\rho$. Let $[\alpha, \beta]$ denote the geodesic in Γ between the vertices α and β.

If $\delta \in [\sigma_i, \tau_i]$ for some $i \geqslant q+1$, then $\nu_C(f) = k m_{\sigma_i}$ for some positive integer k. For the conjugate one has that $\overline{\delta} \in [\overline{\sigma}_i, \overline{\tau}_i]$ and so $\nu_C(\overline{f}) = \nu_{\overline{C}}(f) = k m_{\overline{\sigma}_i}$. Thus, in this case one has that $\nu_C(h) = \nu_C(f) + \nu_C(\overline{f}) = k(m_{\sigma_i} + m_{\overline{\sigma}_i}) = k M_{\sigma_i}$.

In particular for m_{τ_i}, it is known that $m_{\tau_i} = N_i m_{\sigma_i}$ and so $M_{\tau_i} = N_i M_{\sigma_i}$. Moreover, if $\{\varphi = 0\}$ is a curvette at E_{σ_i} there is $e_{\sigma_i}(\varphi) = e_{\tau_{i-1}}(\varphi) = 1$ and therefore

$$e_\rho(\varphi) m_\rho = m_\rho e_{\tau_q}(\varphi) = m_\rho \frac{e_{\tau_q}(\varphi)}{e_{\tau_{q+1}}(\varphi)} \cdot \dots \cdot \frac{e_{\tau_{i-2}}(\varphi)}{e_{\tau_{i-1}}(\phi)} e_{\tau_{i-1}}(\varphi) = N_{q+1} \cdot \dots \cdot N_{i-1} m_\rho \,. \tag{6}$$

The last equality is due to the fact that $e_{\tau_{j-1}}(\varphi)/e_{\tau_j}(\varphi) = e_{j-1}/e_j$ for $j \leqslant i-1$. Thus one has $M_{\sigma_{q+1}} = m_{\sigma_{q+1}} + m_\rho$ and

$$M_{\sigma_i} = m_{\sigma_i} + m_{\overline{\sigma}_i} = m_{\sigma_i} + N_{q+1} \cdot \dots \cdot N_{i-1} m_\rho \quad \text{for } i > q+1 \,. \tag{7}$$

If $\delta \in [\rho, \tau_{q+1}]$, one has $\nu_C(f) \in \langle m_{\sigma_0}, \dots, m_{\sigma_q} \rangle$. On the other hand, $\overline{\delta} \in [\rho, \overline{\tau}_{q+1}]$ and therefore $\nu_C(\overline{f}) = e_\rho(f) m_\rho$. Thus, $\nu_C(h) = \nu_C(f) + \nu_C(\overline{f}) \in \langle M_{\sigma_0}, \dots, M_{\sigma_q} \rangle$.

Finally, let us assume that $\delta \in [\tau_i, \tau_{i+1}]$ for some $i \geqslant q+1$. In this case $\nu_C(f) \in \langle m_{\sigma_0}, \dots, m_{\sigma_i} \rangle$ and so $\nu_C(f) = \sum_{j=0}^{i} k_j m_{\sigma_j}$ with $k_0 \geqslant 0$ and $0 \leqslant k_j < N_j$ for $j = 1, \dots, i$; moreover the integers k_j are uniquely determined by these conditions. On the other hand, $\overline{\delta} \in [\overline{\tau}_i, \overline{\tau}_{i+1}]$ and then $e_{\overline{\tau}_i}(\overline{f}) = e_{\tau_i}(f) \geqslant 1$. By Equation (6), one has

$$e_\rho(f) = e_{\tau_q}(f) = N_{q+1} \cdot \dots \cdot N_i e_{\tau_i}(f) \geqslant N_{q+1} \cdot \dots \cdot N_i$$

and therefore $\nu_C(\overline{f}) = N_{q+1} \cdot \dots \cdot N_i e_{\tau_i}(f) m_\rho \geqslant N_{q+1} \cdot \dots \cdot N_i m_\rho$. By Equation (7), one has

$$\sum_{j=q+1}^{i} k_j m_{\overline{\sigma}_j} = \sum_{j=q+1}^{i} k_j N_{q+1} \cdot \dots \cdot N_{j-1} m_\rho \leqslant m_\rho \sum_{j=q+1}^{i} N_{q+1} \cdot \dots \cdot N_{j-1}(N_j - 1)$$

$$= m_\rho (N_{q+1} \cdot \dots \cdot N_i - 1) < (N_{q+1} \cdot \dots \cdot N_i) m_\rho \leqslant \nu_C(\overline{f}) \,.$$

As a consequence, for some integer $a > 0$, one has

$$\nu_C(h) = \nu_C(f) + \nu_C(\bar{f}) = \sum_0^i k_j m_{\sigma_j} + e_\rho(f) m_\rho$$

$$= \sum_0^q k_j m_{\sigma_j} + \sum_{q+1}^i k_j (m_{\sigma_j} + m_{\bar{\sigma}_j}) + a m_\rho$$

$$= \sum_0^i k_j M_{\sigma_j} + a m_\rho \in \langle M_{\sigma_0}, \dots, M_{\sigma_i} \rangle$$

and the first part of the statement is proved.

The remaining statements of lemma are trivial consequences of the same properties for the set of multiplicities $m_{\sigma_0}, \dots, m_{\sigma_g}$ (the minimal set of generators of the semigroup $S_{\mathbb{C}}$), taking into account that $e_i = \gcd\{M_{\sigma_0}, \dots, M_{\sigma_i}\}$ for $0 \leqslant i \leqslant g$. □

Remark 3. Let $S \subset \mathbb{Z}_{\geqslant 0}$ be a numerical semigroup (i.e., a subsemigroup of $(\mathbb{Z}_{\geqslant 0}, +)$ such that $\#(\mathbb{Z}_{\geqslant 0} \setminus S) < \infty$) and let $\{b_0 < b_1 < \cdots < b_g\}$ be its minimal set of generators. Let $a_i = \gcd(b_0, \dots, b_i)$, $0 \leqslant i \leqslant g$ and $N_i = a_i / a_{i-1}$ for $i = 1, \dots, g$. It is known (see [1] or [12]) that S is the semigroup of values a complex plane branch C' if and only if $N_i b_i \in \langle b_0, \dots, b_{i-1} \rangle$ and $N_i b_i < b_{i+1}$ for $i = 1, \dots, g$. Notice that C' is unique up to topological equivalence as the semigroup is a complete invariant of equisingularity. Thus, Lemma 2 implies that the "real" semigroup $S_C^{\mathbb{R}}$ coincides with the usual semigroup $S_{C'}$ of another plane curve singularity $(C', 0)$. Examples: for C given by $x = t^4$, $y = \alpha t^4 + t^6 + t^7$ with a generic complex α, as the curve C' one can take $x = t^4$, $y = t^{10} + t^{11}$; for C given by $x = t^4$, $y = \alpha t^6 + t^7$ with a generic complex α, as the curve C' one can take $x = t^4$, $y = t^6 + t^{19}$; for C given by $x = t^4$, $y = t^6 + \alpha t^7$ with a generic complex α, as the curve C' one can take the initial curve: $x = t^4$, $y = t^6 + t^7$. In general, if one defines $\beta_0 := M_{\sigma_0}$, $\beta_1 := M_{\sigma_1}$ and $\beta_{i+1} := M_{\sigma_{i+1}} - N_i M_{\sigma_i}$ for $i \geqslant 1$, the semigroup $S_C^{\mathbb{R}}$ is the semigroup of values of the complex branch defined among other curves by $x = t^{\beta_0}, y = \sum_{i \geqslant 1} t^{\beta_i}$.

Theorem 4. *One has*

$$P^S_{\nu_C}(t) = \frac{\prod_{i=1}^g (1 - t^{M_{\tau_i}})}{\prod_{i=0}^g (1 - t^{M_{\sigma_i}})}.$$

Proof. Lemma 2 implies that any element of the semigroup $S_C^{\mathbb{R}}$ can be in a unique way represented as $k_0 M_{\sigma_0} + k_1 M_{\sigma_1} + \cdots + k_g M_{\sigma_g}$ with $k_i \in \mathbb{Z}_{\geqslant 0}$ for $i = 0, \dots, g$ and $k_i < N_i$ for $i \geqslant 1$. This yields the statement (as $M_{\tau_i} = N_i M_{\sigma_i}$). □

Proposition 5. *Let an element* $a \in S_C^{\mathbb{R}}$ *be less than* $M_\rho = m_\rho$, *then one has* $\dim(J(a)/J(a+1)) = 1$. *Moreover, one has* $\dim(J(M_\rho)/J(M_\rho + 1)) = 2$ *(and therefore* $J(M_\rho)/J(M_\rho + 1) \cong \mathbb{C}$*).*

Proof. Let $a = \nu_C(f_1) = \nu_C(f_2)$ with $f_i \in \mathcal{E}_{\mathbb{R}^2,0}$. As $a < m_\rho$, the strict transforms of the curves $\{f_i = 0\}$ intersect the exceptional divisor D only at components preceding E_ρ. One has $\nu_C(f_1) = \nu_C(f_2) < m_\rho$. The multiplicities of the liftings $f_1 \circ \pi$ and $f_2 \circ \pi$ of the germs f_1 and f_2 along the components E_σ with $\sigma \geqslant \rho$ are the same. Therefore on all these components the ratio $\frac{f_1\circ\pi}{f_2\circ\pi}$ is a constant different from zero (and from infinity). As its values at the components E_σ and $E_{\overline{\sigma}}$ are conjugate to each other, this ratio is real. Therefore, $\dim(J(a)/J(a+1)) = 1$.

One has $m_\rho = \sum_{i:\sigma_i<\rho} k_i m_{\sigma_i}$ with $k_i \geqslant 0$. Let us take a real function f_1 such that the strict transform of the curve $\{f_1 = 0\}$ is the union of k_i real curvettes at the components E_{σ_i} for all i such that $\sigma_i < \rho$. Let $\pi_\rho : (X_\rho, D_\rho) \to (\mathbb{C}^2, 0)$ be the modification of $(\mathbb{C}^2, 0)$ produced in the course of the resolution π up to the moment when the component E_ρ is created. Let z be an affine coordinate on E_ρ with real values on the real part of E_ρ and equal to zero at the intersection point of E_ρ with the previous component. Let f_2 be a real function such that the strict transform of $\{f_2 = 0\}$ is a real curvette at E_ρ at infinity. The components E_σ with $\sigma > \rho$ are obtained after blow ups the component E_ρ at nonreal points z_0 and $\overline{z_0}$. The ratio $\psi = \frac{f_1\circ\pi}{f_2\circ\pi}$ on the component E_ρ is equal to cz with real c. (It has a zero at $z = 0$ and a pole at $z = \infty$.) Therefore, its value at the point z_0 (from which the component intersecting the strict transform of the curve C emerges) is not real. The ratio ψ is equal to this constant on all the components from the corresponding connected component of the part of the resolution graph consisting of the vertices $\sigma > \rho$. Therefore, the ratio $\frac{f_1\circ\pi}{f_2\circ\pi}$ on the curve C tends to a nonreal (nonzero) number at the origin. This implies that $J(m_\rho)/J(m_\rho+1) \cong \mathbb{C}$ and $\dim J(m_\rho)/J(m_\rho+1) = 2$. □

Lemma 6. *Let δ be a vertex of the resolution graph Γ lying on the geodesic from ρ to δ_C. Them $M_\delta \in m_\rho + S_C^{\mathbb{R}}$.*

Proof. Let $\varphi \in \mathcal{O}_{\mathbb{C}^2,0}$ be the equation of a curvette at the point δ and let us assume that either $\delta \in [\tau_i, \tau_{i+1}]$, for some $i > q+1$ or $\delta \in [\rho, \tau_{q+1}]$. Then, $M_\delta = \nu_C(\varphi) + \nu_C(\overline{\varphi})$ and by the last part in the proof of Lemma 2, one has that $M_\delta = \sum_0^i k_j M_{\sigma_j} + a m_\rho$ for some $a > 0$. Thus, $M_\delta - m_\rho \in S_C^{\mathbb{R}}$. □

Proposition 7. *Let S_2 be the set of elements a of the semigroup $S_C^{\mathbb{R}}$ such that* $\dim(J(a)/J(a+1)) = 2$. *Then $S_2 = m_\rho + S_C^{\mathbb{R}}$.*

Proof. If $a = m_\rho + b$ with $b \in S_C^{\mathbb{R}}$, then obviously $a \in S_2$: if f_1 and f_2 are functions described above and $\nu(h) = b$, then the classes of $f_1 \cdot h$ and $f_2 \cdot h$ are linear independent over $\mathbb{R}$ in $J(m_\rho + b)/J(m_\rho + b + 1)$.

Assume that $a \notin m_\rho + S_C^{\mathbb{R}}$ and $\dim(J(a)/J(a+1)) = 2$. Let f be a real function with $\upsilon_C(f) = a$, and let the strict transform of the curve $\{f = 0\}$ intersect the exceptional divisor D at points of the subset Γ' of Γ. The subset Γ' cannot contain a vertex from the geodesic from ρ to δ_C (due to Lemma 6). One has

$$f = f' \cdot \prod_{i:\overline{\sigma}_i \neq \sigma_i} f_i' \overline{f_i'},$$

where f' is a real function such that the strict transform of the curve $\{f' = 0\}$ intersects the exceptional divisor D only at points of components E_δ with $\overline{\delta} = \delta$, f_i' is a (complex analytic) function such that the strict transform of the curve $\{f_i' = 0\}$ intersects only at points of the components E_δ from the tail containing E_{σ_i}. One has $\nu_C(f_i'\overline{f_i'}) = k_i M_{\sigma_i}$. If $k_i \geqslant N_i$ then $\nu_C(f_i'\overline{f_i'}) \in m_\rho + S_C^{\mathbb{R}}$ (as $N_i M_{\sigma_i} = M_{\tau_i}$ and Lemma 6). Thus, $k_i < N_i$.

If h is another real function with $\nu_C(h) = a$ and $h = h' \cdot \prod_{i:\overline{\sigma}_i \neq \sigma_i} h_i' \overline{h_i'}$, then $\nu_C(h_i' \overline{h_i'}) = \nu_C(f_i' \overline{f_i'})$ (because of the uniqueness of the representation in terms of the minimal set of generators $\{M_{\sigma_0}, \dots, M_{\sigma_g}\}$). This implies that $\frac{h \circ \pi}{f \circ \pi}$ is constant on the union of the two geodesics from ρ to δ_C and $\overline{\delta}_C$ (ρ included). As its values at complex conjugate points are complex conjugate to each other, this constant is real. Therefore, the elements of the extended semigroup defined by f and h are linear dependent over $\mathbb{R}$. This contradicts the assumption that $\dim(J(a)/J(a+1)) = 2$. □

Theorem 8. *One has:*

$$P_{\nu_C}(t) = P^S_{\nu_C}(t)(1 + t^{m_\rho}) = P^S_{\nu_C}(t)\frac{1 - t^{2m_\rho}}{1 - t^{m_\rho}}; \tag{8}$$

$$P^{\mathbb{R}}_{\nu_C}(t) = P^S_{\nu_C}(t)(1 - t^{m_\rho}).$$

Proof. One has

$$P_{\nu_C}(t) = P^S_{\nu_C}(t) + \sum_{a \in S_2} t^a = P^S_{\nu_C}(t) + \sum_{v \in S^{\mathbb{R}}_C} t^{m_\rho + v} = P^S_{\nu_C}(t) + t^{m_\rho} P^S_{\nu_C}(t);$$

$$P^{\mathbb{R}}_{\nu_C}(t) = P^S_{\nu_C}(t) - \sum_{a \in S_2} t^a = P^S_{\nu_C}(t) - \sum_{v \in S^{\mathbb{R}}_C} t^{m_\rho + v} = P^S_{\nu_C}(t) - t^{m_\rho} P^S_{\nu_C}(t).$$

□

Remark 9. Assume that the group $\mathbb{Z}_2$ of order 2 acts on $(\mathbb{C}^2, 0)$ and $(C', 0) \subset (\mathbb{C}^2, 0)$ is a complex curve such that the dual graph of its equivariant resolution (together with the ages of the vertices) coincides with the resolution graph Γ. In [4], there was defined the equivariant Poincaré series $P^{\mathbb{Z}_2}_{C'}(t)$ as an element of $R_1(\mathbb{Z}_2)[[t]]$ where $R_1(\mathbb{Z}_2) = \mathbb{Z}[\sigma]/(\sigma^2 - 1)$ is the ring of representations of $\mathbb{Z}_2$. One has its reduction under the dimensional homomorphism $\mathrm{red} : R_1(\mathbb{Z}_2) \to \mathbb{Z}$ as an element $\mathrm{red}\, P^{\mathbb{Z}_2}_{C'}(t) \in \mathbb{Z}[[t]]$. Equation (8) and the equation of [4, Theorem 2] imply that $P_{\nu_C}(t) = \mathrm{red}\, P^{\mathbb{Z}_2}_{C'}(t)$. We have no independent explanation of this coincidence.

Remark 10. As it was indicated in Remark 3, the semigroup $S^{\mathbb{R}}_C$ is the semigroup of values of a complex plane curve C'. Therefore, its elements are symmetric with respect to the conductor c of the semigroup in the sense that $a \in S \iff c - 1 - a \notin S$. One has one more symmetry concerning the coefficients of the series $P_{\nu_C}(t)$. This symmetry is with respect to $c + m_\rho$:

$$\dim\left(J(a)/J(a+1)\right) + \dim\left(J(c + m_\rho - 1 - a)/J(c + m_\rho - a)\right) = 2$$

(cf. [6]). (Thus, if one of these dimensions is equal to zero, the other one is equal to 2, and if one of them is equal to 1, then the other one is equal to 1 as well).

Remark 11. From Remark 3, one can easily see that the semigroup Poincaré series $P^S_{\nu_C}(t)$ does not determine the real topology of the curve C, that is, the topology of the curve $C \cup \overline{C}$. On the other hand, each of the series $P_{\nu_C}(t)$ and $P^{\mathbb{R}}_{\nu_C}(t)$ permit to determine not only the semigroup Poincaré series $P^S_{\nu_C}(t)$, but also the value m_ρ and thus the splitting point ρ. Therefore, each of these series determines the real topology of the curve C.

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