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Finite-time stability for discrete-time systems with time-varying delays and nonlinear perturbations using relaxed summation inequality

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Abstract This article is concerned with the problem of delay-dependent finite time stability (FTS) for delayed discrete-time systems with nonlinear perturbations. First, based on a Lyapunov-Krasovskii Functional (LKF), delay-dependent FTS conditions are provided by introducing some free-weighting matrices. Then, a new reduced free-matrix-based inequality (RFMBI) is established to estimate the single summation term. The dimensions of these free matrices integral in our results are less than those obtained in the literature. This reduction in the number of variables does not mean that our method is a particular case but simply that our approach is completely different from the others and therefore our method is more effective. Thus, less conservative design conditions are obtained in this paper in terms of linear matrix inequalities (LMIs) and solved by the LMI Tools of MATLAB to achieve the desired performances. Finally, numerical examples are examined to demonstrate the advantage and effectiveness of the proposed results.

Keywords Finite time stability (FTS) · Delayed discrete-time systems · Nonlinear perturbations · Reduced free-matrix-based inequality

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1 Introduction

Time-delay is known as natural phenomenon that is common in various practical situation such as biology, economy, engineering systems, and so on [8]. Then, the existence of time delay may lead to oscillations, bad performances and even instability for dynamical systems [21]. For this reason, important efforts are devoted to investigate its influence on systems performances [1]. Thus, the researchers have paid great attention to the problems of analysis and synthesis of time-delay systems [10], which makes it possible to perfectly study the stability conditions for these systems using LMIs [3]. The current literary conditions for the stability analysis can be roughly divided into two types : (i) delay-independent stability [23], and (ii) delay-dependent stability [11]. The latter type, that takes into account time-delays and their effects [12], is known to have less conservative results than delay-independent type that is not affected by delay size [26]. In the daily life, the disturbances are very common [7]. In many processes, not only the time-varying delays can affect the systems stability [27], but also the nonlinear perturbations can result in negative influences on these systems [6]. The obtained approaches until now require further development and improvement and therefore there is still room for work and research to ensure the stability of systems taking into account time-varying delays and perturbations and then to get less conservative results.

In order to address the problem of stability for discrete-time systems with time-varying delays and reduce the results conservatism, many approaches have been proposed in the literature such as the Jensen-based inequality (JBI) [13], the Wirtinger-based inequalities (WBIs) [22], the abel-lemma based inequality (ABI) [31], the free-weighting matrix (FWM) approach [6], the auxiliary function-based inequality [19], and the FWM-based inequalities [29]. Thus, great efforts have been made and a lot of works have been published to get a maximum allowable delay as large as possible for a given time-delay system and then to improve the stability criteria over infinite-time interval. On the other hand, the FTS problem is not addressed in these works, which is an important topic of research and study, especially if we take into account that it is highly required. Then, we are motivated by the above-mentioned studies and by the ideas implemented in several works to further research on time-delay systems considering the FTS problem.

To study stability of systems, there are some cases in which large state values are not acceptable and therefore this problem must be addressed. Then, it becomes very important to define a stable system as one whose state, given some initial conditions, remains within prescribed bounds in a short time. For this, there is a need to study another type of stability, which is the stability previously mentioned FTS [14]. Then, a system is said to be finite time stable if its state does not exceed some bounds during a time interval. Thus, important results are obtained for various sort of systems such as linear time-varying systems [24], linear systems with additive time-varying delay [16], neural network systems [33], T-S Fuzzy systems [17], and impulsive systems

[4]. Despite all that, the FTS is not fully covered and has yet to receive a lot of attention especially since this type of stability is required significantly. Then, a new analysis technique over finite-time interval is proposed in this paper to achieve the desired performances and ensure the results conservatism reduction.

In this work, the problem of FTS for discrete-time systems with time-varying delays and perturbations is investigated. To reach this goal in the best possible way, we follow two steps. First, based on a LKF, delay-dependent FTS conditions are provided by introducing some free-weighting matrices. Then, a new RFMBI is established to estimate the single summation term $\sum_{i=k-h_2}^{k-h(k)-1} \eta^T(i)Rn(i)$. The reduced order of free weighting matrices technique is used to reduce the size of certain matrices to n instead of $3n$ as given in [5]. Reducing the variables order is one of the main indices of the effectiveness of our method, especially if we take into account that there is in this article a different approach from those given in the literature. Then, the proposed method offers less conservative results, good transient responses, and feasible control signals. Finally, some examples are proposed to illustrate the effectiveness of our approach.

Notation: Throughout this paper, $\Re^n$ denotes the n-dimensional Euclidean vector space. The real matrix $P > 0$ or $P < 0$ respectively mean that P is positive or negative definite. The superscripts 1 and T stand for the inverse and the transpose of a matrix, respectively. In addition, $sym(A)$ indicates $A + A^T$ for convenience, and $*$ is the symmetry term of symmetry matrix.

2 Problem statement and preliminaries

Consider the following discrete-time system :

$$\begin{aligned} x(k+1) &= Ax(k) + Bx(k-h(k)) + g_1(k, x(k)) + g_2(k, x(k-h(k))) \\ x(\theta) &= \varphi(\theta), \ \ \theta \in [-h_2, -h_2+1, \dots, 0] \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $\varphi(\theta)$ is the initial condition, and A, B are constant matrices with appropriate dimensions. The delay $h(k)$ is assumed to be time dependent and satisfies $0 < h_1 \leq h(k) \leq h_2$, and $g_1(k, x(k))$, $g_2(k, x(k-h(k)))$ are nonlinear function and satisfy the following conditions :

$$\|g_1(k, x(k))\| \leq \rho_1 \|x(k)\|, \ \ \|g_2(k, x(k-h(k)))\| \leq \rho_2 \|x(k-h(k))\| \tag{2}$$

where ρ_1 and ρ_2 are known positive scalars.

Let

$$y(k) = x(k+1) - x(k), \ \ y^T(k)y(k) \leq \varepsilon \tag{3}$$

The purpose of this paper is to derive sufficient conditions that ensure the finite-time stability of the discrete-time system (1). Then, the following definitions and lemmas are useful for the derivation of the main results :

Definition 1 ([32], [15]) The system (1) is finite-time stable with respect to (c_1, c_2, R, N), where $R > 0$ and $0 \leq c_1 < c_2$, if

$$\sup_{\theta \in [-h_2, -h_2+1, ..., 0]} \varphi^T(\theta)\varphi(\theta) \leq c_1 \Rightarrow x^T(k)Rx(k) \leq c_2, \;\; k \in \{1, ..., N\} \tag{4}$$

Lemma 1 *[20] Let $f_1, f_2, \ldots, f_n : \Re^m \to \Re$ have positive values in an open subset D and $\Re^m$, then, the reciprocally convex combination of f_i over D satisfies :*

$$\min_{\alpha_i/\alpha_i>0 \sum_i \alpha_i=1} \sum_i^n \frac{1}{\alpha_i} f_i(t) = \sum_i^n f_i(t) + \max_{g_{ij}(t)} \sum_{i \neq j}^n g_{ij}(t) \tag{5}$$

where

$$\left\{ g_{ij} : \Re^m \to \Re, g_{ij}(t) = g_{ji}(t), \begin{bmatrix} f_i(t) & g_{ij}(t) \\ g_{ji}(t) & f_i(t) \end{bmatrix} \geq 0 \right\} \tag{6}$$

Lemma 2 *[18] Let X and Y be real matrices of appropriate dimensions. For a given scalar $\varepsilon > 0$ and vectors $x, y \in \Re^n$, then*

$$2x^T XYy \leq \varepsilon^{-1} x^T X^T Xx + \varepsilon y^T Y^T Yy \tag{7}$$

Remark 1 As given in [5], the RFMBI is introduced to reduce the size of the matrices Z_i $(i = 1, 2, 3)$ to n instead of $3n$.

3 Finite time stability

In this section, new FTS criteria are obtained by employing the reduced free-matrix-based summation inequality. Then, the lemma below is given to set the stage for the main findings. For simplicity, $h(k)$ is denoted by h_k.

Lemma 3 *(RFMBI) For an n-dimensional real vector sequence $\{x(\alpha_2), \ldots, x(\alpha_1)\}$, if there exist symmetric matrices $Z_i^i \in \Re^{n\times n}$, $i = 0, 1, 2, 3$, matrices $Z_j^i \in \Re^{n\times n}$, $i, j = 0, 1, 2, 3$ $(i < j)$, and scalars α_1, α_2 $(\alpha_2 > \alpha_1)$, satisfying :*

$$\begin{bmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ * & Z_1^1 & Z_1^2 & Z_1^3 \\ * & * & Z_2^2 & Z_2^3 \\ * & * & * & Z_3^3 \end{bmatrix} \geq 0 \tag{8}$$

then, the following inequalities hold :

$$\begin{aligned} -\sum_{i=\alpha_1}^{\alpha_2-1} \eta^T(i) Z_0^0 \eta(i) \leq -\frac{1}{h} \Big(& \xi_0^T \left(Z_1^1 - sym(Z_0^1) \right) \xi_0 + \frac{3(h+1)}{(h-1)} \xi_1^T \Big(Z_2^2 \\ & - sym(Z_0^2) \Big) \xi_1 + \frac{5(h+1)(h+2)}{(h-1)(h-2)} \xi_2^T \Big(Z_3^3 \\ & - sym(Z_0^3) \Big) \xi_0 \Big) \end{aligned} \tag{9}$$

where $\eta(k) = x(k+1) - x(k)$, $h = \alpha_2 - \alpha_1$, *and*

$$\xi_0 = x(\alpha_2) - x(\alpha_1), \;\; \xi_1 = x(\alpha_2) + x(\alpha_1) - \frac{2}{h+1}\sum_{i=\alpha_1}^{\alpha_2} x(i),$$

$$\xi_2 = \xi_0 + \frac{6}{(h+1)(h+2)}\sum_{i=\alpha_1}^{\alpha_2} (\alpha_2 + \alpha_1 - 2i)x(i) \tag{10}$$

Proof Using the constant scalars κ_1 and κ_2, let :

$$f_1(i) = \kappa_1(\alpha_2 + \alpha_1 - 1 - 2i), \;\; f_2(i) = \kappa_2(f_1^2(i) - \frac{1}{3}(h-1)(h+1)) \tag{11}$$

Then, it is easy to verify the following calculus :

$$\sum_{i=\alpha_1}^{\alpha_2-1} f_1(i) = \sum_{i=\alpha_1}^{\alpha_2-1} f_2(i) = \sum_{i=\alpha_1}^{\alpha_2-1} f_1(i)f_2(i) = 0$$
$$\sum_{i=\alpha_1}^{\alpha_2-1} f_1^2(i) = \kappa_1^2\frac{h(h-1)(h+1)}{3}, \;\; \sum_{i=\alpha_1}^{\alpha_2-1} f_2^2(i) = \kappa_2^2\frac{h(h^2-1)(h^2-4)}{45}$$
$$\sum_{i=\alpha_1}^{\alpha_2-1} f_1(i)\eta(i) = -\kappa_1(h+1)\xi_1, \;\; \sum_{i=\alpha_1}^{\alpha_2-1} f_2(i)\eta(i) = \kappa_2\frac{2(h+1)(h+2)}{3}\xi_2 \tag{12}$$

On the other hand, we have $0 \leq \sum_{i=\alpha_1}^{\alpha_2-1} \varpi^T Z \varpi$ where

$$\varpi = \left[\eta^T(i) - \frac{1}{h}\sum_{i=\alpha_1}^{\alpha_2-1} - f_1(i)\chi_1 - f_2(i)\chi_2\right]^T \tag{13}$$

χ_1 and χ_2 are constants vectors to be determined.
Thus, we obtain :

$$\begin{aligned}
-\sum_{i=\alpha_1}^{\alpha_2-1} \eta^T(i)Z_0^0\eta(i) \leq -sym\Bigg(\frac{1}{h}\left(\sum_{i=\alpha_1}^{\alpha_2-1}\eta^T(i)\right) Z_0^1 \left(\sum_{i=\alpha_1}^{\alpha_2-1}\eta(i)\right)\\
-\left(\sum_{i=\alpha_1}^{\alpha_2-1} f_1(i)\eta^T(i)\right) Z_0^2\chi_1 - \left(\sum_{i=\alpha_1}^{\alpha_2-1} f_2(i)\eta^T(i)\right) Z_0^3\chi_2\Bigg)\\
+\frac{1}{h}\left(\sum_{i=\alpha_1}^{\alpha_2-1}\eta^T(i)\right) Z_1^1 \left(\sum_{i=\alpha_1}^{\alpha_2-1}\eta(i)\right) + \sum_{i=\alpha_1}^{\alpha_2-1} f_1^2(i)\chi_1^T Z_2^2\chi_1\\
+\sum_{i=\alpha_1}^{\alpha_2-1} f_2^2(i)\chi_2^T Z_3^3\chi_2
\end{aligned} \tag{14}$$

Then, the inequality (9) can be determined easily taking $\kappa_1 = -\frac{3}{h(h-1)}$, $\kappa_2 = \frac{15}{2h(h-1)(h-2)}$, $\chi_1 = \xi_1$, and $\chi_2 = \xi_2$.

Remark 2 Taking $Z_i^i = R > 0$, $i = 0, \ldots, 3$, the inequality (13) in [30] is obtained, which means that our approach is more general than that given in [30].

Remark 3 Taking $Z_i^3 = 0$, $i = 0, 1, 2, 3$, the variables order of our presented free matrices is reduced to $4.5n^2 + 1.5n$ compared with $24.5n^2 + 3.5n$ that is given in in [5]. On the other hand, reducing the order of the variables does not mean that our approach is a particular case of [5], but rather that Lemma 3 has not been applied in this reference. In the literature, there is not an efficient approach to reduce the order of the free matrices presented in [5] which allows to estimate the unique summation in order to get more convexity, what motivates our work.

Corollary 1 *If there exist symmetric positive definite matrix $R > 0$, and appropriately sized matrices $L_i \in \Re^{n \times n}$, $i = 1, 2, 3$, the following inequality holds :*

$$
\begin{aligned}
-\sum_{i=\alpha_1}^{\alpha_2-1} \eta^T(i) R \eta(i) \geq -\frac{1}{h} \Big(& \xi_0^T \left(L_1^T R^{-1} L_1 - sym(L_1) \right) \xi_0 \\
& + \frac{3(h+1)}{(h-1)} \xi_1^T \left(L_2^T R^{-1} L_2 - sym(L_2) \right) \xi_1 \\
& + \frac{5(h+1)(h+2)}{(h-1)(h-2)} \xi_2^T \left(L_3^T R^{-1} L_3 - sym(L_3) \right) \xi_0 \Big)
\end{aligned} \tag{15}
$$

where $\eta(k) = x(k+1) - x(k)$, $h = \alpha_2 - \alpha_1$, and

$$
\begin{aligned}
\xi_0 &= x(\alpha_2) - x(\alpha_1), \;\; \xi_1 = x(\alpha_2) + x(\alpha_1) - \frac{2}{h+1} \sum_{i=\alpha_1}^{\alpha_2} x(i), \\
\xi_2 &= \xi_0 + \frac{6}{(h+1)(h+2)} \sum_{i=\alpha_1}^{\alpha_2} (\alpha_2 - \alpha_1 - 2i) x(i)
\end{aligned} \tag{16}
$$

Proof Let the following change of variables :

$$
\begin{aligned}
& Z_0^0 = R, \;\; Z_0^1 = L_1^T, \;\; Z_0^2 = L_2^T, \;\; Z_0^3 = L_3^T, \\
& Z_1^1 = L_1^T R^{-1} L_1, \;\; Z_1^2 = L_1^T R^{-1} L_2, \;\; Z_1^3 = L_1^T R^{-1} L_3, \\
& Z_2^2 = L_2^T R^{-1} L_2, \;\; Z_2^3 = L_2^T R^{-1} L_3, \;\; Z_3^3 = L_3^T R^{-1} L_3
\end{aligned} \tag{17}
$$

From the inequality (9), it is easy to see that the condition (15) is automatically verified using the Schur complement [2].

Some results are now derived to ensure FTS for the studied system.

Theorem 1 *The system (1) is finite time stable with respect to (c_1, c_2, R, N) if there exist symmetric positive definite matrices P, Q_1, Q_2, Q_3, R_1, $R_2 \in \Re^{n \times n}$, symmetric matrices Z_i^i, $Y_i^i \in \Re^{n \times n}$, $i = 0, \ldots, 3$, matrices Z_i^j, $Y_i^j \in \Re^{n \times n}$, $i, j = 0, \ldots, 3$*

$(i<j)$, $M_i, S_{ij} \in \Re^{n\times n}$, $i,j=1,2,3$, and scalars λ_i, $i=1,2,\ldots,7$, $\varepsilon_i>0$, $i=1,2,\ldots,6$, $\alpha>1$, such that :

$$0<\lambda_1 I<\tilde{P}<\lambda_2 I, \tag{18}$$

$$0<\tilde{Q}_1<\lambda_3 I,\;\; 0<\tilde{Q}_2<\lambda_4 I,\;\; 0<\tilde{Q}_3<\lambda_5 I, \tag{19}$$

$$0<\tilde{R}_1<\lambda_6 I,\;\; 0<\tilde{R}_2<\lambda_7 I, \tag{20}$$

$$Z_i^i \le sym(Z_i^0),\;\; Y_i^i \le sym(Y_i^0),\;\; i=2,3, \tag{21}$$

$$\begin{bmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ * & Z_1^1 & Z_1^2 & Z_1^3 \\ * & * & Z_2^2 & Z_2^3 \\ * & * & * & Z_3^3 \end{bmatrix} \ge 0,\quad \begin{bmatrix} Y_0^0 & Y_0^1 & Y_0^2 & Y_0^3 \\ * & Y_1^1 & Y_1^2 & Y_1^3 \\ * & * & Y_2^2 & Y_2^3 \\ * & * & * & Y_3^3 \end{bmatrix} \ge 0, \tag{22}$$

$$\begin{pmatrix} Z & S \\ * & Z \end{pmatrix} > 0, \tag{23}$$

$$\begin{pmatrix} \tilde{\Omega}_1+\tilde{\Omega}_2+\tilde{\Omega}_3 & \tilde{\Omega}_4 \\ * & \tilde{\Omega}_5 \end{pmatrix} < 0, \tag{24}$$

$$\sigma_1 c_1 + \sigma_2 c_2 < \alpha^{-N} c_2 \lambda_1 \tag{25}$$

where

$$\begin{aligned}
\tilde{\Omega}_1 &= \Omega - \Pi_1 \Upsilon_1 \Pi_1^T - \Pi_2 \Upsilon_2 \Pi_2^T - (\alpha-1)\left(\begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix}^T ((h_1+1)\Lambda_1) \begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix}\right. \\
&\quad + \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix}^T \Lambda_2 \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix} + \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix}^T \Lambda_3 \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix} \\
&\quad \left. - \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}^T \begin{pmatrix} -P+Q_1+2R_2 & -R_2 & -R_2 \\ * & Q_2+R_2 & 0 \\ * & * & Q_3+R_2 \end{pmatrix} \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}\right), \\
\Omega &= e_1 Q_1 e_1^T + e_2(-Q_1+Q_2)e_2^T + e_3(-Q_2+Q_3)e_3^T - e_4 Q_3 e_4^T \\
&\quad + e_5(P + h_1^2 R_1 + h_{12}^2 R_2)e_5^T + sym(e_1 P e_5^T), \\
\Pi_1 &= [e_1-e_2, e_1+e_2-2e_6, e_1-e_2+6e_9], \\
\Pi_2 &= [e_2-e_3, e_2+e_3-2e_7, e_2-e_3+6\vartheta_{10}, e_3-e_4, e_3+e_4-2e_8, , \\
&\quad e_3-e_4+6\vartheta_{11}], \\
\Upsilon_1 &= -diag\{Z_1^1 - sym(Z_0^1), 3(Z_2^2 - sym(Z_0^2)), 5(Z_3^3 - sym(Z_0^3))\}, \\
Z &= -diag\{Y_1^1 - sym(Y_0^1), 3(Y_2^2 - sym(Y_0^2)), 5(Y_3^3 - sym(Y_0^3))\},
\end{aligned}$$

$$
\begin{aligned}
&\Upsilon_2 = \begin{pmatrix} Z & S \\ * & Z \end{pmatrix}, \ S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}, \ \Lambda_1 = \begin{pmatrix} R_1 & -R_1 \\ * & Q_1 + R_1 \end{pmatrix}, \\
&\Lambda_i = \begin{pmatrix} R_2 & -R_2 \\ * & Q_i + R_2 \end{pmatrix}, \ i = 2,3, \\
&\tilde{\Omega}_2 = 2\left[e_1 M_1 + e_3 M_2 + e_5 M_3\right]\left[e_1(A-I) + e_3 B^T - e_5\right]^T, \\
&\tilde{\Omega}_3 = (\varepsilon_1 + \varepsilon_3 + \varepsilon_5)\rho_1^2 e_1 e_1^T + (\varepsilon_2 + \varepsilon_4 + \varepsilon_6)\rho_1^2 e_3 e_3^T, \\
&\tilde{\Omega}_4 = \left[e_1 M_1^T, e_1 M_1^T, e_3 M_2^T, e_3 M_2^T, e_5 M_3^T, e_5 M_3^T\right], \\
&\tilde{\Omega}_5 = diag\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}, \\
&\sigma_1 = \lambda_2 + h_1\lambda_3 + h_{12}\lambda_4 + h_{12}\lambda_5, \\
&\sigma_2 = \frac{h_1^2(h_1+1)}{2}\lambda_6 + \frac{h_{12}^2(h_2+h_1+1)}{2}\lambda_7, \ \tilde{P} = R^{\frac{-1}{2}} P R^{\frac{-1}{2}}, \\
&\tilde{Q}_i = R^{\frac{-1}{2}} Q_i R^{\frac{-1}{2}}, \ i = 1,2,3, \ \tilde{R}_i = R^{\frac{-1}{2}} R_i R^{\frac{-1}{2}}, \ i = 1,2, \\
&e_i = \left(0_{n\times(i-1)n}, I, 0_{n\times(8-i)n}\right), \ i = 1,\ldots,8.
\end{aligned} \tag{26}
$$

Proof Consider the following LKF :

$$V(k) = V_1(k) + V_2(k) + V_3(k) \tag{27}$$

where

$$
\begin{aligned}
V_1(k) &= x^T(k)Px(k) \\
V_2(k) &= \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_k}^{k-h_1-1} x^T(i)Q_2x(i) + \sum_{i=k-h_2}^{k-h_k-1} x^T(i)Q_3x(i) \\
V_3(k) &= h_1 \sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1} y^T(j)R_1y(j) + h_{12}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y^T(j)R_2y(j)
\end{aligned} \tag{28}
$$

From this LKF, we obtain :

$$
\begin{aligned}
\Delta V_1(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\
&= (x(k)+y(k))^T P(x(k)+y(k)) - x^T(k)Px(k) \\
\Delta V_2(k) &= x^T(k)Q_1x(k) + x^T(k-h_1)(-Q_1+Q_2)x(k-h_1) \\
&\quad + x^T(k-h(k))(-Q_2+Q_3)x(k-h(k)) \\
&\quad + x^T(k-h_2)(-Q_3)x(k-h_2) \\
\Delta V_3(k) &= y^T(k)(h_1^2R_1 + h_{12}^2R_2)y(k) - h_1\sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i) \\
&\quad - h_{12}\sum_{i=k-h_2}^{k-h_1-1} y^T(i)R_2y(i)
\end{aligned} \tag{29}
$$

Let

$$\xi^T(k) = \Big[x^T(k), x^T(k-h_1), x^T(k-h_k), x^T(k-h_2), y(k), \vartheta_1^T(k), \vartheta_2^T(k), \vartheta_3^T(k),$$
$$\vartheta_4^T(k), \vartheta_5^T(k), \vartheta_6^T(k)\Big]^T, \ \ \vartheta_1^T(k) = \frac{1}{h_1+1}\sum_{i=k-h_1}^{k} x^T(i),$$
$$\vartheta_2^T(k) = \frac{1}{h_{k1}+1}\sum_{i=k-h_k-h_1}^{k-h_1} x^T(i), \ \ \vartheta_3^T(k) = \frac{1}{h_{k2}+1}\sum_{i=k-h_2-h_k}^{k-h_k} x^T(i),$$
$$\vartheta_4^T(k) = \frac{1}{(h+1)(h+2)}\sum_{i=k-h_1}^{k}(2k-h_1-2i)x(i),$$
$$\vartheta_5^T(k) = \frac{1}{(h_{k1}+1)(h_{k1}+2)}\sum_{i=k-h_k}^{k-h_1}(2k-h_1-h_k-2i)x(i),$$
$$\vartheta_6^T(k) = \frac{1}{(h_{k2}+1)(h_{k2}+2)}\sum_{i=k-h_k}^{k-h_1}(2k-h_k-h_2-2i)x(i) \tag{30}$$

where $h_{k1} = h_k - h_1$, $h_{k2} = h_2 - h_k$.

Then, we have

$$\Delta V(k) \le \xi^T(k)\Omega\xi(k) - h_1\sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i) - h_{12}\sum_{i=k-h_2}^{k-h_1-1} y^T(i)R_2y(i) \tag{31}$$

On the other hand, we have :

$$-V_2(k) = \xi^T(k)\left(e_1Q_1e_1^T + e_2Q_2e_2^T + e_3Q_3e_3^T\right)\xi(k) + \sum_{i=k-h_1}^{k} x^T(i)Q_1x(i)$$
$$+\sum_{i=k-h_k}^{-h_1} x^T(i)Q_2x(i) + \sum_{i=k-h_2}^{k-h_k} x^T(i)Q_3x(i)$$
$$-V_3(k) \le -\sum_{i=-h_1}^{0}(x(k)-x(k+i))^TR_1(x(k)-x(k+i))$$
$$-\sum_{i=-h_k}^{-h_1}(x(k)-x(k+i))^TR_2(x(k)-x(k+i))$$
$$+(x(k)-x(k+h_1))^TR_2(x(k)-x(k+h_1))$$
$$-\sum_{i=-h_2}^{-h_k}(x(k)-x(k+i))^TR_2(x(k)-x(k+i))$$
$$+(x(k)-x(k+h_k))^TR_2(x(k)-x(k+h_k))$$

$$
\begin{aligned}
&= \sum_{i=-h_1}^{0} (x(k)-x(k+i))^T R_1 (x(k)-x(k+i)) \\
&\quad - \sum_{i=-h_k}^{-h_1} (x(k)-x(k+i))^T R_2 (x(k)-x(k+i)) \\
&\quad +(x(k) - \sum_{i=-h_2}^{-h_k} (x(k)-x(k+i))^T R_2 (x(k)-x(k+i)) \\
&\quad +\xi^T(k)\left(2e_1R_2e_1^T + e_2R_2e_2^T + e_3R_2e_3^T - 2e_1R_2e_2^T - 2e_1R_2e_3^T\right)\xi(k) \qquad (32)
\end{aligned}
$$

Thus, we obtain :

$$
\begin{aligned}
-(V_2(k)+V_3(k)) \leq &- \sum_{i=-h_1}^{0} \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix}^T \Lambda_1 \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix} \\
&- \sum_{i=-h_k}^{-h_1} \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix}^T \Lambda_2 \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix} \\
&- \sum_{i=-h_2}^{-h_k} \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix}^T \Lambda_3 \begin{pmatrix} x(k) \\ x(k+i) \end{pmatrix} + \begin{pmatrix} x(k) \\ x(k-h_1) \\ x(k-h(k)) \end{pmatrix}^T \\
&\times \begin{pmatrix} Q_1+2R_2 & -R_2 & -R_2 \\ * & Q_2+R_2 & 0 \\ * & * & Q_3+R_2 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-h_1) \\ x(k-h(k)) \end{pmatrix} \qquad (33)
\end{aligned}
$$

As given in Theorem 1, it is easy to see that $\Lambda_j > 0$, $j = 1,2,3$.
Using the Jensen inequality [9], we get :

$$
\begin{aligned}
-(V_2(k)+V_3(k)) \leq &-\xi^T(k)\left(\begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix}^T (h_1+1)\Lambda_1 \begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix} + \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix}^T (h_{k1}+1)\right. \\
&\times \Lambda_2 \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix} + \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix}^T (h_{k2}+1)\Lambda_3 \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix} - \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}^T \\
&\left.\times \begin{pmatrix} Q_1+2R_2 & -R_2 & -R_2 \\ * & Q_2+R_2 & 0 \\ * & * & Q_3+R_2 \end{pmatrix} \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}\right)\xi(k) \\
\leq &-\xi^T(k)\left(\begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix}^T (h_1+1)\Lambda_1 \begin{pmatrix} e_1^T \\ e_6^T \end{pmatrix} + \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix}^T \Lambda_2 \begin{pmatrix} e_1^T \\ e_7^T \end{pmatrix}\right. \\
&\left. + \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix}^T \Lambda_3 \begin{pmatrix} e_1^T \\ e_8^T \end{pmatrix} - \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}^T \begin{pmatrix} Q_1+2R_2 & -R_2 & -R_2 \\ * & Q_2+R_2 & 0 \\ * & * & Q_3+R_2 \end{pmatrix} \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix}\right)\xi(k) \qquad (34)
\end{aligned}
$$

Then, we conclude that

$$\Delta V(k)-(\alpha-1)V(k)\leq \xi^T(k)\tilde{\Omega}\xi(k)-h_1\sum_{i=k-h_1}^{k-1}y^T(i)R_1y(i)$$
$$-h_{12}\sum_{i=k-h_2}^{k-h_1-1}y^T(i)R_2y(i) \tag{35}$$

According to Lemma 3, we have :

$$-h_1\sum_{i=k-h_1}^{k-1}y^T(i)R_1y(i)\leq \xi^T(k)\Pi_1\Upsilon_1\Pi_1^T\xi(k) \tag{36}$$

Now, it is easy to see that

$$\begin{aligned}-h_{12}\sum_{i=k-h_2}^{k-h_1-1}y^T(i)R_2y(i) &\leq -h_{12}\sum_{i=k-h_k}^{k-h_1-1}y^T(i)R_2y(i)-h_{12}\sum_{i=k-h_k}^{k-h_k-1}y^T(i)R_2y(i)\\ &\leq \frac{h_{12}}{h_{k1}}\eta_{k1}^T diag\left\{Y_1^1-sym(Y_0^1),3\left(Y_2^2-sym(Y_0^2)\right),\right.\\ &\left.5\left(Y_3^3-sym(Y_0^3)\right)\right\}\eta_{k1}+\frac{h_{12}}{h_{k2}}\eta_{k2}^T diag\left\{Y_1^1-sym(Y_0^1),\right.\\ &\left.3\left(Y_2^2-sym(Y_0^2)\right),5\left(Y_3^3-sym(Y_0^3)\right)\right\}\eta_{k2}\end{aligned} \tag{37}$$

Using $\frac{h_k-h_1}{h_{12}}+\frac{h_2-h_k}{h_{12}}=1$ and Lemma 2 with $\begin{pmatrix} Z & S\\ * & Z\end{pmatrix}>0$, it follows that

$$-h_{12}\sum_{i=k-h_2}^{k-h_1-1}y^T(i)R_2y(i)\leq -\xi^T(k)\Pi_2\Upsilon_2\Pi_2^T\xi(k) \tag{38}$$

where

$$\begin{aligned}\eta_{1k}^T &= \left[x^T(k-h_1)-x^T(k-h_k),x^T(k-h_1)+x^T(k-h_k)-2\vartheta_2(k),\right.\\ &\left.x^T(k-h_1)-x^T(k-h_k)+6\vartheta_5(k)\right]^T,\\ \eta_{2k}^T &= \left[x^T(k-h_k)-x^T(k-h_2),x^T(k-h_k)+x^T(k-h_2)-2\vartheta_3(k),\right.\\ &\left.x^T(k-h_k)-x^T(k-h_2)+6\vartheta_6(k)\right]^T\end{aligned} \tag{39}$$

On the other hand, for any matrices M_1, M_2, M_3 with appropriate dimensions, we get :

$$\begin{aligned}&2\left[x^T(k)M_1+x^T(k-h_k)M_2+y^T(k)M_3\right][(A-I)x(k)+Bx(k-h_k)\\ &-y(k)]+2\left[x^T(k)M_1+x^T(k-h_k)M_2+y^T(k)M_3\left[g_1(k,x(k))\right]\right.\\ &+g_2(k,x(k-h_k))]=0\end{aligned} \tag{40}$$

Applying Lemma 2, we obtain :

$$2x^T(k)M_1\left[g_1(k,x(k))+g_2(k,x(k-h_k))\right] \le (\varepsilon_1^{-1}+\varepsilon_2^{-1})x^T(k)M_1^TM_1x(k) + \varepsilon_1\rho_1^2\|x(k)\|^2+\varepsilon_2\rho_2^2\|x(k-h_k)\|^2 \quad (41)$$

$$2x^T(k-h_k)M_2\left[g_1(k,x(k))+g_2(k,x(k-h_k))\right] \le (\varepsilon_3^{-1}+\varepsilon_4^{-1})x^T(k)M_2^TM_2x(k) + \varepsilon_3\rho_1^2\|x(k)\|^2+\varepsilon_4\rho_2^2\|x(k-h_k)\|^2 \quad (42)$$

$$2y^T(k)M_3\left[g_1(k,x(k))+g_2(k,x(k-h_k))\right] \le (\varepsilon_3^{-1}+\varepsilon_4^{-1})x^T(k)M_2^TM_2x(k) + \varepsilon_5\rho_1^2\|x(k)\|^2+\varepsilon_6\rho_2^2\|x(k-h_k)\|^2 \quad (43)$$

Using the Schur complement, we can deduce that

$$\Delta V(k)-(\alpha-1)V(k) < \xi^T(k)\left(\left(\sum_{i=1}^{3}\tilde{\Omega}_i\right)-\tilde{\Omega}_4\tilde{\Omega}_5^{-1}\tilde{\Omega}_4^T\right)\xi(k) < 0 \quad (44)$$

which implies $V(k+1) < \alpha V(k)$. Therefore, it infers that

$$V(k) < \alpha V(k-1) < \alpha^2 V(k-2) < \ldots < \alpha^k V(0) \quad (45)$$

Furthermore, the initial value of LKF can be obtained as follows :

$$\begin{aligned} V(0) &= x^T(0)Px(0)+\sum_{i=-h_1}^{-1}x^T(i)Q_1x(i)+\sum_{i=-h_k}^{-h_1-1}x^T(i)Q_2x(i) \\ &\quad +\sum_{i=-h_2}^{-h_k-1}x^T(i)Q_3x(i)+h_1\sum_{i=-h_1}^{-1}\sum_{j=i}^{-1}y^T(j)R_1y(j) \\ &\quad +h_{12}\sum_{i=-h_2}^{-h_1-1}\sum_{j=i}^{-1}y^T(j)R_2y(j) \\ &< \sigma_1c_1+\sigma_2c_2 \end{aligned} \quad (46)$$

On the other hand, we know that

$$V(k) \ge x^T(k)Px(k) \ge \lambda_{\min}(\tilde{P})x^T(k)Rx(k) > \lambda_1 x^T(k)Rx(k) \quad (47)$$

Then, we have :

$$x^T(k)Rx(k) < \frac{\alpha^N}{\lambda_1}(\sigma_1c_1+\sigma_2c_2) < c_2 \quad (48)$$

Consequently, the proof is completed.

Remark 4 It should be pointed out that there is difference between finite-time stability and finite-time attractiveness. The first one is about the bound of the system states in a specific time interval, while the latter is about the system state reaching the equilibrium point in a finite-time.

Remark 5 Based on a LKF, new FTS criteria are given in Theorem 1 using some free-weighting matrices. Then, in Theorem 2, less conservative results are obtained by employing the reduced free-matrix-based summation inequality given in Corollary 1. Reducing the variables order is one of the main indices of the effectiveness of our method, especially if we take into account that there is in this article a different approach from those given in the literature.

Theorem 2 *The system (1) is finite time stable with respect to (c_1, c_2, R, N) if there exist symmetric positive definite matrices P, Q_i, R_1, $R_2 \in \Re^{n\times n}$, matrices U_i, V_i, M_i, $S_{ij} \in \Re^{n\times n}$, $i, j = 1, 2, 3$, and scalars λ_i, $i = 1, 2, \ldots, 7$, $\varepsilon_i > 0$, $i = 1, 2, \ldots, 6$, $\alpha > 1$, such that :*

$$0 < \lambda_1 I < \tilde{P} < \lambda_2 I, \tag{49}$$

$$0 < \tilde{Q}_1 < \lambda_3 I,\ \ 0 < \tilde{Q}_2 < \lambda_4 I,\ \ 0 < \tilde{Q}_3 < \lambda_5 I, \tag{50}$$

$$0 < \tilde{R}_1 < \lambda_6 I,\ \ 0 < \tilde{R}_2 < \lambda_7 I, \tag{51}$$

$$\begin{pmatrix} -U_i - U_i^T & U_i^T \\ * & -R_1 \end{pmatrix} \leq 0,\ \ i = 2, 3, \tag{52}$$

$$\begin{pmatrix} -V_i - V_i^T & V_i^T \\ * & -R_2 \end{pmatrix} \leq 0,\ \ i = 2, 3, \tag{53}$$

$$\begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{pmatrix} > 0, \tag{54}$$

$$\begin{pmatrix} \tilde{\Omega}_1 + \tilde{\Omega}_2 + \tilde{\Omega}_3 & \tilde{\Omega}_4 & \Pi_1 \Upsilon_3 & \Pi_2 \Upsilon_4 \\ * & \tilde{\Omega}_5 & 0 & 0 \\ * & * & -R_1 & 0 \\ * & * & * & -R_2 \end{pmatrix} < 0, \tag{55}$$

$$\sigma_1 c_1 + \sigma_2 c_2 < \alpha^{-N} c_2 \lambda_1 \tag{56}$$

where

$$\begin{aligned}
&\Upsilon_1 = sym(diag\{U_1, 3U_2, 5U_3\}),\\
&\Upsilon_2 = sym(diag\{V_1, 3V_2, 5V_3, V_1, 3V_2, 5V_3\}),\\
&\Upsilon_3 = [U_1, \sqrt{3}U_2, \sqrt{5}U_3]^T,\\
&\Upsilon_4 = [V_1, \sqrt{3}V_2, \sqrt{5}V_3, V_1, \sqrt{3}V_2, \sqrt{5}V_3]^T,\\
&\Xi_{11} = \Xi_{22} = sym(diag\{V_1, 3V_2, 5V_3\}),\ \ \Xi_{12} = S,\\
&\Xi_{13} = \Xi_{23} = [V_1, \sqrt{3}V_2, \sqrt{5}V_3]^T,\ \ \Xi_{33} = R_2.
\end{aligned} \tag{57}$$

Proof We can prove Theorem 2 in the same way as Theorem 1 using the summation inequality given in Corollary 1.

Now, we consider the discrete-time system with time-varying delay given by :

$$\begin{aligned}
x(k+1) &= Ax(k) + Bx(k - h(k))\\
x(\theta) &= \varphi(\theta),\ \ \theta \in [-h_2, -h_2 + 1, \dots, 0]
\end{aligned} \tag{58}$$

Based on Theorem 1 and Theorem 2. the following criteria can be easily derived.

Theorem 3 *The system (1) is finite time stable with respect to (c_1, c_2, R, N) if there exist symmetric positive definite matrices P, Q_1, Q_2, Q_3, R_1, $R_2 \in \Re^{n\times n}$, symmetric matrices Z_i^i, $Y_i^i \in \Re^{n\times n}$, $i = 0, \dots, 3$, matrices Z_i^j, $Y_i^j \in \Re^{n\times n}$, $i, j = 0, \dots, 3$ $(i < j)$, M_i, $S_{ij} \in \Re^{n\times n}$, $i, j = 1, 2, 3$, and scalars λ_i, $i = 1, 2, \dots, 7$, $\varepsilon_i > 0$, $i = 1, 2, \dots, 6$, $\alpha > 1$, such that :*

$$0 < \lambda_1 I < \tilde{P} < \lambda_2 I, \tag{59}$$

$$0 < \tilde{Q}_1 < \lambda_3 I,\ \ 0 < \tilde{Q}_2 < \lambda_4 I,\ \ 0 < \tilde{Q}_3 < \lambda_5 I, \tag{60}$$

$$0 < \tilde{R}_1 < \lambda_6 I,\ \ 0 < \tilde{R}_2 < \lambda_7 I, \tag{61}$$

$$Z_i^i \le sym(Z_i^0),\ \ Y_i^i \le sym(Y_i^0),\ \ i = 2, 3, \tag{62}$$

$$\begin{bmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3\\ * & Z_1^1 & Z_1^2 & Z_1^3\\ * & * & Z_2^2 & Z_2^3\\ * & * & * & Z_3^3 \end{bmatrix} \ge 0,\quad \begin{bmatrix} Y_0^0 & Y_0^1 & Y_0^2 & Y_0^3\\ * & Y_1^1 & Y_1^2 & Y_1^3\\ * & * & Y_2^2 & Y_2^3\\ * & * & * & Y_3^3 \end{bmatrix} \ge 0, \tag{63}$$

$$\begin{pmatrix} Z & S\\ * & Z \end{pmatrix} > 0, \tag{64}$$

$$\tilde{\Omega}_1 + \tilde{\Omega}_2 + \tilde{\Omega}_3 < 0, \tag{65}$$

$$\sigma_1 c_1 + \sigma_2 c_2 < \alpha^{-N} c_2 \lambda_1. \tag{66}$$

Proof The proof of Theorem 3 can be directly derived from Theorem 1.

Theorem 4 *The system (1) is finite time stable with respect to* (c_1, c_2, R, N) *if there exist symmetric positive definite matrices* P, Q_i, R_1, $R_2 \in \Re^{n\times n}$, *matrices* U_i, V_i, M_i, $S_{ij} \in \Re^{n\times n}$, $i, j = 1, 2, 3$, *and scalars* λ_i, $i = 1, 2, \ldots, 7$, $\varepsilon_i > 0$, $i = 1, 2, \ldots, 6$, $\alpha > 1$, *such that :*

$$0 < \lambda_1 I < \tilde{P} < \lambda_2 I, \tag{67}$$

$$0 < \tilde{Q}_1 < \lambda_3 I,\ \ 0 < \tilde{Q}_2 < \lambda_4 I,\ \ 0 < \tilde{Q}_3 < \lambda_5 I, \tag{68}$$

$$0 < \tilde{R}_1 < \lambda_6 I,\ \ 0 < \tilde{R}_2 < \lambda_7 I, \tag{69}$$

$$\begin{pmatrix} -U_i - U_i^T & U_i^T \\ * & -R_1 \end{pmatrix} \leq 0,\ \ i = 2, 3, \tag{70}$$

$$\begin{pmatrix} -V_i - V_i^T & V_i^T \\ * & -R_2 \end{pmatrix} \leq 0,\ \ i = 2, 3, \tag{71}$$

$$\begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{pmatrix} > 0, \tag{72}$$

$$\begin{pmatrix} \tilde{\Omega}_1 + \tilde{\Omega}_2 + \tilde{\Omega}_3 & \Pi_1 \Upsilon_3 & \Pi_2 \Upsilon_4 \\ * & -R_1 & 0 \\ * & * & -R_2 \end{pmatrix} < 0, \tag{73}$$

$$\sigma_1 c_1 + \sigma_2 c_2 < \alpha^{-N} c_2 \lambda_1 \tag{74}$$

where

$$\begin{aligned}
&\Upsilon_1 = sym(diag\{U_1, 3U_2, 5U_3\}), \\
&\Upsilon_2 = sym(diag\{V_1, 3V_2, 5V_3, V_1, 3V_2, 5V_3\}), \\
&\Upsilon_3 = [U_1, \sqrt{3}U_2, \sqrt{5}U_3]^T, \\
&\Upsilon_4 = [V_1, \sqrt{3}V_2, \sqrt{5}V_3, V_1, \sqrt{3}V_2, \sqrt{5}V_3]^T, \\
&\Xi_{11} = \Xi_{22} = sym(diag\{V_1, 3V_2, 5V_3\}),\ \ \Xi_{12} = S, \\
&\Xi_{13} = \Xi_{23} = [V_1, \sqrt{3}V_2, \sqrt{5}V_3]^T,\ \ \Xi_{33} = R_2.
\end{aligned} \tag{75}$$

Proof We can follow the same way to prove Theorem 4.

Remark 6 In order to get less conservative results, delay-dependent exponential stability criteria for continuous-time neutral systems is considered in [18]. Then, Jensen inequality, free-weighing matrix, and two delay-partitioning method are used to derive the upper bound of the derivative of LKF. However, these results have conservatism to some extent, which exist room for further improvement. Then, this weakness is well overcome in this article by explicitly taking into account the relaxed summation inequality and system discretization inherent to periodic sampling of the problem. On the other hand, the stability criteria for continuous-time systems given in [26] is delay-independent. It is well known that the delay-dependent criteria are less conservative than delay-independent criteria. Then, Theorem 3 and Theorem 4 show different and good results compared to [18] and [26].

4 Numerical examples

Example 1 Consider the system (1) with the following matrices :

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}$$

Let $R = I$, $\alpha = 1.001$, $\varepsilon = 1.1$, $h_1 = 2$, $c_1 = 4.1$, $c_2 = 60$, and $N = 90$. Applying Theorem 1 and Theorem 2, the maximum upper bound h_2 of time-varying delay is presented in Table 1 for different values of ρ_1 and ρ_2.

Table 1. Comparison of the maximum allowable delay h_2 for different values of ρ_1 and ρ_2.

(ρ_1, ρ_2)	$(0, 0)$	$(0.01, 0.01)$	$(0.01, 0.02)$	$(0.02, 0.02)$	$(0.03, 0.03)$
[15]	12	9	8	7	6
Theorem 1	16	12	11	10	8
Theorem 2	16	12	11	10	8

As it is indicated in Table 1, the obtained values of h_2 in this paper is larger than those obtained in [15], and then the results are significantly improved. On the other hand, it can be clearly seen that the value of h_2 increases when the values of the perturbations bounds ρ_1 and ρ_2 become smaller. Then, it is clear that our approach is less conservative than those given in the literature.

Example 2 Now, we consider the system (58) with the following matrices :

$$A = \begin{bmatrix} 0.6 & 0 \\ 0.35 & 0.7 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}$$

Taking $R = I$, $\alpha = 1.001$, $\varepsilon = 1.1$, $c_1 = 2.1$, $c_2 = 80$, $N = 80$ and applying the FTS results presented in Theorem 3 and Theorem 4, the maximum upper bound h_2 of time-varying delay is given in Table 2 for different values of h_1.

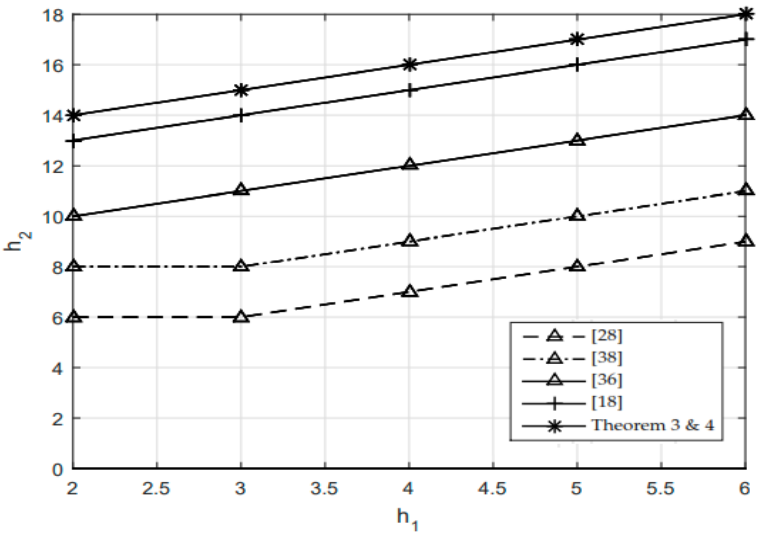

Fig. 1 Comparison of the maximum allowable delay h_2 for different values of h_1

Table 2. Comparison of the maximum allowable delay h_2 for different values of h_1.

h_1	2	3	4	5	6
[24]	6	6	7	8	9
[34]	8	8	9	10	11
[32]	10	11	12	13	14
[15]	13	14	15	16	17
Theorem 3	14	15	16	17	18
Theorem 4	14	15	16	17	18

Then, it can be seen from Table 2 that an increase in the values of h_1 corresponds to an increase in values of the maximum upper bound h_2. Also, the values of h_2 obtained in this paper are better than those given in [24], [34], [32], [15] and then our results are less conservative. Thus, we can say that the use of our summation inequality can gives more interesting results compared than those obtained in the literature. Finally, to further illustrate these results, simulations are given in Figure 1.

5 Conclusion

In this paper, FTS criteria for the discrete-time systems with nonlinear function and interval time-varying delays is discussed. The approach developed here generalizes some of the existing ones, presents less conservative conditions, and sheds new light on this important type of system. Then, based on a LKF and using RFMBI inequality, new approximation of single summation appearing in the derivative of LKF is investigated. The presented work has profound implications for future studies and my can solve the practical control problem. Finally, numerical examples are presented to show the advantage and effectiveness of our approach.

Conflict of interest statement

The authors declare that there is no conflict of interest.

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