An elemental characterization of orthogonal ideals in Lie algebras

Jose Brox, Esther García and Miguel Gómez Lozano

Abstract. In this paper we prove that the ideals generated by two elements x, y in a nondegenerate Lie algebra L over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3}$ are orthogonal if and only if $[x, [y, L]] = 0$.

Key words: Lie algebra, nondegenerate, orthogonal ideals.
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1. Introduction

Elemental characterizations of primeness of nonassociative structures, such as the Beidar, Mikhalev and Slinko's elemental characterization of strong primeness for alternative and linear Jordan algebras [6], its quadratic Jordan systems version [4] by Anquela, Cortés, Loos and McCrimmon, and its analogue for Lie algebras [12], have proved to be very useful in the study of such structures. In particular, they are fundamental in the study of the inheritance of primeness by ideals, and by local algebras and subquotients [1], [3], [2], [15], [16], [11], [9].

A Lie algebra L is prime if it does not have orthogonal nonzero ideals, i.e., if $[I, J] = 0$ for ideals I, J of L implies necessarily that $I = 0$ or $J = 0$. Orthogonality in the associative setting has an easy elemental characterization for semiprime systems: if R is a semiprime associative algebra and $x, y \in R$, then $\mathrm{Id}_R(x) \cdot \mathrm{Id}_R(y) = 0$ if and only $xRy = 0$. This elemental characterization of orthogonality can be exported to other nonassociative settings, as long as they accomplish some requisites:

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1. They are nondegenerate (remember that nondegeneracy is the useful extension of semiprimeness for Jordan and Lie structures).
2. The nondegenerate radical is the intersection of all strongly prime ideals and therefore a nondegenerate structure is a subdirect product of strongly prime ones.
3. Strong primeness can be characterized via elements.

Under these hypotheses it is not difficult to show:

Theorem 2.4. *Let T be a semiprime associative algebra, nondegenerate alternative algebra, or nondegenerate (quadratic) Jordan system, or Lie algebra which is a subdirect product of strongly prime algebras (in particular, if it is a Lie algebra over a field of characteristic zero). Then two elements $x, y \in T$ generate orthogonal ideals if and only if:*

(a) $xTy = 0$ *if T is an associative algebra.*
(b) $x(Ty) = 0$ *or* $(xT)y = 0$ *if T is an alternative algebra.*
(c) $\{x, T, y\} = 0$ *if T is a linear Jordan system.*
(d) $U_x U_T U_y T = 0$ *if T is a quadratic Jordan system.*
(e) $[x, [y, T]] = 0$ *if T is a Lie algebra.*

Hypothesis (2) plays a fundamental role in the proof of this result since it allows to go from the original nondegenerate structure to its strongly prime quotients, and then use the above mentioned elemental characterization of strong primeness.

Lie algebras with a short $\mathbb{Z}$-grading, Lie algebras over fields of characteristic zero, Lie algebras arising from associative algebras and Artinian Lie algebras satisfy (2) [13, 2.10, 3.10, 4.3, 4.7, 5.4], but in general it is an open problem to know if a nondegenerate Lie algebra is a subdirect product of strongly prime Lie algebras.

The aim of this paper is to prove that the property

$$[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] = 0 \text{ if and only if } [x, [y, L]] = 0$$

holds for any nondegenerate Lie algebra L over an arbitrary ring of scalars Φ with $\frac{1}{2}$ and $\frac{1}{3} \in \Phi$. Our proof deeply relies in the ideas and calculations of [12, Proposition 1.4].

2. Preliminaries

2.1. We will be dealing with Lie algebras L over a ring of scalars Φ with $\frac{1}{2}$ and $\frac{1}{3}$. As usual, given a Lie algebra L, $[x, y]$ will denote the Lie bracket, with ad_x the adjoint map determined by x. Typical examples of Lie algebras come from the associative setting: if $(R, +, \cdot)$ is an associative algebra, then $(R, +)$ with product $[x, y] := x \cdot y - y \cdot x$ is a Lie algebra, denoted by R^-. Moreover, if R has an involution $*$, then $\mathrm{Skew}(R, *) := \{x \in R \mid x = -x^*\}$ is a Lie subalgebra of R^-.

2.2. Given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if

$[I, I] \neq 0$ for every nonzero ideal I of L, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for ideals I, J of L. We say that a Lie algebra is *strongly prime* if it is prime and nondegenerate.

2.3. Given a subset S of a Lie algebra L, the *annihilator* or *centralizer* of S in L, $\operatorname{Ann}_L S$, consists of the elements $x \in L$ such that $[x, S] = 0$. By the Jacobi identity, $\operatorname{Ann}_L S$ is a subalgebra of L and an ideal whenever S is so. Clearly, $\operatorname{Ann}_L L = Z(L)$, the center of L. If L is semiprime, then $I \cap \operatorname{Ann}_L I = 0$ for any ideal I of L. Moreover, if L is a Lie algebra with $\frac{1}{2}, \frac{1}{3} \in \Phi$, the annihilator of an ideal I of L, which is nondegenerate as a Lie algebra, has the following nice expression [10, 2.5]:

$$\operatorname{Ann}_L I = \{a \in L \mid [a, [a, I]] = 0\},$$

which implies that $\operatorname{Ann}_L I$ is a nondegenerate ideal of L.

The following theorem is the starting point of our work, since it shows that the elemental characterization of orthogonality we are interested on holds for many families of nondegenerate Lie algebras, and also for semiprime (equiv. nondegenerate) associative algebras, nondegenerate alternative algebras and nondegenerate Jordan systems.

Theorem 2.4. *Let T be a semiprime associative algebra, nondegenerate alternative algebra, or nondegenerate (quadratic) Jordan system, or Lie algebra which is a subdirect product of strongly prime algebras (in particular, if it is a Lie algebra over a field of characteristic zero). Then two elements $x, y \in T$ generate orthogonal ideals if and only if:*

(a) $xTy = 0$ *if T is an associative algebra.*
(b) $x(Ty) = 0$ *or* $(xT)y = 0$ *if T is an alternative algebra.*
(c) $\{x, T, y\} = 0$ *if T is a linear Jordan system.*
(d) $U_x U_T U_y T = 0$ *if T is a quadratic Jordan system (where U denotes the quadratic operator of the Jordan system).*
(e) $[x, [y, T]] = 0$ *if T is a Lie algebra.*

Proof. For any of the (non)associative systems T of the claim, the nondegenerate radical (prime radical in the associative setting, McCrimmon radical in the Jordan setting and Kostrikin radical in the Lie setting) is the intersection of all strongly prime ideals of T [14], [8], [17] (in the alternative case, the intersection of all prime and nondegenerate ideals of a nondegenerate alternative algebra is zero by [5, Corollary 2.5 and Lemma 2.17] and [8]).

Without loss of generality, suppose that T is a Lie algebra with $[x, [y, T]] = 0$. Let us show that the ideals $\operatorname{Id}_T(x)$ and $\operatorname{Id}_T(y)$ generated by x and y are orthogonal, i.e., $[\operatorname{Id}_T(x), \operatorname{Id}_T(y)] = 0$. By hypothesis there exists a family of strongly prime ideals I_α of T such that $\bigcap I_\alpha = 0$ and such that if we denote $T_\alpha := T/I_\alpha$, then there exists a Lie algebra monomorphism $T \to \prod_\alpha T_\alpha$. Now, in each quotient T_α we have that $[\overline{x}, [\overline{y}, \overline{T}_\alpha]] = \overline{0}$ and therefore, by [12, 1.6], $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$ which implies that in each quotient $\overline{[\operatorname{Id}_T(x), \operatorname{Id}_T(y)]} = \overline{0}$ and therefore $[\operatorname{Id}_T(x), \operatorname{Id}_T(y)]$ is contained in each I_α, so $[\operatorname{Id}_T(x), \operatorname{Id}_T(y)] = 0$.

If T is an associative, alternative algebra, a linear Jordan system, or a quadratic Jordan system, the proof follows verbatim by using the elemental characterizations of strong primeness of [6] and [4]. $\square$

3. Main result

In this section we are going to prove that if L is a nondegenerate Lie algebra over an arbitrary ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$, then $[x,[y,L]] = 0$ if and only if $[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] = 0$. Note that in this general setting it is no known if the Kostrikin radical of L is the intersection of all strongly prime ideals of L.

We will use capital letters to denote adjoint operators, i.e., $X := \mathrm{ad}_x$, $A := \mathrm{ad}_a$, $B := \mathrm{ad}_b$, etc.

Theorem 3.1. *Let L be a nondegenerate Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$, and let $x, y \in L$. Then $[x,[y,L]] = 0$ if and only if $[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] = 0$.*

Proof. We will show that $[x,[y,L]] = 0$ implies $[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] = 0$. The converse is trivial.

Let us suppose first that for every $u \in L$ we have $XUY = 0$. Then, by [12, Proposition 1.3] we have that $XY = YX = YUX = 0$ and $[x,y] = 0$. Moreover, for every $a, b \in L$

$$\begin{aligned}
XABY &= X([A,B] + BA)Y = XBAY, \\
XABY &= XA[B,Y] = X[A,[B,Y]] = [X,[A,[B,Y]]] + [A,[B,Y]]X \\
&= -\mathrm{ad}_{[x,[a,[y,b]]]} + ABYX - AYBX - BYAX + YBAX \\
&= YBAX, \\
X^2ABCY &= X[X,A]BCY + XAXBCY \\
&= X[[X,A],B]CY + XB[X,A]CY + XAYBCX \\
&= XC[[X,A],B]Y + XBXACY = XCXABY = 0, \\
XABCY^2 &= 0 \quad \text{(follows symetrically), and} \\
X^2ABCDY^2 &= X[X,A]BCDY^2 = X[[X,A],B]CDY^2 = 0.
\end{aligned}$$

Now, for every $c, d \in L$ we have:

$$\begin{aligned}
&\mathrm{ad}_{\mathrm{ad}_x^2 a} CD\, \mathrm{ad}_{\mathrm{ad}_y^2 a} = (X^2A + AX^2 - 2XAX)CD(Y^2A + AY^2 - 2YAY) \\
&= X^2ACDY^2A + AX^2CDY^2A - 2XAXCDY^2A + X^2ACDAY^2 \\
&+ AX^2CDAY^2 - 2XAXCDAY^2 - 2X^2ACDYAY - 2AX^2CDYAY \\
&+ 4XAXCDYAY = 0.
\end{aligned}$$

Finally, by [12, Proposition 1.5] it is $\mathrm{ad}_x^2 a \in \mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}_y^2 a))$ but since $\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}_y^2 a))$ is a nondegenerate ideal of L by (2.3), i.e., the Lie algebra $L/\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}_y^2 a))$ is nondegenerate, we have that $\overline{\mathrm{ad}_x^2 a} = \overline{0}$ for every

$\overline{a} \in L/\operatorname{Ann}_L(\operatorname{Id}_L(\operatorname{ad}_y^2 a))$, which implies that $\overline{x} = \overline{0}$. Now, $XCD\,\operatorname{ad}_{\operatorname{ad}_y^2(a)} = 0$; repeating this argument and using [12, Proposition 1.5] again $\overline{0} = \overline{y}$ in $L/\operatorname{Ann}_L(\operatorname{Id}_L(x))$, i.e., $y \in \operatorname{Ann}_L(\operatorname{Id}_L(x))$. By induction, using the Jacobi identity, we get that $[\operatorname{Id}_L(x), \operatorname{Id}_L(y)] = 0$.

Now, let us suppose that $XY = 0$. Then for every $a, b \in L$, by [12, Proposition 1.2], we have that $\operatorname{ad}_{\operatorname{ad}_x^3(a)} BY = 0$ and therefore, $\operatorname{ad}_x^3(a) \in \operatorname{Ann}_L(\operatorname{Id}_L(y))$ by the above argument. Similarly, $\operatorname{ad}_y^3(a) \in \operatorname{Ann}_L(\operatorname{Id}_L(x))$. Notice also that $YX = 0$ and $XAY = -YAX$ by [12, Proposition 1.1].

We will now prove some equalities: For every $e, f, g \in L$, $X^2EY^2 = XEY^2 = X^2EFY^2 = X^2EY = 0$ and $XEFY^2 = XFEY^2$, by [12, Proposition 1.2]. Moreover,

$$\begin{aligned} X^2EFY &= X[X,E]FY + XEXFY = X[[X,E],F]Y + XF[X,E]Y \\ &\quad + XEXFY = -Y[[X,E],F]X + XFXEY + XEXFY \\ &= YEXFX + YFXEX - YEFX^2 + XFXEY + XEXFY \\ &= 2YEXFX + 2YFXEX - YEFX^2. \end{aligned}$$

Symmetrically,

$$XEFY^2 = 2XFYEY + 2XEYFY - Y^2FEX, \qquad \text{and}$$

$$\begin{aligned} X^2EFGY^2 &= X[X,E]FGY^2 + XEXFGY^2 \\ &= X[[X,E],F]GY^2 + XF[X,E]GY^2 + XEXFGY^2 \\ &= XG[[X,E],F]Y^2 + XFXEGY^2 + XEXFGY^2 \\ &= XGXEFY^2 + XFXEGY^2 + XEXFGY^2 \\ &= 2XGXEYFY + 2XGXFYEY + 2XFXEYGY \\ &\quad + 2XFXGYEY + 2XEXFYGY + 2XEXGYFY. \end{aligned}$$

Now, using the formulas above:

$$\begin{aligned}
\mathrm{ad}_{\mathrm{ad}_x^2 a}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2 c} &= (X^2A+AX^2-2XAX)B(Y^2C+CY^2-2YCY)\\
&= X^2ABY^2C+AX^2BY^2C-2XAXBY^2C\\
&+X^2ABCY^2+AX^2BCY^2-2XAXBCY^2\\
&-2X^2ABYCY-2AX^2BYCY+4XAXBYCY\\
&= X^2ABCY^2-2XAXBCY^2-2X^2ABYCY\\
&+4XAXBYCY=2XAXBYCY+2XAXCYBY\\
&+2XBXAYCY+2XBXCYAY+2XCXAYBY\\
&+2XCXBYAY-4XAXBYCY-4XAXCYBY\\
&-4XAXBYCY-4XBXAYCY+4XAXBYCY\\
&= -2XAXBYCY+2XCXBYAY-2XAXCYBY\\
&+2XCXAYBY-2XBXAYCY+2XBXCYAY.
\end{aligned}$$

Note that in the last expression the roles of a and c are skew-symmetrical. So if we exchange a and c we obtain: $\mathrm{ad}_{\mathrm{ad}_x^2 a}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2 c}=-\mathrm{ad}_{\mathrm{ad}_x^2 c}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2 b}$. Therefore, if we take $a=\mathrm{ad}_u^2\mathrm{ad}_x^2 v$ and $c=\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'$, for $u,u',v,v'\in L$,

$$\begin{aligned}
\mathrm{ad}_{\mathrm{ad}_x^2 a}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2 c} &= \mathrm{ad}_{\mathrm{ad}_x^2\mathrm{ad}_u^2\mathrm{ad}_x^2 v}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'}\\
&= -\mathrm{ad}_{\mathrm{ad}_x^2\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2\mathrm{ad}_u^2\mathrm{ad}_x^2 v}=0
\end{aligned}$$

because, by [12, Proposition 1.2], $\mathrm{ad}_x^2\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'=0$. Therefore, for every $u,v,u',v'\in L$ we have

$$\mathrm{ad}_{\mathrm{ad}_x^2\mathrm{ad}_u^2\mathrm{ad}_x^2 v}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'}=0.$$

Moreover, since x is an ad-nilpotent element of index ≤ 3 in the quotient algebra $L/\mathrm{Ann}_L(\mathrm{Id}_L(y))$, there exists $z\in\mathrm{Ann}_L(\mathrm{Id}_L(y))$ such that $\mathrm{ad}^2_{\mathrm{ad}_x^2 u}v=\mathrm{ad}_x^2\mathrm{ad}_u^2\mathrm{ad}_x^2 v+z$ by the Jordan identity [7, Lemma 1.7(iii)]. Similarly there exists $z'\in\mathrm{Ann}_L(\mathrm{Id}_L(x))$ such that $\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'=\mathrm{ad}_y^2\mathrm{ad}_{u'}^2\mathrm{ad}_x^2 v'+z'$. Therefore,

$$\mathrm{ad}_{\mathrm{ad}^2_{\mathrm{ad}_x^2 u}v}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'}=\mathrm{ad}_{\mathrm{ad}_x^2\mathrm{ad}_u^2\mathrm{ad}_x^2 v}\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}_y^2\mathrm{ad}_{u'}^2\mathrm{ad}_y^2 v'}=0.$$

Thus, for every $u,v\in L$, $\mathrm{ad}^2_{\mathrm{ad}_x^2 u}v\in\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'))$ and, since the quotient $L/\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'))$ is nondegenerate by (2.3), $\mathrm{ad}_x^2 u\in\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'))$ and $x\in\mathrm{Ann}_L(\mathrm{Id}_L(\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'))$. So

$$\mathrm{ad}_x\mathrm{ad}_b\mathrm{ad}_{\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'}=0.$$

Similarly, for every $u',v'\in L$, $\mathrm{ad}^2_{\mathrm{ad}_y^2 u'}v'\in\mathrm{Ann}_L(\mathrm{Id}_L(x))$ and, since the quotient $L/\mathrm{Ann}_L(\mathrm{Id}_L(x))$ is nondegenerate by (2.3), $\mathrm{ad}_y^2 u'\in\mathrm{Ann}_L(\mathrm{Id}_L(x))$

and $y \in \mathrm{Ann}_L(\mathrm{Id}_L(x))$. By induction, using the Jacobi identity, we get that $[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] = 0$. □

Corollary 3.2. *Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$, and let $x, y \in L$. If $[x, [y, L]] \subset K(L)$, where $K(L)$ denotes the Kostrikin radical of L, then $[\mathrm{Id}_L(x), \mathrm{Id}_L(y)] \subset K(L)$.*

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