CYCLES IN LEAVITT PATH ALGEBRAS BY MEANS OF IDEMPOTENTS

G. ARANDA PINO, J. BROX, M. SILES MOLINA

Abstract. We characterize, in terms of its idempotents, the Leavitt path algebras of an arbitrary graph that satisfies Condition (L) or Condition (NE). In the latter case, we also provide the structure of such algebras. Dual graph techniques are considered and demonstrated to be useful in the approach of the study of Leavitt path algebras of arbitrary graphs. A refining of the so-called Reduction Theorem is achieved and is used to prove that $I(P_c(E))$, the ideal of the vertices which are base of cycles without exits of the graph E, a construction with a clear parallelism to the socle, is a ring isomorphism invariant for arbitrary Leavitt path algebras. We also determine its structure in any case.

Introduction

Leavitt path algebras $L_K(E)$ of row-finite graphs were recently introduced in [2] and [10]. They have become a subject of significant interest, both for algebraists and for analysts working in C^*-algebras. For a field K, the algebras $L_K(E)$ are generalizations of the algebras investigated by Leavitt in [22], and are generated by the quotients of the so-called (CK1) and (CK2) relations applied to path K-algebras associated to graphs E. Moreover, as established in [25], $L_K(E)$ is always an algebra of right quotients of KE. The family of algebras that can be obtained as the Leavitt path algebras of some graph includes, but is by no means limited to, matrix rings $\mathbb{M}_n(K)$ for $n \in \mathbb{N} \cup \{\infty\}$ (where $\mathbb{M}_\infty(K)$ denotes the ring of matrices of countable size with only a finite number of nonzero entries), the Laurent polynomial ring $K[x, x^{-1}]$, the algebraic Toeplitz algebra and the classical Leavitt algebras $L(1, n)$ for $n \geq 2$. Constructions such as direct sums, direct limits and matrices over the previous examples can also be realized in this setting.

Since Leavitt path algebras are constructed from graphs, it is natural to try to understand how the properties of the graph E restrict and shape that of $L_K(E)$. In this approach, maybe the first noticeable restrictions are those related to the cardinality of the graph. In this sense, the first breakthrough was to remove the hypothesis of row-finiteness in the underlying graphs in the original definition. This was first done for Leavitt path algebras in [4] and [27]. It is often the case that the row-finite results are no longer valid for not necessarily row-finite graphs, and when they are, they may come up from totally different proofs, because the existence of infinite emitters (vertices that emit an infinite number of edges) disrupts the application of the (CK2) condition, fundamental to many established proofs, and therefore it causes new phenomena and forces the necessity of finding new tools to circumvent either the application of (CK2) or the appearance of infinite emitters. For example, in [4] it is shown that from a row-infinite countable graph E one can construct a row-finite countable graph

Key words and phrases. Leavitt path algebra, graph C^*-algebra, Condition (L), Condition (NE), primitive idempotent, infinite idempotent, dual graph.

F, a *desingularization* of E, in such a way that $L_K(E)$ and $L_K(F)$ are Morita equivalent and there is a monomorphism of K-algebras from $L_K(E)$ to $L_K(F)$. Finally and very recently, Leavitt path algebras have entered their final stage in terms of cardinality restrictions: by dropping also the countability assumption, arbitrary graphs are now the subject of study, a road started in [20] and [8].

But cardinality is by no means the only graph property relevant to the study of Leavitt path algebras. Because of the handy pictorial representation that the graph provides, a great deal of effort has been focused on trying to figure out the algebraic structure of $L_K(E)$ in terms of the graphical nature of E. Concretely, necessary and sufficient conditions on a graph E have been given so that the corresponding Leavitt path algebra $L_K(E)$ is simple [2], purely infinite simple [3], exchange [14], finite dimensional [6], locally finite (equivalently noetherian) [7], semisimple [5], prime or primitive [15] and von Neumann regular (equivalently π-regular) [8]. Reciprocally, there is some interest on finding ring theoretic characterizations for the Leavitt path algebras of graphs that satisfy properties that are recognizable just by visual inspection, since this implies that if $L_K(E) \cong L_K(F)$ as rings for two graphs E and F, then those graph features are to be satisfied by either or none of the graphs. For example, as was established recently, acyclic graphs are precisely those whose Leavitt path algebras are von Neumann regular rings ([8]); also, graphs whose closed simple paths are never found alone, a graph property known as Condition (K), were characterized in the row-finite [14] and the general case [20] as those whose Leavitt path algebras are exchange rings. In this paper we are interested on two graph properties known as Condition (NE) and Condition (L). The first of them asks for all the cycles of the graph to have no exits, while the second one, in full contrast, demands that every cycle has an exit (an exit for a cycle being an edge that allows us to get "untrapped" from the cycle). Both conditions showed up, jointly with other graph properties, in the characterization of locally noetherian [7] and simple Leavitt path algebras [2], respectively; and necessary and sufficient conditions on $L_K(E)$ in order for Condition (L) to be satisfied by E are known too, but they involve some relation between vertices and ideals (e.g., [15, Proposition 2.8 (ii)]). We present ring theoretic characterizations of both conditions for arbitrary graphs, in terms of idempotents. Concretely, in Theorem 3.2 we establish that E satisfies Condition (NE) if and only if $L_K(E)$ does not present infinite idempotents (and identify the algebraic structure of $L_K(E)$), while in Theorem 4.8 we show that E satisfies Condition (L) if and only if $L_K(E)$ has no non-minimal primitive idempotents.

Conditions (NE) and (L) can be seen as two particular aspects of a more general setting. Consider the set of vertices which are in cycles without exits, $P_c(E)$, and the ideal it generates in $L_K(E)$, $I(P_c(E))$. Then it is to be expected that if E satisfies Condition (NE), the main algebraic features of $L_K(E)$ will be comprised in this ideal, while it is easy to see that E satisfies Condition (L) if and only if $I(P_c(E)) = 0$. So, we study this ideal in Section 5 with the purpose of generalization. Furthermore, in the context of Leavitt path algebras (which are always semiprime, [4, Proposition 6.1]) the socle acquires the form $I(P_l(E))$, where $P_l(E)$ is the set of line points of E ([13, Theorem 5.2]), which comprises the sinks and the vertices in infinite paths without bifurcations or cycles. Hence, there is an a priori eventual relationship between $I(P_c(E))$ and $Soc(L_K(E))$. We prove that this relationship is in fact deep, existing a clear algebraic parallelism between them at other levels as well. In particular, in Theorem 5.10 we show that $I(P_c(E))$ is also a ring isomorphism invariant for Leavitt path algebras of arbitrary graphs.

To achieve our main results we develop two different tools: Condition (L) is determined thanks to an study of primitive idempotents, Condition (NE) by *dual* graph techniques. $I(P_c(E))$ is studied by combining both and by refining, in Theorem 5.8, a useful result about Leavitt path algebras.

In Section 4 we establish, in Proposition 4.3, that the primitive vertices of any Leavitt path algebra are precisely those whose tree does not contain any bifurcations. This contrasts with the path algebras setting even in the finite context, where any vertex is automatically primitive ([19, page 4, (7)]). Since minimal vertices are known to be those whose tree does not contain any bifurcations or cycles without exits, non-minimal primitive vertices allow us to detect the presence of cycles without exits in the Leavitt path algebra.

In Section 2 we follow some ideas introduced by G. Abrams and K. M. Rangaswamy in [8] in the setting of Leavitt path algebras, slightly changing their notation and definitions. Given a graph, we define the *dual graph of any of its subgraphs* and study its properties. The dual construction acts as a localization technique, allowing to isolate any subset X of $L_K(E)$, with E an arbitrary graph, into another Leavitt path algebra $L_K(D_X)$ which shares much of the relevant structure of $L_K(E)$ while having D_X row-finite or even finite (in particular, the behavior of the cycles without exits is respected, as shown in Lemma 5.7). Moreover, the Leavitt path algebra of any graph can be seen as the direct limit of the Leavitt path algebras of the duals of its finite subgraphs. This one belongs to a series of results which serve to study the Leavitt path algebra of a "complex" graph by studying a sequence of Leavitt path algebras of "simpler" graphs. This series was started in [10], where it was shown that a row-finite graph (resp. its Leavitt path algebra) is the direct limit of its finite complete subgraphs (resp. their Leavitt path algebras), and continued in [20], where it is shown that any arbitrary graph (resp. its Leavitt path algebra) is the direct limit of its countable CK-subgraphs (resp. their Leavitt path algebras).

1. Preliminaries

We present the graph-theoretic notation that will be needed in what follows, together with the Leavitt path algebra definition and some basic results about it. Our notation coincides with the standard one encountered along the literature.

Definitions 1.1 (Graph concepts). A *graph* $E = (E^0, E^1, r, s)$ consists of two (disjoint) sets E^0, E^1 of arbitrary cardinal and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For each edge e, $s(e)$ is called *the source of* e and $r(e)$ is called *the range of* e. If $v := s(e)$ and $w := r(e)$, we say that v *emits* e and w *receives* e.

If F is a set of edges and V is a set of vertices of E, we denote $s(F) = \{v \in E^0 \mid v = s(e),\ e \in F\}$ and $s^{-1}(V) = \{e \in E^1 \mid s(e) = v,\ v \in V\}$. We define $r(F)$ and $r^{-1}(V)$ analogously.

We say that E is *countable* if E^0, E^1 are both countable; if both are finite, we say that E is *finite*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. This amounts to saying that each vertex in E emits only a finite number of edges.

A vertex which emits no edges is called a *sink*. A vertex which emits an infinite number of edges is called an *infinite emitter*. A vertex that is neither a sink nor an infinite emitter is said to be *regular*. A vertex which receives no edges is called a *source*. An *isolated vertex* is at the same time a source and a sink.

A *path* μ in a graph E is either a vertex or a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ and $l(\mu) := n$ is the *length* of μ (being 0 the length of a vertex by definition). If $s(\mu) = v$ and $r(\mu) = w$ we say that μ *starts at* v and *ends in* w. We denote by μ^0 the set of its vertices and by μ^1 the set of its edges, that is: $\mu^0 = \{s(e_1), r(e_i) \mid i = 1, \dots, n\}$, $\mu^1 = \{e_i\}_{i=1}^n$. A *bifurcation* for a path μ is a vertex $v \in \mu^0$ such that $|s^{-1}(v)| > 1$. The set of paths of E of length n is denoted by E^n. The set of all paths of E is denoted as $\mathrm{Path}(E)$.

A *cycle* is a path $c = e_1 \dots e_n$ ($e_i \in E^1$) such that $s(c) = r(c) \neq s(e_i)$ for $i \in \{2, \dots, n\}$, i.e., such that it starts at and ends in the same vertex and does not go twice through the same vertex. A *loop* is a cycle of length one. If $v := s(c)$, we say that v *is the base vertex of* c or, equivalently, that c *is based at* v. If E is a graph such that $\mathrm{Path}(E)$ does not contain any cycles, we say that E is *acyclic*. Note that for every cycle $c = e_1 \dots e_n \in \mathrm{Path}(E)$ there are other $n-1$ "equivalent" cycles in $\mathrm{Path}(E)$ formed by cyclic permutation of the edges of c: $e_2 \dots e_n e_1$, $e_3 \dots e_n e_1 e_2$, etc., and that these cycles are all based at different vertices. An edge e is an *exit* for a cycle $c = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. The subset of vertices of E which are base of cycles without exits is denoted by $P_c(E)$.

Recall that E *satifies Condition* (NE) if no cycle of E has exits, while it *satisfies Condition* (L) if everyone of its cycles has an exit.

Given two vertices $v, w \in E^0$, if there is a path $\mu \in \mathrm{Path}(E)$ such that $s(\mu) = v$ and $r(\mu) = w$, we say that v *connects to* w and denote it by $v \geq w$ (this relation is a preorder, but not a partial order if there are cycles).

A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. The *tree* of a vertex v is the set $T(v) = \{w \in E^0 \mid v \geq w\}$, which is the smallest hereditary set of E^0 containing v. A hereditary set is *saturated* if every vertex which (finitely) "feeds" into H and only into H is again in H, that is, if $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. The set of all hereditary and saturated subsets of E is denoted $\mathcal{H}_E$. The *hereditary saturated closure* of $X \subseteq E^0$ is defined as the smallest hereditary and saturated subset of E^0 containing X, and is denoted as $\overline{X}$. It is shown in [17, Remark 3.1] that $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$$\Lambda_0(X) = T(X) := \{v \in E^0 \mid x \geq v \text{ for some } x \in X\},$$
$$\Lambda_n(X) := \Lambda_{n-1}(X) \cup \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\}, \text{ for } n \geq 1.$$

Definition 1.2 (**Leavitt path algebra**). For a graph E and a field K we define the *Leavitt path K-algebra of E*, denoted $L_K(E)$, to be the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e \mid e \in E^1\} \cup \{e^* \mid e \in E^1\}$ which satisfy the following relations:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every $v \in E^0$ which is regular.

Relations (3) and (4) are called, respectively, (CK1) and (CK2) (CK stands for Cuntz-Krieger).

The elements of E^1 are called *real edges* while for $e \in E^1$ we will call e^* a *ghost edge*. We let $r(e^*)$ denote $s(e)$ and $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, by μ^* we denote the element

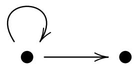
$e_n^* \dots e_1^*$ of $L_K(E)$ and call it a *ghost path*. For any subset $X \subseteq L_K(E)$, we will denote by $I(X)$ the (two-sided) ideal of $L_K(E)$ generated by X.

The following constitute "small", interesting examples of Leavitt path algebras.

(i) The *loop* is the following graph:

It represents the simplest graph (nontrivially) satisfying Condition (NE). Its associated Leavitt path algebra is isomorphic to $K[x, x^{-1}]$, the ring of Laurent polynomials in one variable.

(ii) The *(algebraic) Toeplitz algebra* is the Leavitt path algebra associated to the following graph:



This very simple graph satisfies Condition (L) nontrivially.

We recollect now two fundamental facts about Leavitt path algebras, which are valid in the general case.

Any element in a Leavitt path algebra can be written as a sum of monomials of a specific form ([2, Lemma 1.5]): If $x \in L_K(E)$, then $x = \sum_{i=1}^n k_i p_i q_i^*$, where $n \in \mathbb{N}$, $k_i \in K$ and $p_i, q_i \in \mathrm{Path}(E)$ with $r(p_i) = r(q_i)$ for every $i \in \{1, \dots, n\}$. (†)
Note that this kind of expression is usually not unique (e.g., apply (CK2) to a vertex v with $0 < |s^{-1}(v)| < \infty$).

Leavitt path algebras are also $\mathbb{Z}$-graded algebras ([2, Lemma 1.7]), with grading induced by $deg(v) = 0$ for all $v \in E^0, deg(e) = 1$ and $deg(e^*) = -1$ for all $e \in E^1$. That is, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where $L_K(E)_n = \{\sum kpq^* \mid k \in K,\ p, q \in \mathrm{Path}(E), l(p) - l(q) = n\}$ (note that $E^0 \subseteq L_K(E)_0$).

Later, it will be of importance to know the ring structure of the corner generated by a vertex which is base of a cycle without exits ([13, Lemma 1.5], which is actually valid in full generality):

Proposition 1.3. *Let E be an arbitrary graph and let $v \in P_c(E)$ be the base of the cycle without exits c. Then $vL_K(E)v \cong K[x, x^{-1}]$ as K-algebras (via the identification $x \equiv c, x^{-1} \equiv c^*$).*

Finally, we recall the useful result that follows, which will appear thoroughly in this paper, and whose proof, as done in [12, Proposition 3.1], is also valid in full generality:

Theorem 1.4 (Reduction Theorem). *Let E be an arbitrary graph. Then for every nonzero element $z \in L_K(E)$ there exist $\mu, \nu \in Path(E)$ such that:*

(i) *$\mu^* z \nu = kv$ for some $k \in K \setminus \{0\}$ and $v \in E^0$, or*

(ii) *there exists a vertex $w \in P_c(E)$ such that $\mu^* z \nu$ is a nonzero element in $wL_K(E)w \cong K[x, x^{-1}]$.*

Both cases are not mutually exclusive.

2. Dual graphs

We present the notion of dual of a subgraph in a graph, which is a generalization of the usual notion of dual graph found in the literature, and explore some of its properties. We also propose a new definition of the dual of a graph, which extends the well-behaved properties of the usual one to a wider class of Leavitt path algebras.

Definition 2.1 (Usual dual). Let E be an arbitrary graph. The *usual dual of* E, $D(E)$, is the graph formed from E by taking its length-one paths as the vertices and its length-two paths as the edges; that is,

$$D(E)^0 = \{e \mid e \in E^1\}$$
$$D(E)^1 = \{ef \mid ef \in E^2\}$$
$$s_{D(E)}(ef) = e,\ r_{D(E)}(ef) = f \text{ for all } ef \in E^2.$$

The interest on the usual dual graph notion in the context of Leavitt path algebras lies on the fact that, if E is a row-finite graph without sinks, then there is an algebra isomorphism $L_K(E) \cong L_K(D(E))$ ([1, Proposition 2.11]). The same is true in the context of graph C^*-algebras; i.e., we have $C^*(E) \cong C^*(D(E))$ for E row-finite and with no sinks ([18, Remark 3.3]). Unfortunately, these statements are untrue for the usual dual of a graph with sinks. In what follows, we will propose a new definition of dual graph which generalizes this important property to row-finite graphs with sinks.

Definition 2.2 (Dual of F in E). Let E be a graph and let F be a subgraph of E. Denote $F_1^0 = \{v \in F^0 \mid s_F^{-1}(v) = \emptyset\}, F_1^1 = r_F^{-1}(F_1^0)$ and $F_2^0 = s(F^1) \cap s(E^1 \setminus F^1), F_2^1 = r_F^{-1}(F_2^0)$. We define the graph $D_E(F)$, the *dual of* F *in* E, as follows:

$$D_E(F)^0 = D(F)^0 \cup F_1^0 \cup F_2^0$$
$$D_E(F)^1 = D(F)^1 \cup F_1^1 \cup F_2^1$$
$$s_{D_E(F)}|_{D(F)} = s_{D(F)},\ r_{D_E(F)}|_{D(F)} = r_{D(F)}$$
$$\text{For all } e \in F_i^1 \text{ with } i \in \{1,2\}, s_{D_E(F)}(e) = e \in D(F)^0,\ r_{D_E(F)}(e) = r_F(e) \in F_i^0.$$

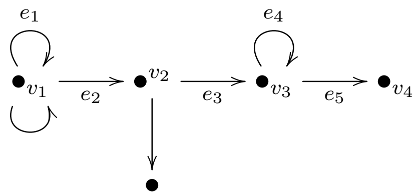
That is, the dual of F in E extends the usual dual of F, by adding to it two kinds of vertices that we will collectively call *vertex-vertices*, together with some edges that we will call *vertex-edges*; concretely, we add:

(i) The sinks of F with their natural connections. That is, for every vertex v which is a sink of F and every edge e which arrives at v in F, we have in $D_E(F)$ a vertex v and an edge (e,v) starting at the vertex e and ending in v.

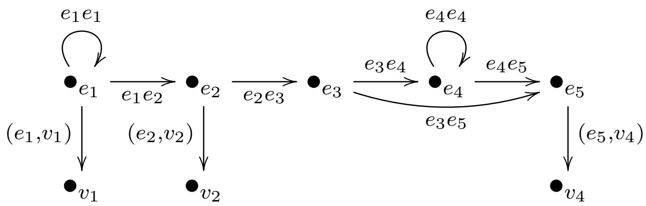
(ii) The "non-full emitters" of F, also with their natural connections. That is, for every vertex v of F which is not a sink of F and emits more edges in E than it does in F, and every edge e which arrives at v in F, we have in $D_E(F)$ a vertex v and an edge (e,v) starting at the vertex e and ending in v. We will call any vertex of this kind, in F, an *intermediate vertex*.

In addition, we will refer as *edge-vertices* and *edge-edges*, respectively, to the vertices and the edges of the dual which come from the usual dual.

Example 2.3 (**Dual of a subgraph in graph**)**.** Consider the following graph E:



Let F be the subgraph of E formed by the vertices $\{v_i\}_{i=1}^4$ and the edges $\{e_i\}_{i=1}^5$. Then the dual graph $D_E(F)$ is



□

We expose, without proof, some elementary properties of the dual graph of a subgraph:

Lemma 2.4. *Let F be a subgraph of a graph E. Then:*

(i) *If F is finite, so is $D_E(F)$.*
(ii) *If F is row-finite, so is $D_E(F)$.*
(iii) *Every loop e in F generates a loop ee (with base e) in $D_E(F)$.*
(iv) *All the vertex-vertices of $D_E(F)$ are sinks, and those are its only sinks.*
(v) *The isolated vertices of F remain isolated in $D_E(F)$.*
(vi) *The intermediate vertices which are also sources in F are isolated in $D_E(F)$.*

Now we can define what we will call the dual of a graph (redefining thus the notion of usual dual) by taking the graph as a subgraph of itself:

Definition 2.5 (**Dual graph**)**.** Given a graph E, we define $d(E) = D_E(E)$ and call it the *dual graph of E.*

When E has no sinks, the usual dual $D(E)$ and the dual $d(E)$ coincide, but they do not when there are sinks present. The advantage of the latter definition has already been stated:

Proposition 2.6 (**Isomorphism with the dual's graph algebra**)**.** *Let E be a row-finite graph. Then:*

(i) $L_K(d(E)) \cong L_K(E)$ *as graded algebras.*
(ii) $C^*(d(E)) \cong C^*(E)$ *as $*$-algebras.*

Proof. We will show that $d(E)$ coincides with the outsplit formed from E by using the maximal partition (see [1, 2.6, 2.9] for the relevant definitions). The proofs follow then from the fact that there is a graded isomorphism (resp. C^*-algebra isomorphism) between the Leavitt path algebra (resp. C^*-algebra) of a graph and that of any of its outsplits ([1, Theorem 2.8] and [18, Theorem 3.2], respectively).

Let E be a row-finite graph and let $\mathcal{P}$ be the partition of E^1 having $m(v) = |s^{-1}(v)|$ for every v that is not a sink (i.e., the partition of E^1 which admits no refinements). Let $E_s(\mathcal{P})$ be the outsplit graph formed from E using the partition $\mathcal{P}$. Since $\mathcal{P}$ is maximal, we have

$$E_s(\mathcal{P})^0 = \{v^e \mid s(e) = v\} \cup \{v \mid v \text{ is a sink}\}$$

$$E_s(\mathcal{P})^1 = \{e^f \mid s(f) = r(e)\} \cup \{e \mid r(e) \text{ is not a sink}\}, \text{ while}$$
$$d(E)^0 = \{e \mid e \in E^1\} \cup \{v \mid v \text{ is a sink}\}$$
$$d(E)^1 = \{ef \mid e, f \in E^1, \ r(e) = s(f)\} \cup \{e \mid r(e) \text{ is not a sink}\}.$$

The maps $\phi^0 : E_s(\mathcal{P})^0 \to d(E)^0$ such that $\phi^0(v^e) = e$, $\phi^0(v) = v$ and $\phi^1 : E_s(\mathcal{P})^1 \to d(E)^1$ such that $\phi^1(e^f) = ef$, $\phi^1(e) = e$ are easily shown to commute with the source and range maps, whence they induce a graph isomorphism from $E_s(\mathcal{P})$ to $d(E)$. □

Remark 2.7. Note that this proof provides us with another way to compute the maximal outsplit of a graph E (other than by definition), namely by constructing its dual $d(E)$, what perhaps is easier and clearer to do, since we just have to put a vertex for every edge of E, an edge for every length-two path of E, a vertex v for every sink of E and an edge (e, v) starting at e and ending in v for every edge e with range v in E (observe that there are no intermediate vertices to consider, since $d(E) = D_E(E)$).

The result above generalizes to any row-finite subgraph, in the sense that the Leavitt path algebra of the dual of the subgraph is a subalgebra of the Leavitt path algebra of the graph:

Proposition 2.8. *Let E be a graph and F be a row-finite subgraph of E. Then there is a graded monomorphism $\theta : L_K(D_E(F)) \to L_K(E)$. In addition, $F^0 \cup F^1 \subseteq \theta(L_K(D_E(F)))$.*

Proof. The proof is essentially a rewriting, in the dual graphs language, of the proofs for [8, Proposition 1] and [8, items (1),(2) of Proposition 2]. The specific construction of the monomorphism will be of use in later sections and for that reason we include it here.
For clarity, denote $G = D_E(F)$. For any vertex-vertex $v \in G^0$, denote

$$u_v = v - \sum_{\{e \in F^1 \ | \ s_F(e) = v\}} ee^* \in L_K(E),$$

understanding an empty sum to be 0. Note that $u_v \neq 0$ because either v is a sink, or it is an intermediate vertex and thus there exists $f \in E^1$ such that $s(f) = v$ and $f \notin F^1$. Note also that $\{u_v \mid v \in G^0 \setminus F^1\}$ is a set of pairwise orthogonal idempotents.
Now, define on the generators and extend to an algebra homomorphism the map $\theta : L_K(G) \to L_K(E)$ in the following way:

If e is an edge-vertex, then $\theta(e) = ee^*$.
If v is a vertex-vertex, then $\theta(v) = vu_v = u_v$.
If ef is an edge-edge, then $\theta(ef) = e\theta(f) = eff^*$.
If (e, v) is a vertex-edge, then $\theta((e, v)) = e\theta(v) = eu_v = e - \sum_{\{f \in F^1 \ | \ s_F(f) = v\}} \theta(ef)$.
If $\mu \in G^1$ then $\theta(\mu^*) = (\theta(\mu))^*$.

Now, to check that θ is compatible with the Leavitt path algebra relations is a matter of simple algebraic manipulations. Moreover, it is not difficult to see that $\theta(v)^* = \theta(v)$ and $\theta(e)^* = \theta(e)$. Note that every generator is mapped to an element of its same degree, so that the homomorphism will be a graded homomorphism. Note also that all the vertices have nonzero images and thus, by the Graded Uniqueness Theorem ([27, Theorem 4.8]), θ will in fact be a graded monomorphism. □

Remark 2.9. It may prove useful for the future to write down, defined on generators, the inverse isomorphism of θ, $\Phi : \theta(L_K(D_E(F))) \to L_K(D_E(F))$, in order to know explicitly where are mapped, in its dual, the vertices and edges of F:

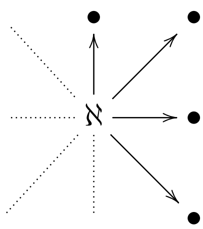
If $v \in F^0$, then $\Phi(v) = \delta_v v \ + \sum_{\{e \in F^1 \ \mid \ s_F(v)=e\}} e$,

where an empty sum is 0 and $\delta_v = \begin{cases} 1, \text{if } v \text{ is a sink or an intermediate vertex of } F \\ 0, \ \text{otherwise.} \end{cases}$

If $e \in F^1$, then $\Phi(e) \ = \sum_{\{f \in D_E(F)^1 \ \mid \ s_{D_E(F)}(f)=e\}} f$.

If $e \in F^1$, then $\Phi(e^*) = (\Phi(e))^*$.

A subgraph F of a graph E is said to be *complete* when for each regular vertex v of F we have $|s_F^{-1}(v)| = |s_E^{-1}(v)|$. Complete subgraphs are precisely those subgraphs that naturally induce a subalgebra $L_K(F)$ of $L_K(E)$. In the same spirit, given a set X of elements of $L_K(E)$ satisfying suitable conditions, Proposition 2.8 allows us to find a subalgebra A of $L_K(E)$ such that $X \subseteq A$ and which is a Leavitt path algebra that inherits several important properties from $L_K(E)$, in the following manner: we decompose every element of X as an expression on some generators, find a subgraph F of E which contains all those generators and, if F is row-finite, we conclude that $X \subseteq \theta(L_K(D_E(F)))$. The mentioned 'suitable conditions' are precisely those which allow the existence of such a row-finite subgraph F. For example, this is trivially achieved if X is finite. Thus, this result is useful for graphs which do not contain "nontrivial" finite complete subgraphs enveloping the generators of our set X of interest, as happens with any subset containing edges of the infinite clock ($\aleph$ being an infinite cardinal):



It is clear that the only finite complete subgraphs of the infinite clock are the empty graph, the graph consisting just of the central vertex, and any subset of the set of sinks.

The following notation will be useful:

Definition 2.10. Given a graph E, and given a subset $X \subseteq L_K(E)$, we express every $x \in X$ in a convenient form (as in (†)). Then we define F_X, *an enveloping subgraph for* X, as the subgraph of E formed by taking all the vertices and all the edges appearing in those expressions, as well as all the sources and ranges of these edges. Concretely, if $X = \{x_l\}_{l \in \Lambda}$, write

$$x_l = \sum_n k_n^l \, v_n^l + \sum_m k_m^l \, p_m^l + \sum_j k_j^l \, p_j^{l*} + \sum_i k_i^l \, p_i^l \, q_i^{l*},$$

where for every $l \in \Lambda$ we have $k_n^l, k_m^l, k_j^l, k_i^l \in K \setminus \{0\}$, $v_n^l \in E^0$ and $p_m^l, p_j^l, p_i^l, q_i^l \in \text{Path}(E)$, which are such that $r(p_i^l) = r(q_i^l)$, and such that we have $p_i^l = e_{i,1}^l \dots e_{i,r_i^l}^l$ and $q_i^l = f_{i,1}^l \dots f_{i,s_i^l}^l$, with $e_{i,m}^l, f_{i,n}^l \in E^1$ and $r_i^l, s_i^l \geq 1$ for every i (i.e., we can assure that these paths are not vertices).

Then F_X is formed by $F_X^1 = \{e_{i,m}^l \mid l \in \Lambda,\ i,\ m \in \{1,\dots,r_i^l\}\} \cup \{f_{i,n}^l \mid l \in \Lambda,\ i,\ n \in \{1,\dots,s_i^l\}\}$ and $F_X^0 = \{v_i^l \mid l \in \Lambda\} \cup s(F_X^1) \cup r(F_X^1)$.

Note that the enveloping subgraph for X is not unique, as its structure depends heavily on the selected expressions (as in (†)) of the elements of X, which are not unique themselves.

In the same spirit, we give the following definition which is central for our purposes:

Definition 2.11. We denote $D_X := D_E(F_X)$ and

$$A_X := \theta(L_K(D_X)),$$

as constructed in Proposition 2.8. In particular, if $X = \{x\}$ is a singleton set, we will forget the braces and simply write F_x, D_x and A_x.

Our last result about dual graphs is the following proposition (the adaptation of [8, Proposition 2, (4)]), which states that the Leavitt path algebra of a graph can be viewed as the direct union $\sqcup$ of the Leavitt path algebras of the duals of its finite subgraphs.

Theorem 2.12. *Let E be an arbitrary graph. Then* $L_K(E) = \bigsqcup_{\{X \subseteq L_K(E) \ \mid\ |X|<\infty\}} A_X$.

Proof. Let $X \subseteq L_K(E)$ be finite, say $X = \{x_n\}_{n=1}^N$, and assume a convenient expression as a sum of monomials (as in (†)) for every x_n . By construction, x_n is in the subalgebra of $L_K(E)$ generated by F_X for every $n \in \{1,\dots,N\}$, which implies by Proposition 2.8 that $x_n \in A_X$ for every $n \in \{1,\dots,N\}$. In addition, since F_X is finite, D_X is finite as well; in particular, $L_K(D_X)$ and thus A_X are finitely generated K-algebras.

Now let X_1, X_2 be two finite subsets of $L_K(E)$ and let T_1, T_2 denote respective finite sets of generators for A_{X_1} and A_{X_2}. Then $T = T_1 \cup T_2$ is, by construction, such that $A_{X_1} \cup A_{X_2} \subseteq A_T$. This proves that the collection $\{A_X \mid X \subseteq L_K(E), |X| < \infty\}$ is an upward directed set of subalgebras of $L_K(E)$. The claim follows now taking into account that $X \subseteq A_X$ for any finite subset X of $L_K(E)$. □

A direct application of Theorem 2.12 would for instance yield the von Neumann regular characterization of Leavitt path algebras given in [8, Theorem 1].

3. Infinite idempotents and Condition (NE)

We characterize in full generality the Leavitt path algebras associated to graphs that satisfy Condition (NE) in terms of idempotents, and establish their structure via dual graph techniques.

The following lemma establishes that the cycles without exits of $D_E(F)$ cannot come from cycles with exits of E, even from those whose exits are “hidden” to F:

Lemma 3.1. *Let E be an arbitrary graph and F be a row-finite subgraph of E. Then there is an injective map from the set of cycles without exits of $D_E(F)$ to the set of cycles without exits of E.*

Proof. Let $c = (e_1, e_2)(e_2, e_3) \dots (e_n, e_1)$ be a cycle without exits of $D_E(F)$, where $c^0 = \{e_1, \dots, e_n\}$ and (e_i, e_j) denotes the edge joining the vertices e_i and e_j. By construction of the dual, any vertex e_i must come from an edge e_i of F. Also by construction, in F, e_n connects (directly) to e_1, e_i connects to e_{i+1} for $i \in \{1, \dots, n-1\}$ and $r(e_i) \neq r(e_j)$ for $i \neq j$, so that $c' = e_1 \dots e_n$ is a cycle of F. Suppose that c' has an exit $e \in E^1$ at a vertex $v = r(e_j)$; then either e is in F^1, which is impossible because in $D_E(F)$ the cycle c would have an exit (e_j, e) at the vertex e_j, or $e \in E^1 \setminus F^1$, in which case v would be an intermediate vertex of F and, in $D_E(F)$, the vertex e_j would have an exit (e_j, v) (with range v), which is also impossible. Thus, every cycle without exits of $D_E(F)$ comes from a cycle without exits of E. That no two of these cycles of $D_E(F)$ come from the same one of E is again clear from the construction of the dual. □

Recall that, given a ring R, an idempotent $e \in R$ is an *infinite idempotent* if eR is isomorphic as a right R-module to a proper direct summand of itself (equivalently, if Re is isomorphic as a left R-module to a proper direct summand of itself).

We remember also that two idempotents $p, q \in R$ are *(Murray-von Neumann) equivalent*, and denote it by $p \sim q$, if there exist $x, y \in R$ such that $p = xy$ and $yx = q$ or, equivalently, if pR and qR are isomorphic as right R-modules (equivalently, if Rp and Rq are isomorphic as left R-modules).

The following characterization of infinite idempotents in terms of elements of the ring is well known: $e \in R$ is an infinite idempotent if and only if there exists a pair of nonzero orthogonal idempotents $x, y \in R$ such that $e = x + y$ and $e \sim x$.

Theorem 3.2. (**Structure Theorem for the Leavitt path algebra of an (NE) graph**) *Let E be any graph. The following conditions are equivalent:*

(i) *E satisfies Condition* (NE).
(ii) $L_K(E) = \bigsqcup_{\{X \subseteq L_K(E) \ \mid\ |X| < \infty\}} A_X$, *where A_X is given in Definition 2.11 and is isomorphic to* $\left(\bigoplus_{i=1}^{r_t} \mathbb{M}_{n_i^t}(K)\right) \oplus \left(\bigoplus_{j=1}^{s_t} \mathbb{M}_{m_j^t}(K[x, x^{-1}])\right)$, *with* r_t, s_t, n_i^t, $m_j^t \in \mathbb{N}$.
(iii) *$L_K(E)$ has no infinite idempotents.*

Proof. (i) ⇒ (ii). Let E be a graph satisfying Condition (NE). By Theorem 2.12, $L_K(E) = \bigsqcup_{\{X \subseteq L_K(E) \ \mid\ |X| < \infty\}} A_X$. Since E satisfies Condition (NE), any of its enveloping subgraphs F_X satisfies it too, and so does D_X by Lemma 3.1. In addition, F_X is finite, which implies that its dual D_X is also finite. Therefore, by the Structure Theorem of noetherian Leavitt path algebras ([7, Theorem 3.8]),

$$L_K(D_X) \cong \left(\bigoplus_{i=1}^{r_X} \mathbb{M}_{n_i^{r_X}}(K)\right) \oplus \left(\bigoplus_{j=1}^{s_X} \mathbb{M}_{m_j^{s_X}}(K[x, x^{-1}])\right),$$

what implies the claim, as $A_X \cong L_K(D_X)$.
(ii) ⇒ (iii). Suppose that

$$L_K(E) \cong \bigsqcup_{t \in T} \left(\left(\bigoplus_{i=1}^{r_t} \mathbb{M}_{n_i^t}(K)\right) \oplus \left(\bigoplus_{j=1}^{s_t} \mathbb{M}_{m_j^t}(K[x, x^{-1}])\right)\right)$$

contained an infinite idempotent. Then there should exist $e, x, y, a, b \in L_K(E)$ such that e is an infinite idempotent, x, y are nonzero orthogonal idempotents, $e = x + y$, $e = ab$ and $x = ba$. But then there should exist, for a t_0 big enough, similar elements in $\left(\bigoplus_{i=1}^{r_{t_0}} \mathbb{M}_{n_i^{r_{t_0}}}(K)\right) \oplus \left(\bigoplus_{j=1}^{s_{t_0}} \mathbb{M}_{m_j^{s_{t_0}}}(K[x, x^{-1}])\right)$. Because of this element-wise characterization of infinite idemponents, we see that any infinite idempotent in a subring would also be an infinite idempotent in any overring of it. Thus, for the aforementioned matrix ring, its classical ring of quotients $\left(\bigoplus_{i=1}^{r_{t_0}} \mathbb{M}_{n_i^{r_{t_0}}}(K)\right) \oplus \left(\bigoplus_{j=1}^{s_{t_0}} \mathbb{M}_{m_j^{s_{t_0}}}(K(x))\right)$ does contain an infinite idempotent. This is impossible because in a semisimple artinian ring there are no infinite idempotents.
(iii) $\Rightarrow$ (i). Suppose that $L_K(E)$ does not satisfy Condition (NE). Then there exists a cycle $c \in L_K(E)$ with exits based at a vertex v, and this implies that v is an infinite idempotent since $v = cc^* + (v - cc^*)$ with $v - cc^* \neq 0$ (because c has exits), $cc^*(v - cc^*) = 0$ and $v = c^*c \sim cc^*$. This is a contradiction with the hypothesis. $\square$

Remark 3.3. In particular, we have shown that Condition (NE) is a ring isomorphism invariant for Leavitt path algebras; that is, if E, F are graphs such that $L_K(E) \cong L_K(F)$ as rings and E satisfies Condition (NE), then F satisfies Condition (NE) too.

4. Primitive idempotents and Condition (L)

Non-minimal primitive idempotents are introduced and an algebraic characterization for Condition (L) is achieved consequently. In this manner, we add the characterization of Condition (L) alone to the already known ones of Condition (L) plus Condition (MT3) in the row-finite context (which give rise to primitive Leavitt path algebras, see [15, Theorem 4.3]) and Condition (L) plus cofinality (which give rise to simple Leavitt path algebras, see [20, Theorem 3.11]). As a corollary, we give a new algebraic characterization of simple Leavitt path algebras.

Proposition 4.1. *Let e be an idempotent in a ring R (not necessarily unital). The following conditions are equivalent:*

(i) *eR is an indecomposable right R-module (equivalently, Re is an indecomposable left R-module).*
(ii) *eRe is a ring without nontrivial idempotents.*
(iii) *e has no decomposition into $a + b$, where a, b are nonzero orthogonal idempotents in R.*

Proof. As in [21, Proposition 21.8]. $\square$

Definition 4.2. Following [21], if an idempotent $0 \neq e \in R$ satisfies any of these conditions, we say that e is a *primitive idempotent*.

We recall that a vertex v is called a *line point* if there are neither cycles nor bifurcations at any vertex $w \in T(v)$. We denote, as usual, the set of all line points of E by $P_l(E)$.

Proposition 4.3. *Let E be an arbitrary graph and let $v \in E^0$. Then v is a primitive idempotent of $L_K(E)$ if and only if its tree $T(v)$ has no bifurcations.*

Proof. Suppose that $T(v)$ has its first bifurcation at w, with μ being the path which connects v to w. Let e and f be two different edges emitted by w; then $ee^* \neq w$ and therefore $0 \neq \mu ee^*\mu^* \neq v$. It is easy to verify that $\mu ee^*\mu^*$ is a (nontrivial) idempotent living in $vL_K(E)v$ and thus, by item (ii) of Proposition 4.1, v cannot be a primitive idempotent.
Now let v be a vertex of E such that $T(v)$ has no bifurcations. Two cases can occur:

Case 1: $T(v)$ does not contain vertices in cycles. In this case, $v \in P_l(E)$, what means that v is minimal ([9, Theorem 1.9]) and therefore primitive.

Case 2: $T(v) \cap P_c(E) \neq \emptyset$. Since $T(v)$ has no bifurcations, there can be only one cycle $c \in L_K(E)$ such that $T(v) \cap c^0 \neq \emptyset$, which in addition has no exits. Furthermore, every vertex of $T(v)$ is either in c^0 or connects to another vertex in c^0 via a path without bifurcations. Thus, by [12, Lemma 2.2] (which is valid in our context), there exists $w \in c^0$ such that $L_K(E)v \cong L_K(E)w$ as left $L_K(E)$-modules. Since w is in a cycle without exits, by Proposition 1.3 we have $wL_K(E)w \cong K[x, x^{-1}]$, which is a ring without nontrivial idempotents. Now Proposition 4.1 gives that w and v are both primitive and finalizes the proof. □

Remark 4.4. If $vL_K(E)v$ is a ring with no nontrivial idempotents (e.g., a domain) then v is a primitive idempotent and, as seen as a consequence of the proof above, we have either $vL_K(E)v \cong K$ (if v is minimal) or $vL_K(E)v \cong K[x, x^{-1}]$ (if it is not).

We have found a close relationship between the primitive and the minimal vertices of the Leavitt path algebra of any graph: the minimal vertices are those whose trees do not contain bifurcations nor bases of cycles, while the primitive vertices see this second condition suppressed. Thus, the following definition is of interest:

Remark 4.5. Hence, $v \in E^0$ is a non-minimal primitive vertex of $L_K(E)$ if and only if $vL_K(E)v \cong K[x, x^{-1}]$. In particular, the vertices in $P_c(E)$ are non-minimal primitive.

Note that while infinite idempotents pass from subrings to rings, this is not the case for non-minimal primitive idempotents.

Proposition 4.3 provides us with a tool to distinguish between cycles with and without exits in a graph, giving us a characterization of Condition (L) in terms of primitive vertices:

Corollary 4.6. *Let E be any graph. The following conditions are equivalent:*

(i) *E satisfies Condition* (L).
(ii) *$L_K(E)$ has no non-minimal primitive vertices.*

Proof. By Proposition 4.3, $L_K(E)$ contains a non-minimal primitive vertex if and only if E contains a cycle without exits. □

As far as we know, a ring-theoretic characterization of Condition (L) is lacking in the literature. We provide one below, extending Corollary 4.6 from the non-minimal primitive vertices to the non-minimal primitive idempotents of the Leavitt path algebra. Hence, we show that Condition (L) is an invariant of ring isomorphisms, in the sense that if E, F are two graphs such that $L_K(E) \cong L_K(F)$ as rings and E satisfies Condition (L), then F satisfies it too.

Proposition 4.7. *If $z \in L_K(E)$ is a primitive idempotent such that we can write $\alpha z\beta = kv$ for $\alpha, \beta \in L_K(E)$ with α or β a monomial, $k \in K \setminus \{0\}$, and some vertex $v \in E^0$, then $L_K(E)z \cong L_K(E)v$. If, moreover, z is non-minimal primitive, then $zL_K(E)z \cong K[x, x^{-1}]$.*

Proof. Consider $a = \frac{1}{k}\alpha z, b = z\beta$ (note that either $va = a$ or $bv = b$ because either α or β is a monomial). Then $ab = v$, and $e := ba = \frac{1}{k}z\beta\alpha z$ is in $zL_K(E)z$. Moreover, $e^2 = baba = bva = ba = e$ and thus $e \sim v$. Since z is a primitive idempotent, $zL_K(E)z$ is a ring without nontrivial idempotents, so that $e \in \{0, z\}$; and since $\alpha e\beta = kv \neq 0$ implies $e \neq 0$, we have $z = e \sim v$, what means, as desired, that $L_K(E)z \cong L_K(E)v$. If in addition z is non-minimal primitive, so is v, and hence $zL_K(E)z \cong vL_K(E)v \cong K[x, x^{-1}]$. □

Theorem 4.8. *Let E be any graph. The following conditions are equivalent:*

(i) *E satisfies Condition* (L).
(ii) *$L_K(E)$ has no non-minimal primitive idempotents.*

Proof. If $L_K(E)$ has no non-minimal primitive idempotents, in particular it has no non-minimal primitive vertices, so that by Corollary 4.6, E satisfies Condition (L).
Now suppose E satisfies Condition (L) and let x be a non-minimal primitive idempotent of $L_K(E)$. By the Reduction Theorem there exist a vertex v, a nonzero scalar k and elements $\mu, \nu \in \mathrm{Path}(E)$ such that $\mu^* x\nu = kv$. Note that, by Corollary 4.6, v cannot be non-minimal primitive. But this is a contradiction since by Proposition 4.7, $L_K(E)v \cong L_K(E)x$. □

The tools developed above will allow us to reformulate, in terms of idempotents, the known simplicity and purely infinite simplicity results for Leavitt path algebras.

In [20, Theorem 3.11], arbitrary Leavitt path algebras $L_K(E)$ which are simple are characterized as those whose graphs simultaneously satisfy these two conditions:

(i) $\mathcal{H}_E = \{\emptyset, E^0\}$.
(ii) E satisfies Condition (L).

Since condition (i) above happens to be equivalent to saying that there are no (two-sided) ideals generated by idempotents in $L_K(E)$ ([10, Proof of Theorem 5.3]), Theorem 4.8 allows us to state the following:

Corollary 4.9. *Let E be an arbitrary graph. Then $L_K(E)$ is simple if and only if it has no non-minimal primitive idempotents and no nontrivial two-sided ideals generated by idempotents.*

If we add a third condition to the characterization of simplicity exposed before the former corollary, we characterize the purely infinite simple Leavitt path algebras; namely, as those whose graphs E satisfy ([4, Theorem 4.3]):

(i) $\mathcal{H}_E = \{\emptyset, E^0\}$.
(ii) E satisfies Condition (L).
(iii) Every vertex of E connects to a cycle.

Note that if E is a finite graph, then condition (iii) can be changed by the condition that there are no minimal idempotents: on the one hand, if every vertex connects to a cycle, there are no minimal vertices and, on the other hand, if there are no minimal vertices then there are no sinks, and since E is finite, every vertex must connect to a cycle. Now, $Soc(L_K(E)) =$

$I(P_l(E))$ ([12, Theorem 4.2]) and $P_l(E) = \emptyset$ (because E is finite and there are no sinks) imply that if there are no minimal vertices, then there are no minimal idempotents at all (the converse is obvious). Therefore, we can establish a result similar to the corollary given above:

Corollary 4.10. *Let E be a finite graph. Then $L_K(E)$ is purely infinite simple if and only if it has no primitive idempotents and no nontrivial ideals generated by idempotents.*

5. The Reduction Theorem and $I(P_c(E))$

$I(P_c(E))$, the ideal generated by the vertices in cycles without exits, bears a clear parallelism with $I(P_l(E))$, the ideal generated by vertices in line points, which is known to be the socle of the Leavitt path algebra $L_K(E)$. That makes this ideal interesting to be studied on its own. In this section we do so by relating non-minimal primitive idempotents with $I(P_c(E))$, what serves us to prove that this ideal is invariant under ring isomorphisms between Leavitt path algebras. We achieve this by a refinement of the Reduction Theorem (see Section 1). First we will work in the row-finite case; next we will build on this one, using again dual graph techniques, to achieve the general case. In the last part we will also provide the structure of $I(P_c(E))$, revealing a bit more of its parallelism with $Soc(L_K(E))$.

5.1. **The row-finite case.** Under certain conditions, we can construct 'quotient' Leavitt path algebras by means of quotient graphs, what will be of use in our next proposition. We recollect their definition:
Let E be a row-finite graph and consider $H \in \mathcal{H}_E$. The quotient graph E/H is defined as

$$(E/H)^0 = E^0 \setminus H$$

$$(E/H)^1 = \{e \in E^1 \mid r(e) \notin H\}$$

$$r_{E/H} := r|_{E^0\setminus H} \text{ and } s_{E/H} := s|_{E^0\setminus H}$$

We note that $I(P_c(E))$ cannot contain any polynomials in cycles with exits:

Lemma 5.1. *If E is an arbitrary graph and c is a cycle with exits of $L_K(E)$, then $p(c, c^*) \notin I(P_c(E))$ for any polynomial p.*

Proof. Suppose on the contrary that there exists a cycle with exits, c, such that $p(c, c^*) \in I(P_c(E))$ for some polynomial p. Write $p(c, c^*) = \sum_i^n k_i c^i + \sum_j^m k'_j (c^*)^j$. As $I(P_c(E)) = I(\overline{P_c(E)})$ (by the first part of [14, Lemma 2.1], which is valid in full generality), $I(P_c(E))$ is a graded ideal by [27, Lemma 5.6] (taking $S = \emptyset$), and therefore every monomial of $p(c, c^*)$ is in $I(P_c(E))$. In particular, for some i, we either have $c^i \in I(P_c(E))$ or $(c^*)^i \in I(P_c(E))$. In any case we get $(c^*)^i c^i = r(c) \in I(P_c(E))$, so that $r(c)$ is in $\overline{P_c(E)}$ (because by [27, Proof of Theorem 5.7 (1)], which is valid in general, $I(H) \cap E^0 = H$ for $H \in \mathcal{H}_E$, taking $S = \emptyset$). Let n be the smallest nonnegative integer having $\Lambda_n(P_c(E)) \cap c^0 \neq \emptyset$. Choose v in this intersection. If $n > 0$ then $\Lambda_{n-1}(P_c(E)) \cap c^0 = \emptyset$ and therefore $\emptyset \neq r(s^{-1}(v)) \subseteq \Lambda_{n-1}(P_c(E))$. In particular $\Lambda_{n-1}(P_c(E)) \cap c^0 \neq \emptyset$, a contradiction, so $n = 0$ and consequently $T(P_c(E)) \cap c^0 = P_c(E) \cap c^0 \neq \emptyset$ (note that $P_c(E)$ is hereditary). But this is another contradiction, because no vertex can be simultaneously a base for a cycle with exits and for a cycle without exits. $\square$

Recall that, when using the Reduction Theorem for an element x, we get in $I(\{x\})$ either a vertex or a polynomial in a cycle without exits; the following proposition gives a sufficient condition to guarantee that we can actually get a vertex.

Proposition 5.2. *If E is a row-finite graph and $x \in L_K(E) \setminus I(P_c(E))$ then there exist $k \in K \setminus \{0\}$, $\mu, \nu \in Path(E)$ and $v \in E^0$ such that $\mu^* x \nu = kv$.*

Proof. Denote $I = I(P_c(E))$. Since $P_c(E)$ is a hereditary subset of E^0, I is a graded ideal, and since E is a row-finite graph, by [14, Lemma 2.3 (i)], $L_K(E)/I$ is graded isomorphic to the Leavitt path algebra with associated graph $F = E/\overline{P_c(E)}$. Denote by $[x]$ the class of the element $x \in L_K(E)$ in $L_K(E)/I$. Suppose $x \notin I$. Then, $0 \neq [x] \in L_K(E)/I$ and by the Reduction Theorem we can find $k \in K \setminus \{0\}$ and two paths $[\mu], [\nu]$ in $L_K(E)/I$ (coming from paths μ, ν of $L_K(E)$) such that either $[\mu]^*[x][\nu] = k[v]$ for some vertex $[v] \in F^0$ (coming from a vertex $v \in E^0$) or $[\mu]^*[x][\nu] = [p]$ for some polynomial $[p]$ based at a cycle without exits $[c]$ of F. But then $[\mu^* x \nu] = [kv]$ or $[\mu^* x \nu] = [p]$ (the conjugation can go inside the class because of the specific construction of the epimorphism between $L_K(E)$ and $L_K(E)/I$) and thus $\mu^* x \nu = kv + y$ or $\mu^* x \nu = p + y'$ in $L_K(E)$, for some $y, y' \in I$. We can write y and y' in the form

$$\sum_n k_n\, v_n + \sum_m k_m\, p_m\, r(p_m) + \sum_j k_j\, r(p_j)\, p_j^* + \sum_i k_i\, p_i\, r(p_i)\, q_i^*,$$

where the set of vertices $V = \{v_n, r(p_m), r(p_j), r(p_i)\}$ is contained in I and all the paths have nonzero lengths. We will study these two cases separately.

Case 1: $[kv] \neq 0$ implies $v \notin I$, so either v is a sink (in which case $vyv = 0$) or there exists $e_1 \in s^{-1}(v)$ such that $e_1 \notin I$. Moreover, $u_1 = r(e_1) \notin I$ either (because $e_1 = e_1 u_1$). Consider $z_1 = vyv$; if $z_1 = 0$ we are done ($v\mu^* x \nu v = kv$). If $z_1 \neq 0$ it must be $z_1 = \sum_{i=1}^N k_i p_i q_i^*$ with $l(p_i), l(q_i) \geq 1$, since $V \subseteq I$ while $v \notin I$. Now consider $z_1' = e_1^* z_1 e_1$: if $z_1' = 0$ we are done ($e_1^* \mu^* x \nu e_1 = ku_1$); if not, we can rearrange z_1 in the form $z_1 = \sum_{i=1}^{N_1} k_i e_1 p_i' q_i'^* e_1^* + z_1''$, with $N_1 \leq N$, $l(p_i') = l(p_i) - 1$, $l(q_i') = l(q_i) - 1$ and $e_1^* z_1'' e_1 = 0$, so that $z_1' = \sum_{i=1}^{N_1} k_i p_i' q_i'^*$ and $e_1^* \mu^* x \nu e_1 = ku_1 + z_1'$. But $z_1' \in I$, $u_1 \notin I$; then either u_1 is a sink (and we are finished) or there exists $e_2 \in s^{-1}(u_1)$ such that $e_2 \notin I$ and therefore we can consider $z_2 = u_1 z_1' u_1$ and $z_2' = e_2^* z_2 e_2$ and reason as above; it is clear that we can extend this process analogously until we arrive to a vertex u_n and a $z_n \in I$ which is either 0 or a linear combination of vertices (when $l(p_i) = l(q_i)$), paths (when $l(p_i) > l(q_i)$) and ghost paths (when $l(p_i) < l(q_i)$), and whence $u_n z_n u_n = 0$ because $u_n \notin I$. Then $e_n^* \dots e_1^* \mu^* x \nu e_1 \dots e_n = ku_n$.

Case 2: Since $[p]$ is a polynomial in a cycle without exits $[c]$ of F, c must be a cycle with exits in E such that all of its exits belong to I (otherwise $[c]$ would be 0 or have an exit in $L_K(E)/I$). Fix one of these exits e and denote $v := s(e), w := r(e)$. We can suppose that c is based at v, because we can cyclically permute $[c] = [e_1 \dots e_l]$ in $L_K(E)/I$ as we please, by sandwiching it between $[e_j^* \dots e_1^*]$ and $[e_1 \dots e_j]$.

First suppose that $z := vy'v = 0$. Write $p = \sum_{i=1}^N (a_i c^{t_i} + b_i (c^*)^{t_i})$ where $t_i \in \mathbb{N}$ ($t_i \neq t_j$ whenever $i \neq j$) and a_1 or b_1 is nonzero (or both). Take $\mu' = c^{t_1}, \nu' = v$ if $a_1 \neq 0$ and $\mu' = v, \nu' = c^{t_1}$ otherwise, to get, respectively, $\mu'^* p \nu' = a_1 v + p'$ or $b_1 v + p''$, where p', p'' are polynomials in $\{c, c^*\}$ without independent term. Now, $e^* \mu'^* p \nu' e = a_1 w$ or $b_1 w$ implies that $e^* \mu'^* v \mu^* x \nu v \nu' e$ is a nonzero multiple of a vertex, as desired.

If $z \neq 0$, as before, it must be of the form $z = \sum_{j=1}^{n} k_j p_j q_j^*$ with $l(p_j), l(q_j) \geq 1$. Write $z = \sum_{j=1}^{n} k_j \, c^{r_j} \, p_j' q_j'^* \, (c^*)^{s_j}$, where $r_j, s_j \in \mathbb{N} \cup \{0\}$ and c is not a left factor of any p_j' or q_j'. Denote $r := \max\{r_1, \ldots, r_n\} + 1$ and $s := \max\{s_1, \ldots, s_n\} + 1$ (note that $r > r_j$ and $s > s_j$). Then

$$(c^*)^r z c^s = \sum_{j=1}^{n} k_j (c^*)^{r-r_j} p_j' q_j'^* c^{s-s_j}.$$

We claim that, in fact, this sum equals to zero. To see it, write $c = e_1 \ldots e_l$ and consider how must $p_j' q_j'^*$ be in order for the j-th term to be nonzero. Since $r - r_j > 0, s - s_j > 0$ and p_j, q_j cannot be of the form $p_j' = cp_j'', q_j' = cq_j''$ (with $p_j'', q_j'' \in \mathrm{Path}(E)$) because c is not a left factor of them, c must be of the form $c = p_j' c', c = q_j' c''$ (with $c', c'' \in \mathrm{Path}(E)$), what implies that $p_j' = e_1 \ldots e_{l_1}, q_j' = e_1 \ldots e_{l_2}$ ($l_1, l_2 < l$, with $l_i = 0$ denoting that the path involved is actually the vertex v). The j-th term of the sum would then read

$$k_j (c^*)^{r-r_j-1} c^* p_j' q_j'^* c (c^{s-s_j-1}) = k_j (c^*)^{r-r_j-1} e_l^* \ldots e_1^* e_1 \ldots e_{l_1} e_{l_2}^* \ldots e_1^* e_1 \ldots e_l c^{s-s_j-1},$$

which equals $k_j (c^*)^{r-r_j-1} e_l^* \ldots e_{l_1+1}^* e_{l_2+1} \ldots e_l c^{s-s_j-1}$ due to (CK1) cancelations. Since it is not 0, it is necessary that $l_1 = l_2$ (this also applies in the special case $l_1 = 0$ or $l_2 = 0$), in which case the j-th summand simplifies to $k_j (c^*)^{r-r_j-1} c^{s-s_j-1}$; and this is a multiple of a power of either c or c^* (depending on the sizes of r, s, r_j and s_j). But then $(c^*)^r z c^s$ would be a nonzero polynomial in $\{c, c^*\}$ with c a cycle with exits, what contradicts the fact that $z \in I(P_c(E))$, because of Lemma 5.1. Hence, $(c^*)^r z c^s = 0$.
To finish the proof, consider the polynomial $(c^*)^r (v\mu^* x \nu v) c^s = (c^*)^r (p + z) c^s = (c^*)^r p c^s$ and apply to it the process defined for the case when $z = 0$. $\square$

Note that the result [14, Lemma 2.3 (i)] used in the previous proof is not valid for arbitrary graphs. In order to consider the direct generalization of this result for arbitrary graphs, one may use the machinery of admissible pairs as explained in Tomforde's paper [26]. Concretely, for H the hereditary and saturated subset generated by the vertices in cycles without exits, $L_K(E)/I(H)$ is isomorphic to the Leavitt path algebra of a graph in which breaking vertices appear, as shown in [26, Theorem 5.7 (2)]; but since the use of infinite emitters and breaking vertices would, in our opinion, obscure the underlying ideas and would enlarge the proofs with some distracting technicalities, we have preferred to switch to a dual graphs approach, which we tackle in the following subsection.

Remark 5.3. Note that in Proposition 5.2 either the vertex is not in the ideal $I(P_c(E))$ or is in a cycle with exits such that all the ranges of the exits are in $I(P_c(E))$.

Examples 5.4. We illustrate the two situations of the previous result and describe $I(P_c(E))$ in each case.

(i) Let E be the graph represented by:

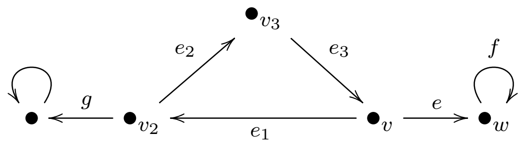
$$\begin{tikzcd} \bullet_u & \bullet_v \arrow[l, "f"'] \arrow[r, "e"] & \bullet_w \arrow[loop above] \end{tikzcd}$$

Consider the idempotent $x = ff^* + w$ of $L_K(E) \backslash I(P_c(E))$, whose class in $L_K(E)/I(P_c(E))$ is $[x] = [ff^*] = [v]$. Now, $[v][x][v] = [v]$ means that we can take $\mu = v = \nu$. In $L_K(E)$,

$\mu^*x\nu = vxv = v(ff^*+w)v = ff^* = v - ee^* \neq v$ $(ee^* \in I(P_c(E)))$. Finally, sandwiching by f^* and f, we get that $f^*\mu^*x\nu f = f^*ff^*f = u \in E^0$.

Using the construction ${}_HE$, one can see that, in this case, $I(P_c(E)) \cong \mathbb{M}_2(K[x,x^{-1}])$ since there is only one path e ending at w.

(ii) Now let E be the graph represented by



and consider the element

$$x = c^5 + c^2 + ee^* + c^3efe^*(c^*)^5 + ce_1e_1^* + ce_1e_2 + ce_1e_2e_2^*e_1^*(c^*)^2$$

of $L_K(E) \setminus I(P_c(E))$, where $c = e_1e_2e_3$. Its class in $L_K(E)/I(P_c(E))$ is $[x] = [c^5 + c^2 + c + ce_1e_2 + c^*]$, since $[e_1e_1^*] = [v], [e_2e_2^*] = [v_2]$. We can take $\mu = v = \nu$ to get $[\mu^*x\nu] = [vxv] = [c^5 + c^2 + c + c^*]$, which is a polynomial $[p]$ in $\{[c],[c]^*\}$ with $[c]$ a cycle without exits. In $L_K(E)$,

$$\mu^*x\nu = vxv = c^5 + c^2 + ce_1e_1^* + ce_1e_2e_2^*e_1^*(c^*)^2 + c^3efe^*(c^*)^5 + ee^* =$$

$$= p + y + c^3efe^*(c^*)^5 + ee^*,$$

where p is the already mentioned polynomial $c^5 + c^2 + c + c^*$ and $y \in I(P_c(E))$ is such that $p + y = c^5 + c^2 + ce_1e_1^* + ce_1e_2e_2^*e_1^*(c^*)^2$, the monomials which give rise to $[p]$ in $L_K(E) \setminus I(P_c(E))$. So, we must have $y = c^5 + c^2 + ce_1e_1^* + ce_1e_2e_2^*e_1^*(c^*)^2 - p = (ce_1e_1^* - c) + (ce_1e_2e_2^*e_1^*(c^*)^2 - c^*) = A + B$. By an application of (CK2), we see, on the one hand, that $c = cv = ce_1e_1^* + cee^*$ so that $A = ce_1e_1 - c = -cee^*$.

On the other hand, by adding $c(c^*)^2 - c(c^*)^2$ to B, we get $B = ce_1e_2e_2^*e_1^*(c^*)^2 - c^* = ce_1e_2e_2^*e_1^*(c^*)^2 - c(c^*)^2 + c(c^*)^2 - c^*$. Taking into account that, by (CK2), $v = e_1e_1^* + ee^* = e_1v_2e_1^* + ee^* = e_1(e_2e_2^* + gg^*)e_1^* + ee^* = e_1e_2e_2^*e_1^* + e_1gg^*e_1^* + ee^*$, we get:

(i) $c(c^*)^2 = cv(c^*)^2 = ce_1e_2e_2^*e_1^*(c^*)^2 + ce_1gg^*e_1^*(c^*)^2 + cee^*(c^*)^2$, what implies that $ce_1e_2e_2^*e_1^*(c^*)^2 - c(c^*)^2 = -ce_1gg^*e_1^*(c^*)^2 - cee^*(c^*)^2$.

(ii) $c^* = vc^* = e_1e_2e_2^*e_1^*c^* + e_1gg^*e_1^*c^* + ee^*c^*$ with $e_1e_2e_2^*e_1^*c^* = e_1e_2e_3e_3^*e_2^*e_1^*c^* = c(c^*)^2$ (because $e_3e_3^* = v_3$), so that $c(c^*)^2 - c^* = -e_1gg^*e_1^*c^* - ee^*c^*$.

(iii) Therefore, $B = -ce_1gg^*e_1^*(c^*)^2 - cee^*(c^*)^2 - e_1gg^*e_1^*c^* - ee^*c^*$.

Then, $vxv = p + A + B + c^3efe^*(c^*)^5 + ee^* = p - cee^* - ce_1gg^*e_1^*(c^*)^2 - cee^*(c^*)^2 - e_1gg^*e_1^*c^* - ee^*c^* + c^3efe^*(c^*)^5 + ee^* = p - ce_1gg^*e_1^*(c^*)^2 - cee^*(c^*)^2 + c^3efe^*(c^*)^5 + z'$. We do not really have to take into account $z' = -cee^* - e_1gg^*e_1^*c^* - ee^*c^* + ee^*$ because $c^*z'c = 0$. Now we take $r = \max\{1,1,3\} + 1 = 4$ and $s = \max\{2,2,5\} + 1 = 6$, so that

$$(c^*)^r v\mu^*x\nu vc^s =$$

$$= (c^*)^4pc^6 - (c^*)^4ce_1gg^*e_1^*(c^*)^2c^6 - (c^*)^4cee^*(c^*)^2c^6 + (c^*)^4c^3efe^*(c^*)^5c^6 =$$

$$= (c^*)^4pc^6 = c + c^2 + c^4 + c^7,$$

because $c^*e = 0$ and $(c^*)^2e_1g = c^*e_3^*e_2^*e_1^*e_1g = c^*e_3^*e_2^*g = 0$. Hence, $t_1 = 1$ for the monomial c and we just have to multiply by $(c^*)^{t_1} = c^*$ on the left to get the polynomial

$c^*(c^*)^4 v\mu^* x\nu v c^6 = c^*(c+c^2+c^4+c^7) = v+c+c^3+c^6$, which has independent term. Finally, $e^*(c^*)^5 v\mu^* x\nu v c^6 e = e^*(v+c+c^3+c^6)e = e^*e + e^*(c+c^3+c^6)e = w$.

Again, using the construction ${}_H E$, one can see that, in this case,

$$I(P_c(E)) \cong \mathbb{M}_\infty(K[x,x^{-1}]) \oplus \mathbb{M}_\infty(K[x,x^{-1}])$$

since there are countably many paths ending at $r(g)$ and w. □

Proposition 5.5. *Let $L_K(E)$ be a Leavitt path algebra:*

(i) *If E is an arbitrary graph and $v \in E^0$ is a non-minimal primitive vertex of $L_K(E)$ then $v \in I(P_c(E))$.*

(ii) *If E is a row-finite graph and $x \in L_K(E)$ is a non-minimal primitive idempotent then $x \in I(P_c(E))$.*

Proof.

(i) Since v is non-minimal primitive, $T(v)$ has no bifurcations and ends in a cycle without exits (Proposition 4.3) with base, say, $w \in E^0$. Then $w \in I(P_c(E))$. The saturated condition on $I(P_c(E)) \cap E^0 = \overline{P_c(E)}$ and the fact that $T(v)$ has no bifurcations give $v \in I(P_c(E))$.

(ii) Suppose $x \notin I(P_c(E))$ is a non-minimal primitive idempotent of $L_K(E)$. Then by Proposition 5.2 there exist $\mu,\nu \in \mathrm{Path}(E)$, $v \in E^0$ and $k \in K\setminus\{0\}$ such that $\mu^* x \nu = kv$. By Proposition 4.7 $L_K(E)v \cong L_K(E)x$, so that v is a non-minimal primitive idempotent and whence $v \in I(P_c(E))$ by (i). Defining, as in Proposition 4.7, $a = \frac{1}{k}\mu^* x, b = x\nu$, we see that $x = e = bva \in I(P_c(E))$ is a contradiction that finishes our proof. □

This proposition implies that $I(P_c(E))$ is an invariant of ring isomorphisms between Leavitt path algebras of row-finite graphs:

Proposition 5.6. *If E, F are row-finite graphs and $\phi : L_K(E) \to L_K(F)$ is an isomorphism of rings, then $\phi(I(P_c(E))) = I(P_c(F))$.*

Proof. $I(P_c(E))$ is generated by the vertices of $P_c(E)$, which are non-minimal primitive idempotents by Remark 4.5. Thus, $\phi(P_c(E))$ is a set of non-minimal primitive idempotents of $L_K(F)$ as well. By Proposition 5.5 above, $\phi(P_c(E)) \subseteq I(P_c(F))$ and so $\phi(I(P_c(E))) \subseteq I(P_c(F))$. By symmetry, we get $\phi(I(P_c(E))) = I(P_c(F))$ as desired. □

5.2. **The general case by means of dual graphs.** The jump to the general case (that is, proving the results of the previous subsection for the case of arbitrary graphs) passes through the dual graph techniques introduced in Section 2 (in particular, θ will denote the monomorphism defined in Proposition 2.8).

To start, we will see that the ideal generated by the vertices in cycles without exits of the dual of a subgraph goes, via θ, into the ideal generated by the vertices in cycles without exits of the whole graph:

Lemma 5.7. *If E is an arbitrary graph and F is a row-finite subgraph of E, then $\theta(I(P_c(D_E(F)))) \subseteq I(P_c(E))$.*

Proof. Since $I(P_c(D_E(F)))$ is generated by the set of vertices $P_c(D_E(F))$, it suffices to show that for one of them, say v, we have $\theta(v) \in I(P_c(E))$. Note that v cannot be a vertex-vertex, since those are sinks; thus $v \equiv e$ must be an edge-vertex. So, $\theta(e) = ee^*$ where, by Lemma 3.1, e is an edge in a cycle without exits of E and therefore $r(e) \in P_c(E)$. Now, $ee^* = er(e)e^* \in I(P_c(E))$ implies the desired result. $\square$

Theorem 5.8 (**Refinement of Reduction Theorem**). *If E is an arbitrary graph and $x \in L_K(E) \setminus I(P_c(E))$ then there exist $k \in K \setminus \{0\}$, $\mu, \nu \in Path(E)$ and $v \in E^0$ such that $\mu^* x\nu = kv$.*

Proof. Take $x \in L_K(E) \setminus I(P_c(E))$ and express it as a convenient sum of monomials (as in (†)). Observe that F_x and D_x are finite and that $x \in A_x \setminus I(P_c(E))$. An application of Lemma 5.7 shows that $\theta^{-1}(x) \notin I(P_c(D_x))$. Now, since D_x is finite and $\theta^{-1}(x) \in L_K(D_x) \setminus I(P_c(D_x))$, by the refinement of the Reduction Theorem for the row-finite case (Proposition 5.2), there exist $k \in K \setminus \{0\}$, $\mu, \nu \in \mathrm{Path}(D_x)$ and $v \in D_x^0$ such that $\mu^*\theta^{-1}(x)\nu = kv$ (‡). We will see that this relation in $L_K(D_x)$ provides us with a similar one in $L_K(E)$.
Note that $r(\mu) = v = r(\nu)$. Write $\mu = (\mu_1, \mu_2)(\mu_2, \mu_3) \ldots (\mu_n, y)$ and $\nu = (\nu_1, \nu_2)(\nu_2, \nu_3) \ldots (\nu_m, y)$, with $(\mu_i, \mu_{i+1}), (\nu_j, \nu_{j+1})$ and $(\mu_n, y), (\nu_m, y)$ edges of D_x, μ_i, ν_j edges of F_x and $y \in F_x^0 \cup F_x^1$. There exist three possible cases for y:

Case (i). The vertex v comes from a sink of F_x. In this case $y = v$ with $\theta(y) = u_v = v$ and $\theta(\mu) = \mu_1\mu_2\mu_2^*\mu_2\mu_3\mu_3^* \ldots \mu_n v = \mu_1\mu_2 \ldots \mu_n$, $\theta(\nu) = \nu_1 \ldots \nu_m$. Thus, the application of θ to (‡) gives

$$(\mu_1 \ldots \mu_n)^* x(\nu_1 \ldots \nu_n) = kv.$$

Case (ii). The vertex v comes from an intermediate vertex of F_x. In this case $y = v$ with $\theta(y) = u_v = v - \sum_{\{s(e)=v, e\in F_x\}} ee^*$ and $\theta(\mu) = \mu_1\mu_2 \ldots \mu_n u_v$, $\theta(\nu) = \nu_1\nu_2 \ldots \nu_m u_v$, so that $(\mu_1\mu_2 \ldots \mu_n u_v)^* x(\nu_1\nu_2 \ldots \nu_m u_v) = ku_v$.
Since v comes from an intermediate vertex of F_x, there exist $f \in E^1 \setminus F_x^1$ such that $s(f) = v$; this means that $u_v f = f$, $f^* u_v^* = f^*$ and $f^* u_v f = r(f)$, which implies that

$$f^*(\mu_1\mu_2 \ldots \mu_n u_v)^* x(\nu_1\nu_2 \ldots \nu_m u_v)f = kf^* u_v f$$

gives us finally

$$(\mu_1\mu_2 \ldots \mu_n f)^* x(\nu_1\nu_2 \ldots \nu_m f) = kr(f).$$

Case (iii). The vertex $v \equiv e$ comes from an edge of F_x. In this case $y = e$ with $\theta(y) = ee^*$ and $\theta(\mu) = \mu_1 \ldots \mu_n ee^*$, $\theta(\nu) = \nu_1 \ldots \nu_m ee^*$, so that $(\mu_1 \ldots \mu_n ee^*)^* x(\nu_1 \ldots \nu_m ee^*) = kee^*$. Now, sandwiching with e^*, e and knowing that $e^*e = r(e)$ we get

$$(\mu_1\mu_2 \ldots \mu_n e)^* x(\nu_1\nu_2 \ldots \nu_m e) = kr(e).$$

$\square$

We are now in position to prove that no non-minimal primitive idempotent of any Leavitt path algebra can live outside the ideal generated by the vertices in cycles without exits. Moreover, we will see immediately that this ideal can be used for classification purposes, as it remains invariant under isomorphisms of rings:

Corollary 5.9. *If E is an arbitrary graph and $x \in L_K(E)$ is a non-minimal primitive idempotent then $x \in I(P_c(E))$.*

Proof. As in Proposition 5.5 but building on Theorem 5.8. □

Theorem 5.10. *If E, F are arbitrary graphs and $\phi : L_K(E) \to L_K(F)$ is an isomorphism of rings, then $\phi(I(P_c(E))) = I(P_c(F))$.*

Proof. As in Proposition 5.6 but building on Theorem 5.8. □

The structure of $I(P_c(E))$ was determined for the Leavitt path algebras of row-finite graphs in [5, Proposition 3.5 (iii)]. We show that this structure is essentially the same for any Leavitt path algebra. We note that in spite of it coming from the row-finite context, it is a particular case of the general definition (specifically, the case with $S = \emptyset$) as stated in [16], where it is used to study the structure of the graded ideals of an arbitrary Leavitt path algebra.

Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \{\mu = \mu_1 \dots \mu_n \mid \mu_i \in E^1, s(\mu_i) \in E^0 \setminus H \text{ for } i \leq n, r(\mu_n) \in H\}.$$

Denote by $\overline{F}_E(H)$ another copy of $F_E(H)$. For $\mu \in F_E(H)$, we write $\overline{\mu}$ to denote a copy of μ in $\overline{F}_E(H)$. Then, we define the graph ${}_HE = ({}_HE^0, {}_HE^1, s', r')$ as follows:

$$({}_HE)^0 = H \cup F_E(H)$$

$$({}_HE)^1 = \{e \in E^1 \mid s(e) \in H\} \cup \overline{F}_E(H).$$

$$\text{For every } e \in E^1 \text{ with } s(e) \in H, s'(e) = s(e) \text{ and } r'(e) = r(e).$$

$$\text{For every } \overline{\mu} \in \overline{F}_E(H), s'(\overline{\mu}) = \mu \text{ and } r'(\overline{\mu}) = r(\mu).$$

We also need to know that an *infinite path* in a graph E is an infinite sequence of edges $\mu = e_1 e_2 \dots$ such that $s(e_{i+1}) = r(e_i)$ for every $i \in \mathbb{N}$, and that an infinite path is said to *end in a cycle* if there exist a cycle c and an $n \in \mathbb{N}$ such that $\mu = e_1 \dots e_n ccc \dots$.

Theorem 5.11. *Let E be an arbitrary graph. Then $I(P_c(E)) \cong \bigoplus_{j \in J} \mathbb{M}_{n_j}(K[x, x^{-1}])$, where n_j is an arbitrary cardinal and J is an arbitrary set.*

Proof. Suppose $I(P_c(E)) \neq 0$, otherwise the result becomes trivial. By [14, Lemma 2.1] (which first part is valid in full generality), $I(P_c(E)) = I(H)$, where $H = \overline{P_c(E)}$, and by [16, Proposition 3.7] (taking $S = \emptyset$ in that proposition), $I(H) \cong L_K({}_HE)$. Thus, we can reduce the problem to study the structure of $L_K({}_HE)$. We claim that it is a locally noetherian Leavitt path algebra with zero socle, and hence the result follows from [5, Theorem 3.7 (iv)] dropping the countability result on the index sets, which comes exclusively as a result of the authors restricting their context to countable graphs. More concretely, the quoted result was proved for row-finite and countable graphs, giving matrices of countable size. For row-finite non-necessarily countable graphs, by following the same proof, matrices of arbitrary size may appear.

To prove this, we will show that ${}_HE$ is row-finite with no sinks, satisfies Condition (NE) and that any of its possible infinite paths must end in a cycle (whence $L_K({}_HE)$ satisfies [5, condition (iii) of Theorem 3.7]). The latter condition is easily seen to be true by construction. Also by construction, ${}_HE$ contains no sinks, as any vertex in ${}_HE$ must connect to some vertex in $P_c(E)$, and hence $Soc(L_K({}_HE)) = I(P_l({}_HE)) = 0$. That ${}_HE$ is row-finite is proved as follows:

as $H = \overline{P_c(E)}$, by the inductive construction of the hereditary saturated closure, H (and subsequently ${}_HE$) could contain an infinite emitter v if and only if $v \in \Lambda_0(P_c(E)) = T(P_c(E))$. But since $P_c(E)$ is hereditary, $T(P_c(E)) = P_c(E)$, and since it contains no bifurcations (recall that it is formed by the vertices which are base of cycles without exits), it cannot contain any infinite emitter.
It remains to show that ${}_HE$ satisfies Condition (NE). Suppose on the contrary that there exists a cycle with exits, c, in ${}_HE$. By the definition of ${}_HE$, c must be a cycle with vertices in H. But this is impossible by Lemma 5.1. □

Note that this structure theorem keeps further the parallelism between $I(P_c(E))$ and $Soc(L_K(E))$, since by [13, Theorem 5.6] (dropping again the countability assumption),

$$Soc(L_K(E)) = I(P_l(E)) \cong \bigoplus_{j\in J} \mathbb{M}_{n_j}(K), \text{ where } n_j \in \mathbb{N} \cup \{\infty\} \text{ and } J \text{ is an arbitrary set.}$$

It is well-known that if $x \in Soc(R)$ for a semiprime ring R, then the right (resp. left) R-module xR (resp. Rx) is semisimple. To finish, we use again the facts showed in the proof above to establish that the $L_K(E)$-modules generated by vertices of $I(P_c(E))$ can be written as a direct sum of indecomposable modules:

Proposition 5.12. *Let E be an arbitrary graph, and let $v \in I(P_c(E))$. Then $vL_K(E)$ (resp. $L_K(E)v$) is completely decomposable as a right (resp. left) $L_K(E)$-module.*

Proof. Since $\overline{P_c(E)}$ has no sinks and no infinite emitters, every $v \in I(P_c(E))$ is regular. Let $s^{-1}(v) = \{e_i\}_{i=0}^n$. By [11, Proof of Lemma 7.3], $vL_K(E) \cong \bigoplus_{i=0}^n r(e_i)L_K(E)$. Knowing that the only infinite paths in $I(P_c(E))$ end in a cycle, we can repeat this process (now with every $r(e_i)$) until every vertex in our sum is the base of a cycle without exits, what happens in a finite number of steps, to get $vL_K(E) \cong \bigoplus_{i=0}^m u_iL_K(E)$ with $u_i \in P_c(E)$. Now, by Proposition 4.3, every u_i is primitive ($T(u_i)$ has no bifurcations) and hence every $u_iL_K(E)$ is an indecomposable left $L_K(E)$-module, proving the claim. The argument for $L_K(E)v$ is analogous. □

Acknowledgments

The second author is supported by the Spanish MEC through the FPU grant AP2009-4848 and was partially supported by the University of Málaga, Spain, through contract CI-09-100 during the development of this paper. All the authors were partially supported by the Spanish MEC and Fondos FEDER through projects MTM2007-60333 and MTM2010-15223, and by the Junta de Andalucía and Fondos FEDER, jointly, through projects FQM-336 and FQM-02467.

The authors thank the referee for his/her careful reading and useful comments.

References

[1] G. Abrams, P. N. Ánh, A. Louly, E. Pardo. The classification question for Leavitt path algebras. *J. Algebra* **320** (2008), 1983–2026.
[2] G. Abrams, G. Aranda Pino. The Leavitt path algebra of a graph. *J. Algebra* **293(2)** (2005), 319–334.

[3] G. Abrams, G. Aranda Pino. Purely infinite Leavitt path algebras. *J. Pure Appl. Algebra* **207** (2006), 553–563.
[4] G. Abrams, G. Aranda Pino. The Leavitt path algebras of arbitrary graphs. *Houston J. Math.* **34** (2008), 423–442.
[5] G. Abrams, G. Aranda Pino, F. Perera, M. Siles Molina. Chain conditions for Leavitt path algebras. *Forum Math.*, **22** (2010), 95–114.
[6] G. Abrams, G. Aranda Pino, M. Siles Molina. Finite-dimensional Leavitt path algebras. *J. Pure Appl. Algebra* **209** (2007), 753–762.
[7] G. Abrams, G. Aranda Pino, M. Siles Molina. Locally finite Leavitt path algebras. *Israel J. Math.* **165** (2008), 329–348.
[8] G. Abrams, K. Rangaswamy. Regularity conditions for arbitrary Leavitt path algebras. *Algebr. Represent. Theory*, **13** (2010), 319–334.
[9] G. Abrams, K. Rangaswamy, M. Siles Molina. The socle series of a Leavitt path algebra. *Israel J. Math.*, in press (2009).
[10] P. Ara, M.A. Moreno, E. Pardo. Nonstable K-Theory for graph algebras. *Algebr. Represent. Theory* **10** (2007), 157–178.
[11] G. Aranda Pino,K.R. Goodearl, F. Perera, M. Siles Molina. Non-simple purely infinite rings. *Amer. J. Math.* **132(2)** (2010) 563–610.
[12] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina. The socle of a Leavitt path algebra. *J. Pure Appl. Algebra* **212** (2008) 500–509.
[13] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina. Socle theory for Leavitt path algebras of arbitrary graphs. *Rev. Mat. Iberoam.*, **26(2)** (2010), 611–638.
[14] G. Aranda Pino, E. Pardo, M. Siles Molina. Exchange Leavitt path algebras and stable rank. *J. Algebra* **305(2)** (2006), 912–936.
[15] G. Aranda Pino, E. Pardo, M. Siles Molina. Prime spectrum and primitive Leavitt path algebras. *Indiana Univ. Math. J.* **58(2)** (2009), 869–890.
[16] G. Aranda Pino, K.M. Rangaswamy, M. Siles Molina. Weakly regular and self-injective Leavitt path algebras over arbitrary graphs. *Algebr. Represent. Theor.* in press (2010).
[17] T. Bates, J. H. Hong, I. Raeburn, W. Szymański. The ideal structure of the C^*-algebras of infinite graphs. *Illinois J. Math* **46(4)** (2002), 1159–1176.
[18] T. Bates, D. Pask. Flow equivalence of graph algebras. *Ergodic Theory Dynam. Systems* **24** (2004), 367–382.
[19] W. Crawley-Boevey. Lectures on representations of quivers. *Unpublished.*
[20] K.R. Goodearl. Leavitt path algebras and direct limits. *Contemp. Math.* **480** (2009), 165–187.
[21] T. Y. Lam. *A First Course in Noncommutative Rings.* Springer-Verlag, Graduate Texts in Mathematics **131** (1991).
[22] W.G. Leavitt. The module type of a ring. *Trans. Amer. Math. Soc.* **42** (1962), 113–130.
[23] I. Raeburn. *Graph algebras.* CBMS Regional Conference Series in Mathematics **103**, Amer. Math. Soc., Providence (2005).
[24] I. Raeburn, W. Szymánski. Cuntz-Krieger algebras of infinite graphs and matrices. *Trans. Amer. Math. Soc.* **356(1)** (2003), 39–59.
[25] M. Siles Molina. Algebras of quotients of path algebras. *J. Algebra* **319** (2008), 5265–5278.
[26] M. Tomforde. Leavitt path algebras with coefficients in a commutative ring. *J. Pure Appl. Algebra* **215** (2011), 471–484.
[27] M. Tomforde. Uniqueness theorems and ideal structure for Leavitt path algebras. *J. Algebra* **318** (2007), 270–299.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga (Spain)
E-mail address: g.aranda@uma.es, brox@agt.cie.uma.es, msilesm@uma.es