Two characterizations of the dense rank

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Abstract

In this paper, we have considered the dense rank for assigning positions to alternatives in weak orders. If we arrange the alternatives in tiers (i.e., indifference classes), the dense rank assigns position 1 to all the alternatives in the top tier, 2 to all the alternatives in the second tier, and so on. We have proposed a formal framework to analyze the dense rank when compared to other well-known position operators, such as the standard, modified and fractional ranks. As the main results, we have provided two different axiomatic characterizations which determine the dense rank by considering position invariance conditions along horizontal extensions (duplication), as well as through vertical reductions and movements (truncation, and upward or downward independency).

Keywords: preferences; linear orders; weak orders; positions; dense rank; duplication; truncation.

1. Introduction

When it is possible to rank order objects (individuals, alternatives, etc.) taking into account some quality or criterion, it is natural to assign positive integer numbers to them in an ascending manner, starting from 1 for the best, 2 for the following one, and so on. Such numbers can be interpreted as rankings[1] or positions in an ordinal sense (first, second, etc.). According to Kendall [18, p. 1], at first sight they might seem not to allow computations: "We cannot substract 'fifth' from 'eighth'; but a meaning can be given [by saying] that the rank according to A is 5 is equivalent to saying that [...] four members are given priority over our particular member, or are *preferred* to it; [...] this is not an ordinal number but a cardinal number, i.e., arises by counting". Italics

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[1] The double use of *ranking* is extended for both the preference relation on a set of objects and for the (ranking) numbers assigned to such objects. In the last case, throughout this paper we use the term *positions*.

(author's own) suggest the formal use of preference relations in the process of assigning positions, which is the main technical tool in our approach along this paper.

However, when comparing alternatives ties can arise for several reasons, among them: imprecision or lack of knowledge in agents' assessing process or, in some contexts, exact coincidence in the considered quality to be evaluated. In such cases, positions to be assigned are not straightforward and might depend on the particular scenario.

For example, in the 2020 Olympic Games (held at Tokyo in 2021 due to the COVID pandemic), the Men's High Jump competition produced a strict tie, both in attempts and top exceeded height, between two athletes. The rules established either a tie-break (jump off) or to assign a shared award in equal terms by consensus agreement of the involved jumpers, and this last option was chosen. In this way, this discipline had two gold medals, no silver and one bronze[2]. This $1-1-3$ way of assigning positions, known as the *standard (competition) rank*, is just one of the possibilities, but not precisely the one our paper is focused on.

Should the jumper currently awarded with the bronze medal have been promoted to silver after the agreement between the two *ex aequo* gold medals? Would the positions to be assigned have been $1-1-2$, without a jump between consecutive positions, instead of the actual ones? Not perhaps in this context of international prestige sports, or whenever awards have a consequent monetary reward. However, such an approach (ours in this paper) makes sense in some other situations.

Consider a process where people can ask (or be asked) for something via the Internet and make accidental mistakes by multiple clicking, or even try strategically to maximize their opportunities (by bots, for example). If the enabled service detects that, from the same origin and in a short time, multiple requests (or answers) are received, it would seem reasonable to identify them as just one and to assign one single output, say $n-n-n-\cdots-n$ (m times), so that the following different petitioner (or voter, etc.) will be given the next list number, $n+1$ (instead of $n+m$).[3]

This way of assigning positions takes into account, in the first instance, not alternatives but ranking levels (gathering indifferent alternatives). Commonly

[2] See the official schedule at `https://olympics.com/en/olympic-games/tokyo-2020/results/athletics/men-s-high-jump`

[3] This example is related to *false-name-proof mechanisms*, i.e., those where individuals do not gain an advantage from participating more than once. Concretely, a voting rule is *false-name-proof* if no voter can benefit by replicating the same vote several times. These kind of rules have been characterized by Fioravanti and Massó [11]. These authors emphasize the importance of the problem, mainly in anonymous online mechanisms where identities cannot be verified, such as "selection processes for Massive Open Online (MOO) Courses or Schools, rating systems for goods and services, Internet auctions (a popular part of Electronic Commerce), or Facebook allowing users to vote". An overview of false-name manipulation in several contexts, and possibilities to prevent it, can be found in Conitzer and Yokoo [7]).

known as *dense rank*[4], it is widely used as a convenient position method in some contexts. For example, the following question, taken from Kyte [19, pp. 562-568], shows how the dense rank naturally appears in a company management scenario where just the rank would not be suitable enough. "Consider this seemingly sensible request: Give me the set of sales people who make the top 3 salaries, that is, find the set of distinct salary amounts, sort them, take the largest three, and give me everyone who makes one of those values. [...] We can simply select all [the individuals] with a dense rank of three or less. [...] In this case, using [standard] rank over dense rank would not have answered our specific query". This happens because the dense rank primarily focuses on obtained salaries, scores, etc., and afterwards on individuals who reach them.

From a theoretical point of view, the dense rank has received less attention than the standard rank (the current case of the Olympic medals) or the *fractional rank* (also called *mid-rank*), where the positions of tied alternatives are computed as if there were no ties, and then the average is assigned: $1.5-1.5-3$, with a kind of gold-silver medal at 50% alloy if applied for the *ex aequo* Olympic winners. What is more, another possibility would exist by taking $2-2-3$, without a gold medal, which corresponds to the *modified (competition) rank*.

In Table 1, the positions of the above Olympic example are shown for all considered ranks.

	Standard rank	Modified rank	Fractional rank	Dense rank
$x \quad y$	1	2	1.5	1
z	3	3	3	2

Table 1: Ranks.

We have intentionally avoided, and will continue to do so, the so called *ordinal rank*, which assigns distinct positions to different objects, even when they are in a tie (for instance, with a random tie-breaking process). In the previous example, although x and y had an equal performance over that of z, the ordinal rank could assign positions $1-2-3$ or $2-1-3$ to the alternatives x, y, z. The reason for discarding these possibilities is that *equality*, a compelling property to appear formally in Definition 6, is violated.

An interesting overview relating the dense rank to these and other ranking methods, with a suitable mathematical treatment, can be found in Vojnović [25, pp. 505-506].

In the literature, the dense rank is not considered at all by Kendall [17, 18, pp. 34-48] when focusing on ties. It is considered, in passing and tacitly, by Fishburn [12, pp. 164-165], related to his *modified equal-spacing procedure*; and it is equivalent to the so-called *ranking level function* introduced by Gärdenfors [15]

[4]Because, as pointed out by Kyte [19, p. 563] "a dense rank returns a ranking number without any gaps". More seldom, it is also called *sequential rank*.

in his analysis of positional voting functions. More recently, in a similar way, Ding *et al.* [9] have proposed a novel method which gathers ambiguous alternatives at different stages in a hierarchical process of aggregation fusion.

Concerning software implementation, although both standard rank and fractional rank exist as `EXCEL` functions, dense rank does not. However, it appears in `SPSS` as the rank case `SEQUENTIAL RANKS TO UNIQUE VALUES`[5], and in `SQL` it is an analytical (or window) function called, precisely, `DENSE_RANK`[6]. As a programming language for storing and processing information, `SQL` is supported by Amazon Web Services (AWS), where it is also possible to find applications of `DENSE_RANK` to the logistics industry, such as the arrangement of products in inventory according to their quantities in order to prioritize restocks[7].

It is important to emphasize that the four considered position operators (standard, modified, fractional and dense ranks) are equivalent from an ordinal point of view. However, from a cardinal approach, they assign positions to the alternatives in a different fashion, still representing the same weak order. This fact crucially affects the results when the positions are the inputs of an aggregation procedure, because they could generate different outcomes[8]. For instance, this is the case of consensus measures in the setting of weak orders. In this way, García-Lapresta and Pérez-Román [14] consider the positions through the fractional rank. In turn, Alcantud et al. [1] and González-Arteaga et al. [16] propose some consensus measures based on the standard rank. Furthermore, as expected, different ways of assigning positions to the alternatives may lead to divergent results when the consensus is measured (see Alcantud et al. [1, Ex. 3.7]).

In a different setting, the rank correlation coefficients introduced by Spearman and Kendall use the fractional rank, although it would be possible to change it by using other position operators, as the standard, modified or dense ranks[9]. Obviously, as happens in consensus measures, the outcomes would be different depending on the rank used.

In the frameworks of positional voting systems (see Gärdenfors [15] and

[5]See `https://www.ibm.com/docs/en/spss-statistics/29.0.0?topic=cases-rank-ties`.

[6]See Kyte [19, pp. 562-568].

[7]See `https://docs.aws.amazon.com/redshift/latest/dg/r_WF_DENSE_RANK.html`.

[8]Continuing with the Olympic example, the most widespread way to establish an international overall ranking is a lexicographic order: first, the total amount of gold medals for each country is taken into account; then, the number of silver medals for breaking possible ties; and, finally, the bronzes, if necessary. This approach only considers ordinal information from the distinct sport disciplines. However, there are other possibilities which use cardinal information from the same ordinal basis, by translating positions into scores. When this option is followed, the international ranking might depend on the use of standard, modified, fractional or dense ranks and their respective scoring conversions.

[9]Kendall [17] himself explicitly considered the use of the standard rank (suggested by Student), but pointed out that "it gives different results if one ranks from the other end of the scale and [...] destroys the useful property that the mean rank of the whole series shall be $\frac{1}{2}(n+1)$". This symmetry is the main reason for the extended use of the fractional rank (mid-rank) in correlation analysis.

García-Lapresta and Martínez-Panero [13]) and scoring rules (see Chevotarev and Shamis [5]), when these voting systems are extended from linear orders to weak orders, the way that positions are defined is crucial to generate a collective ranking of the alternatives. In this way, for practical purposes, Madani et al. [20] show how ranking orders (outputs) are affected by the position operator chosen to aggregate the inputs[10]. From a theoretical point of view, although the fractional rank is commonly used in the most well-known scoring rule, the Borda count (see Black [3], as well as Cook and Seiford [8], among others), Gärdenfors [15] considers other variants and he also defines the *restricted Borda function*, based on the modified rank, and (as mentioned above) the *ranking level function*, related to the dense rank.

It is not the objective of this paper to decide which position operator is the most suitable one in the mentioned and other contexts. Here we aim to see what are the intrinsic positional properties of the dense rank, in particular those not fulfilled by the other usually considered position operators. To do so, this paper proposes a first characterization of the dense rank as the only position operator verifying two independent conditions: sequentiality (the positions of alternatives in linear orders should follow the natural sequential pattern: 1, 2, 3, ...); and duplication, a kind of "clone independency", meaning that new replicated alternatives inherit the positions of the original items and should not modify those of the already existing ones, at any level (also, a weak version of this property is considered in a simply outlined, alternative axiomatization).

A second characterization is also provided, substituting duplication with a condition called UD-independency (upward or downward independency). While duplication entails the preservation of the original positions by cloning (i.e., the addition of indifferent alternatives to the already existing ones, in a horizontal way), UD-independency states that vertical displacements of alternatives from the original situation will not change the positions of the remaining non moving alternatives[11]. As this condition is weaker than duplication (a non immediately evident fact due to the distinct scopes of both properties), in order to achieve a second characterization of the dense rank, truncation (a compelling condition meaning irrelevance of alternatives below when assigning positions) must be added, as well as sequentiality, as in the first characterization.

The rest of the paper is organized as follows. Section 2 introduces the notation and the new framework of position operators to represent the dense rank in a formal way within the setting of weak orders. Section 3 is devoted to explain-

[10] Note that the systematic use by the authors of these ranks for comparing the different outcomes should be taken with reservations. For example, some good properties of the Borda rule (immunity to the inverse order and to the absolute loser paradoxes, among others appearing in Felsenthal [10]), rely on the use of the fractional rank due to the symmetry in the scores assigned to the positions (see chapter 4 in Saari [22]). If we use other ranks, this symmetry vanishes and undesirable effects such as the mentioned paradoxes might occur.

[11] This requirement makes sense in some contexts (related to those in which duplication also does) where one's improvement or setback does not affect others. This is looked at in detail in Section 4.

ing some basic properties required for position operators. Section 4 includes other different properties, their relationships with the basic ones and the characterization theorems of the dense rank. Section 5 presents some concluding remarks, paying special attention to some differences between dense, standard, modified and fractional ranks, and showing how the dense rank is essentially different from the others. Some further research is also outlined. Finally, the Appendix contains the omitted proofs.

2. The dense rank in the setting of weak orders

In this section we provide an in-depth examination of ties and the problem they present when assigning positions, focusing on the dense rank as a way of dealing with this situation. To do so, first of all we provide some notation used throughout the paper.

2.1. Notation

Consider a finite set of alternatives $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$. A *weak order* (or *complete preorder*) on X is a complete[12] and transitive[13] binary relation on X. A *linear order* on X is an antisymmetric[14] weak order on X. With $\mathcal{W}(X)$ and $\mathcal{L}(X)$, we denote the sets of weak and linear orders on X, respectively. Given $R \in \mathcal{W}(X)$, with P and I we denote the asymmetric and symmetric parts of R, respectively: $x_i \, P \, x_j$ if not $x_j \, R \, x_i$; and $x_i \, I \, x_j$ if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$.

Given $R \in \mathcal{W}(X)$ and a permutation σ on $\{1, \ldots, n\}$, we denote by R^σ the weak order obtained from R by relabelling the alternatives according to σ, i.e., $x_i \, R \, x_j \;\Leftrightarrow\; x_{\sigma(i)} \, R^\sigma \, x_{\sigma(j)}$, for all $i, j \in \{1, \ldots, n\}$. As usual, we denote by (i, j) the transposition interchanging these two subindexes and keeping all others unaltered.

Given $R \in \mathcal{W}(X)$ and $Y \subseteq X$, the restriction of R to Y, $R|_Y$, is defined as $x_i \, R|_Y \, x_j$ if $x_i \, R \, x_j$, for all $x_i, x_j \in Y$. Note that $R|_Y \in \mathcal{W}(Y)$.

In turn, $\#Y$ is the cardinality of Y.

2.2. Positions in weak orders

We now introduce the notion of position operator, a suitable mathematical object which allows us to add or withdraw alternatives along the process of assigning positions, in a parallel way to that of voting theory when a variable electorate is considered (see Smith [23]).

[12] A binary relation R on X is complete if $x_i \, R \, x_j$ or $x_j \, R \, x_i$, for all $x_i, x_j \in X$.

[13] A binary relation R on X is transitive if $(x_i \, R \, x_j$ and $x_j \, R \, x_k)$ implies $x_i \, R \, x_k$, for all $x_i, x_j, x_k \in X$.

[14] A binary relation R on X is antisymmetric if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$ implies $x_i = x_j$, for all $x_i, x_j \in X$.

Definition 1. *Given a universe of alternatives U and $X \subseteq U$ finite, a* position operator *O assigns to each $R \in \mathcal{W}(X)$ a function $O_R : X \longrightarrow \mathbb{R}$. We say that $O_R(x_i)$ is the* position *of the alternative $x_i \in X$ in the weak order R.*

Definitions 6, 7 and 8 include some properties that position operators may fulfill.

Once X has been fixed, given $R \in \mathcal{W}(X)$ and $x_i \in X$, we consider the number of alternatives dominated by x_i:

$$p_i = \# \{x_j \in X \mid x_i \, P \, x_j\} \, .$$

Note that $p_i \in \{0, 1, \ldots, n-1\}$.

Assigning positions to the alternatives in a linear order is a trivial task, as shown in the following definition.

Definition 2. *Given $R \in \mathcal{L}(X)$, the* sequential function *on R is the mapping $S_R : X \longrightarrow \{1, \ldots, n\}$ that assigns 1 to the alternative ranked first, 2 to the alternative ranked second, and so on:*

$$S_R(x_i) = \# \{x_j \in X \mid x_j \, R \, x_i\} = n - \# \{x_j \in X \mid x_i \, P \, x_j\} = n - p_i.$$

Unlike the case of linear orders, it is not obvious how to assign positions to the alternatives in weak orders, where ties may appear (we have already dealt with this fact and pointed out some possibilities in Section 1).

According to Kendall [17], there exist different ways of "allocating ranking numbers to tied individuals" [assigning positions to indifferent alternatives, in our context]:

- "The method of allocating ranking numbers to tied individuals in general use is to average the ranks which they cover. For instance, if the observer ties the third and fourth members, each is allotted $3\frac{1}{2}$. This is known as the mid-rank method".
- "An alternative to mid-ranks [is] that the ties should all be ranked as if they were the highest member of the tie", i.e., 3 in the previous example. This is the standard rank, the most common method where just natural numbers are used to assign positions. This is the reason why it is often called simply "the rank", avoiding the adjective "standard".

Of course, although not considered by Kendall [17, 18], it is also possible to assign as position the lowest member in the tie (4 in Kendall's example). This is called the modified rank.

Note that these three cases can be understood as follows. By means of a tie-breaking process, we can reduce the problem of assigning positions to the linear case; then the average, highest or lowest values are given to the tied alternatives (the same for all of them, respectively). Nonetheless, other possibilities exist, and we consider a different way, which does not appear in Kendall [17, 18] either, that is not defined through a tie-breaking procedure: the dense rank.

2.3. Dense rank

Gärdenfors [15] points out that “two alternatives x and y are at the same ranking level in the preference order R iff $x\, I\, y$. This is an equivalence relation and the equivalence classes are called ranking levels” (we use the visual term *tiers*). In other words, all *ex aequo* classified alternatives (at any tier) ought to be equally treated, as just one alternative (equality).

We now formally establish a linear order in the quotient set X/I, whose elements are the induced tiers, where positions are univocally given by the sequence of natural numbers. We then extend this approach from X/I to X, i.e., from tiers to alternatives in each tier, sharing the same positions. Hence, the dense rank is a compelling ranking method which consists of assigning position 1 to the alternatives in the top tier, position 2 to the alternatives in the second tier, and so on.

In this way, consider again the above example taken from Kendall [17], where, among several individuals, there is a tie between the third and fourth members. While the above considered positions range from 3 to 4 (depending on the ranking method used), their dense rank might even reach the value 2 if those other alternatives above them were also in a tie.

Next we introduce the notation in order to formally deal with the dense rank.

Definition 3. *Given $R \in \mathcal{W}(X)$, for each $p \in \{0, 1, \ldots, n-1\}$, we consider the tier gathering all the alternatives that have p alternatives below,*

$$T_p = \{x_i \in X \mid p_i = p\}, \tag{1}$$

and

$$T = \big\{p \in \{0, 1, \ldots, n-1\} \mid T_p \neq \emptyset\big\}. \tag{2}$$

Hereinafter, when we say tiers, we mean non-empty tiers i.e., T_p with $p \in T$.

Remark 1. If $R \in \big(\mathcal{W}(X) \setminus \mathcal{L}(X)\big)$, some T_p will be empty. However, always $T_0 \neq \emptyset$, hence $T \neq \emptyset$. What is more, for any $p \in T$, it holds that

$$\#T = \#\{p' \in T \mid p' > p\} + \#\{p' \in T \mid p' < p\} + 1,$$

where the second member corresponds to the number of tiers above and below T_p, plus 1 for T_p itself.

Definition 4. *Given $R \in \mathcal{W}(X)$ and $x_i \in X$, if $x_i \in T_p$, the* dense rank *of x_i in R is defined as*

$$D_R(x_i) = \#T - \#\{p' \in T \mid p' < p\} = \#\{p' \in T \mid p' > p\} + 1. \tag{3}$$

Example 1. Given the following weak order $R \in \mathcal{W}(\{x_1, \ldots, x_{10}\})$

$$\begin{array}{c} x_3 \\ x_6 \;\; x_8 \\ x_1 \;\; x_4 \;\; x_7 \;\; x_{10} \\ x_2 \;\; x_5 \;\; x_9 \end{array}$$

taking into account Eqs. (1), (2) and (3), we have $T_9 = \{x_3\}$, $T_7 = \{x_6, x_8\}$, $T_3 = \{x_1, x_4, x_7, x_{10}\}$, $T_0 = \{x_2, x_5, x_9\}$, $T_8 = T_6 = T_5 = T_4 = T_2 = T_1 = \emptyset$, $T = \{0, 3, 7, 9\}$ (hence, $\#T = 4$) and

$$\begin{aligned} &D_R(x_3) = 4 - 3 = 1 \\ &D_R(x_6) = D_R(x_8) = 4 - 2 = 2 \\ &D_R(x_5) = D_R(x_4) = D_R(x_7) = D_R(x_{10}) = 4 - 1 = 3 \\ &D_R(x_2) = D_R(x_5) = D_R(x_9) = 4 - 0 = 4. \end{aligned}$$

We now show that indifferent alternatives always belong to the same tier.

Proposition 1. *Given $R \in \mathcal{W}(X)$, for all $x_i, x_j \in X$ it holds*

$$x_i \, I \, x_j \;\Leftrightarrow\; x_i, x_j \in T_p \;\text{ for some }\; p \in T.$$

All the proofs along the paper appear in the Appendix.

While tiers consider alternatives indifferent to others in a horizontal way, we next introduce vertical arrangements, from top to bottom, corresponding to different alternatives in each tier, through the notion of maximal chain, similar to that of the saturated chain in the setting of partial ordered sets (see, for instance, Anderson [2, p. 14]).

Definition 5. *Given $R \in \mathcal{W}(X)$, a* maximal P-chain *in R is any list of alternatives $x_{i_1}, \ldots, x_{i_r} \in X$ such that $x_{i_1} \, P \cdots P \, x_{i_r}$ with maximum length $r = \#T$.*

Remark 2. The existence of maximal P-chains is guaranteed because X is finite. In fact, the maximum length ought to be $r = \#T$: indeed, just selecting one alternative in each tier T_p, with $p \in T$, we have $r \geq \#T$; and this value cannot be surpassed, because if $r > \#T$, by the *pigeonhole principle*[15], at least two alternatives should be in the same tier and, by Proposition 1, they will be indifferent to each other.

Note that if x_{i_l} is the l-th element in a maximal P-chain, then there should be $l - 1$ tiers above, and hence $D_R(x_{i_l}) = (l - 1) + 1 = l$, according to Defini-

[15]Also known as *Dirichlet's box principle*, it states that, when t objects are distributed among $s < t$ boxes (pigeonholes), at least one of them must contain at least two objects. In other words, there is no bijective map of a set of cardinality t into a set of cardinality $s < t$.

tion 4. In this way, the dense rank can also be alternatively obtained through any maximal P-chain as $D_R(x_{i_l}) = l$ for any x_{i_l} in the chain and, again by Proposition 1, extending these positions to all the alternatives that are indifferent to x_{i_l}, tier by tier. In other words, the dense rank is univocally determined by one representative element (precisely, that in the maximal P-chain) for each equivalence class of indifferent alternatives (or tiers). For instance, in Example 1, both x_3, x_8, x_1, x_9 and x_3, x_6, x_{10}, x_2 are maximal P-chains.

3. Basic properties

We now consider some basic properties that position operators on weak orders might (or should) verify. Note that we do not *ex ante* impose compelling requirements in order to assign the same positions to indifferent alternatives, etc.

Definition 6. *Let O be a position operator and $O_R : X \longrightarrow \mathbb{R}$ the function that assigns a position to each alternative of X in the weak order $R \in \mathcal{W}(X)$. We say that the position operator O satisfies the following conditions, when they are fulfilled for all $X \subseteq U$ and $R \in \mathcal{W}(X)$:*

1. Equality*: $x_i \, I \, x_j \;\Rightarrow\; O_R(x_i) = O_R(x_j)$, for all $x_i, x_j \in X$.*
2. Monotonicity*: $x_i \, R \, x_j \;\Leftrightarrow\; O_R(x_i) \leqslant O_R(x_j)$, for all $x_i, x_j \in X$.*
3. Neutrality*: $O_{R^\sigma}(x_{\sigma(i)}) = O_R(x_i)$ for every permutation σ on $\{1, \ldots, n\}$.*
4. Sequentiality*: If $R \in \mathcal{L}(X)$, then $O_R(x_i) = S_R(x_i)$, for every $x_i \in X$.*
5. Truncation*: $O_{R|_{X \setminus T_0}}(x_i) = O_R(x_i)$, for every $x_i \in X \setminus T_0$.*

Remark 3. Equality entails that indifferent alternatives are indistinguishable from a positional point of view. Monotonicity, a stronger condition than equality, means that the better the alternative, the less the position value, and viceversa. Neutrality guarantees an equal treatment of alternatives. Sequentiality formalizes the convention of assigning unit-equidistant positions starting from one if there are no ties. Truncation requires that the deletion of the bottom tier preserves the positions of all the remaining alternatives from the original situation. However, by an iterative process, withdrawing the bottom tier in each step, truncation can also be equivalently formulated, stating that the positions of alternatives at any tier do not depend on those other alternatives staying at tiers below. It is important to note that this property entails that not only removing, but also adding tiers below the original situation, will not change the positions of the already existing alternatives, since by successive deletions, the situation prior to the enlargement could be attained once more.

Remark 4. Since X is finite, $R \in \mathcal{W}(X)$ if and only if there exists a function $u : X \longrightarrow \mathbb{R}$ such that $x \, R \, y \;\Leftrightarrow\; u(x) \geqslant u(y)$, for all $x, y \in X$. We say that u is a *utility representation* of R.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing function and u is a utility representation of $R \in \mathcal{W}(X)$, then $f \circ u$ is also a utility representation of R (see the first three chapters of Bridges and Mehta [4] for further details).

Note that, in the framework of utility functions, the greater, the better. However, in the framework of position operators, the less position value, the better.

Then, the following conditions hold:

1. If O is a monotonic position operator, then $-O_R$ is a utility representation of $R \in \mathcal{W}(X)$.
2. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing function and O is a monotonic position operator, then $f \circ O$ is also a monotonic position operator.

Although equality and neutrality are conditions considered from different scenarios, we now justify that the second is stronger than the first one.

Proposition 2. *If a position operator satisfies neutrality, then it also satisfies equality.*

Remark 5. Standard, modified, fractional and dense ranks satisfy equality, monotonicity, neutrality, sequentiality and truncation. This is straightforward for the first four properties and all these position operators, due to the way they have been defined by extending the unique possibility for the linear case in different ways for tied alternatives.

Truncation fulfillment requires a more detailed explanation. First of all, as commented in Remark 3, truncation implies that, for assigning positions to the alternatives in a tier, the deletion or enlargement of tiers below do not affect such positions. This is precisely what happens with standard, modified and fractional ranks, because only alternatives at the same tier (and not those below) are involved in the tie-breaking process. As for the dense rank, truncation easily follows from Definition 4 (see second identity in Eq. (3)).

Consequently, as we are interested in characterization results, those properties in Definition 6 will not be selective enough for our purposes; so we have to analyze further appropriate conditions capturing the very essence of the dense rank.

4. The characterizations

The first requirement in characterizing the dense rank is sequentiality, which formulates a convention of equidistance between consecutive tiers, extending the case of linear orders, where tiers are singletons. While this is not a very demanding condition, that is not the case for the other properties that appear in Theorems 1 and 2, duplication and UD-independency (jointly with truncation in the last case), respectively, which need more detailed explanations.

The key idea of duplication is that the process of replicating some alternatives (the original ones) keeps not only their initial positions unchanged, but also those of the rest of the non-cloned alternatives. We have already pointed out how this condition makes sense in some scenarios where agent's answers, opinions, etc. can be (strategically or not) replicated. This is why this kind of "twinning" or "clone irrelevance" has been successfully introduced in several disciplines. Concretely in Social Choice, it appears in Tideman's "independence of clones" criterion, which establishes that if one non-winning alternative, but similar to the winner, is added for each voter, then the winner should not vary (see Tideman [24] and Schulze [21]). Furthermore, it also appears in the "duplication" axiom for characterizing some voting rules (see García-Lapresta and Martínez-Panero [13] and the references therein, specially Congar and Merlin [6]). The main difference is that, in the voting context, one alternative is replicated for each voter, or the replication process is even referred to one voter's entire preference over all the alternatives; while in our scenario just one alternative is replicated. Nonetheless, due to this analogy, we will also use the coined term "duplication" in what follows.

A cloned alternative can be understood as one added to those in X, maintaining exactly the same preference relationship as the original item with all the alternatives in X. Hence, *a fortiori*, the clone and its original will be indifferent to each other in the extended structure, and they are supposed to share the same position (as pointed out before, this last requirement can be weakened; this is explained in Remark 12).

Similarly to what happens in the Social Choice context to deal with a variable electorate (for instance, as mentioned, when clones or new voters appear from a previous situation, as in Smith [23]), we have also introduced the notion of position operator (Definition 1) to tackle a similar case concerning solely the duplication of alternatives.

Definition 7. *A position operator O satisfies* duplication *if, whenever $R \in \mathcal{W}(X)$, $R' \in \mathcal{W}(X')$, with $X' = X \cup \{x_{n+1}\}$ such that $x_{n+1} \notin X$, $R'|_X = R$ and $x_{n+1}\, I'\, x_j$ for some $x_j \in X$, then $O_{R'}(x_i) = O_R(x_i)$ for every $x_i \in X$ and $O_{R'}(x_{n+1}) = O_{R'}(x_j)$.*

Remark 6. Note that R' is well defined in X', because from the indifference between x_{n+1} and x_j, we can obtain the relationship between x_{n+1} and the rest of the alternatives in X.

Remark 7. Duplication means that the addition of new alternatives to an existing tier preserves the positions of all the alternatives in the original situation. However, it also means that the deletion of alternatives in a tier does not change the positions of the remaining ones either, whenever, along the withdrawal process, at least one alternative (the original one) stays in the tier; because, if so, by successive replication of the rest by adding the deleted alternatives, we can once again obtain the situation before the deletion. Throughout this second stage, by duplication, the positions of the non-duplicated alternatives are preserved and, consequently, they could not change in the previous deletion.

Next we show that duplication implies the compelling requirement of neutrality. The outline of the proof is as follows[16]: each $x_{\sigma(i)}$ should be a clone of the corresponding x_i, but we cannot directly make such an association, because $x_{\sigma(i)}$ already belongs to X. This is why we introduce new alternatives and denote by $R^{n)}$ the weak order on $X \cup \{x_{n+1}, \ldots, x_{2n}\}$ obtained by the iterative duplication of all the alternatives in X, one by one, replicating each x_j with $x_{n+j} \notin X$ for $j = 1, \ldots, n$.

Proposition 3. *If a position operator satisfies duplication, then it also satisfies neutrality.*

Remark 8. Note that, although duplication only explicitly imposes the restriction that the original item and its clone will have the same positions, as neutrality implies equality (Proposition 2), in fact, this is also necessarily true for all the alternatives in the same tier.

We now present the first characterization theorem.

Theorem 1. *A position operator O satisfies sequentiality and duplication if and only if, for each $X \subseteq U$ finite and $R \in \mathcal{W}(X)$, the function $O_R : X \longrightarrow \mathbb{R}$ assigns to each $x_i \in X$ the dense rank of x_i in R.*

Proposition 4. *Sequentiality and duplication are independent.*

We now give reasons for the main property appearing in the second characterization result (jointly with truncation and sequentiality, already introduced). This new condition entails that vertical displacements of one alternative from/to already existing tiers will not change all the other alternatives' positions, provided that neither the existing tiers disappear nor new tiers are created. As with duplication, this requirement makes sense in some contexts. For instance, a wage increase (or decrease) should not harm (or benefit) other workers' labor conditions (this example is related to the query about the top 3 salaries in Kyte [19, pp. 562-568] mentioned before).

Definition 8. *A position operator O satisfies* UD-independency *(UD standing for* upward or downward*) if, for all $R, R' \in \mathcal{W}(X)$ and $x_i, x_k, x_l \in X$ such that $x_k \, P \, x_l$, $x_l \, I \, x_i$, $x_k \, I' \, x_i$, $x_k \, P' \, x_l$ and $R'|_{X \setminus \{x_i\}} = R|_{X \setminus \{x_i\}}$, then $O_{R'}(x_j) = O_R(x_j)$ for every $x_j \in X \setminus \{x_i\}$.*

[16] A similar argument appears in Congar and Merlin [6] within a voting framework.

Remark 9. UD-independency means that if an alternative jumps upwards to another existing tier (i.e., the movement does not create a new tier) while the departure tier still exists (i.e, it does not become empty), then the positions of all other alternatives remaining in the original situation are preserved. It also means that the same is true for a downwards movement under the same conditions, because we can then replicate the situation before by making the opposite upwards movement. Along this second stage, by UD-independency, the positions of non-moving alternatives are preserved, and consequently they could not change during the previous downwards movement. All in all, this property can be dynamically understood as if x_i were vertically moving, no matter whether upwards or downwards. Note that it does not require anything from its former or current positions during the process. What we demand is that all other alternatives do not change their positions.

We have already pointed out that duplication and UD-independency focus on different approaches: the first, on horizontal changes (extensions or deletions); while the second, on vertical displacements. Nonetheless, the following proposition justifies that duplication is a stronger condition than UD-independency.

Proposition 5. *If a position operator satisfies duplication, then it also satisfies UD-independency.*

The conditions that appear in the second characterization of the dense rank, although apparently disconnected with equality, all together imply this last property. This is justified in the following proposition.

Proposition 6. *If a position operator satisfies sequentiality, truncation and UD-independency, then it also satisfies equality.*

We now present the second characterization theorem.

Theorem 2. *A position operator O satisfies sequentiality, truncation and UD-independency if and only if, for each $X \subseteq U$ finite and $R \in \mathcal{W}(X)$, the function $O_R : X \longrightarrow \mathbb{R}$ assigns to each $x_i \in X$ the dense rank of x_i in R.*

Once this second characterization has been achieved, we can now justify the independence of the three conditions appearing in Theorem 2.

Proposition 7. *Sequentiality, truncation and UD-independency are independent.*

Remark 10. Next, discarding in our analysis the common property considered in both characterizations (sequentiality), we justify the fact that duplication (appearing in Theorem 1) is not decomposable in UD-independency jointly with truncation (appearing in Theorem 2). Otherwise, both given axiomatizations would be essentially the same; but this is not the case, and Theorem 2 is neither equivalent to nor a consequence of Theorem 1. To show this, we demonstrate that although duplication implies UD-independency (Proposition 5), it does not entail truncation. Consider, for instance, the position operator defined as

$$O_R(x_i) = \frac{D_R(x_i)}{\#T}.$$

This position operator satisfies duplication, because it is a particular case of that already defined in Eq. (5) verifying this property, by simply taking f as a contraction (dividing by $\#T$). But this denominator is precisely what prevents O_R from truncation fulfillment, because after deleting the last tier, $\#T$ changes (it is reduced one unit), and hence $O_R(x_i)$ also does so.

Additionally, the relationships of both duplication and the combination of UD-independency plus truncation with neutrality (interesting in themselves) sheds some more light on the above comments on the achieved characterizations, confirming that they are essentially different. Indeed, while duplication implies neutrality (Proposition 3), we show that UD-independency together with truncation do not. Consider again the position operator given by $O_R(x_i) = i$, which satisfies UD-independency and truncation, as justified before. However, it is easy to check that it does not fulfill equality (indifferent alternatives have distinct list numbers) and, due to Proposition 2, it does not satisfy neutrality either.

Remark 11. According to Propositions 3 and 2, duplication implies neutrality, and neutrality implies equality, respectively. Hence, due to Theorems 1 and 2, and taking into account Remark 5, the position operator that defines the dense rank satisfies all the properties appearing in the paper: equality, monotonicity, neutrality, sequentiality, truncation, duplication and UD-independency. Note that the first five mentioned conditions are also satisfied by standard, modified and fractional ranks; while duplication and UD-independency are specific to the dense rank.

In a synoptic way, Fig. 1 shows some relationships between 6 conditions used in the paper (in boxes, those characterizing the dense rank in Theorems 1 and 2).

Remark 12. We have achieved the characterizations appearing in the paper without imposing the compelling requirement of monotonicity. Of course, other axiomatizations of the dense rank would be possible by incorporating this property. To this end, note that sequentiality seems to be essential because it establishes equidistance between contiguous positions. If we add monotonicity, it is possible to obtain a new characterization of the dense rank by relaxing duplication. A weak duplication property can be stated as in Definition 7 by also incorporating a clone x_{n+1}, but without making it inherit the position of

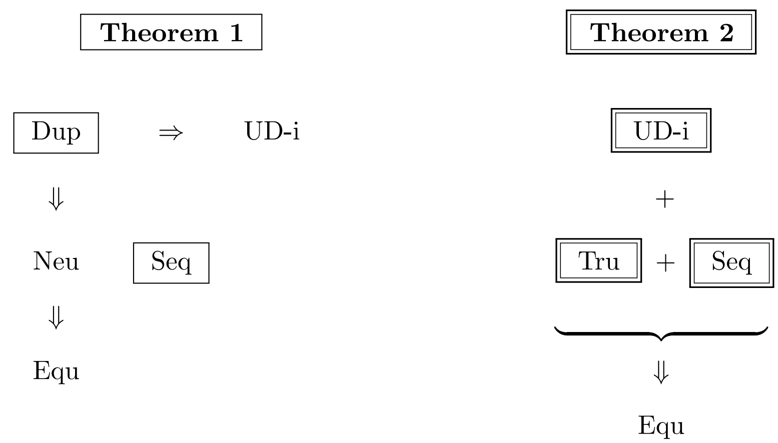

Figure 1: Relationships between conditions.

the original item x_i. As monotonicity implies equality, in Theorem 1, we can substitute duplication for weak duplication plus monotonicity (see Fig. 2).

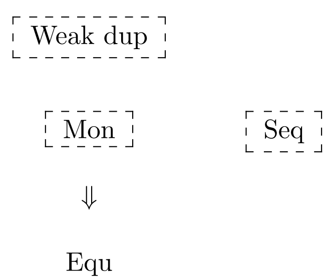

Figure 2: An alternative characterization.

5. Concluding remarks

Along this paper we have focused on the dense rank, but we have also dealt with the standard, modified and fractional ranks. Concerning the duplication property, when the dense rank is used, if a clone is introduced, then the positions of all the already existing alternatives remain. However, it is straightforward to see that, with the standard rank, only the alternatives in the same tier of the clone and those above do not change their positions, while under both modified and fractional ranks, only the alternatives in the tiers above the clone do not change their positions.

Regarding UD-independency, it is easy to check that, when making vertical displacements of alternatives from the k^{th} to the l^{th} tiers or vice-versa, with $k < l$, all the alternatives staying in both tiers or between them (if any) change their positions with the fractional rank; when using the standard rank, there are variations of the positions of those alternatives staying in, or between (if any) $(k+1)^{th}$ and l^{th} tiers; and with the modified rank, the same happens for alternatives staying in, or between (if any) the k^{th} and $(l-1)^{th}$ tiers. However, with the dense rank, all positions remain unchanged except for those of the displaced alternatives.

Other differences can be observed among the dense rank and the standard, modified and fractional ranks. In these last three cases, from the mere knowledge of the positions reached by a list of alternatives, we can induce to some extent (concretely, up to permutations) their preference/indifference arrangement, i.e., the number of alternatives involved in each tier. For instance, in Table 1, under the standard rank, positions 1 and 3 are occupied, and this situation necessarily corresponds to two alternatives on the top and one at the bottom, whatever they are. The same stands for positions 2 and 3 with the modified rank, as well as 1.5 and 3 with the fractional rank. However, if the dense rank is used, just by knowing that positions 1 and 2 have been reached, we cannot determine the structure of the weak order; this information only allows us to assert that there are at least two alternatives at different tiers, but any cardinality at each tier is possible. This fact emphasizes again the fact that the dense rank essentially differs from the standard, modified and fractional ranks (duplication and UD-independency are behind this behavior).

The dense rank makes sense in several contexts (human resources, management, logistics, etc.) and can be used to prevent strategies (with false-name-proof mechanisms). Thus, we advocate for it as an appropriate position operator in some scenarios, although with a different approach from those of the standard, modified and fractional ranks. Characterizations of these last three in this positional setting and a comprehensive framework enclosing all of them are still to be provided.

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Statements and Declarations

Conflict of interest: The authors have no relevant financial or non-financial interests to disclose.

Appendix

Proof of Proposition 1

$\Rightarrow$) As $x_i \, I \, x_j$, if $x_j \, P \, x_k$, we have $x_i \, P \, x_k$; hence, those alternatives dominated by x_j are also dominated by x_i and, consequently, $p_j \leqslant p_i$. Interchanging the roles of the indifferent alternatives, from $x_j \, I \, x_i$, if $x_i \, P \, x_k$, we obtain $x_j \, P \, x_k$ and, again, similarly, $p_i \leqslant p_j$. All in all, the alternatives dominated by x_i coincide with those dominated by x_j. Thus, $p_i = p_j$ and $x_i, x_j \in T_p$, where $p = p_i = p_j$.

$\Leftarrow$) Suppose, by way of contradiction, that $x_i, x_j \in T_p$ for some $p \in T$, i.e., $p_i = p_j = p$, and not $x_i \, I \, x_j$. If $x_i \, P \, x_j$, then $p_i > p_j$; and if $x_j \, P \, x_i$, then $p_j > p_i$. $\square$

Proof of Proposition 2

Let O be a position operator satisfying neutrality and $R \in \mathcal{W}(X)$. Consider $x_i \, I \, x_j$ and let σ be the transposition (i, j). Then, by neutrality, $O_R(x_i) = O_{R^\sigma}(x_{\sigma(i)}) = O_{R^\sigma}(x_j)$. Since $R = R^\sigma$ (due to the symmetry of I), we have $O_R(x_i) = O_R(x_j)$. $\square$

Proof of Proposition 3

Let O be a position operator verifying duplication, $X = \{x_1, \ldots, x_n\}$, $R \in \mathcal{W}(X)$ and a permutation σ on $\{1, \ldots, n\}$ which induces $R^\sigma \in \mathcal{W}(X)$. We have to prove $O_{R^\sigma}(x_{\sigma(i)}) = O_R(x_i)$ for every $i \in \{1, \ldots, n\}$. To do so, duplicate every $x_{\sigma(i)}$ with $y_{\sigma(i)}$ from a set Y such that $\#Y = \#X$ and $Y \cap X = \emptyset$, and let $(R^\sigma)^{n)}$ be the corresponding weak order on $X \cup Y$, obtained by iterated replications. Note that, by duplication, $O_{(R^\sigma)^{n)}}(x_{\sigma(i)}) = O_{(R^\sigma)^{n)}}(y_{\sigma(i)})$. However, each $y_{\sigma(i)}$ also duplicates the original x_i, so that $O_{R^{n)}}(y_{\sigma(i)}) = O_{R^{n)}}(x_i)$. As $y_{\sigma(i)}$ is a clone of both $x_{\sigma(i)}$ and x_i (*caveat*: in different extensions coinciding when restricted to Y), necessarily both the restrictions of $(R^\sigma)^{n)}$ and $R^{n)}$ to X, which are R^σ and R, respectively, must provide the same positions to the respective original items (see Remark 7). Hence, $O_{R^\sigma}(x_{\sigma(i)}) = O_R(x_i)$. $\square$

Proof of Theorem 1

As pointed out in Remark 5, the dense rank satisfies sequentiality due to the way it has been defined, essentially extending the linear case, where tiers are singletons and this property holds, to weak orders where tiers might have a greater cardinality. On the other hand, after replicating an alternative from R to R', already existing tiers are maintained (taking into account Definition 4 before and after the duplication process), so the position for every alternative does not change and duplication is also satisfied.

Conversely, assume both sequentiality and duplication. By concatenation of Propositions 3 and 2, all the alternatives in the same tier have the same position (see Remark 8). Let us now select just one alternative in each tier and withdraw those indifferent to them in a finite number of steps. Then, a maximal P-chain is obtained and positions do not change, as pointed out in Remark 6. In other words, if $C = \{x_{i_1}, \ldots, x_{i_r}\}$ is the set of alternatives involved in a P-chain, then $R|_C$ is a linear order. Now, by sequentiality, the positions of such alternatives ought to follow the list of natural numbers from top to bottom. Finally, again by successive duplication to recover the original $R \in \mathcal{W}(X)$ and taking into account the second paragraph of Remark 2, this exactly corresponds to the dense rank. □

Proof of Proposition 4

1. Let Q be the position operator that assigns to each alternative the quotient between its dense rank and the cardinality of the corresponding tier, i.e.,

$$Q_R(x_i) = \frac{D_R(x_i)}{\# T_p}, \tag{4}$$

where $x_i \in T_p$ (see Eq. (1)).
Q verifies sequentiality, because if $R \in \mathcal{L}(X)$, then $\# T_p = 1$ and $Q_R = D_R$, which satisfies this property (see Remark 5). However, Q does not fulfill duplication: those positions of alternatives $x_i \in T_p$ change after adding a clone, because the denominator of Eq. (4) increases one unit.

2. Let F be the position operator (suggested by Fishburn [12, pp. 164-165]) that assigns an affine linear transformation f to the dense rank:

$$F_R(x_i) = f\big(D_R(x_i)\big), \tag{5}$$

with $f(r) = ar + b$ and $a, b \geqslant 0$.
F satisfies duplication, because the position operator that defines the dense rank does so, as Theorem 1 asserts. However, F only fulfills sequentiality whenever f is the identity function in Eq. (5), because any other affine function will expand or contract the distances among contiguous positions, or it will change 1 as the starting value of the dense rank. In particular, when f is a constant function (as if all possible positions collapse into a common value), duplication trivially holds, but it is obvious that this is not true for sequentiality. □

Proof of Proposition 5

Let O be a position operator, $R, R' \in \mathcal{W}(X)$ and $x_i, x_k, x_l \in X$ such that $x_k \, P \, x_l$, $x_l \, I \, x_i$, $x_k \, I' \, x_i$, $x_k \, P' \, x_l$ and $R'|_{X \setminus \{x_i\}} = R|_{X \setminus \{x_i\}}$. First, withdraw x_i by restricting R to $X \setminus \{x_i\}$ and then extend $R|_{X \setminus \{x_i\}}$ by replicating $x_k \in$

$X \setminus \{x_i\}$ with x_i itself. As the original alternative x_k ought to be indifferent to the clone x_i in such an extension of $R|_{X\setminus\{x_i\}}$, the result is precisely R'. Now, duplication implies that, along the two steps of deletion (see Remark 6) and restoration (at a different tier) of x_i, all the positions of alternatives other than x_i are preserved, so that $O_{R'}(x_j) = O_R(x_j)$ for every $x_j \in X \setminus \{x_i\}$. $\square$

Proof of Proposition 6

By way of contradiction, suppose that a position operator O does not satisfy equality. Then there must exist $R \in \mathcal{W}(X)$ and two alternatives $x_i, x_j \in X$ such that $x_i \, I \, x_j$, $O_R(x_i) = r$ and $O_R(x_j) = s$, with $r \neq s$. Now, take $x_{n+1} \notin X$ and extend R to $R' \in \mathcal{W}(X \cup \{x_{n+1}\})$ by imposing $x_k \, P' \, x_{n+1}$ for all $x_k \in X$ (in other words, add a new bottom tier to those in the original situation with x_{n+1} as the unique alternative). By truncation (see Remark 3), still $O_{R'}(x_i) = r$ and $O_{R'}(x_j) = s$. Next, we select a maximal P'-chain C containing both x_i and x_{n+1}, i.e., $x_{i_1} \, P' \cdots P' \, x_i \, P' \cdots P' \, x_{n+1}$. By applying UD-independency iteratively, we can arrange all the alternatives not appearing in C as indifferent to x_{n+1} (see Remark 9) obtaining a new weak order $R^* \in \mathcal{W}(X \cup \{x_{n+1}\})$ such that $R^*|_C = R'|_C$, and note that $O_{R^*}(x_i) = r$ is still true. Truncating the bottom tier of R^* (that which gathers tied alternatives in R') we obtain the linear order $R^*|_{C\setminus\{x_{n+1}\}}$, and again $O_{R^*|_{C\setminus\{x_{n+1}\}}}(x_i) = r$, being $r \in \mathbb{N}$ by sequentiality.

We undo all this process to recover the original $R \in \mathcal{W}(X)$, with r as the position of x_i, again remaining in each step. Then, all this is repeated with x_j instead of x_i, obtaining that $s \in \mathbb{N}$ is also necessary, due to the same reasons. Even more, as $x_i \, I \, x_j$, they belong to the same tier in R and they share the same number t of tiers above, if any (otherwise, $t = 0$). Hence, x_i and x_j have the same number t of alternatives before, in their respective maximal chains in the parallel processes. Consequently, $r = t+1 = s$, contrary to our assumption. $\square$

Proof of Theorem 2

As sequentiality is a common condition in both characterization results, its fulfillment by the dense rank has already been justified in the proof of Theorem 1. The dense rank also satisfies truncation, as pointed out in Remark 5. In order to complete necessity, notice that, along the process followed in Definition 8, the moving alternative x_i is displaced from an existing tier to another existing one, so that, according to Definition 8, those alternatives in X other than x_k will not change their dense rank positions, which is precisely what UD-independency requires.

For sufficiency, suppose that a position operator O satisfies simultaneously the three abovementioned properties, and let us apply them to $R \in \mathcal{W}(X)$ in order to show that $O_R = D_R$. First, select a maximal P-chain C in R given by $x_{i_1} \, P \cdots P \, x_{i_l}$ and take $x_{n+1} \notin X$, extending R to $R' \in \mathcal{W}(X \cup \{x_{n+1}\})$ by imposing $x_{i_l} \, P' \, x_{n+1}$, so that $C \cup \{x_{n+1}\}$ is also a maximal P'-chain and, by truncation, $O_{R'}(x_{i_m}) = O_R(x_{i_m})$ for each $x_{i_m} \in C$.

Now, the following part of the current proof is similar to that of Proposition 6, this last new tier also playing here a pivotal role, as follows. By applying UD-independency iteratively, we can arrange all the alternatives in $X \setminus C$ as indifferent to x_{n+1} in a new weak order $R^* \in \mathcal{W}(X \cup \{x_{n+1}\})$, which coincides with R' (and hence with R) over C. Thus, $O_{R^*}(x_{i_m}) = O_{R'}(x_{i_m}) = O_R(x_{i_m})$ for each $x_{i_m} \in C$. Truncating the bottom tier of R^* (that which gathers tied alternatives in R), we obtain the linear order $R^*|_C$ and again $O_{R^*|_C}(x_{i_m}) = O_{R^*}(x_{i_m}) = O_{R'}(x_{i_m}) = O_R(x_{i_m})$ for each $x_{i_m} \in C$. Now, by sequentiality in $R^*|_C$, we have $O_{R^*|_C}(x_{i_m}) = O_R(x_{i_m}) = m$, i.e., the alternatives $x_{i_1}, \ldots, x_{i_l} \in C$ have original positions $1, \ldots, l$, respectively. Finally, undoing the previous process for recovering R, each element in C determines the positions of all the alternatives in its tier, by Proposition 6. Hence, we obtain that $O_R = D_R$ (see the second paragraph of Remark 2), in a similar way to what happened at the end of the proof of Theorem 1. $\square$

Proof of Proposition 7

1. The position operator defined by Q_R in Eq. (4) satisfies sequentiality and truncation, but not UD-independency.
   Sequentiality of Q_R has already been proven in Proposition 4. Q_R also fulfills truncation, because the numerator $D_R(x_i)$ does (by Remark 5), and the denominator is not affected by a deleted tier below. However, the position operator defined by Q_R does not satisfy UD-independency, because now, along the x_i moving process in Definition 8, the cardinality of the departure and arrival tiers change, and hence so do those positions of all the alternatives in both tiers.

2. The position operator defined as

$$O_R(x_i) = \begin{cases} D_R(x_i) + n, & \text{if } R \in \big(\mathcal{W}(X) \setminus \mathcal{L}(X)\big), \\ D_R(x_i), & \text{if } R \in \mathcal{L}(X) \end{cases}$$

   satisfies sequentiality and UD-independency, but not truncation.
   By definition, if $R \in \mathcal{L}(X)$, then $O_R = D_R$, which satisfies sequentiality (see Remark 5). As UD-independency is not intended for linear orders (because Definition 8 requires indifference between different alternatives), we use $O_R(x_i) = D_R(x_i) + n$ and this property holds, inherited from that of D_R, taking into account the fact that n remains constant along the vertical moving process. However, a weak order can become linear by truncation, and in such a case, the positions of the remaining alternatives will change.

3. The position operator defined as $O_R(x_i) = i$ satisfies truncation and UD-independency, but not sequentiality.
   In fact, this position operator does not depend on the relationship between alternatives, but just on their list order. Hence, truncation is satisfied because, after deleting some bottom alternatives, those remaining keep

their list numbers. The same holds for UD-independency along the moving process. However, sequentiality does not hold because the list order might not coincide with the sequential positions given by the linear order. $\square$

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