A new technique to estimate the risk-neutral processes in jump-diffusion commodity futures models

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Abstract

In order to price commodity derivatives, it is necessary to estimate the market prices of risk as well as the functions of the stochastic processes of the factors in the model. However, the estimation of the market prices of risk is an open question in the jump-diffusion derivative literature when a closed-form solution is not known. In this paper, we propose a novel approach for estimating the functions of the risk-neutral processes directly from market data. Moreover, this new approach avoids the estimation of the physical drift as well as the market prices of risk in order to price commodity futures. More precisely, we obtain some results that relate the risk-neutral drifts, volatilities and parameters of the jump amplitude distributions with market data. Finally, we examine the accuracy of the proposed method with NYMEX (New York Mercantile Exchange) data and we show the benefits of using jump processes for modelling the commodity price dynamics in commodity futures models. *JEL classification*: G13, G17.

Keywords: commodity futures, jump-diffusion stochastic processes, risk-neutral measure, numerical differentiation, nonparametric estimation.

1. Introduction

The behaviour of many commodity futures has become highly unusual over the past decades. Prices have experienced significant run-ups, and the nature of their fluctuations has changed considerably. This is partly due to financial firms with no inherent exposure to the commodities have adopted strategies of portfolio diversification into commodity futures markets as an asset class, see [2]. However, energy commodities are different from financial assets such as equity and fixed-income securities. For example, changes in market expectations, or even unanticipated macroeconomic developments may cause sudden jumps in energy prices, see [15]. Therefore, traditional modelling techniques are not directly applicable.

In order to price commodity derivatives, the empirical features of the commodity prices need to be considered. First, the spot price and other factors were assumed to follow diffusion processes. For example, Gibson and Schwartz [7] assumed that the spot price and the convenience yield were mean-reverting diffusion processes. Then, Schwartz [20] reviewed one and two-factor models and developed a three-factor mean-reverting diffusion model. Later, Miltersen and Schwartz [16] also considered a three-factor model in order to price commodity futures and futures options. More recently, in the literature, jump-diffusion models have been considered because there are numerous empirical studies which show that commodity prices exhibit jumps, [6], [21] and so on. Hilliard and Reis [12] considered a three-factor model where the spot price follows a jump-diffusion stochastic process. Yan [24] extended existing commodity valuation models to allow for stochastic volatility and simultaneous jumps in the spot price and volatility. Hilliard and Hilliard [13] used the standard geometric Brownian motion augmented by jumps to describe the underlying spot and mean-reverting diffusions for the interest rate and convenience yield state variables for gold and copper prices.

In this paper, we consider a two-factor jump-diffusion commodity model, where one of the factors is the commodity spot price. In the commodity literature, it is very common to use affine models for its simplicity and tractability. They select the simple parametric functions for the model in order to obtain a closed-form solution for the pricing problem. This is mainly important for the market prices of risk, which are assumed to be constant in most of the cases. Then, all the functions can be easily estimated and the commodity derivatives priced. However, there is not any empirical evidence either consensus about affine models are the best models to price commodity futures. Furthermore, the market prices of risk are not observed in the markets. If we considered other more realistic functions for the state variables or the market prices of

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risk or even a nonparametric approach, then, the model would not be affine anymore, a closed-form solution could not be obtained and therefore, the estimation of the market prices of risk would not be possible. In fact, this last problem is an open question in the jump-diffusion commodity literature.

The main contribution of this paper is twofold. First, we obtain some results that allow us to estimate the risk-neutral functions of a two-factor jump-diffusion commodity model directly from commodity spot and futures data on the markets. Therefore, we can obtain a closed-form solution or a numerical approximation for the pricing problem without estimating the market prices of risk, which are not observed and possible to estimate when a closed-form solution is not known. Second, we show the effect of considering jumps in the commodity spot price over the futures prices. We use NYMEX data and a nonparametric approach to estimate the whole functions of our two-factor model. We think that using a nonparametric approach is more realistic than using an affine model.

The remaining of the paper is arranged as follows. In Section 2, we present a two-factor jump-diffusion model to price commodity futures. In Section 3, we prove some results which allow us to estimate the risk-neutral drift, jump intensity and parameters of the distribution of the jump amplitude from spot commodity price and futures data. In Section 4, we estimate our two-factor jump-diffusion model with NYMEX data by means of a nonparametric approach, we price commodity futures and we show its supremacy over a diffusion model. Section 5 concludes.

2. The valuation model

In this section, we present a two-factor commodity futures model. The first factor is the spot price S, and the second factor is δ, which could be, for example, the instantaneous convenience yield or the volatility among other possible variables. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space equipped with a filtration $\mathcal{F}$ satisfying the usual conditions, see [5], [18] or [19]. The factors of the model are assumed to follow this joint jump-diffusion stochastic process:

$$dS(t) = \mu_S(S(t),\delta(t))dt + \sigma_S(S(t),\delta(t))dW_S(t) + J(S(t),\delta(t),Y(t))dN(t), \tag{1}$$

$$d\delta(t) = \mu_\delta(S(t),\delta(t))dt + \sigma_\delta(S(t),\delta(t))dW_\delta(t), \tag{2}$$

where μ_S and μ_δ are the drifts, σ_S and σ_δ the volatilities. The jump amplitude J is a function of the two factors and Y which is a random variable with probability distribution Π. Moreover, W_S and W_δ are Wiener processes and N represents a Poisson process with intensity λ. We assume that the standard Brownian motions are correlated with:

$$Cov(W_S, W_\delta) = \rho t.$$

However, W_S and W_δ are assumed to be independent of N. We also assume that the jump magnitude and the jump arrival time are uncorrelated with the diffusion parts of the processes. We suppose that the functions $\mu_S, \mu_\delta, \sigma_S, \sigma_\delta, J, \lambda$ and Π satisfy suitable regularity conditions: see [5] and [14]. Under the above assumptions, a commodity futures price at time t with maturity at time T, $t \leq T$, can be expressed as $F(t,S,\delta;T)$ and at maturity it is

$$F(T,S,\delta;T) = S.$$

Finally, we assume that there exists a replicating portfolio for the futures price and then, the futures price can be expressed by

$$F(t,S,\delta;T) = E^{\mathcal{Q}}[S(T)|S(t)=S, \delta(t)=\delta], \tag{3}$$

where $E^{\mathcal{Q}}$ denotes the conditional expectation under the $\mathcal{Q}$ measure which is known as the risk-neutral probability measure. The two-factor model (1)-(2) under $\mathcal{Q}$ measure is as follows:

$$dS = \left(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}}E_Y^{\mathcal{Q}}[J]\right)dt + \sigma_S dW_S^{\mathcal{Q}} + Jd\tilde{N}^{\mathcal{Q}}, \tag{4}$$

$$d\delta = \left(\mu_\delta - \sigma_\delta\theta^{W_\delta}\right)dt + \sigma_\delta dW_\delta^{\mathcal{Q}}, \tag{5}$$

where $W_S^{\mathcal{Q}}$ and $W_\delta^{\mathcal{Q}}$ are the Wiener processes under $\mathcal{Q}$ and $Cov(W_S^{\mathcal{Q}}, W_\delta^{\mathcal{Q}}) = \rho t$. The market prices of risk of Wiener processes are $\theta^{W_S}(S,\delta)$ and $\theta^{W_\delta}(S,\delta)$, and $\tilde{N}^{\mathcal{Q}}$ represents the compensated Poisson process, under $\mathcal{Q}$ measure, with intensity $\lambda^{\mathcal{Q}}(S,\delta) = \lambda(S,\delta)\theta^N(S,\delta)$.

3. Exact results and approximations

In the literature, researchers have devoted the greatest attention to affine models such as [20], [12], [24] and [13]. One of the main reasons is that a closed-form solution for the commodity futures price is found. Moreover, this fact allows the application of different estimation techniques, like the Kalman Filter or Maximum Likelihood. However, there is neither evidence nor consensus that affine models are the most suitable for pricing futures contracts.

In the literature, to the best of our knowledge, there is no approach for estimating the market prices of risk for pricing commodity derivatives with jumps, unless a closed-form solution is known. Bandi and Nguyen [1] and Johannes [14] showed how to estimate the functions of a jump-diffusion process by means of their moment equations for interest rate models. However, this approach does not allow us to estimate the market prices of risk, which are necessary to price commodity derivatives but not observable.

In this section, we propose a new approach for estimating the functions of the risk-neutral jump-diffusion stochastic factors of a commodity model directly from market data. Then, we can price futures and the estimation of the market prices of risk can be avoided.

Theorem 1. *Let $F(t,S,\delta;T)$ be the price of the future (3), and S and δ follow the stochastic processes given by (4) and (5), respectively, then:*

$$\begin{aligned} \frac{\partial F}{\partial T}(t,S,\delta;T) &= \left(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J]\right)(T), & (6)\\ \frac{\partial (SF)}{\partial T}(t,S,\delta;T) &= \left(2S\frac{\partial F}{\partial T} + \sigma_S^2 + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J^2]\right)(T), & (7)\\ \frac{\partial (\delta F)}{\partial T}(t,S,\delta;T) &= \left(\delta\frac{\partial F}{\partial T} + S(\mu_\delta - \sigma_\delta\theta^{W_\delta}) + \rho\sigma_S\sigma_\delta\right)(T), & (8) \end{aligned}$$

We prove these results by means of (3). The detailed proof of this theorem can be found in the Appendix. Analogous results, but for diffusion processes, are also shown in [8]. Parallel results for one-factor jump-diffusion interest rate models can be found in [9] and [10].

In order to implement Theorem 1 we rely on numerical differentiation. We use futures prices at a point that is inside the time interval to approximate the slopes at the boundary of the time domain. This fact allows us to consider futures prices with a high spectrum of maturities for the estimation of the functions of the model. More precisely, we obtain a fourth order approximation to the slopes by the well-known difference formula:

$$\frac{\partial g}{\partial T}\mid_{T=t} = \frac{-25g(t) + 48g(t+\Delta) - 36g(t+2\Delta) + 16g(t+3\Delta) - 3g(t+4\Delta)}{12\Delta} + O(\Delta^4). \tag{9}$$

Finally, it is important to remark that after approximating the slopes in Theorem 1, any parametric or nonparametric technique can be applied to estimate them. In this paper, we use the Nadaraya-Watson nonparametric estimator in order to avoid imposing arbitrary restrictions to the different functions of the model. Suppose a data set consists of N observations, $(S_1,\delta_1,Z_1),\cdots,(S_N,\delta_N,Z_N)$, where (S_i,δ_i) are the explanatory variables and Z_i is the response variable. We assume a model of the kind $Z_i = g(S_i,\delta_i) + \epsilon_i$, where $g(S,\delta)$ is an unknown function and ϵ_i is an error term, representing random errors in the observations or variability from sources not included in the (S_i,δ_i) observations. The errors ϵ_i are assumed to be independent and identically distributed with mean 0 and finite variance. The estimate has the closed-form

$$\hat{g}(S,\delta) = \frac{\sum_{i=1}^N w_i(S,\delta)Z_i}{\sum_{i=1}^N w_i(S,\delta)},$$

where $w_i(S,\delta) = K\left(\frac{S-S_i}{h_S}\right)K\left(\frac{\delta-\delta_i}{h_\delta}\right)$ are weight functions (we use the multivariate Gaussian kernel which is also widely used in the literature) and h_S, h_δ the bandwidths, see [11].

4. Empirical application

In this section, we show how practitioners can implement the approach in Section 3 for pricing commodity futures in the markets. Moreover, we analyze the effect of adding jumps to the commodity spot price over the futures prices.

In this empirical application, we use the commodity spot price and the convenience yield as state variables, which are frequently used in the literature, see for example [7] and [20]. We assume that the spot price follows a jump-diffusion process, because commodity prices usually suffer from abrupt changes in the markets, see [6]. However, we assume that the convenience yield is a diffusion process because its behaviour is not affected by extreme changes, see for example [24].

For simplicity and tractability and as usual in the literature, we also assume that the distribution of the jump size under $\mathcal{Q}$ measure is known and equal to the distribution under $\mathcal{P}$ measure. This means that all risk premium related to the jump

Variable	N	Mean	Std. dev.	Max	Min
S_t	2735	5.4732	2.3070	15.3900	1.8200
$S_{t+1} - S_t$	2734	−0.0011	0.2665	2.5000	−1.9100

Table 1: Summary of the statistics on the natural gas spot price and its first differences, January 2004-December 2014.

are artificially absorbed by the change in the intensity of the jump from λ under the physical measure to $\lambda^{\mathcal{Q}}$ under the risk-neutral measure, see [17]. Moreover, we assume in (4) that $J(S, \delta, Y) = Y$, where Y is a random variable which follows a normal distribution $N(0, \sigma_Y)$, see [24], [4] and [23] among others. Therefore, $E_Y^{\mathcal{Q}}[Y] = E_Y[Y] = 0$.

In order to show how the approach in Section 3 can be implemented, we will price natural gas futures with daily data from the NYMEX. Natural gas spot and futures prices were obtained from the Energy Information Administration of the U.S. Department of Energy (E.I.A. database) and Quandl platform. The sample period covers from January 2004 to April 2015. Figure 1 plots the natural gas spot price data and its first differences. We also consider futures prices with maturities equal to $1, 2, 3, 4, 6, 9, 12, 18, 24, 36$ and 44 months. We use data from January 2004 to December 2014 for estimating the risk-neutral functions. Table 1 summarises the in-sample data. We keep data from January to April 2015 as out-of-sample data in order to evaluate the results of our approach.

As it is well known in the literature, the convenience yield is not observed in the markets. Then, following [7], we approximate it by the following result

$$\delta_{T-1,T} = r_{T-1,T} - 12 \ln \left[\frac{F(t, S, \delta; T)}{F(t, S, \delta; T-1)} \right],$$

where $r_{T-1,T}$ denotes the $T-1$ period ahead annualized one month riskless forward interest rate. We obtain this forward interest rate with two daily T-Bill rates with maturities as close as possible to the futures contracts' ones in order to compute $\delta_{1,2}$, the one-month ahead annualized convenience yield. The latter is identified with the instantaneous convenience yield $\delta_{0,1}$ in this study, see [7] for more details. T-Bill rates are obtained from the Federal Reserve h.15 database.

First, we obtain the compensated risk-neutral drift of the spot price. We approximate the partial derivative in (6), using numerical differentiation (9), with futures prices with maturities equal to 1, 2, 3 and 4 months. Then, we estimate it by means of the Nadaraya-Watson estimator. Secondly, in order to obtain the risk-neutral jump intensity, we approximate the partial derivatives $\frac{\partial F}{\partial T}|_{T=t}$ and $\frac{\partial (SF)}{\partial T}|_{T=t}$ in (7), using numerical differentiation (9) with spot prices and futures prices with maturities equal to 1, 2, 3 and 4 months.

In order to estimate the functions of the stochastic processes of S and δ under the physical measure, we use the following result.

Theorem 2. *If S and δ solve (1)-(2), then*

$$\begin{aligned}
M_S^1(S,\delta) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[S(t+\Delta t) - S(t) | S(t) = S, \delta(t) = \delta] = \mu_S(S,\delta) + \lambda(S,\delta) E_Y[Y], && (10)\\
M_S^2(S,\delta) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[(S(t+\Delta t) - S(t))^2 | S(t) = S, \delta(t) = \delta] = \sigma_S^2(S,\delta) + \lambda(S,\delta) E_Y[Y^2], && (11)\\
M_S^k(S,\delta) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[(S(t+\Delta t) - S(t))^k | S(t) = S, \delta(t) = \delta] = \lambda(S,\delta) E_Y[Y^k], \quad k \geq 3, && (12)\\
M_\delta^2(S,\delta) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[(\delta(t+\Delta t) - \delta(t))^2 | S(t) = S, \delta(t) = \delta] = \sigma_\delta^2(S,\delta), && (13)\\
Cov(S,\delta) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[S(t+\Delta t) - S(t))(\delta(t+\Delta t) - \delta(t))^2 | S(t) = S, \delta(t) = \delta] = \rho(S,\delta)\sigma_S(S,\delta)\sigma_\delta(S,\delta). && (14)
\end{aligned}$$

This theorem can be proved with the infinitesimal operator (see [18]), as in [22] and [3], for diffusion processes, and in [14] and [1], for jump-diffusion processes.

As we have previously assumed, the jump size distribution under $\mathcal{Q}$ measure is known and equal to the distribution under $\mathcal{P}$ measure and $Y \rightsquigarrow N(0, \sigma_Y^2)$, then $E_Y^{\mathcal{Q}}[Y] = E_Y[Y] = 0$ and $\sigma_Y^2 = E_Y^{\mathcal{Q}}[Y^2] = E_Y[Y^2]$. Therefore, we estimate σ_Y^2 and the volatility of the spot price σ_S, by means of the moment equations of a jump-diffusion process in Theorem 2, when the jump amplitude follows a normal distribution. More precisely, as $Y \rightsquigarrow N(0, \sigma_Y^2)$, then:

$$E_Y[Y^{2k}] = \sigma_Y^{2k} \prod_{n=1}^{k} (2k-1),$$
$$E_Y[Y^{2k-1}] = 0, \quad k = 1, 2, 3, \ldots$$

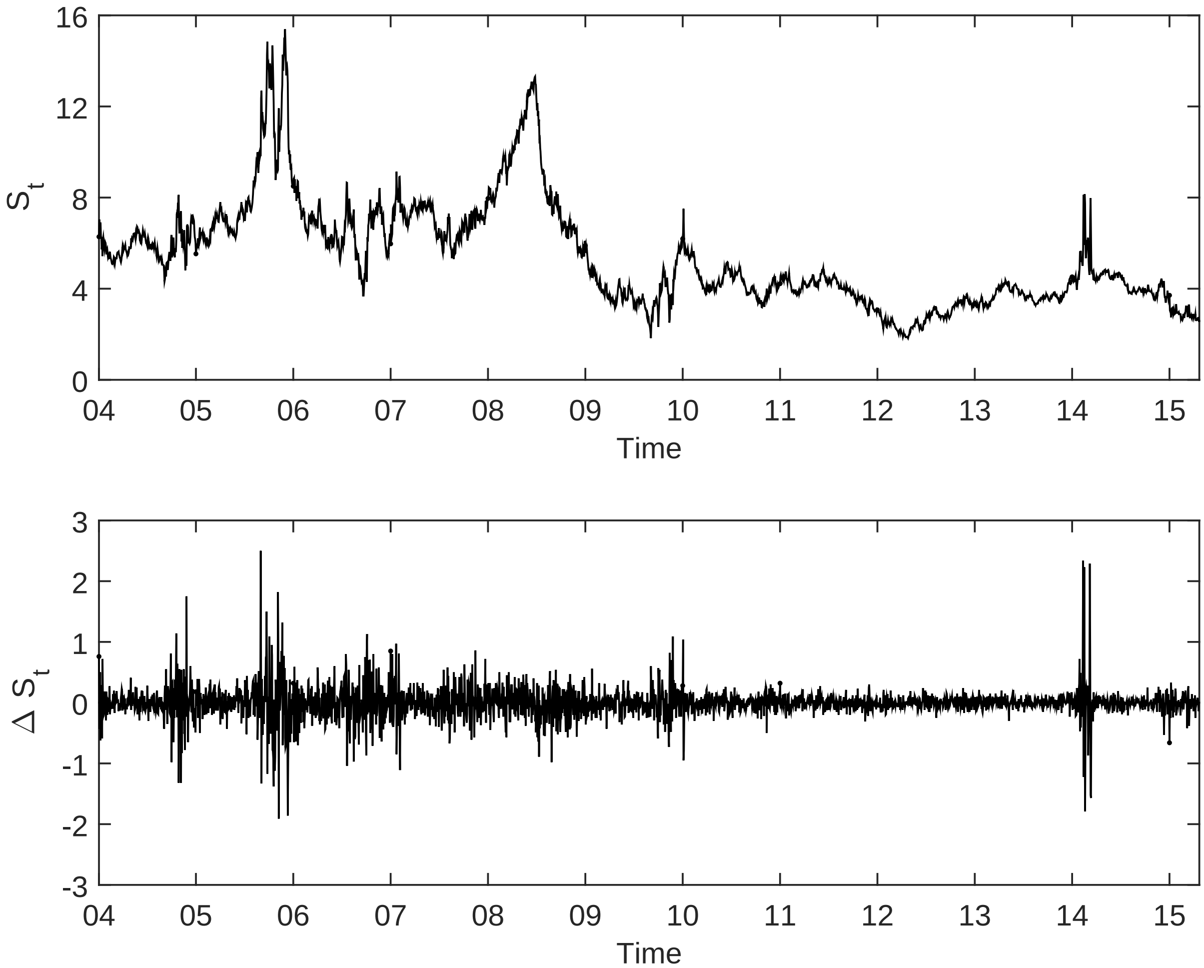

Figure 1: Daily natural gas spot price and its first differences from January 2004 to April 2015.

In order to estimate σ_Y^2 and σ_S^2, We use moments (11) and (12) with $k = 4$ and 6 and the Nadaraya-Watson estimator. Then, we replace these values and the approximations of the partial derivatives $\frac{\partial F}{\partial T}|_{T=t}$ and $\frac{\partial (SF)}{\partial T}|_{T=t}$ in (7) and we estimate the risk-neutral jump intensity of the spot price with the Nadaraya-Watson estimator. As the convenience yield is a diffusion process, we can estimate its risk-neutral drift by means of (8) where $\rho\sigma_S\sigma_X = \mathrm{Cov}(S, \delta)$. In order to estimate this covariance, we use the moment condition (14) and the Nadaraya-Watson estimator, see [3] for more details. Then, we replace the estimated covariance and the approximations of $\frac{\partial F}{\partial T}|_{T=t}$ and $\frac{\partial (\delta F)}{\partial T}|_{T=t}$ in (8) and we get the risk-neutral drift of the convenience yield by means of the Nadaraya-Watson estimator. Finally, the volatility of the convenience yield under $\mathcal{P}$ measure is equal to the volatility under $\mathcal{Q}$ measure, we estimate σ_δ by means of the second order moment (13) and convenience yield data.

In order to price natural gas futures, we use the Monte Carlo simulation approach because it is widely used by practitioners in the markets, especially for multifactorial models because its simplicity and efficiency.

The approach we propose in this paper is a jump-diffusion extension of the one proposed by Gómez-Valle and Martínez-Rodríguez [8] for a diffusion commodity model. Therefore, we can use both approaches for examining the effect of adding jumps to the commodity spot price over the commodity futures prices.

Gómez-Valle and Martínez-Rodríguez [8] assume that the futures price depends on the same two factors: the commodity spot price and the convenience yield. More precisely, they assume that these factors follow this joint diffusion stochastic process under $\mathcal{Q}$ measure:

$$dS = \left(\mu_S - \sigma_S \theta^{W_S}\right) dt + \sigma_S dW_S^{\mathcal{Q}}, \tag{15}$$

$$d\delta = \left(\mu_\delta - \sigma_\delta \theta^{W_\delta}\right) dt + \sigma_\delta dW_\delta^{\mathcal{Q}}, \tag{16}$$

with $Cov(W_S^{\mathcal{Q}}, W_\delta^{\mathcal{Q}}) = \rho t$.

We also estimate model (15)-(16) directly from data in the NYMEX, using the approach by Gómez-Valle and Martínez-Rodríguez [8], and obtain the natural gas futures prices with the method of Monte Carlo.

In order to analyse the effect of adding jumps to the commodity price over the futures prices, we make some comparisons between the futures prices obtained with the jump-diffusion model (JDM) and with the diffusion model (DM). We use the root mean square error (RMSE), the percentage root mean square error (PRMSE) and the mean absolute error (MAE) for the out-of-sample period of time as measures of error:

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(F_t - \hat{F}_t\right)^2},$$

$$PRMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{F_t - \hat{F}_t}{F_t}\right)^2},$$

$$MAE = \frac{1}{n}\sum_{t=1}^{n}|F_t - \hat{F}_t|,$$

where n is the number of observations, F_t is the market futures price and $\hat{F}_t$ is the predicted futures price of the different models.

Table 2: Measures of error: MAE, RMSE and PRMSE, for the out of sample period of time, January 2015-April 2015, with the diffusion and jump-diffusion model.

	DM			**JDM**		
	RMSE	PRMSE	MAE	RMSE	PRMSE	MAE
F1	2.019×10^{-1}	7.2%	1.565×10^{-1}	1.929×10^{-1}	6.9%	1.448×10^{-1}
F6	1.424×10^{-1}	4.9%	1.209×10^{-1}	2.567×10^{-1}	8.8%	2.271×10^{-1}
F9	1.151×10^{-1}	3.7%	9.852×10^{-2}	1.666×10^{-1}	5.4%	1.290×10^{-1}
F12	2.993×10^{-1}	9.1%	2.637×10^{-1}	1.858×10^{-1}	5.7%	1.509×10^{-1}
F18	1.964×10^{-1}	6.1%	1.585×10^{-1}	1.711×10^{-1}	5.3%	1.399×10^{-1}
F24	5.280×10^{-1}	14.8%	4.839×10^{-1}	3.218×10^{-1}	9.0%	2.892×10^{-1}
F36	6.677×10^{-1}	17.8%	6.206×10^{-1}	4.229×10^{-1}	11.3%	3.865×10^{-1}
F44	4.780×10^{-1}	13.2%	4.706×10^{-1}	1.933×10^{-1}	5.3%	1.711×10^{-1}

For the out of sample period of time, in Table 2 we show the values of these different measures of error in the futures prices with different maturities. For short maturities (6 months and 9 months) the DM provides slightly lower errors than the JDM, except for the shortest maturity (1 month). However, for the longest maturities the JDM provides smaller errors than the DM. Moreover, the higher the maturity the higher the differences between the two models.

In Figure 2, we plot the observed futures prices and those estimated with the DM and the JDM along the out-of-sample period of time for different maturities. As seen from the figure, for the different maturities, the observed futures prices are, in general, over the prices obtained with DM and JDM. More precisely, the prices with the DM are nearly always smaller than those with the JDM. Furthermore, the higher the maturity, the higher the differences.

In sum, JDM and DM underprice the futures prices and the prices obtained with the JDM are, in general, closer to the observed prices than those obtained with the DM. Hence this fact supports the use of jump processes when modelling the commodity price dynamics in order to price commodity futures, especially for high maturities.

5. Conclusions

In order to price commodity derivatives with jump-diffusion processes, we provide a novel procedure based on the estimation of the drifts and jump intensities of the risk-neutral processes. This technique is notable because neither the physical drift, nor the market prices of risk have to be estimated. As a consequence, it is not necessary to make arbitrary assumptions about the market prices of risk as usual in the literature, when a closed-form solution is not known. Furthermore, as we estimate the risk-neutral drifts directly from data in the market, we reduce the misspecification error because we don't have

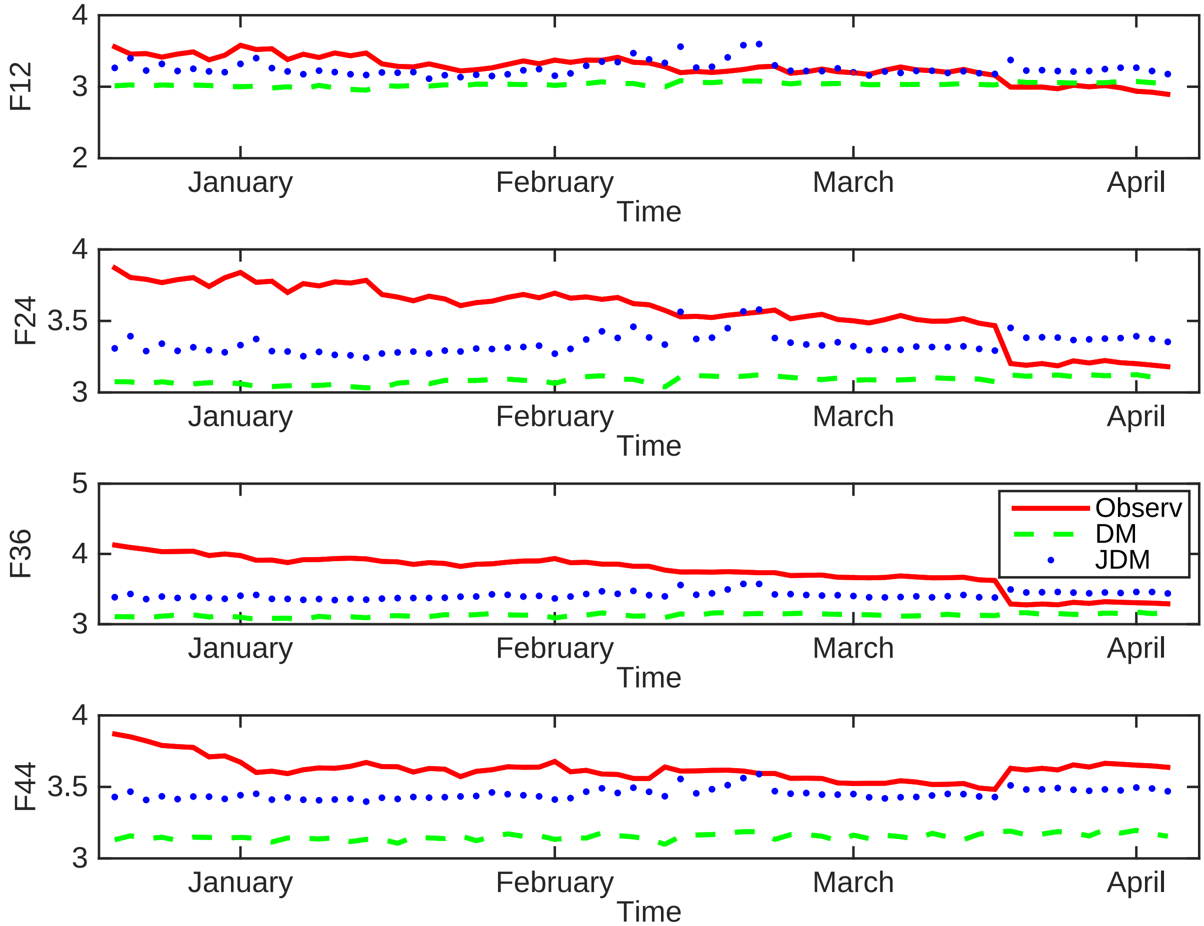

Figure 2: Natural gas futures prices (January 2015-April 2015) with maturities: 12, 24, 36 and 44 months. The observed futures prices on the NYMEX are the red solid line, the DM futures prices are the green dash line and the JDM futures prices are the blue dotted line.

to estimate the physical drifts. Finally, this approach is adaptable: both parametric and nonparametric methods could be used to estimate the required functions.

We show the practical interest of this new approach in an empirical experiment with NYMEX data and we analyse the effect of adding jumps to the commodity spot price over the futures prices.

Acknowledgements

The authors are grateful to the anonymous referees for their constructive suggestions and helpful comments. L. Gómez-Valle and J. Martínez-Rodríguez were supported in part by the GIR Optimización Dinámica, Finanzas Matemáticas y Utilidad Recursiva of the University of Valladolid and the projects: VA191U13 of the Consejería de Educación, JCyL and MTM2014-56022-C2-2-P of the Spanish Ministerio de Economía y Competitividad and European FEDER Funds.

Appendix

In this appendix, we prove the Theorem 1 in Section 3.

Proof of Theorem 1.

We consider the integral form of (4)

$$\begin{aligned} S(T+h)-S(T) &= \int_T^{T+h} \left(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J]\right)(z)dz \\ &+ \int_T^{T+h} \sigma_S dW_S^{\mathcal{Q}}(z) + \int_T^{T+h} J d\tilde{N}(z). \end{aligned}$$

Taking into account (3) and that the expected value of stochastic integral is zero, we calculate the expectation with respect to $\mathcal{Q}$ measure, and we obtain

$$F(t,S,\delta;T+h) - F(t,S,\delta;T) = \int_T^{T+h} E^{\mathcal{Q}}[(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J_S])(z)|S(t)=S,\delta(t)=\delta]dz. \tag{17}$$

If we divide by h and take limits in (17), we get (6).

Using the risk-neutral process (4) and the Ito's product rule, see [5] we have

$$\begin{aligned} d(S^2) &= \left(2S(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J]) + \sigma_S^2 + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J^2]\right)dt \\ &+ 2S\sigma_S dW_S^{\mathcal{Q}} + 2SJd\tilde{N}^{\mathcal{Q}}. \end{aligned} \tag{18}$$

If we consider the integral form of (18), with a similar reasoning to one used for the equality (6), we obtain (7).

Finally, we use the risk-neutral processes (4) and (5) and with the Ito's product rule, we have

$$\begin{aligned} d(S\delta) &= \left(\delta(\mu_S - \sigma_S\theta^{W_S} + \lambda^{\mathcal{Q}} E_Y^{\mathcal{Q}}[J]) + S((\mu_\delta - \sigma_\delta\theta^{W_\delta} + \rho\sigma_S\sigma_\delta\right)dt \\ &+ \delta\sigma_S dW_S^{\mathcal{Q}} + S\sigma_\delta dW_\delta^{\mathcal{Q}} + \delta J d\tilde{N}^{\mathcal{Q}}. \end{aligned} \tag{19}$$

Once more, if we consider the integral form of (19), with a similar reasoning to our analysis above we get (8).

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