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DEPARTAMENTO DE ÁLGEBRA, ANÁLISIS MATEMÁTICO,
GEOMETRÍA Y TOPOLOGÍA

TESIS DOCTORAL:

Monomial multisummability through Borel-Laplace transforms. Applications to singularly perturbed differential equations and Pfaffian systems

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Introducción

Es un hecho bien conocido que las series divergentes aparecen de manera natural en numerosos problemas relacionados con las ecuaciones funcionales, aunque estos problemas involucren exclusivamente series convergentes. Quizás uno de los primeros ejemplos, históricamente hablando, es el que proporciona L. Euler en su tratado *De seriebus divergentibus* [E]. En este artículo, L. Euler estudia, entre otras cosas, la serie numérica

$$1 - 1! + 2! - 3! + 4! - 5! + \cdots = \sum_{n=0}^{\infty} (-1)^n n!, \tag{0-1}$$

que llama *serie hipergeométrica de Wallis.* Para ello, propone hasta cuatro métodos diferentes de sumación, entre ellos

1. Una iteración de las llamadas hoy *transformaciones de Euler* y el cálculo de la "suma hasta el menor término",

2. La introducción de un parámetro x adicional, lo cual nos lleva a considerar la serie de potencias

$$x - x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}, \tag{0-2}$$

que llamaremos *serie de Euler* y que resulta ser una solución de la ecuación diferencial lineal

$$x^2 y'(x) + y(x) = x,$$

resoluble por variación de constantes. La solución de esta ecuación, evaluada en $x = 1$, permite atribuir un valor a la suma de (0-1) (aproximadamente 0.59637164).

En el siglo XVIII estos razonamientos eran interesantes para intentar aproximar el valor de algunas constantes matemáticas, como e ó π: lo importante era atribuir de manera coherente un valor a la suma de este tipo de series sin limitarse a las nociones clásicas de convergencia que se estudian en los primeros cursos de una carrera universitaria.

El interés físico de las series divergentes se puso de manifiesto con los trabajos de G.G. Stokes sobre la función de Airy: esta es una función que aparece en el estudio de las cáusticas en óptica, tales como las del arcoiris. Históricamente, este fue el problema matemático que llevó a G.B. Airy a desarrollar esta función especial. Más precisamente, la función de Airy se define por la siguiente expresión integral:

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(xt + \frac{t^3}{3}\right) dt,$$

y es solución de la ecuación diferencial $y''(x) - xy(x) = 0$. Uno de los métodos que se estudian en cursos elementales para aproximar las soluciones de este tipo de ecuaciones, es el desarrollo en serie de potencias de las mismas. En el caso de la función de Airy el radio de convergencia de su serie de Taylor en el origen es infinito, lo cual aparentemente daría el problema por resuelto. Pero esta serie de potencias resulta ser de convergencia muy lenta, motivo que la hace impracticable a los cálculos. G.G. Stokes tuvo la idea de desarrollar la función Ai en el infinito, lo cual da como resultado una serie divergente en potencias de $x^{1/2}$: "sumando hasta el menor término", dicha serie proporciona datos asombrosamente precisos sobre la función de Airy. Un ejemplo físico más moderno lo hallamos en el campo de la electrodinámica cuántica: en el estudio del momento magnético del electrón aparece una serie de potencias en la que cada término se calcula a partir de diagramas de Feynman. Esta serie resulta ser divergente, y de nuevo la suma de algunos términos (se ignora cuál es el menor término) proporciona valores muy cercanos a los experimentales. Pueden leer detalles de esto, así como de estos problemas, en el artículo de divulgación de J.P. Ramis [R1]. Para una descripción más detallada también se puede consultar [R2].

Hemos mencionado en dos ocasiones la técnica de la "suma hasta el menor término". En numerosas series divergentes que aparecen en problemas físicos, los primeros términos decrecen en valor absoluto, pero luego crecen indefinidamente. La técnica mencionada consiste en truncar la serie en el momento en que los términos empiezan a crecer. Esta técnica es llamada por H. Poincaré "sumación de los astrónomos", en contraposición a la "sumación de los geómetras" (series convergentes en el sentido moderno). Su justificación precisa requiere el uso de las series de tipo Gevrey, tal y como comentaremos más adelante.

Es precisamente H. Poincaré quien da uno de los grandes impulsos a la teoría de la sumación de series divergentes, que diversos matemáticos de prestigio habían despreciado (para N. Abel, eran una "invención del diablo"). Como en numerosos otros problemas de matemáticas, y en palabras de J. Hadamard: "... el mejor y más corto camino entre dos verdades del dominio **real** suele pasar por el dominio **complejo**" [H, pág. 123]. Así, H. Poincaré en su trabajo [P] introduce a finales del siglo XIX la noción de desarrollo asintótico: una función f, holomorfa en un sector $V = V(a,b,r) = \{x \in \mathbb{C} | a < \arg(x) < b, 0 < |x| < r\}$, admite una serie $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ como *desarrollo asintótico* en el origen sobre V si para cada número natural N y cada subsector W de V existe una constante $C_N(W)$ tal que

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C_N(W)|x|^N,$$

sobre W. Cabe notar que esta no es la definición original dada por H. Poincaré pero sí resulta ser equivalente para funciones acotadas en cada subsector de V. Para los detalles de este hecho se puede consultar [FZ].

Los estudios de H. Poincaré, así como de otros matemáticos posteriores, se centran en las series de potencias que aparecen como soluciones de sistemas de ecuaciones diferenciales (lineales o no) holomorfas, en torno a los llamados puntos singulares. Centrándonos en el caso lineal, nos referimos a sistemas de ecuaciones del tipo

$$x^{p+1}y'(x) = A(x)y(x) + b(x),$$

donde $y(x) = (y_1(x), \ldots, y_n(x))^t \in \mathcal{O}(D)^n$, $A \in \mathrm{Mat}(n \times n, \mathcal{O}(D))$, $b \in \mathcal{O}(D)^n$, siendo D un disco en torno de 0. En algunos textos clásicos, como [CL], estos puntos singulares se clasifican en puntos de primera clase (si $p = 0$) y de segunda clase (si $p > 0$), lo cual determina frecuentemente la naturaleza de las soluciones. Aludiendo a estas, los puntos singulares se clasifican como regulares (reducibles a los puntos de primera clase) o irregulares. Es en estos últimos tipos de ecuaciones en los que aparecen fenómenos de divergencia. Así H. Poincaré, M. Hukuhara, H.L. Turritin y W. Wasow entre otros demuestran el siguiente resultado, válido en el caso no lineal:

Teorema.(Teorema fundamental de los desarrollos asintóticos) *Consideremos el sistema de ecuaciones diferenciales holomorfas*

$$x^{p+1}y'(x) = F(x, y(x)), \quad p \in \mathbb{N}^*,$$

que admite el vector de series formales $\hat{y}$ como solución, y en el que la matriz de la parte lineal

$$A := \frac{\partial F}{\partial y}(0, \mathbf{0}),$$

es invertible. Si V es un sector de abertura a lo más π/p, existe una solución $y(x) \in \mathcal{O}(V)^n$ que admite a $\hat{y}$ como desarrollo asintótico en V.

Una prueba de este hecho se puede consultar en [W1].

Este teorema permite dotar de cierto significado geométrico a la serie formal $\hat{y}(x)$, interpretándola como $y(x)$. Pero esta función $y(x)$ dista mucho de ser única, pues hay funciones con desarrollo asintótico nulo que son soluciones de ecuaciones diferenciales.

Un nuevo y crucial impulso a la teoría se produce a finales de los años 70 con los trabajos, por una parte de J. Écalle, sobre las llamadas funciones resurgentes, y por otra parte, de J.P. Ramis, quien introduce y sistematiza la noción de $k-$sumabilidad, la cual generaliza la noción de sumabilidad dada por E. Borel en los años 20 [B]. La definición de desarrollo asintótico dada por H. Poincaré fue precisada en los llamados *desarrollos asintóticos $s-$Gevrey*: en ellos, la constante C_N que allí aparece se sustituye por una del tipo

$$CA^N N!^s,$$

explicitándose la dependencia de N. Resulta que si una serie formal $\hat{y}(x)$ es el desarrollo asintótico $s-$Gevrey, de una función $y(x)$ definida en un sector V de abertura estrictamente superior a $s\pi$, $y(x)$ es la única función con esa propiedad, y es legítimo llamarla la $k-$suma de $\hat{y}$ en V (aquí $k = 1/s$, respetando las notaciones hoy habituales en la teoría). En este contexto el Teorema fundamental de los desarrollos asintóticos fue refinado por J.P. Ramis y Y. Sibuya en 1989, como se enuncia a continuación.

Teorema. *Consideremos el sistema de ecuaciones diferenciales holomorfas*

$$x^{p+1}y'(x) = F(x, y(x)), \quad p \in \mathbb{N}^*,$$

que admite el vector de series formales $\hat{y}$ de tipo $s-$Gevrey como solución. Si V es un sector de abertura a lo más $\min\{\pi s, \pi/p\}$, existe una solución $y(x) \in \mathcal{O}(V)^n$ que admite a $\hat{y}$ como desarrollo asintótico de tipo $s-$Gevrey en V.

La demostración completa de este resultado se puede consultar en [RS2].

Además de introducir la noción de serie $k-$sumable, J.P. Ramis enuncia un resultado sobre la estructura formal de las soluciones de los sistemas lineales con singularidad irregular que equivale a decir que toda solución formal se puede construir a partir de series $k-$sumables, para diversos valores de k (los niveles de la ecuación). Resulta claro a partir de aquí que no toda serie formal solución de una ecuación diferencial holomorfa es $k-$sumable para un único valor de k por lo que se introduce la noción de multisumabilidad en la que intervienen diversos valores de k. La primera prueba de la multisumabilidad de las soluciones de ecuaciones diferenciales lineales es dada por W. Balser, B.L.J. Braaksma, J.P. Ramis y Y. Sibuya en [BBRS]. Posteriormente, B.L.J. Braaksma prueba un resultado similar para las ecuaciones no lineales [Br].

Con esto, tenemos una respuesta parcial al problema de asignar una suma a las series formales obtenidas como soluciones de ecuaciones diferenciales, pero esta respuesta no es constructiva. En el citado texto de E. Borel [B] se describe determinada transformada integral, la hoy llamada *transformada de Borel*, la cual, combinada con la transformada de Laplace permite construir explícitamente la suma en una dirección de una serie $1-$sumable, caso de existir. J.P. Ramis generaliza esta noción introduciendo la noción de *$k-$transformada de Laplace y Borel*, las cuales permiten construir la $k-$suma de una serie $k-$sumable. Asimismo J. Écalle define los operadores de aceleración. Con ayuda de ellos, si $k_1 > k_2 > \cdots > k_m > 0$, y $\hat{y}$ es una serie $(k_1, ..., k_m)-$multisumable en la dirección d, su suma en esta dirección puede computarse como

$$\mathcal{L}_{k_1} \circ \mathfrak{A}_{k_1,k_2} \circ \cdots \circ \mathfrak{A}_{k_{m-1},k_m} \circ \hat{\mathcal{B}}_{k_m}(\hat{y}(x)),$$

donde $\hat{\mathcal{B}}_{k_m}$ representa la k_m-transformada de Borel formal, $\mathfrak{A}_{k,k'}$ es el operador de aceleración de orden (k, k'), $k > k'$, y $\mathcal{L}_{k_1}$ es la k_1-transformada de Laplace. Para que esta maquinaria funcione, es necesario observar que los distintos niveles de $k-$sumabilidad son incompatibles: si $k \neq k'$ toda serie que sea simultáneamente $k'-$sumable y $k-$sumable ha de ser necesariamente convergente.

En este punto de la historia podemos citar las palabras de J.P. Ramis en [R1, pág. 139]:

"Que réserve le futur aux spécialistes des séries divergentes? Les principaux défis concernent ce que l'on nomme les perturbations singulières."

La primera dirección a la que se dirige la presente tesis es hacia los desarrollos asintóticos asociados a problemas de perturbaciones singulares. Un sistema lineal singularmente perturbado es uno del tipo

$$\varepsilon^{\sigma}\frac{\partial y}{\partial x}(x,\varepsilon)=A(x,\varepsilon)y(x,\varepsilon),$$

donde $A(x,\varepsilon)$ es una matriz de funciones holomorfas en un entorno de $(0,0)\in\mathbb{C}^2$. Típicamente una solución formal de este sistema depende del parámetro de perturbación singular ε, y admite un desarrollo en serie de potencias en ε del tipo

$$y(x,\varepsilon)=y_0(x)+y_1(x)\varepsilon+y_2(x)\varepsilon^2+\cdots,$$

con coeficientes $y_j(x)$ holomorfos en un disco común de convergencia, y divergente en ε. Se plantea el problema de la sumabilidad en ε de las soluciones de dichos sistemas, problema al que han contribuido numerosos autores. Citemos algunos logros destacados:

- M. Canalis-Durand prueba el carácter Gevrey de la solución formal de la ecuación de Van der Pol perturbada [CD] y A. Fruchard y R. Schäfke prueban la sumabilidad en ε de dicha solución [FS].
- En el caso general, las soluciones resultan ser no necesariamente $k-$sumables para ningún valor de k. No obstante, M. Canalis-Durand, J.P. Ramis, R. Schäfke y Y. Sibuya [CDRSS] muestran que, en condiciones de invertibilidad de la parte lineal (de hecho, ellos consideran condiciones algo más generales), toda solución formal es de tipo Gevrey y puede representarse como una función holomorfa $y(x,\varepsilon)\in\mathcal{O}(D\times V)$, donde D es un disco en torno al origen y V es un sector, que admite como desarrollo asintótico la serie formal solución.

Un problema adicional se encuentra cuando consideramos perturbaciones singulares de ecuaciones diferenciales con puntos singulares. Por ejemplo, podemos considerar la ecuación de Schrödinger lineal singularmente perturbada

$$\varepsilon^2\frac{\partial^2 y}{\partial x^2}(x,\varepsilon)+P(x)y(x,\varepsilon)=0,$$

donde $P(x)$ es un polinomio. En este caso la singularidad en x está en el infinito, y la singularidad en el parámetro ε, en 0. Este tipo de ecuaciones y sistemas han sido considerados por diversos autores, como W. Wasow [W2]. Para su tratamiento, parece necesario hacer intervenir una noción de desarrollo asintótico en varias variables. La primera noción satisfactoria de esto se debe a H. Majima, quien en [Mj1] define la noción de función con desarrollo asintótico fuerte en un polisector (producto de sectores), la cual permite generalizar a la que tenemos en una variable. La definición de H. Majima es técnica, pero admite diversas equivalencias que la hacen más fácilmente tratable (ver por ejemplo [M2] y [M3]). En el texto [Mj2] H. Majima trata diversos tipos de sistemas de ecuaciones perturbadas empleando esta noción. Entre este tipo encontramos los sistemas de ecuaciones de la forma

$$\varepsilon^{\sigma} x^{p+1} \frac{\partial y}{\partial x}(x,\varepsilon) = F(x,\varepsilon,y),$$

con F una función holomorfa en el origen. Bajo la hipótesis de invertibilidad de la parte lineal de F en el origen, H. Majima prueba que hay soluciones holomorfas en polisectores adecuados, admitiendo un desarrollo asintótico fuerte. Los polisectores que él considera están contenidos en conjuntos de la forma

$$\alpha < \arg(x^p \varepsilon^{\sigma}) < \beta,$$

con $\beta - \alpha < \pi$. Esto induce a pensar que es factible encontrar una noción de desarrollo asintótico en dos variables que haga intervenir la expresión $x^p \varepsilon^{\sigma}$. Es lo que hacen M. Canalis-Durand, J. Mozo Fernández y R. Schäfke en [CDMS], introduciendo la noción de desarrollo asintótico monomial, así como de sumabilidad monomial. Ellos denominan a tales ecuaciones sistemas doblemente singulares y prueban en particular el siguiente resultado:

Teorema. *Considere el sistema de ecuaciones*

$$\varepsilon^{q} x^{p+1} \frac{\partial y}{\partial x}(x,\varepsilon) = F(x,\varepsilon,y),$$

con $p, q \in \mathbb{N}^*$, y en el que suponemos que la parte lineal ($A := \frac{\partial F}{\partial y}(0,0,0)$) es invertible. Entonces el sistema admite una única solución formal $\hat{y}(x,\varepsilon)$, que es $1-$sumable en $x^p \varepsilon^q$.

De forma paralela, W. Balser y J. Mozo Fernández [BM] emplean transformadas de Borel y Laplace en dos variables para, en el caso lineal, mostrar que las soluciones formales de los sistemas anteriores son $(s_1, s_2)-$sumables, donde $ps_1 + qs_2 = 1$. Ello permite la construcción de la suma de las series formales solución de dichos sistemas perturbados, series en dos variables.

No estaba clara la relación entre la sumabilidad monomial y la sumabilidad por medio de transformadas de Borel-Laplace de [BM]. Asimismo se hace necesario tratar de considerar sistemas con parte lineal no invertible, y desarrollar por tanto una noción de multisumabilidad monomial. Esta es la línea en la que se desarrolla el presente trabajo. En él, además de revisar y detallar la noción de sumabilidad con respecto a un monomio, y la de las transformadas de Borel y Laplace generalizadas, se caracteriza la sumabilidad monomial en términos de estas últimas. Con vistas a una definición de multisumabilidad monomial, la cual se propone en el Capítulo 4 para un caso particular, se generalizan los operadores de aceleración y se muestra la incompatibilidad de las nociones de sumabilidad con respecto a dos monomios distintos, al estilo de los teoremas de J.P. Ramis antes mencionados.

Se aplican las técnicas anteriores a los sistemas doblemente singulares, a un tipo especial de ecuación diferencial parcial, así como a los sistemas de pfaffianos del tipo

$$\begin{cases} \varepsilon^q x^{p+1} \dfrac{\partial y}{\partial x} = f_1(x, \varepsilon, y), \\ x^{p'} \varepsilon^{q'+1} \dfrac{\partial y}{\partial \varepsilon} = f_2(x, \varepsilon, y). \end{cases}$$

Es interesante mencionar en este punto que estos sistemas son parte del objeto de estudio de H. Majima en [Mj2], bajo la condición de integrabilidad completa de los mismos. Hemos observado que dicha condición impone fuertes restricciones a dichos sistemas, que en la práctica hace que su estudio se reduzca exclusivamente a casos triviales o muy degenerados, hecho que al parecer no había sido notado por H. Majima ni por otros autores. Detallamos todo esto en el Capítulo 3.

Resumen y resultados principales

De forma más concreta, pasamos a exponer el contenido de los cuatro capítulos que componen esta tesis, con mención expresa de los resultados más destacados obtenidos.

Capítulo 1. Sumabilidad Monomial: El objetivo central del Capítulo 1 es recordar y desarrollar la noción de desarrollo asintótico y sumabilidad en un monomio en dos variables, tal y como fue introducida por M. Canalis-Durand, J.Mozo Fernández y R. Schäfke en [CDMS]. En aras de lograr una exposición lo más autocontenida posible hemos dedicado la primera sección de este capítulo a recopilar los resultados fundamentales de la Teoría de desarrollos asintóticos y sumabilidad en una variable que necesitaremos a lo largo del texto. Trabajando con series de potencias en una variable con coeficientes en un espacio de Banach complejo recordamos la definición de desarrollo asintótico, desarrollos asintóticos de tipo Gevrey y algunos resultados que equivalen a estas nociones (Proposición 1.1.1, Proposición 1.1.2 y Corolario 1.1.3). Luego de enunciar los teoremas de Borel-Ritt y Gevrey-Borel-Ritt y el Lema de Watson (pieza clave para definir sumabilidad como lo hace J.P. Ramis) recordamos efectivamente la definición de $k-$sumabilidad, el método de Borel-Laplace para calcular dichas sumas y el producto de convolución junto con todas las propiedades más relevantes. También destacamos el celebrado Teorema de Ramis-Sibuya y los teoremas tauberianos sobre $k-$sumabilidad debidos a J.P. Ramis. La sección finaliza con un pequeño apunte sobre desarrollos asintóticos en los que las series de potencias que intervienen no tienen necesariamente números naturales como exponentes. Debemos mencionar que no hemos incluido el uso de haces en la teoría puesto que no se utilizará en el texto.

En la segunda sección del primer capítulo abordamos la definición de desarrollo asintótico en un monomio en dos variables x, ε. La razón de la notación de las variables yace en las aplicaciones, en el carácter de variable de perturbación que ejerce la segunda de ellas. Aunque la extensión de la teoría a series con coeficientes en un espacio de Banach complejo y un número arbitrario de variables es factible y los resultados presentados en esta sección se extienden de manera natural a este caso, nos hemos limitado al caso de series con coeficientes complejos en dos variables por ser este más conciso y de menor costo técnico. La sección está dividida en cuatro partes. En la primera de ellas establecemos la definición y propiedades del conjunto de las series formales de tipo Gevrey y en particular del conjunto de las series de tipo $s-$Gevrey en un monomio $x^p\varepsilon^q$ que denotamos por $\hat{R}_s^{(p,q)}$. Se incluyen fórmulas elementales obtenidas al escribir una serie como una serie en el monomio y así se aprovecha para introducir los espacios de funciones donde ciertas series tienen sus coeficientes. Se presta atención en cómo pasar de un monomio arbitrario al monomio simple $x\varepsilon$, en el cual la teoría se escribe más fácilmente. Finalmente se estudia el efecto de introducir una variable z con pesos de manera que reemplazamos el punto (x,ε) por $(z^{s_1/p}x, z^{s_2/q}\varepsilon)$, donde s_1, s_2 son números positivos tales que $s_1+s_2=1$. Una vez establecido el contexto formal, pasamos al contexto analítico en la segunda parte. Se definen los sectores en un monomio que son los dominios fundamentales en la teoría. Estos son conjuntos precisamente de la forma

$$\Pi_{p,q}(a,b,r) = S_{p,q}(d,\alpha,r) = \left\{(x,\varepsilon)\in\mathbb{C}^2 \mid 0<|x|^p, |\varepsilon|^q<r,\ a<\arg(x^p\varepsilon^q)<b\right\},$$

donde $a,b\in\mathbb{R}$, $\alpha=b-a$ es la apertura del sector, $d=(a+b)/2$ es su bisectriz y $0<r\leq+\infty$ es su radio. Tras ver cómo podemos tratar funciones definidas sobre ellos para el caso de $p=q=1$ mediante el cambio de variable $t=x\varepsilon$, se recuerda la definición de desarrollo asintótico en el monomio $x\varepsilon$. A partir de esta definición presentamos dos caracterizaciones de la propiedad de poseer un desarrollo asintótico de este tipo: la primera usando la teoría en una variable, Proposición 1.2.11, y la segunda aproximando por funciones holomorfas, Proposición 1.2.12. Usando las diferentes caracterizaciones se demuestra detalladamente que esta noción de desarrollo asintótico es compatible con las operaciones algebraicas básicas así como con la diferenciación respecto a cualquiera de las variables. Se caracteriza el hecho de poseer un desarrollo asintótico de tipo $s-$Gevrey nulo en el monomio $x\varepsilon$ con tener un decaimiento exponencial de orden $1/s$ en el monomio $x\varepsilon$, en el sector monomial donde se esté trabajando, Proposición 1.2.14. Con este se demuestra el Lema de Watson para el caso monomial. Finalmente todas las consideraciones y resultados se extienden a un monomio arbitrario $x^p\varepsilon^q$. La sección finaliza con el enunciado de los teoremas de Borel-Ritt, Gevrey-Borel-Ritt y el Teorema de Ramis-Sibuya para este tipo de desarrollos.

En la tercera parte de la segunda sección recordamos finalmente la noción de $k-$sumabilidad en un monomio tanto en una dirección d como en general, sus propiedades básicas y cómo calcular la suma pasando a una variable y aplicando el método de Borel-Laplace. En particular se deduce que $R_{1/k,d}^{(p,q)}$ y $R_{1/k}^{(p,q)}$, que denotan el conjunto de series $k-$sumables en el monomio $x^p\varepsilon^q$ en la dirección d y de series $k-$sumables en el monomio $x^p\varepsilon^q$, respectivamente, son álgebras diferenciales con las derivaciones usuales. Además damos una nueva caracterización en la Proposición 1.2.30 de sumabilidad monomial en términos de ciertas subseries obtenidas a partir de la serie que sumamos. También se incluye en la Proposición 1.2.31 el efecto de fijar una de las variables cuando se tiene un desarrollo asintótico monomial. Esta parte finaliza con un ejemplo para ilustrar los razonamientos anteriores. En la cuarta y última parte de la segunda sección proponemos tres fórmulas para calcular la suma de una serie $k-$sumable en algún monomio, dada la dificultad en la práctica de utilizar directamente el paso a una variable. Las Proposiciones 1.2.32 y 1.2.33 explican cómo sumar en x y en ε para el caso $p=q=1$ y el caso general, respectivamente. La Proposición 1.2.34 justifica cómo sumar usando pesos en la variables.

La última sección de este capítulo desarrolla propiedades de tipo tauberiano para la $k-$sumabilidad monomial. Las dos primera y ya conocidas propiedades establecen que: la ausencia de direcciones singulares (direcciones donde no se es sumable) implica convergencia, Proposición 1.3.1, y si $0<k<k'$ entonces $R^{(p,q)}_{1/k}\cap R^{(p,q)}_{1/k'}=\mathbb{C}\{x,\varepsilon\}$, Proposición 1.3.2. Esta última propiedad admite una generalización para el caso de monomios diferentes y es el resultado principal de este capítulo. Para demostrarlo requerimos de varios pasos intermedios. Primero comparamos sumabilidad en un monomio con sumabilidad en una de sus potencias. En este sentido tenemos la siguiente proposición:

Proposición 1.3.3. *Sea $k>0$ un número real, $p,q,M\in\mathbb{N}^*$ números naturales y d una dirección. Entonces $R^{(p,q)}_{1/k,d}=R^{(Mp,Mq)}_{M/k,Md}$.*

Pasando por un caso particular, Proposición 1.3.4, demostramos en su generalidad el siguiente resultado:

Teorema 1.3.5. *Sean $k,l>0$ números reales positivos y sean $x^p\varepsilon^q$ y $x^{p'}\varepsilon^{q'}$ dos monomios. Los siguientes enunciados se verifican:*

1. *Si $p/p'=q/q'=l/k$ entonces $R^{(p,q)}_{1/k}=R^{(p',q')}_{1/l}$.*

2. *Si $p/p'=q/q'$ y $q/q'\neq l/k$ entonces $R^{(p,q)}_{1/k}\cap R^{(p',q')}_{1/l}=\mathbb{C}\{x,\varepsilon\}$.*

3. *Si $p/p'\neq q/q'$ entonces $R^{(p,q)}_{1/k}\cap R^{(p',q')}_{1/l}=\mathbb{C}\{x,\varepsilon\}$.*

con el que finalizamos el Capítulo 1.

Capítulo 2. Métodos de sumabilidad de Borel-Laplace monomiales: El propósito de este capítulo de tres secciones es desarrollar y sistematizar métodos de sumabilidad de tipo Borel-Laplace en dos variables para caracterizar la sumabilidad monomial. Así en la primera sección definimos la transformada de Borel $\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}$, la transformada de Laplace $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$ incluidas sus versiones formales y el producto de convolución $*^{(p,q)}_{k,(s_1,s_2)}$ asociados a un monomio $x^p\varepsilon^q$, un parámetro de sumabilidad k y pesos s_1,s_2 en las variables. Incluimos tres subsecciones para tratar cada transformada, resp. operación por separado. Estos operadores solo los aplicamos a funciones cuyo dominio sea un sector monomial. Todas las propiedades, tales como el comportamiento respecto a desarrollos asintóticos, se focalizan en el caso en que dichos sectores sean en el mismo monomio $x^p\varepsilon^q$. Sin embargo hemos incluido en cada subsección una nota para describir brevemente qué pasa en el caso en que los monomios sean distintos. Señalamos la siguiente interesante fórmula, pieza clave para una de las aplicaciones que tratamos en el siguiente capítulo:

Proposición 2.1.3. *Considere una función acotada* $f \in \mathcal{O}(S_{p,q}(d, \pi/k+2\epsilon_0, R_0))$*. Entonces*

$$\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}\left((x^p\varepsilon^q)^k\left(\frac{s_1}{p}x\frac{\partial f}{\partial x}+\frac{s_2}{q}\varepsilon\frac{\partial f}{\partial \varepsilon}\right)\right)(\xi,\upsilon) = k(\xi^p\upsilon^q)^k\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}(f)(\xi,\upsilon),$$

para todos $s_1, s_2 > 0$ *tales que* $s_1 + s_2 = 1$.

También hemos incluido otras fórmulas de carácter presumible como en la Proposición 2.1.14 que muestra que los operadores de Borel y Laplace son inversos uno del otro (hecho que usa la inyectividad de la transformada de Laplace, Lema 2.1.13) ó como en la Proposición 2.1.15 que afirma que la transformada de Laplace convierte la convolución en el producto usual.

En la segunda sección definimos un método de sumabilidad asociado a un monomio $x^p\varepsilon^q$, un parámetro de sumabilidad k, un peso de las variables s_1, s_2 y una dirección d, utilizando las transformadas antes mencionadas y basados en las mismas líneas que en la teoría de una variable: una serie $\hat{f}$ es $k-(s_1,s_2)-$*Borel sumable en el monomio* $x^p\varepsilon^q$ *en la dirección* d si esta es $1/k-$Gevrey en $x^p\varepsilon^q$, la serie $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$ se puede prolongar analíticamente, digamos φ_{s_1,s_2}, a un sector monomial de la forma $S_{p,q}(d, 2\epsilon, +\infty)$, $\epsilon > 0$, y con crecimiento exponencial del tipo

$$|\varphi_{s_1,s_2}(\xi,\upsilon)| \leq De^{M\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s_2}\}},$$

para algunas constantes positivas D, M. Con esta definición y las propiedades desarrolladas hasta este punto hemos conseguido la caracterización que buscábamos de sumabilidad monomial, la cual consideramos es uno de los resultados más relevantes del trabajo.

Teorema 2.2.1. *Sea* $\hat{f} \in \hat{R}^{(p,q)}_{1/k}$ *una serie de tipo* $1/k-$*Gevrey en el monomio* $x^p\varepsilon^q$*. Entonces es equivalente que:*

1. $\hat{f} \in R^{(p,q)}_{1/k,d}$,

2. *Existen* $s_1, s_2 > 0$ *con* $s_1 + s_2 = 1$ *tales que* $\hat{f}$ *es* $k-(s_1,s_2)-$*Borel sumable en el monomio* $x^p\varepsilon^q$ *en la dirección* d.

3. *Para todo* $s_1, s_2 > 0$ *con* $s_1 + s_2 = 1$, $\hat{f}$ *es* $k-(s_1,s_2)-$*Borel sumable en el monomio* $x^p\varepsilon^q$ *en dirección* d.

En todos los casos las correspondientes sumas coinciden.

Concluimos esta sección utilizando esta caracterización para obtener pruebas alternativas de algunos resultados obtenidos en el primer capítulo. Finalizamos el capítulo con la Sección

2.3 donde exploramos el comportamiento básico de los desarrollos asintóticos monomiales de tipo Gevrey bajo las explosiones de puntos en el plano complejo.

Capítulo 3. Ecuaciones diferenciales analíticas lineales singularmente perturbadas: Dedicamos este capítulo a las aplicaciones de la sumabilidad monomial al estudio de las soluciones formales de ciertos tipos de ecuaciones diferenciales. En la primera sección trabajamos con ecuaciones doblemente singulares de la forma

$$\varepsilon^q x^{p+1}\frac{dy}{dx} = A(x,\varepsilon)y(x,\varepsilon) + b(x,\varepsilon), \tag{3-1}$$

donde p, q son números naturales positivos, $y \in \mathbb{C}^l$, $A \in \mathrm{Mat}(l \times l, \mathbb{C}\{x,\varepsilon\})$ y $b \in \mathbb{C}\{x,\varepsilon\}^l$. Bajo la hipótesis de la invertibilidad de $A(0,0)$ recordamos la demostración del hecho que esta ecuación posee una única solución formal, $1-$Gevrey en el monomio $x^p\varepsilon^q$, Proposición 3.1.2, empleando las normas de Nagumo. Para las propiedades de sumabilidad hemos propuesto una nueva demostración del siguiente teorema:

Teorema 3.1.4. *La única solución formal $\hat{y}$ de la ecuación (3-1) es $1-$sumable en $x^p\varepsilon^q$.*

Las ideas detrás de dicha demostración no son nuevas. La esencia de las mismas se basa en las demostraciones habituales: usar una transformada de Borel apropiada para estudiar por el método del punto fijo las soluciones de la ecuación en convolución que resulta. Una vez construidas dichas soluciones, invocar el Teorema de Ramis-Sibuya para obtener un desarrollo asintótico y así deducir la sumabilidad. Mencionamos además que utilizando la caracterización de sumabilidad monomial a través del método de Borel-Laplace explicado en el Capítulo 2 hemos mejorado el Teorema 3 en [BM], resultado que exponemos en el Corolario 3.1.5.

Finalmente mencionamos que el Teorema 3.1.4 también es válido en el caso no lineal, aunque para dicha situación nos hemos limitado solo a enunciar el resultado en el Teorema 3.1.6.

En la segunda sección, y como lo sugiere la fórmula obtenida en la Proposición 2.1.3, estudiamos las soluciones formales de la ecuación diferencial parcial

$$\frac{s_1}{p}\varepsilon^q x^{p+1}\frac{\partial y}{\partial x} + \frac{s_2}{q}x^p\varepsilon^{q+1}\frac{\partial y}{\partial \varepsilon} = C(x,\varepsilon)y(x,\varepsilon) + \gamma(x,\varepsilon), \tag{3-25}$$

donde p, q son números naturales positivos, s_1, s_2 son números reales positivos que satisfacen $s_1 + s_2 = 1$ y $C \in \mathrm{Mat}(l \times l, \mathbb{C}\{x,\varepsilon\})$, $\gamma \in \mathbb{C}\{x,\varepsilon\}^l$. Siguiendo las ideas aplicadas en la sección anterior pero con las herramientas para sumabilidad monomial hemos obtenidos los siguientes resultados:

Proposición 3.2.1. *Considere la ecuación diferencial parcial (3-25). Si $C(0,0)$ es invertible entonces (3-25) tiene una única solución formal $\hat{y} \in \mathbb{C}[[x,\varepsilon]]^l$. Además $\hat{y} \in (\hat{R}_1^{(p,q)})^l$.*

Teorema 3.2.2. *Considere la ecuación (3-25). Si $C(0,0)$ es invertible entonces la única solución formal $\hat{y}$ dada por la proposición anterior es $1-$summable en $x^p\varepsilon^q$. Sus posibles direcciones singulares son las direcciones que pasan por los valores propios de $C(0,0)$.*

En la última sección pasamos al estudio de sistemas pfaffianos en dos variables de la forma

$$\begin{cases} \varepsilon^q x^{p+1} \dfrac{\partial y}{\partial x} = f_1(x,\varepsilon,y), & \text{(3-35a)} \\ x^{p'}\varepsilon^{q'+1}\dfrac{\partial y}{\partial \varepsilon} = f_2(x,\varepsilon,y), & \text{(3-35b)} \end{cases}$$

donde p, q, p', q' son números naturales positivos, $y \in \mathbb{C}^l$, y f_1, f_2 son funciones analíticas definidas en una vecindad del origen en $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^l$. Recordamos que si $f_1(0,0,0) = f_2(0,0,0) = 0$ y las funciones f_1, f_2 satisfacen sobre su dominio de definición:

$$\begin{aligned} -qx^{p'}\varepsilon^{q'}f_1(x,\varepsilon,y) + x^{p'}\varepsilon^{q'+1}\frac{\partial f_1}{\partial \varepsilon}(x,\varepsilon,y) + \frac{\partial f_1}{\partial y}(x,\varepsilon,y)f_2(x,\varepsilon,y) = \\ -p'x^p\varepsilon^q f_2(x,\varepsilon,y) + x^{p+1}\varepsilon^q\frac{\partial f_2}{\partial x}(x,\varepsilon,y) + \frac{\partial f_2}{\partial y}(x,\varepsilon,y)f_1(x,\varepsilon,y), \end{aligned} \quad \text{(3-36)}$$

el sistema pfaffiano se dice completamente integrable. Bajo esta hipótesis hemos deducido la siguiente proposición sobre el comportamiento de los espectros de las partes lineales de f_1 y f_2 en el origen:

Proposición 3.3.1. *Considere el sistema pfaffiano (3-35a), (3-35b). Si es completamente integrable entonces las siguientes afirmaciones son válidas:*

1. *La matriz $\frac{\partial f_2}{\partial y}(0,0,0)$ es nilpotente si $p = p'$ y $q < q'$, ó $p' = Np$ con $N > 1$, ó $q' = q$ y $p < p'$ ó $q' = Mq$ con $M > 1$.*
2. *La matriz $\frac{\partial f_1}{\partial y}(0,0,0)$ es nilpotente si $p = p'$ y $q' < q$, ó $p = N'p'$ con $N' > 1$, ó $q' = q$ y $p' < p$ ó $q = M'q'$ con $M' > 1$.*
3. *Si $p = p'$ y $q = q'$ para todo valor propio μ de $\frac{\partial f_2}{\partial y}(0,0,0)$ existe un valor propio λ de $\frac{\partial f_1}{\partial y}(0,0,0)$ tal que $q\lambda = p\mu$. El número λ es un valor propio de $\frac{\partial f_1}{\partial y}(0,0,0)$, cuando se restringe a su subespacio invariante $E_\mu = \{v \in \mathbb{C}^n | (\frac{\partial f_2}{\partial y}(0,0,0) - \mu I)^k v = 0$ para algún $k \in \mathbb{N}\}$.*

Teniendo en cuenta estas restricciones, utilizando los resultados de la primera sección y las propiedades tauberianas encontradas en el primer capítulo hemos obtenido el siguiente resultado sobre convergencia y sumabilidad de las soluciones de estos sistemas:

Teorema 3.3.3. *Considere el sistema (3-35a), (3-35b). Las siguientes afirmaciones son válidas:*

1. *Suponga que el sistema tiene una solución formal $\hat{y}$. Si $\frac{\partial f_1}{\partial y}(0,0,0)$ y $\frac{\partial f_2}{\partial y}(0,0,0)$ son invertibles y $x^p\varepsilon^q \neq x^{p'}\varepsilon^{q'}$ entonces $\hat{y}$ es convergente.*

2. *Si el sistema es completamente integrable y* $\frac{\partial f_1}{\partial y}(0,0,0)$ *es invertible entonces el sistema tiene una única solución formal* $\hat{y}$*. Además* $\hat{y}$ *es 1-sumable en* $x^p\varepsilon^q$*.*

3. *Si el sistema es completamente integrable y* $\frac{\partial f_2}{\partial y}(0,0,0)$ *es invertible entonces el sistema tiene una única solución formal* $\hat{y}$*. Además* $\hat{y}$ *es 1-sumable en* $x^{p'}\varepsilon^{q'}$*.*

Finalmente nos restringimos al caso lineal y en el que ambos monomios que aparecen en los sistemas son iguales. Nos referimos a sistemas de la forma

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} & = A(x,\varepsilon)y(x,\varepsilon) + a(x,\varepsilon), \qquad \text{(3-48a)} \\ x^p\varepsilon^{q+1}\dfrac{\partial y}{\partial \varepsilon} & = B(x,\varepsilon)y(x,\varepsilon) + b(x,\varepsilon), \qquad \text{(3-48b)} \end{cases}$$

donde p,q son números naturales positivos, $A,B \in \mathrm{Mat}(l\times l,\mathbb{C}\{x,\varepsilon\})$ y $a,b\in\mathbb{C}\{x,\varepsilon\}^l$. En este contexto tenemos los siguientes resultados sobre convergencia y sumabilidad de sus soluciones:

Proposición 3.3.4. *Las siguientes afirmaciones son válidas:*

1. *Si el sistema (3-48a), (3-48b) es completamente integrable y* $A(0,0)$ *ó* $B(0,0)$ *es invertible entonces el sistema (3-48a), (3-48b) tiene una única solución formal,* $1-$*sumable en* $x^p\varepsilon^q$*.*

2. *Si el sistema tiene una solución formal* $\hat{y}$ *y existen* $s_1,s_2>0$ *tales que* $s_1+s_2=1$ *y* $s_1/pA(0,0)+s_2/qB(0,0)$ *es invertible, entonces* $\hat{y}$ *es* $1-$*sumable en* $x^p\varepsilon^q$*. Sus posibles direcciones singulares son las direcciones que pasan por los valores propios de* $s_1/pA(0,0)+s_2/qB(0,0)$*.*

Teorema 3.3.5. *Considere el sistema (3-48a), (3-48b) y suponga que tiene una solución formal* $\hat{y}$*. Denote por* $\lambda_1(s),...,\lambda_l(s)$ *los valores propios de* $\frac{s}{p}A(0,0)+\frac{(1-s)}{q}B(0,0)$*, donde* $0\le s\le 1$*, y asuma que nunca son cero. Si para cada dirección* d *existe* $s\in[0,1]$ *tal que arg*$(\lambda_j(s))\neq d$ *para todo* $j=1,...,l$ *entonces* $\hat{y}$ *es convergente.*

Capítulo 4. Hacia multisumabilidad monomial: En el último capítulo de esta tesis mostramos los avances logrados hacia una buena noción de multisumabilidad monomial. El capítulo está dividido en tres secciones. En la primera recordamos los operadores de aceleración y la noción de multisumabilidad en dos niveles para una variable, incluyendo fórmulas importantes que serán usadas en la siguiente sección.

En la segunda sección definimos los operadores de aceleración que conectan un monomio $x^p\varepsilon^q$, un parámetro de sumabilidad k y un peso s_1,s_2 con otro monomio $x^{p'}\varepsilon^{q'}$, otro parámetro de sumabilidad l y otro peso s_1',s_2'. Estos operadores los hemos obtenido a partir de calcular formalmente la composición de la transformada de Borel $\mathcal{B}^{(p',q')}_{l,(s'_1,s'_2)}$ con la transformada de Laplace $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$.

Naturalmente existen condiciones sobre estos valores para realizar dicho cálculo, a saber:

$$s'_1 = \frac{s_1 p' q}{s_2 p q' + s_1 p' q}, \quad s'_2 = \frac{s_2 p q'}{s_2 p q' + s_1 p' q},$$
$$s_1(p'q - pq') > \frac{p}{l}(qk - q'l), \quad \min\left\{\frac{p}{p'}, \frac{q}{q'}\right\} < \frac{l}{k}.$$

Así, si $I = (p', q', p, q, l, k, s'_1, s'_2, s_1, s_2)$ donde dichos valores satisfacen las condiciones anteriores, tenemos el operador $\mathfrak{A}_I$. De manera análoga al Capítulo 2, desarrollamos todas las propiedades de estos operadores de aceleración tales como comportamiento respecto a desarrollos asintóticos monomiales y convolución.

Finalmente en la última sección proponemos una definición de multisumabilidad asociada a dos monomios, dos parámetros de sumabilidad y dos pesos, motivados por el siguiente resultado, análogo al de una variable y que hemos demostrado aplicando explosiones de puntos en el plano complejo.

Teorema 4.3.1. Sean $p_0, ..., p_r$, $q_0, ..., q_r$ números naturales positivos y sean $k_0, ..., k_r$ números reales positivos. Sean $\hat{f}_j \in R^{(p_j,q_j)}_{1/k_j} \setminus \mathbb{C}\{x, \varepsilon\}$ series k_j-sumable en el monomio $x^{p_j}\varepsilon^{q_j}$, para $j = 1, ..., r$, respectivamente. Entonces $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_r$ es k_0-sumable en $x^{p_0}\varepsilon^{q_0}$ si y solo si $k_0 p_0 = k_j p_j$ y $k_0 q_0 = k_j q_j$ para todo $j = 1, ..., r$.

Finalizamos el capítulo mostrando que la noción de multisumabilidad propuesta es estable por sumas y productos y que es capaz de sumar series de la forma $\hat{f} + \hat{g}$, donde $\hat{f} \in R^{(p,q)}_{1/k}$ y $\hat{g} \in R^{(p',q')}_{1/l}$.

El tema dista mucho de estar cerrado. Numerosos problemas abiertos se plantean, de los que citamos algunos de ellos:

1. Dar una definición completa de multisumabilidad monomial, que contemple no solo los casos particulares tratados en esta memoria.

2. Demostrar que la propiedad de ser multisumable como aquí lo hemos definido es independiente de los pesos elegidos. Una vía posible es extender el resultado de descomposición de W. Balser de series multisumables como suma de series sumables.

3. Estudio sistemático de los sistemas de ecuaciones lineales singularmente perturbados, sin la hipótesis de invertibilidad de la parte lineal, aplicando operadores de aceleración generalizados a varios niveles.

4. Estudio más general de los sistemas lineales pfaffianos, ya sea con las restricciones que impone la condición de integrabilidad completa o sin ella.
5. Estudio de la ecuación diferencial parcial (3-25) en el caso no lineal con o sin la hipótesis de invertibilidad de la parte lineal.
6. Adaptación de las demostraciones aquí contenidas al caso de varias variables.
7. Hacer uso de la teoría de haces para desarrollar la teoría de manera más intrínseca.

Introduction

It is a well known fact that divergent series appear in a natural way in many problems related with functional equations, even when those problems involve exclusively convergent series. Perhaps one of the first historical examples is the one given by L. Euler in his work *De seriebus divergentibus* [E]. In this paper L. Euler studied, among other things, the numerical series

$$1 - 1! + 2! - 3! + 4! - 5! + \cdots = \sum_{n=0}^{\infty} (-1)^n n!, \tag{0-1}$$

and called it the *Wallis hypergeometric series.* For its study, he proposes four different methods of summation, including

1. An iteration of what are now called *Euler transformations* and the computation of the "sum up to the least term",
2. The introduction of an additional parameter x, what leads us to consider the power series
$$x - x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}, \tag{0-2}$$
that we will call *Euler's series* and that turns out to be a solution of the linear differential equation
$$x^2 y'(x) + y(x) = x,$$
solvable by variation of constants. The solution of this equation evaluated at $x = 1$, allows to attribute a value to the sum of (0-1) (approximately 0.59637164).

In the 18th century those reasonings were interesting to attempt to approximate the value of some mathematical constants, such as e or π: what mattered was to attribute in a coherent way a value to the sum of this kind of series, without restricting to the classical notion of convergence that nowadays is studied in the first courses of a university career.

The physical interest of divergent series was revealed with the works of G.G. Stokes on Airy's function: this is a function that appears in the study of caustics in optics, such as the rainbow. Historically, that was the mathematical problem that took G.B. Airy to develop this special function. More precisely, Airy's function is defined by the following integral expression:

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(xt + \frac{t^3}{3}\right) dt,$$

and it is a solution of the differential equation $y''(x) - xy(x) = 0$. One of the methods studied in elementary courses to approach the solutions of this kind of equations is to develop them into power series. In the case of Airy's function the radius of convergence of its Taylor series at the origin is infinite, what apparently solves the problem. However this series has a very slow convergence, a fact that makes it hardly useful for calculations. G.G. Stokes had the idea of developing the function Ai at infinity, and obtained a divergent power series in $x^{1/2}$: "summing up to the last term", that series gives surprisingly precise values of Airy's function. A more modern physical example can be found in the realm of quantic electrodynamics: in the study of the magnetic moment of the electron appears a power series where every term is calculated from Feynman's diagrams. This series is divergent and again the sum of some terms (the least term is still unknown) gives very close results to the experimental ones. The reader may find details of this and related problems in the divulgence paper of J.P. Ramis [R1]. For a more detailed description the reader may also consult [R2].

We have mentioned twice the technique of the "summation up to the last term". For numerous divergent series coming from physical problems, the first terms decrease in absolute value, but then they grow indefinitely. The mentioned technique consists in truncating the series at the moment when the terms start to grow. This technique was called by H. Poincaré "summation of the astronomers", in contrast with the "summation of the geometers" (convergent series in the modern sense). The precise justification for this requires the use of Gevrey type series, that we will comment further on.

It is precisely H. Poncaré who gives one of the greatest impulses to the theory of summation of divergent series, that many prestigious mathematicians had neglected (for N. Abel those were an "invention of the devil"). As in many mathematical problems and in words of J. Hadamard: "...the shortest and best way between two truths of the **real** domain often passes through the **imaginary** one" [H, page 123]. So it was H. Poincaré in his work [P] who introduced at the end of the 19th century the notion of asymptotic expansion: a function f, holomorphic in a sector $V = V(a, b, r) = \{x \in \mathbb{C} | a < \arg(x) < b, 0 < |x| < r\}$, admits a power series $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ as *asymptotic expansion* at the origin on V if for each natural number N and every subsector W of V there is a constant $C_N(W)$ such that

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C_N(W)|x|^N,$$

on W. We remark that this is not the original definition given by H. Poincaré but it turns out to be equivalent for functions bounded in every subsector of V. For the details of this fact the reader may consult [FZ].

The works of H. Poincaré, as well as of many other mathematicians, focus in power series that come from solutions of systems of holomorphic differential equations (linear or not), around the so called singular points. Restricted to the linear case, we refer to systems of equations of the form

$$x^{p+1}y'(x) = A(x)y(x) + b(x),$$

where $y(x) = (y_1(x), \ldots, y_n(x))^t \in \mathcal{O}(D)^n$, $A \in \mathrm{Mat}(n \times n, \mathcal{O}(D))$, $b \in \mathcal{O}(D)^n$, where D denotes a disc centered at 0. In some classical texts, for instance [CL], those points are classified into first class (if $p = 0$) and second class (if $p > 0$), what usually determines the nature of the solutions. Referring to those, the singular points are classified as regular (reducible to points of first class) or irregular ones. It is for this last type of equations in which divergence phenomena occur. Thus H. Poincaré, M. Hukuhara, H.L. Turritin and W. Wasow among others show the following result, valid in the non-linear case:

Theorem.(Main Asymptotic Existence Theorem) *Let us consider the system of holomorphic differential equations*

$$x^{p+1}y'(x) = F(x, y(x)), \quad p \in \mathbb{N}^*,$$

that admits a vector of formal power series $\hat{y}$ as solution and such that the matrix of the linear part at the origin

$$A := \frac{\partial F}{\partial y}(0, \mathbf{0}),$$

is invertible. If V is a sector of opening at most π/p then there is a solution $y(x) \in \mathcal{O}(V)^n$ that admits $\hat{y}$ as asymptotic expansion on V.

A proof of this fact can be found in [W1].

This theorem provides a geometric meaning to the power series $\hat{y}(x)$, interpreting it as $y(x)$. But the function $y(x)$ is far from being unique because there are functions with null asymptotic expansion that are solutions of differential equations.

A new and crucial impulse to the theory was given at the late seventies with the works, from one side by J. Écalle on the so called resurgent functions and from the other side by J.P. Ramis who introduced and systematized the notion of $k-$summability, that generalizes the notion of summability given by E. Borel in the twenties [B]. The notion of asymptotic expansions was specialized in the so called *asymptotic expansions of $s-$Gevrey type*: in this case the constant C_N in the definition is given by one of the type

$$CA^N N!^s,$$

making explicit the dependence on N. It turns out that if a formal power series $\hat{y}(x)$ is the $s-$Gevrey asymptotic expansion of a function $y(x)$ defined in a sector V of opening strictly greater than $s\pi$ then $y(x)$ is the only function with this property and it is legitimated to call it the $k-$sum of $\hat{y}$ on V (here $k = 1/s$, as in nowadays notations in the theory). In this context the Main Asymptotic Existence Theorem was improved by J.P. Ramis and Y. Sibuya in 1989, as we state below.

Theorem. *Let us consider the system of holomorphic differential equations*

$$x^{p+1}y'(x) = F(x, y(x)), \quad p \in \mathbb{N}^*,$$

that admits a vector of formal power series $\hat{y}$ of $s-$Gevrey type as a solution. If V is a sector of opening at most $\min\{\pi s, \pi/p\}$ *then there is a solution $y(x) \in \mathcal{O}(V)^n$ that admits $\hat{y}$ as $s-$Gevrey asymptotic expansion on V.*

The complete proof of this result can be consulted in [RS2].

Besides of introducing the notion of $k-$summable series, J.P. Ramis formulated a result on the structure of the formal solutions of linear systems with irregular singularities being equivalent to the statement that every formal solution can be built from $k-$summable series for different values of k (the levels of the equation). It is clear from here that not every formal solution of a holomorphic differential equation is $k-$summable for a unique value of k and because of this the notion of multisummability is introduced where different values of k intervene. The first proof of the multisummability of the solutions of linear differential equations was given by W. Balser, B.L.J. Braaksma, J.P. Ramis and Y. Sibuya in [BBRS]. Later B.L.J. Braaksma proved a similar result for non-linear equations [Br].

With this we have a partial answer to the problem of assigning a sum to formal power series obtained as solutions of differential equations, but this answer is not constructive. In the referred text of E. Borel [B] it is described certain integral transformation, nowadays called the *Borel transform* which combined with the Laplace transform allows one to build explicitly the sum in a direction of a $1-$summable series, when it exists. J.P. Ramis generalized this notion introducing the $k-$Borel and $k-$Laplace transform, which makes possible to build the $k-$sum of a $k-$summable series. Likewise J. Écalle defined the acceleration operators. With their aid if $k_1 > k_2 > \cdots > k_m > 0$ and $\hat{y}$ is a $(k_1, ..., k_m)-$multisummable series in a direction d, its sum in that direction can be calculated as

$$\mathcal{L}_{k_1} \circ \mathfrak{A}_{k_1,k_2} \circ \cdots \circ \mathfrak{A}_{k_{m-1},k_m} \circ \hat{\mathcal{B}}_{k_m}(\hat{y}(x)),$$

where $\hat{\mathcal{B}}_{k_m}$ represents the formal k_m-Borel transform, $\mathfrak{A}_{k,k'}$ is the acceleration operator of order (k, k'), $k > k'$ and $\mathcal{L}_{k_1}$ is the k_1-Laplace transform. In order for this machinery to work it is necessary to note that different levels of summability are incompatible: if $k \neq k'$ every power series $k-$summable and $k'-$summable is necessarily convergent.

At this point of the history we can quote the words of J.P. Ramis in [R1, page 139]:

“Que réserve le futur aux spécialistes des séries divergentes? Les principaux défis concernent ce que l'on nomme les perturbations singulières.”

The first direction where the present thesis points to is to asymptotic expansions associated to singularly perturbed problems. A singularly perturbed linear system is one of the form

$$\varepsilon^{\sigma}\frac{\partial y}{\partial x}(x,\varepsilon)=A(x,\varepsilon)y(x,\varepsilon),$$

where $A(x,\varepsilon)$ is a matrix of holomorphic functions in a neighborhood of $(0,0)\in\mathbb{C}^2$. Usually a formal solution of this system depends on the singular perturbation parameter ε and admits an expansion into power series in ε of the form

$$y(x,\varepsilon)=y_0(x)+y_1(x)\varepsilon+y_2(x)\varepsilon^2+\cdots,$$

with coefficients $y_j(x)$ holomorphic in a common disc of convergence and it is divergent in ε. The problem of summability in ε of the solutions of such systems was posed and many authors have contributed to it. We remark some important achievements:

- M. Canalis-Durand proved the Gevrey character of the formal solution of the singularly perturbed Van der Pol's equation [CD] and A. Fruchard and R. Schäfke proved its summability in ε [FS].
- In the general case, the solutions are not necessarily $k-$summable for any value of k. Nonetheless M. Canalis-Durand, J.P. Ramis, R. Schäfke and Y. Sibuya [CDRSS] showed that under the condition of invertibility of the linear part (in fact, they consider slightly more general conditions), every formal solution is of Gevrey type and can be represented by a holomorphic function $y(x,\varepsilon)\in\mathcal{O}(D\times V)$, where D is a disc at the origin and V is a sector, that admits as asymptotic expansion the formal power series solution.

An additional problem is found when we consider singularly perturbed differential equations at singular points. For instance, we can consider the singularly perturbed Schrödinger equation

$$\varepsilon^2\frac{\partial^2 y}{\partial x^2}(x,\varepsilon)+P(x)y(x,\varepsilon)=0,$$

where $P(x)$ is a polynomial. In that case the singularity in x is located at infinity and the singularity in the parameter ε is at 0. This type of equations and systems have been considered for many authors, for example W. Wasow [W2]. For their treatment it seems necessary to use a notion of asymptotic expansions in many variables. The first satisfactory notion was due to H. Majima, who in [Mj1] defined the concept of strongly asymptotically developable functions in a polysector (product of sectors) which led to a generalization of the concept in the one variable case. The definition of H. Majima is technical but admits some equivalences that make it more easily tractable (see for instance [M2] and [M3]). In the text [Mj2] H. Majima considers many types of singularly perturbed systems of equations employing this notion. Among them we find those of the form

$$\varepsilon^\sigma x^{p+1}\frac{\partial y}{\partial x}(x,\varepsilon) = F(x,\varepsilon,y),$$

with F holomorphic at the origin. Under the hypothesis of invertibility of the linear part of F at the origin H. Majima proved that there are holomorphic solutions in adequate polysectors, admitting strong asymptotic expansions. The polysectors he considers are contained in sets of the form

$$\alpha < \arg(x^p\varepsilon^\sigma) < \beta,$$

with $\beta-\alpha < \pi$. This lead to think that it is plausible to find a notion of asymptotic expansion in two variables where the expression $x^p\varepsilon^\sigma$ intervenes. That is what M. Canalis-Durand, J. Mozo Fernández and R. Schäfke did in [CDMS], introducing the concept of monomial asymptotic expansion, as well as monomial summability. They called such equations doubly singular systems and proved in particular the following statement:

Theorem. *Consider the system of equations*

$$\varepsilon^q x^{p+1}\frac{\partial y}{\partial x}(x,\varepsilon) = F(x,\varepsilon,y),$$

with $p,q\in\mathbb{N}^*$*, where we suppose that the linear part* ($A := \frac{\partial F}{\partial y}(0,0,0)$) *is invertible. Then the system admits a unique formal power solution* $\hat{y}(x,\varepsilon)$*,* $1-$*summable in* $x^p\varepsilon^q$*.*

In a parallel way, W. Balser and J. Mozo Fernández [BM] employed Borel and Laplace transformations in two variables, in the linear case, to show that the solutions of the previous systems are $(s_1,s_2)-$summable, where $ps_1+qs_2=1$. This let them build the sum of the formal power series in two variables of such singularly perturbed systems.

The relation between monomial summability and summability through Borel-Laplace transforms in [BM] was not clear. In the same manner it is necessary to treat systems with non-invertible linear part and to develop a notion of monomial multisummability. The present work advances precisely in this direction. In it, besides of recalling and detailing the notion of summability w.r.t. a monomial and the generalized Borel and Laplace transforms, the characterization of monomial summability in terms of the last ones is given. For the sake of a definition of monomial multisumability, which is proposed in Chapter 4 for a particular case, the acceleration operators are generalized and the incompatibility of the notions of summability for different monomials is shown, in the same spirit of Ramis's theorems, mentioned above.

The mentioned techniques are applied to doubly singular systems, to a special type of partial differential equation and to Pfaffian systems of the form

$$\begin{cases} \varepsilon^q x^{p+1} \dfrac{\partial y}{\partial x} = f_1(x,\varepsilon,y), \\ x^{p'}\varepsilon^{q'+1}\dfrac{\partial y}{\partial \varepsilon} = f_2(x,\varepsilon,y). \end{cases}$$

At this point it is interesting to mention that those systems are part of the work of H. Majima in [Mj2], under the hypothesis of complete integrability. We have observed that this condition imposes strong restrictions to such systems and in the practice makes their study to reduce exclusively to trivial cases or highly degenerate ones. This fact apparently had not been noticed by H. Majima or by other authors. We detail all in Chapter 3.

More consistently, we detail now the contents of the four chapters that this thesis has, with express mention of the main results obtained.

Chapter 1. Monomial Summability: The central aim of Chapter 1 is to recall and develop the notion of asymptotic expansion in a monomial in two variables, such as it was introduced by M. Canalis-Durand, J. Mozo Fernández and R. Schäfke in [CDMS]. In order to have an exposition as self-contained as possible we have devoted the first section of this chapter to collect the main results of the theory of asymptotic expansions and summability in one variable that we will need along the text. Working with power series in one variable with coefficients in a complex Banach space we recall the definition of asymptotic expansions, asymptotic expansions of Gevrey type and some characterizations of these notions (Proposition 1.1.1, Proposition 1.1.2 and Corollary 1.1.3). After formulating the Borel-Ritt and Gevrey-Borel-Ritt theorems and Watson's Lemma (key element to define summability as J.P. Ramis does) we recall effectively the notion of $k-$summability, the Borel-Laplace method to calculate such sums and the convolution product all together with their more relevant properties. We also highlight the celebrated Ramis-Sibuya Theorem and the tauberian theorems of $k-$summability due to J.P. Ramis. The section ends with a brief note on asymptotic expansions where the power series involved have not necessarily natural numbers as exponents. We need to mention that we have not included the sheaf theoretical point of view because we will not use it in the text.

In the second section we recall the definition of asymptotic expansion in a monomial in two variables x, ε. The reason of this notation lies in the applications, in the role of perturbation variable of the second one. Although the extension of the theory to series with coefficients in a complex Banach space and in an arbitrary number of variables is possible and the results presented here extend naturally, we have limited to the case of complex number coefficients in two variables for being more concise and of less technical cost. The section is divided into four parts. In the first one we establish the definition and properties of the set of formal power series of Gevrey type and in particular of the set of series of type $s-$Gevrey in a monomial $x^p\varepsilon^q$ that we denote by $\hat{R}_s^{(p,q)}$. Elementary formulas, obtained when writing a power series in a monomial, are included and this process leads to introduce the spaces of functions where certain series have its coefficients. We pay attention to how to pass from an arbitrary monomial to the simple monomial $x\varepsilon$, in which the theory is written more easily. Finally we study the effect of introducing a variable z with weights in the way that the point (x, ε) is replaced by $(z^{s_1/p}x, z^{s_2/q}\varepsilon)$, where s_1, s_2 are positive numbers such that $s_1 + s_2 = 1$. Once the formal setting is established we pass to the analytic one in the second part. We define the sectors in a monomial that are the essential domains in the theory. These sets are precisely given by

$$\Pi_{p,q}(a, b, r) = S_{p,q}(d, \alpha, r) = \left\{(x, \varepsilon) \in \mathbb{C}^2 \mid 0 < |x|^p, |\varepsilon|^q < r,\ a < \arg(x^p\varepsilon^q) < b\right\},$$

where $a, b \in \mathbb{R}$, $\alpha = b - a$ is the opening of the sector, $d = (a + b)/2$ is its bisectrix and $0 < r \leq +\infty$ is its radius. After seeing how we can treat functions defined on them for the case $p = q = 1$ through the change of variable $t = x\varepsilon$, we recall the definition of asymptotic expansion in a monomial $x\varepsilon$. From this definition we present two different characterizations of the property of having such expansions: the first using the theory of one variable, Proposition 1.2.11, and the second one by approximating by holomorphic functions, Proposition 1.2.12. Using the different characterizations we prove in detail that this notion of asymptotic expansion is compatible with the basic algebraic operations as well as with derivation w.r.t. any of the variables. We characterize the fact of having a null $s-$Gevrey asymptotic expansion in the monomial $x\varepsilon$ by exponential decay of order $1/s$ in the monomial $x\varepsilon$ at the origin, Proposition 1.2.14. With the last property we prove the analogous version of Watson's Lemma for the monomial case. Finally all the considerations and results are extended to an arbitrary monomial $x^p\varepsilon^q$. We finish the section by formulating the Borel-Ritt, Gevrey-Borel-Ritt and Ramis-Sibuya theorems for this kind of expansions.

In the third part we finally recall the concept of $k-$summability in a monomial in a direction d as well as in general, the basic properties and how to calculate the sum by passing to one variable and applying the Borel-Laplace method. In particular we deduce that $R^{(p,q)}_{1/k,d}$ and $R^{(p,q)}_{1/k}$, that stand for the set of $k-$summable series in the monomial $x^p\varepsilon^q$ in the direction d and the $k-$summable series in the monomial $x^p\varepsilon^q$, respectively, are differential algebras with the usual derivations. Furthermore we give a new characterization in Proposition 1.2.30 of monomial summability in terms of certain subseries obtained from the series we sum. We also include in Proposition 1.2.31 the result of fixing one of the variables when we have a monomial asymptotic expansion. This part ends with an example to illustrate the previous reasoning. In the last part of the second section we propose three formulas to calculate the sum of a $k-$summable series in a monomial due to the difficulty in practice to use directly the pass to one variable. Propositions 1.2.32 and 1.2.33 explain how to sum in x and in ε for the case $p = q = 1$ and in the general case, respectively. Proposition 1.2.34 justifies how to sum using weights in the variables.

In the last section of this chapter we develop tauberian properties for summability in a monomial. The first two already known properties establish that: absence of singular directions

(directions where a series is not summable) implies convergence, Proposition 1.3.1 and if $0 < k < k'$ then $R^{(p,q)}_{1/k} \cap R^{(p,q)}_{1/k'} = \mathbb{C}\{x,\varepsilon\}$, Proposition 1.3.2. This last property admits a generalization for the case of two different monomials and it is the main result of this chapter. To prove it we require of some intermediate steps. First we compare the summability in a monomial with summability in a power of such monomial. In that sense we have the following proposition:

Proposition 1.3.3. *Let $k > 0$ be a real number, $p, q, M \in \mathbb{N}^*$ be natural numbers and d a direction. Then $R^{(p,q)}_{1/k,d} = R^{(Mp,Mq)}_{M/k,Md}$. .*

Using a particular case, Proposition 1.3.4, we prove the following general result:

Theorem 1.3.5. *Let $k, l > 0$ be positive real numbers and let $x^p\varepsilon^q$ and $x^{p'}\varepsilon^{q'}$ be two monomials. The following statements are true:*

1. *If $p/p' = q/q' = l/k$ then $R^{(p,q)}_{1/k} = R^{(p',q')}_{1/l}$.*
2. *If $p/p' = q/q'$ and $q/q' \neq l/k$ then $R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l} = \mathbb{C}\{x,\varepsilon\}$.*
3. *If $p/p' \neq q/q'$ then $R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l} = \mathbb{C}\{x,\varepsilon\}$.*

with which we end Chapter 1.

Chapter 2. Monomial Borel-Laplace summation methods: The goal of this chapter of three sections is to develop and systematize Borel-Laplace type summability methods in two variables to give another characterization of monomial summability. Thus in the first section we define the Borel transform $\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}$, the Laplace transform $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$ including their formal versions and the convolution product $*^{(p,q)}_{k,(s_1,s_2)}$, all associated with a monomial $x^p\varepsilon^q$, a parameter of summability k and weights s_1, s_2 on the variables. We include three subsections to treat separately each transformation, resp. operation. These operators are only applied to functions whose domain is a sector in a monomial. All the properties, such as the behavior w.r.t. asymptotic expansions, are focused in the case when the mentioned sectors correspond to the same monomial $x^p\varepsilon^q$. However we have included in each subsection a brief note to describe the situation when the monomials are different. We point out the following interesting formula, a key observation to one of the applications we discuss in the next chapter:

Proposition 2.1.3. *Consider a bounded function $f \in \mathcal{O}(S_{p,q}(d, \pi/k + 2\epsilon_0, R_0))$. Then*

$$\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}\left((x^p\varepsilon^q)^k\left(\frac{s_1}{p}x\frac{\partial f}{\partial x} + \frac{s_2}{q}\varepsilon\frac{\partial f}{\partial \varepsilon}\right)\right)(\xi,\upsilon) = k(\xi^p\upsilon^q)^k\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}(f)(\xi,\upsilon),$$

for any $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$.

We also have included other presumable formulas as in Proposition 2.1.14 that shows that the Borel and Laplace operators are inverses one of each other (fact that uses the injectivity of the Laplace transform, Lemma 2.1.13) or as in Proposition 2.1.15 that states that the Laplace transform interchanges the convolution with the usual product.

In the second section we define a summability method associated with a monomial $x^p\varepsilon^q$, a parameter of summability k, a weight on the variables s_1, s_2 and a direction d, using the aforementioned transforms and based in the same lines of the theory in one variable: a series $\hat{f}$ is $k-(s_1,s_2)-$*Borel summable in the monomial* $x^p\varepsilon^q$ *in direction* d if it is $1/k-$Gevrey in $x^p\varepsilon^q$, the series $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$ can be analytically continued, say φ_{s_1,s_2}, to a monomial sector of the form $S_{p,q}(d,2\epsilon,+\infty)$, $\epsilon>0$, and with exponential growth of the form

$$|\varphi_{s_1,s_2}(\xi,\upsilon)| \leq De^{M\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s_2}\}},$$

for some positive constants D, M. Based on this definition and the properties obtained so far we have achieved the searched characterization of monomial summability, what we consider is one of the most remarkable results of the work.

Theorem 2.2.1. *Let* $\hat{f}\in\hat{R}^{(p,q)}_{1/k}$ *be a* $1/k-$*Gevrey series in the monomial* $x^p\varepsilon^q$. *Then the following are equivalent:*

1. $\hat{f}\in R^{(p,q)}_{1/k,d}$,
2. *There are* $s_1,s_2>0$ *with* $s_1+s_2=1$ *such that* $\hat{f}$ *is* $k-(s_1,s_2)-$*Borel summable in the monomial* $x^p\varepsilon^q$ *in direction* d.
3. *For all* $s_1,s_2>0$ *such that* $s_1+s_2=1$, $\hat{f}$ *is* $k-(s_1,s_2)-$*Borel summable in the monomial* $x^p\varepsilon^q$ *in direction* d.

In all cases the corresponding sums coincide.

We finish this section using this characterization to obtain alternative proofs of some results we got in the first chapter. The present chapter ends with Section 2.3 where we explore the basic behavior of monomial asymptotic expansions of Gevrey type under point blow-ups in the complex plane.

Chapter 3. Singularly perturbed analytic linear differential equations: We devote this chapter to the applications of monomial summability to the study of formal solutions of certain type of differential equations. In the first section we work with doubly singular differential equations of the form

$$\varepsilon^q x^{p+1}\frac{dy}{dx} = A(x,\varepsilon)y(x,\varepsilon)+b(x,\varepsilon), \tag{3-1}$$

where p,q are natural numbers, $y\in\mathbb{C}^l$, $A\in\mathrm{Mat}(l\times l,\mathbb{C}\{x,\varepsilon\})$ and $b\in\mathbb{C}\{x,\varepsilon\}^l$. Under the hypothesis of invertibility of $A(0,0)$ we recall the proof that the equation has a unique formal solution and that it is 1−Gevrey in the monomial $x^p\varepsilon^q$, Proposition 3.1.2, using the Nagumo norms. For the monomial summability properties we have proposed an alternative proof of the following theorem:

Theorem 3.1.4. *The unique formal solution $\hat{y}$ of equation (3-1) is* 1−*summable in* $x^p\varepsilon^q$.

The ideas behind the proof are not new. Their essence lies in the typical reasoning: to use an adequate Borel transform to study the solutions of the associated convolution equation, by the fixed point method. Once the solutions are obtained the Ramis-Sibuya theorem is applied to get an asymptotic expansion and deduce summability. We mention that with the characterization of monomial summability through the Borel-Laplace method explained in Chapter 2 we have improved Theorem 3 in [BM], a result that we state in Corollary 3.1.5.

Finally we mention that Theorem 3.1.4 is also valid for the non-linear case, but for that situation we have limited ourselves to enunciate the result in Theorem 3.1.6.

In the second section and as it is suggested by the formula contained in Proposition 2.1.3, we study the formal solutions of the partial differential equation

$$\frac{s_1}{p}\varepsilon^q x^{p+1}\frac{\partial y}{\partial x}+\frac{s_2}{q}x^p\varepsilon^{q+1}\frac{\partial y}{\partial \varepsilon}=C(x,\varepsilon)y(x,\varepsilon)+\gamma(x,\varepsilon), \tag{3-25}$$

where p,q are positive natural numbers, s_1,s_2 are positive real numbers such that $s_1+s_2=1$ and $C\in \mathrm{Mat}(l\times l,\mathbb{C}\{x,\varepsilon\})$, $\gamma\in\mathbb{C}\{x,\varepsilon\}^l$. Following the same ideas applied in the previous section but with the tools of monomial summability we have obtained the following results:

Proposition 3.2.1. *Consider the partial differential equation (3-25). If $C(0,0)$ is invertible then (3-25) has a unique solution $\hat{y}\in\hat{R}^l$. Moreover $\hat{y}\in(\hat{R}_1^{(p,q)})^l$.*

Theorem 3.2.2. *Consider equation (3-25). If $C(0,0)$ is invertible then the unique formal solution $\hat{y}$ given by the previous proposition is* 1−*summable in $x^p\varepsilon^q$. Its possible singular directions are the directions passing through the eigenvalues of $C(0,0)$.*

In the last section we pass to the study of Pfaffian systems in two variables of the form

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} = f_1(x,\varepsilon,y), & \text{(3-35a)}\\ x^{p'}\varepsilon^{q'+1}\dfrac{\partial y}{\partial \varepsilon} = f_2(x,\varepsilon,y), & \text{(3-35b)}\end{cases}$$

where p,q,p',q' are positive natural numbers, $y\in\mathbb{C}^l$, and f_1,f_2 are analytic functions in a neighborhood of the origin in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^l$. We recall that if $f_1(0,0,0)=f_2(0,0,0)=0$ and the functions f_1,f_2 satisfy on its domain of definition the equation:

$$-qx^{p'}\varepsilon^{q'}f_1(x,\varepsilon,y)+x^{p'}\varepsilon^{q'+1}\frac{\partial f_1}{\partial\varepsilon}(x,\varepsilon,y)+\frac{\partial f_1}{\partial y}(x,\varepsilon,y)f_2(x,\varepsilon,y)= \tag{3-36}$$
$$-p'x^p\varepsilon^q f_2(x,\varepsilon,y)+x^{p+1}\varepsilon^q\frac{\partial f_2}{\partial x}(x,\varepsilon,y)+\frac{\partial f_2}{\partial y}(x,\varepsilon,y)f_1(x,\varepsilon,y),$$

the pfaffian system is said to be completely integrable. Under this hypothesis we have deduced the next proposition on the behavior of the spectra of the linear parts of f_1 and f_2 in the origin:

Proposition 3.3.1. *Consider the Pfaffian system (3-35a), (3-35b). If it is completely integrable then the following assertions hold:*

1. *The matrix* $\frac{\partial f_2}{\partial y}(0,0,0)$ *is nilpotent if* $p=p'$ *and* $q<q'$, *or* $p'=Np$ *with* $N>1$, *or* $q'=q$ *and* $p<p'$ *or* $q'=Mq$ *with* $M>1$.
2. *The matrix* $\frac{\partial f_1}{\partial y}(0,0,0)$ *is nilpotent if* $p=p'$ *and* $q'<q$, *or* $p=N'p'$ *with* $N'>1$, *or* $q'=q$ *and* $p'<p$ *or* $q=M'q'$ *with* $M'>1$.
3. *If* $p=p'$ *and* $q=q'$, *for every eigenvalue* μ *of* $\frac{\partial f_2}{\partial y}(0,0,0)$ *there is an eigenvalue* λ *of* $\frac{\partial f_1}{\partial y}(0,0,0)$ *such that* $q\lambda=p\mu$. *The number* λ *is an eigenvalue of* $\frac{\partial f_1}{\partial y}(0,0,0)$, *when restricted to its invariant subspace* $E_\mu=\{v\in\mathbb{C}^n|(\frac{\partial f_2}{\partial y}(0,0,0)-\mu I)^k v=0$ *for some* $k\in\mathbb{N}\}$.

Taking into account these restrictions, we have used the statements of the first section and the tauberian properties found in the first chapter to obtain the following result on the convergence and summability of the solutions of such systems:

Theorem 3.3.3. *Consider the system (3-35a), (3-35b). The following assertions hold:*

1. *Suppose the system has a formal solution* $\hat{y}$. *If* $\frac{\partial f_1}{\partial y}(0,0,0)$ *and* $\frac{\partial f_2}{\partial y}(0,0,0)$ *are invertible and* $x^p\varepsilon^q\neq x^{p'}\varepsilon^{q'}$ *then* $\hat{y}$ *is convergent.*
2. *If the system is completely integrable and* $\frac{\partial f_1}{\partial y}(0,0,0)$ *is invertible then the system has a unique formal solution* $\hat{y}$. *Moreover* $\hat{y}$ *is 1-summable in* $x^p\varepsilon^q$.
3. *If the system is completely integrable and* $\frac{\partial f_2}{\partial y}(0,0,0)$ *is invertible then the system has a unique formal solution* $\hat{y}$. *Moreover* $\hat{y}$ *is 1-summable in* $x^{p'}\varepsilon^{q'}$.

Finally we turn to the linear case and when the two monomials involved are the same. We refer to systems of the form

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} = A(x,\varepsilon)y(x,\varepsilon)+a(x,\varepsilon), & \text{(3-48a)}\\ x^p\varepsilon^{q+1}\dfrac{\partial y}{\partial \varepsilon} = B(x,\varepsilon)y(x,\varepsilon)+b(x,\varepsilon), & \text{(3-48b)}\end{cases}$$

where p, q are positive natural numbers, $A, B \in \mathrm{Mat}(l \times l, \mathbb{C}\{x, \varepsilon\})$ and $a, b \in \mathbb{C}\{x, \varepsilon\}^l$. In this context we have obtained the following results on convergence and summability of their solution:

Proposition 3.3.4. *The following assertions hold:*

1. *If the system (3-48a), (3-48b) is completely integrable and $A(0,0)$ or $B(0,0)$ is invertible then the system (3-48a), (3-48b) has a unique formal solution that is $1-$summable in $x^p\varepsilon^q$.*

2. *If the system has a formal solution $\hat{y}$ and there are $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $s_1/pA(0,0) + s_2/qB(0,0)$ is invertible, then $\hat{y}$ is $1-$summable in $x^p\varepsilon^q$. Its possible singular directions are those passing through the eigenvalues of $s_1/pA(0,0) + s_2/qB(0,0)$.*

Theorem 3.3.5. *Consider the system (3-48a), (3-48b) and suppose it has a formal solution $\hat{y}$. Denote by $\lambda_1(s), ..., \lambda_l(s)$ the eigenvalues of $\frac{s}{p}A(0,0) + \frac{(1-s)}{q}B(0,0)$, where $0 \le s \le 1$, and assume that they are never zero. Then if for every direction d there is $s \in [0,1]$ such that $arg(\lambda_j(s)) \neq d$ for all $j = 1, ..., l$ then $\hat{y}$ is convergent.*

Chapter 4. Toward monomial multisummability: In the last chapter of this thesis we show the progress towards an adequate notion of monomial multisummability. The chapter is divided into three sections. In the first one we recall the acceleration operators and the concept of multisummability for two levels in one variable, including important formulas that will be used in the next section.

In the second section we define the acceleration operators that relate a monomial $x^p\varepsilon^q$, a parameter of summability k and weights s_1, s_2 with another monomial $x^{p'}\varepsilon^{q'}$, another parameter of summability l and other weights s_1', s_2'. Those operators have been obtained from the formal computation of the composition of the Borel transform $\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}$ with the Laplace transform $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$.

But of course there are conditions on those values to be able to make the computation, namely:

$$s_1' = \frac{s_1p'q}{s_2pq' + s_1p'q}, \qquad s_2' = \frac{s_2pq'}{s_2pq' + s_1p'q},$$
$$s_1(p'q - pq') > \frac{p}{l}(qk - q'l), \qquad \min\left\{\frac{p}{p'}, \frac{q}{q'}\right\} < \frac{l}{k}.$$

Then if $I = (p', q', p, q, l, k, s_1', s_2', s_1, s_2)$, where those values satisfy the previous conditions, we have an operator $\mathfrak{A}_I$. As we did in Chapter 2, we develop all the expected properties of these acceleration operators such as the behavior w.r.t. monomial asymptotic expansions and the convolution.

Finally in the last section we propose a definition of monomial multisummability associated with two monomials, two parameters of summability and two pairs of weights, motivated by the following result, analogous to the one in one variable and that we have proved using point blow-ups.

Teorema 4.3.1. Let $p_0, ..., p_r$, $q_0, ..., q_r$ be positive natural numbers and let $k_0, ..., k_r$ be positive real numbers. Let $\hat{f}_j \in R_{1/k_j}^{(p_j,q_j)} \setminus R$ be k_j-summable power series in the monomial $x^{p_j}\varepsilon^{q_j}$, for $j = 1, ..., r$, respectively. Then $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_r$ is k_0-summable in $x^{p_0}\varepsilon^{q_0}$ if and only if $k_0p_0 = k_jp_j$ and $k_0q_0 = k_jq_j$ for all $j = 1, ..., r$.

We finish this chapter showing that the proposed monomial multisummability concept is stable under sums and products and that it is capable of summing series of the form $\hat{f} + \hat{g}$, where $\hat{f} \in R_{1/k}^{(p,q)}$ and $\hat{g} \in R_{1/l}^{(p',q')}$.

The subject is far from being closed. Many open problems are posed and we name some of them:

1. Give a complete definition of monomial multisummability that includes not only the particular cases treated here.

2. Prove that the property of being multisummable as we defined here is independent of the chosen weights. A possible way of doing this is to extend the decomposition result of W. Balser of multisummable series as sums of summable series.

3. Study systematically the singularly perturbed systems of differential equations without the invertibility hypothesis of the linear part at the origin, applying the accelerator operators generalized to many levels.

4. Study in greater generality the linear pfaffian systems with or without the restriction of complete integrability.

5. Study of the partial differential equation (3-25) in the non-linear case with or without the invertibility hypothesis of the linear part at the origin.

6. Adapt the proof given here to the case of many variables.

7. Make use of sheaf theory to develop the theory presented here in a more intrinsic way.

1 Monomial Summability

The aim of this chapter is to recall and develop the notion of asymptotic expansions and summability in a monomial in two variables as was introduced by M. Canalis-Durand, J. Mozo Fernández and R. Schäfke in [CDMS]. In the mentioned paper the authors are motivated by the summability properties that possess the formal power series solutions of certain singularly perturbed systems of holomorphic differential equations that we will also discuss in Chapter 3. The main idea that monomial summability describes is that a source of divergence for some type of series comes from a monomial and then treating this monomial as a new variable lead to a way to associate a sum to the series. Along the chapter we provide complete proofs of the statements related with monomial summability, in many cases following the same lines as in the referred paper.

The chapter is divided into three sections. The first of them is devoted to recall the theory of asymptotic expansions, Gevrey asymptotic expansions and $k-$summability via Ramis definition and via the Bolel-Laplace method, for one complex variable and to put together all the classical results that we will need in the forthcoming sections. It also establishes the notations we will use through the text. The theory is developed for power series with coefficients in a complex Banach space. We remark that we have only included the proofs of Proposition 1.1.1, Proposition 1.1.2 and Corollary 1.1.3 since we did not find any reference where the results are proved in the way we do here. Proposition 1.1.2 and Corollary 1.1.3, surely well-known by the specialists, are not easily traceable in the literature.

The second section is the cornerstone of the chapter since establishes the concept of summability in a monomial in Ramis style. It contains four subsections in order of dependency. In the first one all the necessary formal background is developed, i.e. we introduce the differential subalgebras of Gevrey series of the ring of formal power series in two variables as well as useful formulas that will be used in the text. Special attention is played on the pass from an arbitrary monomial to the simple monomial $x\varepsilon$ via ramification. Also the trick of introducing a new variable by weighting the previous variables is included. The second part of the section treats with the analytic setting for asymptotic expansions in a monomial, i.e. with the analytic maps we will work and its domains: the *monomial sectors.* With the monomial sectors defined we pass to define asymptotic expansions and Gevrey asymptotic expansions in a monomial, first for the simple monomial $x\varepsilon$ and then to an arbitrary monomial. Some equivalent properties of having an asymptotic expansions in a monomial are provided, one of them reducing the notion to the case of one complex variable (the monomial) providing a bridge between the two theories. The stability of monomial asymptotic expansions under the usual algebraic operations including differentiation and also the analogous to the classical theorems as Watson's lemma and Ramis-Sibuya theorem are explored. In the third section the definition of monomial summability is given joint with a way to calculate the sum using the step into one variable. Also an equivalence of summability, not included in [CDMS], using the so called components of a function is provided in Proposition 1.2.30. In order to provide different ways to calculate the sum in a monomial of a series three different formulas are included in the last subsection: one calculating the sum as a series in x, another calculating the sum as a series in ε and the last one by weighting the variables.

The last section contains tauberian properties for monomial summability. The first and already known is the fact that absence of singular directions implies convergence. The second and new one is Theorem 1.3.5 that establishes that a divergent series cannot be summable for two essentially different monomials. This theorem is proved analyzing the different order relations between the exponents of the monomials and the parameters of summability.

1.1 Classical summability

The goal of this section is to quickly recall the well known facts of $k-$summability of Ramis and to establish the notations we are going to use through the text. Most of the results are taken from the books [B1] and [B2]. All the results exposed here are going be used in the next section to be able to recall the notion of monomial summability in two variables.

We need to clarify that the Borel-Laplace method we use here is not precisely the same used in the mentioned books and either in the classical literature. An issue that the classical method faces is that when the formal $k-$Borel transform is applied, the exponents in the series are not necessarily positive integers anymore: the transformation subtracts a k from them. In the mentioned books this problem is remedied by modifying the integral transformations involved. The disadvantage is that the formulas involving differentiation and convolutions increase their complexity. We approach Borel-Laplace method by keeping the classical integral transformations unmodified but instead of calculating the formal $k-$Borel transform to a series we calculate the transform to the series times x^k. At the end when calculating the Laplace transform we divide by x^k to compensate the initial change. Both approaches are equivalent since the resulting function is the $k-$sum of the series in the sense of Ramis. Unfortunately this process does not extend well to the case of multisummaility as will be noticed in Chapter 4.

We will denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers. $\mathbb{N}^*$ will denote the set of natural numbers without 0, $\mathbb{R}_{>0}$ will denote the set of positive real numbers and $\mathbb{R}_{\geq 0}$ will denote the set of non-negative real numbers. For $r > 0$ and $x_0 \in \mathbb{C}$ we will denote by $D_r(x_0)$ the disc of radius r centered at x_0 and by $\overline{D}_r(x_0)$ its closure. If $x_0 = 0$ we will simply write D_r. Also we will set $V = V(a, b, r)$ for the sector in $\mathbb{C}$ (or in $\widetilde{\mathbb{C}^*}$, the

Riemann surface of the logarithm), centered at the origin with opening $\alpha = b - a > 0$, radius r $(0 < r \leq +\infty)$ and bisecting direction $d = (b+a)/2$. We will eventually use also the notation $V(a,b,r) = S(d,\alpha,r)$. If W is a subsector of V we will write $W \Subset V$. We also call sectorial regions to regions G such that there are real numbers d, $\alpha > 0$ and $0 < r \leq +\infty$ such that $G \subset S(d,\alpha,r)$, and for every $0 < \beta < \alpha$ one can find $\rho > 0$ such that $\overline{S}(d,\beta,\rho) \subset G$. As before, d is referred to the bisecting direction and α to the opening of G, respectively. The cartesian product of sectors will be called polysector.

From now on we will work in a fixed but arbitrary complex Banach space E equipped with a norm $\|\cdot\|$. We will denote by $E[[x]]$ the $\mathbb{C}-$vector space of formal power series in the variable x with coefficients in E. We also let $E\{x\}$ denote the subspace of convergent power series and by $E[[x]]_s$, $s > 0$, the subspace of $s-$Gevrey formal power series. Remember that $\hat{f} = \sum_{n=0}^{\infty} a_n x^n$ is $s-$Gevrey if we can find positive constants C, A such that $\|a_n\| \leq CA^n n!^s$, for all $n \in \mathbb{N}$. By Stirling's formula we can replace the term $n!^s$ by $\Gamma(1+sn)$ or by n^{sn}, by changing the constants C, A. Note that when E is a Banach algebra[1], $E[[x]], E\{x\}$ and $E[[x]]_s$ are algebras too. The space of analytic maps (resp. bounded analytic maps) defined on V with values in E will be denoted by $\mathcal{O}(V,E)$ (resp. $\mathcal{O}_b(V,E)$). The last space becomes a Banach space with the supremum norm. Finally we will simply write $\mathcal{O}(D_r) = \mathcal{O}(D_r,\mathbb{C})$ (resp. $\mathcal{O}_b(D_r) = \mathcal{O}_b(D_r,\mathbb{C})$) for these particular cases.

Definition 1.1.1. Let $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in E[[x]]$ be a formal power series and V a sector. An analytic map $f \in \mathcal{O}(V,E)$ is said to have $\hat{f}$ as *asymptotic expansion* at 0 on V and we will use the notation $f \sim \hat{f}$ on V, if for each of its proper subsectors W and each $N \in \mathbb{N}$, there exists *asymptotic constants* $C_N(W) > 0$ such that for all $x \in W$ we have:

$$\left\| f(x) - \sum_{n=0}^{N-1} a_n x^n \right\| \leq C_N(W)|x|^N. \tag{1-1}$$

If we can take $C_N(W) = C(W)A(W)^N N!^s$, the asymptotic expansion is said to be of *$s-$Gevrey type* and we will use the notation: $f \sim_s \hat{f}$ on V. We will also denote by $\mathcal{A}(V,E)$ (resp. $\mathcal{A}_s(V,E)$) the set of analytic maps defined on V with values in E that admits a formal power series as asymptotic expansion on V (resp. asymptotic expansion on V of $s-$Gevrey type).

If $f \sim \hat{f}$ on V, the coefficients of $\hat{f}$ can be characterized as the limits $\lim_{\substack{x\to 0\\ x\in W}} \frac{f^{(n)}(x)}{n!} = a_n$, for any $W \Subset V$, and the existence of this limits is equivalent to $\hat{f}$ being the asymptotic expansion of f on V. Besides we also have that $\|a_n\| \leq C_n(W)$. In particular, if $f \sim_s \hat{f}$ on V, then $\hat{f} \in E[[x]]_s$. In this case, this is equivalent to the condition: for every $W \Subset V$, there are constants $C, A > 0$ such that $\sup_{x\in\overline{W}} \left\| \frac{f^{(n)}(x)}{n!} \right\| \leq CA^n n!^s$, for any $n \geq 0$. For classical examples of asymptotic expansions of special functions the reader may consult the book [O].

[1] Recall that a Banach algebra is a Banach space E in which an operation of multiplication is defined, that makes E an algebra and such that $\|ab\| \leq \|a\|\|b\|$, for all $a, b \in E$.

Before we continue we need the following three characterization of asymptotic expansions (and more importantly, of asymptotic expansions of $s-$Gevrey type) described in the next propositions. The first one is taken from Exercise 1.(a), Section 4.5, Chapter 4, [B2].

Proposition 1.1.1. *Let $f \in \mathcal{O}(V,E)$ be an analytic map defined on V and $\hat{f} = \sum a_n x^n \in E[[x]]$ be a formal power series. The following statements are equivalent:*

1. *$f \sim \hat{f}$ on V,*

2. *Fix an integer $p \geq 1$. For all $M \in \mathbb{N}$ of the form $M = pN$ and every subsector $W \Subset V$ there is $C_M(W)$ such that (1-1) is valid for all $x \in W$.*

The result is also valid for $s-$Gevrey asymptotic expansions: $f \sim_s \hat{f}$ on V if and only if (2) is fulfilled with $C_M(W) = CA^M M!^s$, for some $C, A > 0$ independent of M.

Proof. We only write the proof for the case of Gevrey asymptotic expansions. The general case follows the same lines. The only non-trivial part is that (2) implies (1). We first show that $\hat{f} \in E[[x]]_s$. For $m \in \mathbb{N}$, let $N = [m/p]$ the integer part of m/p, so $Np \leq m < (N+1)p$. Using inequality (1-1) with Np and $(N+1)p$ and some $W \Subset V$ we get from triangle's inequality

$$\left\| a_{Np} + a_{Np+1}x + \dots + a_{Np+p-1}x^{p-1} \right\| \leq CA^{Np}(Np)!^s + CA^{(N+1)p}((N+1)p)!^s |x|^p,$$

for all $x \in W$. Now take $x_0, x_1, \dots, x_{p-1} \in W$ any p distinct points on W with a common radius, say $|x_j| = R$. We may suppose for simplicity that $AR \leq 1$. Let $U = (x_i^j)_{0 \leq i,j \leq p-1}$ the Vandermonde matrix associated with the points $x_0, x_1, \dots, x_{p-1}$. By using the norm $\|\cdot\|_1$ of E^p given by $\|(z_0, \dots, z_{p-1})^t\|_1 = \sum_{j=0}^{p-1} \|z_j\|$, we conclude from the previous inequality that

$$\begin{aligned} \|a_{Np+j}\| &\leq \|(a_{Np}, a_{Np+1}, \dots, a_{Np+p-1})^t\|_1 \\ &= \|U^{-1}U(a_{Np}, a_{Np+1}, \dots, a_{Np+p-1})^t\|_1 \\ &\leq \|U^{-1}\|_1 pCA^{Np}(Np)!^s(1 + A^p(Np+1)^s \cdots (Np+p)^s R^p) \\ &\leq \|U^{-1}\|_1 pCA^{Np}(Np)!^s(1 + (2p)^{sp})^N, \end{aligned}$$

for all $j = 0, 1, \dots, p-1$. Here $\|U\|_1 = \sup_{v \in E^p} \frac{\|Uv\|_1}{\|v\|_1}$ stands for the associated matrix norm of a matrix of complex numbers. We also have used the inequality $Np + j \leq p(N+1) < p2^N$. The previous bound let us conclude that $\hat{f}$ is $s-$Gevrey, say $\|a_n\| \leq DB^n n!^s$, for all $n \in \mathbb{N}$. We also may take $C \leq D$ and $\max\{1, A\} \leq B$.

Now we show that $f \sim_s \hat{f}$ on V. Take $W = W(a', b', r) \Subset V$, $m \in \mathbb{N}$ and let $N = [m/p]$. It follows that for all $x \in W$

$$
\begin{aligned}
\left\| f(x) - \sum_{n=0}^{m-1} a_n x^n \right\| &\le CA^{(N+1)p}((N+1)p)!^s |x|^{(N+1)p} + \sum_{n=m}^{(N+1)p-1} \|a_n\||x|^n \\
&\le DB^m m!^s |x|^m \sum_{n=m}^{(N+1)p} B^{n-m} r^{n-m} \frac{n!^s}{m!^s} \\
&\le DB^m m!^s |x|^m 2^{spN} p^{sp} \sum_{k=0}^{(N+1)p-m} (Br)^k \\
&\le Dp^{sp}(2^s B)^m \frac{(Br)^{p+1}-1}{Br-1} m!^s |x|^m,
\end{aligned}
$$

as we wanted to prove. □

Proposition 1.1.2. *Let V be an open sector in $\mathbb{C}^*$ and $f \in \mathcal{O}(V, E)$. Then $f \in \mathcal{A}(V, E)$ if and only if there is $r > 0$ and a family of analytic maps $f_N \in \mathcal{O}(D_r, E), N \ge 1$, satisfying the condition:*

1. *For every subsector W of V and $N \ge 1$ there is a constant $C_N(W)$ such that $\|f(x) - f_N(x)\| \le C_N(W)|x|^N$ for all $x \in W \cap D_r$,*

Analogously, for $s > 0$, $f \in \mathcal{A}_s(V, E)$ if and only if there is $r > 0$ and a family of analytic maps $f_N \in \mathcal{O}(D_r, E), N \ge 1$, satisfying condition (1) with $C_N(W) = CA^N N!^s$, for some C, A depending only of W, and

2. *There are constants B, D with $\sup_{x\in D_r} \|f_N(x)\| \le DB^N N!^s$, for all $N \ge 1$.*

Proof. As before, we only write the proof for the case of Gevrey asymptotic expansions. If $f \in \mathcal{A}_s(V, E)$ and $\sum_{n=0}^{\infty} a_n x^n$ is its asymptotic expansion then the condition is fulfilled by taking $f_N(x) = \sum_{k=0}^{N-1} a_k x^k$. Conversely, suppose we have a family of such maps. Write them using its Taylor's expansion at the origin, say $f_N(x) = \sum_{m=0}^{\infty} a_m^{(N)} x^m$. Note that condition (2) and Cauchy's inequalities implies that $\|a_m^{(N)}\| \le DB^N N!^s / r^m$ for all $m, N \in \mathbb{N}$. For every positive pair of integers N, k with $k \ge N$ consider the maps $g_{N,k}(x) = x^{-N}(f_N(x) - f_k(x))$. Then the conditions on the f_N imply that for any $W \Subset V$ and $x \in W \cap D_r$ we have:

$$\|g_{N,k}(x)\| \le CA^N N!^s + CA^k k!^s |x|^{k-N}.$$

In particular, the $g_{N,k}$ are bounded on W an thus they have a limit when $x \to 0$ in W. This implies that $a_m^{(N)} = a_m^{(k)}$ for all $k \ge N$ and $m = 0, 1, ..., N-1$ and the $g_{N,k}$ are analytic on D_r. Let $a_m = \lim_{N\to+\infty} a_m^{(N)} = a_m^{(m+1)}$ and $\hat{f} = \sum_{m=0}^{\infty} a_m x^m$. To see that $\hat{f}$ is the s-Gevrey asymptotic expansion of f it is sufficient to establish the inequalities (1-1) for $W \cap D_\rho$ with $\rho < r$. Indeed, for any $x \in W \cap D_\rho$ we get that

$$\left\| f(x) - \sum_{m=0}^{N-1} a_m x^m \right\| \leq \|f(x) - f_N(x)\| + \left\| f_N(x) - \sum_{m=0}^{N-1} a_m x^m \right\|$$
$$= \left(CA^N + \frac{D}{1-\rho/r} \frac{B^N}{r^N} \right) N!^s |x|^N.$$

This concludes the proof.

□

Using the same arguments of the previous proof joint with Proposition 1.1.1 we can obtain another characterization of Gevrey asymptotic expansions. It will be used in the next section (see Proposition 1.2.31).

Corollary 1.1.3. *Let V be an open sector in $\mathbb{C}^*$, $f \in \mathcal{O}(V,E)$ and $p \in \mathbb{N}^*$ fixed. Then $f \in \mathcal{A}_s(V,E)$ if and only if there is $r > 0$ and a family of analytic maps $f_{pN} \in \mathcal{O}(D_r, E), N \geq 1$, satisfying the following conditions:*

1. *For every subsector W of V there are constants $C, A > 0$ such that $\|f(x) - f_{pN}(x)\| \leq CA^{pN}(pN)!^s |x|^{pN}$ for all $x \in W \cap D_r$,*
2. *There are constants B, D with $\sup_{x\in D_r} \|f_{pN}(x)\| \leq DB^{pN}(pN)!^s$, for all $N \in \mathbb{N}$.*

Sometimes when one has to deal with ramifications (a change of variables $t = x^p$) it is useful to be able to express asymptotic expansions in x as asymptotic expansions in t and conversely. Consider $\hat{f} \in E[[x]]$ and decomposed uniquely as

$$\hat{f}(x) = \sum_{j=0}^{p-1} x^j \hat{f}_j(x^p),$$

where each $\hat{f}_j$ can be recovered from $\hat{f}$ by $px^j \hat{f}_j(x^p) = \sum_{l=0}^{p-1} \omega^{l(p-j)} \hat{f}(\omega^l x)$, where ω is a primitive pth root of unity.

This process can also be done for analytic maps. Indeed, take a function $f \in \mathcal{O}(\cup_{0\leq j<p} V_j, E)$, with $V_j = V(a + 2\pi j/p, b + 2\pi j/p, r)$. Note that the domain of f is a *sector in the variable* x^p, i.e., $\cup_{0\leq j<p} V_j = \{x \in \mathbb{C} | 0 < |x|^p < r^p, ap < \arg(x^p) < bp\}$. Then $f(x) = \sum_{j=0}^{p-1} x^j f_j(x^p)$, where each $px^j f_j(x^p) = \sum_{l=0}^{p-1} \omega^{l(p-j)} f(\omega^l x)$ is defined on $V = V(pa, pb, r^p)$. Under these considerations we can obtain the following proposition (see Corollary 2.3.14, [L]).

Proposition 1.1.4. *Let $f \in \mathcal{O}(\cup_{0\leq j<p} V_j, E)$ be an analytic map and $\hat{f} \in E[[x]]_s$ as before. Using the previous notation, it is equivalent:*

1. *For every $j = 0, 1, ..., p-1$, $f \sim_s \hat{f}$ on V_j,*

2. *For every $j = 0, 1, ..., p-1$, $f_j \sim_{ps} \hat{f}_j$ on V.*

On the algebraic properties of asymptotic expansion we recall that $\mathcal{A}(V, E)$ and $\mathcal{A}_s(V, E)$ are $\mathbb{C}-$vector spaces, stable by the derivative d/dx and also $\mathbb{C}$-algebras when E is a Banach algebra. The uniqueness of the asymptotic expansion of a map let us consider the *Taylor's map* defined as $J_E : \mathcal{A}(V, E) \longrightarrow E[[x]]$, $J_E(f) = \hat{f}$ if $f \sim \hat{f}$ on V, and its restriction $J_{E,s} : \mathcal{A}_s(V, E) \longrightarrow E[[x]]_s$ to the case of $s-$Gevrey expansions. Both are linear maps, commuting with d/dx, and morphisms of algebras when E is a Banach algebra. The kernel of the above maps will be denoted by $\mathcal{A}_0(V, E)$ and $\mathcal{A}_{0,s}(V, E)$, respectively. For the last one, those maps are characterized by having exponential decay at 0 of order $k = 1/s$. More precisely, we say that $f \in \mathcal{O}(V, E)$ *has exponential decay at* 0 *in* V *with order* k if for every $W \Subset V$, there are constants $B, C > 0$ such that $\|f(x)\| \leq C \exp(-B/|x|^k)$ for all $x \in W$. For future references we formulate this as a proposition.

Proposition 1.1.5. *Take any $s > 0$. Then $h \in \mathcal{A}_{0,s}(V, E)$ if and only if h has exponential decay at 0 with order $k = 1/s$ on V.*

If V is a sector, the Taylor's map induces exact sequences

$$0 \longrightarrow \mathcal{A}_0(V, E) \longrightarrow \mathcal{A}(V, E) \longrightarrow E[[x]] \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{A}_{0,s}(V, E) \longrightarrow \mathcal{A}_s(V, E) \longrightarrow E[[x]]_s \longrightarrow 0,$$

in the first case for an arbitrary sector, and in the second case only for V with opening less than $s\pi$. These facts are known as the Borel-Ritt and Gevrey-Borel-Ritt theorems and are formulated below.

Theorem 1.1.6 (Borel-Ritt)**.** *Let $\hat{f} \in E[[x]]$ be a formal power series. For every sector V, there exists $f \in \mathcal{A}(V, E)$ such that $f \sim \hat{f}$ on V.*

Theorem 1.1.7 (Gevrey-Borel-Ritt)**.** *Let $k > 0$ and $\hat{f} \in E[[x]]_{1/k}$. For every sector V of opening less than π/k, there exists $f \in \mathcal{A}_{1/k}(V, E)$ such that $f \sim_{1/k} \hat{f}$ on V.*

The key point to be able to define the notion of $k-$summability is the following statement known as Watson's lemma, providing conditions on a sector for the Taylor's map to be injective.

Proposition 1.1.8 (Watson's Lemma)**.** *Let $s > 0$ and $V = V(a, b, r)$ be a sector with opening $b - a > s\pi$. Then $\mathcal{A}_{0,s}(V, E) = \{0\}$.*

In this context we will refer to sectors with opening greater than π/k as *k-wide sectors* and those will be the domains of the sums of the $k-$summable series that we introduce below.

Definition 1.1.2. Let $\hat{f} \in E[[x]]$ be a formal power series, take $k > 0$ and let d be a direction.

1. The formal series $\hat{f}$ is called *$k-$summable on* $V = V(a, b, r)$ if $b - a > \pi/k$ and there exists a map $f \in \mathcal{O}(V, E)$ such that $f \sim_{1/k} \hat{f}$ on V.
2. The formal series $\hat{f}$ is called *$k-$summable in the direction* d if there is a sector V bisected by d such that $\hat{f}$ is $k-$summable on V.
3. The formal series $\hat{f}$ is called *$k-$summable*, if it is $k-$summable in every direction with finitely many exceptions mod. 2π (the singular directions).

We will denote by $E\{x\}_{1/k,d}$ the set of $k-$summable series in direction d and by $E\{x\}_{1/k}$ the set of $k-$summable series. It is clear that both are $\mathbb{C}-$vector spaces, compatible with the derivative and the product (in case of E being a Banach algebra).

One way to calculate explicitly the $k-$sum of a $k-$summable series $\hat{f}$ is the *Borel-Laplace method*. Here we will use the following version of the *formal k-Borel transform*:

$$\begin{aligned}\widehat{\mathcal{B}}_k :& x^k E[[x]] \longrightarrow E[[\xi]],\\ &\sum_{n=0}^{\infty} a_n x^{n+k} \longmapsto \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(1+n/k)} \xi^n,\end{aligned}$$

that establish an isomorphism between the above linear spaces and restricts to an isomorphism between $x^k E[[x]]_{1/k}$ and $E\{\xi\}$. The analytic counterpart is as follows: for $V = S(d, \pi/k + 2\epsilon_0, R_0)$, with $0 < \epsilon_0 < \pi/k$, $R_0 > 0$ and $f \in \mathcal{O}_b(V, E)$, the *k-Borel transform* of f is defined by the integral formula:

$$\mathcal{B}_k f(\xi) = \frac{k}{2\pi i} \int_{\gamma_k} f(x) e^{(\xi/x)^k} \frac{dx}{x^{k+1}},$$

where γ_k denotes a path oriented positively, conformed by three pieces: an arc of circle of radius $R > 0$, $R < R_0$, and of two segments of length R respectively starting and arriving to 0 of arguments $d + \pi/2k + \epsilon'$ and $d - \pi/2k - \epsilon''$, with $0 < \epsilon', \epsilon'' < \epsilon_0$. $\mathcal{B}_k f$ is well-defined, independent of R, ϵ', ϵ'' and analytic in the sector $S(d, 2\epsilon_0, +\infty)$ of infinite radius bisected by d and opening $2\epsilon_0$. Besides if $f(x) = x^\lambda a$ for $\lambda \in \mathbb{C}$ and $a \in E$ then its $k-$Borel transform exists and is given by $\mathcal{B}_k f(\xi) = \frac{\xi^{\lambda-k}}{\Gamma(\lambda/k)} a$, justifying the definition of the formal version.

Among the properties of the Borel transform we emphasize the following two: First, if f is bounded then $\mathcal{B}_k(f)$ has *exponential growth of order at most* k on its domain, i.e., for every subsector of the domain there are constants $C, B > 0$ such that $\|\mathcal{B}_k(f)(\xi)\| \leq Ce^{B|\xi|^k}$ for all

ξ in such subsector. Second, if $f \in \mathcal{O}_b(V,E)$ with $V = S(d,\alpha,R)$, $\alpha > \pi/k$ and $f \sim_{s_1} \hat{f}$ on V then $\mathcal{B}_k(x^k f) \sim_{s_2} \hat{\mathcal{B}}_k(x^k \hat{f})$ on $S(d,\alpha-\pi/k,+\infty)$, where $s_2 = s_1 - 1/k$ if $s_1 > 1/k$ and $s_2 = 0$ otherwise.

The inverse of $k-$Borel transform is the *k-Laplace transform* defined as follows: consider d a direction and $g : [0, e^{id}\infty) \to E$ a continuous function on the half-line in $\mathbb{C}$ with vertex at 0 and direction d. If g has exponential growth of order at most k on its domain, the *k-Laplace transform of f in the direction d* is the function $\mathcal{L}_{k,d}(g)$ defined by:

$$\mathcal{L}_{k,d}(g)(x) = \int_0^{e^{id}\infty} g(\xi) e^{-(\xi/x)^k} d\xi^k.$$

This function is defined in a sectorial region of opening π/k bisected by d and $x^{-k}\mathcal{L}_{k,d}(g)(x)$ is analytic there. If the domain of g contains a sector, d, d' are directions in that sector and $|d-d'| < \pi/k$ then $\mathcal{L}_{k,d}(g) = \mathcal{L}_{k,d'}(g)$ on the intersection of its corresponding domains.

For $g(\xi) = \xi^\lambda a$, $\mathrm{Re}(\lambda) > 0$ and $a \in E$, we have $\mathcal{L}_{k,d}(g)(x) = \Gamma(1+\lambda/k)\, x^{\lambda+k} a$. Then the *formal k-Laplace transform* is defined as the inverse of $\widehat{\mathcal{B}}_k$:

$$\begin{aligned}\widehat{\mathcal{L}}_k : &E[[\xi]] \longrightarrow x^k E[[x]] \\ &\sum_{n=0}^{\infty} a_n \xi^n \longmapsto \sum_{n=0}^{\infty} a_n \Gamma(1+n/k)\, x^{n+k}.\end{aligned}$$

Certainly, $\mathcal{L}_{k,d}$ is also the inverse of $\mathcal{B}_k$, in the sense that if $f \in \mathcal{O}_b(V,E)$, where $V = S(d, \pi/k + 2\varepsilon_0, R)$, $0 < \varepsilon_0 < \pi/k$, then $\mathcal{L}_k(\mathcal{B}_k(f))$ is well-defined and equal to f in the intersection of their domains.

Finally, suppose $g \in \mathcal{O}(V,E)$, where V is a sector of infinite radius and opening α and that g has exponential growth at most k on V. If $g \sim_{s_1} \hat{g}$ on V then $x^{-k}\mathcal{L}_{k,d}(g) \sim_{s_2} x^{-k}\hat{\mathcal{L}}_k(\hat{g})$ in the corresponding sectorial region of opening $\alpha + \pi/k$, where $s_2 = s_1 + 1/k$.

A last remarkable point on the $k-$Borel and $k-$Laplace transforms is their relation with the $k-$convolution product. We recall that given $f, g \in \mathcal{O}(V,E)$, their $k-$convolution is defined by

$$(f *_k g)(x) = \int_0^x f((x^k - t^k)^{1/k}) g(t) d(t^k) = x^k \int_0^1 f(xt^{1/k}) g(x(1-t)^{1/k}) dt,$$

and give as a result an element of $\mathcal{O}(V,E)$. The $k-$convolution is a bilinear, commutative and associative binary operation on $\mathcal{O}(V,E)$. As a particular case, using the Beta function, we obtain the special values

$$\frac{x^{\lambda-k}}{\Gamma\left(\frac{\lambda}{k}\right)} *_k \frac{x^{\mu-k}}{\Gamma\left(\frac{\mu}{k}\right)} = \frac{x^{\lambda+\mu-k}}{\Gamma\left(\frac{\lambda+\mu}{k}\right)}, \qquad \text{for all } \mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0.$$

The relation mention above joint with some other properties useful for the analysis of differential and difference equations are listed in the next the proposition, for future references.

Proposition 1.1.9. *1. $\mathcal{B}_k(f \cdot g) = \mathcal{B}_k(f) *_k \mathcal{B}_k(g)$ and $\mathcal{L}_k(F *_k G) = \mathcal{L}_k(F) \cdot \mathcal{L}_k(G)$, for all functions f, g, F, G where the expressions are meaningful.*

2. $\mathcal{B}_k\left(x^{k+1}\frac{df}{dx}\right)(\zeta) = k\zeta^k\mathcal{B}_k(f)(\zeta)$ and $\mathcal{L}_k(k\zeta^k F)(x) = x^{k+1}\frac{d}{dx}(\mathcal{L}_k(F))(x)$, for all functions f, F where the expressions are meaningful.

3. $\mathcal{B}_k\left(f\left(\frac{z}{(1-cz^k)^{1/k}}\right)\right)(\zeta) = e^{c\zeta^k}\mathcal{B}_k(f)(\zeta)$, for all $c \in \mathbb{C}$ and all functions f where the expressions are meaningful.

With the previous considerations we are able to explain the Borel-Laplace method: in order to sum $\hat{f} \in E[[x]]_{1/k}$, one consider the convergent power series $\widehat{\mathcal{B}}_k(x^k\hat{f})(\xi)$. Choosing a direction d one attempts to make analytic continuation $\varphi(\xi)$ to a small sector W bisected by d. If this is possible and φ has exponential growth of order at most k on W, i.e., $\|\varphi(\xi)\| \leq Ce^{B|\xi|^k}$ for some constants B, C and all $\xi \in W$, then $\hat{f}$ is said to be *k-Borel summable in direction d* and its sum is defined by

$$f(x) = \frac{1}{x^k}\int_0^{e^{id}\infty} \varphi(\xi)e^{-(\xi/x)^k}d\xi^k = \frac{1}{x^k}\mathcal{L}_{k,d}(\varphi)(x).$$

Thanks to the good behavior of the Borel and Laplace transformations w.r.t. Gevrey asymptotic expansion we can easily justify the following theorem.

Theorem 1.1.10 (Ramis)**.** *A power series $\hat{f} \in E[[x]]$ is $k-$Borel summable in a direction d if and only if it is $k-$summable in the direction d and both sums coincide.*

One of the main tools in the theory of asymptotic expansions, very useful for instance in applications to differential equations, is the celebrated theorem due to J.P. Ramis and Y. Sibuya stated below. One proof of this result can be achieved with the Cauchy-Heine's transform.

Theorem 1.1.11 (Ramis-Sibuya)**.** *Let $(V_i)_{i\in I}$ be a finite good covering of a punctured neighborhood of 0 in $\mathbb{C}$ by open sectors and $(f_{i,i+1})_{i\in I}$ be a collection of bounded analytic maps on $(V_i)_{i\in I}$, respectively, admitting an exponential decay at 0 of order $1/s$, for some $s > 0$. Then there exists a collection $(f_i)_{i\in I}$ of analytic maps on $(V_i)_{i\in I}$, with $f_i \in \mathcal{A}_s(V_i, E)$ such that $f_{i,i+1} = f_{i+1} - f_i$. Moreover, the f_i admit the same asymptotic expansion $\hat{f}$ of Gevrey order s.*

The last results we will need are the following tauberian conditions on $k-$summability. The first is that the absence of singular directions implies convergence and the second relates different levels of summability.

Proposition 1.1.12. *Let $k > 0$ and $\hat{f} \in E[[x]]_{1/k}$ be a $1/k-$Gevrey formal power series. If $\hat{f}$ is $k-$summable in every direction then $\hat{f}$ is convergent.*

Theorem 1.1.13 (Ramis)**.** *Let $0 < k < k'$ be positive numbers. Then $E[[x]]_{1/k'} \cap E\{x\}_{1/k} = E\{x\}_{1/k'} \cap E\{x\}_{1/k} = E\{x\}$.*

So far we have only considered formal power series with non-negative integer exponents, but it is possible to extend the results to series with others exponents. Here we follow [Mal] and we describe the case when the set of exponents G satisfy the conditions:

1. $0 \in G$,
2. G is a discrete semigroup of $\mathbb{R}_{\geq 0}$, and we enumerate its elements by λ_n, $n \in \mathbb{N}$, with $\lambda_0 = 0 < \lambda_1 < ... < \lambda_n < \lambda_{n+1} < ...$, and $\lim_{n \to +\infty} \lambda_n = +\infty$,
3. There is $C > 0$ such that $\lambda_{n+1} - \lambda_n \leq C$ for all $n \in \mathbb{N}$.

Actually condition (3) is superfluous: note that $\sup_{n\in\mathbb{N}} \lambda_{n+1} - \lambda_n = \lambda_1$, as a consequence of (1) and (2). The typical example of such G is a semigroup generated by a finite number of positive real numbers $\mu_1, ..., \mu_r$, i.e., every element of G has the form $\sum_{k=1}^{r} n_k \mu_k$ with $n_k \in \mathbb{N}$. In this case the optimal value of C is $\min\{\mu_1, ..., \mu_r\}$.

We will denote by $E[[x^G]]$ the space of formal power series with coefficients in E and exponents in G. Any element of this space is an expression of the form $\hat{f} = \sum_{\lambda \in G} a_\lambda x^\lambda$, with $a_\lambda \in E$. We will also write $E\{x^G\}$ and $E[[x^G]]_s$ for the subspaces of convergent and $s-$Gevrey formal power series with exponents in G, respectively. Then $\hat{f}$ is convergent if there are constants $C, A > 0$ such that $|a_\lambda| < CA^\lambda$ for all $\lambda \in G$ and it is of $s-$Gevrey type if there are constants $C, A > 0$ such that $|a_\lambda| < CA^\lambda \Gamma(1 + s\lambda)$, for all $\lambda \in G$. The sets $E[[x^G]]$, $E\{x^G\}$ and $E[[x^G]]_s$ are $\mathbb{C}-$vector spaces, stable by derivative xd/dx (term by term) and also algebras when E is a Banach algebra (here it is used that G is a semigroup).

The analytic meaning of elements of $E\{x^G\}$ is the following: if $\sum_{n=0}^{\infty} a_{\lambda_n} x^{\lambda_n}$ is convergent and $|a_{\lambda_n}| \leq CA^{\lambda_n}$, by the Weierstraß M-test, it defines an analytic function in any sector of opening less than 2π (condition to chose a determination of the maps x^{λ_n}) and radius less than $1/A$.

Finally we can generalize asymptotic expansions using the above type of series. A possible definition that $f \in \mathcal{O}(V, E)$ admits $\hat{f} \in E[[x^G]]$ as asymptotic expansion on V, with V of opening less than 2π is that there is a determination of the x^λ on V, such that for every $W \Subset V$ and $\lambda \in G$, there are constants $C_\lambda(W)$ with

$$\left\| f(x) - \sum_{\substack{\mu<\lambda \\ \mu \in G}} a_\mu x^\mu \right\| \leq C_\lambda(W)|x|^\lambda,$$

for all $x \in W$. The asymptotic will be of s-Gevrey type if $C_\lambda(W) = CA^\lambda\Gamma(1 + s\lambda)$ for some $C, A > 0$ and in that case necessarily $\hat{f} \in E[[x^G]]_s$. We remark that if there is $p \in \mathbb{N}^*$ such that $G \subset \frac{1}{p}\mathbb{N}$ then the above notion corresponds to asymptotic in the variable $x^{1/p}$, via Proposition 1.1.4.

In this context, maps with null $k-$Gevrey asymptotic expansion are again those which has exponential decay (here is where condition (3) on G is used). Similar results as Watson's lemma, Borel-Ritt and Borel-Ritt-Gevrey theorems hold. Then the notion of $k-$summable series makes sense, up to the restriction $k > 1/2$. The formal $k-$Borel transform $\hat{\mathcal{B}}_k : x^k E[[x^G]] \to E[[\xi^G]]$ extends naturally by $\hat{\mathcal{B}}_k(x^{k+\lambda}a) = \xi^\lambda a/\Gamma(1 + \lambda/k)$, $a \in E$. Finally the Borel-Laplace summation method adapts as follows: $\hat{f}$ is $k-$summable in a direction d if and only if $\hat{\mathcal{B}}_k(\hat{f}) \in E\{\xi^G\}$ extends analytically as φ to a sector of infinite radius bisected by d, with exponential growth at most k. Then the $k-$sum is obtained via the $k-$Laplace transform of φ.

We want to finish the survey with one example taken from [Sa], to illustrate the theory.

Example 1.1.1 (Poincaré)**.** Fix $w \in \mathbb{C}$ with $0 \leq |w| < 1$ and consider the series of meromorphic functions of x:

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(x), \quad \phi_k(x) = \frac{w^k}{1 + kx}. \tag{1-2}$$

Restrict to the non-trivial case $|w| > 0$. For $|x| > 1/N$, $N \geq 1$, $\phi_0 + \phi_1 + \cdots + \phi_N$ is meromorphic with simple poles at -1, $-1/2$,...,$-1/N$, and for $k \geq N+1$, $|\phi_k(x)| \leq \frac{|w|^k}{k|x+1/k|} \leq \left(\frac{1}{N} - \frac{1}{N+1}\right)^{-1} \frac{|w|^k}{k}$, whence the uniform convergence and the analyticity of $\phi_{N+1}+\phi_{N+2}+\cdots$ in this domain. In conclusion ϕ is meromorphic in $\mathbb{C}^*$ with a simple pole at every point of the form $-1/k$, with $k \in \mathbb{N}^*$.

When x approaches 0, ϕ give rise to a divergent series: since $\phi_k(x) = w^k \sum_{n=0}^{\infty}(-1)^n k^n x^n$, for $|x| < 1/k$, when we formally interchange the sums in (1-2) we get:

$$\hat{\phi}(x) = \sum_{n=0}^{\infty}(-1)^n b_n x^n, \quad b_n = \sum_{k=0}^{\infty} k^n w^k.$$

Note that every b_n is a convergent numerical series. However $\hat{\phi}$ is divergent: in fact it is $1-$Gevrey. To prove this assertion, seen b_n as a function of w and setting $w = e^s, \mathrm{Re}(s) < 0$, we see that $b_n = \left(w\frac{d}{dw}\right)^n (b_0) = \left(\frac{d}{ds}\right)^n \left(\frac{1}{1-e^s}\right)$. Then we can recognize the formal $1-$Borel transform of $x\hat{\phi}(x)$ as the Taylor's formula of b_0 in the variable s:

$$\hat{\mathcal{B}}_1(x\hat{\phi}(x))(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n b_n}{n!}\xi^n = \frac{1}{1-e^{s-\xi}} = \frac{e^{\xi}}{e^{\xi}-e^s} = \varphi(\xi).$$

The radius of convergence of the series is $\mathrm{dist}(s, 2\pi i\mathbb{Z})$ and φ can be extended to a meromorphic in $\mathbb{C}$ with simple poles at $s + 2\pi i\mathbb{Z}$.

Now we check the exponential growth of φ. First, if we take $\sigma \in (0, -\mathrm{Re}(s))$ and ξ satisfies $\mathrm{Re}(\xi) \geq -\sigma$, then $|\varphi(\xi)| \leq A(\sigma)$, with $A(\sigma) = (1 - e^{\mathrm{Re}(s)+\sigma})^{-1}$.

For $\delta > 0$, let $C_\delta = \{\xi \in \mathbb{C} | \mathrm{dist}(\xi, s + 2\pi i\mathbb{Z}) \geq \delta\}$. Since $|\varphi(\xi)| = e^{\mathrm{Re}(\xi)}/F(\xi)$ where $F(\xi) = |e^{\xi} - e^s|$, and F is $2\pi i-$periodic, positive on C_δ, $F(\xi) \to +\infty$ as $\mathrm{Re}(\xi) \to +\infty$ and $F(\xi) \to |w|$ as $\mathrm{Re}(\xi) \to -\infty$, we can take $R > 0$ with $F(\xi) \geq |w|/2$ if $\mathrm{Re}(\xi) \geq R$. Then,

$$|\varphi(\xi)| \leq Be^{\mathrm{Re}(\xi)},$$

with $B = B(\delta) = \max\{2/|w|, 1/M(\delta)\}$, and $M(\delta) = \inf\{F(\xi) | \xi \in C_\delta, |\mathrm{Re}(\xi)| \leq R, |\mathrm{Im}(\xi)| \leq \pi\}$ is a well-defined positive number, by compactness.

The above considerations show that $\hat{\varphi}$ is $1-$summable in every direction in $(-\pi/2, \pi/2) \cup \bigcup_{k\in\mathbb{Z}}(\arg(\omega_k), \arg(\omega_{k+1}))$, where $\omega_k = s - 2\pi ik$. Since φ has infinitely many poles, $\hat{\varphi}$ is not $1-$summable.

We can see that the $1-$sum of $\hat{\phi}$ in the direction 0 is precisely ϕ. By the general theory we know that this sum is given by

$$\widetilde{\phi}(x) = \frac{1}{x}\int_0^{+\infty} \frac{e^{\xi-\xi/x}}{e^{\xi}-w} d\xi,$$

and it is defined for $\mathrm{Re}(x) > 0$. If we fix $x > 0$ and consider the previous function as a function $g(w)$ of w, g is analytic in $|w| < 1$. Then a calculation shows that $g^{(k)}(0)/k! = \frac{1}{1+kx}$. By Taylor's formula we see that $g(w) = \sum_{k=0}^{\infty} g^{(k)}(0)w^k/k! = \phi(x)$. Since $x > 0$ was arbitrary, it follows by the identity principle that $\widetilde{\phi}(x) = \phi(x)$ for $\mathrm{Re}(x) > 0$.

1.2 Monomial summability

To establish the notion of monomial summability some previous considerations must be made. In particular, we must settle the type of series we will work with as well as the kind of functions and domains that will play the role of its sums. Those are particular goals behind Subsections 1.2.1 and Subsection 1.2.2. After a brief recall of point blow-ups in $\mathbb{C}^2$ the concept of asymptotic expansion and Gevrey asymptotic expansion in a monomial is exposed. Going into detailed proofs we show the compatibility of this notions with the standard algebraic operations and differentiation. Functions with null $s-$Gevrey monomial asymptotic expansion are characterized by having exponential decay of order $1/s$ at the origin in the monomial. Analogous versions to Watson's lemma, Borel-Ritt and Borel-Ritt-Gevrey theorems as well as the Ramis-Sibuya theorem are also included. With the previous tools the definition of monomial summability and its properties are achieved. The section ends with different ways to calculate these sums, all proposed in Subsection 1.2.4. We note that although it is possible to develop the theory for power series with coefficients in an arbitrary complex Banach space E as in [CDMS], we have opted by restrict our attention to the case $E=\mathbb{C}$.

1.2.1 Formal setting

We will denote by $\hat{R}=\mathbb{C}[[x,\varepsilon]]$ the $\mathbb{C}-$algebra of formal power series in the variables x,ε and by $R=\mathbb{C}\{x,\varepsilon\}$ the algebra of germs of analytic functions at the origin of $\mathbb{C}^2$, i.e., the algebra of convergent power series. Both become differential rings with subring of constants $\mathbb{C}$, when considering the usual derivations $\partial/\partial x$ and $\partial/\partial\varepsilon$. We start by recalling some of its subalgebras that will play an important role on the notes. We remark that the following definitions can be carried on over the formal power series in any number of variables but we restrict ourselves to the case of just two since it is the only one we will use here.

Definition 1.2.1. Let s_1,s_2 non-negative real numbers. A formal power series $\hat{f}\in\hat{R}$ is said to be $(s_1,s_2)-$*Gevrey* if we can find constants C,A such that if $\hat{f}=\sum a_{n,m}x^n\varepsilon^m$ then

$$|a_{n,m}|\leq CA^{n+m}n!^{s_1}m!^{s_2},$$

for all $n,m\in\mathbb{N}^*$. The set of $(s_1,s_2)-$Gevrey formal power series will be denoted by $\hat{R}_{(s_1,s_2)}$.

It is straightforward to check that $\hat{R}_{(s_1,s_2)}$ is a differential subalgebra of $\hat{R}$. It is also closed by composition. Note in particular that $\hat{R}_{(0,0)}=\mathbb{C}\{x,\varepsilon\}$. Other property we note is the following contention:

$$\hat{R}_{(s_1,s_2)}\cap\hat{R}_{(s_1',s_2')}\subseteq\hat{R}_{(s_1'',s_2'')},\tag{1-3}$$

valid for every $(s_1'',s_2'')\in\mathbb{R}^2$ on the segment joining (s_1,s_2) with (s_1',s_2'). This follows from the inequality

$$\min\{a,b\}\leq a^tb^{1-t}\leq\max\{a,b\},\tag{1-4}$$

valid for any $a,b>0$ and $0\leq t\leq 1$.

Another set that will be used in the text is the union of subalgebras $\mathcal{S} = \bigcup_{r>0} \mathcal{S}_r$, where $\mathcal{S}_r = \mathcal{O}_b(D_r)[[x]] \cap \mathcal{O}_b(D_r)[[\varepsilon]]$. Then $\hat{f} \in \mathcal{S}$ if and only if when we write $\hat{f} = \sum_{n=0}^{\infty} f_{n*}(\varepsilon)x^n = \sum_{n=0}^{\infty} f_{*n}(x)\varepsilon^n$, all the f_{n*} and f_{*n} have a common radius of convergence and are bounded. In particular $\mathbb{C}\{x,\varepsilon\} \subset \mathcal{S}$.

Our main interest in this chapter is to present a theory of summability in a monomial. The goal is to sum some type of series that in some sense have a divergence in dimension one "parameterized" analytically (by a monomial). We first consider the monomial $x\varepsilon$, for which the results are easier to write, and then we move on to a general monomial based on the previous results.

Given $\hat{f} \in \mathcal{S}$, using the monomial $x\varepsilon$ and a recursive division process w.r.t. $x\varepsilon$ by ordering the terms of $\hat{f}$ by total degree, we can write it uniquely as

$$\hat{f}(x,\varepsilon) = \sum_{n=0}^{\infty} (b_n(x) + c_n(\varepsilon))(x\varepsilon)^n,$$

where $b_n, c_n \in \mathcal{O}_b(D_r)$ for some $r > 0$ and $c_n(0) = 0$ for all $n \in \mathbb{N}$. More explicitly, if $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$, then $b_n(x) = \sum_{m=0}^{\infty} a_{n+m,n}x^m$ and $c_n(\varepsilon) = \sum_{m=1}^{\infty} a_{n,n+m}\varepsilon^m$. Of course, this formulas also hold for any element of $\hat{R}$ but with b_n and c_n just formal power series. The hypothesis of $\hat{f} \in \mathcal{S}$ is a necessary and sufficient condition to ensure that $b_n, c_n \in \mathcal{O}_b(D_r)$ for some $r > 0$. We note, for future purposes, that

$$f_{n*}(\varepsilon) = \sum_{m=0}^{n} a_{n,m}\varepsilon^m + \varepsilon^n c_n(\varepsilon), \qquad f_{*m}(x) = \sum_{n=0}^{m-1} a_{n,m}x^n + x^m b_m(\varepsilon). \tag{1-5}$$

The previous process allow us to define the map $\hat{T} : \mathcal{S} \to \mathcal{E}[[t]]$, $\hat{T}(\hat{f}) = \sum (b_n(x) + c_n(\varepsilon))t^n$, where $\mathcal{E}$ is the union of the following spaces of analytic functions

$$\mathcal{E} = \bigcup_{r>0} \mathcal{E}_r, \quad \mathcal{E}_r = \{b(x) + c(\varepsilon) \mid b, c \in \mathcal{O}_b(D_r) \text{ and } c(0) = 0\}.$$

Note that every $\mathcal{E}_r$ becomes a Banach space with the supremum norm but unfortunately it is not a Banach algebra with the usual product (it is not closed under this operation). For every $r > 0$, $\hat{T}|_{\mathcal{S}_r}$ is an isomorphism of vector spaces between $\mathcal{S}_r$ and $\mathcal{E}_r[[t]]$. We also remark that $\hat{f} \in \mathbb{C}\{x,\varepsilon\}$ if and only if $\hat{T}(\hat{f}) \in \mathcal{E}\{t\}$.

Definition 1.2.2. Let s be a non-negative real number. A formal power series $\hat{f} \in \mathcal{S}$ is said to be $s-$*Gevrey in the monomial* $x\varepsilon$ if for some $r > 0$, $\hat{T}(\hat{f}) \in \mathcal{E}_r[[t]]$ and it is a $s-$Gevrey series in t. The set of $s-$Gevrey series in the monomial $x\varepsilon$ will be denoted by $\hat{R}_s^{(1,1)} = \mathbb{C}[[x,\varepsilon]]_s^{(1,1)}$.

According to the previous definition we see that $\hat{T}$ maps $\hat{R}_s^{(1,1)}$ into $\mathcal{E}[[t]]_s = \bigcup_{r>0} \mathcal{E}_r[[t]]_s$. We can characterize more explicitly, in terms of the coefficients of a series, the fact of being $s-$Gevrey in the monomial $x\varepsilon$, as it is shown in the next proposition.

Proposition 1.2.1. *Let s be a non-negative real number. For a series $\hat{f}\in\hat{R}$ it is equivalent:*

1. $\hat{f}\in\hat{R}_s^{(1,1)}$,
2. $\hat{f}\in\hat{R}_{(s,0)}\cap\hat{R}_{(0,s)}$,
3. *If* $\hat{f}=\sum a_{n,m}x^n\varepsilon^m$, *then* $|a_{n,m}|\leq CA^{n+m}\min\{n!^s,m!^s\}$ *for some constants* C,A, *for all* $n,m\in\mathbb{N}^*$.

The proof uses Cauchy's formulas and it is left to the reader. Using item (2) of the previous proposition the following statement is clear.

Proposition 1.2.2. *For any* $s\geq 0$, $\hat{R}_s^{(1,1)}$ *is a differential subalgebra of* $\hat{R}$.

We can perform similar constructions replacing $x\varepsilon$ by any other monomial $x^p\varepsilon^q$, with $p,q\in\mathbb{N}^*$. Starting with $\hat{f}\in\mathcal{S}$ and using successive divisions by $x^p\varepsilon^q$ (or equivalently using the filtration of $\hat{R}$ by the sequence of ideals $(x^p\varepsilon^q)^k, k\in\mathbb{N}$), we can write it uniquely as

$$\hat{f}=\sum_{n=0}^{\infty}f_n(x,\varepsilon)(x^p\varepsilon^q)^n,$$

where $f_n(x,\varepsilon)\in\mathcal{E}^{(p,q)}$, and $\mathcal{E}^{(p,q)}$ is the following union of spaces of analytic functions

$$\mathcal{E}^{(p,q)}=\bigcup_{r>0}\mathcal{E}_r^{(p,q)},$$
$$\mathcal{E}_r^{(p,q)}=\left\{\sum_{l=0}^{q-1}\varepsilon^l h_l(x)+\sum_{m=0}^{p-1}x^m g_m(\varepsilon)\;\middle|\; h_l,g_m\in\mathcal{O}_b(D_r),\ g_m^{(j)}(0)=0, 0\leq m<p, 0\leq j<q\right\}.$$

Note that $\mathcal{E}^{(1,1)}=\mathcal{E}$. As before we take only elements in $\mathcal{S}$ to ensure that for some $r>0$, every f_n belongs to $\mathcal{O}_b(D_r)$. Each $\mathcal{E}_r^{(p,q)}$ becomes a Banach space with the supremum norm $\|f\|_r=\sup_{|x|,|\varepsilon|<r}|f(x,\varepsilon)|$ (again it is not a Banach algebra with the usual product).

The previous expressions, if $\hat{f}=\sum a_{n,m}x^n\varepsilon^m$, are given explicitly by:

$$f_n(x,\varepsilon)=\sum_{j=0}^{q-1}\left(\sum_{m=0}^{\infty}a_{np+m,nq+j}x^m\right)\varepsilon^j+\sum_{m=0}^{p-1}\left(\sum_{j=q}^{\infty}a_{np+m,nq+j}\varepsilon^j\right)x^m. \tag{1-6}$$

We define the map $\hat{T}_{p,q}:\mathcal{S}\to\mathcal{E}^{(p,q)}[[t]]$, as $\hat{T}_{p,q}(\hat{f})=\sum_{n=0}^{\infty}f_n(x,\varepsilon)t^n$. As before, for every $r>0$, $\hat{T}_{p,q}|_{\mathcal{S}_r}$ is an isomorphism of vector spaces between $\mathcal{S}_r$ and $\mathcal{E}_r^{(p,q)}[[t]]$. Also $\hat{T}_{1,1}=\hat{T}$.

Definition 1.2.3. Let s be a non-negative real number. A formal power series $\hat{f} \in \mathcal{S}$ is said to be *$s-$Gevrey in the monomial* $x^p\varepsilon^q$ if for some $r > 0$, $\hat{T}_{p,q}(\hat{f}) \in \mathcal{E}_r^{(p,q)}[[t]]$ and it is a $s-$Gevrey series in t. The set of $s-$Gevrey series in the monomial $x^p\varepsilon^q$ will be denoted by $\hat{R}_s^{(p,q)} = \mathbb{C}[[x,\varepsilon]]_s^{(p,q)}$.

The analogous version of Proposition 1.2.1 for the monomial $x^p\varepsilon^q$ reads as follows.

Proposition 1.2.3. *Let $s \geq 0$ a non-negative real number. For a series $\hat{f} \in \hat{R}$ the following assertions are equivalent:*

1. $\hat{f} \in \hat{R}_s^{(p,q)}$,
2. $\hat{f} \in \hat{R}_{(s/p,0)} \cap \hat{R}_{(0,s/q)}$,
3. *If $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$ then $|a_{n,m}| \leq CA^{n+m}\min\{n!^{s/p}, m!^{s/q}\}$ for some constants C, A, for all $n, m \in \mathbb{N}^*$.*

Proof. Statements (2) and (3) are clearly equivalent. It only remains to proof the equivalence between (1) and (3). Suppose first that $\hat{f} = \sum a_{n,m}x^n\varepsilon^m \in \hat{R}_s^{(p,q)}$. Since $\hat{T}_{p,q}(\hat{f})$ is a s-Gevrey series there are constants C, A with $\|f_n\|_r \leq CA^n n!^s$, for some $r > 0$ and for all n. Using this bound together with expressions (1-6) and Cauchy's formulas it follows that there are constants D, B with

$$|a_{np+k,nq+j}| \leq DB^{np+k+nq+j}n!^s,$$

for all $n, k, j \in \mathbb{N}$. Since $n! \leq (np)!^{1/p} \leq (np+k)!^{1/p}$ and $n! \leq (nq)!^{1/q} \leq (nq+j)!^{1/q}$ we get $|a_{np+k,nq+j}| \leq DB^{np+k+nq+j}\min\{(np+k)!^{s/p}, (nq+j)!^{s/q}\}$ as desired.

Conversely, suppose that the coefficients of $\hat{f}$ satisfies $|a_{n,m}| \leq CA^{n+m}\min\{n!^{s/p}, m!^{s/q}\}$ for some C, A. We can directly estimate the growth of the f_n by means of the expression (1-6): if $|x|, |\varepsilon| < r$ and $rA < 1$ we get

$$|f_n(x,\varepsilon)| \leq \sum_{j=0}^{q-1} r^j CA^{np+nq+j}\frac{(nq+j)!^{s/q}}{1-rA} + \sum_{m=0}^{p-1} r^m CA^{np+nq+m}\frac{(rA)^q(np+m)!^{s/p}}{1-rA}.$$

By Stirling's formula we know that $\lim_{n\to+\infty}\frac{(np)!^{1/p}}{p^n n!} = 0$ for any natural number $p \geq 2$. We can conclude that there are $K, B > 0$ such that $|f_n(x,\varepsilon)| \leq KB^n n!^s$ for all $|x|, |\varepsilon| < r$, as we wanted to prove. □

It particular, by taking $s = 0$, the above proposition tell us that $\hat{f} \in \mathbb{C}\{x,\varepsilon\}$ if and only if $\hat{T}_{p,q}(\hat{f}) \in \mathcal{E}^{(p,q)}\{t\}$. Another important property, that follows from item (2), is the following.

Proposition 1.2.4. *For any $s \geq 0$, $\hat{R}_s^{(p,q)}$ is a subalgebra of $\hat{R}$.*

Proposition 1.2.3 also let us relate the Gevrey type in a monomial in terms of another monomial. More concretely we have the following assertion.

Corollary 1.2.5. *If* $\hat{f} \in \hat{R}_s^{(p',q')}$ *then* $\hat{T}_{p,q}(\hat{f})$ *is a* $\max\{p/p', q/q'\}s-$*Gevrey series.*

Proof. If we write $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$, we can take constants $C, A > 0$ such that $|a_{n,m}| \leq CA^{n+m}\min\{n!^{1/p'}, m!^{1/q'}\}^s$ for all $n, m \in \mathbb{N}$. Similar calculations as in the proof of Proposition 1.2.3 let us conclude that for $|x|, |\varepsilon| \leq r < 1/A$ there are constants $K, B >$ such that for all $n \in \mathbb{N}$

$$\sup_{|x|,|\varepsilon|\leq r} |f_n(x,\varepsilon)| \leq KB^n n!^{\max\{p/p',q/q'\}s},$$

as we wanted to prove. □

Remark 1.2.6. Fix $p, q \in \mathbb{N}^*$. Then any formal power series $\hat{f} \in \hat{R}$ can be written uniquely as

$$\hat{f}(x,\varepsilon) = \sum_{\substack{0\leq i<p\\0\leq j<q}} x^i\varepsilon^j\hat{f}_{i,j}(x^p,\varepsilon^q). \tag{1-7}$$

We will say eventually that $\hat{f}_{ij}$ *is the* $(i,j)-$*component of* $\hat{f}$ *in the decomposition* (1-7). Explicitly if $\hat{f} = \sum_{n,m\geq 0} a_{n,m}x^n\varepsilon^m$ then $\hat{f}_{ij}(x^p,\varepsilon^q) = \sum_{k,r\geq 0} a_{kp+i,rq+j}x^{pk}\varepsilon^{qr}$. But we can also determine the $\hat{f}_{i,j}$ by solving the system of linear equations

$$\hat{f}(\omega^m x, \nu^l\varepsilon) = \sum_{\substack{0\leq i<p\\0\leq j<q}} \omega^{mi}\nu^{lj}x^i\varepsilon^j\hat{f}_{i,j}(x^p,\varepsilon^q),$$

where ω, ν are primitive pth and qth roots of unity, respectively, and $m = 0, 1, ..., p-1, l = 0, 1, ..., q-1$. Indeed we can obtain the expressions

$$x^i\varepsilon^j\hat{f}_{i,j}(x^p,\varepsilon^q) = \frac{1}{pq}\sum_{\substack{0\leq m<p\\0\leq l<q}} \omega^{m(p-i)}\nu^{l(q-j)}\hat{f}(\omega^m x, \nu^l\varepsilon). \tag{1-8}$$

Note that the coefficients $\Lambda = (\omega^{mi}\nu^{lj}) \in \mathrm{GL}(pq,\mathbb{C})$ in the above expression are independent of the chosen $\hat{f}$.

Finally from the previous decomposition of $\hat{f}$ we obtain the formula

$$\hat{T}_{p,q}(\hat{f})(t)(x,\varepsilon) = \sum_{\substack{0\leq i<p\\0\leq j<q}} x^i\varepsilon^j\hat{T}_{1,1}(\hat{f}_{ij})(t)(x^p,\varepsilon^q), \tag{1-9}$$

where the maps $\hat{T}_{1,1}$ are taking in the variables $\zeta = x^p, \eta = \varepsilon^q$.

We now state a lemma that let us relate the fact of being $s-$Gevrey in the monomial $x^p\varepsilon^q$ with being $s-$Gevrey in $\zeta\eta$, $\zeta = x^p, \eta = \varepsilon^q$. The proof follows using bounds similar to the ones used in the proof of Proposition 1.2.3 and it is left to the reader.

Lemma 1.2.7. *Let $s \geq 0$ be a non-negative number and $p, q \in \mathbb{N}^*$. Let $\hat{f} \in \hat{R}$ and write it uniquely as $\hat{f}(x,\varepsilon) = \sum_{\substack{0\leq i<p\\ 0\leq j<q}} x^i\varepsilon^j \hat{f}_{i,j}(x^p,\varepsilon^q)$. Then $\hat{f}(x,\varepsilon) \in \hat{R}_s^{(p,q)}$ if and only if $\hat{f}_{ij}(\zeta,\eta) \in \mathbb{C}[[\zeta,\eta]]_s^{(1,1)}$ for all $i = 0, 1, ..., p-1, j = 0, 1, ..., q-1$.*

To finish this section we point out some properties of series obtained from weighting the variables x and ε and that will be essential in our treatment of tauberian properties of monomial summability in this chapter. Consider real parameters $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and a new variable $z \in \mathbb{C}$. It induces the morphism of $\mathbb{C}-$algebras

$$\begin{aligned}\iota_G : \mathbb{C}[[x,\varepsilon]] &\longrightarrow \mathbb{C}[x,\varepsilon][[z^G]],\\ \hat{f}(x,\varepsilon) &\longmapsto \iota_G(\hat{f})(x,\varepsilon)(z) = \hat{f}(z^{s_1/p}x, z^{s_2/q}\varepsilon),\end{aligned}$$

where $G = G_{s_1,s_2}^{(p,q)} := \{ns_1/p + ms_2/q \mid n, m \in \mathbb{N}\}$ denotes the discrete semigroup of $\mathbb{R}_{\geq 0}$ generated by s_1/p and s_2/q. G is the image of $\mathbb{N}^2$ by the map $\ell(n,m) = ns_1/p + ms_2/q$. We remark that ℓ is injective if and only if $s_1/s_2 \notin \mathbb{Q}$ if and only if $s_1 \notin \mathbb{Q}$.

We can find the Gevrey nature of $\iota_G(\hat{f})(x,\varepsilon)$ from the corresponding of $\hat{f}$. As it is expected, if $\hat{f}$ is a $s-$Gevrey series in $x^p\varepsilon^q$, then $\iota_G(\hat{f})(x,\varepsilon)$ is $s-$Gevrey in z. More generally, we have an analogous result of Corollary 1.2.5, relating two monomials.

Proposition 1.2.8. *Let s_1, s_2 be positive real numbers such that $s_1 + s_2 = 1$ and let $G = G_{s_1,s_2}^{(p,q)}$. If $\hat{f} \in \hat{R}_s^{(p',q')}$ then for every $(x,\varepsilon) \in \mathbb{C}^2$, $\iota_G(\hat{f})(x,\varepsilon)(z)$ is a $\max\{p/p', q/q'\}s-$Gevrey series in z.*

Proof. Let $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$ be a $s-$Gevrey series in $x^{p'}\varepsilon^{q'}$. Then there are constants $\widetilde{B}, \widetilde{D} \geq 0$ such that $|a_{n,m}| \leq \widetilde{D}\widetilde{B}^{n+m}\min\{n!^{s/p'}, m!^{s/q'}\}$ for all n, m. By (1-4) and Stirling's formula we know that

$$\begin{aligned}|a_{n,m}| &\leq \widetilde{D}\widetilde{B}^{n+m} n!^{s_1 s/p'} m!^{s_2 s/q'}\\ &\leq \widetilde{D}\widetilde{B}^{n+m} n!^{s_1 s\max\{p/p',q/q'\}/p} m!^{s_2 s\max\{p/p',q/q'\}/q}\\ &\leq DB^{n+m}\Gamma\left(1+\frac{ns_1}{p}s\max\{p/p',q/q'\}\right)\Gamma\left(1+\frac{ms_2}{q}s\max\{p/p',q/q'\}\right)\\ &\leq DB^{n+m}\Gamma\left(1+s\max\{p/p',q/q'\}\left(ns_1/p+ms_2/q\right)\right),\end{aligned}$$

for some constants $B, D > 0$. This implies that for every fixed $(x,\varepsilon) \in \mathbb{C}^2$, $\iota_G(\hat{f})(x,\varepsilon)$ is $s\max\{p/p', q/q'\}$ in z.

□

The previous considerations show in particular that if $\hat{f}$ is convergent then $\iota_G(\hat{f})(x,\varepsilon)$ is convergent, for all $(x,\varepsilon) \in \mathbb{C}^2$. Conversely, the problem of establishing sufficient conditions to ensure the convergence of $\hat{f}$ from the convergence of the series $\iota_G(\hat{f})(x,\varepsilon)$, for (x,ε) in an adequate set can be seen as a particular case of the following more general problem: given a series $\hat{f}(x_1,...,x_n,x) = \sum_{m=0}^{\infty} P_m(x_1,...,x_n)x^m \in \mathbb{C}[x_1,...,x_n][[x]]$, where $\deg(P_m) \leq Am+B$, for some $A > 0, B \geq 0$, establish conditions on a set $C \subset \mathbb{C}^n$ to prove the convergence of $\hat{f}$ from the convergence of $\hat{f}(a_1,...,a_n,x)$ for all $(a_1,...,a_n) \in C$. If C is open or of positive Lebesgue measure or non pluri-polar (in the sense of potential theory) then the answer is positive. For more information, see [Ri]. For our purposes, the following proposition will suffice.

Proposition 1.2.9. *Let $\hat{f} \in \hat{R}$ be a formal power series, s_1, s_2 positive real numbers such that $s_1 + s_2 = 1$ and $p, q \in \mathbb{N}^*$. Also set $G = G_{s_1,s_2}^{(p,q)}$. If there is an open set $U \subset \mathbb{C}^2$ such that $\iota_G(\hat{f})(x,\varepsilon) \in \mathbb{C}\{z^G\}$ for all $(x,\varepsilon) \in \overline{U}$ then $\hat{f} \in R$.*

Proof. If $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$ then $\iota_G(\hat{f})(x,\varepsilon)(z) = \sum_{\lambda\in G} P_\lambda(x,\varepsilon)z^\lambda$, where

$$P_\lambda(x,\varepsilon) = \sum_{\ell(n,m)=\lambda} a_{n,m}x^n\varepsilon^m.$$

We consider first the case where $s_1 \notin \mathbb{Q}$, i.e. ℓ is injective. Then every P_λ has only one summand. If we take $(x_0,\varepsilon_0) \in U$, $x_0 \neq 0, \varepsilon_0 \neq 0$, the convergence of $\iota_G(\hat{f})(x_0,\varepsilon_0)$ means that there are constants $C, A > 0$ such that $|a_{n,m}x_0^n\varepsilon_0^m| < CA^{ns_1/p+ms_2/q}$ for all $n, m \in \mathbb{N}$. Then it is clear that $\hat{f}$ converges.

For the case $s_1 \in \mathbb{Q}$ we need to find uniform bounds for the P_λ in some open set. This can be done as follows: we know that for every $(x,\varepsilon) \in \overline{U}$ there are constants $C_{(x,\varepsilon)}, A_{(x,\varepsilon)} > 0$ such that $|P_\lambda(x,\varepsilon)| \leq C_{(x,\varepsilon)}A_{(x,\varepsilon)}^\lambda$ for all $\lambda \in G$. This implies that the closed sets

$$F_N = \left\{(x,\varepsilon) \in \overline{U} \mid |P_\lambda(x,\varepsilon)| \leq N^\lambda \text{ for all } \lambda \in G\right\},$$

where $N \in \mathbb{N}$, cover $\overline{U}$. By Baire's category theorem, at least one of these closed sets has non-empty interior.

In consequence, we can take $K \in \mathbb{N}$, $(x_0,\varepsilon_0) \in U$, and $r > 0$ with $D_r(x_0) \times D_r(\varepsilon_0) \subset U$ such that $|P_\lambda(x,\varepsilon)| \leq K^\lambda$ for all $(x,\varepsilon) \in D_r(x_0) \times D_r(\varepsilon_0)$ and for all $\lambda \in G$. Since r can be arbitrarily small we may suppose that $r < 2|x_0|, 2|\varepsilon_0|$.

Fix $\lambda \in G$. If we expand P_λ around (x_0, ε_0), say $P_\lambda(x,\varepsilon) = \sum_{n,m=0}^{N,M} b_{n,m}(x-x_0)^n(\varepsilon-\varepsilon_0)^m$, it follows from Cauchy's formulas that $|b_{n,m}| \leq K^\lambda/r^{n+m}$ for all $0 \leq n \leq N$ and $0 \leq m \leq M$, where $N = \max\{n \in \mathbb{N} | \ell(n,m) = \lambda \text{ for some } m\}$ and $M = \max\{m \in \mathbb{N} | \ell(n,m) = \lambda \text{ for some } n\}$ (depending on λ).

Finally we can bound the coefficients $a_{n,m}$ as follows: take any $(n,m) \in \mathbb{N}^2$ and let $\lambda = \ell(n,m)$. Then

$$\begin{aligned}|a_{n,m}| &= \left| \sum_{k=0,l=0}^{N-n,M-m} \binom{n+k}{n}\binom{m+l}{m} b_{n+k,m+l}(-x_0)^k(-\varepsilon_0)^l \right| \\ &\leq \sum_{k=0,l=0}^{N-n,M-m} 2^{n+k+m+l} \frac{K^\lambda}{r^{n+k+m+l}} |x_0|^k |\varepsilon_0|^l \\ &= K^{\ell(n,m)} \left(\frac{2}{r}\right)^{n+m} \left(\frac{(2|x_0|/r)^{N-n+1}-1}{2|x_0|/r-1}\right)\left(\frac{(2|\varepsilon_0|/r)^{M-m+1}-1}{2|\varepsilon_0|/r-1}\right).\end{aligned}$$

Since $\ell(n,m) = \ell(N,m') = \ell(n',M)$, for some $n', m' \in \mathbb{N}$ then $N - n \leq m(s_2p/s_1q)$ and $M - m \leq n(s_1q/s_2p)$. This let us conclude that there are large enough constants B, D such that $|a_{n,m}| \leq DB^{n+m}$ for all $n, m \in \mathbb{N}$, so $\hat{f}$ is convergent.

□

1.2.2 Analytic setting

To be able to sum series of $\mathcal{S}$ we need to determine the domains where the sum will be defined. A relevant way in which the monomial plays a predominant role in the analytic context is that the domains will be *sectors in the monomial.* We first analyze how to work with analytic functions defined over these domains using point blow-ups. Then the notion of asymptotic expansion and Gevrey asymptotic expansion in the monomial $x\varepsilon$ is introduced. We provide two different characterizations of these notions, one using the change of variables $t = x\varepsilon$ and passing to the one variable case, with power series in an adequate complex Banach space and other using approximations with analytic functions. After establishing the compatibility of the monomial asymptotic expansions with the standard algebraic operations and differentiation we carry on again the definition and the mentioned results with an arbitrary monomial $x^p\varepsilon^q$. The subsection ends formulating the similar versions of Watson's lemma, Borel-Ritt, Borel-Ritt-Gevrey and Ramis-Sibuya theorems.

Definition 1.2.4. Fix $p, q \in \mathbb{N}^*$. We will call a *sector in the monomial* $x^p\varepsilon^q$ a set defined as

$$\Pi_{p,q} = \Pi_{p,q}(a,b,r) = \left\{(x,\varepsilon) \in \mathbb{C}^2 \mid 0 < |x|^p < r, 0 < |\varepsilon|^q < r,\ a < \arg(x^p\varepsilon^q) < b\right\},$$

where $a, b \in \mathbb{R}$ with $a < b$ and $r > 0$. The number r is called the *radius*, $b - a$ the *opening* and $(b+a)/2$ the *bisecting direction* of the sector, respectively. Occasionally we will use the notation $\Pi(a', b', r') \Subset \Pi(a, b, r)$ to indicate that $\Pi(a', b', r')$ is a *subsector* of $\Pi(a, b, r)$, that is, if $a < a' < b' < b$ and $0 < r' < r$.

We will also use the notation $S_{p,q}(d, \alpha, r) = \Pi_{p,q}(d - \alpha/2, d + \alpha/2, r)$ to denote the sector in the monomial $x^p\varepsilon^q$ with bisecting direction d, opening α and radius r.

Observe that if $(x, \varepsilon) \in \Pi_{p,q}(a, b, r)$ then $t = x^p\varepsilon^q \in V(a, b, r^2)$. Also for any $a' < b'$ and $a'' < b''$ with $a < pa' + qa'' < pb' + qb'' < b$ we see that $V(a', b', r^{1/p}) \times V(a'', b'', r^{1/q}) \subset \Pi_{p,q}(a, b, r)$.

As in the formal setting we focus first in the case of $p = q = 1$. To take care of analytic functions on a sector in the monomial we may use the charts of the classical blow-up at the origin in $\mathbb{C}^2$. For sake of completeness we recall this notion, of common usage in algebraic geometry.

Consider the point $P = (0, 0) \in \mathbb{C}^2$, and let E_P the following variety

$$E_P = \{((x_1, x_2), [y_1, y_2]) \in \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}} \mid x_1y_2 = x_2y_1\},$$

with the projection $\pi : E_P \to \mathbb{C}^2$ over the first coordinate. Let us observe that if $(x_1, x_2) \in \mathbb{C}^2$ then

$$\pi^{-1}(x_1, x_2) = \begin{cases} ((x_1, x_2), [x_1, x_2]) & \text{if } \ (x_1, x_2) \neq P, \\ \{(0, 0)\} \times \mathbb{P}^1_{\mathbb{C}} & \text{if } \ (x_1, x_2) = P. \end{cases}$$

(E_P, π) is called the *blow-up of the origin in* $\mathbb{C}^2$: the origin has been removed and replaced by a projective line. Each pair of this projective line corresponds to a direction from P. Indeed, consider the straight line $L = \{(\lambda x_1, \lambda x_2) \in \mathbb{C}^2 | \lambda \in \mathbb{C}\}$. As $\pi^{-1}(\lambda x_1, \lambda x_2) = ((\lambda x_1, \lambda x_2), [x_1, x_2])$ if $\lambda \neq 0$, $\overline{\pi^{-1}(L \setminus P)}$ cuts the projective line in $((0, 0), [x_1, x_2])$. This projective line will be called the *exceptional divisor*.

E_P is a bidimensional variety covered by two charts, that we shall describe. Consider $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}_1 \cup \mathbb{C}_2$, where $\mathbb{C}_1 = \{[y_1, 1] \in \mathbb{P}^1_{\mathbb{C}} \mid y_1 \in \mathbb{C}\}$ and $\mathbb{C}_2 = \{[1, y_2] \in \mathbb{P}^1_{\mathbb{C}} \mid y_2 \in \mathbb{C}\}$. Then

$$\begin{aligned} E_P \cap (\mathbb{C}^2 \times \mathbb{C}_1) &= \{((x_2y_1, x_2), [y_1, 1]) \in E_P \mid (x_2, y_1) \in \mathbb{C}^2\}, \\ E_P \cap (\mathbb{C}^2 \times \mathbb{C}_2) &= \{((x_1, x_1y_2), [1, y_2]) \in E_P \mid (x_1, y_2) \in \mathbb{C}^2\}, \end{aligned}$$

both parameterized by $\mathbb{C}^2$. The projection π, in this charts is represented as

$$\begin{aligned} \pi(x_2, y_1) &= (x_2y_1, x_2) \quad \text{in the first chart,} \\ \pi(x_1, y_2) &= (x_1, x_1y_2) \quad \text{in the second chart.} \end{aligned}$$

For the sake of simplifying notation, since in our case the coordinates are (x,ε), we shall consider the two charts of the blow-up with projections $\pi_1(x,\varepsilon)=(x\varepsilon,\varepsilon), \pi_2(x,\varepsilon)=(x,x\varepsilon)$. Then for a sector $\Pi(a,b,r)=\Pi_{1,1}(a,b,r)$ we see that

$$\pi_1(\Pi(a,b,r)) = \left\{(t,\varepsilon)\in\mathbb{C}^2 \mid 0<|t|<r^2, \frac{|t|}{r}<|\varepsilon|<r \text{ and } a<\arg(t)<b\right\},$$
$$\pi_2(\Pi(a,b,r)) = \left\{(x,t)\in\mathbb{C}^2 \mid 0<|t|<r^2, \frac{|t|}{r}<|x|<r \text{ and } a<\arg(t)<b\right\}.$$

We will write the results only for $\pi_1=\pi$. Analogous considerations follow for π_2 due to the symmetric role between x and ε above. Let $f\in\mathcal{O}(\Pi(a,b,r))$ be an analytic function. It induces an analytic function on $\pi(\Pi(a,b,r))$ given by $(t,\varepsilon)\longmapsto f(t/\varepsilon,\varepsilon)$. For fixed t with $0<|t|<r^2$ the function $\varepsilon\longmapsto f(t/\varepsilon,\varepsilon)$ is analytic and single-valued in the annulus $\frac{|t|}{r}<|\varepsilon|<r$ and thus it has a convergent Laurent series expansion on ε:

$$f\left(\frac{t}{\varepsilon},\varepsilon\right)=\sum_{n\in\mathbb{Z}} f_n(t)\varepsilon^n, \tag{1-10}$$

where the functions f_n are given by:

$$f_n(t)=\frac{1}{2\pi i}\int_{|\omega|=\rho}\frac{f\left(\frac{t}{\omega},\omega\right)}{\omega^{n+1}}d\omega, \tag{1-11}$$

for $|t|/r<\rho<r$. In particular $f_n\in\mathcal{O}(V(a,b,r^2))$ and its derivative is given by

$$f_n'(t)=\frac{1}{2\pi i}\int_{|\omega|=\rho}\frac{\frac{\partial f}{\partial x}\left(\frac{t}{\omega},\omega\right)}{\omega^{n+2}}d\omega. \tag{1-12}$$

We can relate the growth order of f with the growth of the f_n by using formula (1-11). More precisely we have the following proposition.

Proposition 1.2.10. *Let $f\in\mathcal{O}(\Pi(a,b,r))$ be an analytic function. Suppose that $|f(x,\varepsilon)|\leq K(|x\varepsilon|)$ for any $(x,\varepsilon)\in\Pi(a',b',\rho)\Subset\Pi(a,b,r)$ and some function $K:(0,\rho^2)\to\mathbb{R}_{>0}$. Then for $t\in V(a',b',\rho^2)$:*

1. *If $n\in\mathbb{N}$, $|f_n(t)|\leq\dfrac{K(|t|)}{\rho^n}$,*
2. *If $n\geq 1$, $|f_{-n}(t)|\leq\dfrac{|t|^nK(|t|)}{\rho^n}$.*

In particular, if $r=\rho=+\infty$ then $f(x,\varepsilon)=f_0(x\varepsilon)$, for all $(x,\varepsilon)\in\Pi(a',b',+\infty)$.

Proof. The inequalities follow directly from Cauchy's formulas. To prove the last part, note that fixing $t\in V$, the inequalities are valid for any $\rho>|t|^{1/2}$. Letting $\rho\to+\infty$ we conclude that $f_n(t)\equiv 0$ for all $n\neq 0$. The result follows from equality (1-10).

□

With the previous considerations we are ready to introduce the notion of asymptotic expansion in the monomial $x\varepsilon$.

Definition 1.2.5. Let f be an analytic function on $\Pi = \Pi(a,b,r)$ and $\hat{f} \in \hat{R}$. We will say that f *has* $\hat{f}$ *as asymptotic expansion at the origin in* $x\varepsilon$ and we will use the notation $f \sim^{(1,1)} \hat{f}$ on $\Pi(a,b,r)$ if: there exists $0 < r' \leq r$ such that $\hat{T}\hat{f} = \sum(b_n + c_n)t^n \in \mathcal{E}_{r'}[[t]]$ and for every $\widetilde{\Pi} = \Pi(a',b',\rho) \Subset \Pi$ with $0 < \rho < r'$ and $N \in \mathbb{N}$ there is a constant $C_N(\widetilde{\Pi}) > 0$ such that for all $(x,\varepsilon) \in \widetilde{\Pi}$:

$$\left| f(x,\varepsilon) - \sum_{n=0}^{N-1} (b_n(x) + c_n(\varepsilon))(x\varepsilon)^n \right| \leq C_N(\widetilde{\Pi})|x\varepsilon|^N. \tag{1-13}$$

The asymptotic expansion is said to be of $s-$*Gevrey type* if we can find $C(\widetilde{\Pi}), A(\widetilde{\Pi})$ independent of N such that $C_N(\widetilde{\Pi}) = C(\widetilde{\Pi})A(\widetilde{\Pi})^N N!^s$. In this case we will use the notation: $f \sim_s^{(1,1)} \hat{f}$ on $\Pi(a,b,r)$. We will denote by $\mathcal{A}^{(1,1)}(\Pi)$ the set of analytic functions defined on Π that admits an asymptotic expansion in the monomial $x\varepsilon$ on Π and by $\mathcal{A}_s^{(1,1)}(\Pi)$ the set of analytic functions defined on Π that admits an asymptotic expansion of $s-$Gevrey type in the monomial $x\varepsilon$ on Π.

We note that we are only using formal series in $\mathcal{S}$. It follows from Proposition 1.2.10 that if $f \sim^{(1,1)} \hat{f}$ on $\Pi(a,b,r)$ then every $f_m(t)$ and $f_{-m}(t)/t^m$, $m \in \mathbb{N}$, associated with f by formula (1-11), admits an asymptotic expansion on $V(a,b,r'^2)$. More precisely, if $W = V(a',b',\rho^2) \Subset V(a,b,r'^2)$ then for every $t \in W$:

$$\left| f_m(t) - \sum_{n=0}^{N-1} a_{n,n+m}t^n \right| \leq C_N(\widetilde{\Pi})\frac{|t|^N}{\rho^m}, \tag{1-14}$$

$$\left| \frac{f_{-m}(t)}{t^m} - \sum_{n=0}^{N-1} a_{n+m,n}t^n \right| \leq C_N(\widetilde{\Pi})\frac{|t|^N}{\rho^m}, \tag{1-15}$$

where $\widetilde{\Pi} = \Pi(a',b',\rho)$. In particular if $f \sim_s^{(1,1)} \hat{f}$ on $\Pi(a,b,r)$ every $f_m(t)$ and $f_{-m}(t)/t^m$, $m \in \mathbb{N}$, admits an $s-$Gevrey asymptotic expansion on $V(a,b,r'^2)$, with the same asymptotic constants for all $m \in \mathbb{Z}$.

We are going to express the notion of asymptotic expansion in the monomial $x\varepsilon$ in terms of the classical notion of asymptotic expansion in an adequate Banach space.

Any $f \in \mathcal{O}(\Pi(a,b,r))$ bounded in some subsector $\Pi(a',b',\rho)$ induces an analytic function $T(f)_\rho : V(a',b',\rho^2) \to \mathcal{E}_{\rho'}$ for all $\rho' < \rho$, by means of the decomposition (1-10):

$$T(f)_\rho(t)(x,\varepsilon) = \sum_{m=0}^{\infty} \frac{f_{-m}(t)}{t^m}x^m + \sum_{m=1}^{\infty} f_m(t)\varepsilon^m. \tag{1-16}$$

The above expression is well-defined since f is bounded on $\Pi(a',b',\rho)$: if $|f(x,\varepsilon)|\leq C$, by Proposition 1.2.10 we see that $|f_m(t)|\leq C/\rho^m$ and $|f_{-m}(t)/t^m|\leq C/\rho^m$ for all $t\in V(a',b',\rho)$ and $T(f)_\rho(t)(x,\varepsilon)$ is absolutely convergent and bounded for $|x|,|\varepsilon|<\rho$. Note that f is determined by $T(f)_\rho$ since $T(f)_\rho(x\varepsilon)(x,\varepsilon)=f(x,\varepsilon)$.

Proposition 1.2.11. *Let f be an analytic function on $\Pi(a,b,r)$, $\hat{f}=\sum a_{n,m}x^n\varepsilon^m\in\hat{R}$ and $0<r'\leq r$ such that $\hat{T}\hat{f}\in\mathcal{E}_{r'}[[t]]$. The following statements are equivalent:*

1. *$f\sim^{(1,1)}\hat{f}$ on $\Pi(a,b,r)$,*
2. *For every $0<\rho<r'$, $T(f)_\rho\sim\hat{T}\hat{f}$ on $V(a,b,\rho^2)$.*

The same result is valid for asymptotic expansions of $s-$Gevrey type. In the last case, if $f\sim_s^{(1,1)}\hat{f}$ on $\Pi(a,b,r)$, then $\hat{f}\in\hat{R}_s^{(1,1)}$.

Proof. We prove that (1) implies (2) The converse is trivial (just put $t=x\varepsilon$). $T(f)_\rho$ is well-defined for $0<\rho<r'$ because f is bounded in every subsector of $\Pi(a,b,r)$. Let $W=V(a',b',\rho'^2)\Subset V(a'',b'',\rho''^2)\Subset V(a,b,\rho^2)$ with $\rho'<\rho''<\rho$. Then using the bounds (1-14) and (1-15) we see that for $t\in W$ and $|x|,|\varepsilon|<\rho''$:

$$\left|T(f)_\rho(t)(x,\varepsilon)-\sum_{n=0}^{N-1}(b_n(x)+c_n(\varepsilon))t^n\right|$$

$$\begin{aligned}
&=\left|\sum_{m=0}^{\infty}\left(\frac{f_{-m}(t)}{t^m}-\sum_{n=0}^{N-1}a_{n+m,n}t^n\right)x^m+\sum_{m=1}^{\infty}\left(f_m(t)-\sum_{n=0}^{N-1}a_{n,n+m}t^n\right)\varepsilon^m\right|\\
&\leq\sum_{m=0}^{\infty}C_N(\widetilde{\Pi})\frac{|t|^N}{\rho''^m}|x|^m+\sum_{m=1}^{\infty}C_N(\widetilde{\Pi})\frac{|t|^N}{\rho''^m}|\varepsilon|^m\\
&\leq C_N(\widetilde{\Pi})\left(\frac{1}{1-|x|/\rho''}+\frac{1}{1-|\varepsilon|/\rho''}\right)|t|^N,
\end{aligned}$$

where $\widetilde{\Pi}=\Pi(a'',b'',\rho'')$. Then taking the supremum for $|x|,|\varepsilon|\leq\rho'$ we obtain the bound

$$\left\|T(f)_\rho(t)-\sum_{n=0}^{N-1}(b_n+c_n)t^n\right\|_{\rho'}\leq\frac{2C_N(\widetilde{\Pi})}{1-\rho'/\rho''}|t|^N,$$

as we wanted to prove.

We also can conclude that $\|b_n+c_n\|_{\rho'}\leq\frac{2C_n(\widetilde{\Pi})}{1-\rho'/\rho''}$, for all $n\in\mathbb{N}$, proving the last part of the statement.

□

The previous proposition let us define the *Taylor's map* for asymptotic expansion in the monomial $x\varepsilon$. The map is defined as $J=J^{(1,1)}:\mathcal{A}^{(1,1)}(\Pi)\to\mathcal{S}$, $J(f)=\hat{f}$, if $f\sim^{(1,1)}\hat{f}$ on Π, and it is the only map that makes the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{A}^{(1,1)}(\Pi) & \xrightarrow{J} & \mathcal{S} \\
{\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \hat{T}} \\
\mathcal{A}(V,\mathcal{E}) & \xrightarrow{J} & \mathcal{E}[[t]]
\end{array}$$

where $V = V(a,b,r^2)$ if $\Pi = \Pi(a,b,r)$, $\mathcal{A}(V,\mathcal{E}) = \bigcup_{r>0}\mathcal{A}(V,\mathcal{E}_r)$, $J = J_{\mathcal{E}}$ is the classical Taylor's map and $T = T_{1,1}$ is defined through (1-16).

For the $s-$Gevrey asymptotic expansions in $x\varepsilon$ we also have the Taylor's map obtained by restriction $J_s = J_s^{(1,1)} : \mathcal{A}_s^{(1,1)}(\Pi) \to \hat{R}_s^{(1,1)}$, and it is the only map that makes the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{A}_s^{(1,1)}(\Pi) & \xrightarrow{J_s} & \hat{R}_s^{(1,1)} \\
{\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \hat{T}} \\
\mathcal{A}_s(V,\mathcal{E}) & \xrightarrow{J} & \mathcal{E}[[t]]_s
\end{array}$$

For asymptotic expansions in a monomial, there is also an analog version of Proposition 1.1.2 which reads as follows.

Proposition 1.2.12. *Let $f \in \mathcal{O}(\Pi)$ be an analytic function. The following assertions are equivalent:*

1. *$f \in \mathcal{A}^{(1,1)}(\Pi)$,*
2. *There is $r > 0$ and a family of bounded analytic functions $f_N \in \mathcal{O}_b(D_r^2), N \geq 1$, such that for every subsector $\widetilde{\Pi}$ of Π there is a constant $A_N(\widetilde{\Pi}) > 0$ such that*

$$|f(x,\varepsilon) - f_N(x,\varepsilon)| \leq A_N(\widetilde{\Pi})|x\varepsilon|^N,$$

for all $(x,\varepsilon) \in \widetilde{\Pi} \cap D_r^2$.

Analogously, $f \in \mathcal{A}_s^{(1,1)}(\Pi)$ if and only if (2) is satisfied with $A_N(\widetilde{\Pi}) = CA^N N!^s$ for some C, A independent of N and additionally there are constants B, D such that $\|f_N\|_r \leq DB^N N!^s$ for all $N \geq 1$.

Proof. We write the proof only for the case of Gevrey asymptotic expansions. If $f \sim_s^{(1,1)} \hat{f}$ on Π, and $\hat{T}(\hat{f}) = \sum(b_n + c_n)t^n \in \mathcal{E}_r[[t]]_s$, with this $r > 0$, it is enough to take $f_N(x,\varepsilon) = \sum_{n=0}^{N-1}(b_n(x) + c_n(\varepsilon))(x\varepsilon)^n$ and the conclusion follows from Definition 1.2.5.

To prove the converse implication write each $f_N(x,\varepsilon) = \sum a_{n,m}^{(N)} x^n \varepsilon^m = \sum_{n=0}^{\infty}(b_{N,n}(x) + c_{N,n}(\varepsilon))(x\varepsilon)^n$, as its Taylor's expansion at the origin valid for $|x|, |\varepsilon| < r$. Note that the condition imposed over the f_N and Cauchy's inequalities implies that $|a_{n,m}^{(N)}| \leq DB^N N!^s / r^{n+m}$ for all $n, m, N \in \mathbb{N}$. Writing the decomposition of the f_N as in (1-10) we see that

$$f_N(t/\varepsilon,\varepsilon)=\sum_{k\in\mathbb{Z}} f_{N,k}(t)\varepsilon^k,\quad f_{N,k}=\sum_{n=0}^{\infty} a^{(N)}_{n,n+k}t^n,\ \frac{f_{N,-k}(t)}{t^k}=\sum_{n=0}^{\infty} a^{(N)}_{n+k,n}t^n.$$

Then for every $\widetilde{\Pi}=\Pi(a',b',\rho)\Subset\Pi$ and $t\in W(a',b',\rho)$ it follows from the hypothesis and Proposition 1.2.10 that

$$|f_k(t)-f_{N,k}(t)|\,,\left|\frac{f_{-k}(t)}{t^k}-\frac{f_{N,-k}(t)}{t^k}\right|\le \frac{CA^N N!^s}{\rho^k}|t|^N. \tag{1-17}$$

Reasoning as in the proof of Proposition 1.1.2 we may conclude that $a^{(N)}_{n,n+k}=a^{(M)}_{n,n+k}$ and $a^{(N)}_{n+k,n}=a^{(M)}_{n+k,n}$ for all $k\ge 0$, $M\ge N$ and $n=0,1,...,N-1$. This implies that $b_{N,n}=b_{M,n}$ and $c_{N,n}=c_{M,n}$ for all $M\ge N$ and $n=0,1,...,N-1$.

Define $\hat{f}=\sum a_{n,m}x^n\varepsilon^m=\sum_{n=0}^{\infty}(b_n(x)+c_n(\varepsilon))(x\varepsilon)^n$, where $b_n=b_{n+1,n}$ and $c_n=c_{n+1,n}$. In other words, $\hat{f}$ is the limit of the Taylor's series of the f_N in the $\mathfrak{m}-$topology of $\hat{R}$, where $\mathfrak{m}$ is the ideal generated by x and ε. It is clear that for $(x,\varepsilon)\in\widetilde{\Pi}\cap D^2_\rho$ with $0<\rho<r$ we have

$$\begin{aligned}\left|f(x,\varepsilon)-\sum_{n=0}^{N-1}(b_n(x)+c_n(\varepsilon))(x\varepsilon)^n\right| &\le CA^N N!^s|x\varepsilon|^N+\left|\sum_{n,m=N}^{\infty} a^{(N)}_{n,m}x^n\varepsilon^m\right|\\ &\le\left(CA^N+\frac{D}{(1-\rho/r)^2}\frac{B^N}{r^{2N}}\right)N!^s|x\varepsilon|^N.\end{aligned}$$

This proves that $f\sim_s^{(1,1)}\hat{f}$ on Π. □

The next step in the study of this type of asymptotic expansions is to study its stability by the usual operations of addition, multiplication and differentiation. This is not straightforward since T and $\hat{T}$ do not behave well under derivatives and of course either with multiplication, since there is no natural product on the range of the maps.

Proposition 1.2.13. *Let Π be a sector in the monomial $x\varepsilon$ and $s>0$. Then $\mathcal{A}^{(1,1)}(\Pi)$ and $\mathcal{A}^{(1,1)}_s(\Pi)$ are differential subalgebras of $\mathcal{O}(\Pi)$ and the Taylor's maps J and J_s are homomorphisms of differential algebras.*

Proof. The compatibility with sums and scalar products follows at once from Definition 1.2.5. To prove the compatibility with derivations we only do it for the case of $\partial/\partial x$: the proof for $\partial/\partial\varepsilon$ is the same replacing x by ε, i.e., using the chart π_2 of the blow-up of the origin and analogous considerations. Suppose that $f\sim^{(1,1)}\hat{f}=\sum a_{n,m}x^n\varepsilon^m$ on $\Pi=\Pi(a,b,r)$ and $\hat{T}(\hat{f})\in\mathcal{E}_{r'}[[t]]$. To show that $\frac{\partial f}{\partial x}\sim^{(1,1)}\frac{\partial\hat{f}}{\partial x}=\sum(n+1)a_{n+1,m}x^n\varepsilon^m$ on Π we show that item (2) of Proposition 1.2.11 holds. First, it follows from equation (1-12) that the Laurent expansion for $\frac{\partial f}{\partial x}$ is given by

$$\frac{\partial f}{\partial x}\left(\frac{t}{\varepsilon},\varepsilon\right)=\sum_{m\in\mathbb{Z}} f'_{m-1}(t)\varepsilon^m.$$

For $0<\widetilde{\rho}<\rho<r'$ and $W=V(a',b',\widetilde{\rho}^2)\Subset V(a,b,\rho)$ take subsectors with $W\Subset\widetilde{W}\Subset V(a'',b'',\rho''^2)\Subset V(a,b,\rho^2)$ and $\alpha>0$ such that $D(t,\alpha|t|)\subset\widetilde{W}$ for all $t\in W$. Note that we can take α as small as we want by enlarging the opening of $\widetilde{W}$. Then for $t\in W$ it follows, from inequalities (1-14) and (1-15) applied to $V(a'',b'',\rho''^2)$ and Cauchy's formulas, that

$$\begin{aligned}\left|f'_m(t)-\sum_{n=0}^{N} n a_{n,n+m}t^{n-1}\right| &= \left|\frac{1}{2\pi i}\int_{|w-t|=\alpha|t|}\frac{f_m(w)-\sum_{n=0}^{N}a_{n,n+m}w^n}{(w-t)^2}dw\right|\\ &\leq C_{N+1}(\overline{\Pi})\frac{(\alpha+1)^{N+1}}{\alpha}\frac{|t|^N}{\rho^m},\end{aligned}$$

and analogously

$$\left|f'_{-m}(t)-\sum_{n=0}^{N}(n+m)a_{n+m,n}t^{n+m-1}\right|\leq C_{N+1}(\overline{\Pi})\frac{(\alpha+1)^{N+m+1}}{\alpha}\frac{|t|^{N+m}}{\rho^m},$$

where $\overline{\Pi}=\Pi(a'',b'',\rho'')$. Finally if we take α such that $\widetilde{\rho}\leq\frac{\rho''}{\alpha+1}$ and $|x|,|\varepsilon|<\widetilde{\rho}$, we get the bound

$$\begin{aligned}&\left|T\left(\frac{\partial f}{\partial x}\right)_\rho(t)(x,\varepsilon)-\sum_{n=0}^{N-1}\left(\sum_{m=0}^{\infty}(n+m+1)a_{n+m+1,n}x^m+\sum_{m=1}^{\infty}(n+1)a_{n+1,n+m}\varepsilon^m\right)t^n\right|\leq\\ &\left|\sum_{m=0}^{\infty}\left(\frac{f'_{-m-1}(t)}{t^m}-\sum_{n=0}^{N-1}(n+m+1)a_{n+m+1,n}t^n\right)x^m+\right.\\ &\qquad\qquad\left.\sum_{m=1}^{\infty}\left(f'_{m-1}(t)-\sum_{n=0}^{N-1}(n+1)a_{n+1,n+m}t^n\right)\varepsilon^m\right|\\ &\leq\left(\frac{C_N(\overline{\Pi})}{\alpha\rho''}(\alpha+1)^{N+1}\frac{1}{1-(\alpha+1)\widetilde{\rho}/\rho''}+\frac{\rho''}{\alpha}C_{N+1}(\overline{\Pi})(\alpha+1)^{N+1}\frac{1}{1-\widetilde{\rho}/\rho''}\right)|t|^N.\end{aligned}$$

This shows that $T(\partial f/\partial x)_\rho\sim\hat{T}(\partial\hat{f}/\partial x)$ on $V(a,b,\rho^2)$, as we wanted to prove. The previous proof also works for the case of $s-$Gevrey asymptotic expansions since the previous bounds remains of $s-$Gevrey type.

To prove the compatibility with multiplication we use the previous proposition. Suppose that $f\sim^{(1,1)}\hat{f}$ and $g\sim^{(1,1)}\hat{g}$ on Π, $\hat{f},\hat{g}\in\mathcal{E}_r[[t]]$ and let $(f_n),(g_n)$ two families of functions in $\mathcal{O}_b(D_r^2)$ such that for every $\widetilde{\Pi}\Subset\Pi$ and $N\in\mathbb{N}$ there are constants $A_N(\widetilde{\Pi}),B_N(\widetilde{\Pi})$ such that $|f(x,\varepsilon)-f_N(x,\varepsilon)|\leq A_N(\widetilde{\Pi})|x\varepsilon|^N$, $|g(x,\varepsilon)-g_N(x,\varepsilon)|\leq B_N(\widetilde{\Pi})|x\varepsilon|^N$, for all $(x,\varepsilon)\in\widetilde{\Pi}\cap D(0,r)$. Then we see that for all $(x,\varepsilon)\in\widetilde{\Pi}\cap D_r^2$ we have

$$|f(x,\varepsilon)g(x,\varepsilon)-f_N(x,\varepsilon)g_N(x,\varepsilon)|\leq(\|g\|_rA_N(\widetilde{\Pi})+\|f_N\|_rB_N(\widetilde{\Pi}))|x\varepsilon|^N.$$

Since limits in the $\mathfrak{m}-$topology commute with the usual product, we see that $J(fg) = \hat{f}\hat{g}$ and the proof is complete. For the $s-$Gevrey case we may take $A_N(\widetilde{\Pi}) = C_1 A_1^N N!^s$, $B_N(\widetilde{\Pi}) = C_2 A_2^N N!^s$, $\|f_N\|_r \leq D_1 B_1^N N!^s$ and $\|g_N\|_r \leq D_2 B_2^N N!^s$, for certain constants $A_j, B_j, C_j, D_j, j = 1,2$ and all $N \in \mathbb{N}$. We already know that $fg \sim^{(1,1)} \hat{f}\hat{g}$ on Π. To ensure that the asymptotic is of $s-$Gevrey type we apply again Proposition 1.2.12 with the family of functions defined by

$$h_N = \sum_{n=1}^{N} (f_n - f_{n-1}) g_{N-n}, \quad f_0 = g_0 = 0.$$

Then the growth of these function is $s-$Gevrey and from the identity

$$fg - h_N = (f - f_N)g + \sum_{n=1}^{N} (f_n - f_{n-1})(g - g_{N-n}),$$

we easily conclude adequate $s-$Gevrey bounds for this expression.

□

As in the classical case we can characterize when a function has null asymptotic expansion in $x\varepsilon$ in terms of its decrease at the origin. More specifically we have the following result.

Proposition 1.2.14. *Let $\Pi = \Pi(a, b, r)$ be a sector in $x\varepsilon$ and $f \in \mathcal{O}(\Pi)$. Then $f \in \mathcal{A}_s^{(1,1)}$ and $J_s(f) = 0$ if and only if for all $\widetilde{\Pi} \Subset \Pi$ there are $C, B > 0$ such that for $(x, \varepsilon) \in \widetilde{\Pi}$:*

$$|f(x,\varepsilon)| \leq C \exp\left(-B/|x\varepsilon|^{1/s}\right).$$

When f satisfies this type of bounds we will say that f has exponential decay of order $1/s$ in the monomial $x\varepsilon$ at the origin.

Proof. Suppose that $J_s(f) = 0$. Then $T(f)_\rho \sim_s 0$ on $V = V(a, b, \rho^2)$ for all $0 < \rho < r$. By the classical result (Proposition 1.1.5) we know that for every subsector $W = V(a', b', \rho'^2) \Subset V$ there are constants C, B with $\|T(f)(t)\|_{\rho'} \leq C \exp\left(-B/|t|^{1/s}\right)$ for all $t \in W$. Then if $\widetilde{\Pi} = \Pi(a', b', \rho)$ and $(x, \varepsilon) \in \widetilde{\Pi}$, $x\varepsilon \in W$ and then $|f(x,\varepsilon)| = |T(f)(x\varepsilon)(x,\varepsilon)| \leq C \exp\left(-B/|x\varepsilon|^{1/s}\right)$ as we wanted to prove.

Conversely, suppose that f has exponential decay of order $1/s$ in the monomial $x\varepsilon$ at the origin. We show that $T(f)_\rho$ has exponential decay of order $1/s$ at the origin. For $\rho < r$ and subsectors $\widetilde{\Pi} = \Pi(a', b', \rho') \Subset \Pi(a'', b'', \rho'') \Subset \Pi(a, b, \rho)$ there are $C, B > 0$ with $|f(x,\varepsilon)| \leq C \exp\left(-B/|x\varepsilon|^{1/s}\right)$ for all $(x, \varepsilon) \in \Pi(a'', b'', \rho'')$. Using Proposition 1.2.10 we obtain the bounds

$$|f_m(t)|, \left|\frac{f_{-m}(t)}{t^m}\right| \leq \frac{C}{\rho''^m} \exp\left(-B/|t|^{1/s}\right),$$

for all $m \in \mathbb{N}$ and $t \in V(a'', b'', \rho''^2)$. Then if $t \in V(a', b', \rho'^2)$ we easily get

$$\|T(f)_\rho(t)(x,\varepsilon)\|_{\rho'} \leq \frac{2C}{1-\rho'/\rho''}\exp\left(-B/|t|^{1/s}\right),$$

proving that $T(f)_\rho \sim_s 0$ on $V(a,b,\rho^2)$. □

Proposition 1.2.15 (Watson's Lemma for $x\varepsilon$)**.** *Let $\Pi = \Pi(a,b,r)$ be a sector in $x\varepsilon$, with opening $b-a > s\pi$ and $f \in \mathcal{A}_s^{(1,1)}(\Pi)$ with $J_s(f)=0$. Then $f \equiv 0$.*

Proof. By Proposition 1.2.14, we know that, in a subsector $\widetilde{\Pi}$ with opening larger than $s\pi$, $|f(x,\varepsilon)| \leq C\exp\left(-B/|x\varepsilon|^{1/s}\right)$. By Proposition 1.2.10 and classical Watson's Lemma 1.1.8 we conclude that $f_n \equiv 0$ for all $n \in \mathbb{Z}$, so $f \equiv 0$ as desired. □

We continue this section describing the corresponding results about asymptotic expansions in a general monomial $x^p\varepsilon^q$. We begin by defining this notion and obtaining an equivalent version in terms of asymptotic expansion in a monomial $\zeta\eta$ in order to recover easily the properties.

Definition 1.2.6. Let f be an analytic function on $\Pi_{p,q}(a,b,r)$ and $\hat{f} \in \hat{R}$. We will say that *f has $\hat{f}$ as asymptotic expansion at the origin in $x^p\varepsilon^q$* and we will use the notation $f \sim^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r)$ if: there exists $0 < r' \leq r$ such that $\hat{T}_{p,q}\hat{f} = \sum f_n t^n \in \mathcal{E}_{r'}^{(p,q)}[[t]]$ and for every $\widetilde{\Pi}_{p,q} = \Pi_{p,q}(a',b',\rho) \Subset \Pi_{p,q}$ with $0 < \rho < r'$ and $N \in \mathbb{N}$ there is a constant $C_N(\widetilde{\Pi}_{p,q}) > 0$ such that for $(x,\varepsilon) \in \widetilde{\Pi}_{p,q}$:

$$\left| f(x,\varepsilon) - \sum_{n=0}^{N-1} f_n(x,\varepsilon)(x^p\varepsilon^q)^n \right| \leq C_N(\widetilde{\Pi}_{p,q})|x^p\varepsilon^q|^N. \tag{1-18}$$

The asymptotic expansion is said to be of *$s-$Gevrey type* if additionally

1. It is possible to chose $C_N(\widetilde{\Pi}_{p,q}) = C(\widetilde{\Pi}_{p,q})A(\widetilde{\Pi}_{p,q})^N N!^s$ for some $C(\widetilde{\Pi}_{p,q}), A(\widetilde{\Pi}_{p,q})$ independent of N.
2. $\hat{f} \in \hat{R}_s^{(p,q)}$.

In this case we will use the notation: $f \sim_s^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r)$. We will denote by $\mathcal{A}^{(p,q)}(\Pi_{p,q})$ the set of analytic functions defined on $\Pi_{p,q}$ that admits an asymptotic expansion in the monomial $x^p\varepsilon^q$ on $\Pi_{p,q}$ and by $\mathcal{A}_s^{(p,q)}(\Pi_{p,q})$ the set of analytic functions defined on $\Pi_{p,q}$ that admits an asymptotic expansion of $s-$Gevrey type in the monomial $x^p\varepsilon^q$ on $\Pi_{p,q}$.

Remark 1.2.16. In contrast to the case of $s-$Gevrey asymptotic expansions in the monomial $x\varepsilon$, here we require by definition that formal series which are $s-$Gevrey asymptotic expansions in a monomial $x^p\varepsilon^q$ of analytic functions, to be $s-$Gevrey in the monomial $x^p\varepsilon^q$.

Remark 1.2.17. The decomposition for formal power series explained in Remark 1.2.6, is valid for analytic functions $f \in \mathcal{O}(\Pi_{p,q}(a,b,r))$ too. Indeed, note that $(x,\varepsilon) \in \Pi_{p,q}(a,b,r)$ if and only if $(\omega x, \nu\varepsilon) \in \Pi_{p,q}(a,b,r)$, for ω and ν pth and qth roots of unity, respectively. Then formula (1-8) is valid in this context and clearly if $\zeta = x^p, \eta = \varepsilon^q$, $f_{ij}(\zeta,\eta) \in \mathcal{O}(\Pi_{1,1}(a,b,r))$ for every $i = 0,1,...,p-1$, $j = 0,1,...,q-1$.

We note that if f satisfies a bound of type $|f(x,\varepsilon)| \leq K(|x^p\varepsilon^q|)$ for all $(x,\varepsilon) \in \Pi_{p,q}(a,b,r)$ and for some function $K : (0,r) \to \mathbb{R}$ then for all i,j:

$$|f_{ij}(\zeta,\eta)| \leq \frac{K(|\zeta\eta|)}{|\zeta|^{i/p}|\eta|^{j/q}}, \tag{1-19}$$

for all $(\zeta,\eta) \in \Pi_{1,1}(a,b,r)$. Conversely, if $|f_{ij}(\zeta,\eta)| \leq K_{ij}(|\zeta\eta|)$ for all $(\zeta,\eta) \in \Pi_{1,1}(a,b,r)$ and some functions $K_{ij} : (0,r) \to \mathbb{R}$ then

$$|f(x,\varepsilon)| \leq \sum_{\substack{0\leq i<p\\0\leq j<q}} |x|^i|\varepsilon|^j K_{ij}(|x^p\varepsilon^q|), \tag{1-20}$$

for all $(x,\varepsilon) \in \Pi_{p,q}(a,b,r)$.

Proposition 1.2.18. *Let $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$ be a sector in $x^p\varepsilon^q$, $f(x,\varepsilon) \in \mathcal{O}(\Pi_{p,q})$. Using the above notation, the following statements are equivalent:*

1. *$f(x,\varepsilon) \in \mathcal{A}^{(p,q)}(\Pi_{p,q})$,*
2. *$f_{ij}(\zeta,\eta) \in \mathcal{A}^{(1,1)}(\Pi_{1,1}(a,b,r))$ for every $i = 0,1,...,p-1$, $j = 0,1,...,q-1$.*

The same result is valid for asymptotic expansions of s-Gevrey type.

Proof. Suppose that $f \sim^{(p,q)} \hat{f}$ on $\Pi_{p,q}$ and $\hat{T}_{p,q}(\hat{f}) \in \mathcal{E}^{(p,q)}_{r'}[[t]]$. We know from formula (1-9) that if $\hat{f} = \sum_{\substack{0\leq i<p\\0\leq j<q}} x^i\varepsilon^j \hat{f}_{ij}(x^p,\varepsilon^q) = \sum_{n=0}^{\infty} f_n(x,\varepsilon)(x^p\varepsilon^q)^n$ then the f_n and $\hat{f}_{ij}$ are related by

$$f_n(x,\varepsilon) = \sum_{\substack{0\leq i<p\\0\leq j<q}} x^i\varepsilon^j(b_{ijn}(x^p) + c_{ijn}(\varepsilon^q)), \quad \hat{f}_{ij}(x^p,\varepsilon^q) = \sum_{n=0}^{\infty}(b_{ijn}(x^p) + c_{ijn}(\varepsilon^q))(x^p\varepsilon^q)^n.$$

We are going to show that $f_{ij} \sim^{(1,1)} \hat{f}_{ij}$ on $\Pi_{1,1}(a,b,r)$ for every possible i and j. From hypothesis we know that for $\widetilde{\Pi}_{p,q} = \Pi_{p,q}(a',b',\rho) \Subset \Pi_{p,q}$ with $0 < \rho < r'$, and $N \in \mathbb{N}$ there is $C_N(\widetilde{\Pi}_{p,q}) > 0$ with

$$\left| f(x,\varepsilon) - \sum_{n=0}^{N-1} f_n(x,\varepsilon)(x^p\varepsilon^q)^n \right| \leq C_N(\widetilde{\Pi}_{p,q})|x^p\varepsilon^q|^N,$$

for all $(x,\varepsilon)\in\widetilde{\Pi}_{p,q}$. Then, for the same (x,ε), using formula (1-19) in Remark 1.2.17 it is clear that

$$\left|f_{ij}(\zeta,\eta)-\sum_{n=0}^{N-1}(b_{ijn}(\zeta)+c_{ijn}(\eta))(\zeta\eta)^n\right|\leq C_N(\widetilde{\Pi}_{pq})|\zeta|^{N-i/p}|\eta|^{N-j/q},$$

for all $(\zeta,\eta)\in\Pi_{1,1}(a',b',\rho)$. Then, using this bound for N and $N+1$ we get

$$\left|f_{ij}(\zeta,\eta)-\sum_{n=0}^{N-1}(b_{ijn}(\zeta)+c_{ijn}(\eta))(\zeta\eta)^n\right|\leq\left(C_{N+1}(\widetilde{\Pi}_{pq})\rho^{1-i/p}\rho^{1-j/q}+\|b_{ijN}+c_{ijN}\|_\rho\right)|\zeta\eta|^N.$$

This concludes the proof in this case. Note that if $f\sim_s^{(p,q)}\hat{f}$, $\hat{f}$ is a $s-$Gevrey series in $x^p\varepsilon^q$. This implies that there are constants $B,D>$ such that $\|b_{ijn}+c_{ijn}\|_\rho\leq DB^nn!^s$ for all $n\in\mathbb{N}$. This shows that the bounds obtained in the proof remains of $s-$Gevrey type.

To prove the converse implication, assume that $f_{ij}\sim^{(1,1)}\hat{f}_{ij}$ on $\Pi_{1,1}=\Pi_{1,1}(a,b,r)$, with $\hat{f}_{ij}(\zeta,\eta)=\sum_{n=0}^\infty(b_{ijn}(\zeta)+c_{ijn}(\eta))(\zeta\eta)^n$ and $\hat{T}_{1,1}(\hat{f}_{ij})\in\mathcal{E}_{r'_{ij}}^{(1,1)}[[t]]$. Then for every i,j, $\widetilde{\Pi}_{1,1}=\Pi_{1,1}(a',b',\rho)\Subset\Pi_{1,1}$ with $0<\rho<r'=\min\{r'_{ij}\}$, and $N\in\mathbb{N}$ there is a constant $C_{ijN}(\widetilde{\Pi}_{1,1})$ such that for all $(\zeta,\eta)\in\widetilde{\Pi}_{1,1}$ we have

$$\left|f_{ij}(\zeta,\eta)-\sum_{n=0}^{N-1}(b_{ijn}(\zeta)+c_{ijn}(\eta))(\zeta\eta)^n\right|\leq C_{ijN}(\widetilde{\Pi}_{1,1})|\zeta\eta|^N.$$

Let $\hat{f}=\sum_{\substack{0\leq i<p\\0\leq j<q}}x^i\varepsilon^j\hat{f}_{i,j}(x^p,\varepsilon^q)$ be the corresponding formal power series. Then $\hat{T}_{p,q}(\hat{f})\in\mathcal{E}_{r'}^{(p,q)}[[t]]$ and it follows from inequality (1-20) that for all $(x,\varepsilon)\in\Pi_{p,q}(a',b',\rho)$

$$\left|f(x,\varepsilon)-\sum_{n=0}^{N-1}f_n(x,\varepsilon)(x^p\varepsilon^q)^n\right|\leq\left(\sum_{\substack{0\leq i<p\\0\leq j<q}}\rho^{i+j}C_{ijN}(\widetilde{\Pi}_{1,1})\right)|x^p\varepsilon^q|^N,$$

as we wanted to prove. In the $s-$Gevrey case the previous bounds are clearly of $s-$Gevrey type too.

□

We want to introduce now the corresponding map $T_{p,q}$, as in the case of the monomial $x\varepsilon$. A possible way to do this and avoid ramifications is the following: given $f\in\mathcal{O}(\Pi_{p,q}(a,b,r))$ bounded in some subsector $\Pi_{p,q}(a',b',\rho)$, we decompose $f(x,\varepsilon)=\sum_{\substack{0\leq i<p\\0\leq j<q}}x^i\varepsilon^jf_{ij}(x^p,\varepsilon^q)$, with $f_{ij}\in\mathcal{O}(\Pi_{1,1}(a,b,r))$, as in Remark 1.2.17. Now, for every i,j, $T_{1,1}(f_{ij})_\rho$ was defined as

$$T_{1,1}(f_{ij})_\rho(t)(\zeta,\eta)=\sum_{m=0}^\infty\frac{f_{ij,-m}(t)}{t^m}\zeta^m+\sum_{m=1}^\infty f_{ij,m}(t)\eta^m,$$

if $f_{ij}(t/\eta,\eta)=\sum_{m\in\mathbb{Z}}f_{ij,m}(t)\eta^m$ is the convergent Laurent expansion of f_{ij} in the annulus $|t|/\rho<|\eta|<\rho$. Now, based on formula (1-9) we define

$$T_{p,q}(f)_\rho(t)(x,\varepsilon) = \sum_{\substack{0\leq i<p\\ 0\leq j<q}} x^i\varepsilon^j T_{1,1}(f_{ij})_\rho(t)(x^p,\varepsilon^q) \tag{1-21}$$

$$= \sum_{j=0}^{q-1}\varepsilon^j\left(\sum_{m=0}^{\infty}\sum_{i=0}^{p-1}\frac{f_{ij,-m}(t)}{t^m}x^{pm+i}\right)+\sum_{i=0}^{p-1}x^i\left(\sum_{m=1}^{\infty}\sum_{j=0}^{q-1}f_{ij,m}(t)\varepsilon^{qm+j}\right). \tag{1-22}$$

As in the case of $x\varepsilon$, the function f is completely determined by the map $T_{p,q}(f)_\rho$ because $T_{p,q}(f)_\rho(x^p\varepsilon^q)(x,\varepsilon) = f(x,\varepsilon)$.

In this context, we have a similar result as the one stated in Proposition 1.2.10 relating the growth of f with the growths of the previous $f_{ij,m}$.

Proposition 1.2.19. *Let $f \in \mathcal{O}(\Pi_{p,q}(a,b,r))$ be an analytic function. Suppose that $|f(x,\varepsilon)| \leq K(|x^p\varepsilon^q|)$ for any $(x,\varepsilon) \in \Pi_{p,q}(a',b',\rho) \Subset \Pi_{p,q}(a,b,r)$ and some function $K : (0,\rho^2) \to \mathbb{R}_{>0}$. Then for $t \in V(a',b',\rho^2)$ and every $0 \leq i < p$ and $0 \leq j < q$ the following bounds hold:*

1. *If $m \in \mathbb{N}$, $|f_{ij,m}(t)| \leq \dfrac{K(|t|)}{|t|^{i/p}\rho^{m+j/q-i/p}}$,*
2. *If $m \geq 1$, $|f_{ij,-m}(t)| \leq \dfrac{|t|^m K(|t|)}{|t|^{j/q}\rho^{m+i/p-j/q}}$.*

In particular, if $r = \rho = +\infty$ then $f(x,\varepsilon) = f_{00}(x^p\varepsilon^q)$, for all $(x,\varepsilon) \in \Pi_{p,q}(a',b',+\infty)$.

Proof. The inequalities follow directly from inequalities (1-19) and Cauchy's formulas. To prove the last part, note that by changing the function K we can assume that p and q are relative primes. Then fixing $t \in V$, the inequalities are valid for any $\rho > |t|^{1/2}$. If $m \geq 1$ then $m + j/q - i/p \geq 1$ and $m + i/p - j/q \geq 1$ and we can let $\rho \to +\infty$ and conclude that $f_{ij,m}(t) \equiv 0$ for all possible i, j. If $m = 0$ then $j/q = i/p$ if and only if $i = j = 0$. Thus the result follows by letting $\rho \to +\infty$ to see that $f_{ij,0}(t) \equiv 0$ for all i, j except for $i = j = 0$. □

We can characterize as before the property of having an asymptotic expansion in $x^p\varepsilon^q$ in terms of classic asymptotic expansion in some Banach space. Indeed, we have the following analog to Proposition 1.2.11, which is an immediate consequence of Proposition 1.2.18 and Proposition 1.2.11.

Proposition 1.2.20. *Let $f \in \mathcal{O}(\Pi_{p,q}(a,b,r))$ be an analytic function, $\hat{f} \in \hat{R}$ and $0 < r' \leq r$ such that $\hat{T}_{p,q}\hat{f} \in \mathcal{E}_{r'}^{(p,q)}[[t]]$. The following statements are equivalent:*

1. *$f \sim^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r)$,*
2. *For every $0 < \rho < r'$, $T_{p,q}(f)_\rho \sim \hat{T}_{p,q}(\hat{f})$ on $V(a,b,\rho^2)$.*

The same result is valid for asymptotic expansions of $s-$Gevrey type.

As before, this allows us to define the *Taylor's map* for asymptotic expansions in the monomial $x^p\varepsilon^q$. The map is given by $J^{p,q}:\mathcal{A}^{(p,q)}(\Pi_{p,q})\to\mathcal{S}$, $J^{p,q}(f)=\hat{f}$ if $f\sim^{(p,q)}\hat{f}$ on $\Pi_{p,q}$, and it is the only map that makes the following diagram commutative:

$$\begin{array}{ccc}\mathcal{A}^{(p,q)}(\Pi_{p,q}) & \xrightarrow{J^{p,q}} & \mathcal{S}\\ {\scriptstyle T_{p,q}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{T}_{p,q}}\\ \mathcal{A}(V,\mathcal{E}^{(p,q)}) & \xrightarrow{J} & \mathcal{E}^{(p,q)}[[t]]\end{array}$$

where $V=V(a,b,r^2)$ if $\Pi_{p,q}=\Pi_{p,q}(a,b,r)$, $\mathcal{A}(V,\mathcal{E}^{(p,q)})=\bigcup_{r>0}\mathcal{A}(V,\mathcal{E}_r^{(p,q)})$, $J=J_{\mathcal{E}^{(p,q)}}$ is the classical Taylor's map and $T_{p,q}$ is defined through (1-21).

For the $s-$Gevrey asymptotic expansions in $x^p\varepsilon^q$ we also have the Taylor's map obtained by restriction $J_s^{p,q}:\mathcal{A}_s^{(p,q)}(\Pi_{p,q})\to\hat{R}_s^{(p,q)}$, and it is the only map that makes the following diagram commutative:

$$\begin{array}{ccc}\mathcal{A}_s^{(p,q)}(\Pi_{p,q}) & \xrightarrow{J_s} & \hat{R}_s^{(p,q)}\\ {\scriptstyle T_{p,q}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{T}_{p,q}}\\ \mathcal{A}_s(V,\mathcal{E}^{(p,q)}) & \xrightarrow{J} & \mathcal{E}^{(p,q)}[[t]]_s\end{array}$$

An alternative way to prove that a function has asymptotic expansion in the monomial $x^p\varepsilon^q$ is by the aid of analytic maps that approximate the functions adequately. The result is described in the next proposition, and it is the generalization of Proposition 1.2.12 for any monomial. The proof is similar to the one of Proposition 1.2.12: every $(i,j)-$component of the given functions is analyzed to obtaining bounds similar to the ones corresponding to (1-17) but in this case from Proposition 1.2.19. The last part of the proof remains unchanged and will not be included here.

Proposition 1.2.21. *Let $f\in\mathcal{O}(\Pi_{p,q})$ be an analytic function. The following assertions are equivalent:*

1. *$f\in\mathcal{A}^{(p,q)}(\Pi_{p,q})$,*
2. *There is $r>0$ and a family of bounded analytic functions $f_N\in\mathcal{O}_b(D_r^2), N\geq 1$, such that for every subsector $\widetilde{\Pi}_{p,q}$ of $\Pi_{p,q}$ there is a constant $A_N(\widetilde{\Pi}_{p,q})>0$ such that*
$$|f(x,\varepsilon)-f_N(x,\varepsilon)|\leq A_N(\widetilde{\Pi}_{p,q})|x^p\varepsilon^q|^N,$$
for all $(x,\varepsilon)\in\widetilde{\Pi}_{p,q}\cap D_r^2$.

Analogously, $f\in\mathcal{A}_s^{(p,q)}(\Pi_{p,q})$ if and only if (2) is satisfied with $A_N(\widetilde{\Pi}_{p,q})=CA^NN!^s$ for some C,A independent of N and additionally there are constants B,D such that $\|f_N\|_r\leq DB^NN!^s$ for all $N\geq 1$.

We also have the compatibility of asymptotic expansion in $x^p\varepsilon^q$ with the basic algebraic operations.

Proposition 1.2.22. *Let $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$ be a sector in the monomial $x^p\varepsilon^q$. Then $\mathcal{A}^{(p,q)}(\Pi_{p,q})$ is a differential subalgebra of $\mathcal{O}(\Pi_{p,q})$ and the Taylor's map $J^{p,q}$ is an homomorphism of differential algebras.*

Proof. The proof can be easily obtained from Proposition 1.2.13 and Proposition 1.2.18. For sums the proof is immediate. For derivatives, for instance to $\partial/\partial x$, if $f \in \mathcal{A}^{(p,q)}(\Pi_{p,q})$ and it decompose as $f(x,\varepsilon) = \sum_{\substack{0\le i<p\\0\le j<q}} x^i\varepsilon^j f_{i,j}(x^p,\varepsilon^q)$ then the decomposition for $\partial f/\partial x$ is given by:

$$\sum_{\substack{0\le i<p-1\\0\le j<q}} \left((i+1)f_{i+1,j}(x^p,\varepsilon^q) + px^p\frac{\partial f_{i+1,j}}{\partial x}(x^p,\varepsilon^q)\right)x^i\varepsilon^j + \sum_{j=0}^{q-1}\left(p\frac{\partial f_{0,j}}{\partial x}(x^p,\varepsilon^q)\right)x^{p-1}\varepsilon^j.$$

Since every $(i,j)-$component of $\partial f/\partial x$ belongs to $\mathcal{A}^{(1,1)}(\Pi_{1,1}(a,b,r))$ (because it is a differential algebra), it follows that $\partial f/\partial x \in \mathcal{A}^{(p,q)}(\Pi_{p,q})$. For products the proof follows the same idea: take into account that the $(i,j)-$components of a product fg can be obtained as sums of products of the components of f and g. □

Finally, a characterization of functions with null asymptotic expansion in $x^p\varepsilon^q$ is given in the next proposition, and it is a consequence of Proposition 1.2.14 and inequalities (1-20) and (1-19).

Proposition 1.2.23. *Let $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$ be a sector in $x^p\varepsilon^q$ and $f \in \mathcal{O}(\Pi_{p,q})$. Then $f \in \mathcal{A}_s^{(p,q)}(\Pi_{p,q})$ and $J_s^{p,q}(f) = 0$ if and only if for all $\widetilde{\Pi}_{p,q} \Subset \Pi_{p,q}$ there are $C, B > 0$ such that for $(x,\varepsilon) \in \widetilde{\Pi}_{p,q}$:*

$$|f(x,\varepsilon)| \le C\exp\left(-B/|x^p\varepsilon^q|^{1/s}\right).$$

When f satisfies this type of bounds we will say that f has exponential decay of order $1/s$ in the monomial $x^p\varepsilon^q$ at the origin.

Proposition 1.2.24 (Watson's Lemma for $x^p\varepsilon^q$)**.** *Let $\Pi_{p,q}(a,b,r)$ be a sector in $x^p\varepsilon^q$, with opening $b-a > s\pi$ and $f \in \mathcal{A}_s^{(p,q)}(\Pi_{p,q}(a,b,r))$ with $J_s^{p,q}(f) = 0$. Then $f \equiv 0$.*

It is also worth to mention the analog to Borel-Ritt's and Borel-Ritt-Gevrey's theorems for this kind of asymptotic expansions.

Theorem 1.2.25 (Borel-Ritt Theorem for $x^p\varepsilon^q$)**.** *Given any $\hat{f} \in \mathcal{S}$ and any $\Pi_{p,q}$ sector in the monomial $x^p\varepsilon^q$, there is $f \in \mathcal{O}(\Pi_{p,q})$ with $f \sim^{(p,q)} \hat{f}$ on $\Pi_{p,q}$.*

Theorem 1.2.26 (Gevrey-Borel-Ritt Theorem for $x^p\varepsilon^q$)**.** *Given any $\hat{f} \in \hat{R}_s^{(p,q)}$ and any sector $\Pi_{p,q}(a,b,r)$ in the monomial $x^p\varepsilon^q$ with opening $b-a < s\pi$, there is $f \in \mathcal{O}(\Pi_{p,q})$ with $f \sim_s^{(p,q)} \hat{f}$ on $\Pi_{p,q}$.*

To finish this section we formulate and prove one of the main tools to obtain what we will call summability in a monomial: the Ramis-Sibuya Theorem for asymptotic expansions in a monomial.

Theorem 1.2.27 (Ramis-Sibuya Theorem for $x^p\varepsilon^q$)**.** *Suppose that a finite family of sectors $\Pi_j = \Pi_{p,q}(a_j, b_j, r)$, $1 \leq j \leq m$, form a covering of $D_r^2 \setminus \{x\varepsilon = 0\}$. Given $f_j : \Pi_j \to \mathbb{C}$ bounded and analytic, assume that for every subsector $\widetilde{\Pi}$ of $\Pi_{j_1} \cap \Pi_{j_2}$ (when not empty) there are constants $\gamma(\widetilde{\Pi}), C(\widetilde{\Pi})$ such that*

$$|f_{j_1}(x,\varepsilon) - f_{j_2}(x,\varepsilon)| \leq C(\widetilde{\Pi}) \exp\left(-\gamma(\widetilde{\Pi})/|x^p\varepsilon^q|^{1/s}\right),$$

for $(x,\varepsilon) \in \widetilde{\Pi}$. Then the functions f_j have a common asymptotic expansion in $x^p\varepsilon^q$ on Π_j of $s-$Gevrey type, respectively.

Proof. Since every f_j is bounded on Π_j, $T_{p,q}(f_j)_\rho$ is bounded on $V_j = V(a_j, b_j, r^2)$, for all $0 < \rho < r$. For every pair j_1, j_2 such that $\Pi_{j_i} \cap \Pi_{j_2} \neq \emptyset$, Proposition 1.2.23 shows that $f_{j_1} - f_{j_2} \sim_s^{(p,q)} 0$ on $\Pi_{j_i} \cap \Pi_{j_2} = \Pi(a,b,r)$. Then for every $0 < \rho < r$, $T_{p,q}(f_{j_1})_\rho - T_{p,q}(f_{j_2})_\rho \sim_s 0$ on $V(a,b,\rho^2)$. Since the Π_j cover $D_r^2 \setminus \{x\varepsilon = 0\}$ the sectors $V_j = V(a_j, b_j, r^2)$ cover $D_{r^2} \setminus \{0\}$. Then by the classical Ramis-Sibuya Theorem, the functions $T_{p,q}(f_j)_\rho$ admit a common asymptotic expansion $\hat{F} \in \mathcal{E}^{p,q}[[t]]_s$ on V_j of $s-$Gevrey type. Then it is clear that $f_j \sim_s^{(p,q)} \hat{T}_{p,q}^{-1}(\hat{F})$ on Π_j. □

1.2.3 Summability in a monomial

As in the classical case, thanks to Watson's Lemma 1.2.24 we can finally define the natural notion of summability in a monomial $x^p\varepsilon^q$ in Ramis style.

Definition 1.2.7. Let $k > 0$ and $\hat{f} \in \mathcal{S}$ be given. We say that $\hat{f}$ is $k-$*summable* in the monomial $x^p\varepsilon^q$ in the direction $d \in S^1$ if there is a sector $\Pi_{p,q}(a,b,r)$ bisected by d with opening $b - a > \pi/k$ and $f \in \mathcal{O}(\Pi_{p,q}(a,b,r))$ with $f \sim_{1/k}^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r)$.

We simply say that $\hat{f}$ is $k-$*summable* in the monomial $x^p\varepsilon^q$ if it is $k-$summable in the monomial $x^p\varepsilon^q$ in every direction $d \in S^1$, with finitely many exceptions mod. 2π.

The set of $k-$summable series in $x^p\varepsilon^q$ in the direction d will be denoted by $R_{1/k,d}^{(p,q)}$ and the set of $k-$summable series in $x^p\varepsilon^q$ will be denoted by $R_{1/k}^{(p,q)}$.

As an immediate consequence of Proposition 1.2.22 we obtain:

Proposition 1.2.28. *Let $k>0$ and $d\in S^1$ be given. Then $R_{1/k,d}^{(p,q)}$ and $R_{1/k}^{(p,q)}$ are differential subalgebras of $\hat{R}_{1/k}^{(p,q)}$.*

It is clear from Proposition 1.2.20 that $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ (resp. $k-$summable in direction d) if and only if $\hat{T}_{p,q}(\hat{f})$ is $k-$summable (resp. $k-$summable in direction d). With this characterization we may apply known theorems of summability in our context. The first consequence of this observation is that we can use the classical Borel-Laplace method of summation to obtain "explicit formulas" for the sum. Indeed, in order to sum $\hat{f}\in R_{1/k,d}^{(p,q)}$ we first sum $\hat{T}_{p,q}(\hat{f})(t)(x,\varepsilon)$ and then we replace t by $x^p\varepsilon^q$. According to Theorem 1.1.10, we first apply the formal Borel transformation $\hat{\mathcal{B}}_k: t^k\mathcal{E}^{(p,q)}[[t]]_{1/k}\to\mathcal{E}^{(p,q)}\{\xi\}$ to $t^k\hat{T}_{p,q}(\hat{f})$, we make analytic continuation and check the exponential growth in the variable ξ and finally we apply the Laplace transform in direction d, $\mathcal{L}_{k,d}$. Thus we have obtained the following proposition.

Proposition 1.2.29. *Let $\hat{f}\in\mathcal{S}_r$, for some $r>0$. Then $\hat{f}$ is $k-$summable in the monomial $x^p\varepsilon^q$ in direction d if and only if $\hat{\mathcal{B}}_k(t^k\hat{T}_{p,q}(\hat{f}))(\xi)(x,\varepsilon)$ can be continued analytically as $\varphi(x,\varepsilon,\xi)$ on $D_r\times D_r\times S(d,\alpha,+\infty)$, for some $\alpha>0$, with exponential growth at most k in ξ, uniform in x and ε. In this case the $k-$sum of $\hat{f}$ is given by*

$$f(x,\varepsilon)=\frac{1}{x^{kp}\varepsilon^{kq}}\int_0^{e^{id}\infty}\varphi(x,\varepsilon,\xi)e^{-(\xi/x^p\varepsilon^q)^k}d\xi^k=\frac{1}{x^{kp}\varepsilon^{kq}}\mathcal{L}_{k,d}(\varphi)(x,\varepsilon,x^p\varepsilon^q).$$

Given $\hat{f}$, $k-$summable in $x^p\varepsilon^q$ in a direction d, when using the decomposition of $\hat{f}$ as in Remark 1.2.17 we note that by Proposition 1.2.18 and its proof $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in direction d with sum f if and only all its components $\hat{f}_{ij}$ are $k-$summable in $\zeta\eta$, $\zeta=x^p,\eta=\varepsilon^q$ in direction d with sum f_{ij} and the components of the k-sum are the $k-$sums of the components, that is, $f(x,\varepsilon)=\sum x^i\varepsilon^j f_{ij}(x^p,\varepsilon^q)$. An alternative proof is offered by the Borel-Laplace method of the previous proposition: if $\hat{f}(x,\varepsilon)=\sum x^i\varepsilon^j\hat{f}_{ij}(x^p,\varepsilon^q)$, by equation (1-9) we see that:

$$\hat{\mathcal{B}}_k(t^k\hat{T}_{p,q}(\hat{f}))(\xi)(x,\varepsilon)=\sum_{i,j}x^i\varepsilon^j\hat{\mathcal{B}}_k(t^k\hat{T}_{1,1}(\hat{f}_{ij}))(\xi)(x^p,\varepsilon^q). \tag{1-23}$$

Since analytic continuation is compatible with standard operations, $\hat{\mathcal{B}}_k(t^k\hat{T}_{p,q})$ can be continued analytically as $\varphi(x,\varepsilon,\xi)$ on $D_r\times D_r\times S(d,\alpha,+\infty)$ if and only if every $\hat{\mathcal{B}}_k(t^k\hat{T}_{1,1}(\hat{f}_{ij}))$ can be continued analytically as $\varphi_{ij}(x,\varepsilon,\xi)$ on $D_r\times D_r\times S(d,\alpha,+\infty)$ and $\varphi(x,\varepsilon,\xi)=\sum x^i\varepsilon^j\varphi_{ij}(x,\varepsilon,\xi)$. Also it is clear that φ has exponential growth at most k in ξ, uniform in x and ε if and only if every φ_{ij} does it. Since the Laplace transform is linear, the result follows.

We now turn back to the case of $p = q = 1$. Let $\hat{f} = \sum a_{n,m} x^n \varepsilon^m \in \mathcal{S}$ and suppose it is $k-$summable in $x\varepsilon$ in direction d with sum f, say defined over $\Pi(a, b, r)$. Inequalities (1-15) and (1-14) show that the series $\hat{f}_{-m}(t)/t^m = \sum_{n=0}^{\infty} a_{n+m,n} t^n$ and $\hat{f}_m(t) = \sum_{n=0}^{\infty} a_{n,n+m} t^n$ are also $k-$summable in direction d with sums $f_{-m}(t)/t^m$ and $f_m(t)$ respectively, defined on $V(a, b, r^2)$ and the series $\sum_{m=0}^{\infty} \frac{f_{-m}(t)}{t^m} x^n + \sum_{m=1}^{\infty} f_m(t)\varepsilon^m$ converges for all $|x|, |\varepsilon| < r$ and has sum $T(f)(t)(x, \varepsilon)$.

We want to give a characterization of $\hat{f}$ being $k-$summable in $x\varepsilon$ in a direction d in terms of the series $\hat{f}_{-m}(t)/t^m$ and $\hat{f}_m(t)$. We note that the corresponding formal Borel transforms of $\hat{T}_{1,1}(\hat{f})$ and the previous series are related by the formula

$$\hat{\mathcal{B}}_k(t^k \hat{T}_{1,1} \hat{f})(\xi)(x, \varepsilon) = \sum_{m=0}^{\infty} \hat{\mathcal{B}}_k \left(t^k \frac{f_{-m}(t)}{t^m} \right) (\xi) x^m + \sum_{m=1}^{\infty} \hat{\mathcal{B}}_k \left(t^k f_m(t) \right) (\xi) \varepsilon^m.$$

From this expression it is clear that the formal $k-$Borel transform of $t^k \hat{T}_{1,1} \hat{f}$ can be analytically continued, say as $\varphi(x, \varepsilon, \xi)$ on $D_r \times D_r \times S(d, \alpha, +\infty)$ for some $\alpha > 0$ if and only if every $\hat{\mathcal{B}}_k \left(t^k \frac{f_{-m}(t)}{t^m} \right)$ and $\hat{\mathcal{B}}_k \left(t^k f_m(t) \right)$ can be analytically continued as $\varphi_{-m}(\xi)/\xi^m$ and $\varphi_m(t)$ on $S(d, \alpha, +\infty)$ respectively, and

$$\varphi(x, \varepsilon, \xi) = \sum_{m=0}^{\infty} \frac{\varphi_{-m}(\xi)}{\xi^m} x^m + \sum_{m=1}^{\infty} \varphi_m(\xi) \varepsilon^m,$$

for all $|x|, |\varepsilon| < r$. By the other hand, if $\varphi(x, \varepsilon, \xi)$ has exponential growth at most k in ξ, uniform in x and ε, say $|\varphi(x, \varepsilon, \xi)| \leq K e^{B|\xi|^k}$, then by Cauchy's inequalities it follows that

$$\left| \frac{\varphi_{-m}(\xi)}{\xi^m} \right| = \left| \frac{1}{m!} \frac{\partial^m \varphi}{\partial x^m}(0, \varepsilon, \xi) \right| \leq \frac{K}{\rho^m} e^{B|\xi|^k}, \quad |\varphi_m(\xi)| = \left| \frac{1}{m!} \frac{\partial^m \varphi}{\partial \varepsilon^m}(x, 0, \xi) \right| \leq \frac{K}{\rho^m} e^{B|\xi|^k}.$$

for all $0 < \rho < r$ and all $m \in \mathbb{N}$. This not only says that the functions $\frac{\varphi_{-m}(\xi)}{\xi^m}$ and $\varphi_m(\xi)$ have exponential growth at most k in $S(d, \alpha, +\infty)$, but also have a common type and the bounding constant K/ρ^m. Conversely, if the functions $\frac{\varphi_{-m}(\xi)}{\xi^m}$ and $\varphi_m(\xi)$ satisfy the above inequalities it follows that

$$|\varphi(x, \varepsilon, \xi)| \leq \frac{2K}{1 - \rho'/\rho} e^{B|\xi|^k},$$

for $|x|, |\varepsilon| < \rho' < \rho$, for all $\rho < r$.

For a general monomial $x^p \varepsilon^q$ and a formal series $\hat{f} = \sum_{i,j} x^i \varepsilon^j \hat{f}_{ij}(x^p, \varepsilon^q) \in \mathcal{S}$, we can apply the above reasoning to each of its components $\hat{f}_{i,j}$. Note that for every possible pair i, j, the corresponding series to $\hat{f}_{ij}$ are $\hat{f}_{ij,-m}(t)/t^m = \sum_{n=0}^{\infty} a_{np+mp+i,nq+j} t^n$ and $\hat{f}_{ij,m}(t) = \sum_{n=0}^{\infty} a_{np+i,nq+mq+j} t^n$. With this notation we can state the following result.

Proposition 1.2.30. *Let $\hat{f} \in \mathcal{S}_r$, for some $r > 0$ and put $\hat{f}(x,\varepsilon) = \sum_{i,j} x^i\varepsilon^j \hat{f}_{ij}(x^p,\varepsilon^q)$. Then $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in the direction d if and only if the following properties holds*

1. *There is $\alpha > 0$ such that all the formal series $\hat{\mathcal{B}}_k\left(t^k\hat{f}_{ij,-m}(t)/t^m\right)$ and $\hat{\mathcal{B}}_k\left(t^k\hat{f}_{ij,m}(t)\right)$ admits analytic continuation, say $\varphi_{ij,-m}(\xi)/\xi^m$ and $\varphi_{ij,m}(\xi)$, to $S(d,\alpha,+\infty)$.*

2. *There are constants $K, B > 0$ such that all the functions $\varphi_{ij,-m}(\xi)/\xi^m$ and $\varphi_{ij,m}(\xi)$ satisfy $|\varphi_{ij,-m}(\xi)/\xi^m| \leq \frac{K}{\rho^m}e^{B|\xi|^k}$, $|\varphi_{ij,m}(\xi)| \leq \frac{K}{\rho^m}e^{B|\xi|^k}$, for all $\xi \in S(d,\alpha,+\infty)$ and all $0 < \rho < r$.*

To finish this section we note that a natural question is what happens when we fix one the variables in the monomial asymptotic expansions. We can see that the asymptotic property remains valid for the non-fixed variable. More precisely we have the following result, whose proof is an immediate consequence of Corollary 1.1.3.

Proposition 1.2.31. *Let $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$ be a monomial sector, $f \in \mathcal{O}(\Pi_{p,q})$ and $\hat{f} \in \mathcal{S}$ such that $f \sim_s^{(p,q)} \hat{f}$ on $\Pi_{p,q}$. Then there is $\rho > 0$ such that for all ε_0 with $|\varepsilon_0| < \rho$ the map $f_{\varepsilon_0}(x) = f(x,\varepsilon_0)$ admits $\hat{f}_{\varepsilon_0}(x) = \hat{f}(x,\varepsilon_0) \in \mathbb{C}[[x]]$ as $s/p-$Gevrey asymptotic expansion on $V(a/p - \arg(\varepsilon_0^q)/p, b/p - \arg(\varepsilon_0^q)/p, \rho)$. In particular, if $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in some direction d then $\hat{f}_{\varepsilon_0}$ is $kp-$summable in direction $d/p - \arg(\varepsilon_0^q)/p$.*

We finish this section with an example of monomial summability based on Example 1.1.1.

Example 1.2.1. Consider the series $\hat{f} = \sum_{n,m\geq 0} a_{n,m}x^n\varepsilon^m$ where $a_{n,m} = (-|n-m|)^{\min\{n,m\}}$ and $a_{0,0} = 0$. We want to study its $1-$summability in $x\varepsilon$. To calculate $\hat{T}_{1,1}(\hat{f})$ note that:

$$b_0(x) = \sum_{m=1}^{\infty} x^m, b_n(x) = (-1)^n \sum_{m=0}^{\infty} m^n x^m, \quad c_n(\varepsilon) = (-1)^n \sum_{m=1}^{\infty} m^n\varepsilon^m.$$

Then we obtain its formal $1-$Borel transform by

$$\hat{\mathcal{B}}_1(t\hat{T}_{1,1}(\hat{f})(t))(\xi)(x,\varepsilon) = \sum_{n=0}^{\infty}(b_n(x)+c_n(\varepsilon))\frac{\xi^n}{n!} = \frac{1}{1-xe^{-\xi}} + \frac{1}{1-\varepsilon e^{-\xi}} - 2 = \varphi(x,\varepsilon,\xi).$$

For a fixed (x,ε) with $|x|, |\varepsilon| < 1$, the radius of convergence of the above series is the minimum between $\mathrm{dist}(u, 2\pi i\mathbb{Z})$ and $\mathrm{dist}(v, 2\pi i\mathbb{Z})$, where $x = e^u, \varepsilon = e^v$ and $\mathrm{Re}(u), \mathrm{Re}(v) < 0$. The domain of definition of the analytic continuation φ is the set conformed by all the triples $(x,\varepsilon,\xi) \in \mathbb{C}^3$ such that $e^\xi \neq x$ and $e^\xi \neq \varepsilon$. On the other hand we see that $\hat{f}_0(t) = 0$ and for any $m \in \mathbb{N}^*$,

$$\hat{f}_m(t) = \hat{f}_{-m}(t)/t^m = \sum_{n=0}^{\infty}(-1)^n m^n t^n = \frac{1}{1+mt}, \quad |t| < 1/m,$$

and its corresponding formal 1-Borel transforms are:

$$\varphi_m(\xi) = \frac{\varphi_{-m}(\xi)}{\xi^m} = e^{-\xi m},$$

that have exponential growth 0 and are all bounded by 1 if $\mathrm{Re}(\xi) > 0$. Also if we fix $0 < \rho < r < 1$ and restrict φ to the domain $D_\rho \times D_\rho \times \{\xi \in \mathbb{C}|\mathrm{Re}(\xi) \geq \log(r)\}$ it follows that $|\varphi(x,\varepsilon,\xi)| \leq \frac{2}{1-\rho/r}$ on that set. In conclusion, it follows from Proposition 1.2.30 that $\hat{f}$ is $1-$summable only in every direction d of $(-\pi/2, \pi/2)$. To calculate its $1-$sum, for instance in direction $d = 0$, we see that the $1-$sum in direction d of $\hat{T}_{1,1}(\hat{f})$ in the space $\mathcal{E}_\rho$ is given by

$$\frac{1}{t}\int_0^{+\infty} \left(\frac{1}{e^\xi - x} + \frac{1}{e^\xi - \varepsilon}\right) e^{\xi - \xi/t} d\xi - 2 = \sum_{n=1}^{\infty} \frac{x^n + \varepsilon^n}{1 + nt}.$$

So the $1-$sum f in $x\varepsilon$ of $\hat{f}$ in direction $d = 0$ is obtained by changing $t = x\varepsilon$ in the above expression, and has domain of definition $\{(x,\varepsilon) \in \mathbb{C} \mid |x|, |\varepsilon| < 1 \text{ and } x\varepsilon \neq -1/n, n \in \mathbb{N}^*\}$.

1.2.4 Some formulas for the sum

As usual we first focus in the case $p = q = 1$. On the problem of computing the sum of a $k-$summable series in a monomial $\hat{f}$, the first issue we face is to calculate the expression $\hat{T}_{1,1}(\hat{f})$. This is not always so easy, so it would be advantageous to have alternative ways to calculate the sum. We will see that in adequate polysectors (products of sectors) this can be done by writing $\hat{f}$ as a series on x with coefficients functions of ε or as a series in ε with coefficients functions of x.

Suppose $\hat{f} = \sum a_{n,m} x^n \varepsilon^m$ is $1/k-$Gevrey in $x\varepsilon$, i.e. there are constants B, D with $|a_{n,m}| \leq DB^{n+m}\min\{n!^{1/k}, m!^{1/k}\}$. When we write $\hat{f} = \sum_{n=0}^{\infty}(b_n(x)+c_n(\varepsilon))(x\varepsilon)^n = \sum_{n=0}^{\infty} f_{n*}(\varepsilon)x^n = \sum_{m=0}^{\infty} f_{*m}(x)\varepsilon^m$, every $f_{n*}(\varepsilon)$ and $f_{*m}(x)$ has radius of convergence at least $1/B$ and $\hat{f} \in \mathcal{O}_b(D_r)[[x]]_{1/k} \cap \mathcal{O}_b(D_r)[[\varepsilon]]_{1/k}$ for $r < 1/B$. Then the *$k-$Borel transforms in x and in ε of* $\hat{f}$, respectively, defined as:

$$\hat{\mathcal{B}}_{k,(1,0)}^{(1,1)}(x^k \hat{f})(\xi_1, \varepsilon) = \sum_{n=0}^{\infty} \frac{f_{n*}(\varepsilon)}{\Gamma(1 + n/k)} \xi_1^n, \quad \hat{\mathcal{B}}_{k,(0,1)}^{(1,1)}(\varepsilon^k \hat{f})(x, \xi_2) = \sum_{m=0}^{\infty} \frac{f_{*m}(x)}{\Gamma(1 + m/k)} \xi_2^m,$$

are convergent for $|\varepsilon| < 1/B$, $|\xi_1| < 1/Bk^{1/k}$ and $|x| < 1/B$,$|\xi_2| < 1/Bk^{1/k}$, respectively. The notation used here will be clear in the next chapter.

Now assume that $\hat{f}$ is $k-$summable in $x\varepsilon$ in direction d with sum f defined on $\Pi(a, b, r)$, with $d = (a + b)/2$ and $b - a > \pi/k$. Consider sectors $V_1 = V(a', b', \rho)$ and $V_2 = V(a'', b'', \rho)$ with $V_1 \times V_2 \subset \Pi(a, b, r)$, that is, $a \leq a' + a'' < b' + b'' \leq b$ and $\rho < r$. Then f defines two analytic functions $f_1 : V_1 \longrightarrow \mathcal{O}_b(V_2)$ and $f_2 : V_2 \longrightarrow \mathcal{O}_b(V_1)$ given by $x \longmapsto f_1(x)(\varepsilon) = f(x, \varepsilon)$ and $\varepsilon \longmapsto f_2(\varepsilon)(x) = f(x, \varepsilon)$, respectively.

If we take V_1 with $\pi/k < b' - a' < b - a$, we can always take V_2 satisfying $b'' < b - b'$ and $a - a' < a''$, i.e., V_1 is a $k-$wide sector and V_2 is a small one. We now can see that $f_1 \in \mathcal{O}(V_1, \mathcal{O}_b(V_2))$ is the classical $k-$sum of $\hat{f}$ on the sector V_1, in the space $\mathcal{O}_b(V_2)$. Indeed, taking any $W \Subset V_1$, $W \times V_2$ is always contained in a subsector $\widetilde{\Pi}$ of Π. So if $x \in W$ and $\varepsilon \in V_2$, applying formulas (1-13) (with $C_N(\widetilde{\Pi}) = CA^N N!^{1/k}$) and (1-5) we obtain:

$$\begin{aligned}\left|f_1(x)(\varepsilon) - \sum_{n=0}^{N-1} f_{n*}(\varepsilon)x^n\right| &= \left|f(x,\varepsilon) - \sum_{n=0}^{N-1}\left(\sum_{m=0}^{n} a_{n,m}x^n\varepsilon^m + c_n(\varepsilon)x^n\varepsilon^n\right)\right| \\ &= \left|f(x,\varepsilon) - \sum_{n=0}^{N-1}(b_n(x)+c_n(\varepsilon))(x\varepsilon)^n + \sum_{m=0}^{N-1}\sum_{n=N}^{\infty} a_{n,m}x^n\varepsilon^m\right| \\ &\le CA^N N!^{1/k}|x\varepsilon|^N + \sum_{m=0}^{N-1}\sum_{n=N}^{\infty} DB^{n+m}m!^{1/k}|x|^n|\varepsilon|^m \\ &\le \left(C(A\rho)^N + D\sum_{m=0}^{N-1}(B\rho)^m\frac{B^N}{1-B\rho}\right)N!^{1/k}|x|^N.\end{aligned}$$

The same calculations works for ε, that is, taking V_2 as a $k-$wide sector and V_1 as a small one we see that $f_2 \in \mathcal{O}(V_2, \mathcal{O}_b(V_1))$ is $k-$summable on V_2. Comparing with the Borel-Laplace transform method we obtain the following proposition.

Proposition 1.2.32. *Let $\hat{f}$ be a $k-$summable series in $x\varepsilon$ on Π with sum f. Consider $V_1 = V(a', b', \rho)$ and $V_2 = V(a'', b'', \rho)$ sectors in $\mathbb{C}$ with $V_1 \times V_2 \subset \Pi$. Then*

1. *If V_1 is a $k-$wide sector, $\hat{f} \in \mathcal{O}_b(V_2)[[x]]$ is $k-$summable on V_1 with sum*

$$f(x,\varepsilon) = f_1(x)(\varepsilon) = \frac{1}{x^k}\int_0^{e^{id'}\infty} \psi_1(\xi_1,\varepsilon)e^{-(\xi_1/x)^k}d\xi_1^k,$$

where $\psi_1(\xi_1,\varepsilon)$ is the analytic continuation of $\hat{\mathcal{B}}^{(1,1)}_{k,(1,0)}(x^k\hat{f})$ to $S((b'+a')/2, \alpha_1, +\infty) \times D_\rho$, for some $\alpha_1 > 0$ and some direction d' on $S((b'+a')/2, \alpha_1, +\infty)$.

2. *If V_2 is a $k-$wide sector, $\hat{f} \in \mathcal{O}_b(V_1)[[\varepsilon]]$ is $k-$summable on V_2 with sum*

$$f(x,\varepsilon) = f_2(\varepsilon)(x) = \frac{1}{\varepsilon^k}\int_0^{e^{id''}\infty} \psi_2(x,\xi_2)e^{-(\xi_2/\varepsilon)^k}d\xi_2^k,$$

where $\psi_2(x,\xi_2)$ is the analytic continuation of $\hat{\mathcal{B}}^{(1,1)}_{k,(0,1)}(\varepsilon^k\hat{f})$ to $D_\rho \times S((b''+a'')/2, \alpha_2, +\infty)$, for some $\alpha_2 > 0$ and some direction d'' on $S((b''+a'')/2, \alpha_2, +\infty)$.

The previous proposition generalizes to any monomial by means of Proposition 1.2.18. Indeed, let $\hat{f}$ be a $k-$summable series in $x^p\varepsilon^q$ on $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$, $b - a > \pi/k$, with sum f.

Since $\hat{f}$ is $1/k-$Gevrey in the monomial $x^p\varepsilon^q$ then $\hat{f}\in\mathcal{O}_b(D_r)[[x]]_{1/pk}\cap\mathcal{O}_b(D_r)[[\varepsilon]]_{1/qk}$, for some $r>0$. Here we use the *$k-$Borel transforms in x^p and in ε^q of $\hat{f}$*, respectively, defined as:

$$\hat{\mathcal{B}}^{(p,q)}_{k,(1,0)}(x^{pk}\hat{f})(\xi_1,\varepsilon)=\sum_{n=0}^{\infty}\frac{f_{n*}(\varepsilon)}{\Gamma\left(1+n/pk\right)}\xi_1^n,\quad \hat{\mathcal{B}}^{(p,q)}_{k,(0,1)}(\varepsilon^{qk}\hat{f})(x,\xi_2)=\sum_{m=0}^{\infty}\frac{f_{*m}(x)}{\Gamma\left(1+m/qk\right)}\xi_2^m,$$

that turn out to be convergent in some polydiscs.

As before, consider sectors $V_1=V(a',b',\rho^{1/p})$, $V_2=V(a'',b'',\rho^{1/q})$ in x^p, ε^q, respectively, satisfying $V_1\times V_2\subset\Pi_{p,q}$ and consider the functions $f_1:V_1\longrightarrow\mathcal{O}_b(V_2)$ and $f_2:V_2\longrightarrow\mathcal{O}_b(V_1)$ that f naturally defines. If $\zeta=x^p$ and $\eta=\varepsilon^q$ then the sectors

$$\widetilde{V_1}=\{\zeta\in\mathbb{C}\mid 0<|\zeta|<\rho,\ pa'<\arg(\zeta)<pb'\},$$
$$\widetilde{V_2}=\{\eta\in\mathbb{C}\mid 0<|\eta|<\rho,\ qa''<\arg(\eta)<qb''\},$$

in the new variables, satisfy $\widetilde{V_1}\times\widetilde{V_2}\subset\Pi_{1,1}(a,b,r)$.

When we decompose f and $\hat{f}$ as $f=\sum_{i,j}x^i\varepsilon^j f_{ij}(x^p,\varepsilon^q)$ and $\hat{f}=\sum_{i,j}x^i\varepsilon^j\hat{f}_{ij}(x^p,\varepsilon^q)$ respectively, we know by Proposition 1.2.18 that $f_{ij}\sim^{(1,1)}_{1/k}\hat{f}_{ij}$ on $\Pi_{1,1}(a,b,r)$, for all $0\leq i<p$ and $0\leq j<q$. Then we can apply the previous proposition to every $\hat{f}_{ij}$ and via Proposition 1.1.4 to obtain the following formulas.

Proposition 1.2.33. *Let $\hat{f}$ be a $k-$summable series in $x^p\varepsilon^q$ on $\Pi_{p,q}$ with sum f. Consider $V_1=V(a',b',\rho^{1/p})$ and $V_2=V(a'',b'',\rho^{1/q})$ sectors in $\mathbb{C}$ with $V_1\times V_2\subset\Pi_{p,q}$. Then*

1. *If V_1 is a $pk-$wide sector, $\hat{f}\in\mathcal{O}_b(V_2)[[x]]$ is $pk-$summable on V_1 with sum*

$$f(x,\varepsilon)=f_1(x)(\varepsilon)=\frac{1}{x^{pk}}\int_0^{e^{id'}\infty}\psi_1(\xi_1,\varepsilon)e^{-(\xi_1/x)^{pk}}d\xi_1^{pk},$$

where $\psi_1(\xi_1,\varepsilon)$ is the analytic continuation of $\hat{\mathcal{B}}^{(p,q)}_{k,(1,0)}(x^{pk}\hat{f})$ to a product of the form $S((b'+a')/2,\alpha_1,+\infty)\times D_{\rho^{1/q}}$, for some α_1 and some direction d' on $S((b'+a')/2,\alpha_1,+\infty)$. Besides, the $i-$component of $x^{pk}f_1(x)$ is given by

$$\sum_{j=0}^{q-1}\varepsilon^j\int_0^{e^{id'}\infty}\psi_{ij,1}(\zeta_1,\varepsilon^q)e^{-(\zeta_1/x^p)^k}d\zeta_1^k,$$

where $\psi_{ij,1}$ is the analytic continuation of $\hat{\mathcal{B}}^{(1,1)}_{k,(1,0)}(\zeta^k\hat{f}_{ij})$ to $S((pb'+pa')/2,p\alpha_1,+\infty)\times D_\rho$.

2. *If V_2 is a $qk-$wide sector, $\hat{f}\in\mathcal{O}_b(V_1)[[\varepsilon]]$ is $qk-$summable on V_2 with sum*

$$f(x,\varepsilon)=f_2(\varepsilon)(x)=\frac{1}{\varepsilon^{qk}}\int_0^{e^{id''}\infty}\psi_2(x,\xi_2)e^{-(\xi_2/\varepsilon)^{qk}}d\xi_2^{qk},$$

where $\psi_2(x,\xi_2)$ is the analytic continuation of $\hat{\mathcal{B}}^{(p,q)}_{k,(0,1)}(\varepsilon^{qk}\hat{f})$ to a product of the form $D_{\rho^{1/p}}\times S((b''+a'')/2,\alpha_2,+\infty)$, for some α_2 and some direction d'' on the domain $S((b''+a'')/2,\alpha_2,+\infty)$. Besides, the $j-$component of $\varepsilon^{qk}f_2(\varepsilon)$ is given by

$$\sum_{i=0}^{p-1} x^i\int_0^{e^{id'}\infty}\psi_{ij,2}(x^p,\zeta_2)e^{-(\zeta_2/\varepsilon^q)^k}d\zeta_2^k,$$

where $\psi_{ij,2}$ is the analytic continuation of $\hat{\mathcal{B}}^{(1,1)}_{k,(0,1)}(\eta^k\hat{f}_{ij})$ to $D_\rho\times S((qb''+qa'')/2,q\alpha_2,+\infty)$.

Example 1.2.2. This example is taken from [CDMS]. Consider the singularly perturbed linear differential equation

$$\varepsilon x^2y'=(1+x)y-x\varepsilon,$$

where $y\in\mathbb{C}$. We will see in Chapter 3 that it has a unique formal solution $\hat{y}\in\hat{R}$ that is 1-summable in $x\varepsilon$ (see Theorem 3.1.4). However we can show here directly that $\hat{y}$ is $1-$Gevrey in $x\varepsilon$.

The formal solution can be easily calculated by inserting the expression $\hat{y}=\sum_{n=0}^{\infty}y_{n*}(\varepsilon)x^n$ into the equation and solving recursively. The solution is given explicitly by

$$\hat{y}=\varepsilon\sum_{n=0}^{\infty}\prod_{l=1}^{n}(l\varepsilon-1)x^{n+1},$$

and it reduces to a polynomial in x when $\varepsilon=1/N$, for $N\in\mathbb{N}^*$. If $|\varepsilon|<R$ then a direct rough estimation shows that

$$\sup_{|\varepsilon|<R}|y_{n+1*}(\varepsilon)|\le R(R+1)^n n^n,$$

for all $n\ge 0$, what shows that $\hat{y}$ is $1-$Gevrey in x.

If we write $\hat{y}=\sum_{n=0}^{\infty}y_{*m}(x)\varepsilon^m$, it follows from the differential equation that

$$y_{*m}(x)=\frac{x^m}{(1+x)^{2m-1}}P_m(x),$$

where $P_m\in\mathbb{Z}[x]$. In fact, $P_0(x)=0$, $P_1(x)=1$, for $m\ge 2$ the polynomial P_m has degree $m-2$ with leading term $(-1)^m$, $P_m(0)=(m-1)!$ and if we write $P_m(x)=\sum_{l=0}^{m-2}(-1)^l a_l^{(m)}x^l$, then $a_l^{(m)}\ge 0$ and they satisfy the recursion formula $a_l^{(m+1)}=(m+l)a_l^{(m)}+(m-l)a_{l-1}^{(m)}$, valid for all $m\ge 2$ and $0\le l\le m-2$. It follows by induction that $a_l^{(m)}\le 2^{m-3}(m-1)!$ if $m\ge 3$. If we take $|x|<\rho<1$ then it is immediate to check that

$$\sup_{|x|<\rho}|y_{*m}(x)|\le\frac{1}{2}\left(\frac{2\rho}{(1-\rho)^2}\right)^m(m-1)!,$$

for all $m\ge 1$, what shows that $\hat{y}$ is $1-$Gevrey in ε.

To obtain a formula for the $1-$sum in $x\varepsilon$ of $\hat{y}$ we can use Proposition 1.2.32. Indeed, a calculation using the binomial series shows that

$$\hat{\mathcal{B}}^{(1,1)}_{1,(1,0)}(x\hat{y})(\xi_1,\varepsilon)=\varepsilon\sum_{n=1}^{\infty}\prod_{l=1}^{n-1}(l\varepsilon-1)\frac{\xi_1^n}{n!}=-\varepsilon(1-\varepsilon\xi_1)^{1/\varepsilon},$$

and the function in the right side is well-defined for $\varepsilon\xi_1\neq 1$ and $\varepsilon\neq 0$. Then the $1-$sum is given in adequate polysectors by the formula

$$f(x,\varepsilon)=\frac{-\varepsilon}{x}\int_0^{e^{id}\infty}(1-\ e\xi_1)^{1/\varepsilon}e^{-\xi_1/x}d\xi_1=\int_0^{e^{i(d+\arg(\varepsilon))}\infty}(1-s)^{1/\varepsilon-1}e^{-s/x\varepsilon}ds,$$

but it has the disadvantage of having the fraction $1/\varepsilon$.

There is still another way we may calculate the sum of a series, $k-$summable in some monomial. This time we introduce a new variable by weighting the variables x and ε, as in the end of Section 1.2.1. For the case $p=q=1$, consider real parameters $s_1,s_2>0$ such that $s_1+s_2=1$, a new variable $z\in\mathbb{C}$ and the morphism ι_G, where $G=G^{(1,1)}_{s_1,s_2}=\{\ell(n,m)=ns_1+ms_2|n,m\in\mathbb{N}\}$.

Suppose that $f\sim^{(1,1)}_{1/k}\hat{f}=\sum a_{n,m}x^n\varepsilon^m$ on a monomial sector $\Pi=\Pi_{1,1}(a,b,r)$. Consider constants B',D' such that $|a_{n,m}|\leq D'B'^{n+m}\min\{\Gamma(1+n/k),\Gamma(1+m/k)\}$, for all n,m. For a fixed $(x,\varepsilon)\in\Pi$ with $|x|,|\varepsilon|<1/B'$, consider the sector $V=V(a-\arg(x\varepsilon),b-\arg(x\varepsilon),\widetilde{r})$, where $\widetilde{r}=\min\{1,(r/|x|)^{1/s_1},(r/|\varepsilon|)^{1/s_2}\}$. It follows that $(z^{s_1}x,z^{s_2}\varepsilon)\in\Pi$ for all $z\in V$.

For a subsector $W\Subset V$ we can always find $\widetilde{\Pi}\Subset\Pi$ such that $(z^{s_1}x,z^{s_2}\varepsilon)\in\widetilde{\Pi}$ for all $z\in W$. Then by hypothesis there are constants $C,A>0$ such that

$$\left|f(z^{s_1}x,z^{s_2}\varepsilon)-\sum_{n=0}^{N-1}(b_n(z^{s_1}x)+c_n(z^{s_2}\varepsilon))(x\varepsilon)^nz^n\right|\leq CA^N\Gamma\left(1+N/k\right)|x\varepsilon|^N|z|^N,\quad (1\text{-}24)$$

for all $N\in\mathbb{N}$ and $z\in W$. We can use this inequalities to show that $f(z^{s_1}x,z^{s_2}\varepsilon)$ admits $\iota_G(\hat{f})$ as asymptotic expansion of $1/k-$Gevrey type in V (in general, in non-integer powers of z). To show this, we use the notation $I_M=\{(n,m)\in\mathbb{N}^2\mid n\leq M \text{ or } m\leq M\}$, where $M\in\mathbb{N}$ and $J_{\lambda_0}=\{(n,m)\in\mathbb{N}^2|\ell(n,m)<\lambda_0\}$ for $\lambda_0\in G$.

Let $\lambda_0\in G$ and let $N=[\lambda_0]$ be its integer part. Note that if $\ell(n,m)<\lambda_0$ then $\min\{n,m\}\leq N$. This shows that $J_{\lambda_0}\subset I_N$. Then if $z\in W$, it follows from inequality (1-24) that

$$\left|f(z^{s_1}x,z^{s_2}\varepsilon)-\sum_{\substack{0\leq\lambda<\lambda_0\\ \lambda\in G}}\sum_{\ell(n,m)=\lambda}a_{n,m}x^n\varepsilon^mz^\lambda\right|=\left|f(z^{s_1}x,z^{s_2}\varepsilon)-\sum_{(n,m)\in J_{\lambda_0}}a_{n,m}x^n\varepsilon^mz^{ns_1+ms_2}\right|$$

$$= \left| f(z^{s_1}x, z^{s_2}\varepsilon) - \sum_{(n,m)\in I_N} a_{n,m}x^n\varepsilon^m z^{ns_1+ms_2} + \sum_{(n,m)\in I_N\setminus J_{\lambda_0}} a_{n,m}x^n\varepsilon^m z^{ns_1+ms_2} \right|$$
$$\leq CA^{N+1}\Gamma\left(1+(N+1)/k\right)|x\varepsilon|^{N+1}|z|^{N+1} + \sum_{(n,m)\in I_N\setminus J_{\lambda_0}} |a_{n,m}||x|^n|\varepsilon|^m|z|^{ns_1+ms_2}$$
$$\leq KL^{\lambda_0}\Gamma(1+\lambda_0/k)|z|^{\lambda_0},$$

for K, L large enough constants. To justify the last inequality note if $(n,m)\in I_N\setminus J_{\lambda_0}$ then $|a_{n,m}|/B'^{n+m}\Gamma(1+N/k)$ is bounded and $|z|^{ns_1+ms_2} < |z|^{\lambda_0}$ because $|z|<1$. In the first summand we also have $|z|^{N+1} < |z|^{\lambda_0}$ since $\lambda_0 < N+1$. The conclusion follows observing that we can replace $\Gamma(1+N/k)$ and $\Gamma(1+(N+1)/k)$ by $\Gamma(1+\lambda_0/k)$ by enlarging, if necessary, the previous constants.

The previous considerations extend to the case of any monomial $x^p\varepsilon^q$. Considering again $s_1, s_2>0$ with $s_1+s_2=1$, but this time we consider the semigroup of $\mathbb{R}_{\geq 0}$ generated by s_1/p and s_2/q and the morphism given by $\hat{f}(x,\varepsilon)\mapsto \hat{f}(z^{s_1/p}x, z^{s_2/q}\varepsilon)$. The asymptotic behavior is similar to the previous case and we state the result in the following proposition.

Proposition 1.2.34. *Let $s_1, s_2>0$ with $s_1+s_2=1$. Suppose that $f\sim_{1/k}^{(p,q)}\hat{f}$ on $\Pi_{p,q}(a,b,r)$. Then there is $\rho>0$ such that for any fixed $(x_0,\varepsilon_0)\in\Pi_{p,q}(a,b,r)$ with $|x_0|, |\varepsilon_0|\leq\rho$, the map $f(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0)$ admits $\hat{f}(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0)$ as asymptotic expansion of $1/k-$Gevrey type in the sector $V(a-arg(x_0^p\varepsilon_0^q), b-arg(x_0^p\varepsilon_0^q), \widetilde{r})$, where $\widetilde{r}=\min\{1, (r/|x_0|^p)^{1/s_1}, (r/|\varepsilon_0|^q)^{1/s_2}\}$.*

The proof follows using Proposition 1.2.18 and the previous case $p=q=1$. We remark that since in general s_1 is not necessarily a rational number, the asymptotic expansion must be understood as explained at the end of Section 1.1. When s_1 is rational, the previous proposition can be understood in the usual sense, up to a ramification (via Proposition 1.1.4).

In particular, the previous proposition implies that when $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in direction d with sum f, for points (x_0,ε_0) with small enough radius, $\hat{f}(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0)$ is $k-$summable in direction $d-\arg(x_0^p\varepsilon_0^q)$ with sum $f(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0)$, and in particular,

$$f(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0) = \frac{1}{z^k}\int_0^{e^{id}\infty}\varphi_{x_0,\varepsilon_0}(\zeta)e^{-(\zeta/z)^k}d\zeta^k, \tag{1-25}$$

where $\varphi_{x_0,\varepsilon_0}$ is the analytic continuation to a sector bisected by d, of

$$\hat{\mathcal{B}}_k(z^k\iota_G(x_0,\varepsilon_0)(\hat{f})(z))(\zeta) = \sum_{\lambda\in G}\sum_{ns_1/p+ms_2/q=\lambda} a_{n,m}x_0^n\varepsilon_0^m\frac{\zeta^\lambda}{\Gamma(1+\lambda/k)},$$

the formal $k-$Borel transform of $z^k\iota_G(x_0,\varepsilon_0)(\hat{f})$.

1.3 Tauberian theorems for monomial summability

The goal of this section is to describe some tauberian theorems for monomial summability as well as relating different levels of summability for different monomials in order to be able to establish in future works a correct definition for a type of monomial multisummability.

As for a fixed monomial, summability in the monomial is equivalent to summability in the classical sense, we obtain analogous results as in Section 1.1. The first classic result is that the absence of singular directions is a tauberian condition.

Proposition 1.3.1. *If $\hat{f} \in R_{1/k}^{(p,q)}$ has no singular directions then $\hat{f} \in R$.*

Proof. Since d is a singular direction of $\hat{f}$ if and only it is for $\hat{T}_{p,q}(\hat{f})$, we conclude that $\hat{T}_{p,q}(\hat{f})$ has no singular directions and by Proposition 1.1.12 it is convergent. It follows from Proposition 1.2.3 that $\hat{f}$ is also convergent. □

The second one says that being summable in a monomial for different levels implies convergence.

Proposition 1.3.2. *Let $0 < k < k'$ be positive real numbers. Then for any monomial $x^p\varepsilon^q$ we have $R_{1/k}^{(p,q)} \cap R_{1/k'}^{(p,q)} = R_{1/k}^{(p,q)} \cap \hat{R}_{1/k'}^{(p,q)} = R$.*

Following the same ideas that in the proof of Proposition 1.1.1 we can relate summability in a monomial with summability in some power of this monomial.

Proposition 1.3.3. *Let $k > 0$ be a real number, $p, q, M \in \mathbb{N}^*$ be natural numbers and d a direction. Then $R_{1/k,d}^{(p,q)} = R_{M/k,Md}^{(Mp,Mq)}$.*

Proof. Let $\hat{f} \in \mathcal{S}$ be a formal power series. We can assume that $M \geq 1$. Note that if we write $\hat{T}_{p,q}(\hat{f})(t)(x,\varepsilon) = \sum_{n=0}^{\infty} f_n(x,\varepsilon)t^n$ then

$$\hat{T}_{Mp,Mq}(\hat{f})(s)(x,\varepsilon) = \sum_{n=0}^{\infty} g_n(x,\varepsilon)s^n, \;\; g_n(x,\varepsilon) = \sum_{j=0}^{M-1} f_{Mn+j}(x,\varepsilon)(x^p\varepsilon^q)^j.$$

Suppose that $\hat{f} \in R_{1/k,d}^{(p,q)}$. Then there is $f \in \mathcal{O}(\Pi_{p,q}(a,b,r))$ such that $f \sim_{1/k}^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r) = \Pi_{Mp,Mq}(Ma, Mb, r^M)$, where $d = (b+a)/2$ and $b - a > \pi/k$. Using the previous decomposition, inequality (1-18) of the definition for $N = ML, L \in \mathbb{N}$ and the limit $\lim_{n\to+\infty} \frac{(Mn)!^{1/M}}{M^n n!} = 0$ we obtain that $f \sim_{M/k}^{(Mp,Mq)} \hat{f}$ on $\Pi_{Mp,Mq}(Ma, Mb, r^M)$, that is, $\hat{f} \in R_{M/k,Md}^{(Mp,Mq)}$.

Conversely, if $f \sim_{M/k}^{(Mp,Mq)} \hat{f}$ on $\Pi_{Mp,Mq}(Ma, Mb, r^M) = \Pi_{p,q}(a,b,r)$, by definition and the previous expressions we see that for all $\widetilde{\Pi}_{Mp,Mq} = \overline{\Pi}_{p,q}(a', b', \rho)$ and $N \in \mathbb{N}$ there are constants C, A such that for all $(x,\varepsilon) \in \overline{\Pi}_{p,q}$ we have

$$\left| f(x,\varepsilon) - \sum_{n=0}^{MN-1} f_n(x,\varepsilon)(x^p\varepsilon^q)^n \right| \leq CA^{MN} N!^{M/k} |x^p\varepsilon^q|^{MN}.$$

Now if we write f and the f_n as explained in Remark 1.2.17, say $f(x,\varepsilon) = \sum_{i,j} x^i\varepsilon^j f_{ij}(x^p,\varepsilon^q)$ and $f_n(x,\varepsilon) = \sum_{i,j} x^i\varepsilon^j(b_{ij,n}(x^p) + c_{ij,n}(\varepsilon^q))$, inequality (1-19) shows that

$$\left| f_{ij}(x,\varepsilon) - \sum_{n=0}^{MN-1} (b_{ij,n}(x^p) + c_{ij,p}(\varepsilon^q))(x^p\varepsilon^q)^n \right| \leq CA^{MN} N!^{M/k} \frac{|x^p\varepsilon^q|^{MN}}{|x|^i|\varepsilon|^j}, \tag{1-26}$$

for all possible i, j and $(x,\varepsilon) \in \overline{\Pi}_{p,q}$. Since $\hat{f} \in \hat{R}_{M/k}^{(Mp,Mq)}$, there is $0 < r' < r$ and constants B, D such that

$$\left| \sum_{l=0}^{M-1} (b_{ij,Mn+l}(x^p) + c_{ij,Mn+l}(\varepsilon^q))(x^p\varepsilon^q)^l \right| \leq DB^n n!^{M/k},$$

for all possible i, j and $|x|, |\varepsilon| < r'$. Using this bounds and (1-26) for $N+1$ we see that

$$\begin{aligned} \left| f_{ij}(x,\varepsilon) - \sum_{n=0}^{MN-1} (b_{ij,n}(x^p) + c_{ij,p}(\varepsilon^q))(x^p\varepsilon^q)^n \right| \leq \\ CA^{M(N+1)}(N+1)!^{M/k}|x^p\varepsilon^q|^{MN}|x|^{Mp-i}|\varepsilon|^{Mq-j} + DB^N N!^{M/k}|x^p\varepsilon^q|^{MN} \\ \leq KL^N N!^{M/k}|x^p\varepsilon^q|^{MN}, \end{aligned}$$

for all possible i, j and $(x,\varepsilon) \in \overline{\Pi}_{p,q}$, where K, L are large enough constants.

If $0 < \rho' < \rho < r'$ and $|x|^p, |\varepsilon|^q < \rho'$ it is straightforward to check that

$$\left\| T_{p,q}(f)_\rho(t) - \sum_{n=0}^{MN-1} f_n t^n \right\|_{\rho'} \leq \frac{2\left(\sum_{i,j} \rho^{i+j}\right)}{1 - \rho'/\rho} KL^N (MN)!^{1/k}|t|^{MN},$$

for all $t \in V(a', b', \rho^2)$ and $N \in \mathbb{N}$. An application of Proposition 1.1.1 shows that $T_{p,q}(f)_\rho \sim_{1/k} \hat{T}_{p,q}(\hat{f})$ on $V(a,b,r^2)$, for all $0 < \rho < r'$. By Proposition 1.2.20, we finally conclude that $f \sim_{1/k}^{(p,q)} \hat{f}$ on $\Pi_{p,q}(a,b,r)$, as we wanted to prove. □

Using Proposition 1.2.34, we can treat a particular case of a summable series in two different monomials.

Proposition 1.3.4. *Let $k, l > 0$ be positive real numbers and let $x^p\varepsilon^q$ and $x^{p'}\varepsilon^{q'}$ two monomials. Suppose that $k \neq l$ and $\min\{p/p', q/q'\} < 1 < \max\{p/p', q/q'\}$. Then $R_{1/k}^{(p,q)} \cap R_{1/l}^{(p',q')} = R$.*

Proof. The hypothesis $\min\{p/p', q/q'\} < 1 < \max\{p/p', q/q'\}$ is a necessary and sufficient condition to ensure that the system of equations

$$s_1 + s_2 = 1, \quad s_1/p = s_1'/p',$$
$$s_1' + s_2' = 1, \quad s_2/q = s_2'/q',$$

has a unique solution conformed by positive numbers. Indeed, the solution is $s_1 = p(q' - q)/(p'q - pq')$, $s_1' = p'(q' - q)/(p'q - pq')$, $s_2 = q(p' - p)/(p'q - pq')$, $s_2' = q'(p' - p)/(p'q - pq')$.

Let $\hat{f} \in R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l}$. From Proposition 1.2.34 we obtain a radius $\rho > 0$ such that for any fixed $(x_0, \varepsilon_0) \in \overline{D}^2_\rho$, the formal series $\hat{f}(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0) = \hat{f}(z^{s_1'/p'}x_0, z^{s_2'/q'}\varepsilon_0)$ in $z^{1/|p'q-pq'|}$ is both $k-$summable and $l-$summable in z. Since $k \neq l$, by Theorem 1.1.13 we conclude that $\hat{f}(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0)$ is convergent. Then using Proposition 1.2.9 we can conclude that $\hat{f}$ is convergent. □

At this point we are ready to formulate and prove the main result so far, comparing summable series in different monomials.

Theorem 1.3.5. *Let $k, l > 0$ be positive real numbers and let $x^p\varepsilon^q$ and $x^{p'}\varepsilon^{q'}$ be two monomials. The following statements are true:*

1. *If $p/p' = q/q' = l/k$ then $R^{(p,q)}_{1/k} = R^{(p',q')}_{1/l}$.*
2. *If $p/p' = q/q'$ and $q/q' \neq l/k$ then $R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l} = R$.*
3. *If $p/p' \neq q/q'$ then $R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l} = R$.*

Proof. We split the proof in cases. First consider the case $p/p' = q/q'$. If $d = \text{g.c.d.}(p, q)$ and $p = dp''$, $q = dq''$ then $p' = np''$, $q' = nq''$ where $n = q'/q'' = p'/p'' \in \mathbb{N}$. By Proposition 1.3.3 we see that

$$R^{(p,q)}_{1/k} = R^{(p'',q'')}_{1/dk} \quad \text{and} \quad R^{(p',q')}_{1/l} = R^{(p'',q'')}_{1/nl}.$$

Then the cases (1) and (2) follows from Proposition 1.3.2.

For the case $p/p' \neq q/q'$ we consider three possibilities:

Case I. Suppose $\max\{p/p', q/q'\} < l/k$. If $\hat{f} \in R^{(p,q)}_{1/k} \cap R^{(p',q')}_{1/l} \subset R^{(p,q)}_{1/k} \cap \hat{R}^{(p',q')}_{1/l}$, we conclude by Corollary 1.2.5 that $\hat{T}_{p,q}(\hat{f})$ is a $\max\{p/p', q/q'\}/l-$Gevrey series. Since $\hat{T}_{p,q}(\hat{f})$ is $k-$summable, by Theorem 1.1.13 we conclude that $\hat{T}_{p,q}(\hat{f})$ and therefore $\hat{f}$ are convergent.

Case II. Suppose $l/k < \min\{p/p', q/q'\}$. This is equivalent to the condition $\max\{p'/p, q'/q\} < k/l$ so this case follows from case (I).

Case III. Suppose $\min\{p/p', q/q'\} \leq l/k \leq \max\{p/p', q/q'\}$. Choose $a, b \in \mathbb{N}^*$ such that $l/k \neq a/b$ and $\min\{p/p', q/q'\} < a/b < \max\{p/p', q/q'\}$, or equivalently

$$\min\{bp/ap', bq/aq'\} < 1 < \max\{bp/ap', bq/aq'\}.$$

Since $R_{1/k}^{(p,q)} = R_{b/k}^{(bp,bq)}$ and $R_{1/l}^{(p',q')} = R_{a/l}^{(ap',aq')}$, this case follows from Proposition 1.3.4. $\square$

2 Monomial Borel-Laplace summation methods

In the previous chapter we have recalled the notion of monomial summability as presented originally in the paper [CDMS] and we have developed many of its properties. As we have seen many of them depend on the theory of one variable, since the notion of monomial summability can be expressed in terms of classical summability (see Proposition 1.2.11 and Proposition 1.2.20). However so far we have no similar tools to the Borel-Laplace method to study it. The aim of this chapter is to develop such methods. The idea behind them is to weight the variables adequately and generalize the formulas in the classical case. The formulas defining the Borel and Laplace transforms used here are essentially the same introduced in the paper [B3] for the case of two variables. The underlying difference between them is the domain of the functions we work with, being adequate connected sets of polysectors in the mentioned paper and monomial sectors used here. Many of the formulas we provide here are already used (sometimes implicitly) in the paper [BM] in the treatment of summability of formal power series solutions of singularly perturbed linear systems of ordinary differential equations given by the authors. In particular they show that those solutions are $(s_1, s_2)-$summable, for adequate values of (s_1, s_2), where this notion of summability is defined precisely using the generalized Borel and Laplace transforms. Once again it is not clear that the domains they consider are sectors in a monomial.

The chapter is divided into three sections. In the first one we define the Borel and Laplace transformations associated to a monomial, a weight in the variables and a parameter of summability, as well as its formal counterparts. Many properties of those transformations are studied, for instance proving that they are inverses one of each other and its behavior w.r.t. Gevrey asymptotic monomial expansions. An interesting affair is that the Borel transform converts a certain vector field into multiplication by the monomial employed. Even many of the properties are natural, the proofs are partly technical and include many calculations. The section ends introducing a convolution product compatible with the previous transformations, i.e. the Laplace transform convert the convolution into the usual product.

Then the method of summability associated with each Borel and Laplace transform is proposed in the second section, following the definitions in the classical case. The main result at that point is that this apparent new method of summability is equivalent to monomial summability. That is the content of Theorem 2.2.1. This new equivalence clarify the relation between monomial summability and $(s_1, s_2)-$summability explained in the first paragraph. The section concludes commenting alternative ways to prove properties obtained in the previous chapter, using the Borel-Laplace method. Finally the matter that concerns us in the last section is the behavior of monomial summability under point blow-ups. We only provide an elementary result that shows the stability of monomial summability under point blow-ups.

2.1 Monomial Borel and Laplace transforms

The goal of this section is to define the Borel transform, the Laplace transform and the convolution product associated with a monomial, a weight of the variables and a parameter of summability. These integral transformations will help us to provide a new way to characterize summability in a monomial. To achieve this purpose it is necessary to develop the fundamental properties of these transformation such as their action on formal power series and their action on Gevrey asymptotic expansions on monomial sectors.

2.1.1 Borel transform

Definition 2.1.1. Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$. The *$k-$Borel transform associated to the monomial $x^p\varepsilon^q$ with weight* (s_1, s_2) of a function f is defined by the formula

$$\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}(f)(\xi,\upsilon) = \frac{(\xi^p\upsilon^q)^{-k}}{2\pi i}\int_\gamma f(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk})e^u du,$$

where γ denotes a Hankel path.

In order to make the above formula meaningful, we are going to restrict our attention to analytic and bounded functions f defined on monomial sectors in $x^p\varepsilon^q$ of the form $S_{p,q}(d, \pi/k + 2\epsilon_0, R_0)$, where $0 < \epsilon_0 < \pi/k$. In this case, $\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}(f)$ will be defined and analytic on the sector $S_{p,q}(d, 2\epsilon_0, +\infty)$. Indeed, if $(\xi, \upsilon) \in S_{p,q}(d, 2\epsilon_0, +\infty)$, take any ϵ such that $(\xi, \upsilon) \in S_{p,q}(d, 2\epsilon, +\infty)$ and $0 < \epsilon < \epsilon_0$. Then take the integral along the path γ oriented positively and given by: the arc of a circle centered at 0 and radius

$$R > \max\{(|\xi|^p/R_0)^{k/s_1}, (|\upsilon|^q/R_0)^{k/s_2}\},$$

with endpoints corresponding to the directions $-\pi/2 - k(\epsilon_0 - \epsilon)$ and $\pi/2 + k(\epsilon_0 - \epsilon)$ and the semi-lines of those directions from this arc to ∞. If u goes along this path we see that $(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk}) \in S_{p,q}(d, \pi/k + 2\epsilon_0, R_0)$ and the integral converges, due to boundedness of f and to the exponential term tending to 0 in those directions. The result is independent of ϵ and R due to Cauchy's theorem.

We note in particular that if $f(x, \varepsilon) = x^\lambda\varepsilon^\mu$, $\lambda, \mu \in \mathbb{C}$, then it follows from Hankel's formula for the Gamma function that

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon)=\frac{\xi^{\lambda-pk}\upsilon^{\mu-qk}}{\Gamma\left(\frac{\lambda s_1/p+\mu s_2/q}{k}\right)}. \tag{2-1}$$

The previous formula let us introduce the *formal* $k-$*Borel transform associated to the monomial* $x^p\varepsilon^q$ *with weight* (s_1,s_2), defined naturally by

$$\hat{\mathcal{B}}_{k,(s_1,s_2)}^{(p,q)}:x^{pk}\varepsilon^{qk}\mathbb{C}[[x,\varepsilon]]\longrightarrow\mathbb{C}[[\xi,\upsilon]]$$
$$\sum_{n,m\geq 0}a_{n,m}x^{n+pk}\varepsilon^{m+qk}\mapsto\sum_{n,m\geq 0}\frac{a_{n,m}}{\Gamma\left(1+\frac{ns_1/p+ms_2/q}{k}\right)}\xi^n\upsilon^m.$$

It follows that $\hat{\mathcal{B}}_{k,(s_1,s_2)}^{(p,q)}$ establish a linear isomorphism between $x^{pk}\varepsilon^{qk}\hat{R}_{(s_1/pk,s_2/qk)}$ and $\mathbb{C}\{\xi,\upsilon\}$. In particular, we see from (1-3) that the image of $x^{pk}\varepsilon^{qk}\hat{R}_{1/k}^{(p,q)}$ is contained in $\mathbb{C}\{\xi,\upsilon\}$.

The reader may note that with the previous definitions we recover the formal Borel transforms in x^p and ε^q introduced in Section 1.2.4 as particular cases by letting $s_1=1$, $s_2=0$ and $s_1=0$, $s_2=1$. However we don't treat that cases here because in the analytic setting the domains of the functions involver change drastically from the product of a sector and a disc to monomial sectors.

Remark 2.1.1. If $f_0\in\mathcal{O}(V)$, $V=S(d,\pi/k+2\epsilon_0,R_0^2)$, is bounded then $f(x,\varepsilon)=f_0(x^p\varepsilon^q)$ defines a bounded analytic function on $S_{p,q}(d,\pi/k+2\epsilon_0,R_0)$ and in this case

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon)=\mathcal{B}_k(f_0)(\xi^p\upsilon^q),$$

for all $s_1,s_2>0$ such that $s_1+s_2=1$. In other words, the $k-$Borel transform associated to the monomial $x^p\varepsilon^q$ with weight (s_1,s_2) reduces to a $k-$Borel transform for functions depending only on $x^p\varepsilon^q$. We point out that the same considerations and the previous formula remain valid for the formal counterpart.

Remark 2.1.2. In relation with the process of weighting the variables introduced in the previous chapter, we can relate the forgoing Borel transform with the classical one as follows: consider a bounded function $f\in\mathcal{O}(S_{p,q})$, $S_{p,q}=S_{p,q}(d,\pi/k+2\epsilon_0,R_0)$, a point $(x_0,\varepsilon_0)\in S_{p,q}$ and $s_1,s_2>0$ with $s_1+s_2=1$. If we set $\widetilde{f}(z)=f(z^{s_1/p}x_0,z^{s_2/q}\varepsilon_0)$ then $\widetilde{f}$ defines a bounded function on $S(d-\arg(x_0^p\varepsilon_0^q),\pi/k+2\epsilon_0,\widetilde{R_0})$, $\widetilde{R_0}=\min\{(R_0/|x_0|^p)^{1/s_1},(R_0/|\varepsilon_0|^q)^{1/s_2}\}$ and in this case

$$(x_0^p\varepsilon_0^q)^k\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\zeta^{s_1/p}x_0,\zeta^{s_2/q}\varepsilon_0)=\mathcal{B}_k(\widetilde{f})(\zeta). \tag{2-2}$$

Note that here $\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)$ is only calculated on points (ξ,υ) where $\xi^{p/s_1}/\upsilon^{q/s_2}$ is constant (the constant given by $x_0^{p/s_1}/\varepsilon_0^{q/s_2}$).

As in the case of one variable, the formal as well as the analytic Borel transforms satisfy an interesting property sending a certain vector field into multiplying by a power of the monomial. Indeed, the vector field rises naturally if we take the derivative of the integrand w.r.t. the variable u in the above definition. This statement we state it as a proposition that will be very useful in the applications to differential equations in the following chapter. The proof is straightforward.

Proposition 2.1.3. *Consider a bounded function $f \in \mathcal{O}(S_{p,q}(d, \pi/k + 2\epsilon_0, R_0))$. Then*

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}\left((x^p\varepsilon^q)^k\left(\frac{s_1}{p}x\frac{\partial f}{\partial x}+\frac{s_2}{q}\varepsilon\frac{\partial f}{\partial \varepsilon}\right)\right)(\xi,\upsilon) = k(\xi^p\upsilon^q)^k\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon),$$

for any $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$.

Proposition 2.1.4. *Consider a bounded function $f \in \mathcal{O}(S_{p,q}(d, \pi/k + 2\epsilon_0, R_0))$ and $t \in \mathbb{C}$. Then*

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}\left(f\left(\frac{x}{(1-t(x^p\varepsilon^q)^k)^{s_1/pk}},\frac{\varepsilon}{(1-t(x^p\varepsilon^q)^k)^{s_2/qk}}\right)\right)(\xi,\upsilon) = e^{t(\xi^p\upsilon^q)^k}\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon),$$

for any $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$.

The formulas in the previous propositions are naturally related: the flow of the vector field $X = \frac{(x^p\varepsilon^q)^k}{k}\left(\frac{s_1}{p}x\frac{\partial}{\partial x}+\frac{s_2}{q}\varepsilon\frac{\partial}{\partial\varepsilon}\right)$ is precisely given by

$$(x,\varepsilon,t) \longmapsto \left(\frac{x}{(1-t(x^p\varepsilon^q)^k)^{s_1/pk}},\frac{\varepsilon}{(1-t(x^p\varepsilon^q)^k)^{s_2/qk}}\right).$$

Then we can deduce Proposition 2.1.3 from Proposition 2.1.4 by differentiating w.r.t. t and evaluating at $t = 0$. These calculations can be justified using the linearity of the Borel transform, the boundedness of the function f and the Dominated Convergence Theorem. Besides in the variable $t = x^p\varepsilon^q$ the vector field X reduces to $\frac{t^{k+1}}{k}\frac{\partial}{\partial t}$, a fact that relates the previous propositions with items (2) and (3) of Proposition 1.1.9.

In regard to the behavior of the Borel transform w.r.t the map $\hat{T}_{p,q}$, we formulate the next remark that will be useful in the forthcoming sections.

Remark 2.1.5. Let $\hat{f} \in \mathcal{S}$ be a formal power series and $\hat{\varphi}_{s_1,s_2} = \hat{\mathcal{B}}_{k,(s_1,s_2)}^{(p,q)}((x^p\varepsilon^q)^k\hat{f})$. Let us write $\hat{f} = \sum a_{n,m}x^n\varepsilon^m$, $\hat{T}_{p,q}(\hat{f}) = \sum_{n\geq 0} f_n t^n$ and $\hat{T}_{p,q}(\hat{\varphi}_{s_1,s_2}) = \sum_{n\geq 0}\varphi_n\tau^n$. Then f_n and φ_n are related by

$$\begin{aligned}\varphi_n(\xi,\upsilon) &= (\xi^p\upsilon^q)^{-n}\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}((x^p\varepsilon^q)^{n+k}f_n)\\ &= \sum_{j=0}^{q-1}\sum_{m=0}^{\infty}\frac{a_{np+m,nq+j}}{\Gamma\left(1+\frac{n}{k}+\frac{s_1m}{pk}+\frac{s_2j}{qk}\right)}\xi^m\upsilon^j + \sum_{m=0}^{p-1}\sum_{j=q}^{\infty}\frac{a_{np+m,nq+j}}{\Gamma\left(1+\frac{n}{k}+\frac{s_1m}{pk}+\frac{s_2j}{qk}\right)}\xi^m\upsilon^j.\end{aligned}$$

Since $\hat{f} \in \mathcal{S}$ we know that all the f_n are analytic in a common disc D_ρ^2. Then we can conclude that the φ_n are all entire functions. We can go further and check from the last expression that there are constants $L, M' > 0$ independent of n, but depending on ρ such that

$$|\varphi_n(\xi, \upsilon)| \leq \frac{L\|f_n\|_\rho}{\Gamma\left(1+\frac{n}{k}\right)} e^{M' \max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}}, \tag{2-3}$$

for all $(\xi, \upsilon) \in \mathbb{C}^2$ and all $n \in \mathbb{N}$. Here $\|f_n\|_\rho = \sup_{(\xi,\upsilon)\in D_\rho^2} |f_n(\xi, \upsilon)|$.

We collect in the following proposition the main properties of this Borel transform, such as the exponential growth and its behavior w.r.t. monomial asymptotic expansions. We remark that the first two parts of the following proposition are properties similar to the classical Borel transform. However statement (3) below provides asymptotic bounds plus an exponential term that will help us understand how to use the forthcoming Laplace transform. The proof of (3) is based in a proof of the behavior of a generalization of the Borel transform for many variables taken from [S].

Proposition 2.1.6. *Let $x^p\varepsilon^q$ be a monomial and $l > 0$ be a positive real number. Consider $f \in \mathcal{O}(S_{p,q})$, $S_{p,q} = S_{p,q}(d, \pi/k + 2\epsilon_0, R_0)$ as before. Then the following statements hold:*

1. *If f is bounded in each monomial subsector of $S_{p,q}$, then $g = \mathcal{B}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k f)$ is analytic on $S_{p,q}(d, 2\epsilon_0, +\infty)$, and for every subsector $\Pi_{p,q} \Subset S_{p,q}(d, 2\epsilon_0, +\infty)$ of infinite radius there are constants $C, M > 0$ such that*

$$|g(\xi, \upsilon)| \leq C e^{M \max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}},$$

for all $(\xi, \upsilon) \in \Pi_{p,q}$.

2. *If $f \sim^{(p,q)}_{1/l} \hat{f}$ on $S_{p,q}$ and $\hat{g} = \hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k \hat{f}) = \sum_{n\geq 0} g_n(\xi, \upsilon)(\xi^p\upsilon^q)^n$ then*

$$g \sim^{(p,q)}_{1/\kappa} \hat{g},$$

on $S_{p,q}(d, 2\epsilon_0, +\infty)$, where $1/\kappa = 1/l - 1/k$ if $l \leq k$ and $1/\kappa = 0$ otherwise.

3. *Furthermore, if the hypotheses of (2) hold, then for every monomial subsector $\Pi_{p,q} \Subset S_{p,q}(d, 2\epsilon_0, +\infty)$ of infinite radius there are constants $B, D, M > 0$ such that*

$$\left| g(\xi, \upsilon) - \sum_{n=0}^{N-1} g_n(\xi, \upsilon)(\xi^p\upsilon^q)^n \right| \leq D B^N \Gamma(1 + N/\kappa) |\xi^p\upsilon^q|^N e^{M \max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}},$$

for all $(\xi, \upsilon) \in \Pi_{p,q}$.

Proof. First note that it is enough to establish the bounds for sector of the form $S_{p,q}(d, 2\epsilon', r)$, with $0 < \epsilon' < \epsilon_0$ and $0 < r \leq +\infty$. Also it is enough to prove statements (2) and (3) because statement (1) can be seen as a particular case of (3) by setting $N = 0$. The key point of the proof relies on choose adequately the radius of the arc of the path γ in the definition. Let $\gamma = \gamma_1 + \gamma_2 - \gamma_3$ given by:

i. γ_1 parameterized by $\gamma_1(\rho) = \rho e^{i(\pi/2+k(\epsilon-\epsilon')/2)}$, $\rho \geq R$,

ii. γ_2 parameterized by $\gamma_2(\theta) = Re^{i\theta}$, $|\theta| \leq \pi/2 + k(\epsilon - \epsilon')/2$,

iii. γ_3 parameterized by $\gamma_3(\rho) = \rho e^{-i(\pi/2+k(\epsilon-\epsilon')/2)}$, $\rho \geq R$,

where $0 < \epsilon' < \epsilon < \epsilon_0$ and R will be chosen appropriately so that if $(\xi, \upsilon) \in S_{p,q}(d, 2\epsilon', r)$ then $(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk}) \in S_{p,q}(d, \pi/k + 2\epsilon, R_0/2)$ for all u on γ.

Note that if $\hat{T}_{p,q}(\hat{f}) = \sum_{n\geq 0} f_n(x, \varepsilon)t^n$ then

$$g_n(\xi, \upsilon) = (\xi^p \upsilon^q)^{-n} \hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^{k+n} f_n)).$$

Also without loss of generality we can assume that the f_n are analytic on $D_{R_0}(0)^2$.

We know that, by hypothesis (2) there are constants $C, A > 0$ such that

$$\left| f(x, \varepsilon) - \sum_{n=0}^{N-1} f_n(x, \varepsilon)(x^p\varepsilon^q)^n \right| \leq CA^N \Gamma(1 + N/l)|x^p\varepsilon^q|^N, \tag{2-4}$$

for all $(x, \varepsilon) \in S_{p,q}(d, \pi/k + 2\epsilon, R_0/2)$ and all $N \in \mathbb{N}$.

If we set $a = \sin(k(\epsilon - \epsilon')/2)/2$ and with R to be chosen and using inequality (2-4), a direct estimate shows that for all $(\xi, \upsilon) \in S_{p,q}(d, 2\epsilon', r)$ and all $N \in \mathbb{N}$ we have

$$\begin{aligned}
&\left| g(\xi, \upsilon) - \sum_{n=0}^{N-1} g_n(\xi, \upsilon)(\xi^p\upsilon^q)^n \right| \\
&= \left| \frac{1}{2\pi i} \int_\gamma \left(f(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk}) - \sum_{n=0}^{N-1} f_n(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk})(\xi^p\upsilon^q)^n u^{-n/k} \right) \frac{e^u}{u} du \right| \\
&\qquad \leq \frac{C}{a} A^N \Gamma(1 + N/l) \frac{|\xi^p\upsilon^q|^N}{R^{N/k}} \left(\frac{e^{-2aR}}{R} + e^R \right).
\end{aligned} \tag{2-5}$$

For $N = 0$ we are denoting $C = C_\epsilon = \sup\{|f(x, \varepsilon)| \mid (x, \varepsilon) \in S_{p,q}(d, \pi/k + 2\epsilon, R_0/2)\} < +\infty$.

To prove statement (2) (i.e. for $r < +\infty$) it is enough to take any fixed $r_0 > 0$ and choose $R \geq \max\{(2r_0/R_0)^{k/s_1}, (2r_0/R_0)^{k/s_2}\}$. Since it is enough to establish the corresponding bounds for large N we can suppose N is large enough and take $R = N/k$. Then it follows from Stirling's formula that there are constants $\widetilde{C}, \widetilde{A} > 0$ such that

$$\left| g(\xi,\upsilon) - \sum_{n=0}^{N-1} g_n(\xi,\upsilon)(\xi^p\upsilon^q)^n \right| \leq \widetilde{C}\widetilde{A}^N \frac{\Gamma(1+N/l)}{\Gamma(1+N/k)} |\xi^p\upsilon^q|^N,$$

for all $(\xi,\upsilon) \in S_{p,q}(d,2\epsilon',r_0)$. One last application of Stirling's formula leads us to the result.

To prove statement (3) we just need to bound (2-5) for $(\xi,\upsilon) \in S_{p,q}(d,2\epsilon',+\infty)\backslash S_{p,q}(d,2\epsilon',r_0)$. Take R_1 and R_2 with $R_1, R_2 < R_0/2$ and let $R(\xi,\upsilon) = \max\{(|\xi|^p/R_1)^{k/s_1}, (|\upsilon|^q/R_2)^{k/s_2}\}$. The following inequalities are clear ((2-6) follows from (1-4)):

$$\frac{|\xi^p\upsilon^q|^k}{R_1^k R_2^k} \leq R(\xi,\upsilon), \tag{2-6}$$

$$R(\xi,\upsilon) \leq \left(\frac{|\xi|^p}{R_1}\right)^{k/s_1} + \left(\frac{|\upsilon|^p}{R_2}\right)^{k/s_2} \leq 2R(\xi,\upsilon). \tag{2-7}$$

If we use $R = R(\xi,\upsilon)$, inequality (2-5) is valid for those (ξ,υ). Using (2-6) and (2-7) it is straightforward to check that (2-5) is bounded by

$$\frac{C}{a}A^N\Gamma(1+N/l)R_1^N R_2^N \left(\frac{e^{-a(|\xi|^p/R_1)^{k/s_1}}}{(|\xi|^p/R_1)^k} + e^{(|\xi|^p/R_1)^{k/s_1}}\right)\left(\frac{e^{-a(|\upsilon|^q/R_2)^{k/s_2}}}{(|\upsilon|^q/R_2)^k} + e^{(|\upsilon|^q/R_2)^{k/s_2}}\right). \tag{2-8}$$

Then it is enough to prove that for $k, a, s > 0$ and $\rho > r_0 > 0$ there exist constants $L, K, M' > 0$ such that for $\tau > r_0$ and $N \in \mathbb{N}$ we have

$$h(N,\tau) := \inf_{0<t<\rho} t^N \left(\frac{e^{-a(\tau/t)^{k/s}}}{(\tau/t)^k} + e^{(\tau/t)^{k/s}}\right) \leq \frac{LK^N}{\Gamma(1+Ns/k)}\tau^N e^{M'\tau^{k/s}}. \tag{2-9}$$

Indeed, if $t = \tau\left(\frac{k}{sN}\right)^{s/k} < \rho$ we can use this t to bound

$$h(N,\tau) \leq \tau^N \left(\frac{k}{sN}\right)^{sN/k}\left(\frac{e^{-asN/k}}{(sN/k)^s} + e^{sN/k}\right),$$

and an application of Stirling's formula lead to the result in this case. On the other hand, if $\tau\left(\frac{k}{sN}\right)^{s/k} \leq \rho$, that is, if $r_0^N \leq \rho^N \leq \tau^N\left(\frac{k}{sN}\right)^{sN/k}$, we can use $t = r_0$ to get

$$h(N,\tau) \leq 2r_0^N e^{(\tau/r_0)^{k/s}} \leq 2\tau^N \left(\frac{k}{sN}\right)^{sN/k} e^{M'\tau^{k/s}},$$

where $M' = 1/r_0^{k/s}$. Again, an application of Stirling's formula let us conclude (2-9).

Finally, returning to inequality (2-8), since R_1 and R_2 can be arbitrarily small, inequality (2-9) can be applied to $\tau = |\xi|^p$, $s = s_1$ and to $\tau = |\upsilon|^q$, $s = s_2$ to conclude that there are large enough constants $\widetilde{D}, \widetilde{B} > 0$ such that

$$\left|g(\xi,\upsilon)-\sum_{n=0}^{N-1}g_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right|\leq \widetilde{D}\widetilde{B}^N\frac{\Gamma(1+N/l)}{\Gamma(1+s_1N/k)\Gamma(1+s_2N/k)}|\xi^p\upsilon^q|^N e^{M(|\xi|^{pk/s_1}+|\upsilon|^{qk/s_2})},$$

for all $(\xi,\upsilon)\in S_{p,q}(d,2\epsilon',+\infty)\setminus S_{p,q}(d,2\epsilon',r_0)$ and $M=\max\{1/r_0^{k/s_1},1/r_0^{k/s_2}\}$. Since $s_1+s_2=1$ we can use Stirling's formula to finally conclude the result.

□

Remark 2.1.7. In this section we have considered the $k-$Borel transform associated to a monomial $x^p\varepsilon^q$ with weight (s_1,s_2) of functions defined on monomial sectors in the same monomial, with adequate opening. We can also analyze the case when the domain of the functions is a monomial sector, for another monomial. More specifically, let $f\in S_{p,q}(d,\alpha,R_0)$ be a bounded analytic function and let us try to apply $\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}$ to f, where $l>0$ and $s_1'+s_2'=1$, $s_1',s_2'>0$.

Following the definition of the Borel transform, we see that if

$$\alpha=\frac{1}{l}\left(s_1'\frac{p}{p'}+s_2'\frac{q}{q'}\right)+2\epsilon_0,\quad 0<\epsilon_0<\frac{1}{l}\left(s_1'\frac{p}{p'}+s_2'\frac{q}{q'}\right),$$

and if we take a Hankel path with a radius $|u|>\max\left\{\frac{|\xi|^{p'l/s_1'}}{R_0^{p'l/ps_1'}},\frac{|\upsilon|^{q'l/s_2'}}{R_0^{q'l/qs_2'}}\right\}$ and arguments satisfying $|\arg(u)|<\frac{\pi}{2}+\epsilon_0 l\left(s_1'\frac{p}{p'}+s_2'\frac{q}{q'}\right)^{-1}$, then $\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}(f)$ is defined and analytic on $S_{p,q}(d,2\epsilon_0,+\infty)$.

We can adapt Proposition 2.1.6 to this case. If f is taken as before then we can calculate $g=\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}((x^{p'}\varepsilon^{q'})^l f)$ and it will have exponential growth of the form $Ce^{M\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}}$.

If additionally $f\sim^{(p,q)}_{1/k}\hat{f}$ on $S_{p,q}(d,\alpha,R_0)$, α as before, and $\hat{g}=\hat{\mathcal{B}}^{(p',q')}_{l,(s_1',s_2')}((x^{p'}\varepsilon^{q'})^l\hat{f})=\sum_{n\geq 0}g_n(\xi,\upsilon)(\xi^p\upsilon^q)^n$ then

$$g\sim^{(p,q)}_{1/\kappa}\hat{g},$$

on $S_{p,q}(d,2\epsilon_0,+\infty)$, where $1/\kappa=1/k-1/l\,(s_1'p/p'+s_2'q/q')$ if this quantity is positive or $1/\kappa=0$ otherwise (in the last case, $\hat{g}$ is convergent). Furthermore, for every monomial subsector $\Pi_{p,q}\Subset S_{p,q}(d,2\epsilon_0,+\infty)$ of infinite radius there are constants $B,D,M>0$ such that

$$\left|g(\xi,\upsilon)-\sum_{n=0}^{N-1}g_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right|\leq DB^N\Gamma(1+N/\kappa)|\xi^p\upsilon^q|^N e^{M\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}},$$

for all $(\xi,\upsilon)\in\Pi_{p,q}$.

The proof of the above statements is, up to minor modifications, the same as the proof of Proposition 2.1.6 and it will not be included here.

2.1.2 Laplace transform

Definition 2.1.2. Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $\alpha \in \mathbb{R}$ such that $|\alpha| < \pi/2$. The *$k-$Laplace transform associated to the monomial $x^p\varepsilon^q$ with weight (s_1, s_2) in direction α* of a function f is defined by the formula

$$\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(f)(x,\varepsilon) = (x^p\varepsilon^q)^k \int_0^{e^{i\alpha}\infty} f(xu^{s_1/pk}, \varepsilon u^{s_2/qk})e^{-u}du.$$

We are going to restrict our attention to analytic functions f defined in monomial sectors of the form $\Pi_{p,q}(a, b, +\infty)$. One may be tempted to impose an exponential growth on f of order k in the monomial $x^p\varepsilon^q$, i.e., to suppose that $|f(\xi, \upsilon)| \leq Ce^{B|x^p\varepsilon^q|^k}$ on the sector $\Pi_{p,q}(a, b, +\infty)$. But since the sector has infinite radius we would conclude from Proposition 1.2.19 that f is a function depending on $x^p\varepsilon^q$ and this will restrict our scope significantly. Instead and as it is suggested by the corresponding Borel transform, a natural condition to impose on f is having an exponential growth of the form

$$|f(\xi, \upsilon)| \leq Ce^{B\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}}, \tag{2-10}$$

valid for all $(\xi, \upsilon) \in \Pi_{p,q}(a, b, +\infty)$ and some constants $B, C > 0$. In particular f is bounded at the origin. In such case, $\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(f)$ is defined and analytic on the domain $D_{k,\alpha}(a, b, B)$ defined by the conditions

$$a - \alpha/k < \arg(x^p\varepsilon^q) < b - \alpha/k, \quad B\max\{|x|^{pk/s_1}, |\varepsilon|^{qk/s_2}\} < \cos\alpha.$$

Note that by changing the constant B in (2-10) we can replace the term $\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}$ by $|\xi|^{pk/s_1} + |\upsilon|^{qk/s_2}$ and vice versa. We also note that by changing the direction α by β we obtain an analytic continuation of $\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(f)$ when the corresponding domains intersects, i.e. when $|\beta - \alpha| < k(b - a)$, a fact that follows directly from Cauchy's theorem. This process leads to an analytic function $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)$ defined in the region

$$\bigcup_{|\alpha|<\pi/2} D_{k,\alpha}(a, b, B).$$

As an example we can consider $f(x, \varepsilon) = x^\lambda\varepsilon^\mu$, $\lambda, \mu \in \mathbb{C}$, $\mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0$. Then using the integral formula for the Gamma function we obtain the expression

$$\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)(x,\varepsilon) = \Gamma\left(1 + \frac{\lambda s_1/p + \mu s_2/q}{k}\right) x^{\lambda+pk}\varepsilon^{\mu+qk}. \tag{2-11}$$

As before we define the *formal* $k-$*Laplace transform associated to the monomial* $x^p\varepsilon^q$ *with weight* (s_1, s_2), as

$$\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)} : \mathbb{C}[[\xi, \upsilon]] \longrightarrow x^{pk}\varepsilon^{qk}\mathbb{C}[[x,\varepsilon]]$$
$$\sum_{n,m\geq 0} a_{n,m}\xi^n\upsilon^m \mapsto \sum_{n,m\geq 0} a_{n,m}\Gamma\left(1+\frac{ns_1/p+ms_2/q}{k}\right)x^{n+pk}\varepsilon^{m+qk}.$$

We see that $\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}$ is nothing but the inverse of $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}$. We will prove later the analytic counterpart, i.e., $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$ and $\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}$ are inverse one of another. Before we do that we need some information about series such that the formal and analytic Laplace transforms coincide an establish the analogous remarks to Remark 2.1.1 and Remark 2.1.2.

Remark 2.1.8. If $f_0 \in \mathcal{O}(V)$, $V = V(a, b, +\infty)$, has exponential growth of order at most k on V, say $|f_0(\zeta)| \leq Ce^{B|\zeta|^k}$ then $f(\xi,\upsilon) = f_0(\xi^p\upsilon^q)$ defines an analytic function on $\Pi_{p,q}(a, b, +\infty)$, due to inequality (1-4) it has exponential growth as in (2-10) for all $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and in this case

$$\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)(x,\varepsilon) = \mathcal{L}_k(f_0)(x^p\varepsilon^q).$$

Expressly, the $k-$Laplace transform associated to the monomial $x^p\varepsilon^q$ with weight (s_1, s_2) reduces to a $k-$Laplace transform for functions depending only on $x^p\varepsilon^q$. We mention that the same considerations and formula remain valid for the formal counterpart.

Remark 2.1.9. Consider a function $f \in \mathcal{O}(\Pi_{p,q})$, $\Pi_{p,q} = \Pi_{p,q}(a, b, +\infty)$, with exponential growth as in (2-10). Fix a point $(\xi_0, \upsilon_0) \in \Pi_{p,q}$ and weights $s_1, s_2 > 0$ with $s_1 + s_2 = 1$. If we set $\widetilde{f}(\zeta) = f(\zeta^{s_1/p}\xi_0, \zeta^{s_2/q}\upsilon_0)$ for $\zeta \in V = V(a-\arg(x_0^p\varepsilon_0^q), b-\arg(x_0^p\varepsilon_0^q), +\infty)$ then $\widetilde{f}$ has exponential growth of order at most k on V and in this case

$$(\xi_0^p\upsilon_0^q)^{-k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)(z^{s_1/p}\xi_0, z^{s_2/q}\upsilon_0) = \mathcal{L}_k(\widetilde{f})(z). \tag{2-12}$$

As usual the expression on the left is only calculated on points (x,ε) where $x^{p/s_1}/\varepsilon^{q/s_2}$ is constant.

Remark 2.1.10. Let $\hat{f} = \sum_{n,m\geq 0} a_{n,m}\xi^n\upsilon^m = \sum_{n\geq 0} f_n(\xi,\upsilon)(\xi^p\upsilon^q)^n$ be a formal power series. A necessary and sufficient condition on $\hat{f}$ so that $\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ is a convergent power series, is that there are constants $C, A > 0$ such that

$$|a_{n,m}| \leq \frac{CA^{n+m}}{\Gamma\left(1+\frac{ns_1/p+ms_2/q}{k}\right)},$$

for all $n, m \geq 0$. This is equivalent to say that $\hat{f}$ defines an entire function f with an exponential growth of the form (2-10). Then $\frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)$ exists, it is analytic in a polydisc at the origin, and it has $\frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ as Taylor's series at the origin.

Now assume that there are constants $l, B, D, M > 0$ such that the family of maps f_n are entire and satisfy the bounds

$$|f_n(\xi, \upsilon)| \leq DB^n\Gamma\left(1 + \frac{n}{l}\right) e^{M\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}}, \tag{2-13}$$

for all $(\xi, \upsilon) \in \mathbb{C}^2$. This is equivalent to require that the coefficient of $\hat{f}$ satisfy bounds of type

$$|a_{np+m,nq+j}| \leq KL^{np+nq+m+j}\frac{\Gamma\left(1+\frac{n}{l}\right)}{\Gamma\left(1 + \frac{s_1 m}{pk} + \frac{s_2 j}{qk}\right)},$$

for all $n, m, j \in \mathbb{N}$ with $m < p$ or $j < q$ (recall formula (1-6)) and some constants $K, L > 0$.

Thus we can conclude that $\hat{f} \in \mathbb{C}[[\xi, \upsilon]]^{(p,q)}_{1/l}$, $\frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f}) \in \mathbb{C}[[x, \varepsilon]]^{(p,q)}_{1/l+1/k}$, all the maps $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f_n)$ are analytic in a common polydisc centered at the origin and

$$\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f}) = \sum_{n\geq 0} \mathcal{L}^{(p,q)}_{k,(s_1,s_2)}((\xi^p\upsilon^q)^n f_n).$$

We focus now in the behavior of the Laplace transform w.r.t. monomial asymptotic expansions. The reader may note that the hypotheses required may seem restrictive, but in fact those appear naturally when we compare with maps coming from the Borel transform.

Proposition 2.1.11. *Let $f \in \mathcal{O}(\Pi_{p,q}(a, b, +\infty))$ be an analytic function. Suppose that the following statements hold:*

1. *$f \sim^{(p,q)}_{1/l} \hat{f}$ on $\Pi_{p,q} = \Pi_{p,q}(a, b, +\infty)$, for some $0 < l \leq +\infty$.*

2. *If $\hat{T}_{p,q}(\hat{f}) = \sum_{n\geq 0} f_n t^n$, then every f_n is an entire function and there are constants $B, D, K > 0$ such that*

$$|f_n(\xi, \upsilon)| \leq DB^n\Gamma\left(1 + \frac{n}{l}\right) e^{K\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}},$$

   for all $n \in \mathbb{N}$ and for all $(\xi, \upsilon) \in \mathbb{C}^2$.

3. *For every monomial subsector $\widetilde{\Pi}_{p,q} \Subset \Pi_{p,q}$ there are constants $C, A, M > 0$ such that for all $N \in \mathbb{N}$*

$$\left|f(\xi, \upsilon) - \sum_{n=0}^{N-1} f_n(\xi, \upsilon)(\xi^p\upsilon^q)^n\right| \leq CA^N\Gamma(1 + N/l)|\xi^p\upsilon^q|^N e^{M\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}},$$

for all $(\xi, \upsilon) \in \widetilde{\Pi}_{p,q}$.

Then $\frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f) \sim^{(p,q)}_{1/k+1/l} \frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ *on* $\bigcup_{|\alpha|<\pi/2} D_{k,\alpha}(a,b,M)$.

Proof. To simplify notation we are going to write $R(\xi,\upsilon) = M\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}$. We note that hypothesis (3) for $N = 0$ is interpreted as f having exponential growth as in (2-10).

Let $h(x,\varepsilon) = \frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)(x,\varepsilon)$ and write $\hat{T}_{p,q}\left(\frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})\right) = \sum_{n\geq 0} h_n\tau^n$. Then, as a consequence of statement (2), we can use the last part of remark 2.1.10 to conclude that

$$h_n(x,\varepsilon)(x^p\varepsilon^q)^n = \frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}((\xi^p\upsilon^q)^n f_n),$$

and additionally that $\frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ is $(1/k+1/l)-$Gevrey in the monomial $x^p\varepsilon^q$.

Now fix α such that $|\alpha| < \pi/2$. It is enough to prove the result for subsectors contained in $D_{k,\alpha}(a,b,M)$. If we take one of those proper subsectors $\overline{\Pi}_{p,q}$, we can find $\delta > 0$ small enough such that

$$R(x,\varepsilon) < \cos\alpha - \delta,$$

for all $(x,\varepsilon) \in \overline{\Pi}_{p,q}$. Now let $\widetilde{\Pi}_{p,q} \Subset \Pi_{p,q}$ such that $(xu^{s_1/pk}, \varepsilon u^{s_2/qk}) \in \widetilde{\Pi}_{p,q}$ if $(x,\varepsilon) \in \overline{\Pi}_{p,q}$ and u is on the semi-line $[0, e^{i\alpha}\infty)$. Using statement (3) for $\widetilde{\Pi}_{p,q}$ we see that

$$\begin{aligned}
&\left| h(x,\varepsilon) - \sum_{n=0}^{N-1} h_n(x,\varepsilon)(x^p\varepsilon^q)^n \right| = \\
&\left| \int_0^{e^{i\alpha}\infty} \left(f(xu^{s_1/pk}, \varepsilon u^{s_2/qk}) - \sum_{n=0}^{N-1} f_n(xu^{s_1/pk}, \varepsilon u^{s_2/qk})(x^p\varepsilon^q)^n u^{n/k} \right) e^{-u} du \right| \\
&\leq \int_0^{+\infty} CA^N\Gamma(1+N/l)|x^p\varepsilon^q|^N \rho^{N/k} e^{-\delta\rho} d\rho \\
&= \frac{C}{\delta}\frac{A^N}{\delta^{N/k}}\Gamma(1+N/l)\Gamma(1+N/k)|x^p\varepsilon^q|^N,
\end{aligned}$$

for all $(x,\varepsilon) \in \overline{\Pi}_{p,q}$. We can conclude that $\frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f) \sim^{(p,q)}_{1/k+1/l} \frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ on $\bigcup_{|\alpha|<\pi/2} D_{k,\alpha}(a,b,M)$ as we wanted to show. □

Remark 2.1.12. As in the previous section, we can also consider the Laplace transform for a monomial and some weights but applied to functions whose domain is another monomial. More specifically, let $f \in \Pi_{p,q}(a,b,+\infty)$ be an analytic function and let us try to apply $\mathcal{L}^{(p',q')}_{l,(s_1',s_2')}$ to f, where $l > 0$ and $s_1' + s_2' = 1$, $s_1', s_2' > 0$.

If we require that f has exponential growth $|f(\xi,\upsilon)|\leq Ce^{B\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}}$, then for each α with $|\alpha|<\pi/2$, $\mathcal{L}^{(p',q')}_{l,\alpha,(s_1',s_2')}(f)$ will be defined and analytic in the region given by

$$B\max\{|x|^{p'l/s_1'},|\varepsilon|^{q'l/s_2'}\}<\cos(\alpha),$$

$$a-\frac{1}{l}\left(s_1'\frac{p}{p'}+s_2'\frac{q}{q'}\right)\alpha<\arg(x^p\varepsilon^q)<b-\frac{1}{l}\left(s_1'\frac{p}{p'}+s_2'\frac{q}{q'}\right)\alpha,$$

and varying α we obtain an analytic function defined in the union of those regions.

If additionally $f\sim^{(p,q)}_{1/k}\hat{f}$ on $\Pi_{p,q}=\Pi_{p,q}(a,b,+\infty)$, and $\hat{f}$ satisfy the requirements:

1. If $\hat{T}_{p,q}(\hat{f})=\sum_{n\geq 0}f_nt^n$, then every f_n is an entire function and there are constants $B,D,K>0$ such that

$$|f_n(\xi,\upsilon)|\leq DB^n\Gamma\left(1+\frac{n}{k}\right)e^{K\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}},$$

for all $n\in\mathbb{N}$ and for all $(\xi,\upsilon)\in\mathbb{C}^2$.

2. For every monomial subsector $\widetilde{\Pi}_{p,q}\Subset\Pi_{p,q}$ there are constants $C,A,M>0$ such that for all $N\in\mathbb{N}$

$$\left|f(\xi,\upsilon)-\sum_{n=0}^{N-1}f_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right|\leq CA^N\Gamma(1+N/k)|\xi^p\upsilon^q|^Ne^{M\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}},$$

for all $(\xi,\upsilon)\in\widetilde{\Pi}_{p,q}$.

then we can conclude that $\frac{1}{(x^{p'}\varepsilon^{q'})^l}\mathcal{L}^{(p',q')}_{l,(s_1',s_2')}(f)\sim^{(p,q)}_{1/k+1/l(s_1'p/p'+s_2'q/q')}\frac{1}{(x^{p'}\varepsilon^{q'})^l}\hat{\mathcal{L}}^{(p',q')}_{l,(s_1',s_2')}(\hat{f})$ on the corresponding monomial sector in $x^p\varepsilon^q$. The proof of the above statements is, up to minor modifications, the same as the proof of Proposition 2.1.11 and it will not be included.

We finish this section proving that the Borel and Laplace transforms, introduced above, are inverse one to the other. To do so, we need the following lemma corresponding to the injectivity of the Laplace transform.

Lemma 2.1.13. *Let $f\in\mathcal{O}(\Pi_{p,q}(a,b,+\infty))$ be an analytic function satisfying the bounds $|f(\xi,\upsilon)|\leq Ce^{B\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s_2}\}}$ on its domain, for some positive constants C,B. If $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)\equiv 0$ then $f\equiv 0$.*

Proof. Take any $n \in \mathbb{N}^*$. In the integral expression defining $\mathcal{L}^{(p,q)}_{k,0,(s_1,s_2)}(f)$ perform the change of variable $e^{-u} = \tau^n$, $0 < \tau \leq 1$. Then from hypothesis we obtain the equality

$$\int_0^1 f\left(x\left(\ln(1/\tau^n)\right)^{s_1/pk}, \varepsilon\left(\ln(1/\tau^n)\right)^{s_2/qk}\right)\tau^{n-1}d\tau = 0,$$

valid for all (x,ε) satisfying $a < \arg(x^p e^q) < b$ and $B^{s_1/k}|x|^p < 1, B^{s_2/k}|\varepsilon|^q < 1$. Fix one point (x_0,ε_0) satisfying those conditions. To show that $f(x_0,\varepsilon_0) = 0$ note that the points $(x_0/(2n)^{s_1/pk}, \varepsilon_0/(2n)^{s_2/qk})$ also satisfy the previous requirements and as a consequence we obtain that

$$\int_0^1 g(\tau)\tau^{n-2}d\tau = 0, \text{ for all } n \in \mathbb{N}^*,$$

where $g(\tau) = \tau f\left(x_0\left(\ln(1/\tau)/2\right)^{s_1/pk}, \varepsilon_0\left(\ln(1/\tau)/2\right)^{s_2/qk}\right)$, for $0 < \tau \leq 1$. From the growth conditions on f we see that $|g(\tau)| \leq C\tau^{1/2}$. Then taking $g(0) = 0$, g defines a complex-valued continuous function on the interval $[0,1]$. By Weierstraß approximation theorem we can find a sequence of polynomials $(P_m(\tau))_{m\in\mathbb{N}}$ that converges uniformly to g on $[0,1]$. Since we have shown that $\int_0^1 g(\tau)P(\tau)d\tau = 0$ for any polynomial P, we conclude that $\int_0^1 \mathrm{Re}(g)(\tau)^2 d\tau = \int_0^1 \mathrm{Im}(g)(\tau)^2 d\tau = 0$. It follows by continuity that $g(\tau) \equiv 0$. Finally evaluating g at $\tau = e^{-2}$ we see that $f(x_0,\varepsilon_0) = 0$ as we wanted to show.

□

Proposition 2.1.14. *Let $k > 0$ be a positive real number and d be a direction. Let $f \in \mathcal{O}(S_{p,q}(d,\pi/k + 2\epsilon_0, R_0))$ be a bounded analytic function, where $0 < \epsilon_0 < \pi/k$ and $R_0 > 0$. Then*

$$\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k f)(x,\varepsilon) = (x^p\varepsilon^q)^k f(x,\varepsilon),$$

for all (x,ε) in the intersection of the domains.

Conversely, if $g \in \mathcal{O}(\Pi_{p,q}(a,b,+\infty))$ in an analytic function with exponential growth of the form $|g(\xi,\upsilon)| \leq Ce^{B\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s_2}\}}$ then

$$\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(g)(\xi,\upsilon) = g(\xi,\upsilon),$$

for all (ξ,υ) in the intersection of the domains.

Proof. Set $g = \mathcal{B}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k f)$ and take $0 < \epsilon' < \epsilon_0$. By Proposition 2.1.6 we can ensure the existence of a constant $C > 0$ such that $|g(\xi,\upsilon)| \leq Ce^{R(\xi,\upsilon)}$, where

$$R(\xi,\upsilon) = \max\{(2|\xi|^p/R_0)^{k/s_1}, (2|\upsilon|^q/R_0)^{k/s_2}\},$$

for all $(\xi,\upsilon) \in S_{p,q}(d,2\epsilon',+\infty)$. For a fixed α with $|\alpha| < \pi/2$, we see that $\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(g)$ is well-defined and analytic for (x,ε) satisfying

$$|\arg(x^p\varepsilon^q) - d + \alpha/k| < \epsilon', \;\; R(x,\epsilon) < \cos\alpha. \tag{2-14}$$

Following the definitions we see that for those (x,ε)

$$\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(g)(x,\varepsilon) = \frac{(x^p\varepsilon^q)^k}{2\pi i}\int_0^{e^{i\alpha}\infty}\int_\gamma f(xu^{s_1/pk}v^{-s_1/pk}, \varepsilon u^{s_2/qk}v^{-s_2/qk})\frac{e^{v-u}}{v}dvdu,$$

where for each u on $[0, e^{i\alpha}\infty)$, γ can be taken as $\gamma = -\gamma_3 + \gamma_2 + \gamma_1$ with: $\gamma_1(\rho) = \rho e^{i(\pi/2+k(\epsilon_0-\epsilon')/2)}$, $\gamma_3(\rho) = \rho e^{-i(\pi/2+k(\epsilon_0-\epsilon')/2)}$, $\rho \geq R(xu^{s_1/p}, \varepsilon u^{s_2/q}) = R(x,\varepsilon)|u|$ and $\gamma_2(\theta) = R(x,\varepsilon)|u|e^{i\theta}$, $|\theta| \leq \pi/2 + k(\epsilon_0 - \epsilon')/2$.

For a fixed u, we can perform the change of variables $w = uv^{-1}$ in the inner integral. Then γ is transformed into $\widetilde{\gamma} = \widetilde{\gamma}_3 + \widetilde{\gamma}_2 - \widetilde{\gamma}_1$ where $\widetilde{\gamma}_1(r) = re^{i(\alpha-\pi/2-k(\epsilon_0-\epsilon')/2)}$, $\widetilde{\gamma}_3(r) = re^{i(\alpha+\pi/2+k(\epsilon_0-\epsilon')/2)}$, $0 < r \leq 1/R(x,\varepsilon)$ and $\widetilde{\gamma}_2(\theta) = e^{i(\alpha-\theta)}/R(x,\varepsilon)$, $|\theta| \leq \pi/2 + k(\epsilon_0 - \epsilon')/2$.

We remark the following properties of $\widetilde{\gamma}$:

i. $\widetilde{\gamma}$ is independent of u.

ii.. If w is on $\widetilde{\gamma}$ then $\mathrm{Re}(e^{i\alpha}/w) < \cos\alpha$. Indeed, if w is on $\widetilde{\gamma}_2$ then $\mathrm{Re}(e^{i\alpha}/w) = R(x,\varepsilon)\cos\theta < R(x,\varepsilon) < \cos\alpha$ due to restrictions (2-14). If w is on $\widetilde{\gamma}_1$ or $\widetilde{\gamma}_3$ then $\mathrm{Re}(e^{i\alpha}/w) = -\sin(k(\epsilon_0-\epsilon)/2)/r < 0 < \cos\alpha$.

iii. The point 1 is inside the interior of the region bounded by $\widetilde{\gamma}$.

With the change of variables we obtain that

$$\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(g)(x,\varepsilon) = \frac{(x^p\varepsilon^q)^k}{2\pi i}\int_0^{e^{i\alpha}\infty}\int_{-\widetilde{\gamma}} f(xw^{s_1/pk}, \varepsilon w^{s_2/qk})e^{u/w-u}\frac{dw}{w}du.$$

Since $\widetilde{\gamma}$ is independent of u, we can apply Fubini's theorem to interchange the order of the integrals, then use (ii) to calculate the inner integral and use (iii) and the Residue theorem (and a limiting process) to conclude that

$$\begin{aligned}
\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(g)(x,\varepsilon) &= \frac{(x^p\varepsilon^q)^k}{2\pi i}\int_{-\widetilde{\gamma}} f(xw^{s_1/pk}, \varepsilon w^{s_2/qk})\left(\int_0^{e^{i\alpha}\infty} e^{u/w-u}du\right)\frac{dw}{w} \\
&= \frac{(x^p\varepsilon^q)^k}{2\pi i}\int_{-\widetilde{\gamma}} \frac{f(xw^{s_1/pk}, \varepsilon w^{s_2/qk})}{w-1}dw \\
&= (x^p\varepsilon^q)^k f(x,\varepsilon),
\end{aligned}$$

as we wanted to prove.

The last part of the proposition follows immediately from the previous lemma. □

2.1.3 The convolution product

In our context there is also a convolution between functions, that shares similar properties with the classical one, and that we develop in the following lines.

Definition 2.1.3. Let k, s_1, s_2 be positive real numbers such that $s_1 + s_2 = 1$ and let $f, g \in \mathcal{O}(\Pi_{p,q})$ be analytic functions on a monomial sector in $x^p\varepsilon^q$. The $k-(s_1,s_2)-$*convolution product of between f and g in the monomial $x^p\varepsilon^q$* is defined through the formula

$$(f *_{k,(s,s_2)}^{(p,q)} g)(x,\varepsilon) = (x^p\varepsilon^q)^k \int_0^1 f(x\tau^{s_1/pk}, \varepsilon\tau^{s_2/qk})g(x(1-\tau)^{s_1/pk}, \varepsilon(1-\tau)^{s_2/qk})d\tau.$$

It is clear from the above formula that $f *_{k,(s,s_2)}^{(p,q)} g$ is also an analytic function defined on $\Pi_{p,q}$. Also it follows from the above definition and some calculations that this binary operation is linear in each variable, commutative and associative.

As an example, we can calculate with the aid of the Beta function the convolution between two monomials plus an exponential term in $(x^p\varepsilon^q)^k$: if $c \in \mathbb{C}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ have positive real part then

$$\frac{x^{\lambda_1}\varepsilon^{\mu_1}e^{c(x^p\varepsilon^q)^k}}{\Gamma\left(\frac{s_1}{pk}\lambda_1 + \frac{s_2}{qk}\mu_1 + 1\right)} *_{k,(s_1,s_2)}^{(p,q)} \frac{x^{\lambda_2}\varepsilon^{\mu_2}e^{c(x^p\varepsilon^q)^k}}{\Gamma\left(\frac{s_1}{pk}\lambda_2 + \frac{s_2}{qk}\mu_2 + 1\right)} = \frac{x^{\lambda_1+\lambda_2+pk}\varepsilon^{\mu_1+\mu_2+qk}e^{c(x^p\varepsilon^q)^k}}{\Gamma\left(\frac{s_1}{pk}(\lambda_1+\lambda_2) + \frac{s_2}{qk}(\mu_1+\mu_2) + 2\right)}.$$

We highlight in the next proposition the main property of the convolution: the $k-$Laplace transform associated to the monomial $x^p\varepsilon^q$ with weight (s_1, s_2) transform this convolution into the usual product.

Proposition 2.1.15. *Let $f, g \in \mathcal{O}(\Pi_{p,q})$ be analytic functions on a monomial sector in $x^p\varepsilon^q$ of infinite radius. Suppose that f, g have exponential growth as in (2-10), say*

$$|f(x,\varepsilon)| \leq C_1 e^{B_1 R(x,\varepsilon)},\ |g(x,\varepsilon)| \leq C_2 e^{B_2 R(x,\varepsilon)}, \quad R(x,\varepsilon) = \max\{|x|^{pk/s_1}, |\varepsilon|^{qk/s_2}\},$$

for all $(x,\varepsilon) \in \Pi_{p,q}$ and some positive constants C_1, C_2, B_1, B_2. Then $f *_{k,(s,s_2)}^{(p,q)} g$ has also exponential growth as in (2-10), its $k-$Laplace transform associated to the monomial $x^p\varepsilon^q$ with weight (s_1, s_2) is well-defined and satisfies

$$\mathcal{L}_{k,(s_1,s_2)}^{(p,q)}(f *_{k,(s,s_2)}^{(p,q)} g)(x,\varepsilon) = \mathcal{L}_{k,(s_1,s_2)}^{(p,q)}(f)(x,\varepsilon) \cdot \mathcal{L}_{k,(s_1,s_2)}^{(p,q)}(g)(x,\varepsilon).$$

Analogously, if $F, G \in \mathcal{O}(S_{p,q}(d, \pi/k + 2\epsilon, R_0))$ are analytic functions then

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(F \cdot G)(\xi,\upsilon) = \mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(F)(\xi,\upsilon) *_{k,(s,s_2)}^{(p,q)} \mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(G)(\xi,\upsilon).$$

Proof. Since $\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}$ and $\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}$ are inverses of each other, it suffices to prove the first part of the proposition. Set $h = f *^{(p,q)}_{k,(s,s_2)} g$. Using the exponential growth of f and g it follows immediately that

$$|h(x,\varepsilon)| \leq C_1C_2|x^p\varepsilon^q|^k e^{BR(x,\varepsilon)} \leq C_1C_2R(x,\varepsilon)e^{BR(x,\varepsilon)},$$

where $B = \max\{B_1, B_2\}$. This proves the first statement. To verify the second one, note that

$$\begin{aligned}
&\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(h)(x,\varepsilon) = (x^p\varepsilon^q)^k \int_0^{e^{i\alpha}\infty} h(xu^{s_1/pk}, \varepsilon u^{s_2/qk})e^{-u}du\\
=&(x^p\varepsilon^q)^{2k} \int_0^{e^{i\alpha}\infty}\int_0^1 f(x(u\tau)^{s_1/pk}, \varepsilon(u\tau)^{s_2/qk})g(x(u(1-\tau))^{s_1/pk}, \varepsilon(u(1-\tau))^{s_2/qk})ue^{-u}d\tau du\\
=&(x^p\varepsilon^q)^{2k} \int_0^{e^{i\alpha}\infty}\int_0^{e^{i\alpha}\infty} f(xv^{s_1/pk}, \varepsilon v^{s_2/qk})g(xw^{s_1/pk}, \varepsilon w^{s_2/qk})e^{-(v+w)}dvdw\\
=&\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f)(x,\varepsilon)\cdot\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(g)(x,\varepsilon),
\end{aligned}$$

where we just performed the change of variables $v = u\tau$, $w = u(1-\tau)$, or equivalently $u = v + w$, $\tau = v/(v + w)$ that establishes a diffeomorphism between $(0, e^{i\alpha}\infty) \times (0, 1)$ and $(0, e^{i\alpha}\infty) \times (0, e^{i\alpha}\infty)$ and used that $ud\tau du = dvdw$. This concludes the proof of the result. □

2.2 Monomial Borel-Laplace summation methods

The goal behind the study of the Borel and Laplace transforms of the previous section is to give another characterization of monomial summability. As in the classical case we can define a summation method based in the above Borel and Laplace transforms and we will see that it turns out to be equivalent to monomial summability.

Definition 2.2.1. Let $k > 0$ be a positive number, $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $x^p\varepsilon^q$ a monomial. Let $\hat{f}$ be a $1/k-$Gevrey series in $x^p\varepsilon^q$ and set $\hat{\varphi}_{s_1,s_2} = \hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$. We will say that $\hat{f}$ is $k-(s_1, s_2)-$*Borel summable in the monomial* $x^p\varepsilon^q$ *in direction* d if $\hat{\varphi}_{s_1,s_2}$ can be analytically continued, say as φ_{s_1,s_2}, to a monomial sector of the form $S_{p,q}(d, 2\epsilon, +\infty)$

having an exponential growth of the form $|\varphi_{s_1,s_2}(\xi,\upsilon)| \leq De^{M\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s_2}\}}$ for some constants $D, M > 0$. In this case the $k-(s_1,s_2)-$*Borel sum of* $\hat{f}$ in direction d is defined as

$$f(x,\varepsilon) = \frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2})(x,\varepsilon).$$

Let us compare this notion with a $k-$summability in a monomial $x^p\varepsilon^q$ in a direction d. Indeed, fix any $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and let $\hat{f} \in R^{(p,q)}_{1/k,d}$ be a formal power series, $k-$summable in the direction d, in the monomial $x^p\varepsilon^q$. Let $f \in \mathcal{O}(S_{p,q}(d, \pi/k+2\epsilon, R_0))$ be an analytic function such that $f \sim^{(p,q)}_{1/k} \hat{f}$ on $S_{p,q}(d, \pi/k+2\epsilon, R_0)$. Set $\varphi_{s_1,s_2} = \mathcal{B}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k f)$ and $\hat{\varphi}_{s_1,s_2} = \hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$.

Since $\hat{f}$ is $1/k-$Gevrey in $x^p\varepsilon^q$, we see that by the contention (1-3), $\hat{\varphi}_{s_1,s_2}$ is a convergent power series, say on D^2_r. We can apply statement (2) of Proposition 2.1.6 to conclude that $\varphi_{s_1,s_2} \sim^{(p,q)}_0 \hat{\varphi}_{s_1,s_2}$ on $S_{p,q}(d, 2\epsilon, +\infty)$. This two properties imply that φ_{s_1,s_2} coincides with the sum of $\hat{\varphi}_{s_1,s_2}$ in the intersection $S_{p,q}(d, 2\epsilon, +\infty) \cap D^2_r$. Since φ_{s_1,s_2} is defined on $S_{p,q}(d, 2\epsilon, +\infty)$ and has exponential growth as in (2-10), we can express those facts by saying that the sum of $\hat{\varphi}_{s_1,s_2}$ can be analytically continued with exponential growth as in (2-10) to $S_{p,q}(d, 2\epsilon, +\infty)$. Therefore $\hat{f}$ is $k-(s_1,s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d and thanks to Proposition 2.1.14 both sums coincide.

Conversely, let $\hat{f}$ be $k-(s_1,s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d and let φ_{s_1,s_2} be the analytic continuation of $\hat{\varphi}_{s_1,s_2}$ to $S_{p,q}(d, 2\varepsilon, +\infty)$, with exponential growth $|\varphi_{s_1,s_2}(\xi,\upsilon)| \leq D'e^{M'R(\xi,\upsilon)}$ as in the previous definition. Here as usual we write $R(\xi,\upsilon) = \max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}$. Also write $\hat{T}_{p,q}(\hat{\varphi}_{s_1,s_2}) = \sum_{n\geq 0}\varphi_n\tau^n$. Since we are taking $\hat{f}$ to be $1/k-$Gevrey in $x^p\varepsilon^q$, we can conclude from Remark 2.1.5 that all the φ_n are entire functions and there are constants $D, B, M > 0$ such that

$$|\varphi_n(\xi,\upsilon)| \leq DB^n e^{MR(\xi,\upsilon)},$$

for all $(\xi,\upsilon) \in \mathbb{C}^2$ and all $n \in \mathbb{N}$. By enlarging the constants we may assume that $D = D'$ and $M = M'$.

To be able to use the Laplace transform and in particular to apply Proposition 2.1.11, it is enough to prove that there are constants $C, A, M > 0$ such that for all $N \in \mathbb{N}$ we have

$$\left|\varphi_{s_1,s_2}(\xi,\upsilon) - \sum_{n=0}^{N-1}\varphi_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right| \leq CA^N|\xi^p\upsilon^q|^N e^{MR(\xi,\upsilon)}, \tag{2-15}$$

for all $(\xi,\upsilon) \in S_{p,d}(d, 2\epsilon, +\infty)$.

Indeed, since $\hat{\varphi}_{s_1,s_2}$ is the convergent Taylor's series of φ_{s_1,s_2} at $(0,0)$ then (2-15) is satisfied for all $|\xi|, |\upsilon| \leq R$ for some $R > 0$. Additionally, due to the growth of the functions φ_n, the series $\sum_{n\geq 0}\varphi_n(\xi,\upsilon)(\xi^p\upsilon^q)^n$ converges in every compact set where $B|\xi^p\upsilon^q)|<1$. Then φ_{s_1,s_2} can be analytically continued there through this series. It follows that if $B|\xi^p\upsilon^q|<1/2$ then inequality (2-15) is also satisfied.

Now, if $(\xi,\upsilon)\in S_{p,d}(d,2\epsilon,+\infty)$ the previous inequalities show that

$$\left|\varphi_{s_1,s_2}(\xi,\upsilon)-\sum_{n=0}^{N-1}\varphi_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right|\leq De^{MR(\xi,\upsilon)}+\sum_{n=0}^{N-1}DB^n|\xi^p\upsilon^q|^n e^{MR(\xi,\upsilon)}.$$

If $1/2\leq B|\xi|^p|\upsilon|^q\leq 2$ the last expression is bounded by

$$De^{MR(\xi,\upsilon)}+D(2^N-1)e^{MR(\xi,\upsilon)}=D2^Ne^{MR(\xi,\upsilon)}\leq D(4B)^N|\xi^p\upsilon^q|^Ne^{MR(\xi,\upsilon)}.$$

On the other hand, if $B|\xi|^p|\upsilon|^q>2$, we can bound by

$$De^{MR(\xi,\upsilon)}+D\frac{B^N|\xi^p\upsilon^q|^N-1}{B|\xi^p\upsilon^q|-1}e^{MR(\xi,\upsilon)}<DB^N|\xi^p\upsilon^q|^Ne^{MR(\xi,\upsilon)},$$

as we wanted to show.

Therefore we are in conditions to apply Proposition 2.1.11 to φ_{s_1,s_2}, $\hat{\varphi}_{s_1,s_2}$ and $1/l=0$ to obtain that

$$f(x,\varepsilon)=\frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2})\sim^{(p,q)}_{1/k}\frac{1}{(x^p\varepsilon^q)^k}\hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}(\hat{\varphi}_{s_1,s_2})=\hat{f},$$

on $\bigcup_{|\alpha|<\pi/2}D_{k,\alpha}(d-\epsilon,d+\epsilon,M)$. In conclusion, $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in direction d and the $k-$sum can be found through the $k-$Laplace transform in $x^p\varepsilon^q$ with weight (s_1,s_2) of the analytic continuation of $\hat{\varphi}_{s_1,s_2}$ to a sector in $x^p\varepsilon^q$ bisected by d of infinite radius.

These considerations, joint to Proposition 2.1.14, prove the following theorem.

Theorem 2.2.1. *Let $\hat{f}\in\hat{R}^{(p,q)}_{1/k}$ be a $1/k-$Gevrey series in the monomial $x^p\varepsilon^q$. Then it is equivalent:*

1. *$\hat{f}\in R^{(p,q)}_{1/k,d}$,*
2. *There are $s_1,s_2>0$ with $s_1+s_2=1$ such that $\hat{f}$ is $k-(s_1,s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d.*
3. *For all $s_1,s_2>0$ such that $s_1+s_2=1$, $\hat{f}$ is $k-(s_1,s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d.*

In all cases the corresponding sums coincide.

To finish this section we can provide a new proofs of formula (1-25) at the end of Section 1.2.4, of Proposition 1.3.1 and Proposition 1.3.3 using the monomial Borel-Laplace methods.

Recall that formula (1-25) provides a way to calculate the $k-$sum in $x^p\varepsilon^q$ in direction d of a series $\hat{f} \in R^{(p,q)}_{1/k,d}$ by weighting the variables. Let f be the sum of $\hat{f}$ in direction d defined in say $S_{p,q} = S_{p,q}(d,\epsilon,r)$, take $(x_0,\varepsilon_0) \in S_{p,q}$ and $s_1, s_2 > 0$ with $s_1 + s_2 = 1$. Consider the variable z varying on $S(d - \arg(x_0^p\varepsilon^q), \epsilon, \widetilde{r})$, $\widetilde{r} = \min\{(r/|x_0|^p)^{1/s_1}, (r/|\varepsilon_0|^q)^{1/s_2}\}$. We know by Theorem 2.2.1 that f can be calculated as

$$f(x,\varepsilon) = \frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2}),$$

where φ_{s_1,s_2} is the analytic continuation of $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$ to a monomial sector bisected by d of infinite radius the required exponential growth. Then using formula (2-2) in Remark 2.1.2 we can conclude that $\hat{\mathcal{B}}_k(z^k\hat{f}(z^{s_1/p}x_0, z^{s_2/q}\varepsilon_0))(\zeta)$ can be analytically continued with exponential at most k to a sector bisected by $d-\arg(x_0^p\varepsilon_0^q)$ of infinite radius by the expression $\varphi_{x_0,\varepsilon_0}(\zeta) = \varphi_{s_1,s_2}(\zeta^{s_1/p}x_0, \zeta^{s_2/q}\varepsilon_0)$. Then using formula (2-12) in Remark 2.1.9 we conclude that

$$\begin{aligned} f(z^{s_1/p}x_0, z^{s_2/q}\varepsilon) &= \frac{1}{z^k(x_0^p\varepsilon_0^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2})(z^{s_1/p}x_0, z^{s_2/q}\varepsilon) \\ &= \frac{1}{z^k}\mathcal{L}_k(\varphi_{x_0,\varepsilon_0})(z), \end{aligned}$$

as we wanted to show. This reasoning prove that the method of finding the sum by weighting the variables as explained in the end of Section 1.2.4 is generalized and strengthen by the monomial Borel-Laplace methods, since the first one only calculate the sum when $x^{p/s_1}/\varepsilon^{q/s_2}$ is constant.

The following reasoning is adapted from [M1]. Recall that Proposition 1.3.1 stated that if $\hat{f} \in R^{(p,q)}_{1/k}$ has no singular directions then $\hat{f} \in R$. Indeed, if $\hat{f} = \sum a_{n,m}x^m\varepsilon^m$ has no singular directions for k-summability in $x^p\varepsilon^q$, then by the previous theorem $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$ defines is an entire function with exponential growth $Ce^{M(|\xi|^{pk/s_1}+|v|^{qk/s_2})}$, for some constants $C, M > 0$. Then using Cauchy estimates, we see that for all $R_1, R_2 > 0$ and all $n, m \in \mathbb{N}$ we have

$$\left|\frac{a_{n,m}}{\Gamma\left(1+\frac{ns_1}{pk}+\frac{ms_2}{qk}\right)}\right| \leq C\frac{e^{MR_1^{pk/s_1}}}{R_1^n}\frac{e^{MR_2^{qk/s_2}}}{R_2^m}.$$

Since the map $x \mapsto e^{Mx^l}/x^n$, $l > 0$, attains a minimum at $x = (n/Ml)^{1/l}$, if we choose $R_1 = (ns_1/Mpk)^{s_1/pk}$ and $R_2 = (ms_2/Mqk)^{s_2/qk}$, we see that

$$|a_{n,m}| \leq C\left[\left(\frac{2Mepk}{ns_1}\right)^{ns_1/pk}\Gamma\left(1+\frac{ns_1}{pk}\right)\right]\left[\left(\frac{2Meqk}{ms_2}\right)^{ms_2/qk}\Gamma\left(1+\frac{ms_2}{qk}\right)\right].$$

Then an application of Stirling's formula in each term in brackets leads us to the conclusion. Note we have used the inequality $\Gamma(1+a+b) \leq 2^{a+b}\Gamma(1+a)\Gamma(1+b)$, valid for all $a, b > 0$.

Finally, Proposition 1.3.3 established that $R_{1/k,d}^{(p,q)} = R_{M/k,Md}^{(Mp,Mq)}$ for all $p, q, M \in \mathbb{N}^*$ and all directions d. The assertions also follows from the previous theorem by noting that for any $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ we have

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)} = \mathcal{B}_{k/M,(s_1,s_2)}^{(Mp,Mq)}, \quad \mathcal{L}_{k,(s_1,s_2)}^{(p,q)} = \mathcal{L}_{k/M,(s_1,s_2)}^{(Mp,Mq)}, \tag{2-16}$$

as well as for the corresponding formal transformations.

2.3 Monomial summability and blow-ups

In this section we shall explore the behavior of monomial asymptotic expansions under point blow-ups. We only analyze what happens when we compose a given series of some Gevrey type in a monomial or summable in a monomial with the usual charts of the blow-up of the origin in $\mathbb{C}^2$. As expected the value of the Gevrey type and the parameter of summability is conserved but the monomial change depending on the chart. This result will be a fundamental tool that provides examples of non-summable series for any parameter and any monomial, see Chapter 4.

As in Section 1.2.2, we consider the charts of the classical blow-up at the origin of $\mathbb{C}^2$: $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}^2$, given by $\pi_1(x, \varepsilon) = (x\varepsilon, \varepsilon)$ and $\pi_2(x, \varepsilon) = (x, x\varepsilon)$.

Lemma 2.3.1. *Let $\hat{f} \in \hat{R}$ be a formal power series. Then the following assertions are true:*

1. *$\hat{f} \in R$ if and only if $\hat{f} \circ \pi_1 \in R$ if and only if $\hat{f} \circ \pi_2 \in R$.*
2. *$\hat{f} \in \hat{R}_s^{(p,q)}$ if and only if $\hat{f} \circ \pi_1 \in \hat{R}_s^{(p,p+q)}$ and $\hat{f} \circ \pi_2 \in \hat{R}_s^{(p+q,q)}$.*

Proof. We only prove the nontrivial implication in the second statement. Let $\hat{f} = \sum a_{n,m} x^n \varepsilon^n$ and write $\hat{f}(x\varepsilon, \varepsilon) = \sum a'_{n,m} x^n \varepsilon^m$ where $a'_{n,m} = 0$ if $m < n$ and $a'_{n,m} = a_{n,m-n}$ if $m \geq n$. Assuming there are constants $C, A > 0$ such that $|a_{n,m}| \leq CA^{n+m} \min\{n!^{s/p}, m!^{s/q}\}$, for all $n, m \in \mathbb{N}$, then by inequality (1-4)

$$|a'_{n,m}| \leq CA^m \min\{n!^{s/p}, (m-n)!^{s/q}\} \leq CA^m n!^{st/p}(m-n)!^{s(1-t)/q},$$

for $m \leq n$ and for all t such that $0 \leq t \leq 1$. If we take $t = p/(p+q)$ then

$$|a'_{n,m}| \leq CA^m n!^{s/(p+q)}(m-n)!^{s/(p+q)} \leq CA^m m!^{s/(p+q)}.$$

This inequality let us conclude that $\hat{f}\circ\pi_1 \in \hat{R}_s^{(p,p+q)}$, as we wanted to show. The proof for π_2 is analogous. □

Proposition 2.3.2. *Let $\hat{f}\in R_{1/k,d}^{(p,q)}$ be a $k-$summable series in $x^p\varepsilon^q$ in direction d with sum f. Then $\hat{f}\circ\pi_1\in R_{1/k,d}^{(p,p+q)}$, $\hat{f}\circ\pi_2\in R_{1/k,d}^{(p+q,q)}$ and have sums $f\circ\pi_1$, $f\circ\pi_2$, respectively.*

Proof. Despite the fact that using the characterization of having asymptotic expansion in $x^p\varepsilon^q$ given by Proposition 1.2.21 the proof follows immediately (if (f_N) is the family of analytic bounded functions that provide the asymptotic expansion to f then $(f_N\circ\pi_j)$ will provide the asymptotic expansion to $f\circ\pi_j$, $j=1,2$), we want to give a proof based on the monomial Borel-Laplace methods.

We only write the proof for π_1, the proof for π_2 can be done in the same way. Consider s_1, s_2 real numbers such that $0<s_1,s_2<1$ and $s_1+s_2=1$. If we also request that $s_1>p/(p+q)$, or equivalently $s_2<q/(p+q)$, then

$$s_1'=\frac{p+q}{q}s_1-\frac{p}{q},\ \ s_2'=\frac{p+q}{q}s_2, \tag{2-17}$$

will satisfy $0<s_1',s_2'<1$ and $s_1'+s_2'=1$. With these numbers it is straightforward to check that

$$\hat{\mathcal{B}}_{k,(s_1',s_2')}^{(p,p+q)}((x^p\varepsilon^{p+q})^k\hat{f}(x\varepsilon,\varepsilon))(\xi,\upsilon)=\hat{\mathcal{B}}_{k,(s_1,s_2)}^{(p,q)}((x^p\varepsilon^q)^k\hat{f}(x,\varepsilon))(\xi\upsilon,\upsilon).$$

If $\hat{f}$ is $k-$summable in $x^p\varepsilon^q$ in direction d, then by Theorem 2.2.1, it is $k-(s_1,s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d, for all $s_1,s_2>0$ with $s_1+s_2=1$. Fix (s_1,s_2) satisfying the conditions of the previous paragraph. If $\hat{\varphi}_{s_1,s_2}=\hat{\mathcal{B}}_{k,(s_1,s_2)}^{(p,q)}((x^p\varepsilon^q)^k\hat{f}(x,\varepsilon))$, then it can be analytically continued, say as φ_{s_1,s_2}, to a monomial sector of the form $S_{p,q}(d,2\epsilon,+\infty)$ having an exponential growth $|\varphi_{s_1,s_2}(\xi,\upsilon)|\leq De^{M\max\{|\xi|^{pk/s_1},|\upsilon|^{qk/s2}\}}$.

Set $\hat{\psi}_{s_1',s_2'}=\hat{\mathcal{B}}_{k,(s_1',s_2')}^{(p,p+q)}((x^p\varepsilon^{p+q})^k\hat{f}(x\varepsilon,\varepsilon))$, where s_1' and s_2' are given by equation (2-17). We remark that $(\xi,\upsilon)\in S_{p,p+q}(d,2\epsilon,+\infty)$ if and only if $(\xi\upsilon,\upsilon)\in S_{p,q}(d,2\epsilon,+\infty)$ (there are no restriction on the norm of the points). Since $\hat{\psi}_{s_1',s_2'}(\xi,\upsilon)=\hat{\varphi}_{s_1,s_2}(\xi\upsilon,\upsilon)$, it follows that $\hat{\psi}_{s_1',s_2'}$ can be analytically continued to $S_{p,p+q}(d,2\epsilon,+\infty)$, by the formula $\psi_{s_1',s_2'}(\xi,\upsilon)=\varphi_{s_1,s_2}(\xi\upsilon,\upsilon)$.

To determine the exponential growth of $\psi_{s_1',s_2'}$, we may use inequality (1-4) to see that

$$|\xi\upsilon|^{pk/s_1}\leq\max\{|\xi|^{pk/s_1t},|\upsilon|^{pk/s_1(1-t)}\},$$

for all $0<t<1$. If we take $t_0=1-ps_2/qs_1$, the condition impose on s_1 implies that $0<t_0<1$, and with this value we see from (2-17) that

$$|\xi\upsilon|^{pk/s_1}\leq\max\{|\xi|^{pk/s_1'},|\upsilon|^{(p+q)k/s_2'}\}.$$

Thus $\psi_{s_1',s_2'}$ satisfies

$$\begin{aligned}|\psi_{s_1',s_2'}(\xi,\upsilon)| \leq & De^{M\left(|\xi\upsilon|^{pk/s_1}+|\upsilon|^{qk/s_2}\right)}\\ \leq & De^{2M\left(|\xi|^{pk/s_1'}+|\upsilon|^{(p+q)k/s_2'}\right)},\end{aligned}$$

for all $(\xi,\upsilon)\in S_{p,p+q}(d,2\epsilon,+\infty)$. The previous reasoning joint with Lemma 2.3.1 prove that $\hat{f}(x\varepsilon,\varepsilon)$ is $k-(s_1',s_2')-$summable in the monomial $x^p\varepsilon^{p+q}$ in direction d and therefore $\hat{f}(x\varepsilon,\varepsilon)$ is $k-$summable in the monomial $x^p\varepsilon^{p+q}$ in direction d.

Since the corresponding sum f of $\hat{f}$ can be calculated as $f(x,\varepsilon)=\frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2})(x,\varepsilon)$ then the $k-(s_1',s_2')-$Borel sum in the monomial $x^p\varepsilon^{p+q}$ of $\hat{f}\circ\pi_1$ and therefore its $k-$sum in the monomial $x^p\varepsilon^{p+q}$ is given by

$$\begin{aligned}\frac{1}{(x^p\varepsilon^{p+q})^k}\mathcal{L}^{(p,p+q)}_{k,(s_1',s_2')}(\psi_{s_1',s_2'})(x,\varepsilon) &= \int_0^{e^{i\alpha}\infty}\psi_{s_1',s_2'}(xu^{s_1'/pk},\varepsilon u^{s_2/(p+q)k})e^{-u}du\\ &= \int_0^{e^{i\alpha}\infty}\varphi_{s_1,s_2}(x\varepsilon u^{1/k\left(s_1'/p+s_2'/(p+q)\right)},\varepsilon u^{s_2/(p+q)k})e^{-u}du\\ &= \int_0^{e^{i\alpha}\infty}\varphi_{s_1,s_2}(x\varepsilon u^{s_1/pk},\varepsilon u^{s_2/(p+q)k})e^{-u}du\\ &= f(x\varepsilon,\varepsilon)=f\circ\pi_1(x,\varepsilon),\end{aligned}$$

as we wanted to show.

□

We want to finish this section giving a quick applications of blow-ups by proving again one of the cases of tauberian Theorem 1.3.5. The idea used here will be applied later again in Theorem 4.3.1 to generalize tauberian Theorem 1.3.5 and will provide examples of series not $k-$summable in a monomial for any monomial and any $k>0$.

Let $k,l>0$ be positive real numbers and let $x^p\varepsilon^q$ and $x^{p'}\varepsilon^{q'}$ two monomials. As in case (3) of the theorem, suppose that $p/p'\neq q/q'$. We want to prove that if $\hat{f}\in R^{(p,q)}_{1/k}\cap R^{(p',q')}_{1/l}$ then $\hat{f}$ is convergent. If $\max\{p/p',q/q'\}<l/k$ or $l/k<\min\{p/p',q/q'\}$ the theorem was proved using the maps $\hat{T}_{p,q}$ or $\hat{T}_{p',q'}$ and classical tauberian Theorem 1.1.13. The remaining case $\min\{p/p',q/q'\}\leq l/k\leq\max\{p/p',q/q'\}$ can be reduced using blow-ups to one of the previous cases. Indeed, to fix ideas suppose that $p/p'<l/k\leq q/q'$. Then take $N\in\mathbb{N}^*$ such that

$$\frac{qk-q'l}{p'l-pk}<N,$$

and consider the series $\hat{f}\circ\pi_2^N$. By Proposition 2.3.2 we conclude that $\hat{f}\circ\pi_2^N\in R^{(p,Np+q)}_{1/k}\cap R^{(p',Np'+q')}_{1/l}$. But the new monomials satisfy $\max\{p/p',Np+q/Np'+q'\}<l/k$ and by the previous case we conclude that $\hat{f}\circ\pi_2^N$ is convergent. Then by Lemma 2.3.1 $\hat{f}$ is convergent as we wanted to prove.

3 Singularly perturbed analytic linear differential equations

In the present chapter we propose some applications of monomial summability and the tauberian theorems obtained in the previous chapters. As has been mentioned along the text, summability in a monomial is useful in the study of formal solutions of certain singularly perturbed problems. We remark that we will only treat problems related with differential equations.

The content of the first section and an essential application is the study of summability properties of formal solutions of a class of singularly perturbed systems of linear ordinary differential equations with holomorphic coefficients. It is mandatory in the sense that it was the initial motivation of the authors of [CDMS] to introduce the concept of summability in a monomial. We will not follow here the lines of that paper but instead we approach the problem by following [BM], restricting our attention to linear systems. The goal is to prove that the solutions of those systems are $1-$summable in the corresponding monomial, under the crucial hypothesis of having invertible constant linear part. Ultimately the proof we provide is based in the Banach's fixed point theorem and the Ramis-Sibuya theorem for monomial summability and has no new ideas in it. Withal it is a self-contained exposition with a well detailed proof. We have however strengthened Theorem 3 in [BM] on $(s_1, s_2)-$summability of the solutions explained there by using the characterization of summability in a monomial in terms of the Borel-Laplace method detailed in the last chapter.

As an application of the formula described in Proposition 2.1.3 we pass in the second section to study the partial differential equation with holomorphic coefficients naturally associated with a monomial and a weight of the variables and depending linearly of the unknown. Once again if the constant linear part is invertible the equation will have a unique solution, $1-$summable in the given monomial. Since the mentioned formula transforms the associated vector field into multiplication by the monomial, the scheme of proof used before can be used once more to provide a correct proof: convert the differential equation into a convolution equation and study its solutions.

The last application we include in the text is the study of formal solutions of pfaffian systems in two independent variables, in such a way that every equation separately corresponds to a singularly perturbed ordinary differential equations with holomorphic coefficients, as the ones studied in the first section but not necessarily linear. A usual condition on those systems is the well-known *integrability condition* that relates the single equations. We start centering our attention in the consequences of such a system to be completely integrable, more particularly in the behavior of their linear parts. In order to obtain results of convergence of solutions we have concluded that generically we cannot assume the completely integrable condition. Indeed, if we want that two different monomial intervene in the equations then the corresponding linear parts will be generically highly degenerated, making impossible the application of the tauberian theorems and the monomial summability results of the first section. We conclude the section with the case of linear pfaffian systems with the same monomial intervening in both equations obtaining monomial summability properties and convergence in a determinate case when the integrability condition is not assumed.

3.1 Monomial summability of solutions of some doubly singular differential equations

The aim of this section is to study the summability properties of formal solutions of a certain class of systems of linear ordinary differential equations with an irregular singularity in the independent variable and additionally a singularity coming from a parameter. More specifically, we are going to consider systems of the form

$$\varepsilon^q x^{p+1}\frac{dy}{dx} = A(x,\varepsilon)y(x,\varepsilon) + b(x,\varepsilon), \tag{3-1}$$

where $p, q \in \mathbb{N}^*$, $y \in \mathbb{C}^l$, $A \in \mathrm{Mat}(l\times l, \mathbb{C}\{x,\varepsilon\})$, $b \in \mathbb{C}\{x,\varepsilon\}^l$. Such systems are denominated *doubly singular systems of ordinary linear differential equations*. We are going to show that under generic conditions, viz. $A(0,0)$ being invertible, there is a unique formal solution of the above system and it is $1-$summable in the monomial $x^p\varepsilon^q$. This result is know, even for the nonlinear case, and a proof can be found in [CDMS]. Nonetheless, we provide an elementary proof in the linear case based in the ideas contained in [BM]. Additionally we recover and strengthen some results exposed there, for instance, the $1-(s_1,s_2)-$Borel summability in the monomial $x^p\varepsilon^q$ of the solutions, for any $s_1, s_2 > 0$ with $s_1+s_2 = 1$.

Consider the system (3-1). Under the above hypotheses, we can choose $r > 0$ such that $A \in \mathrm{Mat}(l\times l, \mathcal{O}_b(D_r^2))$ and $b \in \mathcal{O}_b(D_r^2)^l$ and expand those maps into power series in two, resp. one variables, say

$$\begin{aligned}
A(x,\varepsilon) &= \sum_{n,m\geq 0} A_{n,m}x^m\varepsilon^m = \sum_{n\geq 0} A_{n*}(\varepsilon)x^n = \sum_{m\geq 0} A_{*m}(x)\varepsilon^m,\\
b(x,\varepsilon) &= \sum_{n,m\geq 0} b_{n,m}x^m\varepsilon^m = \sum_{n\geq 0} b_{n*}(\varepsilon)x^n = \sum_{m\geq 0} b_{*m}(x)\varepsilon^m,
\end{aligned}$$

where all of them are convergent for $|x|, |\varepsilon| < r$ and $A_{n*}, A_{*m} \in \mathrm{Mat}(l\times l, \mathcal{O}_b(D_r))$, $b_{n*}, b_{*m} \in \mathcal{O}_b(D_r)^l$ for all $n, m \in \mathbb{N}$.

The proof of existence and uniqueness of formal solutions $\hat{y} \in \hat{R}^l$ of equation (3-1) is classical and it can be done directly inserting the unknown $\hat{y}$ in the equation and finding recursively its coefficients. For the study of its Gevrey order we will require the use of a family of norms and some inequalities that are included for references in the next remark.

Remark 3.1.1. We recall that for any $f \in \mathcal{O}(D_r)$ and $n \in \mathbb{N}$, the *Nagumo norm of order n* of f is defined as

$$\|f\|_n = \sup_{|z|<r} |f(z)|(r-|z|)^n.$$

Of course, the norm depends also of r but the dependence will be avoided if the context is clear. Also the value can be $+\infty$. This family of norms satisfies the following properties

$$\|f+g\|_n \leq \|f\|_n + \|g\|_n, \quad \|fg\|_{n+m} \leq \|f\|_n\|g\|_m, \quad \|f'\|_{n+1} \leq e(n+1)\|f\|_n, \tag{3-2}$$

that make it useful for applications in differential equations. For a proof of these properties, the reader may consult [CDRSS].

We will also use the following inequality satisfied by the Gamma function:

$$\Gamma(1+\alpha)\Gamma(1+\beta) \leq \Gamma(1+\alpha+\beta), \tag{3-3}$$

and valid for all $\alpha, \beta > 0$, and the limit

$$\lim_{N\to+\infty} \frac{(Ns+b)^b\Gamma(1+Ns)}{\Gamma(1+Ns+b)} = 1, \tag{3-4}$$

that is obtained by Stirling's formula, where $b > 0$ is a real number.

Proposition 3.1.2. *Consider the differential equation (3-1). If $A(0,0)$ is invertible then (3-1) has a unique formal solution $\hat{y} \in \hat{R}^l$. Moreover $\hat{y} \in (\hat{R}_1^{(p,q)})^l$.*

Proof. Let us write the unknown formal solution $\hat{y}$ as

$$\hat{y}(x,\varepsilon) = \sum_{n,m\geq 0} y_{n,m}x^m\varepsilon^m = \sum_{n\geq 0} y_{n*}(\varepsilon)x^n = \sum_{m\geq 0} y_{*m}(x)\varepsilon^m. \tag{3-5}$$

The existence and uniqueness of $\hat{y}$ follows directly from replacing the expressions in (3-5) into equation (3-1). Indeed, if we expand respect to x and ε, we obtain the recurrence equations

$$(n-p)y_{n-p,m-q} = \sum_{i=0}^{n}\sum_{j=0}^{m} A_{n-i,m-j}y_{i,j} + b_{n,m}, \tag{3-6}$$

for all $n, m \in \mathbb{N}$. Here and below we set all coefficients with negative indexes as 0. Since $A_{0,0} = A(0,0)$ is invertible, the coefficients $y_{n,m}$ are uniquely determined by these equations. Analogously, if we expand in x, we have

$$(n-p)\varepsilon^q y_{n-p*}(\varepsilon) = \sum_{i=0}^{n} A_{n-i*}(\varepsilon) y_{i*}(\varepsilon) + b_{n*}(\varepsilon), \tag{3-7}$$

for all $n \in \mathbb{N}$. In this case we see that $A_{0*}(\varepsilon)$ is also invertible for $|\varepsilon| < r' \leq r$. By reducing r if necessary we may suppose that $r' = r$. Again, we obtain that the coefficients y_{n*} are uniquely determined by these equations and are analytic on D_r. Finally, if we expand in ε we obtain the family of differential equations

$$x^{p+1} y'_{*m-q}(x) = \sum_{j=0}^{m} A_{*m-j}(x) y_{*j}(x) + b_{*m}(x), \tag{3-8}$$

for all $m \in \mathbb{N}$. As before we may suppose that $A_{*0}(x)$ is invertible for $|x| < r$. We see that the coefficients y_{*m} are uniquely determined by the previous recurrence and are analytic on D_r.

In order to determine the Gevrey order of the entries of $\hat{y}$ in x we use the Nagumo's norm of order 0, and we just write $\|g\| = \|g\|_0 = \sup_{|z|<R} |g(z)|$ if $g \in \mathcal{O}_b(D_R)$. Take any $0 < R < r$ and write $c = \|A_{0*}^{-1}\|$, $z_n = \|y_{n*}\|$, $a_n = \|A_{n*}\|$ and $f_n = \|b_{n*}\|$. If follows from equation (3-7) that these numbers satisfy the inequalities

$$z_n \leq c\left((n-p)R^q z_{n-p} + \sum_{i=0}^{n-1} a_{n-j} z_j + f_n\right). \tag{3-9}$$

If we define recursively w_n by $w_0 = z_0$ and

$$w_n = c\left((n-p)R^q w_{n-p} + \sum_{i=0}^{n-1} a_{n-i} w_i + f_n\right), \tag{3-10}$$

for all $n \geq 1$, then from (3-9) and induction we see that $0 \leq z_n \leq w_n$ for all $n \in \mathbb{N}$. If we define the auxiliary series $\hat{w}(\tau) = \sum_{n\geq 0} w_n \tau^n$, $a(\tau) = \sum_{n\geq 1} a_n \tau^n$ and $f(\tau) = \sum_{n\geq 0} f_n \tau^n$, then $a, f \in \mathbb{C}\{\tau\}$ and equation (3-10) shows that $\hat{w}$ satisfies the differential equation

$$cR^q \tau^{p+1} \frac{dw}{d\tau} = (1 - ca(\tau))w(\tau) - cf(\tau). \tag{3-11}$$

Since $a(0) = 0$, by classical results, this equation has a unique formal solution and it is $1/p-$Gevrey in τ. In conclusion, there are positive constants K, M such that

$$z_n \leq w_n \leq K M^n n!^{1/p},$$

for all $n \in \mathbb{N}$. From Cauchy's formula we obtain the bounds

$$|y_{n,m}| \leq K \frac{M^n}{R^m} n!^{1/p},$$

for all $n, m \in \mathbb{N}$. This shows that $\hat{y} \in (\hat{R}_{(1/p,0)})^l$.

To conclude the proof it remains to estimate the Gevrey order of $\hat{y}$ in ε. As before, take any $0 < R < r$ and write $c = \|A_{*0}^{-1}\|_0$, $z_n = \|y_{*n}\|_n$, $a_n = \|A_{*n}\|_n$ and $f_n = \|b_{*n}\|_n$, where $\|\cdot\|_n$ stands for the Nagumo norm of order n. If follows from equation (3-8) and the properties (3-2) that these numbers satisfy the inequalities

$$z_m \leq c\left(eR^{p+q}(m-q+1)z_{m-q} + \sum_{j=0}^{m-1} a_{m-j}z_j + f_m\right). \tag{3-12}$$

Dividing by $\Gamma(1+m/q) = m/q\Gamma(m/q)$, using the inequality (3-3) and $m-q+1 \leq 2m$, we can conclude that

$$\frac{z_m}{\Gamma\left(1+\frac{m}{q}\right)} \leq c\left(2eqR^{p+q}\frac{z_{m-q}}{\Gamma\left(\frac{m}{q}\right)} + \sum_{j=0}^{m-1}\frac{a_{m-j}}{\Gamma\left(1+\frac{m-j}{q}\right)}\frac{z_j}{\Gamma\left(1+\frac{j}{q}\right)} + \frac{f_m}{\Gamma\left(1+\frac{m}{q}\right)}\right), \tag{3-13}$$

for $m \in \mathbb{N}$. Define recursively w_m by $w_0 = z_0$ and

$$w_m = c\left(2eqR^{p+q}w_{m-q} + \sum_{j=0}^{m-1}\frac{a_{m-j}}{\Gamma\left(1+\frac{m-j}{q}\right)}w_j + \frac{f_m}{\Gamma\left(1+\frac{m}{q}\right)}\right). \tag{3-14}$$

It follows from (3-13) and by induction that $z_m/\Gamma\left(1+\frac{m}{q}\right) \leq w_m$ for all $m \in \mathbb{N}$. Using the series $\hat{w}(\tau) = \sum_{m\geq 0} w_m\tau^m$, $a(\tau) = \sum_{m\geq 1} a_m/\Gamma(1+\frac{m}{q})\tau^m$ and $f(\tau) = \sum_{m\geq 0} f_m/\Gamma(1+m/q)\tau^m$, we see that $a, f \in \mathcal{O}(\mathbb{C})$, and from equation (3-14) it follows that $\hat{w}$ satisfies the functional equation

$$w(\tau) = c(2eqR^{p+q}\tau^q w(\tau) + a(\tau)w(\tau) + f(\tau)). \tag{3-15}$$

Since $a(0) = 0$, this equation has a unique analytic solution at 0 and it must be $\hat{w}$. Then there are positive constants C, D such that

$$w_m \leq CD^m,$$

for all $m \in \mathbb{N}$. Using Cauchy's formula for $\rho < R$ we obtain the bounds

$$|y_{n,m}| \leq C\frac{D^m}{(R-\rho)^m\rho^n}\Gamma\left(1+\frac{m}{q}\right),$$

valid for all $n, m \in \mathbb{N}$. Therefore $\hat{y} \in (\hat{R}_{(0,1/q)})^l$. It follows from Proposition 1.2.3 that $\hat{y} \in (\hat{R}_1^{(p,q)})^l$, as we wanted to prove.

□

As in [BM], we start the analysis of the summability of $\hat{y}$ by studding the behavior and properties of its $p-$Borel transform in the variable x. The use of this variable is justified recalling formulas (1) and (2) in Proposition 1.1.9, since they allow us to pass from a differential equation involving $x^{p+1}\partial/\partial x$ to a convolution equation.

From now on we assume that $A(0,0)$ is invertible. We know from Proposition 3.1.2 that $\hat{y}$ is a $1/p-$Gevrey series in x with coefficients in $\mathcal{O}_b(D_R)^l$, for $0<R<r$. Then the series

$$\hat{\mathcal{B}}_p(x^p\hat{y})(\zeta,\varepsilon)=\sum_{n\geq 0}\frac{y_{n*}(\varepsilon)}{\Gamma(1+n/p)}\zeta^n,$$

defines an analytic function for $|\zeta|<\rho$, $|\varepsilon|<R$, for some $\rho>0$.

Since $\hat{y}$ satisfies equation (3-1), then $\hat{w}(x,\varepsilon)=x^p\hat{y}(x,\varepsilon)$ satisfies the equation

$$\varepsilon^q x^{p+1}\frac{dw}{dx}=(p\varepsilon^q x^p I+A(x,\varepsilon))w(x,\varepsilon)+x^p b(x,\varepsilon), \tag{3-16}$$

where $I=I_l$ denotes the identity matrix of size l. If we apply the $p-$Borel transform to this equation, we see that $F=\hat{\mathcal{B}}_p(\hat{w})$ is a solution of the corresponding convolution equation

$$(p\zeta^p\varepsilon^q I-A_{0*}(\varepsilon))F(\zeta,\varepsilon)=\mathcal{B}_p(\widetilde{A})*_p F(\zeta,\varepsilon)+g(\zeta,\varepsilon), \tag{3-17}$$

where $\widetilde{A}(x,\varepsilon)=p\varepsilon^q x^p I+A(x,\varepsilon)-A_{0*}(\varepsilon)$ and $g=\mathcal{B}_p(x^p b)$. Furthermore we can write

$$g(\zeta,\varepsilon)=\sum_{n\geq 0}\frac{b_{n*}(\varepsilon)}{\Gamma(1+n/p)}\zeta^n,\quad \mathcal{B}_p(\widetilde{A})(\zeta,\varepsilon)=\sum_{n\geq 1}\frac{\overline{A}_{n*}(\varepsilon)}{\Gamma(n/p)}\zeta^{n-p}, \tag{3-18}$$

where $\overline{A}_{n*}(\varepsilon)=A_{n*}(\varepsilon)$ for $n\neq p$ and $\overline{A}_{p*}(\varepsilon)=A_{p*}(\varepsilon)+p\varepsilon^q I$.

We wish to restrict our attention to a domain where we can invert the matrix $p\zeta^p\varepsilon^q I-A_{0*}(\varepsilon)$. Let $\lambda_1,...,\lambda_l$ be the eigenvalues of $A_{0*}(0)$ repeated according to their multiplicity and recall that they are all non-zero by assumption. Also let $\lambda_j(\varepsilon), j=1,...,l$, stand for the eigenvalues of $A_{0*}(\varepsilon)$. Those are algebraic functions of ε and $\lambda_j(0)=\lambda_j$. Using these notations, the matrix $p\zeta^p\varepsilon^q I-A_{0*}(\varepsilon)$ is singular for the points (ζ,ε) satisfying $p\zeta^p\varepsilon^q=\lambda_j(\varepsilon)$ for some $j=1,...,n$.

We can choose $\delta>0$ small enough such that $\delta<|\lambda_j|$, $j=1,...,l$ and such that the open sets $|p\zeta^p\varepsilon^q-\lambda_j|<\delta$ do not intersect if $\lambda_{j_1}\neq\lambda_{j_2}$. We will refer to such δ as *admissible*. It is also possible to choose a positive number $R(\delta) < R$ such that $|\lambda_j(\varepsilon) - \lambda_j| \le \delta/2$ for all $j = 1, ..., l$ and $|\varepsilon| < R(\delta)$. Consider the closed set defined by

$$\Omega_\delta = \{(\zeta, \varepsilon) \in \mathbb{C}^2 \mid |\varepsilon| \le R(\delta),\ \ |p\zeta^p \varepsilon^q - \lambda_j| \ge \delta \text{ for all } j = 1, ..., l\}.$$

It satisfies the following properties:

1. It contains the polydisc at the origin $D_{\rho_1} \times D_{\rho_2}$ for $\rho_1, \rho_2 > 0$ satisfying $\rho_2 \le R(\delta)$ and $p\rho_1^p \rho_2^q < \min\limits_{1 \le j \le l} |\lambda_j| - \delta$.

2. We have $|p\zeta^p\varepsilon^q - \lambda_j(\varepsilon)| \ge \delta/2$ for all $(\zeta, \varepsilon) \in \Omega_\delta$. Therefore the matrix $p\zeta^p\varepsilon^q I - A_{0*}(\varepsilon)$ is invertible on Ω_δ.

3. We can find a number $M = M(\delta) > 0$ such that

$$\left\| \left(p\zeta^p\varepsilon^q I - A_{0*}(\varepsilon)\right)^{-1} \right\| \le M, \qquad \text{for all } (\zeta, \varepsilon) \in \Omega_\delta.$$

Working on Ω_δ, we see that finding solutions of certain type of the convolution equation (3-17) is equivalent to find a fixed point of the operator $\mathcal{H}$ given by

$$\mathcal{H}(F)(\zeta, \varepsilon) = \left(p\zeta^p\varepsilon^q I - A_{0*}(\varepsilon)\right)^{-1} \left(\mathcal{B}_p(\widetilde{A}) *_p F(\zeta, \varepsilon) + g(\zeta, \varepsilon)\right), \qquad (3\text{-}19)$$

and defined in an adequate Banach space E of functions.

We are going to prove that $\hat{\mathcal{B}}_p(\hat{w})$ admits analytic continuation to Ω_δ. To do so we consider a bounded open set $U \subset \Omega_\delta$ and an arbitrary $N \in \mathbb{N}$. Then to prove that $\hat{\mathcal{B}}_p(\hat{w})$ admits analytic continuation to U it is sufficient to prove that $\hat{\mathcal{B}}_p(\hat{w}) - \sum_{n=0}^N \frac{y_{n*}}{\Gamma(1+n/p)}\zeta^n$ admits analytic continuation to U.

If we perform the change of variable $w_N(x, \varepsilon) = w(x, \varepsilon) - \sum_{n=0}^N y_{n*}(\varepsilon)x^{n+p}$ in equation (3-16), then using the recurrences (3-7) we see that w_N satisfies the same differential equation (3-16) but with $b_N(x, \varepsilon) = \sum_{n=N+1}^\infty \left(b_{n*}(\varepsilon) + \sum_{i=0}^N A_{n-i*}(\varepsilon)y_i(\varepsilon) - (n-p)y_{n-p*}(\varepsilon)\right) x^n$ instead of b. Therefore $\hat{\mathcal{B}}_p(w_N)$ satisfies the same convolution equation (3-17) but with g replaced by $g_N = \mathcal{B}_p(x^p b_N)$. The main point here is that $\text{ord}_x g_N > N$.

Let $E_{U,N}$ denote the subspace of functions of $\mathcal{C}(\overline{U}) \cap \mathcal{O}(U)$ such that

$$\|F\|_N = \sup_{(\zeta,\varepsilon)\in\overline{U}} \frac{|F(\zeta, \varepsilon)|}{|\zeta|^N},$$

is finite. $E_{U,N}$ is a Banach space with the norm $\|\cdot\|_N$ and $g_N \in E_{U,N}$. We shall prove that $\mathcal{H}_N : E_{U,N} \to E_{U,N}$, defined as $\mathcal{H}$ but with g_N instead of g, is well-defined and it is a contraction if N is large enough. Indeed, if $F \in E_{U,N}$ then

$$\begin{aligned}|\mathcal{H}_N(F)(\zeta,\varepsilon)| &\leq M\left|\int_0^1 \mathcal{B}_p(\widetilde{A})(\zeta t^{1/p},\varepsilon)F(\zeta(1-t)^{1/p},\varepsilon)\zeta^p dt\right| + M\|g_N\|_N|\zeta|^N\\ &\leq M\int_0^1 \left|\mathcal{B}_p(\widetilde{A})(\zeta t^{1/p},\varepsilon)\right| \|F\|_N|\zeta|^{p+N}(1-t)^{N/p}dt + M\|g_N\|_N|\zeta|^N.\end{aligned}$$

A way to estimate adequately the previous expression it is to bound the following integral:

$$\int_0^1 t^{n/p-1}(1-t)^{N/p}dt = \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(1+\frac{N}{p}\right)}{\Gamma\left(1+\frac{n+N}{p}\right)}.$$

for $n \geq 1$. Note that here we are using the Beta function. The easiest case is when $n > p$, because we can use inequality (3-3) to bound it by

$$\frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(1+\frac{N}{p}\right)}{\frac{n+N}{p}\Gamma\left(\frac{n+N}{p}\right)} \leq \frac{p}{n+N} \leq \frac{p}{1+N}.$$

The case $1 \leq n \leq p$ can be treated using the limit (3-4) as follows: using that limit with $s = 1/p$ and $b = n/p$, we can find for every n a large enough constant $D_{n,p}$ such that

$$\frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(1+\frac{N}{p}\right)}{\Gamma\left(1+\frac{n+N}{p}\right)} \leq \frac{D_{n,p}}{(N+n)^{n/p}}.$$

If $D_p = \max\{p, D_{1,p}, ..., D_{p,p}\}$ then the integral is easily bounded in all cases by $\frac{D_p}{(1+N)^{1/p}}$.

Back to the operator $\mathcal{H}_N(F)$, we now can ensure that

$$|\mathcal{H}_N(F)(\zeta,\varepsilon)| \leq M\left(\frac{D_pK_U}{(N+1)^{1/p}}\|F\|_N + \|g_N\|_N\right)|\zeta|^N, \text{ where } K_U = \sup_{(\zeta,\varepsilon)\in\overline{U}}\sum_{n\geq 1}\frac{|\overline{A}_{n*}(\varepsilon)|}{\Gamma(n/p)}|\zeta|^n.$$

We remark that K_U is finite since U is bounded and $\widetilde{A}$ is analytic on D_R^2. The previous bound is sufficient to ensure that $\mathcal{H}_N(F) \in E_{U,N}$. To show that $\mathcal{H}_N$ is a contraction, we estimate as before to see that if $F, G \in E_{U,N}$ then

$$\|\mathcal{H}_N(F) - \mathcal{H}_N(G)\|_N \leq \frac{MD_pK_U}{(N+1)^{1/p}}\|F-G\|_N.$$

Then it is enough to take N with $\frac{MD_pK_U}{(N+1)^{1/p}} < 1$ to conclude the result.

Applying Banach's fixed point theorem, we can conclude that $\mathcal{H}_N$ has a unique fixed point $F_{U,N} \in E_{U,N}$, that is, equation (3-17) has a unique analytic solution defined on U of the form $F_U(\zeta,\varepsilon) = F_{U,N}(\zeta,\varepsilon) + \sum_{n=0}^{N} \frac{y_{n*}(\varepsilon)}{\Gamma(1+n/p)}\zeta^n$. Now, if we take a polydisc at the origin contained in Ω_δ with sufficiently small poly-radius, the solution provided by the fixed point is precisely $\hat{\mathcal{B}}_p(\hat{w})$, because this is the unique formal solution at $(0,0)$ of equation (3-17). Then if U intersects this polydisc, F_U and $\hat{\mathcal{B}}_p(\hat{w})$ coincide in the intersection, being both solutions of the convolution equation. This let us conclude that $\hat{\mathcal{B}}_p(\hat{w})$ admits analytic continuation to Ω_δ.

We focus now in the exponential growth of the solutions we have obtained. Let $C > 0$ an arbitrary positive constant. Taking $\delta > 0$ as before and S an unbounded open set of Ω_δ, we will denote by $E_{S,C}$ the subspace of functions F in $\mathcal{O}(S)$ such that

$$\|F\|_C = \sup_{(\zeta,\varepsilon)\in S} |F(\zeta,\varepsilon)|e^{-C|\zeta|^p},$$

is finite, i.e., the space of analytic functions on S with exponential growth in ζ of order p and type C. Then $E_{S,C}$ is a Banach space with the norm $\|\cdot\|_C$. Furthermore, it follows from (3-18) that we can find a large enough constant $C' > 0$ such that $g \in E_{\Omega_\delta,C'}$.

Following the same ideas as before we shall prove that $\mathcal{H} : E_{S,C} \to E_{S,C}$, is well-defined and a contraction if $C > C'$ is large enough. For the first assertion, if $F \in E_{S,C}$, then

$$\begin{aligned}|\mathcal{H}(F)(\zeta,\varepsilon)| &\leq M\left|\int_0^1 \mathcal{B}_p(\widetilde{A})(\zeta t^{1/p},\varepsilon)F(\zeta(1-t)^{1/p},\varepsilon)\zeta^p dt\right| + M\|g\|_{C'}e^{C'|\zeta|^p}\\ &\leq M\int_0^1 \left|\zeta^p\mathcal{B}_p(\widetilde{A})(\zeta t^{1/p},\varepsilon)\right| \|F\|_C e^{C|\zeta|^p(1-t)}dt + M\|g\|_{C'}e^{C'|\zeta|^p}.\end{aligned}$$

To estimate adequately the previous expression, we can use the Gamma function to see that

$$\int_0^1 t^{n/p-1}e^{C|\zeta|^p(1-t)}dt = \frac{e^{C|\zeta|^p}}{(C|\zeta|^p)^{n/p}}\int_0^{C|\zeta|^p} u^{n/p-1}e^{-u}du \leq \frac{\Gamma(n/p)}{C^{n/p}|\zeta|^n}e^{C|\zeta|^p},$$

for all $n \geq 1$ and $\zeta \in \mathbb{C}$. Applying these bounds we see that

$$\begin{aligned}|\mathcal{H}(F)(\zeta,\varepsilon)| &\leq \frac{ML}{C^{1/p}}\|F\|_C e^{C|\zeta|^p} + M\|g\|_{C'}e^{C'|\zeta|^p}\\ &\leq M\left(\frac{L}{C^{1/p}}\|F\|_C + \|g\|_{C'}\right)e^{C|\zeta|^p}.\end{aligned}$$

where $L > 0$ is a constant such that

$$\sum_{n\geq 1} |\overline{A}_{n*}(\varepsilon)||z|^{n-1} \leq L,$$

for all $|z|, |\varepsilon| \leq R$ and we take C with $1/C^{1/p} < R$.

Therefore we have proved that $\mathcal{H}(F) \in E_{S,C}$. In the same way, if $F, G \in E_{S,C}$ then

$$\|\mathcal{H}(F) - \mathcal{H}(G)\|_C \leq \frac{ML}{C^{1/p}} \|F - G\|_C.$$

If we take C large enough such that $\frac{ML}{C^{1/p}} < 1$ we can conclude that $\mathcal{H}$ is a contraction, and then it has a unique fixed point. We formulate these results in the next lemma.

Lemma 3.1.3. *Using the previous notation, for every admissible $\delta > 0$ and every open set $S \subset \Omega_\delta$ there exist a unique solution F_S of (3-17) defined on S and there are constants $K = K(S), C = C(S) > 0$ such that $|F_S(\zeta, \varepsilon)| \leq K e^{C|\zeta|^p}$ for all $(\zeta, \varepsilon) \in S$.*

The previous lemma joint with Ramis-Sibuya Theorem 1.2.27 are the keys to tackle the main problem of this section: to prove that $\hat{y}$ is in fact $1-$summable in $x^p\varepsilon^q$. For this, let $d_1, ..., d_l \in [0, 2\pi)$ be the different arguments of $\lambda_1, ..., \lambda_l$, $\arg(\lambda_j) = d_j$, numbered so that $d_1 \leq d_2 \leq \cdots \leq d_l$. They are well-defined since the λ_j are different from zero. These are going to be the *singular directions.* Then for every $d \in [0, 2\pi) \setminus \{d_1, ..., d_l\}$ we are going to construct a bounded solution w_d of equation (3-16) defined in a monomial sector in $x^p\varepsilon^q$ bisected by d with opening greater than π.

Indeed, consider any such d and set

$$\delta = \delta(d) = \frac{1}{4} \min_{\substack{1 \leq j,m \leq l \\ \lambda_j \neq \lambda_m}} \left\{ |\lambda_j|, |\lambda_j - \lambda_m|, \mathrm{dist}(\lambda_m, e^{id}\mathbb{R}_{\geq 0}) \right\}.$$

Then δ is admissible and $\delta(d) \to 0$ as $d \to d_j$ for any $j = 1, ..., l$. Also let

$$\overline{S}_d = \{(\zeta, \varepsilon) \in \mathbb{C}^2 \mid 0 < |\varepsilon| < R(\delta), |\arg(\zeta^p \varepsilon^q) - d| < \theta_d/2\},$$

be the intersection of $\mathbb{C} \times D_{R(\delta)}(0)$ with the monomial sector in $\zeta^p\varepsilon^q$ bisected by d with maximal opening θ_d contained in $\Omega_{\delta(d)}$. Again, $\theta_d \to 0$ as $d \to d_j$ for any $j = 1, ..., l$. From here, by abuse of notation, we are denoting the distance between two directions as the minimal one, modulo 2π. With this convention we see that,

$$0 < \frac{\theta_d}{2} < |d - d_j|, \tag{3-20}$$

for all $j = 1, ..., n$.

As expected, the required solution w_d of equation (3-16) is defined by

$$w_d(x, \varepsilon) = \int_0^{e^{i\alpha}\infty} F_{\overline{S}_d}(xu^{1/p}, \varepsilon) e^{-u} du, \tag{3-21}$$

where α ranks from $-\pi/2$ to $\pi/2$. This is just the Laplace transform of $F_{\overline{S}_d}$ w.r.t. the variable x. For fixed α, the previous formula defines an analytic function in the domain given by

$$-\frac{\theta_d}{2}-\alpha<\arg(x^p\varepsilon^q)-d<\frac{\theta_d}{2}-\alpha,\ \ |x|^p<\frac{\cos(\alpha)}{C_d},\ \ |\varepsilon|<R(\delta),$$

where $C_d=C(\overline{S}_d)$. Moving α from $-\pi/2+\theta_d/4$ to $\pi/2-\theta_d/4$ and choosing a small enough constant such that $0<\sigma_d<\sin(\theta_d/4)/C_d$, we can conclude that w_d is in fact well-defined and bounded in the sector $S_{p,q}(d,\pi+\theta_d/2,\rho_d)$, where $\rho_d=\min\{\sin(\theta_d/4)/C_d-\sigma_d,R(\delta)^q\}>0$.

The next step to be able to apply Theorem 1.2.27 is to consider an adequate finite covering of $D^2_{\rho'}\setminus\{x\varepsilon=0\}$, for some $\rho'>0$, by sets $S_{p,q}(d,\pi+\theta_d/2,\rho_d)$, and estimate the difference $w_d-w_{d'}$, when the corresponding domains intersect.

First we remark that if $d,d'\in[0,2\pi)\setminus\{d_1,...,d_l\}$, and there is no d_j between d and d' then $w_d-w_{d'}=0$ in the intersection of its domains. Indeed, we can suppose $d<d'$ and that d and d' are close enough. Then set $S=\Pi_{p,q}(d-\theta_d/4,d'+\theta_{d'}/4,+\infty)\cap\mathbb{C}\times D_{R(\delta)}(0)$. S is contained in Ω_δ, for $\delta=\min\{\delta(d),\delta(d')\}$ and contain both $\overline{S}_d$ and $\overline{S}_{d'}$. Using Lemma 3.1.3 we can replace $F_{\overline{S}_d}$ and $F_{\overline{S}_{d'}}$ in formula (3-21) by F_S and the result follows.

Now suppose we are given d,d' with $d_{j-1}<d<d_j<d'<d_{j+1}$ for some j. If they are close to d_j we can suppose that $d_j-d,d'-d_j<\pi$ and $d+\theta_d/2<d'-\theta_{d'}/2$. In particular $d'-d>|\theta_d-\theta_{d'}|/4$. Now take any $(x,\varepsilon)\in S_{p,q}(d,\pi+\theta_d/2,\rho_d)\cap S_{p,q}(d',\pi+\theta_{d'}/2,\rho_{d'})$. Then the point satisfies

$$d'-\pi/2-\theta_{d'}/4<\arg(x^p\varepsilon^q)<d+\pi/2+\theta_d/4.$$

Let $\alpha'=d_j-d'+\pi/2+\theta_{d'}/4$ and $\alpha=d_j-d-\pi/2-\theta_d/4$. The above condition is then equivalent to say that the point satisfies

$$\alpha<d_j-\arg(x^p\varepsilon^q)<\alpha'. \tag{3-22}$$

Besides the hypotheses and inequality (3-20) imply that $|\alpha|<\pi/2-\theta_d/4$ and also $|\alpha'|<\pi/2-\theta_{d'}/4$.

With these considerations we can estimate $w_d-w_{d'}$ as follows: Let $\delta=\min\{\delta(d),\delta(d')\}$ and let S be the preimage in the complex $(\zeta,\varepsilon)-$plane under the map $(\zeta,\varepsilon)\mapsto(\eta,\varepsilon),\eta=p\zeta^p\varepsilon^q$, of the largest star-shaped domain w.r.t. the origin, containing the sector defined by $V(d-\theta_d/2,d'+\theta_{d'}/2,+\infty)$ and not the circle $|\eta-\lambda_j|\leq\delta/2$, (without loss of generality we are assuming that λ_j is the λ_m such that $\arg(\lambda_m)=d_j$ and has smallest norm) (see Figure **3-1**). Then S is contained in $\Omega_{\delta/2}$, and contains both $\overline{S}_d$ and $\overline{S}_{d'}$. Using Lemma 3.1.3 and the previous α,α' we see that

$$|w_d(x,\varepsilon)-w_{d'}(x,\varepsilon)|=\left|\int_{e^{i\alpha'}\mathbb{R}_{\geq 0}-e^{i\alpha}\mathbb{R}_{\geq 0}}F_S(xu^{1/p},\varepsilon)e^{-u}du\right|.$$

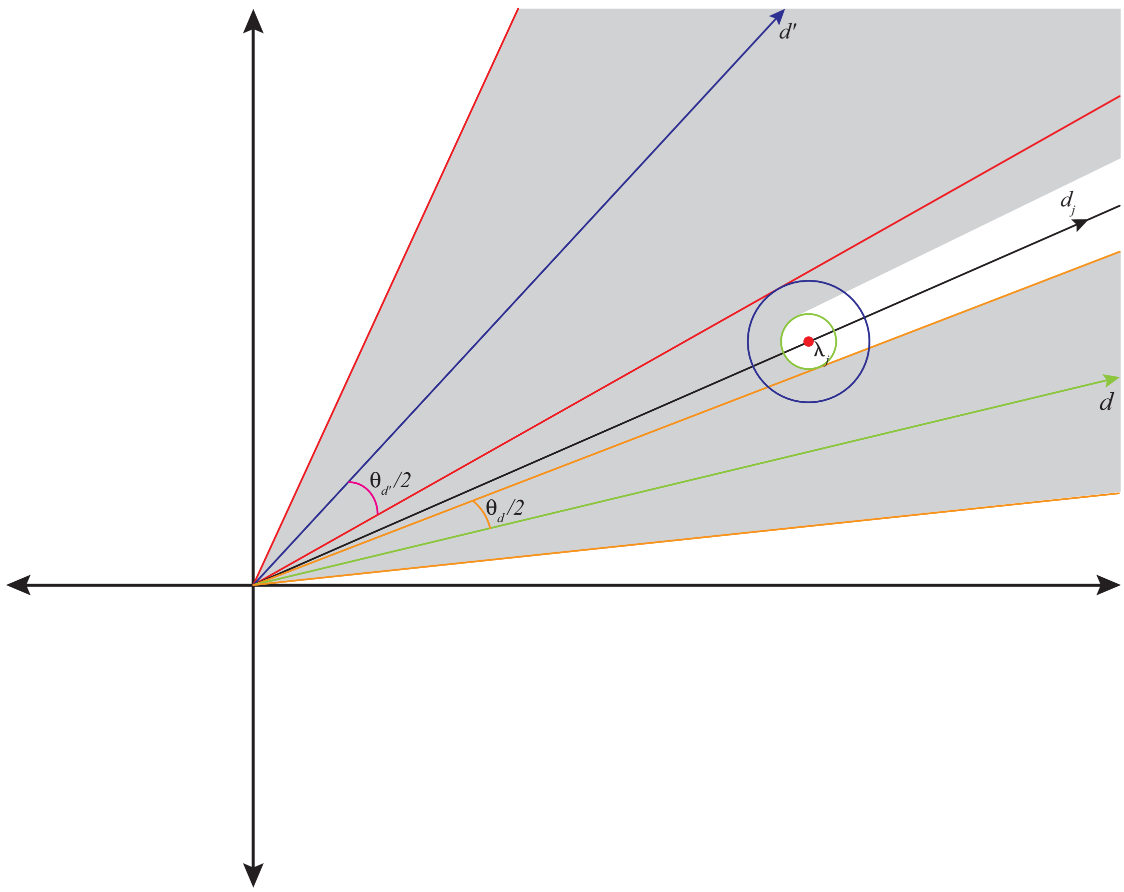

Figura 3-1: S for the case $d < d_j < d'$.

By Cauchy's theorem we can change the above path of integration in the $u-$plane as long as $(xu^{1/p}, \varepsilon) \in S$. In particular we most have $|px^p\varepsilon^q u - \lambda_j| > \delta/2$. Using condition (3-22) we may integrate over a path contained in $V(\alpha, \alpha', +\infty)$ from ∞ down to the vicinity of the point $\lambda_j/px^p\varepsilon^q$, $D_{\delta/p|x^p\varepsilon^q|}(\lambda_j/px^p\varepsilon^q)$, then around this vicinity and back to ∞. Then

$$|w_d(x,\varepsilon) - w_{d'}(x,\varepsilon)| \leq 2\int_{\frac{|\lambda_j|-\delta}{p|x^p\varepsilon^q|}}^{+\infty} K(S)e^{(C(S)|x|^p - \cos(\min\{|\alpha|,|\alpha'|\}))|u|}d|u|,$$

and reducing $\min\{\rho_d, \rho_{d'}\}$ if necessary we may conclude that

$$|w_d(x,\varepsilon) - w_{d'}(x,\varepsilon)| \leq K'_{d,d'}e^{-\frac{M_{d,d'}}{|x^p\varepsilon^q|}},$$

for all (x, ε) in the intersection of the previous sectors, for certain positive constants $K'_{d,d'}$, $M_{d,d'}$.

In conclusion, we have shown that for close non-singular directions d, d', the difference of its corresponding solutions $w_d - w_{d'}$ is zero or have exponential decay of order 1 in the monomial $x^p\varepsilon^q$ at the origin. As we can cover $D^2_{\rho'} \setminus \{x\varepsilon = 0\}$, with $\rho' > 0$ small enough, by a finite number of sectors $S_{p,q}(d,\pi+\theta_d/2,\rho_d)$, where the chosen finite non-singular directions are close enough, we can apply Ramis-Sibuya Theorem 1.2.27 to conclude that the corresponding solutions w_d have a common asymptotic expansion $\widetilde{w}$ in $x^p\varepsilon^q$ of $1-$Gevrey type on the sector $S_{p,q}(d,\pi+\theta_d/2,\rho_d)$. But necessarily $\widetilde{w}=\hat{w}$ because both are formal solutions of equation (3-16). Then we have proved the main result of this section.

Theorem 3.1.4. *The unique formal solution $\hat{y}$ of equation (3-1) is $1-$summable in $x^p\varepsilon^q$.*

As immediate consequences of this theorem, using Proposition 1.2.31 we see that for every fixed ε_0 with small enough norm the series $\hat{y}(x,\varepsilon_0)\in\mathbb{C}[[x]]$ is $p-$summable. Analogously $\hat{y}(x_0,\varepsilon)\in\mathbb{C}[[\varepsilon]]$ is $q-$summable for every fixed x_0 with small enough norm.

We note that to prove the theorem we also could have attempted to analyze directly the properties of analytic continuation and exponential growth of $\hat{\mathcal{B}}^{(p,q)}_{1,(s_1,s_2)}(x^p\varepsilon^q\hat{y})$, for some $s_1,s_2>0$ with $s_1+s_2=1$. Since $\hat{y}$ satisfies (3-1) then $\hat{W}=x^p\varepsilon^q\hat{y}$ satisfies the differential equation.

$$\varepsilon^q x^{p+1}\frac{dW}{dx}=(p\varepsilon^q x^p I+A(x,\varepsilon))W(x,\varepsilon)+x^p\varepsilon^q b(x,\varepsilon), \tag{3-23}$$

Then $\hat{F}=\hat{\mathcal{B}}^{(p,q)}_{1,(s_1,s_2)}(\hat{W})$ is analytic in a neighborhood of the origin and satisfies the equation

$$\xi\frac{\partial}{\partial\xi}\left(\xi^p\upsilon^q\int_0^1 F(\xi t^{s_1/p},\upsilon t^{s_2/q})dt\right)=A(0,0)F(\xi,\upsilon)+\mathcal{B}^{(p,q)}_{1,(s_1,s_2)}(\widetilde{A})*^{(p,q)}_{1,(s_1,s_2)}F(\xi,\upsilon)+g(\xi,\upsilon), \tag{3-24}$$

where $\widetilde{A}(x,\varepsilon)=A(x,\varepsilon)-A(0,0)+px^p\varepsilon^q I$ and $g=\mathcal{B}^{(p,q)}_{1,(s_1,s_2)}(x^p\varepsilon^q b)$. Actually, $\hat{F}$ is the unique formal solution of this equation. From here the scheme of proof used before does not work anymore, due to the derivative in the previous expression. However since we already have the characterization of monomial summability in terms of monomial Borel-Laplace methods (Theorem 2.2.1) we can formulate the following corollary, strengthen Theorem 3 in [BM].

Corollary 3.1.5. *Using the previous notation, for every direction $d\neq d_j$, $j=1,...,l$, $\hat{\mathcal{B}}^{(p,q)}_{1,(s_1,s_2)}(\hat{W})$ admits analytic continuation φ_{d,s_1,s_2} to some $S_{p,q}(d,\beta_d,+\infty)$, for some $\beta_d>0$, and there are constants $C_d,B_d>0$ such that $|\varphi_{d,s_1,s_2}(\xi,\upsilon)|\leq C_d e^{B_d\max\{|\xi|^{p/s_1},|\upsilon|^{q/s_2}\}}$ for all $(\xi,\upsilon)\in S_{p,q}(d,\beta_d,+\infty)$. In particular,*

$$w_d(x,\varepsilon)=\frac{1}{x^p\varepsilon^q}\mathcal{L}^{(p,q)}_{1,s_1,s_2}(\varphi_{d,s_1,s_2})(x,\varepsilon),$$

in the intersections of its domains.

To finish this section we enunciate without proof the generalization of Theorem 3.1.4 to the non-linear case. We refer the reader to [CDMS] for a complete proof.

Theorem 3.1.6. *Consider the singularly perturbed differential equation*

$$\varepsilon^q x^{p+1}\frac{dy}{dx} = f(x,\varepsilon,y),$$

where $y \in \mathbb{C}^l$, $p,q \in \mathbb{N}^*$ f *analytic in a neighborhood of* $(0,0,\underline{0})$ *and* $f(0,0,\underline{0}) = 0$. *If* $\partial f/\partial y(0,0,\underline{0})$ *is invertible then the previous equation has a unique formal solution* $\hat{y}$. *Furthermore it is* $1-$*summable in* $x^p\varepsilon^q$.

3.2 Monomial summability of solutions of a linear partial differential equation

In order to apply directly the Borel-Laplace methods introduced in the previous chapter and in view of Proposition 2.1.3 we can study the partial differential equation

$$\frac{s_1}{p}\varepsilon^q x^{p+1}\frac{\partial y}{\partial x} + \frac{s_2}{q}x^p\varepsilon^{q+1}\frac{\partial y}{\partial \varepsilon} = C(x,\varepsilon)y(x,\varepsilon) + \gamma(x,\varepsilon), \tag{3-25}$$

where $p,q \in \mathbb{N}^*$, $s_1, s_2 > 0$ satisfy $s_1 + s_2 = 1$ and $C \in \mathrm{Mat}(l \times l, \mathbb{C}\{x,\varepsilon\})$, $\gamma \in \mathbb{C}\{x,\varepsilon\}^l$. We remark that in the boundary cases $s_1 = 1, s_2 = 0$ and $s_1 = 0, s_2 = 1$ the equation reduces to equation (3-1), that has been already studied in the previous section.

As usual we choose $r > 0$ such that $C \in \mathrm{Mat}(l \times l, \mathcal{O}_b(D_r^2))$ and $\gamma \in \mathcal{O}_b(D_r^2)^l$. On the existence, uniqueness and Gevrey character of the formal solutions $\hat{y}$ of (3-25) we have as a first result the following proposition.

Proposition 3.2.1. *Consider the partial differential equation (3-25). If* $C(0,0)$ *is invertible then (3-25) has a unique solution* $\hat{y} \in \hat{R}^l$. *Moreover* $\hat{y} \in (\hat{R}_1^{(p,q)})^l$.

Proof. We write $\hat{y}$ as in equation (3-5) and also $C(x,\varepsilon) = \sum_{n,m} C_{n,m}x^n\varepsilon^m$, $\gamma(x,\varepsilon) = \sum_{n,m}\gamma_{n,m}x^n\varepsilon^m$ and analogously when expanding them in powers of x, resp. ε. The existence and uniqueness of $\hat{y}$ follows from replacing the expressions in (3-5) into equation (3-25). Indeed, if we expand respect to x and ε, we obtain the recurrence equations

$$\left(\frac{s_1}{p}(n-p) + \frac{s_2}{q}(m-q)\right) y_{n-p,m-q} = \sum_{i=0}^{n}\sum_{j=0}^{m} C_{n-i,m-j}y_{i,j} + \gamma_{n,m}, \tag{3-26}$$

for all $n, m \in \mathbb{N}$. Here and below we set all coefficients with negative indexes as 0. Since $C(0,0)$ is invertible, the coefficients $y_{n,m}$ are uniquely determined by these equations. Due to the role of x and ε in symmetric in equation (3-25), we only write the calculations for the variable x. Thus if we expand respect to x, we obtain the family of differential equations

$$\frac{s_1}{p}(n-p)\varepsilon^q y_{n-p*}(\varepsilon) + \frac{s_2}{q}\varepsilon^{q+1} y'_{n-p*}(\varepsilon) = \sum_{i=0}^{n} C_{n-i*}(\varepsilon) y_{i*}(\varepsilon) + \gamma_{n*}(\varepsilon), \tag{3-27}$$

for all $n \in \mathbb{N}$. In this case $C_{0*}(\varepsilon)$ is also invertible for $|\varepsilon| < r$ (reducing r if necessary). Again, we obtain that the coefficients y_{n*} are uniquely determined by these equations and are analytic on D_r.

In order to determine the Gevrey order of the entries of $\hat{y}$ in x we use the Nagumo norms. Take any $0 < R < r$ and working on D_R, write $c = \|C_{0*}^{-1}\|_0$, $z_n = \|y_{n*}\|_n$, $c_n = \|C_{n*}\|_n$ and $f_n = \|\gamma_{n*}\|_n$. It follows from equation (3-27) that this numbers satisfy the inequalities

$$z_n \leq c\left((1+e)\,(n-p+1)R^{p+q} z_{n-p} + \sum_{i=0}^{n-1} c_{n-i} z_i + f_n\right). \tag{3-28}$$

Dividing by $\Gamma(1+n/p) = n/p\Gamma(n/p)$, using the inequality (3-3) and $n-p+1 \leq 2n$, we can conclude that

$$\frac{z_n}{\Gamma\left(1+\frac{n}{p}\right)} \leq c\left(2p\,(1+e)\,R^{p+q}\frac{z_{n-p}}{\Gamma\left(\frac{n}{p}\right)} + \sum_{i=0}^{n-1}\frac{c_{n-i}}{\Gamma\left(1+\frac{n-i}{p}\right)}\frac{z_i}{\Gamma\left(1+\frac{i}{p}\right)} + \frac{f_n}{\Gamma\left(1+\frac{n}{p}\right)}\right), \tag{3-29}$$

If we define recursively w_n by $w_0 = z_0$ and

$$w_n = c\left(2p(1+e)R^{p+q} w_{n-p} + \sum_{i=0}^{n-1}\frac{c_{n-i}}{\Gamma\left(1+\frac{n-i}{p}\right)} w_i + \frac{f_n}{\Gamma\left(1+\frac{n}{p}\right)}\right), \tag{3-30}$$

it follows that $z_n/\Gamma(1+n/p) \leq w_n$ for all $n \in \mathbb{N}$. If we set $\hat{w}(\tau) = \sum_{n\geq 0} w_n \tau^n$, $\sigma(\tau) = \sum_{n\geq 1} c_n/\Gamma(1+\frac{n}{p})\tau^n$ and $f(\tau) = \sum_{n\geq 0} f_n/\Gamma(1+n/p)\tau^n$, we see that $\sigma, f \in \mathcal{O}(\mathbb{C})$ and that $\hat{w}$ satisfies the functional equation

$$w(\tau) = c(2p(1+e)R^{p+q}\tau^p w(\tau) + \sigma(\tau)w(\tau) + f(\tau)).$$

Since $\sigma(0) = 0$, this equation has a unique analytic solution at 0 and it must be $\hat{w}$. Then there are positive constants C, D such that

$$w_n \leq CD^n,$$

for all $n \in \mathbb{N}$. Using Cauchy's formula for $\rho < R$ we obtain the bounds

$$|y_{n,m}| \leq C \frac{D^n}{(R-\rho)^n \rho^m} \Gamma\left(1 + \frac{n}{p}\right),$$

valid for all $n, m \in \mathbb{N}$. Therefore $\hat{y} \in (\hat{R}_{(1/p,0)})^l$.

Using a similar reasoning we can conclude also that $\hat{y} \in (\hat{R}_{(0,1/q)})^l$. It follows from Proposition 1.2.3 that $\hat{y} \in (\hat{R}_1^{(p,q)})^l$, as was to be proved. □

We can adapt the model of proof used in Theorem 3.1.4 to this situation, replacing naturally the Borel and Laplace transforms by their monomial counterparts, with weights s_1, s_2. In this context the proof goes easily because none of the variables act as a parameter.

Theorem 3.2.2. *Consider equation (3-25). If $C(0,0)$ is invertible then the unique formal solution $\hat{y}$ given by the previous proposition is $1-$summable in $x^p \varepsilon^q$. Its possible singular directions are the directions passing through the eigenvalues of $C(0,0)$.*

Proof. In order to prove monomial summability we are going to use the characterization given by Theorem 2.2.1. To simplify the notations we are going to write $\mathcal{B} = \mathcal{B}_{1,(s_1,s_2)}^{(p,q)}$, $\hat{\mathcal{B}} = \hat{\mathcal{B}}_{1,(s_1,s_2)}^{(p,q)}$ and $* = *_{1,(s_1,s_2)}^{(p,q)}$ for the corresponding Borel transforms and for the $1-(s_1,s_2)-$convolution, respectively.

The change of variables $w = x^p \varepsilon^q y$ in equation (3-25) leads us to the new equation

$$\frac{s_1}{p} \varepsilon^q x^{p+1} \frac{\partial w}{\partial x} + \frac{s_2}{q} x^p \varepsilon^{q+1} \frac{\partial w}{\partial \varepsilon} = (x^p \varepsilon^q I + C(x,\varepsilon)) w(x,\varepsilon) + x^p \varepsilon^q \gamma(x,\varepsilon), \tag{3-31}$$

which is solved formally by $\hat{w} = x^p \varepsilon^q \hat{y}$. As before $I = I_l$ denotes the identity matrix of size l. If we apply the $1-$Borel transform associated to the monomial $x^p \varepsilon^q$ with weight (s_1, s_2) to this equation, using Propositions 2.1.3 and 2.1.15, we see that $F = \hat{\mathcal{B}}(\hat{w})$ is a solution of the corresponding convolution equation

$$(\xi^p \upsilon^q I - C(0,0)) F(\xi,\upsilon) = \mathcal{B}(\widetilde{C}) * F(\xi,\upsilon) + g(\xi,\upsilon), \tag{3-32}$$

where $\widetilde{C}(x,\varepsilon) = x^p \varepsilon^q I + C(x,\varepsilon) - C(0,0)$ and $g = \mathcal{B}(x^p \varepsilon^q \gamma)$. Furthermore we can write

$$g(\xi,\upsilon) = \sum_{n,m \geq 0} \frac{\gamma_{n,m}}{\Gamma\left(1 + \frac{ns_1}{p} + \frac{ms_2}{q}\right)} \xi^n \upsilon^m, \quad \mathcal{B}(\widetilde{C})(\xi,\upsilon) = \sum_{\substack{n,m \geq 0 \\ (n,m) \neq (0,0)}} \frac{\overline{C}_{n,m}}{\Gamma\left(\frac{ns_1}{p} + \frac{ms_2}{q}\right)} \xi^{n-p} \upsilon^{m-q}, \tag{3-33}$$

where $\overline{C}_{n,m} = C_{n,m}$ for $(n,m) \neq (p,q)$ and $\overline{C}_{p,q} = C_{p,q} + I$.

Let $\lambda_1, ..., \lambda_l$ the eigenvalues of $C(0,0)$ repeated according to their multiplicity and recall that they are all non-zero by assumption. The open set where we can invert the matrix $\xi^p \upsilon^q I - C(0,0)$ is given by

$$\Omega = \{(\xi, \upsilon) \in \mathbb{C}^2 \mid \xi^p \upsilon^q \neq \lambda_j \text{ for all } j = 1, ..., l\}.$$

Working on Ω, we see that finding solutions of certain type of the convolution equation (3-32) is equivalent to find a fixed point of the operator $\mathcal{H}$ given by

$$\mathcal{H}(F)(\xi, \upsilon) = (\xi^p \upsilon^q I - C(0,0))^{-1} \left(\mathcal{B}(\widetilde{C}) * F(\xi, \upsilon) + g(\xi, \upsilon)\right), \tag{3-34}$$

and defined in an adequate Banach space E of functions.

We are going to prove that $\mathcal{B}(\hat{w})$ admits analytic continuation to Ω. To do so we consider an arbitrary bounded open set U such that $\overline{U} \subset \Omega$ and an arbitrary $N \in \mathbb{N}$. Then it is sufficient to prove that $\mathcal{B}(\hat{w}) - \sum_{n=0}^{N} \sum_{m=0}^{\infty} \frac{y_{n,m}}{\Gamma(1+ns_1/p+ms_2/q)} \xi^n \upsilon^m$ admits analytic continuation to U.

If we perform the change of variable $w_N(x, \varepsilon) = w(x, \varepsilon) - \sum_{n=0}^{N} \varepsilon^q y_{n*}(\varepsilon) x^{n+p}$ in equation (3-31), then using the recurrences (3-27) we see that w_N satisfies the same differential equation (3-31) but with γ replaced by a γ_N with $\text{ord}_x \gamma_N > N$. Therefore $\mathcal{B}(w_N)$ satisfies the same convolution equation (3-32) but with g replaced by $g_N = \mathcal{B}(x^p \varepsilon^q \gamma_N)$ and $\text{ord}_x g_N > N$.

Let $E_{U,N}$ denote the subspace of functions of $\mathcal{C}(\overline{U}) \cap \mathcal{O}(U)$ such that

$$\|F\|_N = \sup_{(\xi,\upsilon)\in\overline{U}} \frac{|F(\xi, \upsilon)|}{|\xi|^N},$$

is finite. $E_{U,N}$ is a Banach space with the norm $\|\cdot\|_N$ and $g_N \in E_{U,N}$. We shall prove that $\mathcal{H}_N : E_{U,N} \to E_{U,N}$, defined as $\mathcal{H}$ but with g_N instead of g, is well-defined and it is a contraction if N is large enough. Indeed, if $F \in E_{U,N}$ then

$$\begin{aligned}
&|\mathcal{H}_N(F)(\xi, \upsilon)| \\
&\quad \leq M_U \left| \int_0^1 \mathcal{B}(\widetilde{C})(\xi t^{s_1/p}, \upsilon t^{s_2/q}) F(\xi(1-t)^{s_1/p}, \upsilon(1-t)^{s_2/q}) \xi^p \upsilon^q dt \right| + M_U \|g_N\|_N |\xi|^N \\
&\quad \leq M_U \int_0^1 \left|\mathcal{B}(\widetilde{C})(\xi t^{s_1/p}, \upsilon t^{s_2/q})\right| \|F\|_N |\xi|^{p+N} |\upsilon|^q (1-t)^{Ns_1/p} dt + M_U \|g_N\|_N |\xi|^N,
\end{aligned}$$

where $M_U > 0$ is a constant such that

$$\left\|(\xi^p \upsilon^q I - C(0,0))^{-1}\right\| \leq M_U, \qquad \text{for all } (\xi, \upsilon) \in \overline{U}.$$

A way to estimate adequately the previous expression it is to bound the following integral:

$$\int_0^1 t^{ns_1/p+ms_2/q-1}(1-t)^{Ns_1/p}dt = \frac{\Gamma\left(\frac{ns_1}{p}+\frac{ms_2}{q}\right)\Gamma\left(1+\frac{Ns_1}{p}\right)}{\Gamma\left(1+\frac{(n+N)s_1}{p}+\frac{ms_2}{q}\right)}.$$

If $ns_1/p + ms_2/ > 1$, inequality (3-3) shows that

$$\frac{\Gamma\left(\frac{ns_1}{p}+\frac{ms_2}{q}\right)\Gamma\left(1+\frac{Ns_1}{p}\right)}{\left(\frac{(n+N)s_1}{p}+\frac{ms_2}{q}\right)\Gamma\left(\frac{(n+N)s_1}{p}+\frac{ms_2}{q}\right)} \leq \frac{p}{Ns_1}.$$

For the case $ns_1/p + ms_2/ \leq 1$ we use the limit (3-4) with $s = s_1/p$ and $b = ns_1/p + ms_2/q$. It implies that there is a constant D_{n,m,p,q,s_1,s_2} such that

$$\frac{\Gamma\left(\frac{ns_1}{p}+\frac{ms_2}{q}\right)\Gamma\left(1+\frac{Ns_1}{p}\right)}{\Gamma\left(1+\frac{(n+N)s_1}{p}+\frac{ms_2}{q}\right)} \leq \frac{D_{n,m,p,q,s_1,s_2}}{\left(\frac{(N+n)s_1}{p}+\frac{ms_2}{q}\right)^{ns_1/p+ms_2/q}}.$$

If $D = D_{p,q,s_1,s_2} = \max\{D_{n,m,p,q,s_1,s_2}, p/s_1, (p/s_1)^{ns_1/p+ms_2/q} | ns_1/p + ms_2/ \leq 1\}$ (it is a finite constant because the maximum is taken over a finite number of values) the integral is bounded in any case by

$$\frac{D}{N^{\min\{s_1/p, s_2/q\}}}.$$

Back to the operator $\mathcal{H}_N$, we now can ensure that

$$|\mathcal{H}_N(F)(\xi,\upsilon)| \leq M_U\left(\frac{DK_U}{N^{\min\{s_1/p,s_2/q\}}}\|F\|_N + \|g_N\|_N\right)|\xi|^N,$$

where

$$K_U = \sup_{(\xi,\upsilon)\in\overline{U}} \sum_{\substack{n,m\geq 0\\(n,m)\neq(0,0)}} \frac{|\overline{C}_{n,m}|}{\Gamma\left(\frac{ns_1}{p}+\frac{ms_2}{q}\right)}|\xi|^n|\upsilon|^m.$$

We remark that K_U is finite since U is bounded and $\widetilde{C}$ is analytic at $(0,0)$. The previous bound is sufficient to ensure that $\mathcal{H}_N(F) \in E_{U,N}$. To show that $\mathcal{H}_N$ is a contraction, we estimate as before to see that if $F, G \in E_{U,N}$ then

$$\|\mathcal{H}_N(F) - \mathcal{H}_N(G)\|_N \leq \frac{DM_UK_U}{N^{\min\{s_1/p,s_2/q\}}}\|F-G\|_N.$$

Then it is enough to take N with $\frac{DM_UK_U}{N^{\min\{s_1/p,s_2/q\}}} < 1$ to conclude the result.

Applying Banach's fixed point theorem, we can conclude that $\mathcal{H}_N$ has a unique fixed point $F_{U,N} \in E_{U,N}$, that is, equation (3-32) has a unique analytic solution defined on U of the form $F_U(\zeta,\varepsilon) = F_{U,N}(\zeta,\varepsilon) + \sum_{n=0}^{N}\sum_{m=0}^{\infty} \frac{y_{n,m}}{\Gamma(1+ns_1/p+ms_2/q)}\xi^n\upsilon^m$. Now, if we take a polydisc at the origin contained in Ω with sufficiently small poly-radius, the solution provided by the fixed point is precisely $\hat{\mathcal{B}}(\hat{w})$, because this is the unique formal solution at $(0,0)$ of equation (3-32). Then if U intersects this polydisc, F_U and $\hat{\mathcal{B}}(\hat{w})$ coincide in the intersection, being both solutions of the convolution equation. This let us conclude that $\hat{\mathcal{B}}(\hat{w})$ admits analytic continuation to Ω.

It remains to prove that the above solutions have the adequate exponential growth. Let $C>0$ an arbitrary positive constant. Let S an unbounded open set such that $\overline{S}\subset\Omega$. We will denote by $E_{S,C}$ the subspace of functions F in $\mathcal{O}(S)$ such that

$$\|F\|_C = \sup_{(\xi,\upsilon)\in S} |F(\xi,\upsilon)|e^{-CR(\xi,\upsilon)},$$

is finite, where $R(\xi,\upsilon) = \max\{|\xi|^{p/s_1}, |\upsilon|^{q/s_2}\}$. Then $E_{S,C}$ is a Banach space with the norm $\|\cdot\|_C$. Furthermore, it follows from (3-33) that we can find a large enough constant $C'>0$ such that $g\in E_{\Omega,C'}$.

Following the same ideas as before we shall prove that $\mathcal{H}: E_{S,C}\to E_{S,C}$, is well-defined and a contraction if $C>C'$ is large enough. For the first assertion, if $F\in E_{S,C}$, then

$$\begin{aligned}
&|\mathcal{H}(F)(\xi,\upsilon)| \\
&\quad\le M_S\left|\int_0^1 \mathcal{B}(\widetilde{C})(\xi t^{s_1/p},\upsilon t^{s_2/q})F(\xi(1-t)^{s_1/p},\upsilon(1-t)^{s_2/q})\xi^p\upsilon^q dt\right| + M_S\|g\|_{C'}e^{C'R(\xi,\upsilon)} \\
&\quad\le M_S\int_0^1 \left|\xi^p\upsilon^q\mathcal{B}(\widetilde{C})(\xi t^{s_1/p},\upsilon t^{s_2/q})\right| \|F\|_C e^{CR(\xi,\upsilon)(1-t)}dt + M_S\|g\|_{C'}e^{C'R(\xi,\upsilon)},
\end{aligned}$$

where $M_S>0$ is a constant such that

$$\left\|(\xi^p\upsilon^q I - C(0,0))^{-1}\right\| \le M_S, \qquad \text{for all } (\xi,\upsilon)\in\overline{S}.$$

To estimate adequately the previous expression, we can use the Gamma function to see that

$$\begin{aligned}
\int_0^1 t^{ns_1/p+ms_2/q-1}e^{CR(\xi,\upsilon)(1-t)}dt &= \frac{e^{CR(\xi,\upsilon)}}{(CR(\xi,\upsilon))^{ns_1/p+ms_2/q}}\int_0^{CR(\xi,\upsilon)} u^{ns_1/p+ms_2/q-1}e^{-u}du \\
&\le \frac{\Gamma(ns_1/p+ms_2/q)}{(CR(\xi,\upsilon))^{ns_1/p+ms_2/q}}e^{CR(\xi,\upsilon)},
\end{aligned}$$

for all $n,m\in N$, $(n,m)\neq(0,0)$ and $\xi,\upsilon\in\mathbb{C}^*$. Applying these bounds we see that

$$\begin{aligned}
|\mathcal{H}(F)(\zeta,\varepsilon)| &\le M_S L\left(\frac{1}{C^{s_1/p}}+\frac{1}{C^{s_2/q}}\right)\|F\|_C e^{CR(\xi,\upsilon)} + M_S\|g\|_{C'}e^{C'R(\xi,\upsilon)} \\
&\le M_S\left(L\left(\frac{1}{C^{s_1/p}}+\frac{1}{C^{s_2/q}}\right)\|F\|_C + \|g\|_{C'}\right)e^{CR(\xi,\upsilon)},
\end{aligned}$$

where $L > 0$ is a constant such that

$$\sum_{m\geq 1} |\overline{C}_{0m}||\upsilon|^{m-1}, \sum_{\substack{n\geq 1\\ m\geq 0}} |\overline{C}_{nm}||\xi|^{n-1}|\upsilon|^m \leq L,$$

for all $|\xi|, |\upsilon| \leq R$ and $1/C^{s_2/q}, 1/C^{s_1/p} < R$.

Therefore we have proved that $\mathcal{H}(F) \in E_{S,C}$. In the same way, if $F, G \in E_{S,C}$ then

$$\|\mathcal{H}(F) - \mathcal{H}(G)\|_C \leq M_S L \left(\frac{1}{C^{s_1/p}} + \frac{1}{C^{s_2/q}}\right) \|F - G\|_C.$$

If we take C large enough such that $M_S L\left(\frac{1}{C^{s_1/p}} + \frac{1}{C^{s_2/q}}\right) < 1$ we can conclude that $\mathcal{H}$ is a contraction, and then it has a unique fixed point. This means that (3-32) has a unique solution in S with the exponential growth above.

If we choose any direction $d \neq \arg(\lambda_j)$, $j = 1, ..., l$ and $\theta_d > 0$ small enough such that $S = S_{p,q}(d, 2\theta_d, +\infty) \subset \Omega$, then we have proved in particular that $\hat{\mathcal{B}}(\hat{w})$ can be analytically continued to S with exponential growth as required in Definition 2.2.1 for $k = 1$. Then by Theorem 2.2.1, $\hat{y}$ es $1-$summable in $x^p\varepsilon^q$ in direction d as we wanted to prove. □

3.3 Monomial summability of solutions of a class of Pfaffian systems

The last application we will give in this text is the study of the convergence and the monomial summability properties of formal solutions of a class of Pfaffian systems in two independent variables. In the first place we explore the consequences of such a system to be completely integrable focusing in the behavior of their linear parts. Then we pass to the study of the mentioned formal solutions and prove their convergence in generic cases in the situation of non-integrability.

The more general situation we are going to analyze here is the study of formal solutions of the systems of singular partial differential equations or *Pfaffian system with normal crossings* of the form

$$\begin{cases} \varepsilon^q x^{p+1} \dfrac{\partial y}{\partial x} = f_1(x, \varepsilon, y), & \text{(3-35a)}\\ x^{p'} \varepsilon^{q'+1} \dfrac{\partial y}{\partial \varepsilon} = f_2(x, \varepsilon, y), & \text{(3-35b)} \end{cases}$$

where $p, q, p', q' \in \mathbb{N}^*$, $y \in \mathbb{C}^l$, and f_1, f_2 are analytic functions defined on a neighborhood of the origin in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^l$. If $f_1(x, \varepsilon, 0) = f_2(x, \varepsilon, 0) = 0$ and the functions f_1, f_2 satisfy the following *integrability condition* on its domains of definition:

$$-qx^{p'}\varepsilon^{q'}f_1(x,\varepsilon,y)+x^{p'}\varepsilon^{q'+1}\frac{\partial f_1}{\partial\varepsilon}(x,\varepsilon,y)+\frac{\partial f_1}{\partial y}(x,\varepsilon,y)f_2(x,\varepsilon,y)= \tag{3-36}$$

$$-p'x^{p}\varepsilon^{q}f_2(x,\varepsilon,y)+x^{p+1}\varepsilon^{q}\frac{\partial f_2}{\partial x}(x,\varepsilon,y)+\frac{\partial f_2}{\partial y}(x,\varepsilon,y)f_1(x,\varepsilon,y),$$

then the system will be referred as *completely integrable Pfaffian system with normal crossings.* The normal crossing refers to the singular locus $x\varepsilon=0$ where the differential equation changes to an implicit one. This is a plausible condition to impose since it helps to relate the solutions of both equations (note that the condition can be obtained from the equality of mixed derivatives). We shall see that the condition of complete integrability imposes serious restrictions on f_1 and f_2 and the results we present require hypotheses that completely integrable systems may not satisfy. Fortunately the complete integrability condition is not always necessary for the existence of solutions as we will see through examples.

Let us write $f_1(x,\varepsilon,y)=A(x,\varepsilon)y+\sum_{|J|\geq 2}f_{1,J}y^J$ and $f_2(x,\varepsilon,y)=B(x,\varepsilon)y+\sum_{|J|\geq 2}f_{2,J}y^J$, as a Taylor's series in y around $0\in\mathbb{C}^l$. We may suppose that A and B have entries in $\mathcal{O}(D_r^2)$ for some $r>0$. Then replacing these expressions into equation (3-36) and equaling to zero the common terms in each y^J , we see in particular that A and B satisfy

$$x^{p'}\varepsilon^{q'}\left(\varepsilon\frac{\partial A}{\partial\varepsilon}-qA\right)-x^{p}\varepsilon^{q}\left(x\frac{\partial B}{\partial x}-p'B\right)+[A,B]=0, \tag{3-37}$$

where $[,]$ is the usual Lie bracket of matrices. In particular, by taking $x=0$ and $\varepsilon=0$ we conclude that $[A(0,0),B(0,0)]=0$, $[A(x,0),B(x,0)]=0$ and $[A(0,\varepsilon),B(0,\varepsilon)]=0$ for all $|x|,|\varepsilon|<r$.

We can extract more information about $A(0,0)=A_{0,0}$ and $B(0,0)=B_{0,0}$ and their spectra from equation (3-37), depending on p,q,p' and q'. We are going to consider only some possible cases. For this let us put

$$A(x,\varepsilon)=\sum_{n,m\geq 0}A_{n,m}x^m\varepsilon^m=\sum_{n\geq 0}A_{n*}(\varepsilon)x^n=\sum_{m\geq 0}A_{*m}(x)\varepsilon^m,$$
$$B(x,\varepsilon)=\sum_{n,m\geq 0}B_{n,m}x^m\varepsilon^m=\sum_{n\geq 0}B_{n*}(\varepsilon)x^n=\sum_{m\geq 0}B_{*m}(x)\varepsilon^m.$$

Then replacing the previous expressions in equation (3-37) and grouping by common powers we see that

$$0 =(m-q-q')A_{n-p',m-q'} - (n-p-p')B_{n-p,m-q} + \sum_{i=0}^{n}\sum_{j=0}^{m}[A_{i,j}, B_{n-i,m-j}], \tag{3-38}$$

$$0 =\varepsilon^{q'}\left(\varepsilon A'_{n-p'*}(\varepsilon) - qA_{n-p'*}(\varepsilon)\right) - (n-p-p')\varepsilon^{q}B_{n-p*}(\varepsilon) + \sum_{i=0}^{n}[A_{i*}(\varepsilon), B_{n-i*}(\varepsilon)], \tag{3-39}$$

$$0 =(m-q-q')x^{p'}A_{*m-q'}(x) - x^{p}\left(xB'_{*m-q}(x) - p'B_{*m-q}(x)\right) + \sum_{j=0}^{m}[A_{*j}(x), B_{*m-j}(x)], \tag{3-40}$$

for all $n, m \in \mathbb{N}$ and $|x|, |\varepsilon| < r$.

We consider an arbitrary eigenvalue μ_0 of $B_{0,0}$. If this is the only eigenvalue of $B_{0,0}$ we proceed to the cases described below. If it is not unique then we can always find an adequate constant invertible matrix P_0 such that

$$P_0 B_{0,0} P_0^{-1} = \begin{pmatrix} \overline{B}_0^{11}(0) & 0 \\ 0 & \overline{B}_0^{22}(0) \end{pmatrix},$$

in such a way that the only eigenvalue of $\overline{B}_0^{11}(0)$ is μ_0 and $\overline{B}_0^{11}(0)$ and $\overline{B}_0^{22}(0)$ have no common eigenvalues. We can even find $\rho > 0$ small enough and $P \in \mathrm{GL}(l, \mathcal{O}(D_\rho))$ such that $P(0) = P_0$ and

$$\overline{B}_{0*}(\varepsilon) = P(\varepsilon)B_{0*}(\varepsilon)P(\varepsilon)^{-1} = \begin{pmatrix} \overline{B}_0^{11}(\varepsilon) & 0 \\ 0 & \overline{B}_0^{22}(\varepsilon) \end{pmatrix},$$

so that for every $|\varepsilon| < \rho$, the matrices $\overline{B}_0^{11}(\varepsilon)$ and $\overline{B}_0^{22}(\varepsilon)$ have no common eigenvalues (see Theorem 25.1, [W1]). This last property joint with the fact that B_{0*} and A_{0*} commute let us conclude that

$$\overline{A}_{0*}(\varepsilon) = P(\varepsilon)A_{0*}(\varepsilon)P(\varepsilon)^{-1} = \begin{pmatrix} \overline{A}_0^{11}(\varepsilon) & 0 \\ 0 & \overline{A}_0^{22}(\varepsilon) \end{pmatrix},$$

where $[\overline{A}_0^{jj}(\varepsilon), \overline{B}_0^{jj}(\varepsilon)] = 0$, $j = 1, 2$. Let us also write

$$\overline{A}_{1*}(\varepsilon) = P(\varepsilon)A_{1*}(\varepsilon)P(\varepsilon)^{-1} = \begin{pmatrix} \overline{A}_1^{11}(\varepsilon) & \overline{A}_1^{12}(\varepsilon) \\ \overline{A}_1^{21}(\varepsilon) & \overline{A}_1^{22}(\varepsilon) \end{pmatrix},$$

$$\overline{B}_{1*}(\varepsilon) = P(\varepsilon)B_{1*}(\varepsilon)P(\varepsilon)^{-1} = \begin{pmatrix} \overline{B}_1^{11}(\varepsilon) & \overline{B}_1^{12}(\varepsilon) \\ \overline{B}_1^{21}(\varepsilon) & \overline{B}_1^{22}(\varepsilon) \end{pmatrix},$$

in the same block-decomposition as $\overline{A}_{0*}(\varepsilon)$ and $\overline{B}_{0*}(\varepsilon)$. We consider the following cases regarding p, p', q and q':

Case I. Suppose $p = 1$ and $1 < p'$. Then the equation (3-39) for $n = 1$ reduces to

$$p'\varepsilon^q B_{0*}(\varepsilon) + [A_{0*}(\varepsilon), B_{1*}(\varepsilon)] + [A_{1*}(\varepsilon), B_{0*}(\varepsilon)] = 0. \tag{3-38}$$

If necessary, after multiplying equation (3-38) by $P(\varepsilon)$ to the left and by $P(\varepsilon)^{-1}$ to the right, the equation obtained in the position $(1,1)$ according to the previous block-decomposition is

$$p'\varepsilon^q \overline{B}_0^{11}(\varepsilon) + [\overline{A}_0^{11}(\varepsilon), \overline{B}_1^{11}(\varepsilon)] + [\overline{A}_1^{11}(\varepsilon), \overline{B}_0^{11}(\varepsilon)] = 0. \tag{3-39}$$

Applying the trace in the previous equation we see that $\mathrm{tr}(p'\varepsilon^q\overline{B}_0^{11}(\varepsilon)) = 0$ and thus $\mathrm{tr}(\overline{B}_0^{11}(0)) = 0$. Since μ_0 is the only eigenvalue of $\overline{B}_0^{11}(0)$ we conclude that $\mu_0 = 0$. Since μ_0 was arbitrary then $B_{0,0}$ is nilpotent.

Case II. Suppose that $p = p' = 1$. Here equation (3-39) for $n = 1$ is given by

$$\varepsilon^{q'}\left(\varepsilon A'_{0*}(\varepsilon) - qA_{0*}(\varepsilon)\right) + \varepsilon^q B_{0*}(\varepsilon) + [A_{0*}(\varepsilon), B_{1*}(\varepsilon)] + [A_{1*}(\varepsilon), B_{0*}(\varepsilon)] = 0. \tag{3-40}$$

If necessary, multiplying equation (3-40) by $P(\varepsilon)$ to the left and by $P(\varepsilon)^{-1}$ to the right, the equation obtained in the position $(1,1)$ according to the previous block decomposition is

$$\varepsilon^{q'+1}(PA'_{0*}P^{-1})^{(1,1)}(\varepsilon) - q\varepsilon^{q'}\overline{A}_0^{11}(\varepsilon) + \varepsilon^q\overline{B}_0^{11}(\varepsilon) + [\overline{A}_0^{11}(\varepsilon), \overline{B}_1^{11}(\varepsilon)] + [\overline{A}_1^{11}(\varepsilon), \overline{B}_0^{11}(\varepsilon)] = 0, \tag{3-41}$$

where $(PA'_{0*}P^{-1})^{(1,1)}$ indicates the matrix in position $(1,1)$ of $PA'_{0*}P^{-1}$. Taking the trace in this equation we see that

$$\varepsilon^{q'+1}\mathrm{tr}\left((PA'_{0*}P^{-1})^{(1,1)}(\varepsilon)\right) - q\varepsilon^{q'}\mathrm{tr}\left(\overline{A}_0^{11}(\varepsilon)\right) + \varepsilon^q\mathrm{tr}\left(\overline{B}_0^{11}(\varepsilon)\right) = 0. \tag{3-42}$$

If $q < q'$ we conclude that $\mu_0 = 0$ and since this eigenvalue was arbitrary then $B_{0,0}$ is nilpotent. If instead $q = q'$ we conclude that

$$q\ \mathrm{tr}\left(\overline{A}_0^{1,1}(0)\right) = \mathrm{tr}\left(\overline{B}_0^{1,1}(0)\right) = l_1\mu_0,$$

where l_1 is the size of $\overline{B}_0^{1,1}$. We have two cases here:

1. $\overline{A}_0^{1,1}(0)$ has only one eigenvalue λ_0. In this case we can conclude that $q\lambda_0 = \mu_0$.
2. $\overline{A}_0^{1,1}(0)$ has at least two different eigenvalues. Let λ_0 be one of them. We apply again the previous process. Take an adequate constant invertible matrix T_0 such that

$$T_0\overline{A}_0^{1,1}(0)T_0^{-1} = \begin{pmatrix} C_0^{11}(0) & 0 \\ 0 & C_0^{22}(0) \end{pmatrix},$$

in such a way that the only eigenvalue of $C_0^{11}(0)$ is λ_0 and $C_0^{11}(0)$ and $C_0^{22}(0)$ have no common eigenvalues. Find $0 < \rho' \leq \rho$ small enough and $T \in \mathrm{GL}(l_1, \mathcal{O}(D_{\rho'}))$ such that $T(0) = T_0$ and

$$C_0(\varepsilon) = T(\varepsilon)\overline{A}_0^{1,1}(\varepsilon)T(\varepsilon)^{-1} = \begin{pmatrix} C_0^{11}(\varepsilon) & 0 \\ 0 & C_0^{22}(\varepsilon) \end{pmatrix},$$

so that for every $|\varepsilon| < \rho'$, the matrices $C_0^{11}(\varepsilon)$ and $C_0^{22}(\varepsilon)$ have no common eigenvalues. Then

$$D_0(\varepsilon) = T(\varepsilon)\overline{B}_0^{1,1}(\varepsilon)T(\varepsilon)^{-1} = \begin{pmatrix} D_0^{11}(\varepsilon) & 0 \\ 0 & D_0^{22}(\varepsilon) \end{pmatrix}.$$

As before, considering the equation obtained from the position $(1,1)$ in equation (3-41), taking the trace and evaluating at $\varepsilon = 0$ we conclude that

$$q \text{ tr } C_0^{1,1}(0) = \text{ tr } D_0^{1,1}(0),$$

but the only eigenvalue of $C_0^{1,1}(0)$ is λ_0 and the only one of $D_0^{1,1}(0)$ is μ_0, and in this case we can also conclude that $q\lambda_0 = \mu_0$.

Case III. Suppose $p > 1$ and $p' = Np$ for some $N \in \mathbb{N}^*$. The idea is to apply rank reduction to be able to use the previous cases. Indeed, consider the ramification $t = x^p$ and let us write

$$\begin{aligned} A(x,\varepsilon) &= A_0(x^p,\varepsilon) + xA_1(x^p,\varepsilon) + \cdots + x^{p-1}A_{p-1}(x^p,\varepsilon), \\ B(x,\varepsilon) &= B_0(x^p,\varepsilon) + xB_1(x^p,\varepsilon) + \cdots + x^{p-1}B_{p-1}(x^p,\varepsilon). \end{aligned}$$

Then replacing these expressions in equation (3-37) and equaling to zero the terms containing each power x^i, $i = 0, 1, ..., p-1$ we see that

$$t^N\varepsilon^{q'}\left(\varepsilon\frac{\partial A_i}{\partial \varepsilon} - qA_i\right) - pt\varepsilon^q\left(t\frac{\partial B_i}{\partial t} - \left(N - \frac{i}{p}\right)B_i\right) + \sum_{j=0}^{i}[A_j, B_{i-j}] + \sum_{j=i+1}^{p-1} t[A_j, B_{p-j+i}] = 0. \tag{3-43}$$

Define the following matrices

$$\widetilde{A}(t,\varepsilon) = \begin{pmatrix} A_0 & tA_{p-1} & \cdots & tA_1 \\ A_1 & A_0 - t\varepsilon^q I & \cdots & tA_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{p-1} & A_{p-2} & \cdots & A_0 - (p-1)t\varepsilon^q I \end{pmatrix},$$

$$\widetilde{B}(t,\varepsilon) = \begin{pmatrix} B_0 & tB_{p-1} & \cdots & tB_1 \\ B_1 & B_0 & \cdots & tB_2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{p-1} & B_{p-2} & \cdots & B_0 \end{pmatrix}.$$

The matrices $\widetilde{A}, \widetilde{B} \in \mathrm{Mat}(pl \times pl, \mathbb{C}\{t, \varepsilon\})$ are obtained from the system (3-35a), (3-35b) as follows: if we write $y(x,\varepsilon) = y_0(x^p,\varepsilon) + xy_1(x^p,\varepsilon) + \cdots + x^{p-1}y_{p-1}(x^p,\varepsilon)$ and put $Y = (y_0, ..., y_{p-1})^t$ then $\widetilde{A}$ and $\widetilde{B}$ correspond to the linear parts of the Pfaffian system satisfied by Y.

Using equations (3-43) it is possible to check that $\widetilde{A}, \widetilde{B}$ satisfy the differential equation

$$t^N \varepsilon^{q'} \left(\varepsilon \frac{\partial \widetilde{A}}{\partial \varepsilon} - q\widetilde{A} \right) - pt\varepsilon^q \left(t\frac{\partial \widetilde{B}}{\partial t} - N\widetilde{B} \right) + [\widetilde{A}, \widetilde{B}] = 0. \tag{3-44}$$

So we are in a similar situation as the initial equation (3-37) and we can apply Case (I) and Case (II) in this case. If $N > 1$ then by Case (I) we conclude that $B_{0,0}$ is nilpotent. If $N = 1$, that is, if $p = p'$ and $q < q'$ then $B_{0,0}$ is nilpotent. Finally, if $p = p'$ and $q = q'$ then for each eigenvalue μ of $B_{0,0}$ there is an eigenvalue λ of $A_{0,0}$ such that $q\lambda = p\mu$.

We can repeat the same considerations if we start from an arbitrary eigenvalue λ_0 of $A_{0,0}$. By abuse of notation we suppose in this case that the only eigenvalue of $\overline{A}_0^{11}(0)$ is λ_0, that $\overline{A}_0^{11}(0)$ and $\overline{A}_0^{22}(0)$ have no common eigenvalues and that $P(\varepsilon)$ block-diagonalize $A_{0*}(\varepsilon)$ and $B_{0*}(\varepsilon)$ as above, with $\overline{A}_0^{11}(\varepsilon)$ and $\overline{A}_0^{22}(\varepsilon)$ with no common eigenvalues, for $|\varepsilon| < \rho$. Then the corresponding cases read as follows:

Case I$'$. Suppose $p' = 1$ and $1 < p$. . Then the equation (3-39) for $n = 1$ reduces to

$$\varepsilon^{q'} \left(\varepsilon A'_{0*}(\varepsilon) - qA_{0*}(\varepsilon) \right) + [A_{0*}(\varepsilon), B_{1*}(\varepsilon)] + [A_{1*}(\varepsilon), B_{0*}(\varepsilon)] = 0. \tag{3-45}$$

If necessary, after multiplying equation (3-45) by $P(\varepsilon)$ to the left and by $P(\varepsilon)^{-1}$ to the right, the equation obtained in the position $(1,1)$ according to the given block-decomposition is

$$\varepsilon^{q'+1}(PA'_{0*}P^{-1})^{(1,1)}(\varepsilon) - q\varepsilon^{q'} A_{0*}^{(1,1)}(\varepsilon) + [\overline{A}_0^{11}(\varepsilon), \overline{B}_1^{11}(\varepsilon)] + [\overline{A}_1^{11}(\varepsilon), \overline{B}_0^{11}(\varepsilon)] = 0. \tag{3-46}$$

Applying the trace in the previous equation, evaluating at $\varepsilon = 0$ and recalling that λ_0 is the only eigenvalue of $\overline{A}_0^{11}(0)$ we conclude that $\lambda_0 = 0$. Since λ_0 was arbitrary then $A_{0,0}$ is nilpotent.

Case II$'$. Suppose that $p = p' = 1$. Proceeding as in Case (II), if $q' < q$ we can conclude from equation (3-42) that $\lambda_0 = 0$ and since it was an arbitrary eigenvalue of $A_{0,0}$ then $A_{0,0}$ is nilpotent.

Case III′. Suppose $p' > 1$ and $p = N'p'$ for some $N' \in \mathbb{N}^*$. Here we apply rank reduction as in Case (III) but with the ramification $t = x^{p'}$. In this case the corresponding matrices $\widetilde{A}$ and $\widetilde{B}$ satisfy the differential equation

$$t\varepsilon^{q'}\left(\varepsilon\frac{\partial\widetilde{A}}{\partial\varepsilon} - q\widetilde{A}\right) - p't^{N'}\varepsilon^{q}\left(t\frac{\partial\widetilde{B}}{\partial t} - \widetilde{B}\right) + [\widetilde{A}, \widetilde{B}] = 0. \tag{3-47}$$

If $N' > 1$ then by Case (I′) we conclude that $1/p'A_{0,0}$ and thus $A_{0,0}$ are nilpotent. If $N' = 1$ and $q' < q$ then by Case (II′) we also conclude that $A_{0,0}$ is nilpotent.

Finally if we rewrite equation (3-37) as

$$\varepsilon^{q}x^{p}\left(x\frac{\partial B}{\partial x} - p'B\right) - \varepsilon^{q'}x^{p'}\left(\varepsilon\frac{\partial A}{\partial\varepsilon} - qA\right) + [B, A] = 0,$$

we can then change the roles of x and ε and deduce similar conclusions from the previous cases. Gathering all these results we can establish the following proposition. The cases left out require a more careful analysis than the one done here.

Proposition 3.3.1. *Consider the Pfaffian system (3-35a), (3-35b). If it is completely integrable then the following assertions hold:*

1. *The matrix* $\frac{\partial f_2}{\partial y}(0,0,0)$ *is nilpotent if* $p = p'$ *and* $q < q'$*, or* $p' = Np$ *with* $N > 1$*, or* $q' = q$ *and* $p < p'$ *or* $q' = Mq$ *with* $M > 1$.
2. *The matrix* $\frac{\partial f_1}{\partial y}(0,0,0)$ *is nilpotent if* $p = p'$ *and* $q' < q$*, or* $p = N'p'$ *with* $N' > 1$*, or* $q' = q$ *and* $p' < p$ *or* $q = M'q'$ *with* $M' > 1$.
3. *If* $p = p'$ *and* $q = q'$*, for every eigenvalue* μ *of* $\frac{\partial f_2}{\partial y}(0,0,0)$ *there is an eigenvalue* λ *of* $\frac{\partial f_1}{\partial y}(0,0,0)$ *such that* $q\lambda = p\mu$*. The number* λ *is an eigenvalue of* $\frac{\partial f_1}{\partial y}(0,0,0)$*, when restricted to its invariant subspace* $E_\mu = \{v \in \mathbb{C}^n | (\frac{\partial f_2}{\partial y}(0,0,0) - \mu I)^k v = 0 \text{ for some } k \in \mathbb{N}\}$.

Finally we turn to the study of formal solutions of the Pfaffian system (3-35a), (3-35b). To motivate the results we are going to present we start by commenting the better known case $q = 0$ and $p' = 0$ that we do not treat here (each equation took separately is not singularly perturbed). As mentioned by H. Majima in [Mj2], the study of those systems in the completely integrable case, i.e. of completely integrable Pfaffian systems with irregular singular points was opened by R. Gérard and Y. Sibuya in [GS] and by K. Takano in [T]. Among the study of existence and uniqueness of formal solutions, of their asymptotical behavior (with different notions of asymptotic introduced in [GS]) and of the analytic reduction of those systems perhaps one of the most remarkable results is the following:

Theorem 3.3.2 (Gérard-Sibuya). *Consider the completely integrable Pffafian system (3-35a), (3-35b), with $q = p' = 0$. If $\frac{\partial f_1}{\partial y}(0,0,0)$ and $\frac{\partial f_2}{\partial y}(0,0,0)$ are invertible then the Pfaffian system admits a unique analytic solution y at the origin such that $y(0,0) = 0$.*

At first glance the result is in conflict comparing it with the usual results in one variable, but one may think that these completely integrable systems are quite rigid and impose many conditions reducing the complexity of their solutions. The first proof of Theorem 3.3.2 can be found in [GS]. Due to the nature of the result Y. Sibuya reproved it with different methods, see [S2] for a proof using summability theory and see [S1], [S3] for a proof in the linear case using algebraic tools. For a more recent proof the reader may also consult [S].

Returning to the general case, we mention that H. Majima in [Mj2] using his theory of strongly asymptotic expansions of functions of several variables has studied the systems (3-35a), (3-35b) and its generalization to more independent variables in the completely integrable case. Unfortunately the lack of examples in his exposition make it more complicated to assimilate. Using the tools we have developed here we can provide information on the solutions of those systems. Indeed, we can apply Theorem 3.1.6 and tauberian Theorem 1.3.5 to prove easily the convergence of solutions under generic conditions, when they exist. More specifically we have the following theorem.

Theorem 3.3.3. *Consider the system (3-35a), (3-35b). The following assertions hold:*

1. *Suppose the system has a formal solution $\hat{y}$. If $\frac{\partial f_1}{\partial y}(0,0,0)$ and $\frac{\partial f_2}{\partial y}(0,0,0)$ are invertible and $x^p\varepsilon^q \neq x^{p'}\varepsilon^{q'}$ then $\hat{y}$ is convergent.*
2. *If the system is completely integrable and $\frac{\partial f_1}{\partial y}(0,0,0)$ is invertible then the system has a unique formal solution $\hat{y}$. Moreover $\hat{y}$ is 1-summable in $x^p\varepsilon^q$.*
3. *If the system is completely integrable and $\frac{\partial f_2}{\partial y}(0,0,0)$ is invertible then the system has a unique formal solution $\hat{y}$. Moreover $\hat{y}$ is 1-summable in $x^{p'}\varepsilon^{q'}$.*

Proof. To prove (1) note that if we consider equation (3-35a) as a singularly perturbed ordinary differential equation and $\frac{\partial f_1}{\partial y}(0,0,0)$ is invertible then by Theorem 3.1.6 it has a unique formal solution $\hat{y}_1$, 1−summable in $x^p\varepsilon^q$. In the same way if $\frac{\partial f_2}{\partial y}(0,0,0)$ is invertible then (3-35b) has a unique formal solution $\hat{y}_2$, 1−summable in $x^{p'}\varepsilon^{q'}$. If we assume that the system has a formal solution $\hat{y}$ we are assuming that $\hat{y} = \hat{y}_1 = \hat{y}_2$. If $x^p\varepsilon^q \neq x^{p'}\varepsilon^{q'}$ it follows from the tauberian Theorem 1.3.5 that $\hat{y}$ converges.

The proofs of (2) and (3) are analogous so we only prove (2). If we suppose that $\frac{\partial f_1}{\partial y}(0,0,0)$ is invertible we already know that by Theorem 3.1.6 the equation (3-35a) has a unique formal solution $\hat{y} \in (R_1^{(p,q)})^l$. It only remains to see that $\hat{y}$ is also a solution of (3-35b). We consider $\hat{w} = x^{p'}\varepsilon^{q'+1}\frac{\partial \hat{y}}{\partial \varepsilon} - f_2(x,\varepsilon,\hat{y})$. Then using the integrability condition (3-36) it is straightforward to check that $\hat{w}$ is a solution of the linear differential equation with formal coefficients:

$$x^{p+1}\varepsilon^q \frac{\partial w}{\partial x} = \left(p' x^p \varepsilon^q I_l + \frac{\partial f_1}{\partial y}(x, \varepsilon, \hat{y}) \right) w,$$

Since $\frac{\partial f_1}{\partial y}(0, 0, \underline{0})$ is invertible, the above equation has a unique formal solution, and since 0 is a solution then $\hat{w} = 0$ as we wanted to show. □

The reader may note that the reason why we do not assume in the first statement of the previous theorem that the system is completely integrable is because Proposition 3.3.1 indicates that the conditions imposed could never be satisfied. In particular we can not take for granted that Theorem 3.3.3 is a generalization of the Gérard-Sibuya Theorem 3.3.2. On the other side the following example exhibits a simple situation of a non-completely integrable system where the hypotheses of the previous theorem hold, showing in particular its not vacuity.

Example 3.3.1. Consider a constant vector $c \in \mathbb{C}^l$, arbitrary $p, q, p', q' \in \mathbb{N}^*$ and the Pfaffian system

$$\begin{cases} x^{p+1}\varepsilon^q \; \frac{\partial y}{\partial x} = y - c, \\ \varepsilon^{q'+1} x^{p'} \frac{\partial y}{\partial \varepsilon} = y - c. \end{cases}$$

It has a unique formal solution given by $\hat{y} = c$ and it is convergent. Also the system is not completely integrable except by the case $p = p' = q = q'$.

Example 3.3.2. This trivial example describes the Pfaffian systems coming from differential equations in one independent variable. Consider the differential equation $z^{r+1}\frac{dw}{dz} = f(z, w)$, where $r \in \mathbb{N}^*$, $w \in \mathbb{C}^l$ and f is an analytic function defined in a neighborhood of the origin in $\mathbb{C} \times \mathbb{C}^l$ such that $f(0, w) = 0$. If we set $y(x, \varepsilon) = w(x^p \varepsilon^q)$, where $p, q \in \mathbb{N}^*$ then it induces the completely integrable system

$$\begin{cases} \varepsilon^{rq} x^{rp+1} \frac{\partial y}{\partial x} = p f(x^p \varepsilon^q, y), \\ x^{rp} \varepsilon^{rq+1} \frac{\partial y}{\partial \varepsilon} = q f(x^p \varepsilon^q, y). \end{cases}$$

It has the same monomial in the singular part and illustrates the situation of statement (3) of Proposition 3.3.1. It follows from Theorem 3.3.3 or directly from the classical theory in one variable that if $\frac{\partial f}{\partial w}(0, 0)$ is invertible then the system has a unique solution $\hat{y}$, $1-$summable in $x^{pr}\varepsilon^{qr}$. Furthermore $\hat{y}(x, \varepsilon) = \hat{w}(x^p \varepsilon^q)$, where $\hat{w}$ is $r-$summable and it is the only solution of the initial differential equation.

The statement (1) of Theorem 3.3.3 give us positive information on the convergence of formal solutions of the system (3-35a), (3-35b) only when the monomials involved are different. However, thanks to Theorem 3.2.2 we still can obtain a convergence result for the case of systems (3-35a), (3-35b) when the functions f_1, f_2 are affine in y and the monomial in both equation is equal. So we now focus in Pfaffian systems of the form

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} = A(x,\varepsilon)y(x,\varepsilon) + a(x,\varepsilon), & \text{(3-48a)} \\ x^p\varepsilon^{q+1}\dfrac{\partial y}{\partial \varepsilon} = B(x,\varepsilon)y(x,\varepsilon) + b(x,\varepsilon), & \text{(3-48b)} \end{cases}$$

where $p, q \in \mathbb{N}^*$ and $A, B \in \mathrm{Mat}(l \times l, \mathbb{C}\{x,\varepsilon\})$, $a, b \in \mathbb{C}\{x,\varepsilon\}^l$. Note we can pass from system (3-48a), (3-48b) to an equation of the form (3-25) by multiplying (3-48a) by s_1/p, (3-48b) by s_2/p and adding them. In that case $C(x,\varepsilon) = \frac{s_1}{p}A(x,\varepsilon) + \frac{s_2}{q}B(x,\varepsilon)$ and $\gamma(x,\varepsilon) = \frac{s_1}{p}a(x,\varepsilon) + \frac{s_2}{q}b(x,\varepsilon)$. As an immediate consequence of Theorem 3.3.3 and Theorem 3.2.2 we have the following proposition.

Proposition 3.3.4. *The following assertions hold:*

1. *If the system (3-48a), (3-48b) is completely integrable and $A(0,0)$ or $B(0,0)$ is invertible then the system (3-48a), (3-48b) has a unique formal solution that is $1-$summable in $x^p\varepsilon^q$.*

2. *If the system has a formal solution $\hat{y}$ and there are $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $s_1/pA(0,0) + s_2/qB(0,0)$ is invertible, then $\hat{y}$ is $1-$summable in $x^p\varepsilon^q$. Its possible singular directions are the directions passing through the eigenvalues of $s_1/pA(0,0) + s_2/qB(0,0)$.*

The reader should note again that in the second statement we do not assume that the system is completely integrable because from Proposition 3.3.1 we can show that the process explained above is useless in that case. Indeed, if the system is completely integrable then $A(0,0)$ and $B(0,0)$ commute. Let $\mu_1, ..., \mu_m$ be the different eigenvalues of B with algebraic multiplicities $l_1, ..., l_m$ respectively. To unify notation set $l_0 = 0$. After a linear change of coordinates we can assume that $A(0,0)$ and $B(0,0)$ are in block-diagonal

$$A(0,0) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, \quad B(0,0) = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix},$$

where the matrices A_j, B_j have size l_j, all are upper-triangular and the only eigenvalue of B_j is μ_j, for all $j = 1, ..., m$. Let $\lambda_1, ..., \lambda_l$ be the eigenvalues of $A(0,0)$, counting repetitions. If we number them in such a way that A_1 has eigenvalues $\lambda_1, ..., \lambda_{l_1}$, A_2 has eigenvalues

$\lambda_{l_1+1}, ..., \lambda_{l_1+l_2}$ and so on then from the form of the matrices we conclude that $\frac{s_1}{p}A(0,0) + \frac{s_2}{q}B(0,0)$ has eigenvalues $\frac{s_1\lambda_k}{p} + \frac{s_2\mu_j}{q}$, where $j = 1, ..., m$ and $l_{j-1} + 1 \leq k \leq l_{j-1} + l_j$. But statement (3) of Proposition 3.3.1 tell us that $q\lambda_k = p\mu_j$ for all $j = 1, ..., m$ and $l_{j-1} + 1 \leq k \leq l_{j-1} + l_j$. In particular the spectrum of the matrix $\frac{s_1}{p}A(0,0) + \frac{s_2}{q}B(0,0)$ is independent of $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$.

In the non-integrable case there are not imposed relations between $A(0,0)$ and $B(0,0)$ and then there are more possible situations for the spectrum of $\frac{s_1}{p}A(0,0)+\frac{s_2}{q}B(0,0)$. In particular there is a case when we can conclude convergence due to the absence of singular directions and it is explained in the next theorem.

Theorem 3.3.5. *Consider the system (3-48a), (3-48b) and suppose it has a formal solution* $\hat{y}$*. Denote by* $\lambda_1(s), ..., \lambda_l(s)$ *the eigenvalues of* $\frac{s}{p}A(0,0) + \frac{(1-s)}{q}B(0,0)$*, where* $0 \leq s \leq 1$*, and assume that they are never zero. Then if for every direction* d *there is* $s \in [0,1]$ *such that* $arg(\lambda_j(s)) \neq d$ *for all* $j = 1, ..., l$ *then* $\hat{y}$ *is convergent.*

Proof. Let d be a direction. If we take $s \in [0,1]$ such that $\arg(\lambda_j(s)) \neq d$ for all $j = 1, ..., l$ we know by Proposition 3.3.4 that d is not a singular direction for $1-$summability in $x^p\varepsilon^q$ of $\hat{y}$. Then $\hat{y}$ has no singular directions and by tauberian Proposition 1.3.1 $\hat{y}$ is convergent. □

We finish this section with a simple example where the hypotheses of the theorem hold.

Example 3.3.3. Consider the Pfaffian system given by

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} = \begin{pmatrix} x^p+\varepsilon+1 & -x^p-x \\ 1 & 1-x \end{pmatrix} y + \begin{pmatrix} x^p\varepsilon-\varepsilon-1 \\ x\varepsilon-\varepsilon-2 \end{pmatrix}, & \text{(3-49a)} \\ x^p\varepsilon^{q+1}\dfrac{\partial y}{\partial \varepsilon} = \begin{pmatrix} i-x+\varepsilon & -x-\varepsilon \\ -i-\varepsilon^q & i+\varepsilon^q \end{pmatrix} y + \begin{pmatrix} (2-i)x+x^2+\varepsilon^2-i \\ x\varepsilon^q+ix-i\varepsilon \end{pmatrix}, & \text{(3-49b)} \end{cases}$$

where $p, q \in \mathbb{N}^*$. It is not completely integrable but nonetheless it admits a unique formal solution

$$\hat{y} = (x+1, \varepsilon+1)^t,$$

and it is convergent. This can be seen an a consequence of the previous theorem: the only eigenvalue of $\frac{s}{p}A(0,0) + \frac{(1-s)}{q}B(0,0)$ is given by $\lambda(s) = s/p + i(1-s)/q$, $0 \leq s \leq 1$. If d is a direction and $d \notin [0, \pi/2]$ then it is non-singular because $d(s) = \arg(\lambda(s)) \in [0, \pi/2]$. If instead $d \in [0, \pi/2]$ there is only one s with $d = d(s)$ so taking any $s' \neq s$ we see that $d \neq d(s')$ and d is also non-singular.

4 Toward monomial multisummability

The aim of this chapter is to propose a definition of monomial multisumability for two levels, i.e. a method of summability that mixes two monomial summability methods. In order to do this we have developed acceleration operators associated to two monomials, two parameters of summability and two weights, when restricted to adequate cases when the calculations are possible.

The chapter is divided into three sections. In the first one we have recalled the classical acceleration operators and the notion of multisummability for two levels, in one variable. In the second one we have formally calculated the composition of a Borel transform associated to a monomial, a parameter of summability and a weight of the variables and a Laplace transform associated to another monomial, a parameter of summability and a weight of the variables. The resulting operator is an acceleration operator for monomial summability. In this section we have developed all the properties of such operator as their behavior w.r.t. monomial asymptotic expansions and convolutions.

In the last section we prove that the sum of divergent monomial summable series cannot be monomial summable at least that they all belong to the same space of monomial summable series. In order to sum series obtained in that way we propose a definition of monomial multisummability for two levels using the monomial acceleration operators. Finally we show that this notion is stable by sums and products.

4.1 Classical acceleration operators and multisummability

The goal of this section is to quickly recall the notion of multisummability (for two levels) of formal power series. There are many equivalent ways to introduce the concept of multisummability, for instance using cohomological methods as in [MR], through iterated Laplace transforms as in [B1] or using acceleration operators as was originally done by J. Ecalle in [Ec]. Here we only explain the point of view of the acceleration operators following mainly the exposition in [B1]. Many of the formulas used here as well as relevant results in the theory are contained in the paper [MrR].

Nowadays it is well known that $k-$summability is not a strong enough method to sum all the formal power series solutions of systems of linear or nonlinear meromorphic ordinary differential equations. A more sophisticated summation process called *multisummability* had become necessary. The first example of this situation was provided by J.P. Ramis and Y. Sibuya in 1984 and it consists in the sum of two divergent power series of different levels of summability. This is the kind of series that multisummability for two levels will sum. We reproduce the example here because we will use the same idea for the case of two monomials.

Consider the series $\hat{E}(x) := \sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$, called the *Euler series*. It is $1-$summable and satisfies the differential equation $D_1(\hat{E}(x)) = x$, where $D_1 = x^2 d/dx + 1$. Then $\hat{E}(x^2)$ is $2-$summable and satisfies $D_2(\hat{E}(x^2)) = 2x^2$, where $D_2 = x^3\frac{d}{dx} + 2$. If we calculate the *left least common multiple* of D_1 and D_2 (in the non-commutative ring $\mathbb{C}(x)[d/dx]$), i.e. the monic differential operator D of minimal degree in d/dx that can be factored as $D = L_1 D_1 = L_2 D_2$, for some $L_1, L_2 \in \mathbb{C}(x)[d/dx]$ we find that

$$D = x^5(2-x)\frac{d^2}{dx^2} - x^2(2x^3 - 5x^2 - 4)\frac{d}{dx} + 2(x^2 - x + 2),$$

$L_1 = x^3(2-x)\frac{d}{dx} + 2x^2 - 2x + 4$ and $L_2 = x^2(2-x)\frac{d}{dx} + x^2 - x + 2$. Then $\hat{f}(x) = \hat{E}(x) + \hat{E}(x^2)$ satisfies the differential equation

$$D(\hat{f}) = D(\hat{E}(x)) + D(\hat{E}(x^2)) = L_1(x) + L_2(2x^2) = -3x^4 + 10x^3 + 2x^2 + 4x,$$

and also naturally satisfies $d^5/dx^5 D(\hat{f}) = 0$. However $\hat{f}$ is not $k-$summable for any value of k, as the following proposition shows.

Proposition 4.1.1. *Let $0 < k_m < \cdots < k_2 < k_1$ be positive numbers, $m \geq 2$, and $\hat{f}_i \in \mathbb{C}\{x\}_{1/k_i}$ for every $i = 1, ..., m$. If the $\hat{f}_i$ are not convergent then $\hat{f} = \hat{f}_1 + \hat{f}_2 + \cdots + \hat{f}_m$ cannot be $k-$summable for any $k > 0$.*

The reader may note that the previous proposition is indeed equivalent to the part of Theorem 1.1.13 that establishes that $\mathbb{C}\{x\}_{1/k'} \cap \mathbb{C}\{x\}_{1/k} = \mathbb{C}\{x\}$ for all $0 < k < k'$.

To be able to define multisummability we need to recall the following family of special functions. For a real number $\alpha > 1$ and $z \in \mathbb{C}$ the *acceleration function* corresponding to α is defined by the integral formula

$$\mathcal{C}_\alpha(z) = \frac{1}{2\pi i}\int_\gamma e^{v - zv^{1/\alpha}} dv,$$

where the integral is taken over a Hankel path γ. It is well known that $\mathcal{C}_\alpha$ is an entire function and that for every $0 < \theta < \pi/\beta$ there are constants $c_1 = c_1(\alpha, \theta), c_2 = c_2(\alpha, \theta) > 0$ with

$$|C_\alpha(z)| \leq c_1 e^{-c_2|z|^\beta},$$

for all $z \in \mathbb{C}$ with $|\arg(z)| \leq \theta/2$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. By calculating the power series expansion of C_α it follows that

$$C_\alpha(1/z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-n}}{n!\Gamma\left(\frac{-n}{\alpha}\right)} = z^\alpha \mathcal{B}_\alpha(e^{-1/t})(z),$$

for all $z \in \mathbb{C}^*$. Using the Laplace transform $\mathcal{L}_\alpha$ we can express this equality as

$$\mathcal{L}_\alpha \left(\frac{1}{z^\alpha} C_\alpha \left(\frac{1}{z} \right) \right) (w) = e^{-1/w},$$

or after a change of variables,

$$\int_0^{e^{id}\infty} C_\alpha \left(wu^{-1/\alpha} \right) \frac{e^{-u}}{u} du = e^{-w}, \tag{4-1}$$

an equality valid for $|d| < \pi/2$ and $w \in \mathbb{C}$ satisfying $\left|\arg(w) - \frac{d}{\alpha}\right| < \frac{\pi}{2\beta}$.

Finally an application of property (1) of Proposition 1.1.9 leads us to the following formula:

$$\int_0^1 C_\alpha \left(\frac{z}{t^{1/\alpha}} \right) C_\alpha \left(\frac{w}{t^{1/\alpha}} \right) \frac{dt}{t(1-t)} = C_\alpha \left(z + w \right), \tag{4-2}$$

valid for all $z, w \in \mathbb{C}$, very useful in the study of convolutions.

The acceleration functions allow us to introduce the *acceleration operators* in the same way as the exponential function lead us to the Laplace transform. In this case the exponential kernel in the Laplace transform is replaced by a function $\mathcal{C}_\alpha$, for some $\alpha > 1$. More specifically, let $0 < k_2 < k_1$ be positive numbers, let κ be determined by $\frac{1}{\kappa} = \frac{1}{k_2} - \frac{1}{k_1}$ and let d be a direction. The *acceleration operator of index* (k_1, k_2) *in the direction* d is defined by the integral formula

$$\mathfrak{A}_{k_1,k_2,d}(f)(z) = \frac{1}{z^{k_1}} \int_0^{e^{id}\infty} f(u) \mathcal{C}_{k_1/k_2}((u/z)^{k_2}) du^{k_2}, \tag{4-3}$$

for functions $f : [0, e^{id}\infty) \to \mathbb{C}$, with exponential growth at infinity at most κ. The resulting function is defined in a sectorial region of opening π/κ bisected by d and $z^{k_1-k_2}\mathfrak{A}_{k_1,k_2,d}(f)(z)$ is analytic there. If the domain of f contains a sector, d, d' are directions in that sector and $|d - d'| < \pi/\kappa$ then $\mathfrak{A}_{k_1,k_2,d}(f)(z) = \mathfrak{A}_{k_1,k_2,d'}(f)(z)$ on the intersection of their corresponding domains.

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ it can be proved that:

$$\mathfrak{A}_{k_1,k_2,d}(z^\lambda)(z) = \frac{\Gamma\left(\frac{\lambda+k_2}{k_2}\right)}{\Gamma\left(\frac{\lambda+k_2}{k_1}\right)} z^{\lambda+k_2-k_1},$$

what lead us to define the *formal acceleration operator of index* (k_1, k_2):

$$\hat{\mathfrak{A}}_{k_1,k_2} : \mathbb{C}[[z]] \longrightarrow z^{k_2-k_1}\mathbb{C}[[z]]$$
$$\sum_{n=0}^{\infty} a_n z^n \longmapsto \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+k_2}{k_2}\right)}{\Gamma\left(\frac{n+k_2}{k_1}\right)} a_n z^{n+k_2-k_1}.$$

Note that $\hat{\mathcal{L}}_{k_1} \circ \hat{\mathfrak{A}}_{k_1,k_2} \circ \hat{\mathcal{B}}_{k_2}(z^{k_2}\hat{f}) = z^{k_2}\hat{f}$ for any $\hat{f} \in \mathbb{C}[[z]]$. In the analytic context, we can assure that if f has exponential growth at most k_2 on V then $\mathfrak{A}_{k_1,k_2}(f)$ is analytic and of exponential growth at most k_1 on the corresponding sectorial region and the following formula holds:

$$\mathcal{L}_{k_1}(\mathfrak{A}_{k_1,k_2}(f))(z) = \mathcal{L}_{k_2}(f)(z),$$

where both expressions are defined.

The behavior of acceleration operators w.r.t. asymptotic expansions can be described as follows: Suppose $f \in \mathcal{O}(V)$, where V is a sector of infinite radius and opening ϑ, and that f has exponential growth at most κ on V. If $f \sim_{s_1} \hat{f}$ on V, then $z^{k_1-k_2}\mathfrak{A}_{k_1,k_2,d}(f) \sim_{s_2} z^{k_1-k_2}\hat{\mathfrak{A}}_{k_1,k_2}(\hat{f})$ on the corresponding sectorial region of opening $\vartheta + \pi/\kappa$, where $s_2 = s_1 + \frac{1}{\kappa}$ and d is a direction in V.

Finally, we recall that acceleration operators behave well under convolution: if f, g have exponential growth at infinity at most κ, then so does $f *_{k_2} g$ and

$$\mathfrak{A}_{k_1,k_2}(f *_{k_2} g) = \mathfrak{A}_{k_1,k_2}(f) *_{k_1} \mathfrak{A}_{k_1,k_s}(g). \tag{4-4}$$

At this point we are ready to introduce the notion of multisummability using the acceleration operators. The definition just asks for natural conditions to be able to use the acceleration operators. We are going to do it only in the case of two levels of multissumability because that is the case we are going to treat in the attempt of a generalization using monomials in the next sections.

Definition 4.1.1. Let $0 < k_2 < k_1$ positive real numbers and let $\underline{k} = (k_1, k_2)$. A pair of directions $\underline{d} = (d_1, d_2)$ is said to be $\underline{k}-$*admissible* if it satisfies

$$|d_1 - d_2| \leq \frac{\pi}{2\kappa}, \quad \text{where } \frac{1}{\kappa} = \frac{1}{k_2} - \frac{1}{k_1}.$$

If $\underline{k}$ and $\underline{d}$ satisfy these conditions then we say they are *admissible*.

Given $\underline{k} = (k_1, k_2)$ and $\underline{d} = (d_1, d_2)$ it follows that $\underline{d}$ is $\underline{k}-$admissible if and only if the intervals $I_j = \left[d_j - \frac{\pi}{2k_j}, d_j + \frac{\pi}{2k_j}\right]$, $j = 1, 2$, satisfy $I_1 \subset I_2$.

Definition 4.1.2. Let $0 < k_2 < k_1$ be positive real numbers, let $\underline{k} = (k_1, k_2)$ and $\underline{d} = (d_1, d_2)$ admissible. Suppose a formal power series $\hat{f} \in \mathbb{C}[[x]]_{1/k_2}$ satisfies the following conditions:

1. $\hat{\mathcal{B}}_{k_2}(\hat{f})$ can be analytically continued analytically, say as φ, into a small sector of infinite radius bisected by d_2 with exponential growth at most κ. Then we can calculate $\mathfrak{A}_{k_1,k_2}(\varphi)$.

2. $\mathfrak{A}_{k_1,k_2}(\varphi)$ extends analytically, say as ψ, into a small sector of infinite radius bisected by d_1 with exponential growth at most k_1.

Then $f(x) = \mathcal{L}_{k_1,d_1}(\psi)(x)$ is well-defined in a sector bisected by d_1 with opening greater than π/k_1. In that case we say that $\hat{f}$ is $\underline{k}-$*multisummable* in the multidirection $\underline{d}$. The function f is called the $\underline{k}-$*multisum* of $\hat{f}$ in the multidirection $\underline{d}$.

The set of $\underline{k}-$multisummable power series in the multidirection $\underline{d}$ will be denoted by $\mathbb{C}\{x\}_{\underline{k},\underline{d}}$.

The reader may note that we have suppressed the additional factor x^{k_2} in the previous definition as compared with the definition of summability. This is because in this case even if we included such factor, we need then to apply the acceleration operator that also modifies the exponents but in this case we cannot add another such factor to compensate the change, at least not maintaining the relation $\hat{\mathcal{L}}_{k_1} \circ \hat{\mathfrak{A}}_{k_1,k_2} \circ \hat{\mathcal{B}}_{k_2}(z^{k_2}\hat{f}) = z^{k_2}\hat{f}$. One advantage of not adding this factors is that using convolutions it can be proved directly from the definition that $\mathbb{C}\{x\}_{\underline{k},\underline{d}}$ is closed by the usual product.

Given $\underline{k} = (k_1, k_2)$ and $\underline{d} = (d_1, d_2)$ admissible and $\hat{f} \in \mathbb{C}[[x]]_{1/k_2}$ we may wonder when $\hat{f} \in \mathbb{C}\{x\}_{\underline{k},\underline{d}}$. If this happens we say that $\underline{d}$ is a *non-singular multidirection of* $\hat{f}$ for $\underline{k}-$summability. If instead $\underline{d}$ is singular it can be for several reasons: first if $\hat{\mathcal{B}}_{k_2}(\hat{f})$ cannot be analytically continued to a small sector of infinite radius bisected by d_2 or it can but with exponential growth greater than κ. In that case any $\underline{k}-$admissible $\underline{d}$ with d_2 as second component is a singular multidirection. Then we say that $\underline{d}$ is *singular at level* 2. Second, if $\hat{\mathcal{B}}_{k_2}(\hat{f})$ can be analytically continued, say as φ, into a small sector of infinite radius bisected by d_2 with exponential growth at most κ but $\mathfrak{A}_{k_1,k_2}(\varphi)$ cannot be extended analytically into a small sector of infinite radius bisected by d_1 or it can but with exponential growth greater than k_1. Then we say that $\underline{d}$ is *singular at level* 1.

Once we have identified all singular multidirections at level 2 with common second component and also identifying admissible multidirections modulo 2π we say that $\hat{f}$ is $\underline{k}-$*multisummable* if only remain a finite number of singular multidirections. The set of $\underline{k}-$multisummable formal power series will be denoted by $\mathbb{C}\{x\}_{\underline{k}}$.

If $\underline{k}$ and $\underline{d}$ are admissible then $\mathbb{C}\{x\}_{\underline{k},\underline{d}}$ and $\mathbb{C}\{x\}_{\underline{k}}$ are differential algebras and the map that assigns to each element of $\mathbb{C}\{x\}_{\underline{k},\underline{d}}$ its sum, is a homomorphism of differential algebras. Using the properties of the acceleration operators it follows that $\mathbb{C}\{x\}_{1/k_j,d_j} \subset \mathbb{C}\{x\}_{\underline{k},\underline{d}}$ for $j = 1, 2$, and the sum operator coincide in both spaces. In particular if $\hat{f}_j \in \mathbb{C}\{x\}_{1/k_j,d_j}$,

$j = 1, 2$, then $\hat{f}_1 + \hat{f}_2 \in \mathbb{C}\{x\}_{\underline{k},\underline{d}}$. Conversely we have the following decomposition theorem due to W. Balser.

Theorem 4.1.2 (Balser). *Given $\underline{k} = (k_1, k_2)$ and $\underline{d} = (d_1, d_2)$ admissible, assume that $1/k_2 - 1/k_1 < 2$. Then for $\hat{f} \in \mathbb{C}\{x\}_{\underline{k},\underline{d}}$ there are $\hat{f}_j \in \mathbb{C}\{x\}_{1/k_j,d_j}$ such that $\hat{f} = \hat{f}_1 + \hat{f}_2$ and the $\underline{k}-$sum of $\hat{f}$ is given by the sum of the k_1-sum of $\hat{f}_1$ and the k_2-sum of $\hat{f}_2$.*

The definitions of multisummability in a multidirection and multisummability can be generalized to any number of levels and all the previous properties hold. The most remarkable result in this theory is the fact that all the formal power series solutions of systems of nonlinear meromorphic ordinary differential equations are multisummable. The first complete proof of this fact was given by Braaksma [Br]. Another complete proof using similar reasonings can be found in chapter 8 of [B1]. A different proof based on cohomological arguments is due to Ramis and Sibuya [RS1].

4.2 Monomial acceleration operators

In this section we define an analogue to the acceleration operators adapted to monomials using the Borel and Laplace transformations defined in Chapter 2. The aim of these operators is to lead us to a definition of monomial multisummability. Along the section we develop all its properties, similar to the ones of the classical acceleration operators such as its action on formal power series, its compatibility with the corresponding Laplace transforms, their behavior w.r.t monomial asymptotic expansions and with convolutions.

As in the classical case, we want to obtain an analogue to the acceleration operators for monomials. Following the same lines as in the one variable case we formally calculate the composition between a Borel and Laplace transforms of different indexes. More specifically, let $p, q, p', q' \in \mathbb{N}^*$ be positive natural numbers and let $s_1, s_2, s_1', s_2' > 0$ be positive real numbers such that $s_1 + s_2 = 1$ and $s_1' + s_2' = 1$. Then a simple calculation shows that for a function f we have

$$\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}\left(\mathcal{L}^{(p,q)}_{k,d,(s_1,s_2)}(f)\right)(\xi,\upsilon) =$$
$$\frac{(\xi^p\upsilon^q)^k}{2\pi i(\xi^{p'}\upsilon^{q'})^l}\int_\gamma\int_0^{e^{id}\infty} u^{-k/l\left(s_1'p/s_1+s_2'q/s_2\right)} f(\xi u^{-s_1'/p'l}\upsilon^{s_1/pk}, \upsilon u^{-s_2'/q'l}\upsilon^{s_2/qk})e^{u-v}dvdu,$$

where d is a direction such that $|d| < \pi/2$ and γ is a Hankel path.

A possible way to proceed is to request that $\Lambda := \frac{s_1}{s_1'}\frac{p'}{p} = \frac{s_2}{s_2'}\frac{q'}{q}$. This equation can always be solved for fixed s_1, s_2 or fixed s_1', s_2'. For instance, in the first case the solution is given by

$$s_1' = \frac{s_1 p' q}{s_2 p q' + s_1 p' q} \qquad s_2' = \frac{s_2 p q'}{s_2 p q' + s_1 p' q}. \tag{4-5}$$

Then, for each u, we can consider the change of variables $w = u^{-1}\upsilon^{\Lambda l/k}$. Some calculations, including the formal interchange in the order the integrals, lead us to the following formula

$$\begin{aligned}\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}\left(\mathcal{L}^{(p,q)}_{k,d,(s_1,s_2)}(f)\right)(\xi,\upsilon) &= \frac{(\xi^p\upsilon^q)^k}{(\xi^{p'}\upsilon^{q'})^l}\int_0^{e^{id}\infty} f(\xi w^{s_1'/p'l}, \upsilon w^{s_2'/q'l}) C_{\Lambda l/k}(w^{k/\Lambda l}) d(w^{k/\Lambda l}) \\ &= \frac{(\xi^p\upsilon^q)^k}{(\xi^{p'}\upsilon^{q'})^l}\int_0^{e^{id}\infty} f(\xi\tau^{s_1/pk}, \upsilon\tau^{s_2/qk}) C_{\Lambda l/k}(\tau) d\tau,\end{aligned}$$

provided that $\Lambda l/k > 1$. Using equations (4-5), this inequality is equivalent to have

$$s_1(p'q - pq') > \frac{p}{l}(qk - q'l). \tag{4-6}$$

In order to be able to choose $0 < s_1 < 1$ satisfying the above inequality we compare $p/p', q/q'$ and l/k and check all the possible cases:

1. Suppose $\max\left\{\frac{p}{p'}, \frac{q}{q'}\right\} < \frac{l}{k}$. Then any $0 < s_1 < 1$ satisfies (4-6).
2. Suppose $\min\left\{\frac{p}{p'}, \frac{q}{q'}\right\} < \frac{l}{k} \leq \max\left\{\frac{p}{p'}, \frac{q}{q'}\right\}$. Then if $p/p' < l/k \leq q/q'$, we can take any s_1 satisfying $0 \leq \frac{p(qk-q'l)}{l(p'q-pq')} < s_1 < 1$. If $q/q' < l/k \leq p/p'$, we can take any s_1 satisfying $0 < s_1 < \frac{p(qk-q'l)}{l(p'q-pq')} \leq 1$.
3. Suppose $\frac{l}{k} \leq \min\left\{\frac{p}{p'}, \frac{q}{q'}\right\}$. Then there is no $0 < s_1 < 1$ satisfying (4-6).

We remark that in the case of the same monomial, i.e., $p = p'$, $q = q'$, we have $s_1' = s_1$, $s_2' = s_2$ and $\Lambda = 1$. Then inequality (4-6) is just $l > k$.

The previous considerations justify the following definition of an acceleration operator.

Definition 4.2.1. Let $p, q, p', q' \in \mathbb{N}^*$ be positive natural numbers and $k, l > 0$ be positive real numbers such that $\min\left\{\frac{p}{p'}, \frac{q}{q'}\right\} < \frac{l}{k}$. Let $s_1, s_2 > 0$ be positive real numbers satisfying $s_1 + s_2 = 1$ and such that $s_1(p'q - pq') > \frac{p}{l}(qk - q'l)$. Let s_1', s_2' be given by (4-5) and set $I = (p', q', p, q, l, k, s_1', s_2', s_1, s_2)$. The *acceleration operator in direction θ, associated to the monomials $x^p\varepsilon^q$, $x^{p'}\varepsilon^{q'}$, with index (l,k) and weights (s_1, s_2), (s_1', s_2')*, or simply the *acceleration operator associated to I in direction θ*, of a function f is defined through the formula

$$\mathfrak{A}_{I,\theta}(f)(\xi,\upsilon) = \frac{(\xi^p\upsilon^q)^k}{(\xi^{p'}\upsilon^{q'})^l}\int_0^{e^{i\theta}\infty} f(\xi\tau^{s_1/pk}, \upsilon\tau^{s_2/qk}) C_{\Lambda l/k}(\tau) d\tau, \quad \text{where } \Lambda = \frac{s_1}{s_1'}\frac{p'}{p} = \frac{s_2}{s_2'}\frac{q'}{q}.$$

To determine the type of functions such that the above integral is meaningful we take into account the exponential behavior of $C_{\Lambda l/k}$. If we assume that f has an exponential growth of the form $|f(\xi,\upsilon)|\leq Ce^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}}$, for some $C,M>0$, then the norm of the integrand can be bounded by

$$|f(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})C_{\Lambda l/k}(\tau)|\leq Cc_1e^{M\max\{|\xi|^{\kappa_1}|\tau|^{\kappa_1 s_1/pk},|\upsilon|^{\kappa_2}|\tau|^{\kappa_2 s_2/qk}\}-c_2|\tau|^{1/(1-k/\Lambda l)}},$$

as long as $|d|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)$ and $(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})$ belongs to the domain of f. Then it is natural to request that

$$\frac{pk}{\kappa_1 s_1}=\frac{qk}{\kappa_2 s_2}=1-\frac{k}{\Lambda l}.$$

In conclusion, we may work with functions f having exponential growth as

$$|f(\xi,\upsilon)|\leq Ce^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}},\quad \frac{1}{\kappa_1}:=\frac{s_1}{pk}-\frac{s_1'}{p'l},\quad \frac{1}{\kappa_2}:=\frac{s_2}{qk}-\frac{s_2'}{q'l}, \tag{4-7}$$

for some constants $C,M>0$ and all (ξ,υ) in the domain of f. On the domain of f we can assert the following statements:

1. If $f\in\mathcal{O}(\Pi_{p,q}(a,b,+\infty))$, and has exponential growth as in (4-7), then for each θ satisfying $|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)$, $\mathfrak{A}_I$ is defined on the region $D'_{I,\theta}(a,b,M)$ given by

$$a-\theta/k<\arg(\xi^p\upsilon^q)<b-\theta/k,\quad M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}<c_2(\Lambda l/k,\theta).$$

   Note that changing the direction θ we obtain an analytic continuation of $\mathfrak{A}_{I,\theta}$. This process leads to an analytic function $\mathfrak{A}_I(f)$ defined in the region

$$\bigcup_{|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)} D'_{I,\theta}(a,b,M),$$

   which is a sectorial region in the monomial $\xi^p\upsilon^q$ of opening $b-a+\pi\left(\frac{1}{k}-\frac{1}{\Lambda l}\right)$.

2. If $f\in\mathcal{O}(\Pi_{p',q'}(a,b,+\infty))$, and has exponential growth as in (4-7), then for each θ satisfying $|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)$, $\mathfrak{A}_I$ is defined on the region $D''_{I.\theta}(a,b,M)$ given by

$$a-\Lambda\theta/k<\arg(\xi^{p'}\upsilon^{q'})<b-\Lambda\theta/k,\quad M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}<c_2(\Lambda l/k,\theta).$$

   As before, changing the direction θ we obtain an analytic continuation of $\mathfrak{A}_{I,\theta}$. This process leads to an analytic function $\mathfrak{A}_I(f)$ defined in the region

$$\bigcup_{|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)} D''_{I,\theta}(a,b,M),$$

   which is a sectorial region in the monomial $\xi^{p'}\upsilon^{q'}$ of opening $b-a+\pi\left(\frac{\Lambda}{k}-\frac{1}{l}\right)$.

The first natural property of the monomial acceleration operators is that they coincide with the composition of the corresponding monomial Borel and Laplace transforms, for functions having the adequate exponential growth.

Proposition 4.2.1. *Let $f \in \mathcal{O}(\Pi)$ be an analytic function, where Π is a monomial sector on infinite radius in $\xi^p \upsilon^q$ or $\xi^{p'} \upsilon^{q'}$ and let I be as in Definition 4.2.1. Suppose f has exponential growth $|f(\xi,\upsilon)| \leq Ce^{B\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}}$ for some $C, B > 0$ and all $(\xi,\upsilon) \in \Pi$. Then $\mathfrak{A}_I(f)$ is analytic in a corresponding monomial sector of infinite radius, has exponential growth of the form $|\mathfrak{A}_I(f)(\xi,\upsilon)| \leq De^{M\max\{|\xi|^{p'l/s'_1}, |\upsilon|^{q'l/s'_2}\}}$ for some $D, M > 0$ and satisfies*

$$\mathcal{L}^{(p',q')}_{l,(s'_1,s'_2)}(\mathfrak{A}_I(f)) = \mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(f).$$

Proof. Set $\alpha = \Lambda l/k$ and β such that $1/\alpha + 1/\beta = 1$. Also to simplify notation write $R(\xi,\upsilon) = \max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}$. To check that $\mathfrak{A}_I(f)$ has the mentioned exponential growth we bound it directly, as

$$\begin{aligned} |\mathfrak{A}_I(f)(\xi,\upsilon)| &\leq \frac{|\xi^p\upsilon^q|^k}{|\xi^{p'}\upsilon^{q'}|^l|} \int_0^{+\infty} Ce^{BR(\xi,\upsilon)-c_2t^\beta} dt \\ &= \frac{|\xi^p\upsilon^q|^k}{|\xi^{p'}\upsilon^{q'}|^l|} \frac{C}{R(\xi,\upsilon)} \int_0^{+\infty} e^{Bu-c_2u^\beta/R(\xi,\upsilon)^\beta} dt. \end{aligned}$$

Take any positive number δ and set $u_0 = \left(\frac{B+\delta}{c_2}\right)^{1/\beta-1} R(\xi,\upsilon)^{\beta/\beta-1}$. Note that if $u_0 \leq u$ then $B - c_2u^{\beta-1}/R(\xi,\upsilon)^\beta \leq -\delta$. By bounding the integral from 0 to u_0 and then from u_0 to $+\infty$ we see that

$$\begin{aligned} |\mathfrak{A}_I(f)(\xi,\upsilon)| &\leq \frac{|\xi^p\upsilon^q|^k}{|\xi^{p'}\upsilon^{q'}|^l} \frac{C}{R(\xi,\upsilon)} \left(\int_0^{u_0} e^{Bu} du + \int_{u_0}^{+\infty} e^{-\delta u} du\right) \\ &\leq \frac{|\xi^p\upsilon^q|^k}{|\xi^{p'}\upsilon^{q'}|^l} \frac{C}{R(\xi,\upsilon)} \left(\frac{1}{B}e^{Bu_0} + \frac{1}{\delta}\right). \end{aligned}$$

Since $R(\xi,\upsilon)^{\beta/\beta-1} = \max\{|\xi|^{p'l/s'_1}, |\upsilon|^{q'l/s'_2}\}$, the result follows.

The proof of the last part of the statement follows by calculating the left side of the equality, interchanging the order of integrals and using formula (4-1). ☐

Using the previous proposition and formulas (2-1) and (2-11) it can be seen that for $\lambda, \mu \in \mathbb{C}$ such that $\mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0$ we have

$$\mathfrak{A}_I(\xi^\lambda \upsilon^\mu)(\xi,\upsilon) = \frac{\Gamma\left(1+\frac{\lambda s_1}{pk}+\frac{\mu s_2}{qk}\right)}{\Gamma\left(\frac{k}{\Lambda l}+\frac{\lambda s_1'}{p'l}+\frac{\mu s_2'}{q'l}\right)}\xi^{\lambda+pk-p'l}\upsilon^{\mu+qk-q'l},$$
$$= \frac{\Gamma\left(1+\frac{\lambda s_1}{pk}+\frac{\mu s_2}{qk}\right)}{\Gamma\left(\frac{k}{\Lambda l}\left(1+\frac{\lambda s_1}{pk}+\frac{\mu s_2}{qk}\right)\right)}\xi^{\lambda+pk-p'l}\upsilon^{\mu+qk-q'l}.$$

The previous formula suggests the definition of the *formal acceleration operator associated with I* as:

$$\hat{\mathfrak{A}}_I : \mathbb{C}[[\xi,\upsilon]] \longrightarrow \frac{\xi^{pk}\upsilon^{qk}}{\xi^{p'l}\upsilon^{q'l}}\mathbb{C}[[\xi,\upsilon]]$$
$$\sum_{n,m\geq 0} a_{n,m}\xi^n\upsilon^m \longmapsto \sum_{n,m\geq 0} a_{n,m}\frac{\Gamma\left(1+\frac{\lambda s_1}{pk}+\frac{\mu s_2}{qk}\right)}{\Gamma\left(\frac{k}{l\Lambda}+\frac{\lambda s_1'}{p'l}+\frac{\mu s_2'}{q'l}\right)}\xi^{n+pk-p'l}\upsilon^{m+qk-q'l}.$$

It is a liner isomorphism and satisfy $\hat{\mathfrak{A}}_I = \hat{\mathcal{B}}^{(p',q')}_{l,(s_1',s_2')} \circ \hat{\mathcal{L}}^{(p,q)}_{k,(s_1,s_2)}$, giving us the formal counterpart of the previous proposition.

Remark 4.2.2. Let $\hat{f} = \sum_{n,m\geq 0} a_{n,m}\xi^n\upsilon^m = \sum_{n\geq 0} f_n(\xi,\upsilon)(\xi^p\upsilon^q)^n$ be a formal power series and I as in Definition 4.2.1. A necessary and sufficient condition on $\hat{f}$ so that $\hat{\mathfrak{A}}_I(\hat{f})$ is a convergent power series, is that there are constants $K, A > 0$ such that

$$|a_{n,m}| \leq \frac{KA^{n+m}}{\Gamma\left(1+\frac{n}{\kappa_1}+\frac{m}{\kappa_2}\right)},$$

for all $n, m \geq 0$, where κ_1, κ_2 are given by (4-7). This is equivalent to say that $\hat{f}$ defines an entire function f with an exponential growth of the form (4-7). Then $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I(f)$ exists, it is analytic in a polydisc at the origin, and it has $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})$ as Taylor's series at the origin.

Now assume that there are constants $s, B, D, M > 0$ such that the family of maps f_n are entire and satisfy the bounds

$$|f_n(\xi,\upsilon)| \leq DB^n\Gamma\left(1+sn\right)e^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}}, \tag{4-8}$$

for all $(\xi,\upsilon)\in\mathbb{C}^2$. This is equivalent to require that the coefficient of $\hat{f}$ satisfy bounds of type

$$|a_{np+m,nq+j}| \leq KL^{np+nq+m+j}\frac{\Gamma\left(1+sn\right)}{\Gamma\left(1+\frac{m}{\kappa_1}+\frac{j}{\kappa_2}\right)},$$

for all $n, m, j \in \mathbb{N}$ with $m < p$ or $j < q$ (recall formula (1-6)) and some constants $K, L > 0$.

Thus we can conclude that $\hat{f} \in \mathbb{C}[[\xi, \upsilon]]_s^{(p,q)}$, $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f}) \in \mathbb{C}[[\xi, \upsilon]]_{s+1/k-1/\Lambda l}^{(p,q)}$, all the maps $\mathfrak{A}_I((\xi^p\upsilon^q)^n f_n)$ are analytic in a common polydisc centered at the origin and

$$\hat{\mathfrak{A}}_I(\hat{f}) = \sum_{n\geq 0} \mathfrak{A}_I((\xi^p\upsilon^q)^n f_n).$$

In the same way, if $\hat{f} = \sum_{n\geq 0} f'_n(\xi, \upsilon)(\xi^{p'}\upsilon^{q'})^n$ and assuming that there are constants $s, B, D, M > 0$ such that the family of maps f'_n are entire and satisfy the bounds

$$|f'_n(\xi, \upsilon)| \leq DB^n\Gamma(1 + sn)\, e^{M\max\{|\xi|^{\kappa_1}, |\upsilon|^{\kappa_2}\}}, \tag{4-9}$$

for all $(\xi, \upsilon) \in \mathbb{C}^2$, or equivalently, to require that the coefficient of $\hat{f}$ satisfy bounds of type

$$|a_{np'+m,nq'+j}| \leq KL^{np'+nq'+m+j} \frac{\Gamma(1 + sn)}{\Gamma\left(1 + \frac{m}{\kappa_1} + \frac{j}{\kappa_2}\right)},$$

for all $n, m, j \in \mathbb{N}$ with $m < p'$ or $j < q'$ and some constants $K, L > 0$, we can conclude that $\hat{f} \in \mathbb{C}[[\xi, \upsilon]]_s^{(p',q')}$, $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f}) \in \mathbb{C}[[\xi, \upsilon]]_{s+\Lambda/k-1/l}^{(p',q')}$, all the maps $\mathfrak{A}_I((\xi^{p'}\upsilon^{q'})^n f'_n)$ are analytic in a common polydisc centered at the origin and

$$\hat{\mathfrak{A}}_I(\hat{f}) = \sum_{n\geq 0} \mathfrak{A}_I((\xi^{p'}\upsilon^{q'})^n f'_n).$$

As in the study of the Laplace transform we center our attention to the behavior of the acceleration operators w.r.t. monomial asymptotic expansions. Since these operators relate two monomials, it is natural to obtain results of asymptotic expansions for each monomial. The following two propositions are the analogue to Proposition 2.1.11 in this context and the proofs follow the same lines. Thus we only write the proof of the first one.

Proposition 4.2.3. *Let $f \in \mathcal{O}(\Pi_{p,q}(a, b, +\infty))$ be an analytic function. Suppose that the following statements hold:*

1. *$f \sim_s^{(p,q)} \hat{f}$ on $\Pi_{p,q} = \Pi_{p,q}(a, b, +\infty)$, for some $s \geq 0$.*

2. *If $\hat{T}_{p,q}(\hat{f}) = \sum_{n\geq 0} f_n t^n$, then every f_n is an entire function and there are constants $B, D, K > 0$ such that*

$$|f_n(\xi, \upsilon)| \leq DB^n\Gamma(1 + sn)\, e^{K\max\{|\xi|^{\kappa_1}, |\upsilon|^{\kappa_2}\}},$$

for all $n \in \mathbb{N}$ and for all $(\xi, \upsilon) \in \mathbb{C}^2$.

3. *For every monomial subsector $\widetilde{\Pi}_{p,q} \Subset \Pi_{p,q}$ there are constants $C, A, M > 0$ such that for all $N \in \mathbb{N}$*

$$\left|f(\xi,\upsilon)-\sum_{n=0}^{N-1} f_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right| \leq CA^N\Gamma(1+sN)|\xi^p\upsilon^q|^N e^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}},$$

for all $(\xi,\upsilon)\in\widetilde{\Pi}_{p,q}$.

Then $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I(f)\sim^{(p,q)}_{s+1/k-1/\Lambda l}\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})$ *on* $\bigcup_{|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)} D'_{I,\theta}(a,b,M)$.

Proof. To simplify notation we are going to write $R(\xi,\upsilon)=M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}$. We note that hypothesis 3. for $N=0$ is interpreted as f having exponential growth as in (4-7).

Let $h(\xi,\upsilon)=\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I(f)(\xi,\upsilon)$ and write $\hat{T}_{p,q}\left(\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})\right)=\sum_{n\geq 0}h_n\tau^n$. Then, as a consequence of statement (2), we can use Remark 4.2.2 to conclude that

$$h_n(\xi,\upsilon)(\xi^p\upsilon^q)^n=\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I((\xi^p\upsilon^q)^n f_n),$$

and additionally that $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})$ is $(s+1/k-1/\Lambda l)-$Gevrey in the monomial $\xi^p\upsilon^q$.

Now fix θ such that $|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)$. It is enough to prove the result for subsectors contained in $D'_{I,\theta}(a,b,M)$. If we take one of those proper subsectors $\overline{\Pi}_{p,q}$, we can find $\delta>0$ small enough such that

$$R(\xi,\upsilon)<c_2(\Lambda l/k,\theta)-\delta,$$

for all $(\xi,\upsilon)\in\overline{\Pi}_{p,q}$. Now let $\widetilde{\Pi}_{p,q}\Subset\Pi_{p,q}$ such that $(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})\in\widetilde{\Pi}_{p,q}$ if $(\xi,\upsilon)\in\overline{\Pi}_{p,q}$ and τ is on the semi-line $[0,e^{i\theta}\infty)$. Using statement 3. for $\widetilde{\Pi}_{p,q}$ we see that

$$\begin{aligned}
&\left|h(\xi,\upsilon)-\sum_{n=0}^{N-1}h_n(\xi,\upsilon)(\xi^p\upsilon^q)^n\right|=\\
&\left|\int_0^{e^{i\theta}\infty}\left(f(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})-\sum_{n=0}^{N-1}f_n(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})(\xi^p\upsilon^q)^n\tau^{n/k}\right)C_{\Lambda l/k}(\tau)d\tau\right|\\
&\leq\int_0^{+\infty}CA^N\Gamma(1+N/l)|\xi^p\upsilon^q|^N\rho^{N/k}e^{-\delta\rho^{1/(1-k/\Lambda l)}}d\rho\\
&=\left(1-\frac{k}{\Lambda l}\right)\frac{C}{\delta^{1-k/\Lambda l}}\frac{A^N}{\delta^{N\left(\frac{1}{k}-\frac{1}{\Lambda l}\right)}}\Gamma(1+sN)\Gamma\left(1+N\left(\frac{1}{k}-\frac{1}{\Lambda l}\right)+1-\frac{k}{\Lambda l}\right)|\xi^p\upsilon^q|^N,
\end{aligned}$$

for all $(\xi,\upsilon)\in\overline{\Pi}_{p,q}$. We can conclude that $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I(f)\sim^{(p,q)}_{s+1/k-1/\Lambda l}\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})$ on on $\bigcup_{|\theta|<\frac{\pi}{2}\left(1-\frac{k}{\Lambda l}\right)} D'_{I,\theta}(a,b,M)$, as we wanted to show. □

Proposition 4.2.4. *Let* $f\in\mathcal{O}(\Pi_{p',q'}(a,b,+\infty))$ *be an analytic function. Suppose that the following statements hold:*

1. $f \sim_s^{(p',q')} \hat{f}$ *on* $\Pi_{p',q'} = \Pi_{p',q'}(a,b,+\infty)$, *for some* $s \geq 0$.
2. *If* $\hat{T}_{p',q'}(\hat{f}) = \sum_{n\geq 0} f'_n t^n$, *then every* f'_n *is an entire function and there are constants* $B, D, K > 0$ *such that*
$$|f'_n(\xi,\upsilon)| \leq DB^n\Gamma(1+sn)\, e^{K\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}},$$
for all $n \in \mathbb{N}$ *and for all* $(\xi,\upsilon) \in \mathbb{C}^2$.
3. *For every monomial subsector* $\widetilde{\Pi}_{p',q'} \Subset \Pi_{p',q'}$ *there are constants* $C, A, M > 0$ *such that for all* $N \in \mathbb{N}$
$$\left| f(\xi,\upsilon) - \sum_{n=0}^{N-1} f'_n(\xi,\upsilon)(\xi^{p'}\upsilon^{q'})^n \right| \leq CA^N\Gamma(1+sN)\left|\xi^{p'}\upsilon^{q'}\right|^N e^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}},$$
for all $(\xi,\upsilon) \in \widetilde{\Pi}_{p',q'}$.

Then $\frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\mathfrak{A}_I(f) \sim_{s+\Lambda/k-1/l}^{(p',q')} \frac{(\xi^{p'}\upsilon^{q'})^l}{(\xi^p\upsilon^q)^k}\hat{\mathfrak{A}}_I(\hat{f})$ *on* $\bigcup_{|\theta|<\frac{\pi}{2}(1-\frac{k}{\Lambda l})} D''_{I,\theta}(a,b,M)$.

To conclude this section we prove the relation between the convolution product and monomial acceleration operators, i.e., the analogue to formula (4-4) in this context.

Proposition 4.2.5. *Let* $f, g \in \mathcal{O}(\Pi_{p,q})$ *be analytic functions on a monomial sector in* $\xi^p\upsilon^q$ *of infinite radius. Suppose* f, g *have exponential growth as in (4-7). Then so does* $f *_{k,(s_1,s_2)}^{(p,q)} g$, $\mathfrak{A}_I(f *_{k,(s_1,s_2)}^{(p,q)} g)$ *is well defined and we have*
$$\mathfrak{A}_I(f) *_{l,(s'_1,s'_2)}^{(p',q')} \mathfrak{A}_I(g) = \mathfrak{A}_I(f *_{k,(s_1,s_2)}^{(p,q)} g).$$

Proof. The fact that $f *_{k,(s_1,s_2)}^{(p,q)} g$ has exponential growth as in (4-7) follows by a direct estimate. To verify the equality note that by definition we have for some adequate θ that:
$$\mathfrak{A}_I(f) *_{l,(s'_1,s'_2)}^{(p',q')} \mathfrak{A}_I(g)(\xi,\upsilon) = \frac{(\xi^p\upsilon^q)^{2k}}{(\xi^{p'}\upsilon^{q'})^l}\int_0^1\int_0^{e^{i\theta}\infty}\int_0^{e^{i\theta}\infty}(t(1-t))^{k/\Lambda l-1}$$
$$f(\xi t^{s'_1/p'l}u^{s_1/pk}, \upsilon t^{s'_2/q'l}u^{s_2/qk})g(\xi(1-t)^{s'_1/p'l}v^{s_1/pk}, \upsilon(1-t)^{s'_2/q'l}v^{s_2/qk})C_{\Lambda l/k}(u)C_{\Lambda l/k}(v)dudvdt.$$

By performing the change of variables $w = ut^{k/\Lambda l}$, $z = v(1-t)^{k/\Lambda l}$, interchanging the order of integrals and applying formula (4-2) we get
$$\frac{(\xi^p\upsilon^q)^{2k}}{(\xi^{p'}\upsilon^{q'})^l}\int_0^{e^{i\theta}\infty}\int_0^{e^{i\theta}\infty} f(\xi w^{s_1/pk}, \upsilon w^{s_2/qk})g(\xi z^{s_1/pk}, \upsilon z^{s_2/qk})C_{\Lambda l/k}(w+z)dwdz.$$

Then fixing z, making the change $\zeta = w + z$ and after interchanging the order of the integrals we obtain the expression:

$$\begin{aligned}\frac{(\xi^p \upsilon^q)^{2k}}{(\xi^{p'} \upsilon^{q'})^l} \int_0^{e^{i\theta}\infty} \int_0^{\zeta} f(\xi(\zeta - z)^{s_1/pk}, \upsilon(\zeta - z)^{s_2/qk}) g(\xi z^{s_1/pk}, \upsilon z^{s_2/qk}) C_{\Lambda l/k}(\zeta) dz d\zeta = \\ = \frac{(\xi^p \upsilon^q)^k}{(\xi^{p'} \upsilon^{q'})^l} \int_0^{e^{i\theta}\infty} (f *^{(p,q)}_{k,(s_1,s_2)} g(\zeta)) C_{\Lambda l/k}(\zeta) d\zeta \\ = \mathfrak{A}_I (f *^{(p,q)}_{k,(s_1,s_2)} g)(\xi, \upsilon),\end{aligned}$$

as we wanted to show.

□

4.3 A definition of monomial multisummability

To motivate the definition of *monomial multisummability* we will propose here, we can prove an analogous result to Proposition 4.1.1 in the context of monomial summability. It is proved applying point blow-ups and it provides examples of power series in $\mathcal{S}$ that are not $k-$summable for any monomial and for any $k > 0$. The reader may note that the following result is a generalization of tauberian Theorem 1.3.5.

Theorem 4.3.1. *Let $p_0, ..., p_r$, $q_0, ..., q_r$ be positive natural numbers and let $k_0, ..., k_r$ be positive real numbers. Let $\hat{f}_j \in R^{(p_j,q_j)}_{1/k_j} \setminus R$ be k_j-summable power series in the monomial $x^{p_j}\varepsilon^{q_j}$, for $j = 1, ..., r$, respectively. Then $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_r$ is k_0-summable in $x^{p_0}\varepsilon^{q_0}$ if and only if $k_0 p_0 = k_j p_j$ and $k_0 q_0 = k_j q_j$ for all $j = 1, ..., r$.*

Proof. We prove the theorem by induction on r. If $r = 1$ the statement is just the tauberian Theorem 1.3.5. Suppose the theorem is true for $r - 1$ and let us prove it for r. If the conditions $k_0 p_0 = k_j p_j$ and $k_0 q_0 = k_j q_j$ hold for all $j = 1, ..., r$ then by Proposition 1.3.3 we see that $R^{(p_0,q_0)}_{1/k_0} = R^{(p_j,q_j)}_{1/k_j}$ for all $j = 1, ..., r$ and the statement is clear.

Conversely, suppose that $\hat{f}_0 \in R^{(p_0,q_0)}_{1/k_0}$. We may assume (reindexing the power series if necessary) that $k_0 p_0 \leq ... \leq k_r p_r$. The following situations cover all the possible cases:

I. We have the strict inequalities $k_0 p_0 < ... < k_r p_r$ and $k_0 q_0 < ... < k_r q_r$. If we apply $\hat{T}_{p_0,q_0}$ to $\hat{f}_0$ we see from Proposition 1.2.20 and Corollary 1.2.5 that $\hat{T}_{p_0,q_0}(\hat{f}_0)$ is k_0-summable and a sum of $\max\{p_0/p_j, q_0/q_j\}/k_j-$Gevrey series, $j = 1, ..., r$. Since $\max\{p_0/p_j, q_0/q_j\}/k_j < 1/k_0$ for all $j = 1, ..., r$ then by Theorem 1.1.13, $\hat{T}_{p_0,q_0}(\hat{f}_0)$ and so $\hat{f}_0$ are convergent. Using the induction hypothesis on the series $\hat{g}_0 = \hat{f}_0 - \hat{f}_1$ and $\hat{g}_j = \hat{f}_{j+1}$, $j = 1, ..., r-1$, we obtain a contradiction.

II. We have the equalities $k_0p_0 = \cdots = k_rp_r$. Then we compare the numbers k_jq_j for $j = 0, 1, ..., r$. Suppose there are $i \neq j$ such that $k_iq_i = k_jq_j$ then by Proposition 1.3.3 $R_{1/k_i}^{(p_i,q_i)} = R_{1/k_j}^{(p_j,q_j)}$. Reindexing the series we may suppose $i = 0, j = 1$. Then using the induction hypothesis on the series $\hat{g}_0 = \hat{f}_0 - \hat{f}_1$ and $\hat{g}_j = \hat{f}_{j+1}$, $j = 1, ..., r-1$, we see that $k_0p_0 = k_jp_j$ and $k_0q_0 = k_jq_j$ for all $j = 1, ..., r$, as we wanted to show.

Otherwise $k_iq_i \neq k_jq_j$ for all i, j. Reindexing the series we can assume we have the strict inequalities $k_0q_0 < ... < k_rq_r$. Composing with π_2 we obtain series $\hat{f}_j \circ \pi_2$ satisfying $\hat{f}_j \circ \pi_2 \in R_{1/k_j}^{(p_j+q_j,q_j)}$ for all $j = 0, 1, ..., r$. But now the new numbers satisfy $k_0(p_0 + q_0) < ... < k_r(p_r + q_r)$ and $k_0q_0 < ... < k_rq_r$ (strict inequalities). Arguing as in case (I) we can conclude that $\hat{f}_0 \circ \pi_2$ is convergent and using Proposition 2.3.1 we see that $\hat{f}_0$ is also convergent. Finally using the induction hypothesis on the series $\hat{g}_0 = \hat{f}_0 - \hat{f}_1$ and $\hat{g}_j = \hat{f}_{j+1}$, $j = 1, ..., r-1$, we obtain a contradiction.

III. We have $k_0p_0 < k_rp_r$ but some of the numbers in between are equal. We can write

$$
\begin{aligned}
k_0p_0 = \cdots = k_{i_0}p_{i_0} < k_{i_0+1}p_{i_0+1} = \cdots = k_{i_1}p_{i_1} < k_{i_1+1}p_{i_1+1} = \cdots \\
< k_{i_m+1}p_{i_m+1} = \cdots = k_rp_r,
\end{aligned}
$$

where the indexes $i_0, i_1, ..., i_m$ indicate when we have a strict inequality. More precisely, if $i_l + 1 \leq j \leq i_{l+1}$ then $k_jp_j = k_{i_{l+1}}p_{i_{l+1}}$ and if $j = i_l$ then $k_{i_l}p_{i_l} < k_{i_l+1}p_{i_l+1}$. Now consider $N \in \mathbb{N}^*$ satisfying

$$
N > \max_{0 \leq l \leq m} \frac{k_{i_l}q_{i_l} - k_{i_l+1}q_{i_l+1}}{k_{i_l+1}p_{i_l+1} - k_{i_l}p_{i_l}}.
$$

Composing N times the given series with π_1 we obtain series $\hat{f}_j \circ \pi_1^N \in R_{1/k_j}^{(p_j,q'_j)}$, $j = 0, 1, ..., r$, where $q'_j = Np_j + q_j$. By the election of N we have the strict inequalities

$$
k_{i_l}q'_{i_l} < k_{i_l+1}q'_{i_l+1}, \text{ for all } l = 0, ..., m.
$$

Furthermore the order relations between $k_{i_l+1}q'_{i_l+1}, ..., k_{i_{l+1}}q'_{i_{l+1}}$ are the same as the ones between $k_{i_l+1}q_{i_l+1}, ..., k_{i_{l+1}}q_{i_{l+1}}$. If for some l a pair of numbers among $k_{i_l+1}q'_{i_l+1}, ...,$ $k_{i_{l+1}}q'_{i_{l+1}}$ are equal the corresponding spaces coincide and we can use the induction hypothesis to get a contradiction. If not, all the numbers $k_{i_l+1}q'_{i_l+1}, ..., k_{i_{l+1}}q'_{i_{l+1}}$ are different, for all $l = 0, 1, ..., m$. We can even assume, by reindexing the series with index in the set $\{i_l + 1, ..., i_{l+1}\}$, for every possible l, that these numbers are ordered by the index, as

$$
k_{i_l+1}q'_{i_l+1} < \cdots < k_{i_{l+1}}q'_{i_{l+1}},
$$

where all the inequalities are strict. In other words, we have achieve to the situation $k_0q'_0 < \cdots < k_rq'_r$, where all the inequalities are strict. Finally composing with π_2 we obtain series $\hat{f}_j \circ \pi_1^N \circ \pi_2 \in R_{1/k_j}^{(p_j+q_j',q_j')}$, $j = 0, 1, ..., r$, and the corresponding numbers satisfy the strict inequalities $k_0(p_0 + q_0') < \cdots < k_r(p_r + q_r')$ and $k_0 q_0' < \cdots < k_r q_r'$. As in case (I) we conclude that $\hat{f}_0 \circ \pi_1^N \circ \pi_2$ is convergent and then $\hat{f}_0$ is convergent. Using the induction hypothesis on the series $\hat{g}_0 = \hat{f}_0 - \hat{f}_1$ and $\hat{g}_j = \hat{f}_{j+1}$, $j = 1, ..., r-1$, we obtain a contradiction.

Since the only non-contradictory case is when $k_0 p_0 = k_j p_j$ and $k_0 q_0 = k_j q_j$ for all $j = 1, ..., r$, this is the only possible order relation among those numbers and the statement of the theorem is true for r. The result follows by the principle of induction.

□

Example 4.3.1. Consider two different monomials $x^p \varepsilon^q$ and $x^{p'} \varepsilon^{q'}$ and $a, b \in \mathbb{C}^*$ and define the series

$$\hat{f}(x, \varepsilon) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{p a^{n+1}} \varepsilon^{q(n+1)} x^{p(n+1)} = \frac{1}{p} \hat{E} \left(\frac{1}{a} x^p \varepsilon^q \right),$$
$$\hat{g}(x, \varepsilon) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{p' b^{n+1}} \varepsilon^{q'(n+1)} x^{p'(n+1)} = \frac{1}{p'} \hat{E} \left(\frac{1}{b} x^{p'} \varepsilon^{q'} \right),$$

where $\hat{E}$ denotes the Euler series. The series $\hat{f}$ is $1-$summable in $x^p \varepsilon^q$ and satisfies the differential equation $D_1(\hat{f}) = x^p \varepsilon^q$, where $D_1 = \varepsilon^q x^{p+1} \partial/\partial_x + ap$. In the same way, the series $\hat{g}$ is $1-$summable in $x^{p'} \varepsilon^{q'}$ and satisfies the differential equation $D_2(\hat{g}) = x^{p'} \varepsilon^{q'}$, where $D_2 = \varepsilon^{q'} x^{p'+1} \partial/\partial_x + bp'$. By Theorem 4.3.1 the series $\hat{h} = \hat{f} + \hat{g}$ is not $k-$summable in any monomial, for any $k > 0$. We want to explore what kind of differential equation it satisfies. As in the example of J.P. Ramis and Y. Sibuya mentioned in the first section we calculate a differential operator D of degree 2 that can be factored as $D = L_1 D_1 = L_2 D_2$, for some $L_1, L_2 \in \mathbb{C}(x, \varepsilon)[\partial/\partial_x]$. If we call $P = \max\{p, p'\}$ and $Q = \max\{q, q'\}$ then a possible such operator is given by $D = A(\partial/\partial_x)^2 + B\partial/\partial_x + C$ where

$$\begin{aligned}
A &= p'bx^{3P-p'+2}\varepsilon^{3Q-q'} - apx^{3P-p+2}\varepsilon^{3Q-q},\\
B &= p'(p'+1)bx^{3P-p'+1}\varepsilon^{3Q-q'} + (bp')^2x^{3P-2p'+1}\varepsilon^{3Q-2q'} - p(p+1)ax^{3P-p+1}\varepsilon^{3Q-q}\\
&\quad - (ap)^2x^{3P-2p+1}\varepsilon^{3Q-2q},\\
C &= pp'ab(p'-p)x^{3P-p-p'}\varepsilon^{3Q-q-q'} + p(p')^2ab^2x^{3P-2p'-p}\varepsilon^{3Q-2q'-q} - a^2bp^2p'x^{3P-2p-p'}\varepsilon^{3Q-2q-q'}.
\end{aligned}$$

The operators L_1 and L_2 are given by

$$\begin{aligned}
L_1 =& (p'bx^{3P-p-p'+1}\varepsilon^{3Q-q-q'} - apx^{3P-2p+1}\varepsilon^{3Q-2q})\partial_x + bp'(p'-p)x^{3P-p-p'}\varepsilon^{3Q-q-q'}\\
&+ (bp')^2x^{3P-2p'-p}\varepsilon^{3Q-2q'-q} - abpp'x^{3P-2p-p'}\varepsilon^{3Q-2q-q'},\\
L_2 =& (p'bx^{3P-2p'+1}\varepsilon^{3Q-2q'} - apx^{3P-p-p'+1}\varepsilon^{3Q-q-q'})\partial_x + ap(p'-p)x^{3P-p-p'}\varepsilon^{3Q-q-q'}\\
&- (ap)^2x^{3P-2p-p'}\varepsilon^{3Q-2q-q'} + abpp'x^{3P-p-2p'}\varepsilon^{3Q-q-2q'}.
\end{aligned}$$

Then it follows that $\hat{h}$ satisfies the differential equation of second order

$$D(y) = 2b(p')^2 x^{3P-p'}\varepsilon^{3Q-q'} - 2ap^2 x^{3P-p}\varepsilon^{3Q-q} + (bp')^2 x^{3P-2p'}\varepsilon^{2q'} - (ap)^2 x^{3P-2p}\varepsilon^{3Q-2q}.$$

We can differentiate $3P - \min\{p, p'\} + 1$ times w.r.t. x to obtain a homogenous differential equation satisfied by $\hat{h}$. We note that the term multiplying the highest derivative of y in this new equation is still A. In order to factor a common monomial in A and that the resulting factor is invertible in $(0,0)$ it is necessary and sufficient that $3P - p' + 2 \leq 3P - p + 2$ and $3Q - q' \leq 3Q - q$ or $3P - p' + 2 \geq 3P - p + 2$ and $3Q - q' \geq 3Q - q$. These inequalities are equivalent to require that

$$\max\left\{\frac{p}{p'}, \frac{q}{q'}\right\} \leq 1 \quad \text{or} \quad \max\left\{\frac{p'}{p}, \frac{q'}{q}\right\} \leq 1.$$

Since the monomials are different, we will see that we are in adequate conditions to apply monomial multisummability to the series $\hat{h}$.

As in the classical theory of multisummability, we want to define a summability method for series in $\mathcal{S}$ capable to sum the series described in Theorem 4.3.1, at least for two summands and that combines the monomial summability of the monomials involved. Indeed, if we take $\hat{f} \in R_{1/k}^{(p,q)}$ and $\hat{g} \in R_{1/l}^{(p',q')}$ and set $\hat{h} = \hat{f} + \hat{g}$ we distinguish between the following cases:

1. If $p/q = p'/q'$ then we can suppose that we are working with the same monomial, so suppose that $x^p\varepsilon^q = x^{p'}\varepsilon^{q'}$ and $k < l$. In particular, the domains of the sum will be a monomial sectors in that monomial. Then we can use the operator $\hat{T}_{p,q}$ to study the classical multisummability of the series $\hat{T}_{p,q}(\hat{h})$.
2. If $p/q \neq p'/q'$ the monomials are essentially different. Then the *monomial multisum* of $\hat{h}$ would be defined in the intersection of the domains of the sum of $\hat{f}$ and the sum of $\hat{g}$, i.e. in sets of the form $\Pi_{p,q} \cap \Pi_{p',q'}$, for some monomial sectors. At this point the path changes drastically in view of the nature of this sets.

When restricting our attention to directions d_1 and d_2 of monomial summability of $\hat{g}$ and $\hat{f}$, respectively, the condition of (d_1, d_2) being $(l,k)-$admissible in the sense of Definition 4.1.1 is only meaningful in case (1). Then we need to adapt this condition for the general case to a condition where the domains of the different sums intersect. Taking into account this remark, a straight generalization of classical multisummability is available with the aid of the monomial acceleration operators presented in the previous section.

Definition 4.3.1. Let $I = (p', q', p, q, l, k, s_1', s_2', s_1, s_2)$ be as in Definition 4.2.1. We will say that $\hat{f} \in \mathcal{S}$ is *I-multisummable in the multidirection* (d_1, d_2) if the following conditions are satisfied

1. $\hat{f}$ is $1/k-$Gevrey in the monomial $x^p\varepsilon^q$,

2. $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$, being convergent in a neighborhood of the origin, can be analytically continued, say as φ_{s_1,s_2}, to a monomial sector of the form $S_{p,q}(d_1,\theta_1,+\infty)$, for some $\theta_1>0$, with exponential growth of the form $Ce^{M\max\{|\xi|^{\kappa_1},|\upsilon|^{\kappa_2}\}}$, for some constants $C,M>0$.

3. $\mathfrak{A}_I(\varphi_{s_1,s_2})$, being defined in a sectorial region in the monomial $\xi^p\upsilon^q$ bisected by d_1, can be analytically continued, say as $\psi_{s_1',s_2'}$, to the intersection of monomial sectors of the form $S_{p,q}(d_1,\theta_1',+\infty)$ and $S_{p',q'}(d_2,\theta_2',+\infty)$, for some $\theta_1',\theta_2'>0$, with exponential growth of the form $C'e^{M'\max\{|\xi|^{p'l/s_1'},|\upsilon|^{q'l/s_2'}\}}$, for some constants $C',M'>0$.

Then the $I-$*multisum of* $\hat{f}$ *in the multidirection* (d_1,d_2) is defined as

$$f(x,\varepsilon)=\mathcal{L}^{(p',q')}_{l,(s_1',s_2')}(\psi_{s_1',s_2'})(x,\varepsilon),$$

and it is an analytic function in a set of the form $S_{p,q}(d_1,\theta_1''+\pi/l\Lambda,r)\cap S_{p',q'}(d_2,\theta_2''+\pi/l,r)$, where $\theta_1''<\theta_1'$, $\theta_2''<\theta_2'$ and r is small enough.

The set of $I-$multisummable power series in the multidirection (d_1,d_2) will be denoted by $R_{I,(d_1,d_2)}=\mathbb{C}\{x,\varepsilon\}_{I,(d_1,d_2)}$.

From this definition we can deduce the following two properties guaranteeing the stability of the set $R_{I,(d_1,d_2)}$ by sums and products and that series of the form $\hat{f}+\hat{g}$, where $\hat{f}\in R^{(p,q)}_{1/k,d_1}$ and $\hat{g}\in R^{(p',q')}_{1/l,d_2}$ belong to $R_{I,(d_1,d_2)}$ under the assumption that the domains of their sums intersects. The first property follows from the linearity of the operators involved for the addition and from Proposition 4.2.5 for the product.

Proposition 4.3.2. *Let I be as in Definition 4.2.1. If $\hat{f},\hat{g}\in R_{I,(d_1,d_2)}$ then $\hat{f}+\hat{g}\in R_{I,(d_1,d_2)}$ and $\hat{f}\hat{g}\in R_{I,(d_1,d_2)}$.*

Proposition 4.3.3. *Let I be as in Definition 4.2.1. If $\hat{f}\in R^{(p,q)}_{1/k,d_1}$ and $\hat{g}\in R^{(p',q')}_{1/l,d_2}$ and the domains of their sums intersect then $\hat{f}+\hat{g}\in R_{I,(d_1,d_2)}$.*

The proof of the last proposition reduces to prove that $\hat{f}$ and $\hat{g}$ belong to $R_{I,(d_1,d_2)}$. For $\hat{f}$ the proof follows using Proposition 4.2.1. For $\hat{g}$ and its Gevrey order we can use Remark 4.2.2 to conclude that $\mathfrak{A}_I(\mathcal{B}^{(p,q)}_{k,(s_1,s_2)}(\hat{g}))=\mathcal{B}^{(p',q')}_{l,(s_1',s_2')}(\hat{g})$ is analytic at the origin and use the $l-$sumability of $\hat{g}$ in $x^{p'}\varepsilon^{q'}$ in direction d_2 to conclude that $\hat{g}\in R_{I,(d_1,d_2)}$. The hypothesis of the intersection of the domains is used to ensure the third condition of the definition.

We remark that in the proofs of the above propositions we also have seen that the monomial multisum of the series involved is obtained from the monomial sum, accordingly to each case.

Conclusions and future work

We want to briefly summarize in this last part the main conclusions of this thesis and to indicate some possible lines of work.

We have recalled and developed in detail the notion of monomial asymptotic expansions and we have focused in the special case of expansions of Gevrey type and monomial summability, as in the article [CDMS] on which is based our work. Many simple properties have been written to support the stronger results, including formulas to calculate the monomial sum. The first remarkable result is the tauberian property that establishes the incompatibility of non-equivalent monomial summation methods, described in the Theorem 1.3.5.

In the absence of a systematic approach to monomial summability using integral transformations we have developed Borel and Laplace operators adapted to a monomial but using weights in the variables, to be able to use the monomial sectors as natural domains of the functions on which the operators act. Based in the classical theory we have defined a summability method using these operators (adequate Gevrey type plus analytic continuation of the Borel transform with good exponential growth) and proved in the Theorem 2.2.1 that it is equivalent to monomial summability.

The natural scenario to apply monomial summability is the field of singularly perturbed analytic differential equations and so we did. The applications we have included treat three types of equations: doubly singular analytic linear differential equations, a partial differential equations induced naturally by a property of the monomial Borel transform and pfaffian systems in which every single equation is doubly singular. In all of them we have obtained properties of existence and uniqueness of formal solutions joint with monomial summability properties under the key hypothesis of the invertibility of the linear part at the origin of the analytic function involved: Proposition 3.1.2, Theorem 3.1.4, Proposition 3.3.4, Theorem 3.2.2. In the case of the pfaffian systems also properties of the spectra of the linear parts at the origin of those functions have been deduced from the classical integrability condition, Proposition 3.3.1. In the non integrable case, we have deduced from the tauberian theorems the convergence of formal solutions under mild conditions, Theorem 3.3.3 and Theorem 3.3.5.

Finally, after building examples of non-monomial summable series, Theorem 4.3.1, we have proposed a notion of monomial multisummability for two levels, by using acceleration operators adapted to monomials. We have defined and developed such operators in the same way we did it for the Borel and Laplace transformations in the second chapter. This is just a first step into a vast, technical and intricate theory far from being understood. By the key application of point blow-ups to prove Theorem 4.3.1 we can inquire that a more careful study of this geometric tool will be necessary to understand monomial multisummability.

Many open questions still remain unanswered and some tools have to be improved. We have already mentioned the necessity of extend the concept of monomial multisummability to an arbitrary number of levels. Of course a natural thought is to be able to handle these concepts also in many complex variables. We have the certainty that the results will extend with no difficulty, up to increasing technicality. The real problems underlay in the nature of the domains of the multisum: intersection of many monomial sectors. We hope this summation method will be as useful for doubly singular equations as the usual multisummability is to analytic differential equations at singular points, dropping the invertibility hypothesis. Besides we also can explore in more detail the pfaffian systems we have treated here. Of course another path to unravel is the sheaf theoretical approach to this theory.

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