Positional voting rules generated by aggregation functions and the role of duplication

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Abstract

In this paper, we consider a typical voting situation where a group of agents show their preferences over a set of alternatives. Under our approach, such preferences are codified into individual positional values which can be aggregated in several ways through particular functions, yielding positional voting rules and providing a social result in each case. We show that scoring rules belong to such class of positional voting rules. But if we focus our interest on OWA operators as aggregation functions, other well-known voting systems naturally appear. In particular, we determine those ones verifying duplication (i.e., clone irrelevance) and present a proposal of an overall social result provided by them.

Keywords: positional voting rules, scoring rules, aggregation functions, OWA operators, duplication.

1. Introduction

There exists in Social Choice a long tradition controversy between the positional and non-positional approaches to voting theory, coming early from

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Borda and Condorcet, respectively. Gärdenfors [26] established a comprehensive framework to understand this opposition, and considered that "positionalist voting functions are those social choice functions where the positions of the alternatives in the voters' preference orders crucially influence the social ordering of the alternatives". Of course, this assertion can be understood in different ways[1], being Borda, plurality and antiplurality rules the most popular cases of positional voting rules. All of them are specific cases of scoring rules[2], where the alternatives are socially ordered taking into account the sum of individual scores according to the agents' preferences. In fact, from Riker [41] on, many authors have identified scoring rules with positional voting methods. However, as we will show, scoring rules are not exclusive in capturing positional features of voting.

Our proposal is based on aggregation functions, mainly through *OWA operators*[3]. This tool has been revealed as a unifying way to face different issues appearing in several fields, Social Choice Theory among them (on this particular matter, see Wang *et al.* [46], Llamazares [35], García-Lapresta *et al.* [21] and Kacprzyk *et al.* [30], among others). As will be shown along the paper, this approach sheds light on some aspects avoided in the scoring context[4].

One of these interesting properties, not satisfied by the scoring rules, is duplication. This property entails irrelevance of clone voters in the final outcome and might not seem suitable at all in voting scenarios. Nonetheless, its fulfillment should be convenient in several contexts; for example, when multiple votes are allowed for each voter (what happens if some Internet mechanisms are used) or whenever that streams of opinion, rather than individual opinions, should be

[1]See Pattanaik [39], especially Section 3.

[2]See Chebotarev and Shamis [12] for a referenced survey on scoring rules and their characterizations.

[3]The initials in OWA stand for *ordered weighted averaging* (see Yager [47], Yager and Kacprzyk [49] and Yager *et al.* [50]). This kind of operators have been characterized by Fodor *et al.* [19].

[4]This argument, supported along the present paper, already preliminarily appeared in García-Lapresta and Martínez-Panero [24].

taken into account. Even more, it will be established that duplication arises in some specific positional voting rules induced by OWA operators interestingly related to decision under complete ignorance.

It is worth pointing out that there exist in the literature other possible alternative approaches extending the framework of scoring rules, such as *flexible scoring rules* introduced by Baharad and Nitzan [1]. On their hand, Xia and Conitzer [54, 55] have proposed what they call *generalized scoring rules*, extending well known voting rules such as Copeland, maximin, Bucklin and, of course, scoring rules. And recently Llamazares and Peña [36, 37] have employed *cumulative standing functions* representing in an interesting comprehensive way scoring rules such as plurality, antiplurality, Borda rule, k-approval voting, etc. Even more, these last authors also include in their proposal other voting rules based on variable scoring vectors or taking into account the support behind the candidates.

The rest of the paper is organized as follows. In Section 2 we introduce the basic notation for the voters' preferences over the alternatives and their related positions. Section 3 is devoted to voting rules as aggregation functions; we show that scoring rules are specific cases of such positional rules, and then we focus our attention on OWA operators and show their connections with some well-known voting rules appearing in the literature. The need of taking into account a variable electorate leads us to use *extended OWA operator*s (EOWA operators) and, with this background, in Section 4 we define duplication and then we characterize those OWA-generated positional voting rules satisfying this property. An illustrative example is also presented, and a proposal of an overall social order based on the characterized rules is obtained in a unifying way. Finally, some concluding remarks are included in Section 5.

2. Preliminaries

Consider a set of voters $V = \{1, \ldots, m\}$, with $m \geq 2$ (occasionally, just for completion reasons, we will also consider the trivial case $m = 1$). These voters show their preferences on a set of alternatives $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$. With $L(X)$ we denote the set of *linear orders* on X, and with $W(X)$ the set of *weak orders* (or *complete preorders*) on X. Given $R, \succcurlyeq \in W(X)$, with $\succ$ and $\sim$ we denote the asymmetric and the symmetric parts of R, respectively. A *profile* is a vector $\boldsymbol{R} = (R_1, \ldots, R_m)$ of weak orders, where R_v represents the preferences of the voter $v \in V$. Vectors in $\mathbb{R}^n$ are denoted as $\boldsymbol{a} = (a_1, \ldots, a_n)$. Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, with $\boldsymbol{a} \leq \boldsymbol{b}$ we mean $a_i \leq b_i$ for every $i \in \{1, \ldots, n\}$.

Definition 1. *Given $R \in W(X)$, the* position *of alternative $x_i \in X$ is defined as*

$$p(x_i) = n - \#\{x_j \in X \mid x_i \succ x_j\} - \frac{1}{2}\#\{x_j \in (X \setminus \{x_i\}) \mid x_j \sim x_i\}. \qquad (1)$$

It is worth mentioning that this proposal of assessing positions to the alternatives is equivalent to the way of scoring in extended versions of the Borda count for weak orders, where the scores of the tied alternatives are obtained as the average of the corresponding ones after a linearization process (see Smith [42], Black [8] and Cook and Seiford [14]).

Example 1. Consider $R \in W(\{x_1, \ldots, x_7\})$ given by

R
x_7
$x_3\ x_4$
x_2
$x_1\ x_5\ x_6$

Then,

$$p(x_1) = p(x_5) = p(x_6) = \frac{5+6+7}{3} = 6 = 7 - 0 - \frac{1}{2}2,$$

$$p(x_2) = 4 = 7 - 3 - \frac{1}{2}0,$$

$$p(x_3) = p(x_4) = \frac{2+3}{2} = 2.5 = 7 - 4 - \frac{1}{2}1,$$

$$p(x_7) = 1 = 7 - 6 - \frac{1}{2}0.$$

Consequently, R is codified by the *positions vector*

$$(p(x_1), p(x_2), p(x_3), p(x_4), p(x_5), p(x_6), p(x_7)) = (6,\ 4,\ 2.5,\ 2.5,\ 6,\ 6,\ 1).$$

In the particular case of linear orders, positions for each alternative vary from 1 to n with step 1 and the coordinates of each positions vector are permutations of $\mathcal{P}_l = \{1, 2, \ldots, n\}$. In the general case of weak orders, it is easy to check that possible positions range from 1 to n with step 0.5, i.e., in this case the positional vectors take their coordinates in

$$\mathcal{P} = \{1 + \lambda \cdot 0.5 \mid \lambda \in \{0, 1, \ldots, 2(n-1)\}\} = \{1,\ 1.5,\ 2, \ldots,\ n - 0.5,\ n\},$$

although not all n-dimensional vectors with these values as coordinates do represent a weak order[5].

Taking into account the positions of the alternatives, every profile $\boldsymbol{R} \in W(X)^m$ has associated a *position matrix* containing the positions of the alternatives for all the voters

$$\begin{pmatrix} p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ p_2(x_1) & p_2(x_2) & \cdots & p_2(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ p_m(x_1) & p_m(x_2) & \cdots & p_m(x_n) \end{pmatrix},$$

where $p_v(x_i)$ is the position of x_i for voter v. Thus, row v contains the positions of the alternatives according to voter v, and column i contains the positions of the alternative x_i.

3. The aggregation process

Definition 2. *Given a domain $\mathcal{D} \subseteq W(X)^m$, a* voting rule on $\mathcal{D}$ *is a mapping $F : \mathcal{D} \longrightarrow W(X)$ that satisfies the following conditions:*

[5]The set of of all admisible positions vectors in the previous sense has been characterized by García-Lapresta and Pérez-Román [25].

1. Anonymity*: For every permutation* π *on* $\{1,\ldots,m\}$ *and every profile* $\boldsymbol{R}\in\mathcal{D}$*,*
$$F\left(R_{\pi(1)},\ldots,R_{\pi(m)}\right)=F\left(R_1,\ldots,R_m\right).$$
2. Neutrality*: For every permutation* σ *on* $\{1,\ldots,n\}$ *and every profile* $\boldsymbol{R}\in\mathcal{D}$*,*
$$F\left(R_1^\sigma,\ldots,R_m^\sigma\right)=\left(F(R_1,\ldots,R_m)\right)^\sigma,$$
where R_v^σ *and* $(F(R_1,\ldots,R_m))^\sigma$ *are the orders obtained from* R_v *and* $F(R_1,\ldots,R_m)$*, respectively, by relabeling the alternatives according to* σ*, i.e.,* $x_{\sigma(i)}\,R_v^\sigma\,x_{\sigma(j)} \;\Leftrightarrow\; x_i\,R_v\,x_j$ *and* $x_{\sigma(i)}\,(F(R_1,\ldots,R_m))^\sigma\,x_{\sigma(j)} \;\Leftrightarrow\; x_i\,F(R_1,\ldots,R_m)\,x_j$*.*
3. Unanimity*: For every profile* $\boldsymbol{R}\in\mathcal{D}$ *and all* $x_i,x_j\in X$*,*
$$(\forall v\in V \;\; x_i\,R_v\,x_j) \;\Rightarrow\; x_i\,F(\boldsymbol{R})\,x_j.$$

Anonymity means a symmetric consideration for the voters; neutrality means a symmetric consideration for the alternatives; and unanimity means that if all the individuals consider an alternative as good as another one, then the social preference coincides with the individual preferences on this issue.

The previous framework considering voting rules, where the outcome is a social order (as in Smith [42]), is not unique at all in Social Choice Theory. Other possible approaches can be taken into account, such as social choice correspondences, where the result is the (nonempty) subset of the best alternatives (as in Young [52, 53]; see also Laslier [31] for further rank-based and pairwise-based approaches), or even social choice functions, where a single alternative is assigned to each profile[6].

3.1. Aggregation functions

In our proposal, we have adapted the notion of aggregation function from $[0,1]^m$ to m-tuples of $[1,\infty)^m$. In fact, for our purposes, it suffices to deal

[6] As pointed out by Courtin *et al.* [15], differences in the axiomatic treatment arise depending on the type of social mechanism considered.

with more restricted domains, namely, $\mathcal{P}^m = \{1, 1.5, 2, \ldots, n - 0.5, n\}^m$ or $\mathcal{P}_l^m = \{1, 2, \ldots, n\}^m$. On aggregation functions in the standard unit interval, see Calvo *et al.* [10], Beliakov *et al.* [7, 6] and Grabisch *et al.* [27].

Definition 3. *Let D be a domain being $D = \mathcal{P}^m$ or $D = \mathcal{P}_l^m$. An* aggregation function *on D is a mapping $A : D \longrightarrow \mathbb{R}$ verifying the following conditions:*

1. Boundary conditions*: $A(1, \ldots, 1) = 1$ and $A(n, \ldots, n) = n$.*
2. Monotonicity*: $\boldsymbol{a} \leq \boldsymbol{b} \Rightarrow A(\boldsymbol{a}) \leq A(\boldsymbol{b})$, for all $\boldsymbol{a}, \boldsymbol{b} \in D$.*

If, additionally, A satisfies idempotency*, i.e., $A(a, \ldots, a) = a$ for every $a \in \mathcal{P}$ (resp. $a \in \mathcal{P}_l$), then A is called* averaging aggregation function.

It is easy to see that averaging aggregation functions satisfy *compensativeness*:

$$\min\{a_1, \ldots, a_m\} \leq A(a_1, \ldots, a_m) \leq \max\{a_1, \ldots, a_m\},$$

for every $(a_1, \ldots, a_m) \in D$.

Typical averaging aggregation functions are the arithmetic mean, trimmed means, the median, the maximum, the minimum, etc. In fact, we can gather all these aggregation functions as specific cases of OWA operators.

Definition 4. *A* weighting vector *of dimension m is a vector $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0, 1]^m$ such that $\sum_{i=1}^{m} w_i = 1$.*

Definition 5. *Given $D = \mathcal{P}^m$ or $D = \mathcal{P}_l^m$ and a weighting vector $\boldsymbol{w}$ of dimension m, the* OWA operator on D associated with $\boldsymbol{w}$ *is the mapping $A_{\boldsymbol{w}} : D \longrightarrow \mathbb{R}$ defined as*

$$A_{\boldsymbol{w}}(a_1, \ldots, a_m) = \sum_{i=1}^{m} w_i \cdot a_{[i]},$$

where $a_{[i]}$ is the i-th greatest number of $a_1, \ldots, a_m$.

As noted before, some well-known aggregation functions are specific cases of OWA operators.

With appropriate weighting vectors $\boldsymbol{w} = (w_1, \ldots, w_m)$ we obtain

1. The *maximum*, for $\boldsymbol{w} = (1, 0, \ldots, 0)$.
2. The *minimum*, for $\boldsymbol{w} = (0, \ldots, 0, 1)$.
3. The *arithmetic mean*, for $\boldsymbol{w} = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$.
4. The *k-trimmed means*:
   - If $k = 1$, $\boldsymbol{w} = \left(0, \frac{1}{m-2}, \ldots, \frac{1}{m-2}, 0\right)$.
   - If $k = 2$, $\boldsymbol{w} = \left(0, 0, \frac{1}{m-4}, \ldots, \frac{1}{m-4}, 0, 0\right)$.
   - $\ldots$.
5. The *median*:
   (a) If m is odd, $w_i = \begin{cases} 1, & \text{if } i = \frac{m+1}{2}, \\ 0, & \text{otherwise.} \end{cases}$

   (b) If m is even, $w_i = \begin{cases} \frac{1}{2}, & \text{if } i \in \{\frac{m}{2}, \frac{m}{2}+1\}, \\ 0, & \text{otherwise.} \end{cases}$
6. The *mid-range*, for $\boldsymbol{w} = (0.5, 0, \ldots, 0, 0.5)$.

For $m = 3$, the set of weighting vectors

$$\{(\alpha, \beta, 1 - \alpha - \beta) \mid 0 \leq \alpha \leq 1\,,\ 0 \leq \beta \leq 1\,,\ \alpha + \beta \leq 1\}$$

can be identified with the triangle $\{(\alpha, \beta) \in [0,1]^2 \mid \alpha + \beta \leq 1\}$ (see Figure 1).

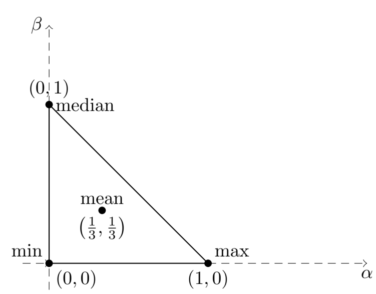

Figure 1: The main OWA operators for $m = 3$.

Note that the vertices of the triangle, $(0,0)$, $(1,0)$ and $(0,1)$, correspond to the minimum, the maximum and the median, respectively; and the baricenter of the triangle, $\left(\frac{1}{3}, \frac{1}{3}\right)$, corresponds to the arithmetic mean.

3.2. Positional voting rules

Definition 6. *Given the aggregation function* $A : \mathcal{P}^m \longrightarrow \mathbb{R}$ *and a profile* $\boldsymbol{R} \in W(X)^m$*, the* aggregated position *of the alternative* $x_i \in X$ *is defined as*

$$p_A(x_i) = A(p_1(x_i), \ldots, p_m(x_i)),$$

where $p_v(x_i)$ *is the position of* x_i *for voter* $v \in V$*.*

For every aggregation function $A : \mathcal{P}^m \longrightarrow \mathbb{R}$, we consider the mapping $F_A : W(X)^m \longrightarrow W(X)$ defined as $F_A(\boldsymbol{R}) = \succcurlyeq_A$, where

$$x_i \succcurlyeq_A x_j \;\Leftrightarrow\; p_A(x_i) \le p_A(x_j).$$

Remark 1. In the previous situation, it is easy to check that F_A is a voting rule.

Definition 7. *For every aggregation function* $A : \mathcal{P}^m \longrightarrow \mathbb{R}$*,* F_A *is the* positional voting rule associated with A.

3.2.1. Scoring rules as positional voting rules

As pointed out before, scoring rules appear to be the paradigm of the positional approach to voting theory. In what follows we define this class of rules, which encloses well-known voting rules such as plurality, antiplurality and Borda rules, among others. Then, we will show that all of them are positional voting rules in the aforementioned sense, but it will be shown that the reverse is not true.

Definition 8. *A* scoring vector *of dimension* $n \in \mathbb{N}$ *is a vector* $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$ *such that* $s_1 \ge \cdots \ge s_n$ *and* $s_1 > s_n$*.*

Now, suppose that voters' preferences over the alternatives are linear orders (i.e., weak orders where ties among distinct alternatives are avoided), gathered in a profile $\boldsymbol{R} \in L(X)^m$. Given a scoring vector $\boldsymbol{s} = (s_1, \ldots, s_n)$, for each voter, s_1 points are assigned to the top-ranked alternative, s_2 points to the second-ranked alternative, and so on. Formally, in terms of positions, the *individual* *score* of voter $v \in V$ for the alternative x_i is $r^v_{\boldsymbol{s}}(x_i) = s_{p_v(x_i)}$. The *collective score* for the alternative x_i is $r_{\boldsymbol{s}}(x_i) = \sum_{v=1}^{m} r^v_{\boldsymbol{s}}(x_i)$. The alternative(s) with the largest total score is (are) the winner(s).

Definition 9. *Given a scoring vector* $\boldsymbol{s}$ *of dimension* n*, the* scoring rule associated with $\boldsymbol{s}$ *is the mapping* $F_{\boldsymbol{s}} : L(X)^m \longrightarrow W(X)$ *defined as* $F_{\boldsymbol{s}}(\boldsymbol{R}) = \succcurlyeq_{\boldsymbol{s}}$, *where* $x_i \succcurlyeq_{\boldsymbol{s}} x_j \Leftrightarrow r_{\boldsymbol{s}}(x_i) \geq r_{\boldsymbol{s}}(x_j)$.

Remark 2. It is straightforward to check that $F_{\boldsymbol{s}}$ is a voting rule for every scoring vector $\boldsymbol{s}$. In what follows, notice that when dealing with positional voting rules, the smallest value(s) correspond(s) to the best position(s), just the opposite happening with scoring rules, where the highest score determines the winner(s).

The following cases give the scoring rules associated with various voting rules appearing in the literature.

- k–approval voting[7], $k \in \{1, 2, \ldots, n-1\}$: $\boldsymbol{s} = (1, \ldots, 1, 0, \ldots, 0)$, with k 1's. As important specific cases of this, we have
  - Plurality: for $k = 1$, $\boldsymbol{s} = (1, 0, \ldots, 0)$.
  - Antiplurality: for $k = n-1$, $\boldsymbol{s} = (1, \ldots, 1, 0)$.
- Borda rule: $\boldsymbol{s} = (n-1, n-2, \ldots, 1, 0)$.
- Best-worst voting rules[8]: $\boldsymbol{s} = (1, s, \ldots, s, 0)$, with $s \in (0, 1)$.

  Note that the excluded cases $s = 0$ and $s = 1$ would correspond again to plurality and antiplurality, respectively.

[7] Notice that while k–approval voting is a scoring rule, approval voting (where each voter can approve of as many alternatives as wished) is not. However, it can be understood as a *flexible* scoring rule following the extended framework proposed by Baharad and Nitzan [1].

[8] Best-worst voting rules were introduced and axiomatically characterized in the scoring context by García-Lapresta *et al.* [22].

Given the scoring rule associated with the scoring vector $(s_1, \ldots, s_n)$, $a, b \in \mathbb{R}$ such that $a > 0$, the new scoring rule associated with the scoring vector $(s'_1, \ldots, s'_n)$, where $s'_i = as_i + b$ for every $i \in \{1, \ldots, n\}$, is equivalent to the previous one, in the sense that they provide the same social outcomes. In this way, every scoring vector $(s_1, \ldots, s_n)$ can be normalized, i.e., it is equivalent to $(s'_1, \ldots, s'_n)$ with $s'_1 = 1$ and $s'_n = 0$, by simply taking $s'_i = (s_i - s_n)/(s_1 - s_n)$.

For $n = 4$, the set of normalized scoring vectors $\{(1, s, t, 0) \mid 0 \leq t \leq s \leq 1\}$ can be identified with the triangle $\{(s, t) \in [0, 1]^2 \mid t \leq s\}$ (see García-Lapresta *et al.* [20]). As shown in Figure 2, the vertices of the triangle, $(0, 0)$, $(1, 0)$ and $(1, 1)$, correspond to plurality, 2–approval voting and antiplurality, respectively; the baricenter of the triangle, $\left(\frac{2}{3}, \frac{1}{3}\right)$, corresponds to the Borda rule; and the segment conecting $(0, 0)$ and $(1, 1)$ corresponds to the set of best-worst voting rules.

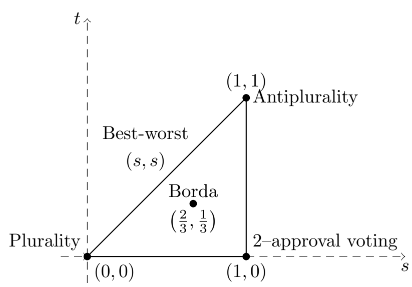

Figure 2: The best-known scoring rules for $n = 4$.

The next result shows how to construct an aggregation function from any scoring vector so that their associated positional voting rule and the corresponding scoring rule are the same.

Proposition 1. *All the scoring rules are positional voting rules.*

Proof: Let $\boldsymbol{s} = (s_1, \ldots, s_n) \in [0, 1]^n$ be a normalized scoring vector, i.e., $1 = s_1 \geq s_2 \geq \cdots \geq s_{n-1} \geq s_n = 0$. Let $A_{\boldsymbol{s}} : \mathcal{P}_l^m \longrightarrow \mathbb{R}$ be the mapping defined as

$$A_{\boldsymbol{s}}(a_1, \ldots, a_m) = 1 + \frac{n-1}{m} \sum_{v=1}^{m} \varphi_{\boldsymbol{s}}(a_v),$$

with $\varphi_{\boldsymbol{s}}(i) = 1 - s_i$ for every $i \in \{1, \dots, n\}$. It is easy to see that $A_{\boldsymbol{s}}$ is an aggregation function on $\mathcal{P}_l^m$.

We also have

$$
\begin{aligned}
& x_i \succcurlyeq_{A_{\boldsymbol{s}}} x_j \;\Leftrightarrow\; p_{A_{\boldsymbol{s}}}(x_i) \leq p_{A_{\boldsymbol{s}}}(x_j) \;\Leftrightarrow \\
& A_{\boldsymbol{s}}(p_1(x_i), \dots, p_m(x_i)) \leq A_{\boldsymbol{s}}(p_1(x_j), \dots, p_m(x_j)) \;\Leftrightarrow \\
& 1 + \frac{n-1}{m} \sum_{v=1}^{m} \varphi_{\boldsymbol{s}}(p_v(x_i)) \leq 1 + \frac{n-1}{m} \sum_{v=1}^{m} \varphi_{\boldsymbol{s}}(p_v(x_j)) \;\Leftrightarrow \\
& r_{\boldsymbol{s}}(x_i) = \sum_{v=1}^{m} r_{\boldsymbol{s}}^v(x_i) \geq \sum_{v=1}^{m} r_{\boldsymbol{s}}^v(x_j) = r_{\boldsymbol{s}}(x_j) \;\Leftrightarrow\; x_i \succcurlyeq_{\boldsymbol{s}} x_j .
\end{aligned}
$$

Hence, $\succcurlyeq_{A\boldsymbol{s}} \; = \; \succcurlyeq_{\boldsymbol{s}}$ and, consequently, the scoring rule associated with $\boldsymbol{s}$ coincides with the positional voting rule associated with $A_{\boldsymbol{s}}$. ■

Now we will show that our positional approach actually does extend the scoring context. We mean that, although sharing similar patterns, it is not true that every positional voting rule associated with an aggregation function can be represented by a scoring rule (in the sense that both provide the same social order).

Example 2. Consider the profile given by

R_1	R_2	R_3
x_1	x_1	x_2
x_3	x_3	x_3
x_2	x_2	x_1

where the associated position matrix is

$$
\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} .
$$

Aggregating through the maximum ($A = \max$), i.e., under the maximin voting rule, the following social order is obtained: $x_3 \succ_A x_1 \sim_A x_2$. Let now consider a generic scoring rule with associated (normalized) scoring vector $\boldsymbol{s} = (1, s, 0)$, $0 \leq s \leq 1$. Then, the collective scores obtained for the alternatives are

$$r_{\boldsymbol{s}}(x_1) = 2, \quad r_{\boldsymbol{s}}(x_2) = 1, \quad r_{\boldsymbol{s}}(x_3) = 3s.$$

According to the results, every scoring rule provides $x_1 \succ_{\boldsymbol{s}} x_2$. In any case, none of them represents the positional voting rule associated with the maximum, where $x_1 \sim_A x_2$, as pointed out before. Consequently, the class of positional voting rules does not coincide with the class of scoring rules.

Remark 3. Notice an interesting feature of the maximin rule that can be observed in the above situation: A kind of *clone irrelevance*, i.e., the influence of voter's copies is irrelevant in the final outcome. Thus, in Example 2, the presence or absence of voter 2 (a clone of voter 1) does not affect the result. This is a determinant fact for excluding the maximin voting rule from scoring rules, because it is against Young's [53] continuity[9], an axiom appearing in his characterization of scoring rules.

3.2.2. OWA-generated positional voting rules

Taking into account some of the OWA operators mentioned above, we obtain positional voting rules which are connected to (or even replicate) well-known procedures appearing in the literature:

- The **arithmetic mean** as aggregation function induces the Borda rule. And it is worth mentioning that the arithmetic mean is also the basis for the *Range Voting* method (Smith [43]), in a decisional context where the alternatives receive numerical assessments one by one.
- The **median** instead of the arithmetic mean, and linguistic terms instead of numerical values, are used in the *Majority Judgment* voting system in-

[9]This is an archimedean-type property which states that if two disjoint sets of voters U and V select x and y as winners, respectively, then x should be a winner for the superset $(nU) \cup V$ for n sufficiently large, where (nU) means n copies of those voters and their votes. Clearly, as exposed, this is not fulfilled by the maximin voting rule.

troduced by Balinski and Laraki [3, 4]. Extensions of this procedure using centered OWA operators (Yager [48]) and distances appear in García-Lapresta and Martínez-Panero [23] and in Falcó and García-Lapresta [17], respectively. Again, in a different scenario, Bucklin's method selects the candidates with highest median ranking as winners (see Tideman [44] and Felsenthal [18]), and similarly Basset and Persky [5] also proposed to select the alternative with best median evaluation (see also Laslier [33], who has coined the term *maxmed* for this voting scheme).

- The **maximum** leads to a voting rule in which each alternative is evaluated according to the worst reached position. Those with the best assigned value are then elected. Such a *maximin* voting rule, which advocates the maximin principle of normative economics[10], is called *fallback bargaining*[11] by Brams and Kilgour [9]. It has been characterized in the voting context by Congar and Merlin [13] (see also Llamazares and Peña [36]).

  The same underlying idea appears in the *leximin* voting system proposed by Laslier [32] (see also Laslier [33]), and in the Simpson-Kramer method (see Levin and Nalebuff [34]), although in different decisional frameworks. Furthermore, the procedure obtained through the maximum as aggregation operator is also related to the *Coombs* method (where the alternative with the largest number of last positions is sequentially withdrawn), as well as to the antiplurality rule (see Baharad and Nitzan [2] and Congar and Merlin [13]).

- The **minimum** entails a voting rule called *maximax*[12] by Congar and

[10] Rawls [40, p. 328]: "the basic structure is perfectly just when the prospects of the least fortunate are as great as they can be".

[11] Concretely, the maximin rule corresponds to the case of *fallback bargaining with unanimity*, also called *Kant-Rawls social compromise* after Hurwicz and Sertel [29].

[12] The apparent discordance leading the maximum to the maximin voting rule, as well as the minimum to the maximax, relies on our positional approach where, contrary to the scoring context, the smallest value is associated with the best position, as pointed out before. It is

Merlin [13], also characterized by them. Its conception is similar to that of the *Hare* rule, also known as *alternative vote* (where the alternative with the fewest first positions is sequentially withdrawn). It is also related to the most used (and criticized) voting system: plurality rule (see Laslier [32] and Congar and Merlin [13]).

- The **mid-range** is related to the basic 1-*best*-1-*worst* voting rule (see García-Lapresta *et al.* [22]).

Remark 4. There can exist a unique "translation" between weighting vectors and normalizated scoring vectors. This is what happens with the Borda rule: as a scoring rule, it is associated with $\left(1, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1} 1, 0\right)$, while as OWA-generated positional voting rule corresponds to $\left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$. However, in some cases such translation does not exist (see Remark 3, where it is shown that maximin rule can not be captured through any scoring rule; and it is also true for the maximax rule). Even more, depending on the situation, there can be several possibilities of translation from the same scoring rule into OWA-generated positional voting rules, being not compatible among them. This fact is shown in what follows.

Consider the profile given by

R_1	R_2	R_3
x_1	x_2	x_3
x_3	x_3	x_2
x_2	x_1	x_1

The result under plurality rule is $x_1 \sim x_2 \sim x_3$, and it is easy to check that this social order is also the same under the OWA-generated positional voting rule associated with the weighting vector $(0, 0, 1)$, i.e., the maximax rule.

On the other hand, consider the new profile

also worth to note in what follows that in the scoring context there are as many scores as alternatives, whereas in the positional scenario, when dealing with OWA operators, there are as many weights as voters.

R_1	R_2	R_3
x_1	x_1	x_2
x_2	x_3	x_3
x_3	x_2	x_1

Now plurality rule gives $x_1 \succ x_2 \succ x_3$ and, after some computation, the same social order can be obtained through every OWA-generated positional voting rule associated with any weighting vector $(\alpha, \beta, 1 - \alpha - \beta)$ such that $\beta > 0$.

Thus, these two profiles show that, even when existing weights associated with scores, they could have not common values. Concretely, in words referred to our example, being the first weighting vector incompatible with those obtained to capture plurality in the second profile, it can be argued that there not exists "the" OWA-generated positional voting rule associated with plurality (similar reasons for the antiplurality rule also stand).

Remark 5. One can ask if OWA-generated positional voting rules which are not scoring rules are those considering in the aggregation process only the information about the best and/or worst positioned alternatives, as happening with maximin and maximax rules. The answer is no. In both of the profiles appearing in Remark 4, the maxmed rule (whose weighting vector is $(0, 1, 0)$) cannot be represented under any scoring rule. For example, in the first profile, under a normalized scoring vector $(1, s, 0)$ with $s \in [0, 1]$, the obtained social order is $x_1 \sim x_2 \sim x_3$ when $s = 0$ (plurality, as aforementioned) and $x_3 \succ x_2 \succ x_1$ when $s > 0$. However, if maxmed were applied, then the result would be $x_3 \sim x_2 \succ x_1$, incompatible with any of these scoring rules.

3.3. Extended notions

Sometimes it is necessary to take into account a variable electorate (for instance, as mentioned, to deal with the clonation or appearance of new voters, as happening in Example 2). This is the reason why we introduce some extended notions of those already defined throughout the paper.

Definition 10. *An* extended voting rule *is a mapping*

$$\widetilde{F} : \bigcup_{m \in \mathbb{N}} W(X)^m \longrightarrow W(X)$$

such that $F_m = \widetilde{F}|_{W(X)^m}$ *is a voting rule for each dimension* $m = 2, 3, \ldots,$ *and* $F_1(R) = R$.

Definition 11. *An* extended OWA operator (EOWA) *is a sequence of OWA operators* $\widetilde{A} = (A_{\boldsymbol{w}^m})_{m \in \mathbb{N}}$ *with associated weighting vectors* $\boldsymbol{w}^m = (w_1^m, \ldots, w_m^m)$, *one for each dimension* $m \in \mathbb{N}$.

Following Calvo and Mayor [11] and Mayor and Calvo [38] (see also Beliakov *et al.* [7, pp. 54-56]) and Beliakov *et al.* [6, pp. 73-76]), we can show graphically an EOWA operator as a weighting triangle where the entries in each row add up to one.

$$\begin{array}{ccccccccc}
 & & & & w_1^1 & & & & \\
 & & & w_1^2 & & w_2^2 & & & \\
 & & w_1^3 & & w_2^3 & & w_3^3 & & \\
 & w_1^4 & & w_2^4 & & w_3^4 & & w_4^4 & \\
w_1^5 & & w_2^5 & & w_3^5 & & w_4^5 & & w_5^5 \\
\ldots\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots
\end{array}$$

4. Duplication

Here we formally introduce the aforementioned clone irrelevance property which, broadly speaking, requires that new voters replicating the same preferences of already existing voters will not affect the social outcome. At first sight such statement might seem a vulneration of the very essence of democracy, but it can make sense in some contexts. For example, in bargaining, when the above mentioned fallback method is used to find a compromise among the bargainers because they "fall back in lockstep to less and less preferred positions until they agree on outcome" (see Brams and Kilgour [9]). But it also make sense in voting scenarios such as the Internet, where agents can cast their votes more than one time[13]; or wherever that, rather than merely the total amount of votes, different currents of opinion or electoral bodies (such as minorities) should be taken into account.

This property appears as *duplication* in Congar and Merlin [13], where they consider this axiom in order to capture situations of complete ignorance in some voting contexts (see references therein) and characterize the maximin procedure.

Definition 12. *An extended voting rule* $\widetilde{F}$ *satisfies* duplication *if*

$$F_{m+1}(\boldsymbol{R}, R_i) = F_m(\boldsymbol{R}),$$

for every profile $\boldsymbol{R} = (R_1, \ldots, R_i, \ldots, R_m) \in W(X)^m$ *and every* $i \in \{1, \ldots, m\}$.

4.1. A characterization result

It is interesting to find those procedures satisfying duplication, and the following result shows the answer for positional voting rules associated with EOWA operators.

Theorem 1. *Given an EOWA operator* $\widetilde{A} = (A_{\boldsymbol{w}^m})_{m \in \mathbb{N}}$, *the extended voting rule* $\widetilde{F}_{\widetilde{A}}$ *satisfies duplication if and only if* $\widetilde{A}$ *is a rational convex combination of the maximum and the minimum EOWA operators, i.e., there exists* $\alpha \in [0,1] \cap \mathbb{Q}$ *such that* $\boldsymbol{w}^m = \alpha(1, 0, \ldots, 0) + (1 - \alpha)(0, \ldots, 0, 1)$ *for every* $m \in \mathbb{N}$.

Proof: It is straightforward that positional voting rules associated with $A_{\boldsymbol{w}^m}$, where $\boldsymbol{w}^m = (1, 0, \ldots, 0)$ (i.e., maximin), $\boldsymbol{w}^m = (0, 0, \ldots, 1)$ (i.e., maximax),

[13] See Yokoo and Matsubara [51], where they analyze the effect of false name bids in Internet auctions as well as Wagman and Conitzer [45], where these authors deal with *false-name-proof* voting mechanisms, i.e., those where no agent benefits from participating more than once.

and convex combinations of them, $\boldsymbol{w}^m = (\alpha, 0, \ldots, 0, 1-\alpha)$, with $\alpha \in [0,1]$, satisfy duplication[14].

For the reciprocal, we first prove that if duplication holds for an extended voting rule $\widetilde{F}_{\widetilde{A}}$, where $\widetilde{A} = (A_{\boldsymbol{w}^m})_{m\in\mathbb{N}}$, then all intermediate weights in each dimension m, $w_2, \ldots, w_{m-1}$, should be zero. Our reasoning will deal with a profile consisting in all circular permutations[15] of three alternatives, but the argument is extensible to $m > 3$. Thus, consider the profile

R_1	R_2	R_3
x_1	x_2	x_3
x_2	x_3	x_1
x_3	x_1	x_2

where the associated position matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

As every alternative occupies each position exactly once, a global tie arises and the aggregated position for each is $p_A(x_i) = 3w_1^3 + 2w_2^3 + w_3^3$, $i = 1,2,3$, so that $x_1 \sim_{A_{\boldsymbol{w}^3}} x_2 \sim_{A_{\boldsymbol{w}^3}} x_3$, where $\boldsymbol{w}^3 = (w_1^3, w_2^3, w_3^3)$.

Now suppose that voter 1 is replicated, becoming the new situation

R_1	R_2	R_3	$R_4 = R_1$
x_1	x_2	x_3	x_1
x_2	x_3	x_1	x_2
x_3	x_1	x_2	x_3

[14]Notice that, in the previous argument, α does not need to be rational. However, as pointed out by Fagin and Wimmers [16] "in some situations we have to restrict our attention to rational weights". One of these situations naturally arises when "we simply allow multiple copies of voters", which is exactly our case.

[15]These circular permutations yield a *Condorcet cycle*.

where the new associated position matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then, the aggregated positions for each alternative are

$$\begin{aligned} p_A(x_1) &= 3w_1^4 + 2w_2^4 + w_3^4 + w_4^4, \\ p_A(x_2) &= 3w_1^4 + 2w_2^4 + 2w_3^4 + w_4^4, \\ p_A(x_3) &= 3w_1^4 + 3w_2^4 + 2w_3^4 + w_4^4. \end{aligned}$$

Taking into account duplication, the tie among all three alternatives holds; hence

$$\begin{aligned} x_1 \sim_A x_2 &\Leftrightarrow w_3^4 = 0, \\ x_1 \sim_A x_3 &\Leftrightarrow w_2^4 + w_3^4 = 0, \\ x_1 \sim_A x_3 &\Leftrightarrow w_2^4 = 0. \end{aligned}$$

Then, $w_2^4 = w_3^4 = 0$. Once proven that central weights are null (this fact will be taken into account in what follows), what remains is to show that lateral weights in each side of the triangle should the same at any level, i.e., there exists $\alpha \in [0, 1]$ such that $w_1^m = \alpha$ and $w_m^m = 1 - \alpha$, for every $m \geq 2$. To do this, consider $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $p < q$, expressed as an irreductible fraction, and any profile with m voters and $q + 1$ alternatives where the alternative x_1 is at least the best for one voter and the worst for another one, while x_2 occupies the position $p + 1$ for all of them. A sketch of such *ad hoc* profile would be

position	R_1	$\cdots$	R_i	$\cdots$	R_j	$\cdots$	R_m
1	$\cdots$	$\cdots$	x_1	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$p+1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$q+1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	x_1	$\cdots$	$\cdots$

The aggregated positions for the selected alternatives would be

$$p_A(x_1) = \frac{p}{q}(q+1) + \left(1 - \frac{p}{q}\right) = p+1,$$

$$p_A(x_2) = \frac{p}{q}(p+1) + \left(1 - \frac{p}{q}\right)(p+1) = p+1,$$

so that $x_1 \sim_A x_2$, being A the voting rule corresponding to any EOWA with such weights.

But now, if we replicate any subset of voters becoming the new weights $\beta \neq \alpha$ and hence $1-\beta \neq 1-\alpha$, then the new aggregated positions would be

$$p_A(x_1) = \beta(q+1) + (1-\beta) \neq p+1,$$

$$p_A(x_2) = \beta(p+1) + (1-\beta)(p+1) = p+1,$$

so that $x_1 \sim_A x_2$ does not hold. Hence, if lateral weights change from one dimension to another, duplication fails. ■

In conclusion, under duplication we obtain the class of weighting triangles

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \alpha & & 1-\alpha & & \\ & \alpha & & 0 & & 1-\alpha & \\ \alpha & & 0 & & 0 & & 1-\alpha \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

As specific cases we have:

- $\alpha = 1$: maximum (maximin rule),
- $\alpha = 0$: minimum (maximax rule),
- $\alpha = 0.5$: mid-range.

It is worth mentioning that duplication is related to the *Hurwicz criterion* [28] used in decision making under complete uncertainty, where the value of a decision is a convex combination of its lowest possible expected value (pessimistic assessment) and of its highest one (optimistic assessment). On the other hand, although duplication might seem to be antidemocratic, Congar and Merlin [13] show that at least it is compatible with the basic democratic principle of anonymity and advocate it as a way "to protect the opinion of a minority against the will of the majority".

4.2. An illustrative example

Consider three voters that arrange three alternatives according to the following profile

R_1	R_2	R_3
x_2	$x_2\ x_3$	$x_1\ x_3$
x_3	x_1	x_2
x_1		

with associated position matrix

$$\begin{pmatrix} 3 & 1 & 2 \\ 3 & 1.5 & 1.5 \\ 1.5 & 3 & 1.5 \end{pmatrix}.$$

If we choose the OWA operator $A_{\boldsymbol{w}(\alpha)}$ associated with the weighting vector $\boldsymbol{w}(\alpha) = (\alpha, 0, 1-\alpha)$, with $\alpha \in [0,1]$, then the corresponding aggregated positions for the alternatives would be

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_1) = 3\,\alpha + 1.5\,(1-\alpha) = 1.5\,\alpha + 1.5,$$

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_2) = 3\,\alpha + 1\,(1-\alpha) = 2\,\alpha + 1,$$

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_3) = 2\,\alpha + 1.5\,(1-\alpha) = 0.5\,\alpha + 1.5.$$

According to the possible values of α, the corresponding social orders are the following

$\alpha = 0$	$0 < \alpha < \frac{1}{3}$	$\alpha = \frac{1}{3}$	$\frac{1}{3} < \alpha < 1$	$\alpha = 1$
x_2	x_2	$x_2 \quad x_3$	x_3	x_3
$x_1 \quad x_3$	x_3	x_1	x_2	$x_1 \quad x_2$
	x_1		x_1	

As one could expect, different social orders appear depending on α.

In the following subsection we propose an integrating method to obtain a unified result for each alternative taking into account the different outcomes when α ranges from 0 to 1.

4.3. Overall positions and social order

For the general case with n alternatives and using in a first stage the positional voting rule associated with the OWA operator of weighting vector $\boldsymbol{w}(\alpha) = (\alpha, 0, \ldots, 0, 1-\alpha)$, it is possible to assign the corresponding social position $p_{A_{\boldsymbol{w}(\alpha)}}(x_i)$ to the alternative x_i. Thus, we can introduce the function $\mu_i : [0,1] \longrightarrow \mathbb{R}$ defined as $\mu_i(\alpha) = p_{A_{\boldsymbol{w}(\alpha)}}(x_i)$. Such function is always a degree one polynomial (a linear function), and hence Riemann integrable. This fact allows us to define the *overall position of* x_i as

$$p(x_i) = \int_0^1 \mu_i(\alpha)\, d\alpha.$$

Easy computations lead to the following results in the previous example:

$$p(x_1) = \int_0^1 \mu_1(\alpha)\, d\alpha \quad = \quad \int_0^1 (1.5\,\alpha + 1.5)\, d\alpha \quad = 9/4,$$

$$p(x_2) = \int_0^1 \mu_2(\alpha)\, d\alpha \quad = \quad \int_0^1 (2\,\alpha + 1)\, d\alpha \quad = 2,$$

$$p(x_3) = \int_0^1 \mu_3(\alpha)\, d\alpha \quad = \quad \int_0^1 (0.5\,\alpha + 1.5)\, d\alpha \quad = 7/4.$$

Thus, the overall social order is $x_3 \succ x_2 \succ x_1$.

In conclusion, for each $\alpha \in [0,1]$ the corresponding positional voting rule associated with $A_{\boldsymbol{w}(\alpha)}$ only takes into account the best and worst positions for each alternative, yielding different social orders in each case. However, the possible criticism on the influence of the choice of α in the result can be mitigated under this overall approach, where a social order is obtained not corresponding with any predetermined α, but amalgamating all allowable values for this parameter.

5. Concluding remarks

In this paper we have presented a general framework for positional voting rules which includes all scoring rules as especial cases. To this aim, we need an aggregation process for obtaining a collective position from individual ones for each alternative. This is the reason why we have mainly used OWA operators, as they provide a comprehensive way to deal with this kind of information. More concretely, we have analyzed how the maximum and the minimum OWA operators induce the so called maximin and maximax voting rules, respectively, recently characterized by Congar and Merlin [13]. Of course, these rules are not scoring rules (they satisfy duplication, a property radically opposed to the continuity verified by the scoring rules) although all of them share interesting features due to their positional nature. A comprehensive diagram showing our knowledge of the logical relationship among these rules appears in Figure 3.

We have focused on the duplication property appearing in the above mentioned characterization. On our part, once introduced suitable extended notions

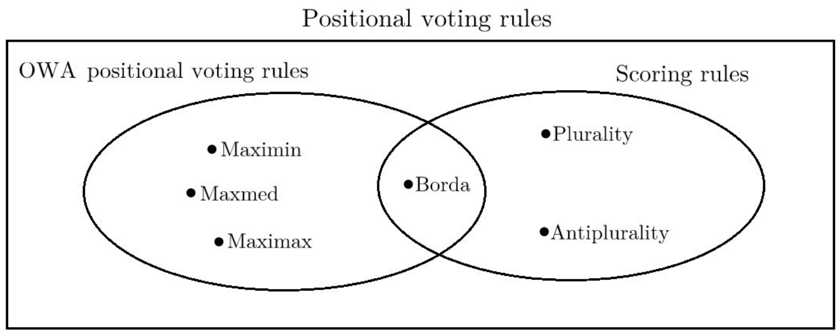

Figure 3: Relationship among positional voting rules

to take into account a variable electorate, we have characterized all EOWA-generated positional voting rules satisfying that property.

Some questions remain unanswered. As proven in Prop. 1, all scoring rules are positional voting rules. Even more, some scoring rules are positional voting rules generated by OWA operators. This is the case of the Borda rule, associated with the arithmetic mean. However, a characterization of the family of scoring rules that are generated by OWA operators (in other words, the relationship among scores and weights) is to be found. Additionally, families of aggregation functions other than OWA operators, such as quasiarithmetic means, could be also taken into account.

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