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Pairwise dichotomous cohesiveness measures

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Abstract In a framework where experts or agents express their opinions in a dichotomous way, we analyze the cohesiveness of their opinions on a fixed set of issues in a population. A parametric family of related measures are introduced and axiomatically characterized. They are ordinally equivalent when the population is fixed, and some further properties are proved. In order to argue that this restricted dichotomous situation is nevertheless versatile, the paper ends with several empirical illustrations based on real forecasts (for the 2012 American presidential election) and elections (with real data from referenda in two countries and from elections in several scientific societies).

Keywords Approval voting · Dichotomous opinions · Cohesiveness measures · Cohesiveness of opinions

Address(es) of author(s) should be given

1 Introduction

Reaching consensus in group decision making problems, as well as its measurement, are prominent and active research areas in Decision Making Theory (see Alonso et al (2009), Herrera-Viedma et al (2011), and Xu and Cai (2013), among others) and in Social Choice Theory (see Alcalde and Vorsatz (2013), Alcantud et al (2013) and Bosch (2005)). This increasing attention is motivated by real-life problems where one would like to assess how much consensus an arrangement or configuration conveys to the group.

In this paper we contribute to the question: how can the similarity of the opinions in a group be measured? In social debates one is often faced with assertions like "the more agreement on a list of issues there is in a collectivity, the greater its cohesion". However, how can we tell whether or not there is a high degree of agreement in the collectivity? We aim at contributing to that topic. To be precise, we focus on the case where the opinions of the group on the issues are dichotomous. Alcantud et al (2013) first posed this problem, which is both theoretically tractable and empirically versatile. In fact that paper mentions real-world situations where Approval Voting (AV) is applied as a first field of application. This is the case of many organizations and scientific societies: e.g., the Mathematical Association of America (MAA), the American Mathematical Society (AMS), the Institute for Operational Research and Management Sciences (INFORMS), the American Statistical Association (ASA), or the Society for Social Choice and Welfare as a sample. A second example from Alcantud et al (2013), Section 5, concerns the predictions made by three polling agencies about the results of the 2012 presidential elections in several states of the USA. Now by contrast with the voting situations, which are further explored in this paper, the alternatives (either party R or D would win the state) are evaluated by each outfit, and no comparison among them is involved or could be reasonably induced. Another problem that is usually modelled in terms of dichotomous opinions is group identification or the qualification problem (see Dimitrov (2011) for a recent survey). [1] As to normative analyses, Alcantud et al (2013) prove axiomatic characterizations of two families of elementary dichotomous cohesiveness measures.

The generic problem under inspection has been explored from the perspective of ordinal preferences in a few studies. An early antecedent in statistics is Kendall (1962), who proposed a well-known measure of the degree of similarity between two rankings, namely, Kendall's τ. In the seminal Bosch (2005) and the later work by Alcalde and Vorsatz (2013), particular formulations for absolute measures of consensus or cohesiveness whitin a group are proposed and axiomatized. Bosch dealt with some simple measures and then Alcalde and Vorsatz made a more elaborated analysis which included the characterization of a family of measures (and of a focal subfamily). In both cases it is assumed that the agents linearly order the alternatives. However in our analysis the

[1] We appeal to the standard model where each person either qualifies or disqualifies each member of the collective. Ju (2013) refers to a more general domain condition that allows neutral opinions.

agents are not asked to compare the alternatives but grade them instead. This nuance is important because axioms and properties designed for the analysis of preferences could be meaningless for analyzing dichotomous evaluations. Irrespective of the allowed description of the agents' attitudes, they partake of a list of internal characteristics whose cohesiveness could be used to assess similarity of the members of a group or group cohesion (Alcalde and Vorsatz, 2013, Section 1).

To fulfil our purpose we appeal to dichotomous consensus measures or DCMs, a technical tool introduced in Alcantud et al (2013).

Our first objective is to define some noteworthy examples of DCMs and explore their performance.

Firstly we introduce a measure of the cohesiveness of dichotomous opinions with a probabilistic interpretation. We call it *Pairwise dichotomous cohesiveness measure* (PDCM), a specific type of dichotomous cohesiveness measure (i.e., mappings from the class of dichotomous opinions on the set of options to the unit interval for which 1 is attained exactly when all opinions on each option are coincident). In our construction we first compute the degree to which the agents coincide in their evaluations of each alternative, and then we take the average of these values (one for each alternative). As is the case of the Shapley value or the Banzhaf power index, this has a probabilistic interpretation: it is the probability that for a randomly chosen option, two randomly chosen agents of the society have the same opinion upon it.

Secondly, we correct the PDCM in order to procure two appealing properties. Many authors assume that a perfectly divided society should have a null cohesiveness measure; this property is violated by PDCM. Besides, it could be the case that the size of the society is not known but only proportions of answers are available (e.g., the example in Subection 5.4 below). This setup would not permit to evaluate cohesiveness by PDCM. We prove that a minor modification of the PDCM, namely the *Modified* PDCM (MPDCM), permits to overcome these difficulties at the cost of losing its probabilistic interpretation.

Thirdly, we introduce an index, namely the *Proportional* PDCM (PPDCM), that shares nice features with the previous cases for large populations: it depends on proportions only, and when the number of agents is large it approximates the value of PDCM so it is virtually the probability measured by PDCM.

Our second objective is to define and axiomatically characterize a family of DCMs that abstracts the spirit of the previous ones. Three necessary and sufficient conditions are employed, and the aforementioned central instances are characterized by stipulating the content of one of them.

Finally, the paper provides several empirical illustrations. They show the applicability of our indexes to real situations with different characteristics, like forecasts (for the 2012 American presidential election) and elections (with data from real referenda in Switzerland and Italy, and from elections in scientific societies as the 2012 Council elections of the Society for Social Choice and Welfare).

The paper is structured as follows. Section 2 is devoted to introduce basic terminology, as well as the general concept of (normal) dichotomous cohesiveness measure. Section 3 introduces and discusses our three noteworthy specifications, namely PDCM, MPDCM, and PPDCM. Section 4 introduces our generic family of DCMs. We prove that our definition generates normal DCMs and includes the three aforementioned indexes. Then we provide three axioms that jointly characterize such family of DCMs. We also prove that all members of the family are ordinally equivalent in the common case where profiles with the same population size are compared. Our empirical illustrations are given in Section 5. We conclude in Section 6. An Appendix provides other relevant properties of our family of measures.

2 Notation and definitions

The remaining of this Section borrows notation and definitions from Alcantud et al (2013).

Let $X = \{x_1, ..., x_k\}$ be the finite set of k options, alternatives or candidates. It is assumed that X contains al least two alternatives, i.e., that the cardinality of X is greater or equal than 2, $|X| \geqslant 2$. Abusing notation, on occasions we refer to option x_s as option s for convenience. A population of agents or experts is a finite subset $\mathbf{N} = \{1, 2, ..., N\}$ of natural numbers.

We consider that each expert can vote for or approve of as many options, alternatives or candidates as he/she wishes, thus showing extreme and dichotomous opinions. In order to formalize these assessments Alcantud et al (2013) propose the following model:

Definition 1 A *dichotomous profile* of a society $\mathbf{N}$ on the set of alternatives X is an $N \times k$ matrix

$$M = \begin{pmatrix} M_{11} & \dots & M_{1k} \\ \vdots & \ddots & \vdots \\ M_{N1} & \dots & M_{Nk} \end{pmatrix}_{N \times k}$$

where M_{ij} is the opinion of the expert i over the alternative x_j, in the sense

$$M_{ij} = \begin{cases} 1 \text{ if expert } i \text{ approves the alternative } x_j, \\ 0 \text{ otherwise.} \end{cases}$$

We write $\mathbb{M}_{N \times k}$ for the set of all such $N \times k$ matrices. For convenience, $(1)_{N \times k}$ denotes the $N \times k$ matrix whose cells are universally equal to 1.

Remark 1 Alcantud et al (2013) discuss alternative formal modelizations of the same information on the agents' opinions that we do not use here.

Any permutation σ of the experts $\{1,2,...,N\}$ determines a profile M^σ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile M. Similarly, any permutation π of the alternatives $\{1,2,...,k\}$ determines a profile ${}^\pi M$ by permutation of the columns of M: column i of the profile ${}^\pi M$ is column $\pi(i)$ of the profile M.

For each dichotomous profile M, its restriction to a *subprofile* on the alternatives in $I \subseteq X$, denoted M^I, arises from exactly selecting the columns of M that are associated with the respective alternatives in I (in the same order). In particular, and dropping brackets for simplicity, $M^{\{j\}} = M^j$ is column j of M, and $M^{i,j}$ is the two-column submatrix of M that consists of columns i and j (in the same order). An s-restricted profile of M is the restriction of M to a subprofile on s alternatives.

Any partition $\{I_1,...,I_t\}$ of $\{1,2,...,k\}$, that we identify with a partition of X, generates a *decomposition of* M into subprofiles M^{I_1}, ... , M^{I_t}.[2]

For each dichotomous profile M on k alternatives, by n_0^j we denote the number of agents that disapprove of alternative j, and by n_1^j we denote the number of experts that approve of alternative j. When k equals 1, i.e., for profiles on one alternative, we drop the super-index to write n_0 and n_1. Of course, $N = n_0^j + n_1^j$ for each j.

A dichotomous profile M is *unanimous* if the set of approved alternatives is the same across experts. In matrix terms, if the columns of $M \in \mathbb{M}_{N\times k}$ are constant.

An *extension* of a dichotomous profile M of the society $\mathbf{N}$ on $X = \{x_1,...,x_k\}$ is a dichotomous profile $\tilde{M}$ on $\tilde{X} = \{x_1,...,x_k,,x_{k+1},...,,x_{k'}\}$ such that the restriction of $\tilde{M}$ to the first k alternatives of $\tilde{X}$ coincides with M.

An *expansion* of a dichotomous profile M of the society $\mathbf{N}$ on $X = \{x_1,...,x_k\}$ is a dichotomous profile $\bar{M}$ of a society $\bar{\mathbf{N}} = \{1,...,N,N+1,\ldots,\bar{N}\}$ on $X = \{x_1,...,x_k\}$ such that the restriction of $\bar{M}$ to the first N experts of $\mathbf{N}$ coincides with M.

Finally, a *replication* of a dichotomous profile M of the society $\mathbf{N}$ on $X = \{x_1,...,x_k\}$ is the dichotomous profile $M \uplus M \in \mathbb{M}_{2N\times k}$ obtained by duplicating each row of M, in the sense that rows t and $N+t$ of $M \uplus M$ are row t of M, for each $t = 1,\ldots,N$.

The main analytical tool for our purpose is the following:

Definition 2 A *dichotomous cohesiveness measure* (also DCM for simplicity) for the group $\mathbf{N} = \{1,...,N\}$ is a mapping $\mu : \mathbb{M}_{N\times k} \to [0,1]$ that assigns a number $\mu(M) \in [0,1]$ to each dichotomous profile M, with the property:

i) $\mu(M) = 1$ if and only if M is unanimous.

[2] A partition of a set S is a collection of pairwise disjoint non-empty subsets of S whose union is S.

A dichotomous cohesiveness measure is a collection of dichotomous cohesiveness measures for each group **N**.

Henceforth we restrict our attention to DCMs that are normal, in the following sense:

Definition 3 A DCM is *normal* if it further verifies:

ii) *Anonymity*: $\mu(M^\sigma) = \mu(M)$ for each permutation σ of the agents and $M \in \mathbb{M}_{N\times k}$
iii) *Neutrality*: $\mu({}^\pi M) = \mu(M)$ for each permutation π of the alternatives and $M \in \mathbb{M}_{N\times k}$

The inspiring Alcantud et al (2013) explained that further properties must be imposed on DCMs in order to avoid weird/odd behaviors. Moreover, DCMs that verify certain lists of properties are identified. Some of these properties capture very basic notions of cohesiveness as unanimity or perfect agreement, which are often restrictive. In the following Sections we investigate more sophisticated approaches to the measurement of cohesiveness under dichotomous opinions. They are intended to fit a wider range of applications.

All too often the dichotomous profile M cannot be fully retrieved. For example, when the available data are aggregated and the underlying dichotomous profile cannot be used to assess the cohesiveness of the opinions. This is usually the case when the agents are *individuals* rather than agencies (as in Subsection 5.1), political entities (as in Subsection 5.2), companies, fictitious or representative agents, ... In this case, what matters is the *number* of experts that approves of each alternative irrespective of the specific votes or ballots cast by each expert. In technical terms, one needs to consider the following restriction called Property (A).

Definition 4 With each dichotomous profile $M \in \mathbb{M}_{N\times k}$ we associate $P_M = (n_1^1, ..., n_1^k)$. The DCM μ verifies Property (A) when $M, M' \in \mathbb{M}_{N\times k}$ and $P_M = P_{M'}$ entail $\mu(M) = \mu(M')$.

Furthermore, in certain polls or surveys the data do not give information on the size of the populations, and only proportions of agents that approve or disapprove each alternative are available. This fact motivates the following restriction called Property (B).

Definition 5 With each dichotomous profile $M \in \mathbb{M}_{N\times k}$ we associate $V_M = (\frac{n_1^1}{N}, ..., \frac{n_1^k}{N})$. The DCM μ verifies Property (B) when $M \in \mathbb{M}_{N\times k}$, $M' \in \mathbb{M}_{N'\times k}$, $V_M = V_{M'}$ entail $\mu(M) = \mu(M')$.

3 Dichotomous Cohesiveness Measures: some noteworthy examples

In this Section we propose three prominent dichotomous cohesiveness measures. The first one has a probabilistic interpretation, since it is similar in spirit to Hays (1960)'s seminal work of a cohesiveness measure for profiles of linear orders. Our second proposal is a simple modification of the former in order to procure two appealing properties. Finally, the third one approximates the first measure when only proportions are available.

3.1 Pairwise Dichotomous Cohesiveness Measure (PDCM)

Definition 6 The *pairwise dichotomous cohesiveness measure* (PDCM) for the group $\mathbf{N} = \{1, ..., N\}$ is the mapping $\mathcal{C}_p : \mathbb{M}_{N\times k} \to [0, 1]$ given by

$$\mathcal{C}_p(M) = 1 - \frac{\sum_{j=1}^{k} n_0^j \cdot n_1^j}{k \cdot C_N^2} \tag{1}$$

for each dichotomous profile M on k alternatives, where k denotes the cardinality of the set of alternatives.

In words, PDCM measures the cohesiveness of a dichotomous profile as the probability that for a randomly chosen option, two randomly chosen agents of the group have the same opinion upon it. Some basic combinatorics prove that Eq. (1) captures such probability, which is expressed as the following sum of probabilities of k mutually exclusive conditional events:

$$\sum_{j=1}^{k} \frac{1}{k} \left(\frac{n_0^j(n_0^j - 1)}{N(N-1)} + \frac{(N - n_0^j)(N - n_0^j - 1)}{N(N-1)} \right),$$

or equivalently,

$$\frac{1}{k}\sum_{j=1}^{k} \frac{N^2 - 2Nn_0^j + 2(n_0^j)^2 - N}{N(N-1)} = \frac{1}{k}\sum_{j=1}^{k}\left(1 - \frac{2Nn_0^j - 2(n_0^j)^2)}{N(N-1)}\right) =$$

$$= 1 - \frac{1}{k}\sum_{j=1}^{k} \frac{Nn_0^j - (n_0^j)^2}{\frac{N(N-1)}{2}} = 1 - \frac{\sum_{j=1}^{k} n_0^j \cdot n_1^j}{k \cdot C_N^2}.$$

In addition, it is easy to check that Definition 6 provides a *normal* DCM and that PDCM satisfies Property (A) but fails to satisfy Property (B).

3.2 Modified Pairwise Dichotomous Cohesiveness Measure (MPDCM)

A possible handicap of PDCM is that it contradicts two properties that in certain contexts could be natural for a dichotomous cohesiveness measure, namely balancedness and replication invariance. The first one claims that cohesiveness is null when there is an exact half-half distribution of the opinions in the society. The second one claims that replicating a society preserves the index of cohesiveness. In technical terms:

1. A DCM μ is *balanced* if $\mu(M) = 0$ whenever $n_0^j = n_1^j$ for each $j = 1, ..., k$. To prove that PDCM is not balanced, observe that if $M \in \mathbb{M}_{N\times k}$ verifies $n_0^j = n_1^j$ for each j, then $\mathcal{C}_p(M) = 1 - 2\frac{k(n_0^1)^2}{kN(N-1)} = 1 - 2\frac{(\frac{N}{2})^2}{N(N-1)} > 0$.

2. Let us say that a DCM μ verifies *replication invariance* if $\mu(M \uplus M) = \mu(M)$ for each $M \in \mathbb{M}_{N\times k}$. [3] Note that PDCM does not verify replication invariance, because for each $N > 2$ and each non-unanimous $M \in \mathbb{M}_{N\times k}$ it is easy to show that $\mathcal{C}_p(M \uplus M) - \mathcal{C}_p(M) = \frac{1-\mathcal{C}_p(M)}{2N-1} \neq 0$.

In order to overcome these possible drawbacks we propose a technical modification of PDCM, namely the *modified pairwise dichotomous cohesiveness measure* (MPDCM), that satisfies the two properties above at the cost of losing the probabilistic and natural interpretation of PDCM. For each group $\mathbf{N} = \{1, ..., N\}$, it is defined as follows:

Definition 7 The *modified* PDCM (MPDCM) for the group $\mathbf{N} = \{1, ..., N\}$ is the mapping $\mathcal{C}_m : \mathbb{M}_{N\times k} \to [0, 1]$ given by

$$\mathcal{C}_m(M) = 1 - \alpha \cdot \frac{2 \cdot \sum_{j=1}^{j=k} n_0^j \cdot n_1^j}{k \cdot N(N-1)},$$

where $\alpha = 2 \cdot \frac{N-1}{N}$ for each dichotomous profile M on k alternatives. Therefore

$$\mathcal{C}_m(M) = 1 - \frac{4}{k} \frac{\sum_{j=1}^{j=k} n_0^j \cdot n_1^j}{N^2} = 1 - \frac{4}{k} \sum_{j=1}^{j=k} \frac{n_0^j}{N} \frac{n_1^j}{N}. \qquad (2)$$

It is easy to check that Definition 7 provides a *normal* DCM too. Observe that Eq. (2) defines MPDCM in terms of the *proportions* of agents that approve or disapprove each alternative, *i.e.* it satisfies Property (B). This fact decides the choice between MPDCM and PDCM when the data do not give information on the size of population, as as in certain polls or surveys.[4] Furthermore, it ultimately yields the property of replication invariance for MPDCMs. It is trivial to check for balancedness too.

3.3 Proportional Pairwise Dichotomous Cohesiveness Measure (PPDCM)

The evaluation of the cohesiveness of a profile under PDCM permits to interpret it as a probability but requires to know the size of the population. However, it can be approximated when the population is 'large' ($N \approx N - 1$) by a simplified formula, that only depends on proportions like in the case of MPDCM. This motivates the definition of the following DCM, namely *proportional pairwise dichotomous consensus measure*:

[3] This is the adapted version of the axiom with the same name in Alcalde and Vorsatz (2013) in the analysis of coherence in societies with linear orders.

[4] Subsection 5.4 below refers to this case in more detail.

Definition 8 The *proportional* PDCM (PPDCM) for the group $\mathbf{N} = \{1, ..., N\}$ is the mapping $\mathcal{C}'_p : \mathbb{M}_{N\times k} \to [0, 1]$ given by

$$\mathcal{C}'_p(M) = 1 - \frac{2}{k}\sum_{j=1}^{k} \frac{n_0^j}{N} \cdot \frac{n_1^j}{N} \tag{3}$$

A straightforward computation gives $\mathcal{C}_m(M) = 2\mathcal{C}'_p(M) - 1$ for each $M \in \mathbb{M}_{N\times k}$, henceforth we deduce $\mathcal{C}_m(M) \approx 2\mathcal{C}_p(M) - 1$ because $\mathcal{C}'_p(M) \approx \mathcal{C}_p(M)$ when N is 'sufficiently large'.

This is worth mentioning because $\mathcal{C}'_p$ verifies Property (B) and at the same time provides a reasonable approximation of PDCM –which does not verify it– for otherwise non-tractable cases like the Italian referenda example in Subsection 5.4.

4 A Family of Dichotomous Cohesiveness Measure: Axiomatic characterization

In this Section we propose and analyze a parametric family of dichotomous cohesiveness measures, that includes the three measures introduced in the previous Section. The family is parameterized by a function that weights the disagreement in terms of the size of the population.

Definition 9 The f-pairwise dichotomous cohesiveness measure (f-PDCM) for the group $\mathbf{N} = \{1, \ldots, N\}$ is the mapping $\mathcal{C}_f : \mathbb{M}_{N\times k} \to [0, 1]$ given by:

$$\mathcal{C}_f(M) = 1 - f(N)g(N, P_M), \tag{4}$$

where $f : \mathbb{N} \to (0, \frac{2(N-1)}{N}]$ and $g : \mathbb{M}_{N\times k} \to [0, 1]$ is the *disagreement quote* given by:

$$g(N, P_M) = g(N, \{n_1^j\}_{j=1}^k) = \sum_{j=1}^{k} \frac{(N - n_1^j)n_1^j}{k \cdot C_N^2}.$$

Note that the measures $\mathcal{C}_p$, $\mathcal{C}_m$ and $\mathcal{C}'_p$ respectively correspond to the cases $f(N) = 1$, $f(N) = \frac{2(N-1)}{N}$ and $f(N) = \frac{N-1}{N}$. Since $\mathcal{C}_p$ has a probabilistic interpretation, we can infer that the role of function $f(N)$ in Eq. (9) is to weight the disagreement quote, and to quantify the impact of the lack of agreement in the community.

The next lemma shows that each $\mathcal{C}_f$ is indeed a normal DCM.

Lemma 1 Every f-PDCM is a normal dichotomous cohesiveness measure.

Proof Firstly, we prove that $\mathcal{C}_f(M) \in [0, 1]$ for all profile $M \in \mathbb{M}_{M\times k}$. Since f and g are positive we infer

$$\mathcal{C}_f(M) = 1 - f(N)g(N, P_M) \leq 1.$$

On the other hand, we have

$$\mathcal{C}_f(M) \geq 0 \Leftrightarrow 1 - f(N)g(N, P_M) \geq 0 \Leftrightarrow f(N) \leq \frac{1}{g(N, P_M)}.$$

Now, a simple computation reveals that $\max_{M \in \mathbb{M}_{N\times k}} g(N, P_M) \leq \frac{N}{2(N-1)}$. Hence

$$\mathcal{C}_f(M) \geq 0 \Leftrightarrow f(N) \leq \frac{2(N-1)}{N}$$

which is the upper bound for f in Definition 9.

Anonymity and neutrality are easily checked. Finally, to show $\mathcal{C}_f(M) = 1$ if and only if M is unanimous, we observe that

$$\mathcal{C}_f(M) = 1 \Leftrightarrow g(N, P_N) = 0 \Leftrightarrow P_N = \begin{cases} (0, \dots, 0) \\ \text{or} \\ (N, \dots, N) \end{cases} \Leftrightarrow M \text{ is unanimous.}$$

This completes the proof. □

It is trivial that each f-PDCM verifies Property (A).

Now we proceed to list three properties that permit to completely characterize the f-PDCM family (cf., Theorem 1 below). Later on in Appendix 1, some further properties are discussed.

Let μ be a normal dichotomous cohesiveness measure.

1. We say that μ verifies 1-*reducibility* if for each dichotomous profile M on X,

$$\mu(M) = \frac{1}{k} \sum_{j=1}^{k} \mu(M^j).$$

   1-reducibility means that the cohesiveness of a dichotomous profile is the average of the cohesiveness measures of all its subprofiles on one alternative. As explained in the Introduction, this says that we can first compute a degree of coincidence in the evaluations of each alternative by the agents (the proportion of pairs of agents whose opinions coincide), and then aggregate these values by taking their average.

2. We say that the *cost of leaving unanimity* for μ is equal to $c(N)$ if the following holds true: when M is a unanimous profile on one alternative, and exactly one agent changes her/his opinion on it then the evaluation by μ is reduced by $c(N)$. The amount $\mu(M) - \mu(M') = 1 - \mu(M') = c(N)$ is called cost of leaving unanimity.

3. We say that μ verifies *proportionality* if the following holds true. Let $M, M', M'', M''' \in \mathbb{M}_{N\times 1}$ be dichotomous profiles on one alternative. Assume that when moving from M to M' (or from M' to M'', or from M'' to M''') exactly one agent that expresses a 0 opinion has changed to a 1 opinion. Then the difference in cohesiveness from the 1^{st} to the 4^{th} profiles

is three times as large as the difference in cohesiveness from the 2^{nd} to the 3^{rd} profile. Formally:

Let n_1, n_1', n_1'', n_1''' denote the number of agents that approve the unique alternative with the dichotomous profiles M, M', M'', M''', respectively. Assume that $n_1' = n_1 + 1$, $n_1'' = n_1' + 1$ and $n_1''' = n_1'' + 1$. Then,

$$\frac{\mu(M') - \mu(M'')}{\mu(M) - \mu(M''')} = \frac{1}{3}.$$

We can now characterize the family of f-pairwise dichotomous cohesiveness measures, in the following terms:

Theorem 1 Let μ be a dichotomous cohesiveness measure on X. Then $\mu = \mathcal{C}_f$ if and only if μ verifies 1-reducibility, proportionality, and its cost of leaving unanimity is $\frac{2}{N} f(N)$.

Proof Let us first prove necessity. In order to show 1-reducibility, note that for a profile $M \in \mathbb{M}_{N\times k}$

$$g(N, P_M) = \frac{1}{k} \sum_{i=1}^{k} \frac{n_1^i (N - n_1^i)}{C_N^2} = \frac{1}{k} \sum_{i=1} g(N, P_{M^i})$$

where M^i denotes the subprofile of M on the alternative i. Therefore

$$\mathcal{C}_f(M) = 1 - f(N) g(N, P_M) = 1 - \frac{1}{k} \sum_{i=1}^{k} f(N) g(N, P_{M^i}) = \frac{1}{k} \sum_{i=1}^{k} \mathcal{C}_f(M^i).$$

To prove that the cost of leaving unanimity for $\mathcal{C}_f$ is $\frac{2}{N} f(N)$, observe that when M is a unanimous profile on one alternative and M' is the new dichotomous profile where exactly one agent changes her/his opinion, a direct computation yields:

$$\mathcal{C}_f(M') = 1 - f(N)\frac{2}{N} = \mathcal{C}_f(M) - f(N)\frac{2}{N}.$$

In order to check for proportionality, and assuming the notation in its definition, we need to show

$$\frac{\mathcal{C}_f(M') - \mathcal{C}_f(M'')}{\mathcal{C}_f(M) - \mathcal{C}_f(M''')} = \frac{1}{3},$$

where M, M', M'', $M''' \in \mathbb{M}_{N\times 1}$ and

$$P_M = (n_1), \quad P_{M'} = (n_1 + 1), \qquad P_{M''} = (n_1 + 2), \qquad P_{M'''} = (n_1 + 3).$$

A straightforward computation gives:

$$\frac{\mathcal{C}_f(M') - \mathcal{C}_f(M'')}{\mathcal{C}_f(M) - \mathcal{C}_f(M''')} = \frac{g(N, n_1 + 2) - g(N, n_1 + 1)}{g(N, n_1 + 3) - g(N, n_1 + 3)}$$
$$= \frac{N - 2n_1 - 3}{3N - 6n_1 - 9} = \frac{1}{3}.$$

Let us now prove sufficiency. Due to 1-reducibility, we only need to solve the case of a single alternative, therefore let us first assume $k = 1$, and let M a dichotomous profile on one alternative. From proportionality one obtains that μ can be expressed as a function of n_1, and that it verifies the following difference equation:

$$\frac{\mu(n_1+1)-\mu(n_1+2)}{\mu(n_1)-\mu(n_1+3)} = \frac{1}{3}.$$

According to Greene and Knuth (1982), page 13, the solution of such equation is $\mu(n_1) = a + b \cdot n_1 + c \cdot n_1^2$ for some parameters a, b, c. Let us disclose their values.

Because μ is DCM, it fullfils unanimity, therefore $\mu(0) = a = 1$ (case $n_1 = 0$: unanimous rejection) and $\mu(N) = a + b \cdot N + c \cdot N^2 = 1$ (case $n_1 = N$: unanimous approval). Moreover, since the cost of leaving unanimity of μ is $\frac{2}{N}f(N)$, we obtain $\mu(n_1) = 1 - \frac{2}{N}f(N)$ when either $n_1 = 1$ or $n_1 = N - 1$ holds true. These equalities yield the following system of equations:

$$\begin{cases} b + c & = -\frac{2}{N}f(N) \\ b + c \cdot N & = 0. \end{cases}$$

From these equations we easily deduce

$$\mu(n_1) = 1 - f(N)\frac{2n_1(N-n_1)}{N(N-1)} = 1 - f(N)g(N, P_M).$$

In case of a general number of alternatives k, 1-reducibility applies and we obtain the desired conclusion $\mu = \mathcal{C}_f$. □

In particular, the previous Theorem provides an axiomatic characterization of the pairwise dichotomous measures introduced at Section 3. We rewrite this result as the following corollary.

Corollary 1 Let μ be a dichotomous cohesiveness measure on X. Then $\mu = \mathcal{C}_p$ (resp. $\mu = \mathcal{C}_m$, resp. $\mu = \mathcal{C}'_p$) if and only if μ verifies 1-reducibility, proportionality, and its cost of leaving unanimity is $\frac{2}{N}$, (resp. $\frac{4(N-1)}{N^2}$, resp. $\frac{2(N-1)}{N^2}$).

The choice of a prominent element of the family as a cohesiveness index is essentially technical because they are equivalent in the following sense:

Definition 10 Let μ, μ' be DCMs. We say that μ, μ' are equivalent if $\mu(M) \geqslant \mu(M') \Leftrightarrow \mu'(M) \geqslant \mu'(M')$, for each $M, M' \in \mathbb{M}_{N \times k}$.

Lemma 2 All f-DCMs are equivalent.

Proof It is a consequence of the fact that irrespective of the choice of f,

$$\mathcal{C}_f(M) \geq \mathcal{C}_f(M') \Leftrightarrow g(N, P_M) \leq g(N, P_{M'}), \quad \forall M, M' \in \mathbb{M}_{N \times k}.$$

□

Moreover, there exists the following relationship between the PDCM and any generic f-PDCM.

Lemma 3 For every $f : \mathbb{N} \to (0, \frac{2(N-1)}{N}]$,

$$\mathcal{C}_f(M) = f(N)\mathcal{C}_p(M) + (1 - f(N)) \tag{5}$$

Proof By Definition 9 and using Definition 6 twice we have

$$\begin{aligned}\mathcal{C}_f(M) &= 1 - f(N)g(N, P_M) = (\mathcal{C}_p(M) + g(N, P_M)) - f(N)(g(N, P_M) \\ &= \mathcal{C}_p(M) + (1 - f(N))(1 - \mathcal{C}_p(M)),\end{aligned}$$

and we conclude by grouping terms in $\mathcal{C}_p$. □

Therefore in the case of a fixed population, like the Swiss cantons exercise in Subsection 5.2 below, when we compare different dichotomous profiles on the basis of their cohesiveness appraisal by $\mathcal{C}_p$ we obtain the same ordering that if we proceed on the basis of any other $\mathcal{C}_f$ instead.

5 Discussion and real examples

In this Section we show the flexibility and applicability of our proposals. After discussing the informational basis of the model we exemplify their use in various real examples. They capture situations of different nature, both with individual and collective agents.

5.1 Predictions for the 2012 American presidential election

In this Subsection we compute our indexes for forecasts made for the 2012 American presidential election. The information as to the evaluations is complete thus in this exercise the corresponding approval profile is perfectly defined. To be concrete, the exercise consists of measuring the degree of agreement about the electoral predictions among the USA states made by several agencies. We focus on the forecasts made by the following polling agencies: New York Times[5], Real Clear politics[6], Dave Leip's Atlas of U.S. Presidential Elections[7], Election Projection[8], CNN[9] and The Huffington Post[10]. These agencies have been selected because they conducted surveys for each state.

Table 1 gathers the results of the respective surveys. For convention, '1', respectively '0', means a prediction that the Democrats, respectively the Republicans, would win the state. One can observe there are eight "swing states"

[5] Source: http://elections.nytimes.com/2012/results/president/exit-polls
[6] Source: http://www.realclearpolitics.com/polls/
[7] Source: http://uselectionatlas.org/PRED/
[8] Source: http://www.electionprojection.com/2012elections/president12.php
[9] Source: http://edition.cnn.com/ELECTION/2012/
[10] Source: http://elections.huffingtonpost.com/pollster

where the predictions vary with the agency: namely, Nevada, Colorado, Florida, Iowa, New Hampshire, Ohio, Virginia and Wisconsin. We obtain that the evaluation of the cohesiveness in the predictions is $0,9386$ according to PDCM, $0,8976$ according to MPDCM and $0,9488$ according to PPDCM. When we restrict the analysis to the aforementioned eight conflicting forecasts, the evaluations decrease to $0,6083$, $0,3472$ and $0,6736$ respectively.

5.2 Referenda in Switzerland: cohesiveness across cantons

In this Subsection we compute our indexes for results of referenda in Switzerland. Since we are using these data for illustrative purposes only, we restrict our inspection to referenda in 2012-2013. A more extensive discussion with data since 1991 is available in Alcantud and Muñoz-Torrecillas (2013). In this reference some related indexes are used to perform an aggregate, intertemporal analysis in order to give quantitative support to the existence of political periods with different characteristics.

We use aggregate information: for each referendum, we observe if each canton voted 'yes' or 'no'. Therefore as in Subsection 5.1, the information on the evaluations is complete and the corresponding dichotomous profile M is fully known and can be retrieved from the Swiss Federal Statistical Office. In view that both PDCM and MPDCM verify Property (A) we supply the corresponding V_M instead, which is easier to retrieve and display: for each referendum, the number of cantons (out of a total of 26) that voted 'yes' is needed. These figures are given in Table 2. The evaluation of the cohesiveness according to PDCM is $0,8562$, while it is $0,7234$ according to MPDCM and $0,8617$ according to PPDCM.

5.3 Elections in the Society for Social Choice and Welfare and other scientific societies

Below are but a few examples both of own elaboration and from previously published material, where voters are individual members of scientific societies.

1. Consider the data in Table 3. It captures the number of votes that each candidate received in the 2012 Council elections of the Society for Social Choice and Welfare (SSCW), where the Approval Voting mechanism was used for the renewal of 8 seats. A total of $N = 44$ votes were emitted. The evaluation of the cohesiveness according to measures introduced at Section 3 are $0,5693$ for PDCM, $0,1582$ for MPDCM and $0,5791$ for PPDCM.

2. Consider the analysis of the 2006 Public Choice Society election of a new president in Brams et al (2006). As in the case of SSCW, the voting rule was Approval Voting. The authors explain that "nominations were solicited from the membership in the fall of 2005, and five candidates agreed to run". They base their analysis on the outcome of the 36 ballots that indicated approval of between one and five candidates. There was another blank

ballot. Their conclusion is that "AV found a cohesiveness choice (candidate A)". Some computations show that the evaluation of the cohesiveness according to PDCM is $0,5225$, to MPDCM is $0,0709$ while it is $0,5354$ according to MPDCM. This indicates less cohesiveness in the *reported* opinions than in the case of the Council elections of the SSCW.[11]

3. Table 3 in Brams and Fishburn (2005) shows data from the 1988 IEEE election, where a total of $N = 54204$ voters cast their ballots under Approval Voting. There were 1100 blank ballots, and 523 voters approved all candidates. The evaluation of the cohesiveness according to PDCM is $0,5539$, which virtually coincides with the PPDCM index since the size of population is large enough, while it is $0,1078$ according to MPDCM.

5.4 'Large' populations and a case study: Italian referenda

Table 4 collects data from the eight Italian referenda since 2006 to the present. To compute PDCM one should know the (size of the) group for which it is applied. This is not known from publicly available data because abstentions, modifications of the censuses, et cetera do not permit to infer the total number of voters involved. Nevertheless for our purposes it seems clearly 'large'. Of course MPDCM and PPDCM can be computed directly. We obtain that their respective values are $0,5578$ and $0,7778$. The last one approximates the value of the PDCM index.

6 Concluding remarks and future research

In this paper we define measures of the cohesiveness of dichotomous opinions that are based on pairwise comparisons. For a fixed population, our indexes are ordinally equivalent: they can be interchangeably used to compare profiles in terms of their cohesiveness.

To motivate their appeal we have began by discussing intuitions and properties of three particular cases. Our first index (PDCM) has a probabilistic interpretation: it is the probability that for a randomly chosen option, two randomly chosen members of the society have the same opinion upon it. The second one (MPDCM) corrects the PDCM in order to procure two appealing properties. For the particular case where the number of agents is unknown, may vary with the issue, and only proportions of agents with the same opinion are available, a third proposal (PPDCM) is suggested. This index gives a nice approximation of PDCM when the population is large enough.

An inspection of the common traits of these related indexes suggests that a family of measures incorporating these three measures can be abstracted.

[11] We do not intend to draw conclusions from these figures. It is known that the adequacy of AV for the choice of committees and of single candidates is different. For sophisticated voters as the members of these societies are, the expression of their opinions can easily be influenced by the type of election.

We axiomatically characterize such new class by means of three properties. As corollaries, respective characterizations for the three particular indexes (PDCM, MPDCM, and PPDCM) follow. We also provide some further properties of our family of measures.

Several real empirical illustrations show the versatility of the model and the applicability of our indexes to a variety of real situations.

From another point of view, in contrast with the approach from social choice that we have followed in this work, Alcantud and de Andrés Calle (2014) introduce the idea of degree of consensus from a fuzzy viewpoint. To this purpose, the seminal contribution by Bosch (2005) and other related works are redefined in terms of fuzzy sets. This is an unexplored area where further research can be conducted.

7 Appendix 1: Further properties of f-PDCMs

As announced in Section 3, we proceed to prove additional properties for the family of f-PDCMs. Hence let us fix $f : (0, \frac{2(N-1)}{N}] \to \mathbb{R}$.

1. *Convexity.* For each dichotomous profile $M \in \mathbb{M}_{N\times k}$, and each decomposition of M into two subprofiles M_1 and M_2 with k_1 and k_2 columns respectively,

$$\mathcal{C}_f(M) = \frac{k_1\, \mathcal{C}_f(M_1) + k_2\, \mathcal{C}_f(M_2)}{k}$$

   Convexity means that the measure of a dichotomous profile is a weighted average of the measures of any decomposition into subprofiles, their weights being given by their respective relative sizes.

   Assuming the notation of the previous paragraph, note that convexity is a consequence of the fact that

$$g(N, P_M) = \frac{1}{k}\sum_{i=1}^{k} \frac{n_1^i (N - n_1^i)}{C_N^2} = \frac{k_1}{k} g(N, P_{M_1}) + \frac{k_2}{k} g(N, P_{M_2}).$$

   It is clear that convexity implies 1-reducibility.

2. *Reversal invariance.* The complementary dichotomous profile of M, namely $M^c = (1)_{N\times k} - M$, produces the same cohesiveness as M, i.e.,

$$\mathcal{C}_f(M) = \mathcal{C}_f(M^c)$$

   This fact is trivial because $n_0^j \cdot n_1^j = (n_0^c)^j \cdot (n_1^c)^j$ where $(n_0^c)^j$ resp., $(n_1^c)^j$, denotes the number of agents that disapprove of, resp. approve of, alternative j in M^c.

3. *Convergence to unanimity.* If we repeatedly introduce alternatives with the property that all agents agree on their acceptability then cohesiveness approaches 1. Formally: Suppose that alternatives $k+1, ..., k+t$ are added to the set of alternatives X, and that each alternative is either unanimously approved or unanimously disapproved by all agents. If the introduction of new alternatives does not affect the agents' assessments of past sets of alternatives, then the cohesiveness measurement of the extended dichotomous profiles $\tilde{M}^{(t)}$ approaches 1 when t tends to infinity.

   We want to show $\lim_{t\to\infty} \mathcal{C}_f(\tilde{M}^{(t)}) = 1$ under the aforementioned conditions. Since

$$\begin{aligned}\lim_{t\to\infty} \mathcal{C}_f(\tilde{M}^{(t)}) &= \lim_{t\to\infty}\left(1 - f(N+t)\frac{\sum_{j=1}^{k+t} n_0^j n_1^j}{(k+t)\cdot C_N^2}\right)\\ &= \lim_{t\to\infty}\left(1 - f(N+t)\left(\frac{\sum_{j=1}^{k} n_0^j n_1^j}{(k+t)\cdot C_N^2} - \frac{\sum_{j=k+1}^{t} 0N}{(k+t)\cdot C_N^2}\right)\right)\\ &= \lim_{t\to\infty}\left(1 - f(N+t)\frac{2\sum_{j=1}^{k} n_0^j n_1^j}{(k+t)N(N-1)}\right)\end{aligned}$$

   then the thesis ensues because f is bounded.

4. *Replication monotonicity.* If M is not a unanimous dichotomous profile then $\mathcal{C}_f(M \uplus M) > \mathcal{C}_f(M)$ whenever

$$\frac{f(2N)}{f(N)} < \frac{2N-1}{2(N-1)}, \tag{6}$$

   i.e., replicating a non-unanimous profile produces a higher evaluation by the f-PDCMs that satisfies (6). In fact

$$\mathcal{C}_f(M \uplus M) > \mathcal{C}_f(M) \Leftrightarrow \frac{f(2N)}{f(N)} < \frac{g(N, P_M)}{g(2N, P_{M\uplus M})} = \frac{2N-1}{2(N-1)}.$$

   In particular PDCM and PPDCM verify replication monotonicity. Of course, each DCM μ verifies $\mu(M \uplus M) = \mu(M) = 1$ when M is unanimous due to Definition 2 i).

5. *Replication invariance.* If M is not a unanimous dichotomous profile then $\mathcal{C}_f(M \uplus M) = \mathcal{C}_f(M)$ whenever

$$\frac{f(2N)}{f(N)} = \frac{2N-1}{2(N-1)}. \tag{7}$$

   The proof mimics the previous one. In particular, MPDCM satisfies Replication invariance.

Note that this property and Replication monotonicity are mutually exclusive thus PDCM and PPDCM fail to verify Replication invariance.

6. *Monotonicity in dominant opinions.* Let $M \in \mathbb{M}_{N\times k}$ be fixed. Suppose $n_1^j \geqslant n_0^j$ (resp., $n_0^j \geqslant n_1^j$), i.e., for alternative j the dominant opinion is approval (resp., rejection). If $\widehat{M}$ is identical to M except in the opinion of an agent on alternative j, and this agent's opinion shifts from rejection to approval (resp., from approval to rejection), then $\mathcal{C}_f(\widehat{M}) > \mathcal{C}_f(M)$. To prove it,

$$\mathcal{C}_f(\widehat{M}) > \mathcal{C}_f(M) \Leftrightarrow \mathcal{C}_f(\widehat{M}) - \mathcal{C}_f(M) > 0$$

By convexity, we only need to solve the case of a single alternative, therefore let us first assume $k = 1$, $\widehat{n}_1 = n_1 + 1$ and $\widehat{n}_0 = n_0 - 1$. Then it is necessary to verify:

$$\begin{aligned} &\left[1 - f(N)g(N, P_{\hat{M}})\right] - \left[1 - f(N)g(N, P_M)\right] > 0 \\ &\Leftrightarrow f(N)\left[g(N, P_M) - g(N, P_{\hat{M}})\right] > 0 \end{aligned}$$

Since f is positive, this latter inequality reduces to $\frac{2\cdot(2n_1 - N + 1)}{N(N-1)} > 0$ or simply $n_1 - n_0 + 1 > 0$. The assumption $n_1 > n_0$ yields this fact.

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Table 1 Forecasts for the 2012 USA presidential election

	State									
	Ca	Co	DC	Del	Hawaii	Ill	Mass	Md	Me	NJ
New York Times	1	1	1	1	1	1	1	1	1	1
Real Clear politics	1	1	1	1	1	1	1	1	1	1
Dave Leip's Atlas	1	1	1	1	1	1	1	1	1	1
Election Projection	1	1	1	1	1	1	1	1	1	1
CNN	1	1	1	1	1	1	1	1	1	1
The Huffington Post	1	1	1	1	1	1	1	1	1	1
Total (n_1^j)	6	6	6	6	6	6	6	6	6	6

	State									
	NY	Ore	RI	Vet	Wash	Mich	Minn	NM	Nev	Pa
New York Times	1	1	1	1	1	1	1	1	1	1
Real Clear politics	1	1	1	1	1	1	1	1	1	1
Dave Leip's Atlas	1	1	1	1	1	1	1	1	1	1
Election Projection	1	1	1	1	1	1	1	1	1	1
CNN	1	1	1	1	1	1	1	1	0	1
The Huffington Post	1	1	1	1	1	1	1	1	1	1
Total (n_1^j)	6	6	6	6	6	6	6	6	5	6

	State									
	Colo	Fla	Iowa	NH	Ohio	Va	Wis	Ariz	NC	Alaska
New York Times	1	1	1	1	1	1	1	0	0	0
Real Clear politics	1	0	1	1	1	1	1	0	0	0
Dave Leip's Atlas	1	0	1	1	1	1	1	0	0	0
Election Projection	1	0	1	1	1	1	1	0	0	0
CNN	0	1	0	0	0	0	0	0	0	0
The Huffington Post	0	1	1	1	1	1	1	0	0	0
Total (n_1^j)	4	3	5	5	5	5	5	0	0	0

	State										
	Ala	Ark	Ga	Idaho	Ind	Kan	Ky	La	Mo	Miss	Mont
New York Times	0	0	0	0	0	0	0	0	0	0	0
Real Clear politics	0	0	0	0	0	0	0	0	0	0	0
Dave Leip's Atlas	0	0	0	0	0	0	0	0	0	0	0
Election Projection	0	0	0	0	0	0	0	0	0	0	0
CNN	0	0	0	0	0	0	0	0	0	0	0
The Huffington Post	0	0	0	0	0	0	0	0	0	0	0
Total (n_1^j)	0	0	0	0	0	0	0	0	0	0	0

	State									
	ND	Neb	Okla	SC	SD	Tenn	Tex	Utah	W.Va	Wyo
New York Times	0	0	0	0	0	0	0	0	0	0
Real Clear politics	0	0	0	0	0	0	0	0	0	0
Dave Leip's Atlas	0	0	0	0	0	0	0	0	0	0
Election Projection	0	0	0	0	0	0	0	0	0	0
CNN	0	0	0	0	0	0	0	0	0	0
The Huffington Post	0	0	0	0	0	0	0	0	0	0
Total (n_1^j)	0	0	0	0	0	0	0	0	0	0

Table 2 Votes in 2012 and 2013 in Switzerland. Source: http://www.bfs.admin.ch

2012

Initiative No.	1	2	3	4	5	6	7	8	9	10	11	12
No. cantons ‘yes’	15	5	0	26	6	0	0	0	26	10	1	24

2013

Initiative No.	1	2	3	4	5	6	7	8	9	10	11
No. cantons ‘yes’	11	26	25	0	26	0	22	21	0	3	0

Table 3 The 2012 Council elections of the Society for Social Choice and Welfare

Candidates [a]	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Votes	42	18	23	21	23	40	26	33	23	18	13	19	16	15

[a] The 14 candidates are anonymous and randomly ordered.

Table 4 Results of the Italian referenda in the 2006-2011 period. Source: Italian Secretary of State, Archivio Storico delle Elezioni (http://elezionistorico.interno.it/)

	2006	2009		
	I	I	II	III
Approve (%)	38.71	77.63	77.68	87.00
Disapprove (%)	61.29	22.37	22.32	13.00
Valid votes	25,753,782	10,372,226	10,362,230	10,908,329

	2011			
	I	II	III	IV
Approve (%)	95.35	95.80	94.05	94.62
Disapprove (%)	4.65	4.20	5.95	5.38
Valid votes	27,200,859	27,277,283	27,265,741	27,197,124