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ELEMENTARY PARTICLES AND FIELDS
Theory

From $osp(1|32) \oplus osp(1|32)$ to the M-Theory Superalgebra: a Contraction Procedure*

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Abstract—We show the impossibility to obtain the D'auria—Fré-type superalgebras that allow for an underlying gauge theoretical structure of $D = 11$ supergravity from the superalgebra $osp(1|32)_+ \oplus osp(1|32)_-$, by means of a Weimar-Woods contraction.

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1. INTRODUCTION

In the original paper where supergravity theory in $D = 11$ was introduced, Cremmer, Julia, and Scherk (CJS) [1] raised the question of the identification of its underlying gauge symmetry group. They conjectured that the theory could admit a geometrical interpretation in terms of the simple supergroup $OSp(1|32)$. The evidence in favor of this suggestion was the fact that its graded Lie algebra $osp(1|32)$ contains an internal $o(8)$ subalgebra, which is also a subalgebra of the internal invariance group of a $D = 4$ reduction of the $D = 11$ model.

However, the lack of understanding about how this connection could be realized, was caused because of the presence of a three-form field $A_3 = A_{\mu\nu\rho}(x)\, dx^\mu \wedge dx^\nu \wedge dx^\rho$ in the action found in [1]. While the graviton $e^a = e^a_\mu dx^\mu$, the gravitino $\psi^\alpha = \psi^\alpha_\mu dx^\mu$ and the spin connection $w^{ab} = w^{ab}_\mu dx^\mu$ one-forms can be considered as the gauge fields of a Lie superalgebra, the antisymmetric $A_{\mu\nu\rho}(x)$ gauge field cannot be associated to a symmetry operator in an easy way. D'Auria and Fré [2], addressed this problem by looking at the free differential algebra (FDA) satisfied by the above forms in the absence of curvatures. The FDA formalism does not consist only of one-forms, so it is the natural extension of the Lie algebras, being particulary suitable to account the three-form field mentioned above. D'Auria and Fré's idea was to express A_3 in terms of linear combinations of exterior products of one-forms, treated as gauge fundamental fields belonging to a certain superalgebra which had to be found. For this to be possible, it was necessary to introduce a set of additional one-forms, two of them bosonic fields $B^{ab} = B^{ab}_\mu dx^\mu$ and $B^{a_1\dots a_5} = B^{a_1\dots a_5}_\mu dx^\mu$, and one extra fermionic contribution $\psi'^\alpha = \psi'^\alpha_\mu dx^\mu$, which play a central role in the new algebra. Consequently, the composite nature of the three-form field A_3 required extending the underlying gauge group of $D = 11$ CJS supergravity into a new superalgebra with larger algebraic dimension (hereafter $\mathfrak{E}^{(528|32+32)}$).

Two superalgebras were obtained which allowed the decomposition of A_3 in that way. The question about how these superalgebras could be related to a simple supergroup was studied in [3], where the $osp(1|32)$, as well as the $su(32|1)$ and the conformal $osp(1|64)$ superalgebras, were ruled out as algebras that could lead to the D'Auria—Fré ones by contraction.

Nevertheless, the semisimple superalgebra $osp(1|32) \oplus osp(1|32)$ was not in the above list, but this is the algebra that was later considered by Hořava [4] as a prospective candidate to construct a Chern—Simons (CS) M-theory group on a holographic scenario. The reason for this choice relies on the fact that the CS action must be parity invariant in the Hořava—Witten construction [5], which is based on the properties of the heterotic string theory [6], and a single $osp(1|32)$ superalgebra does not yield parity invariance [7—10]. Rather, parity invariance will require a non-minimal extension of the $osp(1|32)$ superalgebra into an algebra with 64 supercharges.

Another implication of Hořava's suggestion is that $D = 11$ CJS supergravity would be a low-energy limit of a CS theory based on $osp(1|32) \oplus osp(1|32)$.

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In [4, 11], it was assumed that the supersymmetry group in the low-energy limit had to be a contraction of $osp(1|32) \oplus osp(1|32)$. The contraction problem was considered in [11], where the superalgebras obtained, although with the same structure, did not coincide with those originally found by D'Auria and Fré. The question remained of interpreting this discrepancy.

The two superalgebras found in [2] were shown to be two particular elements of an infinite set $\mathfrak{G}(s) = \mathfrak{E}^{(528|32+32)}(s) \rtimes so(10,1)$ parametrized by one real parameter s [12, 13] (see Section 3). Moreover, all values of s except $s = 0$, allow for the decomposition of the three-form field A_3 in terms of combinations of one-forms dual to the generators of the algebra. In fact, this particular case $\mathfrak{G}(0)$ for which it is not possible make such decomposition can be obtained via an expansion procedure from $osp(1|32)$. An expansion of a given Lie (super)algebra is obtained by a suitable rescaling of the Maurer–Cartan (MC) dual one-forms in terms of a parameter λ [14–16]. The resulting (super)algebra is expressed by the coefficients of each power in λ in the resulting MC equations. In this way, superalgebras with an infinite number of generators are obtained, so this process does not preserve the dimension of the start handle algebra. Certain conditions can be imposed in order to ensure that, by cutting the expansion in λ up to a finite power, the resulting equations are the MC equations of a finite (super)algebra [17].

In all the examples obtained so far, the resulting expansions can be viewed as extensions followed by contractions, and this will presumably be true in general. However, the inverse statement is obviously false: contractions are more general than expansions in the sense that the latter remember the structure of the original (super)algebra, whereas the former procedure leads to more possibilities. Nevertheless, taking two copies of the same algebra would reduce the freedom associated to contraction, and it is not clear whether a contraction of $osp(1|32) \oplus osp(1|32)$ leads to an expansion $\mathfrak{G}(0)$ or to the other class of the set $\mathfrak{G}(s \neq 0)$.

For the above reasons, we have made a detailed computation of all possible contractions of $osp(1|32) \oplus osp(1|32)$ leading to a superalgebra with the generic structure

$$\left(\mathfrak{E}^{(528|32+32)}(s) \oplus \mathcal{L}^{(473)}\right) \rtimes so(10,1), \qquad (1)$$

where $\mathcal{L}$ is an arbitrary superalgebra, not necessarily abelian, that mixes in a trivial way with the D'Auria–Fré superalgebra, and has to be present because the contraction procedure is dimension preserving, and the dimensions of $osp(1|32) \oplus osp(1|32)$ and $\mathfrak{G}(s)$ do not match.

Our study states that it is only possible to obtain the expansion case ($s = 0$) by contraction procedure from the $osp(1|32) \oplus osp(1|32)$ superalgebra. Therefore, none of the Lie superalgebras, suitable for decomposing the three-form A_3 of $D = 11$ supergravity in terms of MC one-forms, can be obtained by contraction from the direct sum of two $osp(1|32)$ algebras.

The paper is organized as follows: in Section 2 a brief review of $osp(1|32)$ is presented. In Section 3 we show the main properties of the D'Auria–Fré family superalgebras. Lie algebra and Maurer–Cartan one-form languages are used in both sections. Section 4 is devoted to showing the main details in the contraction procedure from the two copies of $osp(1|32)$. Finally, we collect some conclusions in the last section.

2. THE SUPERALGEBRA $osp(1|32)$

The orthosymplectic supergroup $OSp(1|32)$ defines the minimal grading of the symplectic $Sp(32)$ bosonic group which, in turn, is the maximal group preserving the Majorana property of the $SO(10,1)$ spinors. Since its algebraic counterpart $osp(1|32)$ verifies the inclusion $so(10,1) \subset sp(32) \subset osp(1|32)$, its bosonic generators $P_a, J_{ab}, Z_{a_1\ldots a_5}$ could be directly associated to the even symmetry operators of the $D = 11$ superPoincaré bosonic extended superalgebra [18–20].

The orthosymplectic Lie algebra $osp(1|32)$ can be defined, in a certain basis $\{Z_{\alpha\beta},\ Q_\gamma\}$, by the following anticommutator and commutator relations:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \eta Z_{\alpha\beta}, \\ [Z_{\alpha\beta}, Q_\gamma] &= C_{\alpha\gamma} Q_\beta + C_{\beta\gamma} Q_\alpha, \\ [Z_{\alpha\beta}, Z_{\gamma\delta}] &= C_{\alpha\gamma} Z_{\beta\delta} + C_{\beta\gamma} Z_{\alpha\delta} \\ &\quad + C_{\alpha\delta} Z_{\beta\gamma} + C_{\beta\delta} Z_{\alpha\gamma}, \end{aligned} \qquad (2)$$

where $Z_{\alpha\beta}$ is a symmetric matrix in the spinorial indices ($\alpha, \beta, \gamma = 1, \ldots, 32$), which are raised and lowered by the 32×32 skewsymmetric charge conjugation matrix $C_{\alpha\beta}$. We point out that, in the first relation, the parameter η may take the values ± 1, but these choices do not make any difference in the complex Lie algebra in contrast with the real case, where they determine two nonisomorphic superalgebras denoted by $osp_+(1|32)$, $osp_-(1|32)$, as it happens in the the case of $osp(1|2)$ (see [21]).

Decomposing $Z_{\alpha\beta}$ in the basis of the $\mathrm{Spin}(1,10)$ gamma-matrices, we can express it in terms of the usual tensorial generators Z_a, Z_{ab}, $Z_{a_1\ldots a_5}$, as

$$\begin{aligned} Z_{\alpha\beta} &= \frac{1}{1!\cdot 32}\Gamma^a_{\alpha\beta} Z_a + \frac{1}{2!\cdot 32}\Gamma^{ab}_{\alpha\beta} Z_{ab} \\ &\quad + \frac{1}{5!\cdot 32}\Gamma^{a_1\ldots a_5}_{\alpha\beta} Z_{a_1\ldots a_5}, \end{aligned} \qquad (3)$$

where the notation $\Gamma^{a_1\ldots a_n}_{\alpha\beta} = (\Gamma^{a_1\ldots a_n}C^{-1})_{\alpha\beta}$, has been used to denote generically the antisymmetrized products of $D = 11$ Dirac matrix.

Using this last relation (3) in (2), we obtain the commutator and anticommutator relations [22, 23]

$$
\begin{aligned}
&[Z_a, Z_b] = \frac{1}{8}J_{ab},\\
&[Z^a, J_{b_1b_2}] = \frac{1}{4}\delta^a_{[b_1}\delta^k_{b_2]}Z_k,\\
&[J^{a_1a_2}, J_{b_1b_2}] = \frac{1}{2}\delta^{[a_1}_{[k_1}\delta^{a_2]}_{[b_1}\delta^{k_2]}_{b_2]}J^{k_1}{}_{k_2},\\
&[Z_a, Z_{b_1\ldots b_5}]\\
&= \frac{i}{8\cdot 5!}\epsilon_{c_5\ldots c_1k_1k_2\ldots k_6}\delta^{[k_1}_a\delta^{k_2}_{[b_1}\ldots\delta^{k_6]}_{b_5]}Z^{c_1\ldots c_5},\\
&[J^{a_1a_2}, Z_{b_1\ldots b_5}] = \frac{5}{4}\delta^{[a_1}_{[k_1}\delta^{a_2]}_{[b_1}\delta^{k_2}_{b_2}\ldots\delta^{k_5]}_{b_5]}Z^{k_1}{}_{k_2\ldots k_5},\\
&[Z^{a_1\ldots a_5}, Z_{b_1\ldots b_5}]\\
&= \frac{i}{8}\delta^{[a_1}_{[k_1}\ldots\delta^{a_5]}_{k_5}\delta^{k_6}_{[b_1}\ldots\delta^{k_{10}]}_{b_5]}\epsilon^{k_1\ldots k_5}{}_{k_6\ldots k_{10}c}Z^c\\
&\quad + \frac{5i}{4!}\delta^{[a_1}_{[k_1}\delta^{a_2}_{k_2}\delta^{a_3}_{k_3}\delta^{a_4}_{[b_2}\delta^{a_5]}_{b_1}\delta^{[k_4}_{b_3}\delta^{k_5}_{b_4}\delta^{k_6]}_{b_5]}\\
&\quad\times \epsilon^{k_1k_2k_3}{}_{k_4k_5k_6c_5c_4c_3c_2c_1}\\
&\quad\times Z^{c_1c_2c_3c_4c_5} + 75\,\delta^{[a_1}_{[k_1}\delta^{a_2}_{[b_4}\ldots\delta^{a_5]}_{b_1}\delta^{k_2]}_{b_5]}J^{k_1}{}_{k_2},\\
&[Z_a, Q_\alpha] = \frac{1}{16}(\Gamma_a)_\alpha{}^\beta Q_\beta,\\
&[J_{ab}, Q_\alpha] = -\frac{1}{16}(\Gamma_{ab})_\alpha{}^\beta Q_\beta,\\
&[Z_{a_1\ldots a_5}, Q_\alpha] = \frac{1}{16}(\Gamma_{a_1\ldots a_5})_\alpha{}^\beta Q_\beta,\\
&\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta}Z_a + \frac{1}{2!}\Gamma^{ab}_{\alpha\beta}J_{ab}\\
&\quad + \frac{1}{5!}\Gamma^{a_1\ldots a_5}_{\alpha\beta}Z_{a_1\ldots a_5}, \qquad (4)
\end{aligned}
$$

where the square brackets in the r.h.s. denote antisymmetrization with height one. Remember that the bosonic generators J_{ab} and $Z_{a_1\ldots a_5}$ are associated to even symmetry operators of the $D = 11$ super-Poincaré bosonic extended superalgebra.

It is convenient to resort to a dual point of view to deal with Lie algebras, in agreement with the FDA context developed by D'Auria and Fré. If we use the dual algebra spanned by the MC one-forms $\Pi_{\alpha\beta}$, Π_α dual to the algebraic generators $Z_{\alpha\beta}$, Q_α, which verify the identities

$$
\Pi^{\alpha\beta}(Z_{\gamma\delta}) = 2\delta^{(\alpha}_\gamma\delta^{\beta)}_\delta \equiv \delta^\alpha_\gamma\delta^\beta_\delta + \delta^\beta_\gamma\delta^\alpha_\delta,\quad \Pi^\alpha(Q_\beta) = \delta^\alpha_\beta,
$$

the Eqs. (2) can be rewritten in a compact form by the MC close relations [24]

$$
d\Pi_{\alpha\beta} = -(\Pi_{\alpha\gamma}\wedge\Pi^\gamma{}_\beta) - \eta(\Pi_\alpha\wedge\Pi_\beta), \qquad (5)
$$
$$
d\Pi_\alpha = -\Pi_{\alpha\gamma}\wedge\Pi^\gamma,
$$

which provide the 528 bosonic MC one-forms of the $sp(32)$ algebra through the symmetric spin-tensor $\Pi_{\alpha\beta}$, and the 32 fermionic MC one-forms Π_α.

Expressing the spinorial symmetric one-form $\Pi_{\alpha\beta}$ in terms of the MC one-forms Π^a, Π^{ab}, $\Pi^{a_1\ldots a_5}$ dual to the algebraic generators Z_a, Z_{ab}, $Z_{a_1\ldots a_5}$ respectively, the couple of MC equations (5) can be split as

$$
\begin{aligned}
d\Pi^a &= -\frac{1}{8}\left(\Pi^b\wedge\Pi_b{}^a\right) - \frac{1}{2}\Gamma^a_{\alpha\beta}\left(\pi^\alpha\wedge\pi^\beta\right)\\
&\quad - \frac{i}{16(5!)^2}\epsilon^{ab_1\ldots b_5}{}_{c_1\ldots c_5}(\Pi_{b_1\ldots b_5}\wedge\Pi^{c_1\ldots c_5}),\\
d\Pi^{ab} &= -\frac{1}{8}\left(\Pi^a\wedge\Pi^b\right) - \frac{1}{8}\left(\Pi^{ac}\wedge\Pi_c{}^b\right)\\
&\quad - \frac{1}{2}\Gamma^{ab}_{\alpha\beta}\left(\pi^\alpha\wedge\pi^\beta\right) - \frac{1}{4!\cdot 8}\left(\Pi^a{}_{c_1\ldots c_4}\wedge\Pi^{c_1\ldots c_4b}\right),\\
d\pi^\alpha &= \frac{1}{16}(\Gamma_a)_\beta{}^\alpha\left(\pi^\beta\wedge\Pi^a\right) - \frac{1}{2\cdot 16}(\Gamma_{ab})_\beta{}^\alpha\\
&\quad\times\left(\pi^\beta\wedge\Pi^{ab}\right) + \frac{1}{5!\cdot 16}(\Gamma_{a_1\ldots a_5})_\beta{}^\alpha\left(\pi^\beta\wedge\Pi^{a_1\ldots a_5}\right),\\
d\Pi^{a_1\ldots a_5} &= -\frac{i}{5!\cdot 8}\epsilon_{cb_1\ldots b_5}{}^{a_1\ldots a_5}\left(\Pi^c\wedge\Pi^{b_1\ldots b_5}\right)\\
&\quad - \frac{5}{8}\left(\Pi^{[a_1}{}_b\wedge\Pi^{ba_2\ldots a_5]}\right) - \frac{1}{2}\Gamma^{a_1\ldots a_5}_{\alpha\beta}\left(\pi^\alpha\wedge\pi^\beta\right)\\
&\quad - \frac{i}{2\cdot(4!)^2}\epsilon^{a_1\ldots a_5b_1b_2b_3}{}_{c_1c_2c_3}\\
&\quad\times\left(\Pi_{b_1\ldots b_5}\wedge\Pi^{b_5b_4c_1c_2c_3}\right). \qquad (6)
\end{aligned}
$$

which provide the same information as its algebraic counterpart described by the (anti)commutator relations (4).

3. THE D'AURIA–FRÉ SUPERALGEBRAS

The two solutions found originally by D'Auria and Fré can be identified as two examples of an infinite family of superalgebras $\mathfrak{G}(s) = \mathfrak{E}^{(528|32+32)}(s) \rtimes so(10,1)$, which solved in general the problem posed by them. All of these superalgebras contain a set of 528 bosonic and $32 + 32 = 64$ fermionic generators, plus the Lorentz generators J_{ab}, and are defined through the (anti)commutator relations

$$
\begin{aligned}
&[Z_a, Q_\alpha] = \tau_2(s-1)(\Gamma_a)_\alpha{}^\beta Q'_\beta,\\
&[Z_{ab}, Q_\alpha] = \tau_2(\Gamma_{ab})_\alpha{}^\beta Q'_\beta,\\
&[Z_{a_1\ldots a_5}, Q_\alpha] = \tau_2\left(\frac{s}{6!} - \frac{1}{5!}\right)(\Gamma_{a_1\ldots a_5})_\alpha{}^\beta Q'_\beta,\\
&\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta}Z_a + \frac{1}{2!}\Gamma^{ab}_{\alpha\beta}Z_{ab} + \frac{1}{5!}\Gamma^{a_1\ldots a_5}_{\alpha\beta}Z_{a_1\ldots a_5},\\
&\{Q'_\alpha, \cdot\} = 0, \qquad (7)
\end{aligned}
$$

where the fermionic generator Q'_α does not introduce a new grading in the extended algebra because of its central character. We point out that Z_{ab} and $Z_{a_1\ldots a_5}$ are the central generators of the M algebra [25, 26], so this set of superalgebras are just a class of fermionic central extensions of the M-theory superalgebra. This point of view has been used as a way to try to understand the underlying symmetry structure of the M theory from the knowledge of the gauge symmetry of $D=11$ supergravity [12].

In the above equations the real parameter τ_2 is always different from zero and it can be included in the normalization of the additional central charged Q'_α, so it is thus inessential. Then, only one free parameter s remains which labels all the equivalent, but non-isomorphic, members belonging to this uniparameter family $\mathfrak{E}^{(528|32+32)}(s)$ [12]. Note that this factorization also includes the case when $\tau_2 \to 0$ and so $s \to \infty$, such that $\tau_2 \cdot s$ remains finite. In particular, the two specific D'Auria–Fré solutions take the values $\mathfrak{E}(3/2)$ and $\mathfrak{E}(-1)$ under the above notation.

In a similar way as in the previous section, introducing the MC one-forms Π^a, $\Pi'^{\,ab}$, $\Pi^{a_1\ldots a_5}$, π^α, π'^α dual to the algebraic generators Z_a, Z_{ab}, $Z_{a_1\ldots a_5}$, Q_α, Q'_α, respectively, the family of superalgebras $\mathfrak{E}^{(528|32+32)}(s)$ can be equivalently descibed by the MC equations

$$\begin{aligned} d\Pi^a &= -\frac{1}{2}\Gamma^a_{\alpha\beta}(\pi^\alpha \wedge \pi^\beta), \\ d\Pi'^{ab} &= -\frac{1}{2}\Gamma^{ab}_{\alpha\beta}(\pi^\alpha \wedge \pi^\beta), \\ d\Pi^{a_1\ldots a_5} &= -\frac{1}{2}\Gamma^{a_1\ldots a_5}_{\alpha\beta}(\pi^\alpha \wedge \pi^\beta), \\ d\pi^\alpha &= 0, \\ d\pi'^\alpha &= -\tau_2\Bigg((s-1)(\Gamma_a)_\beta{}^\alpha(\Pi^a \wedge \pi^\beta) \\ &\quad + \frac{1}{2!}(\Gamma_{ab})_\beta{}^\alpha(\Pi'^{\,ab} \wedge \pi^\beta) \\ &\quad + \left(\frac{s}{6!} - \frac{1}{5!}\right)\left(\Gamma_{a_1\ldots a_5})_\beta{}^\alpha(\Pi^{a_1\ldots a_5} \wedge \pi^\beta\right)\Bigg), \end{aligned} \quad (8)$$

where the real parameter s is only involved in the last relation.

In this parametrization, all the algebras in (7) and (8) can be used to write the three-form A_3 of $D=11$ supergravity as a composite one, except of the case $s=0$. This particular value corresponds to the only superalgebra $\mathfrak{G}(0) = \mathfrak{E}^{(528|32+32)}_{(s=0)} \rtimes so(10,1)$ for which the Lorentz group $SO(10,1)$ can be enlarged to $Sp(32)$, and is ruled out on the searching of the local symmetry of the $D=11$ supergravity [12, 13, 15].

Hence, it appears that the real factor s plays an important role in the study of the connection of the $D=11$ supergravity with the $osp(1|32) \oplus osp(1|32)$ superalgebra.

4. CONTRACTIONS OF $osp(1|32) \oplus osp(1|32)$

In [4], the author tried to explore the possible relation of the M theory with Chern–Simons supergravities. He focused the attention on the fact that M theory is parity invariant. However, the Chern–Simons action based on the eleven dimensional anti-de Sitter group $Osp(1|32)$ is not compatible with such invariance. Thus, in order to respect parity invariance, Horava pointed out that the gauge group will contain extra bosonic charges and an extra supercharge Q'_α, so that the complete set of bosonic and fermionic generators lead to an algebraic structure isomporphic to $osp(1|32) \oplus osp(1|32)$. Morever, he suggested that this algebra contracts to the D'Auria–Fré superalgebra $\mathfrak{G}(s \neq 0)$, because in low-energy limit we have to recover the $D=11$ CJS supergravity. In this section we study whether this is the case.

At a first step, we are interested in writing the explicit relations of the $osp(1|32) \oplus osp(1|32)$ superalgebra in a generic basis. In general, we consider a change of basis from the basis of generators $\{X_i\}$ and $\{\bar{X}_i\}$ of the component Lie algebras $\mathcal{G}$ and $\bar{\mathcal{G}}$, to a new one $\{Y_i, \bar{Y}_i\}$ of $\mathcal{G} \oplus \bar{\mathcal{G}}$. In our case, we have $\mathcal{G} = osp_+(1|32)$ and $\bar{\mathcal{G}} = osp_-(1|32)$. Since these two superalgebras are actually two non-isomorphic real versions of the same complex algebra (see Section 2), we can take $\mathcal{G} = \bar{\mathcal{G}} = osp_+(1|32)$ by considering complex factors $\left\{a_i^j, b_i^j, c_i^j, d_i^j\right\}$ on the mix process. Consequently, we can take two copies of (4) and consider linear combinations of the generators given by

$$\begin{aligned} Y_i &= a_i^j X_j + b_i^j \bar{X}_j, \\ \bar{Y}_i &= c_i^j X_j + d_i^j \bar{X}_j. \end{aligned} \quad (9)$$

This process can also be done applied to the Maurer–Cartan one-forms $\{\Pi_i,\ \bar{\Pi}_i\}$ dual to the algebraic generators $\{X_i,\ \bar{X}_i\}$. In fact, this is how we have done our calculations. Then, the linear combination (9) leads to a new set of Maurer–Cartan one-forms denoted $\left\{\rho_+^{(n)},\ \rho_-^{(n)}\right\}$, which can be written generically as

$$\begin{aligned} \rho_+^{(n)} &= \alpha_{(n)}\Pi^{(n)} + \beta_{(n)}\bar{\Pi}^{(n)}, \\ \rho_-^{(n)} &= \gamma_{(n)}\Pi^{(n)} + \delta_{(n)}\bar{\Pi}^{(n)}, \end{aligned} \quad (10)$$

where $\{\alpha_{(n)};\beta_{(n)};\gamma_{(n)};\delta_{(n)}\}$ are a set of 16 complex scalars and $n=(1,2,5,\alpha)$ denotes the number of Lorentz indices for the bosonic one-forms $\rho^a_\pm,\rho^{ab}_\pm,\rho^{a_1\ldots a_5}_\pm$ or the spinorial index for the fermionic one $\rho^\alpha_\pm\equiv\psi^\alpha_\pm$.

We have to emphasize that the above linear combinations cannot be arbitrary in order to keep the Lorentz transformation law inside the $osp(1|32)\oplus osp(1|32)$. This means that we have to take combinations of pairs of one-forms Π_a and $\bar{\Pi}_a$, Π_{ab} and $\bar{\Pi}_{ab}$, etc. separately. Moreover, we must ensure that these linear combinations have to be invertible in order to really perform a change of basis, so

$$\det\begin{pmatrix}\alpha_{(n)} & \beta_{(n)}\\ \gamma_{(n)} & \delta_{(n)}\end{pmatrix}\neq 0.$$

Thus, the $osp(1|32)\oplus osp(1|32)$ superalgebra may be written explicity in terms of these complex scalar coefficients $\{\alpha_{(n)};\beta_{(n)};\gamma_{(n)};\delta_{(n)}\}$, their inverse relations $\{\alpha'_{(n)};\beta'_{(n)};\gamma'_{(n)};\delta'_{(n)}\}$ and the structure constants (6) (see appendix on reference [27]).

Now let us perform a generalized Weimar-Woods contraction on these equations. Generalized, or Weimar-Woods [28, 29], contractions can be constructed as follows: let $\mathcal{G}$ be a Lie (super)algebra given, as a vector space, by the direct sum

$$\mathcal{G}=V_0\oplus V_1\oplus\cdots\oplus V_n, \qquad (11)$$

and such that the (graded) commutators obey

$$[V_p,V_q]\subset\bigoplus_{l=0}^{p+q}V_l. \qquad (12)$$

In particular, V_0 is a subalgebra of $\mathcal{G}$. Let $\{X_{p,\alpha_p}\}$, $p=0,\ldots,n$, $\alpha_p=1,\ldots,\dim V_p$, be a basis of $\mathcal{G}$ relative to the splitting (11), then expression (12) can be written explicitly as

$$[X_{p,\alpha_p},X_{q,\beta_q}]=C^{r,\gamma_r}_{p,\alpha_p;q,\beta_q}X_{r,\gamma_r},$$
$$C^{r,\gamma_r}_{p,\alpha_p;q,\beta_q}=0\quad\forall r>p+q.$$

If ω^{p,α_p} are the one-forms dual to the vector fields X_{p,α_p}, i.e., $\omega^{p,\alpha_p}(X_{q,\beta_q})=\delta^p_q\delta^{\alpha_p}_{\beta_q}$, the MC equations of $\mathcal{G}$ are

$$d\omega^{r,\gamma_r}=-\frac{1}{2}\sum_{p+q\leq r}C^{r,\gamma_r}_{p,\alpha_p q,\beta_q}\omega^{p,\alpha_p}\wedge\omega^{q,\beta_q}. \qquad (13)$$

It turns out that the same vector space (11), but now with modified MC equations given by (13) with the sum only extended to $p+q=r$, defines a new Lie (super)algebra $\mathcal{G}_c$, known as the Weimar-Woods contracted/super)algebra relative to the splitting (11). This contracted algebra can be obtained by rescaling in terms of parameter λ the forms ω^{p,α_p} as $\omega^{p,\alpha_p}\to\lambda^p\omega^{p,\alpha_p}$ in the starting MC equations, and then taking the limit $\lambda\to 0$. This is the procedure that we use in this paper. The case $n=1$ corresponds to the original, İnönü–Wigner [30, 31], contractions.

Given a starting algebra and a set of structure constants, in practice one has to study a system of equations of the rescaled exponents for which the contraction limit ($\lambda\to 0$) is well defined and reproduces the algebraic structure desired. However, in our case the starting structure constants of $osp(1|32)\oplus osp(1|32)$ are written in terms of $\{\alpha_{(n)};\beta_{(n)};\gamma_{(n)};\delta_{(n)}\}$ and $\{\alpha'_{(n)};\beta'_{(n)};\gamma'_{(n)};\delta'_{(n)}\}$, so the set of exponents of λ in the rescaling

$$\rho^a_+\Rightarrow\lambda^n\rho^a_+,\quad \rho^{ab}_+\Rightarrow\lambda^p\rho^{ab}_+,$$
$$\rho^{a_1\ldots a_5}_+\Rightarrow\lambda^r\rho^{a_1\ldots a_5}_+,\quad \psi^\alpha_+\Rightarrow\lambda^v\psi^\alpha_+,$$
$$\rho^a_-\Rightarrow\lambda^m\rho^a_-,\quad \rho^{ab}_-\Rightarrow\lambda^q\rho^{ab}_-,$$
$$\rho^{a_1\ldots a_5}_-\Rightarrow\lambda^t\rho^{a_1\ldots a_5}_-,\quad \psi^\alpha_-\Rightarrow\lambda^w\psi^\alpha_-, \qquad (14)$$

are not the only coefficients to be determined. Hence, we have to add a set of extra conditions, in terms of the coefficients of the linear combinations (10), that ensures that the structure constants after the contraction limit $\lambda\to 0$ reproduce the structure (1).

We list below the steps that we have followed:

1. First, we have performed the change of scale (14) on the MC equations of the $osp(1|32)\oplus osp(1|32)$ ([27]). The resulting MC equations could be rewritten by sums of terms with the following structure

$$\lambda^{E(m,n,p,q,r,t,u,w)}C(\alpha_{(n)},\beta_{(n)},\gamma_{(n)},\delta_{(n)})\,(\rho_\pm\wedge\rho_\pm),$$

where, apart from the exterior product of two one-forms, there is a power of λ that depends on the scaling factors of (14), and a coefficient (structure constant) that depends on the parameters of the linear combination (10) and their inverse relations.

2. We then have chosen the one-form $\rho^{ab}_\pm$ dual to the Lorentz generator J_{ab} which fixes the tensorial transformation on the resulting algebras, as well as the fermionic one-form field associated to the central supercharge Q'_α. Without loss of generality, we have made the election ρ^{ab}_+ as the boost generator and ψ'^α as the central fermionic one.

3. Next, we have carried out a suitable election, via visual inspection, of the products of MC one-forms $\rho^a_+,\rho^{ab}_-,\rho^{a_1\ldots a_5}_+,\psi^\alpha$ that reproduce the D'Auria–Fré structure (8), plus the Lorentz transformations. On the other hand, the remaining MC one-forms ρ^a_-, $\rho^{a_1\ldots a_5}_-$ belonging to the bosonic algebra $\mathcal{L}$ are identified by exclusion procedure.

4. The above step fixes the exponents E which have to be vanished, as well as the values of the structure constants C in terms of the real parameter s. Moreover, the study of the compatibility in terms of the scaling factors (14) for all the 112 powers associated to the 112 terms of the MC equations of the $osp(1|32) \oplus osp(1|32)$, leads to:

(a) terms with $E = 0$ that should not appear in the limit, so we have to impose the vanishing of their associated structure constant $C = 0$,

(b) terms whose powers are negative $E < 0$, so we have to ensure that their algebraic coefficient also vanishes $C = 0$, consistently with the Weimar-Woods approach.

5. Finally, we have imposed that the linear combination in (10) is invertible.

These steps lead to a system of equations and inequations in terms of the complex scalar coefficients $\{\alpha_{(n)}; \beta_{(n)}; \gamma_{(n)}; \delta_{(n)}\}$, their inverse matrix relations $\left\{\alpha'_{(n)}; \beta'_{(n)}; \gamma'_{(n)}; \delta'_{(n)}\right\}$ and the real parameter s, performed by

fifteen conditions $C(\alpha_{(n)}, \beta_{(n)}, \gamma_{(n)}\delta_{(n)}) \neq 0$, associated to the third step,

six equations $C(\alpha_{(n)}, \beta_{(n)}, \gamma_{(n)}, \delta_{(n)}) = 0$ for the fourth step,

and four inequalities needed to express the admissible basic changes of the fifth step.

The results of this problem rely on heavy algebraic manipulations which have been performed by using a symbolic manipulation program (Mathematica). The resulting system of equations and inequations have a solution only when $s = 0$, i.e., the expansion case of $osp(1|32)$ for which the three-form of $D = 11$ supergravity A_3 cannot be written in terms of Maurer–Cartan one-forms. We have also considered the case $s \to \infty$ and checked that there is no solution. We do not include here the detailed expressions of the explicit computing, but they are available from the authors upon request.

5. CONCLUSIONS

The main result of this paper is the proof that it is not possible to obtain by generalized Weimar-Woods contraction from $osp_+(1|32) \oplus osp_-(1|32)$ any of the algebras found in [12], which allow a gauge group interpretation of the three-form field in the sense of [2]. In other words, $D = 11$ supergravity cannot be connected with the semi-simple supergroup $OSp_+(1|32) \otimes OSp_-(1|32)$ by trivializing the three-form field A_3.

But we cannot claim that the conjecture made in [4], according to which $D = 11$ supergravity can be obtained as a low-energy limit of a Chern–Simons theory based on $osp_+(1|32) \oplus osp_-(1|32)$, is incorrect. This is due to the fact that there is no reason why the λ^q term in the expansion of the CS action should be invariant under the contracted algebra.

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