Optimal Environmental Policy for a Polluting Monopoly with Abatement Costs: Taxes versus Standards*

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Abstract

In this paper we characterize the optimal environmental policy for a polluting monopoly that devotes resources to abatement activities when damages are caused by a stock pollutant. With this aim, we calculate the stagewise feedback Stackelberg equilibrium of a (differential) policy game where the regulator is the leader and the monopolist the follower. Our analysis shows that the first-best policy consists of applying a Pigouvian tax and a subsidy on production equal to the difference between the price and the marginal revenue. However, for a stock pollutant the Pigouvian tax is not equal to the marginal damages but to the difference between the *social* and *private* valuation of the pollution stock. On the other hand, if a second-best emission tax is used, the tax is lower than the Pigouvian tax and the difference decreases with the price elasticity of the demand. Finally, we find that taxes and standards are equivalent in a second-best setting. In the second part of the paper, we solve a linear-quadratic differential game and we obtain that the first-best tax increases with the pollution stock whereas the subsidy decreases but that the tax is negative for low values of the pollution stock regardless of the importance of the environmental damages i.e. for low values of the pollution stock the social valuation of the stock is lower than the private valuation. Moreover, when a second-best policy is applied the steady-state pollution stock is lower than the steady-state pollution stock corresponding to the efficient outcome.

Keywords: monopoly, abatement, emission tax, emission standard, stock pollutant, differential games

JEL Classification System: H23, L12, L51, Q52, Q55

1 Introduction

Regulation of a polluting monopoly is an interesting case of environmental regulation since the market allocation can be inefficient because two market failures are operating at the same time but in an opposite direction. On one hand, the firm has *market power* that tends to reduce output. On the other hand, there is a *negative externality* that tends to increase output. Obviously, there will exist a level of marginal environmental damages for which the market allocation is efficient. Thus, for low marginal damages the optimal policy will consist of a subsidy and for high marginal damages it will be a tax. Moreover, when this is the case the tax will be lower than the marginal damages. These well known results showed by Buchanan (1969) apply when the only way to adjust emissions is changing production. Interestingly, when the firm devotes resources to abatement activities the first-best policy is a combination of a subsidy on production and a Pigouvian tax on emissions equal to the marginal damages and the tax cannot be negative. However, as was established by Barnett (1980), if a second-best emission tax is applied the tax will be lower than marginal damages and it could be negative for low values of marginal damages.

This analysis of the regulation of a polluting firm with market power for a *flow pollutant* was extended by Benchekroun and Long (1998, 2002) to the case of a *stock pollutant* bringing the analysis to a dynamic framework. The first paper shows that there exists a stationary tax rule that guides polluting oligopolists to achieve the efficient production path. The optimal tax depends on the current pollution stock, and it may be negative when the pollution stock is low and there are just a few firms in the markets. This result is established for a model where the only way to control emissions is reducing production as in Buchanan's (1969) paper. In our paper, we extend their analysis of emission taxation to cover the possibility that the firm devotes resources to abatement activities as in Barnett's (1980) paper.

In the first part of the paper, we calculate the *stagewise feedback Stackelberg equilibrium* (SFSE) of a differential policy game between a welfare maximizing regulator and a profit maximizing monopolist to characterize first the first-best policy. As in the static model it consists of applying a Pigouvian tax along with a subsidy on production that closes the gap between the price and the marginal revenue. However, for a stock pollutant we obtain that the Pigouvian tax is not equal to the marginal damages but to the difference between the *social* and *private* valuations of the pollution stock explained by the different effect that an increase in the stock has on the discounted present value of net social welfare and on the discounted present value of net profits. We also confirm Barnett's result that if a second-best emission tax is applied, the tax is lower than the Pigouvian tax and that the difference between these two taxes decreases with the price elasticity of the demand. Finally, we show that an emission standard implements the same SFSE than that supported by a tax. The two regulated market equilibria coincide and the tax and the standard are two equivalent policy instruments.

In order to investigate the dynamics of the model, we solve in the second part of the paper a linear-quadratic (LQ) differential game for an end-of-the-pipe abatement technology. We obtain that the tax increases with the pollution stock whereas the subsidy decreases. However, as in Benchekroun and Long (1998) the tax is negative for low values of the pollution stock. The characterization of the Pigouvian tax in a dynamic framework we have obtained in the first part of the paper allows us to explain the rationale of this result. Something that was not in Benchekroun and Long's (1998) paper. For low values of the pollution stock, the private valuation of the stock is larger than the social valuation because the firm takes into account that more production and emissions will bring higher taxes for the rest of the game whereas the social valuation is low because for low values of the pollution stock, damages are small. In this case, the difference between the social and private valuation of the stock is negative and the first-best policy is a subsidy on emissions. Something that cannot occur for a flow pollutant. This difference causes that the level of production is bellow the efficient level is spite of the subsidy on production and the optimal policy is a subsidy on emissions instead of a tax tat indirectly operates as subsidy on production. Notice that this result does not depend on the importance of marginal damages. Even if the marginal damages curve is very steep, the optimal tax is negative for low values of the pollution stock. The same occurs for the second-best emission tax with the difference that the term of the tax that corrects the market power of the firm is also negative strengthening the negative effect of the private valuation on the tax. In this framework, it is very interesting to know that a standard could implement the same equilibrium because applying a standard the regulator could avoid an environmental policy that recommends a subsidy on emissions. A policy that could be difficult to implement because of the political protest of the environmentalist groups.

Finally, we ends the paper evaluating the effects of the second-best environmental policy. The main result is that the steady-state pollution stock for the efficient (first-best) solution is larger then the steady-state pollution stock the second-best solution and consequently steady-state emissions are also larger. This result is explained because if a subsidy on production is not applied, the firm will produce less. An incentive that is not compensated by the second-best tax resulting finally in less production and less emissions. In fact, we find that if damages are not very low the optimal strategy for emissions for the second-best solution is below the optimal strategy for emissions for the efficient outcome, i.e. for any level of pollution, emissions are larger when the first-best policy is used. Moreover, we find that the optimal temporal path of the pollution stock for the efficient solution is above the optimal temporal path for the second-best solution except when the initial pollution stock is very large. If this is the case, the pollution stock when a second-best policy is used can be initially larger than the pollution stock when the first-best policy applies although finally this relationship will invert.

1.1 Literature Review

The literature addressing the regulation of firms with market power in the context of stock dynamics includes Bergstrom et al. (1981), Karp and Livernois (1992) and Karp (1992) for the case of a non-renewable resource and Xepapadeas (1992), Kort (1996), Benchekroun and Long (1998, 2002), Stimming (1999) Feenstra et al. (2001) and Yanese (2009) for the case of polluting firms. Bergstrom et al. (1981) show that there exists a continuum of tax schemes that induce the monopolist to exploit efficiently a non-renewable resource. However, these tax schemes are not subgame perfect. Karp and Livernois (1992) design a subgame perfect tax scheme that is efficiently inducing and is unique and Karp (1992) extends this result to the case of a common property oligopoly. The rest of paper addressing the case of polluting firms, except Benchekroun and Long (1998, 2002) and Yanase (2009), assume that damages are caused by a flow pollutant and focus on the investment in abatement technology. The environmental policy is *exogenously* determined and the research assesses the effects of a stricter environmental policy and the comparison of taxes vs emission standards. However, in Yanese (2009), as in Benchekroun and Long (1998, 2002), the environmental policy is *endogenously* determined. The author examines a non-cooperative (differential) policy game between national governments in a model of international pollution control of a stock pollutant in which duopolists compete myopically in quantities in a third country with product differentiation, and expense resources in abatement activities. The comparison of the Markov perfect Nash equilibrium of the game for different policy instruments establishes that an emission tax produces more pollution and lower welfare than those generated by a standard. However, our policy game, although for a different setting, does not give the same prediction, the tax is equivalent to the standard.

More recently, Wirl (2014) has investigated a differential policy game between a monopoly that provides a clean technology for a polluting competitive industry and a regulator that uses an emission tax or standard to control a flow pollutant.[1] Thus, when the regulator uses a tax, the monopoly is forced to provide the clean technology at a price equal to the emission tax. For this model, the Markov perfect Nash equilibria (MPNE) when the regulator applies a tax and when it uses a standard are compared. This comparison shows that for a MPNE the efficient solution cannot be implemented using only one instrument but that the tax and the standard are equivalent as it occurs in our model.[2] However, our model is different in several aspects. Mainly, Wirl focuses on a flow pollutant whereas we consider a stock pollutant. Moreover, we assume that

[1] Golombek et al.(2010) consider that the supply of abatement equipment services are monopolistic competitive. They show in a two-period model that the first-best outcome can be reached through a technology subsidy and carbon taxes.

[2] In contrast with our model, in Wirl's (2014) policy game the tax does not depend on the investment and vice versa and consequently the stagewise feedback Stackelberg equilibrium coincides with the Markov perfect Nash equilibrium.

the polluting industry is a monopoly that can devote resources directly to abatement activities. Thus, to best our knowledge, our paper derives for the first time the optimal environmental policy for a polluting monopoly with a stock pollutant and abatement costs.

Another branch of the literature has focused on the effects of taxes and standards on welfare for a stock pollutant under uncertainty. This issue has been addressed by Hoel and Karp (2001, 2002), Karp and Zhang (2005, 2006) and, more recently, by Masoudi and Zaccour (2014). In these papers, it is assumed that a competitive representative firm takes aggregate emissions as exogenous and behave non-strategically. Since the firm takes both the current and future policies as exogenous, it solves a sequence of static optimization problems. Then, the regulator optimization problem becomes a stochastic optimal control problem where the tax or the standard maximizes social welfare. As is well know since the publication of Weitzman's (1974) paper, the uncertainty breaks the equivalence between the tax and the standard.

Finally, we would like to comment the papers by Saltari and Travaglini (2011) and Karp and Zhang (2012) where both the abatement capital stock and the pollution stock are taken into account in the analysis of the firm's investment decisions. Saltari and Travaglini (2011), following the approach adopted by the previous literature, assume that the environmental policy is exogenously determined. In their model, the pollution stock affects negatively the production function, and a single competitive firm has to decide for a given price about the use of a polluting factor and the abatement investment taking has given the dynamics of the pollution stock that evolves according to a geometric Brownian motion. Karp and Zhang (2012) extend Hoel and Karp (2002) allowing the representative firm to invest in abatement capital. They analyze a two-stage game where in the first stage the firm selects in each period the emissions that maximize its current profits and in the second stage the firm and the regulator play a simultaneous non-cooperative game. In this second stage, the firm decides the level of investment and the regulator the level of the policy instrument. They solve the game numerically using a linear-quadratic model for an application to climate change. Both papers focus on comparing taxes vs emission standards.

The remainder of the paper is organized as follows. Section 2 presents the ingredients of the differential game and the efficient conditions. Section 3 characterizes the first-best policy and Section 4 analyzes the second-best environmental policy using taxes and standards. In Section 5 we calculate the first-best and second-best environmental policy for a linear-quadratic differential game and evaluate the effects of the second-best policy. Section 6 offers some concluding remarks and points out lines for future research.

2 The Model and the Efficient Conditions

We consider a monopoly that faces a market demand represented by the decreasing inverse demand function $P(q(t))$ where $q(t)$ is the output at time t. The production process generates pollution emissions, however the firm can devote resources to abatement activities represented by $w(t)$. In this case, both the emission and cost functions depend on output and abatement effort. The emission function is represented by $s(q(t), w(t))$ with $\partial s/\partial q$ positive and $\partial s/\partial w$ negative, and the cost function by $C(q(t), w(t))$ with $\partial C/\partial q$ and $\partial C/\partial w$ positive. The second-order partial derivatives are assumed to be positive or zero. The focus of the paper is on a stock pollutant that evolves according to the following differential equation

$$\dot{x}(t) = s(t) - \delta x(t) = s(q(t), w(t)) - \delta x(t), \qquad x(0) = x_0 \geq 0, \tag{1}$$

where $x(t)$ stands for the pollution stock and $\delta > 0$ for the decay rate of pollution. The environmental damages are given by the function $D(x(t))$ that is assumed strictly convex. Thus, the policy game we analyze in this paper is a *differential game* between a welfare maximizing regulator and a profit maximizing monopolist. Before analyzing it, we first derive the first-order conditions that characterize the efficient outcome.

The efficient conditions are obtained from the maximization of the discounted present value of net social welfare defined as the difference between gross consumer surplus minus costs and environmental damages.[3]

[3]Time argument will be eliminated when no confusion arises.

$$\max_{\{q,w\}} \int_0^\infty e^{-rt} \left\{ \int_0^q P(y)dy - C(q,w) - D(x) \right\} dt$$
$$s.t.\ \dot{x} = s(q,w) - \delta x, \quad x(0) = x_0 \geq 0,$$

where r is the time discount rate.

Solving by dynamic programming, the solution to this dynamic optimization problem must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation

$$rW(x) = \max_{\{q,w\}} \left\{ \int_0^q P(y)dy - C(q,w) - D(x) + W'(x)(s(q,w) - \delta x) \right\},$$

where $W(x)$ represents the maximum discounted present value of net social welfare for the current value, x, of the pollution stock.

The maximization of the right-hand side of the HJB equation yields the following first-order conditions (FOCs)

$$P = \frac{\partial C}{\partial q} - W'(x)\frac{\partial s}{\partial q}, \tag{2}$$

$$\frac{\partial C}{\partial w} = W'(x)\frac{\partial s}{\partial w}. \tag{3}$$

The first FOC establishes that the price must be equal to the marginal costs that include the marginal cost of production plus the *social* valuation (shadow price) of the pollution stock. The latter is given by the reduction in the present value of the net social welfare because of an increase in the pollution stock caused by an increase in production. Notice that this shadow price is multiplied by the effect of an increase in production on emissions. On the other hand, the second FOC requires that the marginal cost of abatement is equal to the marginal benefit defined by the increase in the present value of the net social welfare caused by a reduction in the stock because of an increase in abatement. Notice that $W'(x)$, is a marginal cost when we are considering an increase in production, and it stands for a marginal benefit when we are evaluating an increase in abatement.

To implement these conditions as a regulated market equilibrium we propose, following Barnett (1980), a subsidy to correct the market power of the firm together a tax on emissions to control the external diseconomy. In the next section, we calculate the *stagewise feedback Stackelberg equilibrium* (SFSE) of a (differential) policy game where the regulator is the leader and the monopolist the follower and show that using this policy mix the regulated market equilibrium will be efficient.

3 The First-Best Policy

The SFSE is based on the assumption that the regulator moves first in each moment. To find the regulator's optimal policy, we apply backward induction, substituting the monopolist's reaction function in the regulator's HJB equation, and computing the optimal strategy by maximizing the right-hand side of this equation. The resulting outcome is a stagewise feedback Stackelberg solution, which is a Markov-perfect equilibrium. For this kind of equilibria no commitment is required for the entire temporal horizon. For our model, this equilibrium is time consistent in the sense defined in the seminal paper by Kydland and Prescott (1997) and also satisfies subgame perfection.

The output and abatement selection occurs in the second stage. The monopolist chooses its output and abatement to maximize the discounted present value of net profits

$$\max_{\{q,w\}} \int_0^\infty e^{-rt} \{P(q)q - C(q,w) - \tau s(q,w) + vq\} \, dt,$$

subject to differential equation (1) where τ is the emission tax and v stands for a subsidy on production. We assume that the firm acts strategically at this stage because it is aware that the dynamics of the stock will be taken into account by the regulator to set up the optimal policy.

The solution to this dynamic optimization problem must satisfy the following HJB equation

$$rV(x) = \max_{\{q,w\}} \{P(q)q - C(q,w) - \tau s(q,w) + vq + V'(x)(s(q,w) - \delta x)\}, \tag{4}$$

where $V(x)$ stands for the maximum discounted present value of net profits for the current value, x, of the pollution stock.

From the FOCs for the maximization of the right-hand side of the HJB equation, we get

$$P'q + P + v = \frac{\partial C}{\partial q} + (\tau - V'(x)) \frac{\partial s}{\partial q}, \tag{5}$$

$$\frac{\partial C}{\partial w} = -(\tau - V'(x))\frac{\partial s}{\partial w}. \tag{6}$$

The left-hand side of the first FOC stands for the marginal revenue plus the subsidy and the right-hand side represents the marginal costs that include the marginal cost of production, the tax and the *private* valuation (shadow price) of the pollution stock. The latter is given by the reduction in the present value of the firm's net profits because of an increase in the pollution stock caused by an increase in production. Observe that the last two terms are multiplied by the effect of an increase in production on emissions. On the other hand, the left-hand side of the second FOC represents the marginal cost of abatement and on the right-hand side appear the marginal benefits that includes the marginal reduction in taxes because of the reduction in emissions caused by an increase in abatement that is given by the tax rate, and the increase in the present value of the firm's profits because of the reduction in the stock caused by an increase in abatement. Notice that $V'(x)$ is a marginal cost when we are considering an increase in production, and it stands for a marginal benefit when we are evaluating an increase in abatement. The same occurs for the tax rate. Conditions (5) and (6) implicitly define the firm's strategies: $q(\tau, v, x)$ and $w(\tau, v, x)$.

In the first stage, the regulator selects the emission tax rate and subsidy by unit of output that maximize net social welfare defined as the sum of consumer surplus and monopoly net profits plus tax revenues minus subsidies and environmental damages

$$\max_{\{\tau,v\}} \int_0^\infty e^{-rt} \left\{ \int_0^q P(y)dy - Pq(\tau,v,x) + \pi(q(\tau,v,x), w(\tau,v,x), \tau, v) \right.$$
$$\left. + \tau s(q(\tau,v,x), w(\tau,v,x)) - vq(\tau,v,x) - D(x) \right\} dt,$$

where π stands for the firm net profits. Notice that consumer expense and firm revenue on one hand and firm tax expense and subsidies and regulator tax revenue and subsidy expenses on the other hand, cancel out. Therefore, this optimization problem can be rewritten as

$$\max_{\{\tau,v\}} \int_0^\infty e^{-rt} \left\{ \int_0^{q(\tau,v,x)} P(y)dy - C(q(\tau,v,x), w(\tau,v,x)) - D(x) \right\} dt.$$

The solution to this dynamic optimization problem must satisfy the following HJB equation

$$rW(x) = \max_{\{\tau,v\}} \left\{ \int_0^q P(y)dy - C(q(\tau,v,x), w(\tau,v,x)) - D(x) \right.$$
$$\left. +W'(x)(s(q(\tau,v,x), w(\tau,v,x)) - \delta x) \right\}. \quad (7)$$

From the FOCs for the maximization of the right-hand side of the HJB equation, we get

$$\left(P - \frac{\partial C}{\partial q} + W'(x)\frac{\partial s}{\partial q}\right)\frac{\partial q}{\partial \tau} - \left(\frac{\partial C}{\partial w} - W'(x)\frac{\partial s}{\partial w}\right)\frac{\partial w}{\partial \tau} = 0, \quad (8)$$

$$\left(P - \frac{\partial C}{\partial q} + W'(x)\frac{\partial s}{\partial q}\right)\frac{\partial q}{\partial v} - \left(\frac{\partial C}{\partial w} - W'(x)\frac{\partial s}{\partial w}\right)\frac{\partial w}{\partial v} = 0. \quad (9)$$

Assuming that both output and abatement are affected by the tax and subsidy, it is immediate that these conditions are satisfied if the efficient conditions hold.[4] Thus, using the efficient conditions along with FOCs (5) and (6) we can characterize the *first-best policy.* Conditions (3) and (6) allows us to define the optimal tax

$$\tau^* = -W'(x) + V'(x), \quad (10)$$

and conditions (2) and (5) the optimal subsidy: $v^* = -P'q$,and we can conclude that

Proposition 1 *The first-best policy consists of applying a Pigouvian tax equal to the difference between the social and private valuations of the pollution stock and a subsidy that closes the gap between the price and marginal revenue of output.*

Observe that in a dynamic context, the Pigouvian tax is not equal to the marginal environmental damages but to the difference between the social and private shadow price that environmental damages cause. Moreover, we can highlight that each instrument corrects one distortion. On one hand, the tax adjusts the output and abatement to correct the external diseconomy caused by the firm's emissions. On the other hand, the subsidy corrects the market power of the firm doing that the marginal revenue be equal to the price. Finally, we would like to point out that although for the subsidy is not explicitly

[4]Notice that the only requirement we impose on the strategies is that their partial derivatives do not be all zero.

recognized the dependence with respect to the pollution stock this is the case because we are using dynamic programming to solve the optimization problem. This means that we derive two policy rules that depends on the pollution stock. The properties of these policy rules will depend critically on the properties of the value functions as we show in Section 5 for a linear-quadratic differential game.

4 The Second-Best Environmental Policy

A subsidy on production is a policy instrument an environmental regulator could not apply because it could be not recognized as an environmental policy instrument, in other words, for political or institutional constraints, the environmental regulator could be restricted to select policy instruments in a menu that does not include subsidies on production. For this reason, we find interesting to investigate which would be the optimal environmental policy in a second-best setting. In this paper, we will focus on two classical environmental policy instruments: a tax on emissions and an emission standard. We will begin studying the case of an emission tax.[5]

4.1 The Second-Best Emission Tax

The study of this case follows the steps of the analysis developed in the previous section omitting the possibility the regulator applies a subsidy on output. Thus, the subsidy must be cancelled in the FOC (5) yielding

$$P'q + P = \frac{\partial C}{\partial q} + (\tau - V'(x)) \frac{\partial s}{\partial q}. \tag{11}$$

However, there are no changes in the FOC (6). On the other hand, the dependence of production and abatement with respect to the subsidy must be eliminated from the HJB equation (7) and then the two FOCs that characterize the maximization of the right-hand side of the regulator HJB equation reduces to one, the condition (8).

[5] In a companion paper, Martín-Herrán and Rubio (2016), we analyze the time consistency of the second-best emission tax.

Doing again the expressions between parenthesis in the condition (8) equal to zero the efficient conditions are obtained. However, these conditions are incompatible with conditions (6) and (11) that characterize the maximization of the discounted present value of net profits by the firm and the SFSE is inefficient. The problem is that it is necessary to correct two conditions to induce the firm to act efficiently but the regulator is using only one policy instrument. The tax can be defined to yield that the condition (6) be equal to the efficient condition (3) but then the condition (11) does not give the efficient condition (5). As expected, a second-best emission tax does not implement the efficient outcome and has a cost in welfare terms.

Conditions (11) and (6) can be rewritten as

$$P - \frac{\partial C}{\partial q} = (\tau - V'(x))\,\frac{\partial s}{\partial q} - P'q,$$

and

$$\frac{\partial C}{\partial w} = -(\tau - V'(x))\frac{\partial s}{\partial w}.$$

Eliminating $P - \partial C/\partial q$ and $\partial C/\partial w$ in (8) using these expressions yields

$$\left(-P'q + (\tau - V'(x) + W'(x))\frac{\partial s}{\partial q}\right)\frac{\partial q}{\partial \tau} + (\tau - V'(x) + W'(x))\frac{\partial s}{\partial w}\frac{\partial w}{\partial \tau} = 0, \tag{12}$$

and taking common factor gives

$$\tau^{sb} = \frac{-\frac{P}{|\eta|}\frac{\partial q}{\partial \tau}}{\frac{\partial s}{\partial q}\frac{\partial q}{\partial \tau} + \frac{\partial s}{\partial w}\frac{\partial w}{\partial \tau}} - (W'(x) - V'(x)), \tag{13}$$

where η is the price elasticity of demand and the superscript sb stands for the second-best tax. Expression (13) is the dynamic version of the condition that characterizes the second-best emission tax derived by Barnett (1980) for a flow pollutant. When the firm has not market power, the price elasticity is infinite (13) gives the Pigouvian tax. However, for a monopoly the absolute value of the elasticity is positive and consequently the first term on the right-hand side of (13) is negative provided that $\partial q/\partial \tau$ is negative and $\partial w/\partial \tau$ is positive and then we can conclude that[6]

[6] Although these are the expected signs for these partial derivatives, they depend on the sign of cross effects (cross second-order partial derivatives) of the cost and emission functions.

Proposition 2 *The second-best emission tax is lower than the difference between the social and the private valuations of the pollution stock whenever the tax reduces the gross and net emissions.*

Moreover, expressions (12) and (13) allow us to interpret condition (8). Notice that according to (13), $\tau^{sb} - V'(x) + W'(x)$ must be negative. Then, if the tax has a negative impact on emissions and production, the first term in brackets in (12) must be positive and consequently the first term between parenthesis in (8) must be positive and the second term negative. Thus, the optimal tax balances the reduction in consumer surplus net of the social shadow price of the pollution stock caused by the decrease in output induced by the increase in the tax rate with the increase in welfare caused by the augmentation in abatement provoked by the increase in the tax rate. This increase in welfare is explained by the fact that at the equilibrium the marginal cost of abatement is lower that the social shadow price of the pollution stock that represents the increase in the discounted present value of welfare because of a decrease in the stock. Thus, when the abatement increases the reduction in the social shadow price of the pollution stock more than compensated the augmentation in abatement costs.

4.2 The Emission Standard

The study of the emission standard is interesting not only because is the classical example of an environmental policy based on the regulation of quantities instead of regulating prices represented by the emission tax but also because we cannot discard the possibility that the tax operates as a subsidy. Notice that the first term on the right-hand side of (13) is negative and this could yield a negative tax, i.e. a subsidy on emissions. In the next section, we find that this is the case, for low values of the pollution stock we obtain that both the first-best and second-best emission tax are negative. In this case, it would be interesting to know what is achieved if the regulator applies a standard to avoid applying a subsidy on emissions, a policy that could be difficult to justify as an environmental policy.

Next, we solve a stagewise feedback Stackelberg equilibrium where in the first stage the regulator chooses the standard and in the second stage the monopolist selects the abatement that given the standard determines production.

As the profits do not depend on the stock and the dynamics is driven by the standard, the monopolist maximizes (instantaneous) profits in each moment. Moreover, given the dependence of emissions on production and abatement, we can define implicitly the production as a function of the standard, $\bar{s}$, and abatement. Thus, the profits function can be written as follows

$$\pi = P\left(q(\bar{s}, w)\right) q(\bar{s}, w) - C(q(\bar{s}, w), w). \tag{14}$$

The FOC yields

$$\left(P'q + P - \frac{\partial C}{\partial q}\right)\frac{\partial q}{\partial w} = \frac{\partial C}{\partial w}. \tag{15}$$

This condition establishes that the marginal cost of investment must be balanced with the marginal benefits represented by the left-hand side of the condition. When abatement increases for a given standard, production must be adjusted to satisfy the standard what increases profits by the difference between the marginal revenue and the marginal cost of production.[7]

Condition (15) implicitly defines a function of abatement with respect to the standard, $w(\bar{s})$, that does not depend of the pollution stock so that production can be also written as a function of the standard: $q(\bar{s}, w(\bar{s}))$.

In the first stage, the regulator maximizes the discounted present value of net social welfare with respect to the standard

$$\max_{\{\bar{s}\}} \int_0^{\infty} e^{-rt}\left\{\int_0^{q(\bar{s})} P(y)dy - C(q(\bar{s}), w(\bar{s})) - D(x)\right\} dt$$

$$s.t.\ \dot{x} = \bar{s} - \delta x, \quad x(0) = x_0 \geq 0.$$

The solution to this dynamic optimization problem must satisfy the following HJB equation

$$rW(x) = \max_{\{\bar{s}\}}\left\{\int_0^{q(\bar{s})} P(y)dy - C(q(\bar{s}), w(\bar{s})) - D(x) + W'(x)(\bar{s} - \delta x)\right\}.$$

[7]Notice that $\partial q/\partial w = -\partial s/\partial w/\partial s\partial q > 0$. Thus an increase in abatement for a given standard gives an increase in output.

In this case, the maximization of the right-hand side of the HJB equation gives the following FOC

$$\left(P - \frac{\partial C}{\partial q}\right)\frac{\partial q}{\partial \bar{s}} - \frac{\partial C}{\partial w}\frac{\partial w}{\partial \bar{s}} = -W'. \tag{16}$$

Assuming that an increase in the standard reduces the abatement, the left-hand side represents the marginal benefits of the standard that has two components.[8] The first one stands for the increase in benefits because of the increase in output when the standard augments whereas the second component stands for the reduction in abatement costs.[9] On the other hand, the social shadow price of the pollution stock gives the marginal cost of increasing the standard.

Next, we show that this equilibrium coincides with the SFSE when the regulator uses a tax. First, it can be shown that conditions (6) and (11) that characterize the SFSE when a tax is used summarize in condition (15). From condition (6) we obtain

$$\tau - V' = -\frac{\frac{\partial C}{\partial w}}{\frac{\partial s}{\partial w}},$$

that by substitution in (11) gives

$$P'q + P = \frac{\partial C}{\partial q} - \frac{\frac{\partial C}{\partial w}}{\frac{\partial s}{\partial w}}\frac{\partial s}{\partial q},$$

where according to footnote 9

$$-\frac{\frac{\partial s}{\partial q}}{\frac{\partial s}{\partial w}} = \frac{1}{\frac{\partial q}{\partial w}}.$$

Then, the condition can be written as follows

$$\left(P'q + P - \frac{\partial C}{\partial q}\right)\frac{\partial q}{\partial w} = \frac{\partial C}{\partial w},$$

that is condition (15).

Moreover, it is easy to show using the emission function that the strategies obtained in the first stage to calculate the optimal tax are equivalent to the strategies used to calculate the optimal standard resulting in two dynamic optimization problems that yield the same

[8]Notice that $\partial q/\partial \bar{s} = 1/\partial s \partial q > 0$. Thus an increase in the standard for a given level of abatement results in an increase in output.

[9]Although we expect that an increase in the standard reduces the abatement, it must be pointed out that this will depend on the sign of the emission function cross effects.

production, abatement and emissions for the same levels of pollution stocks. From (6) and (11), we can write the production and abatement as functions of $\tau - V'$: $q(\tau - V')$ and $w(\tau - V')$ so that by substitution in the emission functions we obtain the following expression

$$s = s(q(\tau - V'), w(\tau - V')) = s(\tau - V'),$$

that can be used to write $\tau - V'$ as a function of emissions: $\tau - V' = g(s)$. Then, by substitution we obtain

$$\begin{aligned} q(\tau - V') &= q(g(s)) = q(s), \\ w(\tau - V') &= w(g(s)) = w(s), \end{aligned}$$

where $q(s)$ and $w(s)$ are used to calculate the optimal standard and $q(\tau - V')$ and $w(\tau - V')$ to calculate the optimal tax.

Thus, we can conclude that

Proposition 3 *The tax and the standard are two equivalent policy instruments in the sense that they yield the same outcomes for the policy game, i.e. the same production, abatement, emissions, net social welfare and gross profits.*

This is an interesting result for two reasons. The first reason is that the coincidence between the two equilibria occurs even when the rationality behind the behavior of the firm is different. When the regulator uses an emission tax, we face a differential game where the firm maximizes the discounted present value of net profits, i.e. the firm takes into account how its decisions today can influence its decisions tomorrow taking the dynamics of the pollution stock as a constraint in its optimization problem. However, when the regulator applies a standard the firm acts *myopically.* However, the outcome is the same in both cases. The regulator taking into account the reaction function $w(\bar{s})$ defined by the FOC (15) or taking into account the strategies $q(\tau, x)$ and $w(\tau, x)$ defined by the FOCs (6) and (11) implements the same regulated market equilibrium in terms of production, abatement, emissions, net social welfare and gross profits. The second reason is that given this coincidence, the regulator can replicate the same outcome obtained using a tax applying a standard avoiding the concerns the tax can originate because it implies a subsidy on emissions for low values of the pollution stock. The alternative of postponing the environmental regulation until the level of the pollution stock is large enough to justify a tax is clearly dominated in terms of welfare by the use of a standard that can be implemented for any level of the pollution stock. Notice that the market is inefficient for any level of the pollution stock hence postponing the regulation of the market will lead to a lower discounted present value of net social welfare.

In the next section we solve a linear quadratic (LQ) (differential) policy game for illustrating these results and evaluate the effects of a second-best environmental policy.

5 The LQ Policy Game

The LQ differential game we analyze in this section is an extension of the LQ differential game studied by Benchekroun and Long (1998) to include abatement activities. It considers a monopolist that faces a linear (inverse) demand function given by $P(t) = a - q(t)$, where $P(t)$ is the price and $q(t)$ is the output at time t. The production process generates pollution emissions. After an appropriate choice of measurement units we can say that each unit of output generates one unit of pollution. The emissions can be reduced without declining output if the monopoly employs an abatement technology. The abatement technology is assumed to be the end-of-the-pipe type.[10] For this type of abatement technology the emission function is $s(q(t), w(t)) = q(t) - w(t)$ where $w(t)$ stands for the emission reduction achieved operating the abatement technology. On the other hand, we assume an additive and separable cost function $C(q(t), w(t)) = cq(t) + \gamma w(t)^2/2$. The abatement technology has decreasing returns to scale, with the parameter γ measuring the extent of such decreasing returns, and the production technology presents constant returns to scale, with the parameter c standing for the marginal cost of production. In

[10] This assumption has been extensively used in the literature among others by Petrakis and Xepapadeas (2003), Poyago-Theotoky and Teerasuwannajak (2002) in a static context. In a dynamic context has been used for instance by Yanese (2009).

this case, the stock of pollution follows the dynamic equation

$$\dot{x}(t) = s(q(t), w(t)) - \delta x(t) = q(t) - w(t) - \delta x(t), \quad x(0) = x_0 \geq 0. \tag{17}$$

The disutility from environmental deterioration is given by the damage function $D(x(t)) = dx(t)^2/2,\ d > 0$.

Next, characterize the first-best policy calculating the SFSE of a differential game between the regulator and the monopolist when the regulator uses an emission tax and a subsidy on output.

5.1 The First-Best Policy

For this LQ policy game, the FOCs (5) and (6) define the dependence of production and abatement with respect to the tax and subsidy

$$q(\tau, v, x) = \frac{1}{2}\left(a + v - c - (\tau - V'(x))\right), \tag{18}$$

$$w(\tau, v, x) = \frac{1}{\gamma}(\tau - V'(x)). \tag{19}$$

Therefore, $\partial q/\partial \tau = -1/2,\ \partial w/\partial \tau = 1/\gamma,\ \partial q/\partial v = 1/2$ and $\partial w/\partial v = 0$. For a given level of the pollution stock, the tax reduces output but increases abatement whereas the subsidy has a positive effect on production but not effect on abatement. Nevertheless, Prop. 1 holds and the optimal tax and subsidy are given by

$$\tau^*(x) = -W'(x) + V'(x), \tag{20}$$

$$v^*(x) = a - c + W'(x). \tag{21}$$

Next, substituting $\tau^* - V'$ in (18) and (19) and v^* in (18), we obtain the optimal strategies for production and abatement

$$q^*(x) = a - c + W'(x), \tag{22}$$

$$w^*(x) = -\frac{1}{\gamma}W'(x). \tag{23}$$

Then, emissions can be obtained as the difference between production (gross emissions) and abatement

$$s^*(x) = q^*(x) - w^*(x) = a - c + \frac{\gamma + 1}{\gamma}W'(x). \tag{24}$$

Now, substituting production and abatement by the efficient strategies (22) and (23) in the regulator's HJB equation

$$rW(x) = aq - \frac{1}{2}q^2 - cq - \frac{\gamma}{2}w - \frac{d}{2}x^2 + W'(x)(q - w - \delta x) \tag{25}$$

the following nonlinear differential equation is obtained

$$rW(x) = \frac{1}{2}(a-c)^2 + (a-c)W'(x) + \frac{\gamma+1}{2\gamma}W'(x)^2 - \frac{d}{2}x^2 - \delta x W'(x). \tag{26}$$

In order to find the solution for this equation, we guess a quadratic representation for the value function W :

$$W^*(x) = \frac{A_r^*}{2}x^2 + B_r^* x + C_r^*,$$

which implies that $dW^*(x)/dx = A_r^* x + B_r^*$ and where $A_r^*,\ B_r^*$ and C_r^* are unknowns to be determined.

The substitution of $W^*(x)$ and $dW^*(x)/dx$ into (26) gives a system of Riccati equations that must be satisfied for every x. Selecting the stable solution of this system, we obtain the following values for the coefficients of the regulator's value function

$$A_r^* = \frac{\gamma(r+2\delta) - (\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5}}{2(\gamma+1)} < 0, \tag{27}$$

$$B_r^* = \frac{(a-c)\gamma A_r^*}{\gamma(r+\delta) - (\gamma+1)A_r^*} < 0, \tag{28}$$

that allow us to calculate the optimal strategies for production, abatement and emissions

$$q^*(x) = \frac{(a-c)(\gamma(r+\delta) - A_r^*)}{\gamma(r+\delta) - (\gamma+1)A_r^*} + A_r^* x, \tag{29}$$

$$w^*(x) = \frac{(a-c)A_r^*}{(\gamma+1)A_r^* - \gamma(r+\delta)} - \frac{A_r^*}{\gamma}x, \tag{30}$$

$$s^*(x) = \frac{\gamma(a-c)(r+\delta)}{\gamma(r+\delta) - (\gamma+1)A_r^*} + \frac{\gamma+1}{\gamma}A_r^* x. \tag{31}$$

Using the optimal strategy for emissions and the differential equation (17), and taking into account the first Riccati equation for A_r^*, we obtain the steady-state pollution stock

$$x_{SS}^* = \frac{(a-c)\gamma(r+\delta)}{(\gamma+1)d + \gamma\delta(r+\delta)}. \tag{32}$$

This expression defines a negative relationship between the steady-state pollution stock and d, the slope of the marginal damages function. Thus, we find that the larger the marginal damages the lower the accumulation of emissions at the steady state.

According to the optimal strategies, production and emissions decrease with the pollution stock whereas abatement is increasing. Thus, there will exist a level for the pollution stock for which emissions are zero. To calculate this value we use the equation $s^*(x) = 0$ that yields[11]

$$x_s^* = \frac{(a-c)\gamma(r+\delta)}{(\gamma+1)(\delta A_r^* + d)} > 0. \tag{33}$$

The difference of this value with the steady-state pollution stock is

$$x_{SS}^* - x_s^* = \frac{(a-c)\gamma(r+\delta)((\gamma+1)\delta A_r^* - \gamma\delta(r+\delta))}{((\gamma+1)d + \gamma\delta(r+\delta))(\gamma+1)(\delta A_r^* + d)} < 0,$$

since $\delta A_r^* + d$ is positive. Then, we can conclude that

Proposition 4 *The optimal strategies of the efficient solution satisfy the nonnegativity constraint in the interval $[0, x_s^*]$ that includes the steady-state pollution stock. In this interval, production and emissions decrease and abatement increases with the pollution stock.*

Finally, we characterize the dynamics of the pollution stock. Substituting emissions given by (31) in the dynamics of the pollution stock defined by (17), we obtain the following differential equation for the pollution stock

$$\dot{x} = \frac{(a-c)\gamma(r+\delta)}{\gamma(r+\delta) - (\gamma+1)A_r^*} + \left(\frac{\gamma+1}{\gamma}A_r^* - \delta\right)x,$$

whose solution is

$$x^*(t) = (x_0 - x_{SS}^*)e^{\alpha^* t} + x_{SS}^*, \text{ with } \alpha^* = \frac{\gamma+1}{\gamma}A_r^* - \delta < 0, \tag{34}$$

for x_0 in the interval $[0, x_s^*]$. Then, the dynamics of the model can be summarized as follows

Remark 1 *If x_0 is lower than x_{SS}^*, abatement increases asymptotically to its steady-state value whereas production and emissions decrease. However, if x_0 is larger than x_{SS}^*, the dynamics is the contrary and abatement decreases asymptotically to its steady-state value whereas production and emissions increase.*

[11] As $s(x)$ is a decreasing linear function whose intersection point with the vertical axis is positive, the intersection point with the horizontal axis must be positive which implies that $\delta A_r^* + d$ is positive.

Once, we have the solution for the regulator's value function, we can calculate the optimal subsidy using the expression (21).[12] However, for calculating the optimal tax we need to solve the monopolist's HJB equation. With this aim, we substitute the tax and the subsidy given by (20) and (21), and abatement and production defined by (22) and (23) in the monopolist's HJB equation

$$rV(x) = (a-q)q - cq - \frac{\gamma}{2}w^2 - \tau(q-w) + vq + V'(x)(q-w-\delta x),$$

and we obtain the following differential equation

$$rV(x) = (a-c+W'(x))^2 + \frac{1}{2\gamma}W'(x)^2 - \delta x V'(x). \tag{35}$$

In order to solve this equation, we also guess a quadratic representation

$$V^*(x) = \frac{A_m^*}{2}x^2 + B_m^* x + C_m^*,$$

that yields $dV^*(x)/dx = A_m^* x + B_m^*$. The substitution of $V^*(x)$ and $dV^*(x)/dx$ along with $dW^*(x)/dx$ into (35) gives a system of Riccati equations whose solution is

$$A_m^* = \frac{2\gamma+1}{\gamma(r+2\delta)}(A_r^*)^2 > 0, \tag{36}$$

$$B_m^* = \frac{(a-c)\gamma((r+2\gamma(r+\delta))A_r^* - d)}{(\gamma+1)(r+\delta)(\gamma(r+\delta)-(\gamma+1)A_r^*)} < 0, \tag{37}$$

where $A_r^* < 0$ is given by (27). Then, eliminating $V'(x)$ and $W'(x)$ in (20) using the coefficients of the value functions, the optimal tax is obtained.[13]

Proposition 5 *The optimal policy is given by the following rule*

$$\tau^*(x) = \frac{(a-c)(\gamma(r+\delta)A_r^* - (A_r^*)^2)}{(r+\delta)(\gamma(r+\delta)-(\gamma+1)A_r^*)} + \frac{d+(A_r^*)^2}{r+2\delta}x, \tag{38}$$

where $A_r^* < 0$ is given by (27). τ^* increases with the pollution stock but is negative for $x = 0$, i.e. the optimal policy consists of setting up a subsidy for low values of the pollution stock.

[12] The subsidy coincides in our model with the production quantity because the slope of the demand function is -1. This establishes that the subsidy decreases with the pollution stock.

[13] Notice that the monopolist's value function has a minimum for a positive value of the pollution stock. It is easy to verify that this minimum is larger than x_s^* ensuring that $dV^*(x)/dx$ is negative in the interval $[0, x_s^*]$.

This result states that even when a subsidy on production is used to correct the market failure caused by the market power of the firm, the Pigouvian tax in this dynamic setting could be negative resulting in a subsidy on emissions for low values of the pollution stock. In fact, we cannot discard the possibility that a subsidy on emissions applies at the steady state. Basically, the sign of the environmental policy at the steady state will depend on the importance of the marginal damages, in particular, on the value of parameter d. The negative sign of the optimal policy also says us that the difference between the social and private valuations of the pollution stocks is not necessary positive for all values of the pollution stock. The previous proposition establishes that for low values of the pollution stock, its private shadow price is larger than its social shadow price, in other words, the firm's valuation is larger than the regulator's valuation. This divergence between both valuations comes mainly from the fact that when the pollution stock is low, the term that reflects the environmental damages in the regulator's HJB equation is low even if parameter d is large. The consequence is that regardless of the value of this parameter the social shadow price of the stock is below the private shadow price for low values of the pollution stock. An interesting result that clarifies that fact that the private valuation not necessarily is below the social valuation for all values of the pollution stock.

5.2 The Second-Best Emission Tax

When the regulator can only use a tax to control emissions the expressions (18) and (19) define the dependence of the production and abatement with respect to tax just cancelling v in (18).

$$q(\tau, v, x) = \frac{1}{2}\left(a - c - (\tau - V'(x))\right), \tag{39}$$

$$w(\tau, v, x) = \frac{1}{\gamma}(\tau - V'(x)). \tag{40}$$

Again, $\partial q/\partial \tau = -1/2$ and $\partial w/\partial \tau = 1/\gamma$, and consequently Prop. 2 holds, i.e. the second-best emission tax with commitment is lower than the difference between the social and the private valuations of the pollution stock. Taking into account these partial derivatives, condition (8) along with (39) and (40) define the optimal tax[14]

$$\tau^{sb}(x) = V'(x) - \frac{\gamma(a-c) + 2(\gamma+2)W'(x)}{\gamma+4}. \tag{41}$$

Next, substituting $\tau - V'$ in (39) and (40) the optimal strategies for output and abatement read

$$q^{sb}(x) = \frac{\gamma+2}{\gamma+4}(a - c + W'(x)), \tag{42}$$

$$w^{sb}(x) = -\frac{\gamma(a-c) + 2(\gamma+2)W'(x)}{\gamma(\gamma+4)}. \tag{43}$$

Notice that as both output and abatement depend on $\tau - V'$, finally the optimal strategies of these two control variables are independent of the first derivative of the monopolist's value function. The emissions can be obtained as the difference between output (gross emissions) and abatement

$$s^{sb}(x) = q(x) - w(x) = \frac{\gamma(\gamma+3)(a-c) + (\gamma+2)^2 W'(x)}{\gamma(\gamma+4)}. \tag{44}$$

Now, substituting the optimal strategies (42) and (43) in the regulator's HJB equation (25), the following nonlinear differential equation is obtained

$$rW(x) = \frac{\gamma+3}{2(\gamma+4)}(a-c)^2 + \frac{\gamma+3}{\gamma+4}(a-c)W'(x) + \frac{(\gamma+2)^2}{2\gamma(\gamma+4)}W'(x)^2 - \frac{d}{2}x^2 - \delta x W'(x). \tag{45}$$

In order to solve this equation, we guess a quadratic representation for the value function W:

$$W^{sb}(x) = \frac{A_r^{sb}}{2}x^2 + B_r^{sb}x + C_r^{sb}, \tag{46}$$

which implies that $dW^{sb}(x)/dx = A_r^{sb}x + B_r^{sb}$ and where A_r^{sb}, B_r^{sb} and C_r^{sb} are unknowns to be determined.

The substitution of $W^{sb}(x)$ and $dW^{sb}(x)/dx$ into (45) yields a system of Riccati equations that must hold for every x. Choosing the stable solution of this system, we derive the following values for the coefficients of the regulator's value function

$$A_r^{sb} = \frac{\gamma(\gamma+4)(r+2\delta) - (\gamma(\gamma+4)(4d(\gamma+2)^2 + \gamma(\gamma+4)(r+2\delta)^2))^{0.5}}{2(\gamma+2)^2} < 0, \tag{47}$$

$$B_r^{sb} = \frac{(a-c)\gamma(\gamma+3)A_r^{sb}}{\gamma(\gamma+4)(r+\delta) - (\gamma+2)^2 A_r^{sb}} < 0, \tag{48}$$

[14] Where the superscript sb stands for the second-best emission tax.

that using (42), (43) and (44) allows us to calculate the optimal strategies for production, abatement and emissions

$$
\begin{aligned}
q^{sb}(x) &= \frac{(a-c)(\gamma+2)(A_r^{sb}-\gamma(r+\delta))}{(\gamma+2)^2A_r^{sb}-\gamma(\gamma+4)(r+\delta)} + \frac{(\gamma+2)A_r^{sb}}{\gamma+4}x, && (49)\\
w^{sb}(x) &= \frac{(a-c)(\gamma(r+\delta)+(\gamma+2)A_r^{sb})}{(\gamma+2)^2A_r^{sb}-\gamma(\gamma+4)(r+\delta)} - \frac{2(\gamma+2)A_r^{sb}}{\gamma(\gamma+4)}x, && (50)\\
s^{sb}(x) &= \frac{(a-c)\gamma(\gamma+3)(r+\delta)}{\gamma(\gamma+4)(r+\delta)-(\gamma+2)^2A_r^{sb}} + \frac{(\gamma+2)^2A_r^{sb}}{\gamma(\gamma+4)}x, && (51)
\end{aligned}
$$

Next, we calculate the steady-state pollution stock substituting emissions in the differential equation (17) by (51), taking into account the first Riccati equation for A_r^{sb}.

$$
x_{SS}^{sb} = \frac{(a-c)\gamma(\gamma+3)(r+\delta)}{(\gamma+2)^2d+\gamma\delta(\gamma+4)(r+\delta)}. \tag{52}
$$

As in the previous subsection, this expression clearly establishes an inverse relationship between the pollution stock at the steady state and d, the slope of the marginal damages curve.

According to the optimal strategies, production and emissions decrease with the pollution stock whereas abatement is increasing. Thus, there is a level for the pollution stock for which emissions are zero. We calculate this value doing $s^{sb}(x) = 0$. The solution to this equation is

$$
x_s^{sb} = \frac{\gamma(a-c)(\gamma+3)(r+\delta)}{(\gamma+2)^2(\delta A_r^{sb}+d)} > 0. \tag{53}
$$

The difference of this value with the steady-state pollution stock is

$$
x_{SS}^{sb} - x_s^{sb} = \frac{(a-c)\gamma(\gamma+3)(r+\delta)((\gamma+2)^2\delta A_r^{sb}-\gamma(\gamma+4)(r+\delta)\delta)}{((\gamma+2)^2d+\gamma(\gamma+4)(r+\delta)\delta)(\gamma+2)^2(\delta A_r^{sb}+d)} < 0,
$$

since δA_r^{sb} is positive. Moreover, it is easy to show substituting A_r^{sb} in the expression $\gamma(r+\delta)+(\gamma+2)A_r^{sb}$ of the optimal strategy (50) that $w^{sb}(0)$ is positive provided that d is larger than the critical value

$$
d_w = \frac{\gamma(r+\delta)((r+\delta)(\gamma+2)+(\gamma+4)(r+2\delta))}{(\gamma+4)(\gamma+2)}. \tag{54}
$$

Then, we can conclude that

Proposition 6 *If d is larger than d_w, the optimal strategies of the second-best solution satisfy the nonnegativity constraint in the interval $[0, x_s^{sb}]$ that includes the steady-state pollution stock. In this interval, production and emissions decrease and abatement increases with the pollution stock.*

Next, we characterize the dynamics of the pollution stock. Substituting emissions defined by (51) in the dynamics of the pollution stock given by (17), we get the following differential equation

$$\dot{x} = \frac{(a-c)\gamma(\gamma+3)(r+\delta)}{\gamma(\gamma+4)(r+\delta)-(\gamma+2)^2 A_r^{sb}} + \left(\frac{(\gamma+2)^2}{\gamma(\gamma+4)} A_r^{sb} - \delta\right) x,$$

whose solution is

$$x^{sb}(t) = (x_0 - x_{SS}^{sb}) e^{\alpha^{sb} t} + x_{SS}^{sb}, \text{ with } \alpha^{sb} = \frac{(\gamma+2)^2}{\gamma(\gamma+4)} A_r^{sb} - \delta < 0, \tag{55}$$

for x_0 in the interval $[0, x_s^{sb}]$. Then, the dynamics of the model coincides with that summarized in Remark 1 for the efficient solution.

The features of the model allow to calculate the optimal strategies for production, abatement and net emissions without solving the monopolist's HJB equation. However, the next step, the calculation of the regulator optimal policy, cannot be given without solving this equation. With this aim, we substitute the tax given by (41), the output defined by (42) and the abatement specified by (43) in the monopolist's HJB equation

$$rV(x) = (a-q)q - cq - \frac{\gamma}{2} w^2 - \tau(q-w) + V'(x)(q - w - \delta x),$$

obtaining the following differential equation

$$\begin{aligned} rV(x) &= \frac{2\gamma^2 + 9\gamma + 8}{2(\gamma+4)^2}(a-c)^2 + \frac{2(\gamma+3)(\gamma+2)}{(\gamma+4)^2}(a-c)W'(x) \\ &+ \frac{(\gamma+2)^3}{\gamma(\gamma+4)^2} W'(x)^2 - V'(x)\delta x. \end{aligned} \tag{56}$$

In order to solve this equation, we also guess a quadratic representation

$$V^{sb}(x) = \frac{A_m^{sb}}{2} x^2 + B_m^{sb} x + C_m^{sb}, \tag{57}$$

that yields $dV^{sb}(x)/dx = A_2^{sb}x + B_2^{sb}$. The substitution of $V^{sb}(x)$ and $dV^{sb}(x)/dx$ along with $dW^{sb}(x)/dx$ into (56) yields a system of Riccati equations whose solution is

$$A_m^{sb} = \frac{2(\gamma+2)^3}{\gamma(r+2\delta)(\gamma+4)^2}(A_r^{sb})^2 > 0, \tag{58}$$

$$B_m^{sb} = \frac{2(a-c)\gamma\,(\gamma+3)\,(\gamma+2)A_r^{sb}}{(\gamma+4)(\gamma(\gamma+4)(r+\delta)-(\gamma+2)^2A_r^{sb})} < 0, \tag{59}$$

where $A_r^{sb} < 0$ is given by (47). Then eliminating $V'(x)$ and $W'(x)$ in (41) using the coefficients of the value functions, the optimal policy is obtained.[15]

Proposition 7 *The optimal policy is given by the following rule*

$$\tau^{sb}(x) = -\frac{\gamma(a-c)}{\gamma+4} + \frac{2(\gamma+2)d}{(r+2\delta)(\gamma+4)}x. \tag{60}$$

τ^{sb} *increases with the pollution stock but the optimal policy consists of setting up a subsidy for low values of the pollution stock.*

We observe that as occurs with the first-best policy, the tax becomes a subsidy for low values of the pollution stock. According to (13), in a second-best setting the negative sign could appears even when the social shadow price of the pollution stock is larger than the private shadow price because the tax has an additional negative term since in this setting we have only one instrument to address two market distortions.

5.3 The Emission Standard

As explained in Subsection 4.2, a standard could be applied to avoid an environmental policy that recommends a subsidy on emissions above all if the standard can implement the same outcome getting the same level of net social welfare. Following the procedure presented in Subsection 4.2 first we write the output as a function of the standard and abatement: $q = \bar{s} + w$, where $\bar{s}$ represents the standard selected by the regulator. Then by substitution in the profit function we obtain the following expression

$$\pi = (a - (\bar{s}+w))(\bar{s}+w) - c(\bar{s}+w) - \frac{\gamma}{2}w^2.$$

[15] As occurred in the previous subsection the monopolist's value function has a minimum for a positive value of the pollution stock. It is easy to check that this minimum is larger than x_s^{sb} ensuring that dV^{sb}/dx is negative in the interval $[0, x_s^{sb}]$.

The FOC for the maximization of profits is

$$a - 2(\bar{s} + w) = c + \gamma w, \tag{61}$$

that requires that the marginal revenue must be equal to marginal cost of production plus the marginal cost of abatement. This FOC defines the reaction function of the firm

$$w(\bar{s}) = \frac{a - c - 2\bar{s}}{\gamma + 2}, \quad \frac{dw}{d\bar{s}} = -\frac{2}{\gamma + 2} < 0. \tag{62}$$

that establishes that abatement is a *strategic substitute* of the standard. Using this reaction function we can obtain by substitution the reaction function for production

$$q(\bar{s}) = \bar{s} + w(\bar{s}) = \frac{a - c + \gamma\bar{s}}{\gamma + 2}, \quad \frac{dq}{d\bar{s}} = \frac{\gamma}{\gamma + 2} > 0. \tag{63}$$

that defines production as a *strategic complement* of the standard.

Given the firm's reaction functions for production and abatement, the regulator selects the standard that maximizes the discounted present value of the net social welfare.

The HJB equation for this optimization problem is

$$rW(x) = \max_{\{\bar{s}\}} \left\{ aq(\bar{s}) - \frac{1}{2}q(\bar{s})^2 - cq(\bar{s}) - \frac{\gamma}{2}w(\bar{s})^2 - \frac{d}{2}x^2 + W'(x)(\bar{s} - \delta x) \right\}, \tag{64}$$

and the FOC can be written as follows

$$(a - q(\bar{s}) - c)\frac{dq}{d\bar{s}} - \gamma w(\bar{s})\frac{dw}{d\bar{s}} = -W'(x), \tag{65}$$

that has the same interpretation than the FOC (16). Notice that as abatement is a strategic substitute of the standard, an increase in the standard will reduce the abatement costs. Finally, substituting $q(\bar{s})$, $dq/d\bar{s}$, $w(\bar{s})$ and $dw/d\bar{s}$ in (65) we obtain that the optimal strategy for the standard is equal to the optimal strategy for emissions (44) and that the substitution of this strategy in (62) and (63) yields the optimal strategies for production and abatement (42) and (43). Then, as the HJB equation is the same for both policy instruments, we would obtain the same optimal strategies than those derived when the regulator applies a tax on emissions.

5.4 The Effects of the Second-Best Environmental Policy

Next, we investigate the effects that the constraint on the menu of policy instruments has on the main variables of the model. Notice that the effects will be the same regardless a tax or a standard is applied. We begin the analysis calculating the difference between the steady-state values of the pollution stock. The result is

$$x_{SS}^{*} - x_{SS}^{sb} = \frac{(a-c)\gamma(r+\delta)(d+\gamma\delta(r+\delta))}{((\gamma+1)d+\gamma\delta(r+\delta))((\gamma+2)^2 d+\gamma\delta(\gamma+4)(r+\delta))} > 0, \qquad (66)$$

that allows us to conclude that

Proposition 8 *The steady-state pollution stock for the efficient solution is larger than the steady-state pollution stock for the second-best solution, i.e.* $x_{SS}^{*} > x_{SS}^{sb}$ *and consequently the steady-state emissions are also larger.*

The intuition behind this result is as follows. The elimination of the subsidy on production will lead to the firm to produce less. An incentive that is not compensated by the tax resulting finally in less production (gross emissions) and less emissions what explains a lower accumulation of emissions when a second-best policy is used. Moreover, the elimination of the subsidy reduces net social welfare. To give support to this intuition we compare the optimal strategies for production (29) and (49). The following proposition summarizes the result of this comparison.

Proposition 9 *The intersection point and the absolute value of the slope for the production optimal strategy of the efficient solution are larger than those corresponding to the second-best solution. However, the pollution stock for which production is zero is lower.*

Proof. See Appendix. ■

However, the optimal strategy for production when a second-best policy is used by the regulator is not completely below the optimal strategy corresponding to the efficient solution so that emissions could be larger for the second-best solution for large enough values of the pollution stock. Nevertheless, it must be taken into account that as abatement is increasing, emissions will be zero for a pollution stock lower that the pollution stock for which production is zero. In other words, it is possible that for the interval of pollution stock values for which emissions are positive, gross emissions for the second-best solution be lower than gross efficient emissions. To verify this relationship holds, next we compare the optimal strategies for emissions.

The comparison gives the following result

Proposition 10 *The intersection point and the absolute value of the slope for the emission optimal strategy of the efficient solution are large than those corresponding to the second-best solution. Moreover, the pollution stock for which emissions are zero is also larger provided that damages are not very low.*

Proof. See Appendix.[16] ■

▶ FIGURE 1

Fig. 1 shows the relationship between the optimal strategies for emissions and the steady-state values of the pollution stock and emissions. The figure confirms that, regardless of the changes in the abatement optimal strategy, the final effect of the changes in the production optimal strategy leads to less emissions for any pollution stock in the interval $[0, x_s^{sb}]$. This gives support to the idea that the cancellation of the subsidy on production is not compensated by the changes in the optimal strategy of the tax, and the firm reduces production and gross emissions resulting in a reduction of emissions that leads to a lower steady-state pollution stock for the second-best solution.

We end this subsection comparing the dynamics of the pollution stock. Using (34) and (55), the comparison yields the following results

Proposition 11 *If x_0 is lower or equal to x_{SS}^{e}, the optimal temporal path of the pollution stock for the efficient solution is above the optimal temporal path for the second-best solution. However, if x_0 is larger than x_{SS}^{e}, the pollution stock for the second-best solution*

[16] We would like to point out that the condition that appears in the proposition is a sufficient condition so that the pollution stock for which emissions are zero for the efficient solution could be larger for any level of environmental damages.

could be initially larger than the pollution stock for the efficient solution although finally this relationship will reverse.

Proof. See Appendix. ■

In the Appendix, we present a condition the initial pollution stock must satisfied to conclude that the pollution stock for the second-best solution is initially larger than the pollution stock for the efficient solution but we do not have to expect that this relationship occurs except for high initial levels of the pollution stock. Obviously, if the firm starts to operate on a pristine (nonpolluted) environment and the regulation is applied from the beginning of the productivity activities, we should expect a level of the pollution stock below the efficient level when a second-best environmental policy is applied for the entire temporal horizon. Even if the regulation is not applied from the moment the firm starts to operate, we should expect a lower pollution stock if a second-best environmental policy is applied provided that the initial stock is below its steady-state value.

6 Conclusions

In this paper we have characterized the optimal environmental policy for a polluting monopoly that devotes resources to abatement activities so that emissions can be reduced without affecting necessarily production. As we focus on the case of a stock pollutant we have used a differential game between a welfare maximizing regulator and a profit maximizing monopolist and to characterize the optimal policy we have calculated the stagewise feedback Stackelberg equilibrium (SFSE) of the game. Our analysis shows that the first-best policy consists of applying a Pigouvian tax on emissions and a subsidy on production equal to the difference between the price and the marginal revenue. However, we find that the Pigouvian tax is not equal to the marginal environmental damages but to the difference between the *social* and *private* valuation of the pollution stock. On the other hand, if a second-best emission tax is used to control emissions, the tax is lower than the Pigouvian tax and the difference between these two taxes decreases with the price elasticity of the demand. Finally, we find that taxes and standards are equivalent in a second-best setting, both implement the same SFSE.

In the second part of the paper, we solve a linear-quadratic differential game for an end-of-pipe abatement technology. When a first-best policy is applied, we find that the tax increases with the pollution stock whereas the subsidy decreases but that the tax is negative for low values of the pollution stock regardless of the importance of the environmental damages. Something that cannot occur for a flow pollutant. The explanation of this result is given by the fact that the private valuation of the pollution stock is larger than the social valuation resulting in a subsidy instead of a tax. The private valuation is larger than the social valuation because the firm knows that more production and emissions will yield larger taxes for the rest of the game whereas for low values of the pollution stock marginal damages are low and an increase in the pollution stock will not have a big effect on social welfare. This divergence generates that production is below the efficient level despite the subsidy on production and that the optimal policy is a subsidy on emissions instead of a tax. The same occurs when a second-best tax is used. Notice that in this case, the negative effect of the private valuation is reinforced but the negative term in the tax that corrects the market power of the firm. Both the first-best policy and a second-best tax could face difficulties to be applied because of political constraints given that they are recommending that emissions should be subsidized and in the case of the first-best policy that also production should be subsidized. In a second-best setting, this difficulty could be overcome applying a standard that yields the same outcome than that obtained applying a tax. Finally, we find that the steady-state pollution stock when a first-best policy is applied is larger than the steady-state pollution for the second-best solution. This difference in the accumulated emissions is drove by the fact that the subsidy on production is not applied when a second-best policy is being used what leads to the firm to produce and emit more. Our result clearly establishes that this effect is not compensated by the tax and the result is a lower steady-state pollution stock when a second-best policy is applied. Moreover, we find that if the initial pollution stock is not very large the optimal temporal path of the pollution stock for the first-best policy is above the optimal temporal path for a second-best policy.

A limitation of our analysis is that we have assumed the simplest form of the emission function, i.e. one that is additively separable in production (gross emissions) and abatement. To address this limitation, an interesting extension would be to consider that abatement expenditures can reduce the emissions-to-output ratio. Moreover, we could also consider that the abatement capital can be adjusted through investment. This second approach would allow to study the dynamic interaction between the accumulation of emissions and the accumulation of abatement capital. A further step in this line of research would be to analyze the environmental policy when the abatement technology is subject to stochastic innovation. Finally, it would be also interesting to know how the environmental policy would change if the market structure is an oligopoly.

Appendix

Proof of Proposition 9

Let's suppose that $q^{sb}(0) \geq q^*(0)$. Then according to (29) and (49) we have that

$$\frac{(\gamma+2)(\gamma(r+\delta)-A_r^{sb})}{\gamma(\gamma+4)(r+\delta)-(\gamma+2)^2 A_r^{sb}} \geq \frac{\gamma(r+\delta)-A_r^*}{\gamma(r+\delta)-(\gamma+1)A_r^*}.$$

Cross multiplying and adding terms we obtain the following inequality

$$-2\gamma^2(r+\delta)^2-\gamma(r+\delta)(\gamma^2+2\gamma-2)A_r^*+(\gamma+1)\gamma(r+\delta)(\gamma+2)A_r^{sb}$$
$$-(\gamma+2)A_r^* A_r^{sb} \geq 0, \tag{67}$$

where $\gamma^2+2\gamma-2$ is zero for $\gamma = 0.732$. Thus, if $\gamma \leq 0.732$ we have a contradiction because all terms in (67) are negative. Suppose now that $\gamma > 0.732$ and that

$$-\gamma(r+\delta)(\gamma^2+2\gamma-2)A_r^*+(\gamma+1)\gamma(r+\delta)(\gamma+2)A_r^{sb} \geq 0 \tag{68}$$

in (67). Then, substituting A_r^* and A_r^{sb} by (27) and (47) respectively the previous inequality yields

$$(\gamma+1)\frac{\gamma(\gamma+4)(r+2\delta)-(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2))^{0.5}}{\gamma+2}$$
$$\geq (\gamma^2+2\gamma-2))\frac{\gamma(r+2\delta)-(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}}{\gamma+1},$$

that can be rewritten as follows

$$\gamma(\gamma^3+5\gamma^2+7\gamma+6)(r+2\delta)+(\gamma^2+2\gamma-2)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}$$

$$\geq (\gamma+1)^2(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5},$$

that squaring and simplifying terms gives

$$(\gamma^3+5\gamma^2+7\gamma+6)(r+2\delta)(\gamma^2+2\gamma-2)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}$$

$$\geq \gamma(\gamma^2+2\gamma-2)(\gamma^3+5\gamma^2+7\gamma+6)(r+2\delta)^2$$

$$+2d(\gamma+1)(\gamma^6+11\gamma^5+46\gamma^4+97\gamma^3+116\gamma^2+76\gamma+12).$$

Squaring again and ordering terms, we get the following contradiction

$$-4\gamma(\gamma^3+5\gamma^2+7\gamma+6)(\gamma^2+2\gamma-2)(\gamma+1)^3(\gamma^4+8\gamma^3+22\gamma^2+30\gamma+24)(r+2\delta)^2 d$$

$$-4d^2(\gamma+1)^2(\gamma^6+11\gamma^5+46\gamma^4+97\gamma^3+116\gamma^2+76\gamma+12)^2 \geq 0.$$

Thus, we have to conclude that (68) is negative and then (67) yields another contradiction and we can conclude that $q^{sb}(0) < q^*(0)$.

Next, we compare the slope of the strategies. Suppose that

$$\frac{(\gamma+2)A_r^{sb}}{\gamma+4} \leq A_r^*,$$

that substituting the A coefficients implies

$$\frac{\gamma(\gamma+4)(r+2\delta)-(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5}}{\gamma+2}$$

$$\leq (\gamma+4)\frac{\gamma(r+2\delta)-(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}}{\gamma+1},$$

that reordering terms yields

$$(\gamma+2)(\gamma+4)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}$$

$$\leq \gamma(\gamma+4)(r+2\delta)+(\gamma+1)(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5},$$

squaring to eliminate the quadratic root on the left-hand side of the inequality and simplifying term we obtain

$$2\gamma(\gamma^2+5\gamma+4)(r^2\gamma^2+4r^2\gamma+4r\gamma^2\delta+16r\gamma\delta+4\gamma^2\delta^2+6d\gamma^2+16\gamma\delta^2+24d\gamma+24d)$$

$$\leq 2\gamma(\gamma+4)(r+2\delta)(\gamma+1)(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5},$$

squaring again a contradiction is obtained

$$16d\gamma^2(\gamma^3+7\gamma^2+14\gamma+8)^2$$

$$(2r^2\gamma^2+8r^2\gamma+8r\gamma^2\delta+32r\gamma\delta+8\gamma^2\delta^2+9d\gamma^2+32\gamma\delta^2+36d\gamma+36d)\leq 0,$$

and we can conclude that

$$0>\frac{(\gamma+2)A_r^{sb}}{\gamma+4}>A_r^*.$$

Finally, we compare the values of the pollution stock for which production is zero. These values are

$$x_q^*=\frac{(a-c)(\gamma(r+\delta)-A_r^*)}{\gamma(d+\delta A_r^*)},\quad x_q^{sb}=\frac{(a-c)(\gamma(r+\delta)-A_r^{sb})}{\gamma(d+\delta A_r^{sb})},$$

and the difference is given by the following expression

$$x_q^{sb}-x_q^*=\frac{a-c}{\gamma}\frac{(\gamma(r+\delta)\delta+d)(A_r^*-A_r^{sb})}{(d+\delta A_r^{sb})\gamma(d+\delta A_r^*)}. \tag{69}$$

As the denominator is positive, the sign of the difference depends on the relationship between A coefficients. Using (27) and (47), the difference between the A coefficients can be written as follows

$$A_r^*-A_r^{sb}=\frac{1}{2}\left(\frac{(\gamma+1)(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5}}{(\gamma+1)(\gamma+2)^2}\right.$$
$$\left.-\frac{(\gamma+2)^2(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}+\gamma^2(r+2\delta)}{(\gamma+1)(\gamma+2)^2}\right). \tag{70}$$

As the denominator is positive, we focus on the sign of the numerator investigating the dependence of this sign with respect to parameter d. It is easy to check that for $d=0$, $A_r^*=A_r^{sb}$. Next, we calculate the first derivative of the numerator with respect to d.

$$2(\gamma+1)(\gamma+2)^2\left(\frac{\gamma^{0.5}(\gamma+4)^{0.5}}{(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5}}-\frac{\gamma}{(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5}}\right).$$

This derivative is zero for $d=0$. Calculating the second derivative we can find out whether this extreme is a maximum or a minimum. The second derivative of the numerator with respect to d is

$$-\frac{4(\gamma+1)\gamma^{0.5}(\gamma+4)^{0.5}(\gamma+2)^4}{(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{1.5}}+\frac{4(\gamma+2)^2(\gamma+1)^2\gamma^2}{(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{1.5}},$$

that takes the value

$$\frac{4(\gamma+1)(\gamma+2)^2}{(r+2\delta)^3(\gamma+4)} > 0$$

for $d = 0$. Thus, $d = 0$ is a minimum and the numerator of (70) is an increasing function for all $d > 0$ which allows us to conclude that $A_r^* - A_r^{sb}$ is positive and consequently that $x_q^{sb} - x_q^* > 0$.

Proof of Proposition 10

Suppose that $s^{sb}(0) \geq s^*(0)$. Then, according to optimal strategies (31) and (51) we obtain the following inequality

$$\frac{\gamma+3}{\gamma(\gamma+4)(r+\delta)-(\gamma+2)^2 A_r^{sb}} \geq \frac{1}{\gamma(r+\delta)-(\gamma+1)A_r^*}.$$

Cross multiplying and adding term it yields

$$-\gamma(r+\delta)-(\gamma+3)(\gamma+1)A_r^* + (\gamma+2)^2 A_r^{sb} \geq 0.$$

Next, we eliminate the A coefficients using (27) and (47) obtaining the following expressions

$$(\gamma+3)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5} \geq r\gamma + (\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2))^{0.5},$$

that squaring and simplifying terms yields the following contradiction

$$-\gamma^2(2\gamma+7)(r+2\delta)^2 - 4\gamma(\gamma^2+5\gamma+7)d - r^2\gamma^2$$

$$\geq 2r\gamma(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2))^{0.5} > 0,$$

and we can conclude that $s^{sb}(0) < s^*(0)$.

Next, we compare the slope of the strategies. Suppose that

$$\frac{(\gamma+2)^2 A_r^{sb}}{\gamma(\gamma+4)} \leq \frac{(\gamma+1)A_r^*}{\gamma},$$

that substituting the A coefficients gives

$$(\gamma(\gamma+4)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2))^{0.5} \geq (\gamma+4)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{0.5},$$

squaring to cancel the quadratic root in both sides of the inequality and simplifying term we obtain the following contradiction: $\gamma \leq 0$ and we can conclude that

$$0 > \frac{(\gamma+2)^2 A_r^{sb}}{\gamma(\gamma+4)} > \frac{(\gamma+1)A_r^*}{\gamma}.$$

Finally, we compare the values of the pollution stock for which emissions are zero. Using (33) and (53) the difference between the pollution stock is

$$x_s^{sb} - x_s^* = \frac{\gamma(a-c)(r+\delta)((\gamma+3)(\gamma+1)\delta A_r^* - (\gamma+2)^2\delta A_r^{sb} - d)}{(\gamma+2)^2(\gamma+1)(\delta A_r^{sb}+d)(\delta A_r^*+d)}. \tag{71}$$

As the denominator is positive and the the first three factors of the numerator are also positive, the sign of the difference depends on the sing of the fourth factor. Eliminating the A coefficients we find that

$$(\gamma+3)(\gamma+1)\delta A_r^* - (\gamma+2)^2\delta A_r^{sb} - d$$

$$= \frac{\delta}{2}\left(-\gamma(r+2\delta) - (\gamma+3)(\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5}\right.$$
$$\left. +(\gamma(\gamma+4)(4d(\gamma+2)^2 + \gamma(\gamma+4)(r+2\delta)^2))^{0.5}\right) - d. \tag{72}$$

It is easy to verify that this expression is zero for $d = 0$. Next, we investigate whether there exists another positive value for d for which the difference is zero. If this is the case, the value of d must satisfy that

$$-(\gamma+3)(\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5} + (\gamma(\gamma+4)(4d(\gamma+2)^2 + \gamma(\gamma+4)(r+2\delta)^2))^{0.5}$$
$$= \gamma(r+2\delta) + \frac{2}{\delta}d, \tag{73}$$

where the right-hand side is an increasing linear function of d with a slope equal to $2/\delta$. To know the behavior of the left-hand side with respect to d, we calculate its first derivative

$$-\frac{2\gamma(\gamma+1)(\gamma+3)}{(\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5}} + \frac{2\gamma^{0.5}(\gamma+2)^2(\gamma+4)^{0.5}}{(4d(\gamma+2)^2 + \gamma(\gamma+4)(r+2\delta)^2)^{0.5}}$$

that takes a value equal to $2/(r+2\delta)$ for $d = 0$. It is easy to check that this derivative is also positive for $d > 0$. Suppose that it is negative or zero. Then it must satisfy that

$$\frac{2\gamma^{0.5}(\gamma+2)^2(\gamma+4)^{0.5}}{(4d(\gamma+2)^2 + \gamma(\gamma+4)(r+2\delta)^2)^{0.5}} \leq \frac{2\gamma(\gamma+1)(\gamma+3)}{(\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5}},$$

what implies that

$$\gamma^{0.5}(\gamma+2)^2(\gamma+4)^{0.5}(\gamma^2(r+2\delta)^2 + 4(\gamma+1)\gamma d)^{0.5}$$

$$\leq \gamma(\gamma+1)(\gamma+3)(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{0.5},$$

that squaring and adding terms gives the following contradiction

$$\gamma^2(r+2\delta)^2(2\gamma^3+16\gamma^2+39\gamma+28)+4d\gamma(\gamma+2)^2(\gamma^3+6\gamma^2+12\gamma+7)=0.$$

Thus, we can conclude that the left-hand side is an increasing function of d with a slope equal to $2/(r+2\delta)$ for $d=0$. Next, we calculate the second derivative to find out the curvature of the function. The second derivative is

$$\frac{4\gamma^2(\gamma+1)^2(\gamma+3)}{(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{1.5}}-\frac{4\gamma^{0.5}(\gamma+2)^4(\gamma+4)^{0.5}}{(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{1.5}}.$$

Let's suppose that this derivative is positive or zero

$$\frac{\gamma^2(\gamma+1)^2(\gamma+3)}{(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^{1.5}}\geq\frac{\gamma^{0.5}(\gamma+2)^4(\gamma+4)^{0.5}}{(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^{1.5}}.$$

Squaring, the inequality can be written as follows

$$\gamma^4(\gamma+1)^4(\gamma+3)^2(4d(\gamma+2)^2+\gamma(\gamma+4)(r+2\delta)^2)^3$$

$$\geq \gamma(\gamma+2)^8(\gamma+4)(\gamma^2(r+2\delta)^2+4(\gamma+1)\gamma d)^3,$$

developing the different terms and grouping terms we obtain that the following polynomial in d should be positive or zero

$$-4^3\gamma^4(\gamma+2)^6(\gamma+1)^3(\gamma^2+5\gamma+7)d^3$$

$$-48\gamma^5(\gamma+4)(\gamma+2)^4(\gamma+1)^2(2\gamma^2+8\gamma+7)(r+2\delta)d^2$$

$$+12\gamma^6(\gamma+4)(\gamma+2)^2(\gamma+1)(\gamma^5+6\gamma^4-23\gamma^2-51\gamma-28)(r+2\delta)^4d$$

$$+\gamma^7(\gamma+4)(2\gamma^7+23\gamma^6+100\gamma^5+191\gamma^4+98\gamma^3-183\gamma^2-280\gamma-112)(r+2\delta)^4\geq 0,$$

where

$$\gamma^5+6\gamma^4-23\gamma^2-51\gamma-28>0 \text{ for } \gamma>2.1853,$$

and

$$2\gamma^7+23\gamma^6+100\gamma^5+191\gamma^4+98\gamma^3-183\gamma^2-280\gamma-112=0 \text{ for } \gamma>1.115.$$

Thus a contradiction is obtained for $\gamma \leq 1.115$ and we can conclude that the left-hand side of (73) is an increasing concave function that is below the increasing linear function on the right-hand side for all $d > 0$ since the first derivative of the concave function for $d = 0,\ 2/(r+2\delta)$, is lower than the first derivative of the linear function for $d = 0,\ 2/\delta$. The result is that the expression (72) will be negative and hence the difference $x_s^{sb} - x_s^*$ will be also negative. However, the polynomial equation could have, according to Descartes' rule of signs, a positive real root for $\gamma > 1.115$. If this is the case, the left-hand side of (73) presents a inflexion point for the positive root and the function is first convex to become a concave function for the values of d larger than the root. Then, there are two possibilities. The left-hand side of (73) is still below the increasing linear function on the right-hand side for all $d > 0$ or the linear function crosses the function on the left-hand side of (73). In the first case, we have the same result than the one obtained for $\gamma \leq 1.115$. In the second case, the equation (73) will have two positive roots. Between the roots the difference (72) will be positive and consequently $x_s^{sb} > x_s^*$, but for the rest of values of d, the difference will be negative and hence $x_s^{sb} < x_s^*$. Thus, we can conclude that for values of d larger than the higher positive root, $x_s^{sb} < x_s^*$. In any case, we can state that for large enough values of d, $x_s^{sb} < x_s^*$.

Proof of Proposition 11

Using the temporal trajectories for the pollution stock given by (34) and (55) we obtain the following expression

$$x^*(t) - x^{sb}(t) = (x_0 - x_{SS}^*)e^{\alpha^* t} - (x_0 - x_{SS}^{sb})e^{\alpha^{sb} t} + x_{SS}^* - x_{SS}^{sb},$$

where $x_{SS}^* > x_{SS}^{sb}$ and $x^*(0) = x^{sb}(0) = x_0$. Thus, if there exists an intersection point between these trajectories, it must satisfy that

$$x_{SS}^* - x_{SS}^{sb} = (x_0 - x_{SS}^{sb})e^{\alpha^{sb} t} - (x_0 - x_{SS}^*)e^{\alpha^* t}, \tag{74}$$

where the right-hand side presents an extreme for

$$e^{(\alpha^{sb} - \alpha^*)t} = \frac{(x_0 - x_{SS}^*)\alpha^*}{(x_0 - x_{SS}^{sb})\alpha^{sb}}, \tag{75}$$

that yields the following value for t

$$t^* = \frac{1}{\alpha^{sb} - \alpha^*} \ln \frac{(x_0 - x_{SS}^*)\alpha^*}{(x_0 - x_{SS}^{sb})\alpha^{sb}}. \tag{76}$$

The extreme is positive provided that $x_0 < x_{SS}^{sb}$ since we have established in the proof of Prop. 10 that

$$\frac{(\gamma+2)^2 A_r^{sb}}{\gamma(\gamma+4)} > \frac{(\gamma+1)A_r^*}{\gamma}$$

what implies that

$$\alpha^{sb} = \frac{(\gamma+2)^2 A_r^{sb}}{\gamma(\gamma+4)} - \delta > \alpha^* = \frac{(\gamma+1)A_r^*}{\gamma} - \delta.$$

Then,

$$\frac{(x_0 - x_{SS}^*)\alpha^*}{(x_0 - x_{SS}^{sb})\alpha^{sb}} > 1$$

and the logarithm and the extreme will be positive. If $x_0 > x_{SS}^*$, $(x_0 - x_{SS}^*)\alpha^*/(x_0 - x_{SS}^{sb})\alpha^{sb}$ could be greater or lower than the unity depending on the value of the initial stock. To analyze this dependence, we calculate the first derivative of this expression with respect to x_0

$$\frac{\alpha^*\alpha^{sb}(x_{SS}^* - x_{SS}^{sb})}{(x_0 - x_{SS}^{sb})^2(\alpha^{sb})^2} > 0.$$

Thus, the expression is zero when $x_0 = x_{SS}^*$ and increases with x_0. The second derivative establishes that the function is convex and the l'Hôpital rule allows us calculate the limit when x_0 tends to infinite

$$\lim_{x_0 \to +\infty} \frac{(x_0 - x_{SS}^*)\alpha^*}{(x_0 - x_{SS}^{sb})\alpha^{sb}} = \lim_{x_0 \to +\infty} \frac{\alpha^*}{\alpha^{sb}} > 1.$$

Then, we can conclude that there will exist a value for x_0 given by the following expression

$$x_0 = \frac{x_{SS}^*\alpha^* - x_{SS}^{sb}\alpha^{sb}}{\alpha^* - \alpha^{sb}},$$

such that if

$$x_{SS}^* < x_0 < \frac{x_{SS}^*\alpha^* - x_{SS}^{sb}\alpha^{sb}}{\alpha^* - \alpha^{sb}},$$

the extreme defined by (76) is negative and in the contrary case positive. Next, we study whether the extreme is a maximum or a minimum. The second derivative of the right-hand side of (74) can be written as follows

$$e^{\alpha^* t}((x_0 - x_{SS}^{sb})(\alpha^{sb})^2 e^{(\alpha^{sb}-\alpha^*)t} - (x_0 - x_{SS}^*)(\alpha^*)^2),$$

that evaluated for the extreme according to (75) yields

$$e^{\alpha^* t}((x_0 - x_{SS}^{sb})(\alpha^{sb})^2 \frac{(x_0 - x_{SS}^*)\alpha^*}{(x_0 - x_{SS}^{sb})\alpha^{sb}} - (x_0 - x_{SS}^*)(\alpha^*)^2)$$

$$= e^{\alpha^* t}(x_0 - x_{SS}^*)\alpha^e(\alpha^{sb} - \alpha^*),$$

that is positive if $x_0 < x_{SS}^*$ and negative in the contrary case. Thus, we can conclude that if $x_0 < x_{SS}^*$ the extreme is a minimum and if $x_0 > x_{SS}^*$ the extreme is a maximum. Moreover, doing the second derivative equal to zero we find that the right-hand side of (74) has an inflexion point for

$$e^{(\alpha^{sb} - \alpha^*)t} = \frac{(x_0 - x_{SS}^*)(\alpha^*)^2}{(x_0 - x_{SS}^{sb})(\alpha^{sb})^2},$$

that comparing with (75) must be on the right of the extreme since $\alpha^*/\alpha^{sb} > 1$.

Taking into account these results, if $x_0 < x_{SS}^{sb}$, the right-hand side of (74) presents a minimum for a positive value of t and consequently is increasing on the right of this minimum. But as

$$\lim_{t \to +\infty} \left((x_0 - x_{SS}^{sb})e^{\alpha^{sb} t} - (x_0 - x_{SS}^*)e^{\alpha^* t}\right) = 0,$$

and the function has an inflexion point for a value of t larger than the minimum we have to conclude that for the minimum the right-hand side of (74) is negative and from this minimum converges asymptotically to zero, taking negative values when t is larger than the minimum. hence, we obtain that

$$x_{SS}^* - x_{SS}^{sb} > (x_0 - x_{SS}^{sb})e^{\alpha^{sb} t} - (x_0 - x_{SS}^*)e^{\alpha^* t}, \text{ for all } t > 0,$$

and we can state that if $x_0 < x_{SS}^{sb}$, $x^*(t) > x^{sb}(t)$ for all $t > 0$.

If $x_{SS}^* < x_0$, we have two cases:

i)

$$x_{SS}^* < x_0 < \frac{x_{SS}^*\alpha^* - x_{SS}^{sb}\alpha^{sb}}{\alpha^* - \alpha^{sb}}.$$

In this case, the right-hand side of (74) presents a maximum for a value of t negative and consequently on the right of the maximum the function is decreasing and changes from being concave to be convex to converge asymptotically to zero. Thus as the function is decreasing for $t > 0$, we obtain the same result than for $x_0 < x_{SS}^{sb}$.

ii)

$$x^*_{SS} < \frac{x^*_{SS}\alpha^* - x^{sb}_{SS}\alpha^{sb}}{\alpha^* - \alpha^{sb}} < x_0.$$

Now, the right-hand side of (74) presents a maximum for a value of t positive and as on the right of the maximum the function is decreasing and changes from being concave to be convex to converge asymptotically to zero, there will exist a value for t, $t' > t^*$, for which the two temporal gives the same value for the pollution stock. Then, for $t < t'$ we have that $x^*(t) < x^{sb}(t)$ and for $t > t'$, $x^*(t) > x^{sb}(t)$.

Finally, if $x_0 \in (x^{sb}_{SS},\ x^*_{SS})$, $x^*(t) > x^{sb}(t)$ for all t since $x^*(t)$ is an increasing function of time whereas $x^{sb}(t)$ is a decreasing function and $x^*(0) = x^{sb}(0) = x_0$.

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