Bounded Rational Formation of Beliefs: Explaining Biased Assessment of Probabilities in Insurance and Hedging Markets†

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Abstract

We show that, jointly with *moral hazard* and *adverse selection*, insurance and hedging activities present an additional problem related to the accuracy of agents' beliefs. In particular, we show that contrary to the market selection hypothesis, departure from accurate beliefs is, in some circumstances, beneficial for agents. To prove it, we provide a standard canonical general equilibrium model of insurance markets showing that *bounded rational moderate optimistic* traders with incorrect beliefs obtain, period by period, higher actual expected utility and higher actual expected wealth than agents with accurate beliefs. Our findings are then consistent with the empirical evidence showing a systematic and coherent optimistic bias of agents in the assessment of probabilities.

JEL classification: D51, D84, D90.
Keywords: General Equilibrium Model; Arrow-Debreu Securities; Bounded Rationality; Accuracy of Beliefs.

1 Introduction

Once uncertainty was consistently introduced into economic analysis in the early 1950s by Arrow (1953), Debreu (1953, 1959) and Savage (1952, 1954), insurance and hedging activities soon became a central issue in economic theory. As a result of this interest, two important problems inherent to insurance markets were identified during the 1960s and the 1970s, namely *moral hazard* and *adverse selection.* In both problems, the ultimate cause is the existence of informational asymmetries. In a *moral hazard* problem, the insurer is unable to determine

† Financial support from Education and Science Department, Spanish Government, research project ECO2009-10231 is gratefully acknowledged. The suggestions of Alan Kirman, Guido A. Baldi and the participants at the 1st Conference of the Herbert Simon Society noticeably improved the final outcome and are highly appreciated. The authors also acknowledge the kind assistance of A.F. Hynds.

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the change in the probability of the risky event motivated by the subscription of the insurance contract[1]. In an *adverse selection* problem, the insurer cannot differentiate between the risky and non-risky insurance buyers, therefore assigning the same probability for the risky event to all the individuals[2]. As a consequence of these information asymmetries, the insurance/hedging market becomes inefficient, fails to reduce individual risks, and does not replicate the social optimum, reasons that justify the great interest in these questions in today's economics.

In this paper, we show that together with these two well known problems, insurance and hedging markets present additional matters worth examining, directly related to the accuracy of agents' subjective beliefs. More specifically, two questions arise regarding the rationality of agents beliefs: What are the implications of the accuracy degree in agents' subjective beliefs?; and, Why agents continuously and consistently incur in an optimistic bias when assessing probabilities?

Indeed, these unknowns are at the core of a central puzzle in today's economics. On the one hand, there exist strong theoretical arguments justifying the so called *market selection hypothesis*, without any doubt the main theoretical result concerning the relevance of accurate beliefs: In canonical models of insurance markets, essentially Arrow-Debreu general equilibrium economies with complete markets, agents with accurate beliefs are selected by the market over those with incorrect beliefs, since they earn higher expected wealth. This *market selection hypothesis*, proposed by Alchian (1950), Friedman (1953), Cootner (1964) and Fama (1965) during the 1950s and 1960s, and more recently by Sandroni (2000, 2005) and Blume and Easley (2006), is nevertheless contradicted by empirical evidence. As documented by Arnould and Grabowski (1981), DeJoy (1989), Camerer and Kunreuther (1989), De Long, Shleifer, Summers and Waldmann (1990, 1991), Shleifer and Summers (1990), Palomino (1996), Alpert and Raiffa (1982), Weinstein (1980), Buehler, Griffin and Ross (1994), and Puri and Robinson (2007), there is substantial evidence revealing that agents habitually overestimate the probability of good outcomes, consistently and continuously maintain an optimistic bias in their assessment of probabilities, and can earn higher expected wealth than agents with correct beliefs by adopting biased beliefs.

The introduction of a bounded rational mechanism of belief formation can help to solve this puzzle of paramount importance in current economic theory. To provide the intuition behind this possibility, let us consider a standard general equilibrium model of insurance markets in which insurance and hedging activities operate on the basis of the household heterogeneity condition and the law of large numbers. In simple words, it is assumed that when an agent offers an Arrow-Debreu security, there is another agent wishing to demand it; or, equivalently, that when a state of nature is relatively better for a household, it is simultaneously relatively worse for at least another household. Without any loss of generality to our purposes, let us assume that there exist two states of nature (1 and 2) and two households (A and B), and that state of nature 1 is the relatively better for household A and the relatively worse for household B. Therefore, there exist two Arrow-Debreu securities: Arrow-Debreu security 1,

[1] See Arrow (1965, 1971).

[2] See the seminal article by Akerloff (1970), and also Laffont (1985) and Hirshleifer and Riley (1992).

which provides 1 unit of good at state of nature 1, and Arrow-Debreu security 2, which provides one unit of god at state of nature 2. To hedge against risk, household A sells Arrow-Debreu security 1 and buys Arrow-Debreu security 2, whilst household B sells Arrow-Debreu security 2 and buys Arrow-Debreu security 1.

Let us consider that there does not exist social risk, that both agents are rational and have the true probabilities as beliefs, and that therefore they totally hedge against risk. From this initial situation, let us now assume that household A becomes optimistic assigning a higher probability of occurrence to his/her relatively better state[3] of nature 1, and therefore a lower probability to his/her relatively worse state of nature 2. As is well known, agents optimally allocate more wealth to the states of nature that they consider more likely to occur; consequently, in this general equilibrium framework, household A will sell a lower quantity of Arrow-Debreu security 1 in order to increase her/his wealth at state of nature 1, now more probable, and will buy a lower amount of Arrow-Debreu security 2 in order to decrease her/his wealth at state of nature 2, now less likely. The price of the Arrow-Debreu security 1 will then increase since the supply of Arrow-Debreu security 1 has decreased, whilst the lower demand of Arrow-Debreu security 2 will originate a fall in the Arrow-Debreu security 2 price. Obviously, these changes in prices benefit the optimistic agent A and are prejudicial to agent 2 despite his/her more accurate beliefs. Indeed, given that the now more optimistic household A sells Arrow-Debreu security 1 at a higher price and buys Arrow-Debreu security 2 at a lower price, he/she is able to increase his/her wealth at the expense of the rational agent B, who buys Arrow-Debreu security 1 at a higher price and sells Arrow-Debreu security 2 at a lower price. This reallocation of wealth alters the initial equilibrium characterized by the total removal of uncertainty, leading to a non-efficient sharing of risk from the social point of view. From the individual perspective, however, optimism is clearly beneficial for agents, and it can be assumed therefore that there is a natural –and rational– tendency for individuals to depart from accurate beliefs and to become optimistic. Indeed, this incentive toward an optimistic change in the agents' subjective beliefs can be understood as a mechanism of belief formation, worthy of a deeper analysis.

The first question is: How strong and how valid is this incentive to rational optimism? By applying standard economic arguments, we can deduce that the success of becoming more *rationally optimistic* depends on several aspects. The first is the capability of the optimistic agent to exert influence on the Arrow-Debreu security prices. As we have seen, the benefits of optimism are a consequence of the modification in the relative price of the Arrow-Debreu security prices, a modification caused by the alteration of the individual behavior. If we assume competitive markets, then the only way optimism can produce an aggregate reallocation of wealth is through some kind of spreading of this sentiment. In this respect, by their nature, hedging and insurance markets are prone to experience contagion of sentiments: on the one hand, although the original underlying market can be atomized, the existence of cooperatives, mutual associations, and in general of big insurance and hedging companies, can lead to the concentration of buyers and sellers, opening the possibility to exert

[3]This intuitive formalization of optimism has been proposed, among others, by Hirshleifer and Riley (1992), p. 250.

influence on the Arrow-Debreu security prices. On the other hand, hedging and insurance markets are markets in which information and news flow very easily and quickly between agents, leading to simultaneous and concordant decisions of a big number of Arrow-Debreu security buyers or sellers, decisions that therefore can alter the Arrow-Debreu relative prices. In this respect, we will assume throughout this paper that there exists a real possibility of optimism (and pessimism) having influence over the Arrow-Debreu security prices, focusing on the particularities and effects of this influence.

The second aspect to consider when analyzing this incentive towards *rational optimism* is its confrontation against reality. When an individual assigns a higher probability to her/his good states, she/he can get better Arrow-Debreu security prices, as we have seen. Nevertheless, if the actual probability of her/his good state is lower than his/her subjective probability, and, accordingly the true probability of his/her bad state is greater than that subjectively assigned, the real outcomes will imply more than expected bad states and fewer than expected good states. Given that the *rationally optimistic* agent has decided to allocate a lower wealth to her/his bad states, more probable than expected, and a greater wealth to her/his good states, less likely than expected, she/he is betting against reality and losing expected wealth. On this point and as we will show in the following sections, only when optimism is moderate the positive effects derived from the more favorable Arrow-Debreu security prices surpass the negative effects of the uncertainty misperception. Indeed, under extreme optimism, outcomes are worse than under accurate beliefs, since the costs of the distorted expectations are higher than the benefits of optimism.

Finally, there exists an important additional issue concerning the way in which the agents envisage the uncertainty scheme and formulate the objective for this change in beliefs. Both aspects are the consequence of their ability to perceive and process information and to make the associated calculations. In this respect, we will consider that these tasks are carried out by bounded rational agents, in the sense proposed by Simon (1957, 2000). More specifically and as Simon (2000) summarizes, we will assume that *"the choices the people make are determined not only by some consistent overall goal and the properties of the external world, but also by the knowledge that decision makers do and don't have of the world, their ability or inability to evoke that knowledge when it is relevant, to work out the consequences of their actions, to conjure up possible courses of action, to cope with uncertainty (including uncertainty derived from the possible responses of other actors), and to adjudicate among their many competing ways. Rationality is bounded because these abilities are severely limited."*

Following Simon (2000), we will distinguish between the psychology of the decision maker –the procedural rationality– and his/her ultimate goal/utility function and external environment formulation –the substantive rationality. With respect to the former, we will adopt an evolutionary perspective: the decision maker will pursue his/her survival, consciously or unconsciously compelled by the external environment and the interaction with other agents. With respect to the substantive rationality and the external environment formulation, it will be assumed that only relatively simple stationary uncertainty schemes can be rationalized by the agents in order to adopt the optimal decisions. This is a very logical assumption, since non-stationary stochastic processes – therefore governed by time varying probability functions– do not allow probabilistic regularities to be reasonably identified and anticipated, and lead to the impossibility of taking insurance decisions and to the non-existence of plausible explanations of insurance activities. On this point and as usual in the economic literature on insurance and hedging activities, we will assume that the considered insurance and hedging markets entail only stationary uncertainty. The second aspect concerning substantive rationality refers to the way the agents formulate their goals when taking decisions under uncertainty and evaluating the implications of changing their subjective beliefs, i.e. to the objective functions considered in the mechanism of belief formation. In this respect and as we will clarify in section 2, our model introduces another suggestion posed by Simon (2000) when dealing with environments with uncertainty, namely the necessity of taking into account, under a bounded rational perspective, how the involved agents react to the action of any of them.

In particular, we will assume that, for a given set of subjective beliefs concerning the underlying stationary scheme of uncertainty, agents maximize their logarithmic expected utility. The reason for this formulation is that expected utility maximization with logarithmic utility is an evolutionary dominant objective, coherent with our procedural rationality assumption of agents consciously or unconsciously pursuing their survival. As proved by Sinn (2003), this is a dominant preference in the sense that a population following any other preference for decision-making under risk, will disappear relative to the population following this preference with a probability that approaches certainty[4]. It is worth noting that this assumption does not contradict bounded rationality, since it not only incorporates two universally accepted psychophysical laws –namely Webers and Fechners Psychophysical Laws– as bounded rationality preconizes, but also implies very tractable problems under simple stationary uncertainty schemes.

Concerning the mechanism of belief formation, we will assume that the agents fix as criterion the maximization of their *one-period-ahead* true expected utility. This is a logical criterion, not only heuristically reasonable but also compatible with rational decision-making individuals with limited cognitive and calculation abilities. More specifically, we will assume that each household transfers probability from a bad state to a good state in order to improve her/his future one-period ahead true expected utility. From the economic point of view, the election of this criterion obeys several reasons, all of them related to the paradigm of bounded rationality proposed by Simon (1957). First, it is obvious that agents do not exactly know the true probabilities of the different states of nature, and they proceed to adjust their subjective probabilities heuristically. Second, we can also admit that households continuously revise their beliefs in the light of the observed actual evidence, and, logically, this revision pursues the improvement of the real outcome that the consumer perceives. The consideration of the true/objective expected utility appears then as a very logical criterion for the agents, since it informs on the real gains the agent is obtaining by changing her/his beliefs. Moreover, given the usual assumption on the strict quasi-concavity of the Bernoulli logarithmic utility function, the improvement of the true/objective expected utility necessarily implies the improvement of the true/objective expected wealth/consumption, an additional objective fact that could be assessed by households when modulating their subjective beliefs,

[4] On the psychlogical and biological aspects of expected logarithmic utility, see also Karni and Schmeidler (1986), Cooper (1987), Robson (1996, 2001), and Sinn and Weichenrieder (1993).

and that, as the former, is compatible and coherent with our assumption on the psychology of agents pursuing survival. In addition, the maximization of the one-period-ahead true expected utility is a problem the agents can tackle from the perspective of bounded rationality: given the stationarity of the uncertainty, it can be considered as a measurable and identifiable objective as time passes; and it is a simple one-period problem, whose solution (carried out consciously or unconsciously) does not entail excessive calculation abilities. Note that a completely rational agent would maximize his/her true expected utility for the total time horizon he/she is considering, a very complicated multi-period problem difficult to visualize, compute and solve and that we consider to be beyond the capabilities of real households (even the abilities of expert mathematicians). Finally, as explained above and as we will clarify in the following sections, this mechanism of belief formation explicitly takes into account the reaction of an agent to the decision adopted by the remaining agents, a salient feature of bounded rationality when dealing with uncertainty according to Simon (2000).

The rest of the paper is as follows. After this introduction, section 2 describes the economy -a canonical economy with insurance markets- and defines the bounded rational optimal expectations general equilibrium in the sense we have commented above. Section 3 presents the main theoretical results concerning the role played by biased beliefs, and these results are applied to some illustrative examples in section 4. Finally, section 5 concludes and provides directions for future research.

2 The Economy and its Bounded Rational Optimal Expectations General Equilibrium

In essence, our economy is defined by the usual assumptions in a canonical general equilibrium model of insurance/hedging markets[5]. More specifically, our economy is an infinite-period exchange economy with I households, denoted by i, $i = 1, 2, \dots, I$, and one physical good. Let $t = 0, 1, \dots, \infty$ denote the period of time. Uncertainty is originated by the occurrence each period t of one of a finite set of possible states of nature, characterized by a specific distribution across households of good endowments. Let L be the number of states of nature that can happen at any period, and let l be each particular state, $l = 1, 2, \dots, L$. Let us denote by $\mathcal{S}$ the subsequent uncertainty/event tree; by s each node or history in the uncertainty/event tree; by sl, $l = 1, 2, \dots, L$, the nodes immediately subsequent to s; by $s-1$ the node immediately preceding s; by $w^i(s)$ the endowment of good for household i at node s; and by $t(s)$ the period of time at which history/node s happens. Concerning the probability of occurrence of each node $s \in \mathcal{S}$, we depart from the canonical model in considering the existence of subjective probabilities that can differ from the objective/true probabilities. On this point, let $\pi(s)$ and $\pi^1(s)$ be, respectively, the objective and the subjective probabilities of occurrence for node s. Therefore, assuming stationary stochastic processes, for any node/history $s \in \mathcal{S}$, $s := (s - t(s), s - t(s) + 1, \dots, s - 1, s)$,

$$\pi(s) = \pi(s|s-1)\pi(s-1|s-2)\dots\pi(s-t(s)+1|s-t(s))\pi(s-t(s)),$$

[5] The reader can find the basis of the theory of insurance markets in Laffont (1989) and Hirshleifer and Riley (1992).

$$\pi^i(s) = \pi^i(s|s-1)\pi^i(s-1|s-2)\ldots\pi^i(s-t(s)+1|s-t(s))\pi^i(s-t(s)).$$

Insurance/hedging activities operate on the basis of a complete system of Arrow-Debreu securities, in which all the consumers competitively trade. At each node $s \in \mathcal{S}$, there exist L possible states of nature with respect to the next period $t(s)+1$, and therefore L Arrow-Debreu securities. We will denote by a^i_{sl} the amount bought (if positive) or sold (if negative) by household i of the Arrow-Debreu security l providing one unit of good if state of nature l happens after node s and 0 units otherwise, and by q_{sl} the price of this Arrow-Debreu security l.

From the methodological point of view, the main characteristic of our model is the definition of the general equilibrium through a double maximization condition. In a first step and as usual, agents maximize their discounted expected utility given their respective subjective beliefs. In a second step, they choose the optimal subjective probabilities in order to maximize their real one-period-ahead expected utility. Our approach therefore follows the idea of *optimal expectations* posed by Brunnermeier and Parker (2005), but with two distinctive features. First, we conclude the existence of benefits from *bounded rational optimism* through the analysis of the general equilibrium solution objective properties, perfectly observable, and not from the introduction of an additional function maximized by each agent. Second, in our model, agents consistently and continuously keep presenting biased optimistic beliefs because they get, at the next period, a higher actual expected utility and a higher actual expected wealth. These are objective criteria ensuring a systematic and consistent bias in the assessment of probabilities, different and alternative from that proposed by Brunnermeier and Parker (2005), and closer to the idea of bounded rationality. Indeed, in Brunnermeier and Parker (2005), the mechanism of belief formation is based on the improvement, under optimism, of a well-being function defined as an average of real and subjective utilities. This well-being function is defined as the expected time-average of the felicity of the agent, a mix of real historical and future subjective expected utilities[6]. This implies that the problem that the households must solve in Brunnermeier and Parker (2005) in order to calculate their optimal expectations, involves not only the maximization in the subjective probabilities of an infinite-period problem, but also to do so across repeated realizations of uncertainty, actually a very complicated problem demanding a huge capacity and ability of idealization and calculation. Our proposal is, in this respect, much more simple and concordant with the real abilities of real households, since it entails the maximization in the subjective probabilities of a one-period function, namely the one-period-ahead true expected utility, a relatively easygoing problem and much easier to visualize by the households. Our mechanism of belief formation is not only closer to the idea of bounded rationality and the real capabilities of households, but also provides a complete explanation of why agents with bounded rational optimal expectations are not driven out of the market by agents with accurate beliefs. Indeed, in Brunnermeier and Parker (2005) there is no formal analysis of the evolutionary consequences of their results, whilst in our model, agents with bounded rational optimism survive in the economy because they earn more expected wealth than agents with accurate beliefs. In any case, our *optimal expectations* approach joins and complements that proposed by Brunnermeier and Parker (2005) and

[6]It is a function similar to that in Caplin and Leahy (2000)

adds to the literature identifying theoretical arguments implying the consistent persistence of incorrect optimistic beliefs[7].

Summing up and to clarify concepts, we will define the optimal expectations general equilibrium in two steps. In a first stage we will impose the usual general equilibrium conditions, so the I households will maximize their expected discounted utility given their respective subjective beliefs, and markets will clear. In a second step, we will consider that the consumers choose the subjective beliefs/probabilities that maximize their future one-period-ahead true expected utility, something which ensures the improvement of their one-period-ahead true expected wealth. Therefore, in this second step, the consumers will fix their subjective beliefs as those beliefs that make their one-period ahead true expected utility maxima and improve their true expected wealth when the economy is in a general equilibrium consistent with their subjective beliefs.

2.1 Subjective General Equilibrium

Let us consider the problem solved by each household in our economy,

$$\left.\begin{array}{rc} & \max_{C_s^i, a_{sl}^i} \sum_{s\in\mathcal{S}} \beta_i^{t(s)} \pi^i(s) U^i(C_s^i) \\ s.t. & C_s^i + \sum_{l=1}^{L} q_{sl} a_{sl}^i \leq w^i(s) + a_s^i \\ & C_s^i \geq 0 \\ & s \in \mathcal{S} \end{array}\right\} \qquad (1)$$

where β_i, U^i and C_s^i are, respectively, the time discount factor; the Bernoulli utility function; and the consumption at node s for household i. Under the usual assumptions ($U(\cdot)$ increasing and strictly quasiconcave), the former problem has solutions in C_s^i and a_{sl}^i, which depend on the whole set of Arrow-Debreu security prices and on the whole set of subjective probabilities, that is, on q_{sl} and $\pi^i(s)$, where $s \in \mathcal{S}$ and $l = 1, 2, \ldots, L$. Let q and π^i be, respectively, the sets $q := \{q_{sl} | s \in \mathcal{S}, l = 1, 2, \ldots, L\}$ and $\pi^i := \{\pi^i(s) | s \in \mathcal{S}\}$. We will therefore denote the solution functions by $C_s^i(q, \pi^i)$ and $a_{sl}^i(q, \pi^i)$. These functions must also verify the market equilibrium conditions

$$\left.\begin{array}{c} \sum_{i=1}^{I} C_s^i(q, \pi^i) = \sum_{i=1}^{I} w^i(s), \\ \sum_{i=1}^{I} a_{sl}^i(q, \pi^i) = 0, \\ s \in \mathcal{S}, \quad l = 1, 2, \ldots, L, \end{array}\right\} \qquad (2)$$

and then the subjective general equilibrium is defined as:

Definition 1 (Subjective General Equilibrium) *Set of functions $\hat{C}_s^i(q, \pi^i)$ and $\hat{a}_{sl}^i(q, \pi^i)$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, $i = 1, 2, \ldots, I$, and set of prices $\hat{q}_{sl}$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, solving the household's problems (1) and verifying the market clearing conditions (2).*

[7]See De long, Shleifer, Summers and Waldmann (1990, 1991), Shleifer and Summmers (1990), Blume and Easley (1992, 2002, 2006) and Palomino (1996).

2.2 Bounded Rational Optimal Expectations Equilibrium

Under the canonical assumptions –increasing and strictly quasiconcave Bernoulli utility functions and strictly positive endowments– the former subjective general equilibrium exists, and leads to the equilibrium prices of the Arrow-Debreu securities. For each Arrow-Debreu security, its equilibrium price depends on the subjective probabilities of all the households, and therefore so do the general equilibrium optimal consumptions. Let π^{SE} be the set of all the subjective probabilities, i.e. $\pi^{SE} := \{\pi^i | i = 1, 2, \ldots, I\}$. Then, the solution functions defining the subjective general equilibrium are $\hat{C}^i_s(\pi^{SE})$ and $\hat{a}^i_{sl}(\pi^{SE})$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, $i = 1, 2, \ldots, I$.

Let us now consider as reference any node s and its immediately subsequent nodes sl, $l = 1, 2, \ldots, L$. In our framework of insurance/hedging markets, materialized in the existence at node s of a complete set of L Arrow-Debreu securities, households trade in these Arrow-Debreu securities to transfer wealth across states of nature. Therefore, at node s, each household will sell Arrow-Debreu security l if node sl is a *relatively good* node, and will buy Arrow-Debreu security l if node sl is a *relatively bad* node. Consequently, $a^i_{sl} < 0$ if node sl is a *relatively good* node, and $a^i_{sl} > 0$ if node sl is a *relatively bad* node. Let $\pi^i(sl|s)$ be the subjective beliefs of household i at node s concerning nodes sl. According to our reasonings, when household i becomes more optimistic, she/he transfers subjective probability from her/his relatively bad states to her/his relatively good states, causing a favorable alteration in the relative price of the associated Arrow-Debreu securities –determined in the subjective general equilibrium– through changes in the Arrow-Debreu supplies and demands. As a consequence, there also appear changes in the solution allocation defining the subjective general equilibrium, and the question to elucidate is then when these changes are beneficial for the optimistic agent. In other words, since in the subjective general equilibrium the final allocation depends on the set of all the subjective beliefs, it is possible for each consumer to modify in her/his own benefit the equilibrium solutions by modulating her/his subjective probabilities.

In this respect and as we explained above, we will assume that each household transfers probability from a bad state to a good state in order to improve her/his future one-period-ahead true expected utility at the subjective general equilibrium. In this respect, let $E_s[U^i(\hat{C}^i)]$ denote, at node s, the household i true expected one-period-ahead utility at the subjective general equilibrium, i.e.

$$E_s[U^i(\hat{C}^i)] = \pi(s1|s)U^i(\hat{C}^i_{s1}) + \pi(s2|s)U^i(\hat{C}^i_{s2}) + \ldots + \pi(sL|s)U^i(\hat{C}^i_{sL}).$$

The preceding ideas led to the following definition:

Definition 2 (Bounded Rational Optimal Expectations General Equilibrium) *An optimal expectations general equilibrium is a set of functions $\hat{C}^i_s(q, \pi^i)$ and $\hat{a}^i_{sl}(q, \pi^i)$, a set of prices $\hat{q}_{sl}$, and a set of subjective probabilities $\hat{\pi}^i(sl|s)$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, $i = 1, 2, \ldots, I$, such that*

- $\hat{C}^i_s(q, \pi^i)$, $\hat{a}^i_{sl}(q, \pi^i)$, *and* $\hat{q}_{sl}$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, *constitute a subjective general equilibrium, and*
- $\hat{\pi}^i(sl|s)$ *solve the household's problems*

$$\max_{\pi^i(sl|s)} E_s[U^i(\hat{C}^i)].$$

On the basis of this definition of optimal expectations general equilibrium, we will provide in the following section some useful mathematical results ensuring the existence of bounded rational moderate optimism of agents in insurance and hedging markets. In addition, we will numerically and/or algebraically solve four canonical general equilibrium models of insurance markets, showing how their optimal expectations general equilibrium imply the presence of a bounded rational optimistic bias in beliefs, which in addition are evolutionary compatible.

3 Theoretical Results

By their nature, in general and with very few exceptions, the class of general equilibrium models of insurance/hedging markets that we have considered do not have an algebraic solution, and must be numerically solved. This inconvenience forces us to prove the existence of optimism in agents by making use of very simple models with algebraic solutions, or alternatively, by ensuring the verification of simple mathematical properties in the optimal expectations general equilibrium solutions. In the following lines we present three set of sufficient conditions, easily testable, which ensure and/or prove the existence of a bounded rational bias in the optimal expectations equilibrium. As we will show in section 4, these conditions are verified by all the standard models of insurance markets with algebraic solution.

Given node s and household i, let us subdivide the set of the L immediate subsequent nodes/states into two subsets, namely the subset G^i_s of the household i relatively good states, and the subset B^i_s of the household i relatively bad states. Taking into account the sign of the household i Arrow-Debreu security holdings, these sets are defined as follow:

$$G^i_s := \{l \in L | a^i_{sl} < 0\}, \qquad B^i_s := \{l \in L | a^i_{sl} > 0\}.$$

Obviously, for household i, the total true probabilities of being at a good state or at a bad state, respectively $\Pi(G^i_s)$ and $\Pi(B^i_s)$, are given by

$$\Pi(G^i_s) = \sum_{l \in G^i_s} \pi(sl|s), \qquad \Pi(B^i_s) = \sum_{l \in B^i_s} \pi(sl|s).$$

Let $\hat{C}^i_{G^i_s}$ and $\hat{C}^i_{B^i_s}$ be the set of the subjective general equilibrium optimal consumptions at the good and bad states, respectively

$$\hat{C}^i_{G_s} := \{\hat{C}^i_{sl} | l \in G^i_s\}, \qquad \hat{C}^i_{B_s} := \{\hat{C}^i_{sl} | l \in B^i_s\}.$$

From the definitions of the two aggregate true probabilities $\Pi(G^i_s)$ and $\Pi(B^i_s)$ and of the sets $\hat{C}^i_{G^i_s}$ and $\hat{C}^i_{B^i_s}$, the household i true expected one-period ahead utility at the subjective general equilibrium $E_s[U^i(\hat{C}^i)]$ can be written

$$E_s[U^i(\hat{C}^i)] = \Pi(G^i_s) V(\hat{C}^i_{G^i_s}) + \Pi(B^i_s) W(\hat{C}^i_{B^i_s}),$$

where

$$V(\hat{C}^i_{G^i_s}) = \frac{\sum_{l \in G^i_s} \pi(l|s) U^i(\hat{C}^i_{sl})}{\Pi(G^i_s)},$$

$$W(\hat{C}^i_{B^i_s}) = \frac{\sum_{l\in B^i_s} \pi(l|s)U^i(\hat{C}^i_{sl})}{\Pi(B^i_s)}.$$

Let us now consider any good state/node $g \in G^i_s$ and any bad state/node $b \in B^i_s$. Let us assume that household i increases $\pi^i(sg|s)$ in detriment of $\pi^i(sb|s)$, becoming more optimistic. Since the subjective general equilibrium optimal consumptions $\hat{C}^i_{sl}$ depend on the subjective probabilities, so do the functions $V(\hat{C}^i_{G^i_s})$ and $W(\hat{C}^i_{B^i_s})$, and then a change $dE_s[U^i(\hat{C}^i)]$ in the one-period ahead true expected utility appears. This change is given by

$$dE_s[U^i(\hat{C}^i)] = \Pi(G^i_s)[\frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} d\pi^i(sg|s) + \frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)} d\pi^i(sb|s)] +$$

$$\Pi(B^i_s)[\frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} d\pi^i(sg|s) + \frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)} d\pi^i(sb|s)].$$

Since $d\pi^i(sg|s) = -d\pi^i(sb|s) > 0$,

$$dE_s[U^i(\hat{C}^i)] =$$

$$\left\{\Pi(G^i_s)[\frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} - \frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)}] + \Pi(B^i_s)[\frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} - \frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)}\right\} d\pi^i(sg|s)],$$

and then

$$dE_s[U^i(\hat{C}^i)] > 0 \Leftrightarrow$$

$$\left\{\Pi(G^i_s)[\frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} - \frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)}] + \Pi(B^i_s)[\frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} - \frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)}\right\} > 0.$$

- **Sufficient Conditions 1**: Let us assume that $V(\hat{C}^i_{G^i_s})$ is increasing and concave in $\pi^i(sg|s)$ and decreasing and concave in $\pi^i(sb|s)$, and that $W(\hat{C}^i_{B^i_s})$ is increasing and concave in $\pi^i(sb|s)$ and decreasing and concave in $\pi^i(sg|s)$. Then, if $\forall i$

$$\left.\frac{\frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)} - \frac{\partial V(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)}}{\frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sb|s)} - \frac{\partial W(\hat{C}^i_{G^i_s})}{\partial \pi^i(sg|s)}}\right|_{\pi^i(sg|s)=\pi(sg|s)} > \frac{\Pi(B^i_s)}{\Pi(G^i_s)},$$

  all the households are optimistic at the bounded rational optimal expectations general equilibrium.

Proof The proof is immediate by applying basic arguments of algebra and calculus. □

- **Sufficient Conditions 2**: Let $\pi^i_s = (\pi^i(s1|s), \dots, \pi^i(sL|s))$, $i = 1, \dots, I$. Let us assume that the optimal consumption functions at the subjective general equilibrium $\hat{C}^i_{sl}(\pi^{SE})$, $s \in \mathcal{S}$, $l = 1, 2, \dots, L$, are concave in π^i_s. Then,

a) If
$$\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi^i(sl|s)=\pi(sl|s)} > 0$$
for some $i = 1, \ldots, I$, and some $l = 1, \ldots, L$, the bounded rational optimal expectation equilibrium implies non-accurate biased beliefs.

b) If $\forall i$
$$\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi^i(sl|s)=\pi(sl|s)} > 0 \qquad \forall l \in G_s^i,$$
and
$$\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi^i(sl|s)=\pi(sl|s)} < 0 \qquad \forall l \in B_s^i,$$
the bounded rational optimal expectation equilibrium implies biased optimistic beliefs for all the agents.

Proof Let π_s^{SE} be the vector of the subjective probabilities assigned at node $s \in \mathcal{S}$ to the occurrence of the different states of the world at the following period for all households, i.e. $\pi_s^{SE} := (\pi_s^1, \ldots, \pi_s^i, \ldots, \pi_s^I) \in \mathbb{R}^{I \times L}$, where $\pi_s^i = (\pi^i(s1|s), \ldots, \pi^i(sL|s))$ for all $i = 1, \ldots, I$. Let π_s be the vector, at node $s \in \mathcal{S}$, of the true probabilities of occurrence of the different states of nature, i.e. $\pi_s := (\pi(s1|s), \ldots, \pi(sL|s))$. As we know, the solution functions defining the subjective general equilibrium are $\hat{C}_s^i(\pi^{SE})$ and $\hat{a}_{sl}^i(\pi^{SE})$, $s \in \mathcal{S}$, $l = 1, 2, \ldots, L$, $i = 1, 2, \ldots, I$. At any node $s \in \mathcal{S}$, the one-period-ahead true expected utility associated to the subjective general equilibrium is given by the expression
$$E_s[U^i(\hat{C}^i)] =$$
$$\pi(s1|s)U^i(\hat{C}_{s1}^i(\pi^{SE})) + \pi(s2|s)U^i(\hat{C}_{s2}^i(\pi^{SE})) + \ldots + \pi(sL|s)U^i(\hat{C}_{sL}^i(\pi^{SE})).$$

At each node $s \in \mathcal{S}$, let us now consider the game in which each household i is described by her/his one-period-ahead true expected utility at the subjective general equilibrium $E_s[U^i(\hat{C}^i]$ and by his/her set of available actions, namely her/his set of subjective probabilities, defined by
$$A^i = \{\pi_s^i = (\pi^i(s1|s), \ldots, \pi^i(sL|s)) \in \mathbb{R}^L : \pi^i(sl|s) \in [0,1]\ \forall l = 1, \ldots, L\}.$$
The cartesian product $A = A^1 \times \cdots \times A^I$ is the action space of the game, and an element of A is a vector $\pi_s^{SE} := (\pi_s^1, \ldots, \pi_s^i, \ldots, \pi_s^I) \in \mathbb{R}^{I \times L}$, which specifies the subjective probabilities chosen by all agents. The actions of all agents jointly determine the result of this game in the form of a payoff to each agent, $E_s[U^i(\hat{C}^i]$, $i = 1, \ldots, I$, which depends on both her/his actions and those of all the other agents.

In order to establish the existence of a Nash equilibrium for the game in normal form $\Gamma = ((E_s[U^i(\hat{C}^i], A^i) : \ i = 1, \ldots, I)$, it suffices to show that the game satisfies the following conditions of Nash's Theorem:

(i) For each $i = 1, \ldots, I$, the payoff function $E_s[U^i(\hat{C}^i] : A \longrightarrow \mathbb{R}$ is continuous and quasi-concave in $\pi_s^{SE} := (\pi_s^1, \ldots, \pi_s^i, \ldots, \pi_s^I) \in \mathbb{R}^{I \times L}$, for a given $\pi_s^{SE-i} := (\pi_s^1, \ldots, \pi_s^{i-1}, \pi_s^{i+1}, \ldots, \pi_s^I) \in \mathbb{R}^{(I-1) \times L}$,

and

(ii) For each $i = 1, \ldots, I$, the action space A^i is a nonempty, compact, and convex subset of $\mathbb{R}^L$.

First, notice that A^i is a compact and convex subset of $\mathbb{R}^L$. Next, consider the payoff function for agent i, $E_s[U^i(\hat{C}^i]$. Since each composite function $U^i(\hat{C}^i_{sl}(\pi^{SE}))$ is concave in A (this is because U^i is an increasing and concave function, and $\hat{C}^i_{sl}$ is assumed to be concave in π^i_s), and any nonnegative linear combination of concave functions is a concave function, $E_s[U^i(\hat{C}^i]$ is concave and therefore quasi-concave. Thus, the conditions of Nash's Theorem are satisfied, and it follows that the game has one Nash equilibrium $\pi^\star_s = (\pi^{1^\star}_s, \ldots, \pi^{L^\star}_s)$ in A, which means that it is a feasible action (i.e. $\pi^\star_s \in A$) and it is the best response to the joint actions of all the other agents, that is

$$\bar{\pi}^{i^\star}_s \in \arg\max_{\pi^i_s \in A^i} E_s[U^i(\hat{C}^i(\pi^i_s, \pi^{SE-i^\star}_s).$$

Hence, given that the rest of the agents play $\pi^{SE-i^\star}_s$, there is no incentive for the ith agent to deviate unilaterally from the equilibrium action $\pi^{i^\star}_s$, or equivalently, for all $i = 1, \ldots, I$, $\pi^{i^\star}_s$ is a solution for the problem

$$\max_{\pi^i_s \in A_i} E_s[U^i(\hat{C}^i(\pi^i_s, \pi^{SE-i^\star}_s))], \tag{3}$$

where the actions $\pi^{SE-i^\star}_s$ chosen by the other agents are given.

For our purposes, we are interested in knowing the properties of the Nash equilibrium with respect to the accuracy of the households beliefs. In particular, we would like to know if the optimal probabilities $\pi^\star_s$ coincide with the true probabilities π_s, or, on the contrary, if the Nash equilibrium $\pi^\star_s$ implies a departure from accurate beliefs.

In this sense, we will prove that, under a very weak assumption which is satisfied in practice, then $\pi^\star_s \neq \pi_s$, and the true probability π_s is not a Nash equilibrium, contradicting the *market selection* hypothesis. To do this, we consider the following result for a nonlinear maximization problem with nonnegative constraints:

> If $\pi^{i^\star}_s$ is an optimal solution for problem (3), then
>
> it follows that $\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}(\pi^{i^\star}_s, \pi^{SE-i^\star}_s) \leq 0$ for all $l = 1, \ldots, L$.

Thus, whenever $\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}(\pi_s) > 0$ for some $i = 1, \ldots, I$, and some $l = 1, \ldots, L$, the true probability profile π_s will not be a Nash equilibrium, and therefore $\pi^\star_s \neq \pi_s$ and part a) is proved.

To prove part b), it is enough to consider the sign of the partial derivatives

$$\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi^i(sl|s)=\pi(sl|s)} > 0 \qquad \forall l \in G^i_s,$$

and

$$\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi^i(sl|s)=\pi(sl|s)} < 0 \qquad \forall l \in B^i_s,$$

since, to improve the one-period-ahead true expected utility, all the individuals must increase their subjective probabilities at the good states and decrease their subjective probabilities at the bad states.□

4 Particular Cases

4.1 Model 1

Let us consider the simplest canonical model of insurance markets. The economy is a one-period exchange economy with two households, denoted by A and B, and one physical good. Uncertainty is originated by the occurrence at the unique period of one of two possible states of nature, characterized by a specific distribution across households of good endowments. On this point, let $l = 1, 2$ be the states of nature, and let w_l^A and w_l^B be the endowments at node l of household A and B, respectively. Without any loss of generality, we will assume that state of nature $l = 1$ is the relatively good state for household A, and that state of nature $l = 2$ is the relatively good state for household B. Therefore $w_1^A > w_2^A$ and $w_2^B > w_1^B$. On the basis of this uncertainty scheme, households trade in two Arrow-Debreu securities with real prices q_1 and q_2.

Let the Bernoulli utility function of the two households be $U(C) = \ln(C)$. The subjective general equilibrium is therefore defined as follows:

Definition 3 (Subjective General Equilibrium) *Set of functions $\hat{C}_l^A(q_1, q_2, \pi_1^A, \pi_2^A)$, $\hat{C}_l^B(q_1, q_2, \pi_1^B, \pi_2^B)$, $\hat{a}_l^A(q_1, q_2, \pi_1^A, \pi_2^A)$ and $\hat{a}_l^B(q_1, q_2, \pi_1^B, \pi_2^B)$, and set of prices $\hat{q}_l$, $l = 1, 2$, solving the household's problems*

$$\left.\begin{array}{lc} & \max_{C_l^A, a_l^A} \pi_1^A \ln(C_1^A) + (1 - \pi_1^A) \ln(C_2^A) \\ s.t. & q_1 a_1^A + q_2 a_2^A = 0 \\ & C_1^A = w_1^A + a_1^A \\ & C_2^A = w_2^A + a_2^A \\ & C_1^A, C_2^A \geq 0 \end{array}\right\},$$

$$\left.\begin{array}{lc} & \max_{C_l^B, a_l^B} \pi_1^B \ln(C_1^B) + (1 - \pi_1^B) \ln(C_2^B) \\ s.t. & q_1 a_1^B + q_2 a_2^B = 0 \\ & C_1^B = w_1^B + a_1^B \\ & C_2^B = w_2^B + a_2^B \\ & C_1^B, C_2^B \geq 0 \end{array}\right\},$$

and verifying the market clearing conditions

$$C_1^A + C_1^B = w_1^A + w_1^B, \qquad C_2^A + C_2^B = w_2^A + w_2^B,$$
$$a_1^A + a_1^B = 0, \qquad a_2^A + a_2^B = 0.$$

This subjective general equilibrium has an algebraic solution, given by the following functions:

$$\hat{C}_1^A(\pi_1^A, q_1, q_2) = \frac{\pi_1^A (q_1 w_1^A + q_2 w_2^A)}{q_1}, \qquad \hat{C}_2^A(\pi_1^A, q_1, q_2) = \frac{(1 - \pi_1^A)(q_1 w_1^A + q_2 w_2^A)}{q_2},$$

$$\hat{C}_1^B(\pi_1^B, q_1, q_2) = \frac{\pi_1^B(q_1 w_1^B + q_2 w_2^B)}{q_1}, \qquad \hat{C}_2^B(\pi_1^B, q_1, q_2) = \frac{(1-\pi_1^B)(q_1 w_1^B + q_2 w_2^B)}{q_2},$$

$$\hat{a}_1^A(\pi_1^A, q_1, q_2) = \hat{C}_1^A(\pi_1^A, q_1, q_2) - w_1^A, \qquad \hat{a}_2^A(\pi_1^A, q_1, q_2) = \hat{C}_2^A(\pi_1^A, q_1, q_2) - w_2^A,$$

$$\hat{a}_1^B(\pi_1^B, q_1, q_2) = \hat{C}_1^B(\pi_1^B, q_1, q_2) - w_1^B, \qquad \hat{a}_2^B(\pi_1^B, q_1, q_2) = \hat{C}_2^B(\pi_1^B, q_1, q_2) - w_2^B,$$

$$\hat{q}_1 = \pi_1^A w_2^A + \pi_1^B w_2^B, \qquad \hat{q}_2 = (1-\pi_1^A) w_1^A + (1-\pi_1^B) w_1^B.$$

After substituting the equilibrium prices $\hat{q}_1$ and $\hat{q}_2$ into the optimal consumptions, we get the subjective general equilibrium consumptions as a function of the whole set of subjective beliefs and endowments:

$$\hat{C}_1^A(\pi_1^A, \pi_1^B) = \frac{\pi_1^A[\pi_1^B(w_1^A w_2^B - w_1^B w_2^A) + w_2^A(w_1^a + w_1^B)]}{\pi_1^A w_2^A + \pi_1^B w_2^B},$$

$$\hat{C}_2^A(\pi_1^A, \pi_1^B) = \frac{(1-\pi_1^A)[\pi_1^B(w_1^A w_2^B - w_1^B w_2^A) + w_2^A(w_1^a + w_1^B)]}{(1-\pi_1^A) w_1^A + (1-\pi_1^B) w_1^B},$$

$$\hat{C}_1^B(\pi_1^A, \pi_1^B) = \frac{\pi_1^B[(1-\pi_1^A)(w_1^A w_2^B - w_1^B w_2^A) + w_1^B(w_2^a + w_2^B)]}{\pi_1^A w_2^A + \pi_1^B w_2^B},$$

$$\hat{C}_2^B(\pi_1^A, \pi_1^B) = \frac{(1-\pi_1^B)[(1-\pi_1^A)(w_1^A w_2^B - w_1^B w_2^A) + w_1^B(w_2^a + w_2^B)]}{(1-\pi_1^A) w_1^A + (1-\pi_1^B) w_1^B}.$$

It is immediate to prove that $\hat{C}_1^A$ is increasing and concave in π_1^A, and that $\hat{C}_2^A$ is decreasing and concave in π_1^A. We can apply the set of sufficient conditions 1 since the optimal expectations general equilibrium problem

$$\max_{\pi_1^A} E[\ln(\hat{C}^A)] = \pi_1 \ln(\hat{C}_1^A(\pi_1^A, \pi_1^B) + (1-\pi_1)\ln(\hat{C}_2^A(\pi_1^A, \pi_1^B)$$

has a unique interior solution, given by the reaction function

$$\hat{\pi}_1^A(\pi_1^B, \pi_1) = \frac{\sqrt{\pi_1^B w_2^B}}{2[\pi(\pi_1^B(w_1 A w_2^B - w_1 B w_2^A) + w_1 B w_2^A) + w_1^B w_2^A(\pi_1^B - 1)]}$$

$$[w_1^b(1-\pi_1^B)\sqrt{[4\pi_1(\pi_1^B)(w_1^a w_2^B - w_1^B w_2^A) + w_2^A(w_1^A + w_1^B) - 4\pi_1^2 * (\pi_1^B * (w_1^A w_2^B - w_1^B w_2^A) + (w_2^a(w_1^a + w_1^B))) - \pi_1^B w_1^B w_2^B(\pi_1^B - 1)]} +$$

$$\sqrt{\pi_1^B w_2^B}(2 * \pi_1^B w_1^A - w_1^B(\pi_1^B - 1))]$$

and

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^A}\right|_{\pi_1^A = \pi_1} > 0 \Leftrightarrow \frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1(1-\pi_1^B)}{\pi_1^B(1-\pi_1)}.$$

Therefore, the last inequality provides a necessary and sufficient condition ensuring the bounded rational optimism of household A, i.e. ensuring that $\hat{\pi}_1^A > \pi_1$.

By applying the same arguments to household B, when the inequality

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial (1-\pi_1^B)}\right|_{\pi_1^B = \pi_1} > 0 \Leftrightarrow \frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1^A(1-\pi_1)}{\pi_1(1-\pi_1^A)},$$

holds, household B is necessarily bounded rational optimistic and $1 - \hat{\pi}_1^B > 1 - \pi_1$.

The questions to elucidate are then, first, whether the optimal expectations general equilibrium can imply the existence of optimism for both agents; and second, whether this bounded rational optimistic departure from accurate beliefs has a limit. With respect to the first question, it is enough to show that the second set of sufficient conditions holds, or, equivalently, that the two above inequalities are compatible when there exist insurance markets. Since, when $\pi_1^A = \pi_1^B = \pi_1$, we get

$$\frac{w_1^A w_2^B}{w_1^B w_2^A} > 1 = \frac{\pi_1^A(1-\pi_1)}{\pi_1(1-\pi_1^A)} = \frac{\pi_1(1-\pi_1^B)}{\pi_1^B(1-\pi_1)},$$

then

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^A}\right|_{\pi_1^A=\pi_1} > 0, \quad \left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial (1-\pi_1^B)}\right|_{\pi_1^B=\pi_1} > 0,$$

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial (1-\pi_1^A)}\right|_{\pi_1^A=\pi_1} < 0, \quad \left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^B}\right|_{\pi_1^B=\pi_1} < 0,$$

and the second set of sufficient conditions is verified. Therefore, at the bounded rational optimal expectations equilibrium, both agents are bounded rational optimistic agents.

Alternatively, let us assume that $\frac{w_1^A w_2^B}{w_1^B w_2^A} > 1$ and that there exist insurance markets. Since $\frac{\pi_1^A(1-\pi_1)}{\pi_1(1-\pi_1^A)}$ is continuous and increasing in π^A, and $\frac{\pi_1(1-\pi_1^B)}{\pi_1^B(1-\pi_1)}$ is continuous and decreasing in π^B, given that when $\pi^A = \pi$ and $\pi^B = \pi$ both fractions are equal to 1, there always exist $\pi^B < \pi$ and $\pi^A > \pi$ for which

$$\frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1(1-\pi_1^B)}{\pi_1^B(1-\pi_1)}$$

and

$$\frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1^A(1-\pi_1)}{\pi_1(1-\pi_1^A)}.$$

Concerning the second question, it is obvious that the reaction functions $\hat{\pi}_1^A(\pi_1^B, \pi_1)$ and $\hat{\pi}_1^B(\pi_1^A, \pi_1)$ provide the values above which optimism is prejudicial for agents. The reader can carry out similar analyses for results 2 and 3, and can obtain the conditions under which these results are verified. In the following table we provide some numerical results showing that moderate optimism is beneficial, extreme optimism is prejudicial, and also showing that there exist bounded rational optimal expectations equilibria implying moderate optimism for both agents. As we pointed out in the introductory section, it is worth noting that the Nash equilibrium does not replicate the social optimum and entails the appearance of macroeconomic risk.

Table 1. Model 1.

$\pi_1 = 0.5$; $\pi_2 = 0.5$; $w_1^A = w_2^B = 120$; $w_2^A = w_1^B = 80$.

π_1^B	$\hat{\pi}_1^A$	Moderate optimism	Extreme optimism	True expected utility and consumption of agent A, optimal optimism of agent A	True expected utility and consumption of agent A, accurate beliefs of agent A	True expected utility and consumption of agent B, optimal optimism of agent A
0.45	0.56	[0.5,0.56]	(0.56,1]	4.60001 100.092013	4.59629 99.2430	4.598150 99.90798
0.5	0.58	[0.5,0.58]	(0.58,1]	4.61164876 100.971455	4.60517019 100	4.59209226 99.0285448
0.6	0.6	[0.5,0.6]	(0.6,1]	4.6443909 104	4.62941936 102.950311	4.56434819 96

Nash Equilibrium.

$\pi_1^{A\star}$ (optimistic)	$\pi_2^{B\star}$ (optimistic)	$\pi_2^{A\star}$ (optimistic)	$\pi_1^{B\star}$ (optimistic)	True Expected Utility and Consumption, Nash equilibrium	True Expected Utility and Consumption, accurate beliefs
0.5505	0.5505	0.4495	0.4495	4.60004349 100	4.60517019 100

4.2 Model 2

Let us now consider the simplest dynamic canonical model of insurance markets. The economy is a two-period exchange economy with two households, denoted by A and B, and one physical good. Uncertainty is originated by the occurrence at the second period of one of two possible states of nature, characterized by a specific distribution across households of good endowments. Let $s = 0, 1, 2$ denote, respectively, the initial state of nature and the two possible future nodes at period $t = 1$, and let w_s^A and w_s^B be the endowments at node s of households A and B, respectively. Without any loss of generality, we will assume that node/state of nature $s = 1$ is the relatively good state for household A, and that state of nature $s = 2$ is the relatively good state for household B. Therefore $w_1^A > w_2^A$ and $w_2^B > w_1^B$. On the basis of this uncertainty scheme, households trade in two Arrow-Debreu securities with real prices q_1 and q_2.

Let the Bernoulli utility function of the two households be $U(C) = \ln(C)$, and let the time discount factor β be the same for the two households. The subjective general equilibrium is therefore defined as follows:

Definition 4 (Subjective General Equilibrium) *Set of functions* $\hat{C}_s^A(q_1, q_2, \pi_1^A)$, $\hat{C}_s^B(q_1, q_2, \pi_1^B)$, $\hat{a}_l^A(q_1, q_2, \pi_1^A)$ *and* $\hat{a}_l^B(q_1, q_2, \pi_1^B)$, *and set of prices* $\hat{q}_l$, $s = 0, 1, 2$

and $l = 1, 2$*, solving the household's problems*

$$\left.\begin{array}{lc} & \max_{C_s^A, a_l^A} \ln(C_0^A) + \beta[\pi_1^A \ln(C_1^A) + (1-\pi_1^A)\ln(C_2^A)] \\ s.t. & C_0^A + q_1 a_1^A + q_2 a_2^A \leq w_0^A \\ & C_1^A = w_1^A + a_1^A \\ & C_2^A = w_2^A + a_2^A \\ & C_0^A, C_1^A, C_2^A \geq 0 \end{array}\right\},$$

$$\left.\begin{array}{lc} & \max_{C_s^B, a_l^B} \ln(C_0^B) + \beta[\pi_1^B \ln(C_1^B) + (1-\pi_1^B)\ln(C_2^B)] \\ s.t. & C_0^B + q_1 a_1^B + q_2 a_2^B \leq w_0^B \\ & C_1^B = w_1^B + a_1^B \\ & C_2^B = w_2^B + a_2^B \\ & C_0^B, C_1^B, C_2^B \geq 0 \end{array}\right\},$$

and verifying the market clearing conditions

$$C_0^A + C_0^B = w_0^A + w_0^B, \qquad C_1^A + C_1^B = w_1^A + w_1^B, \qquad C_2^A + C_2^B = w_2^A + w_2^B,$$

$$a_1^A + a_1^B = 0, \qquad a_2^A + a_2^B = 0.$$

This subjective general equilibrium has an algebraic solution, given by the following functions:

$$\hat{C}_0^A(\pi_1^A, q_1, q_2) = \frac{w_0^A + q_1 w_1^A + q_2 w_2^A)}{1+\beta)},$$

$$\hat{C}_1^A(\pi_1^A, q_1, q_2) = \frac{\beta\pi_1^A(w_0^A + q_1 w_1^A + q_2 w_2^A)}{q_1(1+\beta)},$$

$$\hat{C}_2^A(\pi_1^A, q_1, q_2) = \frac{\beta(1-\pi_1^A)(w_0^A + q_1 w_1^A + q_2 w_2^A)}{q_2(1+\beta)},$$

$$\hat{C}_0^B(\pi_1^B, q_1, q_2) = \frac{w_0^B + q_1 w_1^B + q_2 w_2^B)}{(1+\beta)},$$

$$\hat{C}_1^B(\pi_1^B, q_1, q_2) = \frac{\beta\pi_1^B(w_0^B + q_1 w_1^B + q_2 w_2^B)}{q_1(1+\beta)},$$

$$\hat{C}_2^B(\pi_1^B, q_1, q_2) = \frac{\beta(1-\pi_1^B)(w_0^B + q_1 w_1^B + q_2 w_2^B)}{q_2(1+\beta)},$$

$$\hat{a}_1^A(\pi_1^A, q_1, q_2) = \hat{C}_1^A(\pi_1^A, q_1, q_2) - w_1^A, \qquad \hat{a}_2^A(\pi_1^A, q_1, q_2) = \hat{C}_2^A(\pi_1^A, q_1, q_2) - w_2^A,$$

$$\hat{a}_1^B(\pi_1^B, q_1, q_2) = \hat{C}_1^B(\pi_1^B, q_1, q_2) - w_1^B, \qquad \hat{a}_2^B(\pi_1^B, q_1, q_2) = \hat{C}_2^B(\pi_1^B, q_1, q_2) - w_2^B,$$

$$\hat{q}_1 = \frac{\beta[\beta(w_0^A + w_0^B)(\pi_1^A w_2^A + \pi_1^B w_2^B) + (w_2^A + w_2^B)(\pi_1^A w_0^A + \pi_1^B w_0^B)]}{(w_1^A + w_1^B)(w_2^A + w_2^B) - \beta[\pi_1^A(w_1^A w_2^B - w_1^B w_2^A) + \pi_1^B(w_1^B w_2^A - w_1^A w_2^B) - (w_1^A + w_1^B)(w_2^A + w_2^B)]},$$

$$\hat{q}_2 = \frac{\beta[\beta(w_0^A + w_0^B)((1-\pi_1^A)w_1^A + (1-\pi_1^B)w_1^B) + (w_1^A + w_1^B)((1-\pi_1^A)w_0^A + (1-\pi_1^B)w_0^B)]}{\beta[\pi_1^A(w_1^A w_2^B - w_1^B w_2^A) + \pi_1^B(w_1^B w_2^A - w_1^A w_2^B) - (w_1^A + w_1^B)(w_2^A + w_2^B)] - (w_1^A + w_1^B)(w_2^A + w_2^B)}.$$

After substituting the equilibrium prices $\hat{q}_1$ and $\hat{q}_2$ into the optimal consumptions, we get the subjective general equilibrium consumptions as a function of the whole set of subjective beliefs and endowments:

$$\hat{C}_0^A(\pi_1^A, \pi_1^B) = \frac{\beta(w_0^A + w_0^B)[\pi_1^B(w_1^A w_2^B - w_1^B w_2^A) + w_2^A(w_1^A + w_1^B)] + w_0^A(w_1^A + w_1^B)(w_2^A + w_2^B)}{(w_1^A + w_1^B)(w_2^A + w_2^B) - \beta[\pi_1^A(w_1^a w_2^B - w_1^B w_2^A) + \pi_1^B(w_1^B w_2^A - w_1^A w_2^B) - (w_1^A + w_1^B)(w_2^A + w_2^B)]},$$

$$\hat{C}_0^B(\pi_1^A, \pi_1^B) = \frac{\beta(w_0^A + w_0^B)[\pi_1^A(w_1^A w_2^B - w_1^B w_2^A) - w_2^B(w_1^A + w_1^B)] - w_0^B(w_1^A + w_1^B)(w_2^A + w_2^B)}{\beta[\pi_1^A(w_1^a w_2^B - w_1^B w_2^A) + \pi_1^B(w_1^B w_2^A - w_1^A w_2^B) - (w_1^A + w_1^B)(w_2^A + w_2^B)] - (w_1^A + w_1^B)(w_2^A + w_2^B)},$$

$$\hat{C}_1^A(\pi_1^A, \pi_1^B) = \frac{\pi_1^A[\beta(w_0^A + w_0^B)(\pi_1^B(w_1^A w_2^B - w_1^B w_2^A) + w_2^A(w_1^A + w_1^B)) + w_0^A(w_1^A + w_1^B)(w_2^A + w_2^B)]}{\beta(w_0^A + w_0^B)(\pi_1^A w_2^A + \pi_1^B w_2^B) + (w_2^A + w_2^B)(\pi_1^A w_0^A + \pi_1^B w_0^B)},$$

$$\hat{C}_1^B(\pi_1^A, \pi_1^B) = \frac{\pi_1^B[w_0^A(w_1^A + w_1^B)(w_2^A + w_2^B) - \beta(w_0^A + w_0^B)(\pi_1^A(w_1^A w_2^B - w_1^B w_2^A) - w_2^B(w_1^A + w_1^B))]}{\beta(w_0^A + w_0^B)(\pi_1^A w_2^A + \pi_1^B w_2^B) + (w_2^A + w_2^B)(\pi_1^A w_0^A + \pi_1^B w_0^B)},$$

$$\hat{C}_2^A(\pi_1^A, \pi_1^B) = \frac{(\pi_1^A - 1)[\beta(w_0^A + w_0^B)(\pi_1^B(w_1^A w_2^B - w_1^B w_2^A) + w_2^A(w_1^A + w_1^B)) + w_0^A(w_1^A + w_1^B)(w_2^A + w_2^B)]}{\beta(w_0^A + w_0^B)((\pi_1^A - 1)w_1^A + (\pi_1^B - 1)w_1^B) + (w_1^A + w_1^B)((\pi_1^A - 1)w_0^A + (\pi_1^B - 1)w_0^B)},$$

$$\hat{C}_2^B(\pi_1^A, \pi_1^B) = \frac{(1 - \pi_1^B)[\beta(w_0^A + w_0^B)(\pi_1^A(w_1^A w_2^B - w_1^B w_2^A) - w_2^B(w_1^A + w_1^B)) - w_0^A(w_1^A + w_1^B)(w_2^A + w_2^B)]}{\beta(w_0^A + w_0^B)((\pi_1^A - 1)w_1^A + (\pi_1^B - 1)w_1^B) + (w_1^A + w_1^B)((\pi_1^A - 1)w_0^A + (\pi_1^B - 1)w_0^B)}.$$

It is immediate to prove that $\hat{C}_1^A$ is increasing and concave in π_1^A, and that $\hat{C}_2^A$ is decreasing and concave in π_1^A. The bounded rational optimal expectations general equilibrium problem

$$\max_{\pi_1^A} E[\ln(\hat{C}^A)] = \pi_1 \ln(\hat{C}_1^A(\pi_1^A, \pi_1^B) + (1-\pi_1)\ln(\hat{C}_2^A(\pi_1^A, \pi_1^B)$$

has a unique interior solution[8]. Given that

$$(\frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1(1-\pi_1^B)}{\pi_1^B(1-\pi_1)}) \wedge (\pi_1^B \leq \pi_1 + \frac{\pi_1}{2\beta}) \wedge (\frac{w_1^A w_2^A}{w_1^B w_2^B} \geq \frac{\pi_1^B(1+2\beta) - 2\beta\pi_1}{\pi_1(1+2\beta) - 2\beta\pi_1^B}) \Rightarrow$$

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^A}\right|_{\pi_1^A = \pi_1} > 0,$$

we can apply the first set of sufficient conditions to ensure the presence of bounded rational optimism for household A, i.e. to ensure that $\hat{\pi}_1^A > \pi_1$.

By applying the same arguments to household B, when the set of inequalities

$$(\frac{w_1^A w_2^B}{w_1^B w_2^A} > \frac{\pi_1^A(1-\pi_1)}{\pi_1(1-\pi_1^A)}) \wedge \pi_1^A \leq \pi_1 + \frac{\pi_1}{2\beta}) \wedge (\frac{w_1^A w_2^A}{w_1^B w_2^B} \geq \frac{\pi_1(1+2\beta) - 2\beta\pi_1^A}{\pi_1^A(1+2\beta) - 2\beta\pi_1})$$

holds, household B is necessarily bounded rational optimistic and $1 - \hat{\pi}_1^B > 1 - \pi_1$.

As in the previous model, the questions to elucidate are the same: first, whether this bounded rational optimistic departure from accurate beliefs has a limit; and second, whether the bounded rational optimal expectations general equilibrium can imply the existence of bounded rational optimism for both agents. In this respect, we reach the same conclusions as in the previous case by applying the same reasonings. The reader can carry out similar analyses for the

[8] We do not provide this solution/reaction function given its large expression. The interested reader can contact the authors.

second set of sufficient conditions, and can obtain the conditions under which these results are verified. In table 2 we provide some numerical results showing that moderate optimism is beneficial, extreme optimism is prejudicial, and that there exist optimal expectations equilibria implying bounded rational moderate optimism for both agents. Again, the Nash Equilibrium does not replicate the social optimum and entails the inefficient appearance of macroeconomic risk.

Table 2. Model 2.

$\pi_1 = 0.5$; $\pi_2 = 0.5$; $w_0^A = w_0^B = 100$; $w_1^A = w_2^B = 120$; $w_2^A = w_1^B = 80$; $\beta = 0.9$.

π_1^B	$\hat{\pi}_1^A$	Moderate optimism	Extreme optimism	True expected utility and consumption of agent A, optimal optimism of agent A	True expected utility and consumption of agent A, accurate beliefs of agent A	True expected utility and consumption of agent B, optimal optimism of agent A
0.45	0.52	[0.5,0.52]	(0.52,1]	4.6019606 99.9254853	4.60166296 99.7758811	4.60115094 100.004768
0.5	0.54	[0.5,0.54]	(0.54,1]	4.60656332 100.380389	4.60517019 100	4.60216579 99.7804768
0.6	0.56	[0.5,0.56]	(0.56,1]	4.6257734 102.164123	4.61944202 101.938822	4.58241105 97.8358767

Nash Equilibrium.

$\pi_1^{A\star}$ (optimistic)	$\pi_2^{B\star}$ (optimistic)	$\pi_2^{A\star}$ (optimistic)	$\pi_1^{B\star}$ (optimistic)	True Expected Utility and Consumption, Nash equilibrium	True Expected Utility and Consumption, accurate beliefs
0.5237	0.5237	0.4763	0.4763	4.60404554 100	4.60517019 100

4.3 Model 3

Let us now consider the simplest canonical model of insurance markets when the number of states of nature is greater than 2. The economy is a one period exchange economy with two households, denoted by A and B, and one physical good. Uncertainty is originated by the occurrence at the unique period of one of three possible states of nature, characterized by a specific distribution across households of good endowments. On this point, let $l = 1, 2, 3$ be the states of nature, and let w_l^A and w_l^B be the endowments at node l of household A and B, respectively. Without any loss of generality, we will assume that state of nature $l = 1$ is a relatively good state for household A, that state of nature $l = 3$ is a relatively good state for household B. Therefore, for household A, $w_1^A \geq w_2^A$ and $w_1^A > w_3^A$, or $w_1^A > w_2^A$ and $w_1^A \geq w_3^A$. Analogously, for household B, $w_3^B > w_1^B$ and $w_3^B \geq w_2^B$, or $w_3^B \geq w_1^B$ and $w_3^B > w_2^B$. On the basis of this uncertainty scheme, households trade in three Arrow-Debreu securities with real prices q_1, q_2 and q_3.

Let the Bernoulli utility function of the two households be $U(C) = \ln(C)$. The subjective general equilibrium is therefore defined as follows:

Definition 5 (Subjective General Equilibrium) *Set of functions* $\hat{C}_l^A(q_1, q_2, \pi_1^A, \pi_2^A)$, $\hat{C}_l^B(q_1, q_2, \pi_1^B, \pi_2^B)$, $\hat{a}_l^A(q_1, q_2, \pi_1^A, \pi_2^A)$ *and* $\hat{a}_l^B(q_1, q_2, \pi_1^B, \pi_2^B)$, *and set of prices*

$\hat{q}_l$, $l = 1, 2, 3$, solving the household's problems

$$\left.\begin{array}{lc} & \max_{C_l^A, a_l^A} \pi_1^A \ln(C_1^A) + \pi_2^A \ln(C_2^A) + (1 - \pi_1^A - \pi_2^A) \ln(C_3^A) \\ s.t. & q_1 a_1^A + q_2 a_2^A + q_3 a_3^A = 0 \\ & C_1^A = w_1^A + a_1^A \\ & C_2^A = w_2^A + a_2^A \\ & C_3^A = w_3^A + a_3^A \\ & C_1^A, C_2^A, C_3^A \geq 0 \end{array}\right\},$$

$$\left.\begin{array}{lc} & \max_{C_l^B, a_l^B} \pi_1^B \ln(C_1^B) + \pi_2^B \ln(C_2^B) + (1 - \pi_1^B - \pi_3^B) \ln(C_3^B) \\ s.t. & q_1 a_1^B + q_2 a_2^B + q_3 a_3^B = 0 \\ & C_1^B = w_1^B + a_1^B \\ & C_2^B = w_2^B + a_2^B \\ & C_3^B = w_3^B + a_3^B \\ & C_1^B, C_2^B, C_3^B \geq 0 \end{array}\right\},$$

and verifying the market clearing conditions

$$C_1^A + C_1^B = w_1^A + w_1^B, \qquad C_2^A + C_2^B = w_2^A + w_2^B, \qquad C_3^A + C_3^B = w_3^A + w_3^B,$$

$$a_1^A + a_1^B = 0, \qquad a_2^A + a_2^B = 0, \qquad a_3^A + a_3^B = 0.$$

This subjective general equilibrium has an algebraic solution, given by the following functions:

$$\hat{C}_1^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) = \frac{\pi_1^A(q_1 w_1^A + q_2 w_2^A + q_3 w_3^A)}{q_1},$$

$$\hat{C}_1^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) = \frac{\pi_1^B(q_1 w_1^B + q_2 w_2^B + q_3 w_3^B)}{q_1},$$

$$\hat{C}_2^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) = \frac{\pi_2^A(q_1 w_1^B + q_2 w_2^B + q_3 w_3^A)}{q_2},$$

$$\hat{C}_2^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) = \frac{\pi_2^B(q_1 w_1^B + q_2 w_2^B + q_3 w_3^B)}{q_2},$$

$$\hat{C}_3^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) = \frac{(1 - \pi_1^A - \pi_2^A)(q_1 w_1^A + q_2 w_2^A + q_3 w_3^A)}{q_3},$$

$$\hat{C}_1^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) = \frac{(1 - \pi_1^B - \pi_2^B)(q_1 w_1^B + q_2 w_2^B + q_3 w_3^B)}{q_3},$$

$$\hat{a}_1^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) = \hat{C}_1^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) - w_1^A,$$

$$
\begin{aligned}
\hat{a}_2^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) &= \hat{C}_2^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) - w_2^A, \\
\hat{a}_3^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) &= \hat{C}_3^A(\pi_1^A, \pi_2^A, q_1, q_2, q_3) - w_3^A, \\
\hat{a}_1^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) &= \hat{C}_1^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) - w_1^B, \\
\hat{a}_2^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) &= \hat{C}_2^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) - w_2^B, \\
\hat{a}_3^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) &= \hat{C}_3^B(\pi_1^B, \pi_2^B, q_1, q_2, q_3) - w_3^B.
\end{aligned}
$$

$$\hat{q}_1 = 1,$$

$$\hat{q}_2 = \frac{\pi_1^A\pi_2^B(w_1^Aw_3^B - w_1^Bw_3^A) + \pi_1^B\pi_2^A(w_1^Bw_3^A - w_1^Aw_3^B) - (w_1^A + w_1^B)(\pi_2^Aw_3^A + \pi_2^Bw_3^B)}{\pi_1^A[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)] + \pi_1^B[w_3^B(w_2^A + w_2^B) - \pi_2^A(w_2^Aw_3^B - w_2^Bw_3^A)]},$$

$$\hat{q}_3 = \frac{\pi_1^A[\pi_2^B(w_1^Aw_2^B - w_1^Bw_2^A) - w_1^A(w_2^A + w_2^B)] - \pi_1^B[\pi_2^A(w_1^Aw_2^B - w_1^Bw_2^A) + w_1^B(w_2^A + w_2^B)] - (w_1^A + w_1^B)[w_2^A(\pi_2^A - 1) + w_2^B(\pi_2^B - 1)]}{\pi_1^A[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)] + \pi_1^B[w_3^B(w_2^A + w_2^B) - \pi_2^A(w_2^Aw_3^B - w_2^Bw_3^A)]}.$$

After substituting the equilibrium prices $\hat{q}_1$, $\hat{q}_2$ and $\hat{q}_3$ into the optimal consumptions, we get the subjective general equilibrium consumptions as a function of the whole set of subjective beliefs and endowments:

$$\hat{C}_1^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) = \frac{\pi_1^A\{\pi_1^B(w_2^A + w_2^B)(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)]\}}{\pi_1^A[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)] + \pi_1^B[w_3^B(w_2^A + w_2^B) - \pi_2^A(w_2^Aw_3^B - w_2^Bw_3^A)]},$$

$$\hat{C}_1^B(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) = \frac{\pi_1^B\{\pi_1^A(w_2^A + w_2^B)(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) - w_3^B(w_2^A + w_2^B)]\}}{\pi_1^A[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)] + \pi_1^B[w_3^B(w_2^A + w_2^B) - \pi_2^A(w_2^Aw_3^B - w_2^Bw_3^A)]},$$

$$\hat{C}_2^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) = \frac{\pi_2^A\{\pi_1^B(w_2^A + w_2^B)(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)]\}}{\pi_1^A\pi_2^B(w_1^Bw_3^A - w_1^Aw_3^B) + \pi_1^B\pi_2^A(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)(\pi_2^Aw_3^A + \pi_2^Bw_3^B)},$$

$$\hat{C}_2^B(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) = \frac{\pi_2^B\{\pi_1^A(w_2^A + w_2^B)(w_1^Bw_3^A - w_1^Aw_3^B) - (w_1^A + w_1^B)[\pi_2^A(w_2^Aw_3^B - w_2^Bw_3^A) - w_3^B(w_2^A + w_2^B)]\}}{\pi_1^A\pi_2^B(w_1^Bw_3^A - w_1^Aw_3^B) + \pi_1^B\pi_2^A(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)(\pi_2^Aw_3^A + \pi_2^Bw_3^B)},$$

$$\hat{C}_3^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) =$$

$$\frac{(1 - \pi_1^A - \pi_2^A)\{\pi_1^B(w_2^A + w_2^B)(w_1^Aw_3^B - w_1^Bw_3^A) + (w_1^A + w_1^B)[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) + w_3^A(w_2^A + w_2^B)]\}}{\pi_1^A[\pi_2^B(w_1^Aw_2^B - w_1^Bw_2^A) - w_1^A(w_2^A + w_2^B)] - \pi_1^B[\pi_2^A(w_1^Aw_2^B - w_1^Bw_2^A) + w_1^B(w_2^A + w_2^B)] - (w_1^A + w_1^B)[w_2^A(\pi_2^A - 1) + w_2^B(\pi_2^B - 1)]},$$

$$\hat{C}_3^B(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B) =$$

$$\frac{(1 - \pi_1^B - \pi_2^B)\{\pi_1^A(w_2^A + w_2^B)(w_1^Bw_3^A - w_1^Aw_3^B) - (w_1^A + w_1^B)[\pi_2^B(w_2^Aw_3^B - w_2^Bw_3^A) - w_3^B(w_2^A + w_2^B)]\}}{\pi_1^A[\pi_2^B(w_1^Aw_2^B - w_1^Bw_2^A) - w_1^A(w_2^A + w_2^B)] - \pi_1^B[\pi_2^A(w_1^Aw_2^B - w_1^Bw_2^A) + w_1^B(w_2^A + w_2^B)] - (w_1^A + w_1^B)[w_2^A(\pi_2^A - 1) + w_2^B(\pi_2^B - 1)]}.$$

Regarding the optimal expectation for the probability of occurrence of state of nature 1 for household A, it is immediate to prove that $\hat{C}_1^A$ is increasing and concave in π_1^A, and that $\hat{C}_2^A$ and $\hat{C}_3^A$ are decreasing and concave in π_1^A. Given the complexity of the algebraic solutions, these are the unique general mathematical properties we can easily obtain for this model. However, numerically and for endowment values compatible with the existence of insurance markets, all the qualitative results are verified. For instance, it is immediate to numerically obtain that the bounded rational optimal expectations general equilibrium problem

$$\max_{\pi_1^A} E[\ln(\hat{C}^A)] = \pi_1 \ln(\hat{C}_1^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B)) + \pi_2 \ln(\hat{C}_2^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B)) +$$

$$(1 - \pi_1 - \pi_2) \ln(\hat{C}_3^A(\pi_1^A, \pi_2^A, \pi_1^B, \pi_2^B))$$

has a unique interior solution, and that the inequality

$$\left. \frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^A} \right|_{\pi_1^A = \pi_1} > 0$$

holds for a feasible set of endowment values. Therefore, under a set of sufficient conditions, we can apply our theoretical results and ensure the existence of bounded rational optimism for household A , i.e. we can ensure that $\hat{\pi}_1^A > \pi_1$. By applying the same arguments to household B, it is possible to numerically find a set of sufficient conditions ensuring the bounded rational optimism of household B, i.e. that $\hat{\pi}_3^B > \pi_1$. Again, the questions to elucidate are, first, whether this optimistic rational departure from accurate beliefs has a limit; and second, whether the optimal expectations general equilibrium can imply the existence of bounded rational optimism for both agents. As for the former models, we provide some numerical results showing that moderate optimism is beneficial, extreme optimism is prejudicial, and that the second set of sufficient conditions holds and there exist optimal expectations equilibria implying bounded rational moderate optimism for both agents.

Table 3. Model 3.

$\pi_1 = 0.33$; $\pi_2 = 0.34$; $\pi_3 = 0.33$;
$w_1^A = w_3^B = 120$; $w_2^A = w_2^B = 100$; $w_3^A = w_1^B = 80$.

π_1^B	π_2^B	π_2^A	$\hat{\pi}_1^A$	Moderate optimism	Extreme optimism	True expected utility and consumption of agent A, optimal optimism of agent A	True expected utility and consumption of agent A, accurate beliefs of agent A	True expected utility and consumption of agent B, optimal optimism of agent A
0.25	0.34	034	0.35	(0.33,0.35]	(0.35,1]	4.59448736 99.7110656	4.5938222 99.3743675	4.60020555 100.288934
0.33	0.34	0.34	0.374	(0.33,0.374]	(0.374,1]	4.6095859 100.590554	4.60517019 100	4.59774862 99.4094462
0.5	0.34	0.34	0.44	(0.33,0.44]	(0.44,1]	4.68393925 108.575798	4.65866386 107.798296	4.51024943 91.4242015

Nash Equilibrium. Partial derivatives $\left.\frac{\partial E_s[U^i(\hat{C}^i]}{\partial \pi^i(sl|s)}\right|_{\pi(sl|s)}$.

$\frac{\partial E_s[U^i(\hat{C}^A]}{\partial \pi_1^A}$	$\frac{\partial E_s[U^i(\hat{C}^A]}{\partial \pi_2^A}$	$\frac{\partial E_s[U^i(\hat{C}^A]}{\partial \pi_3^A}$	$\frac{\partial E_s[U^i(\hat{C}^B]}{\partial \pi_1^B}$	$\frac{\partial E_s[U^i(\hat{C}^B]}{\partial \pi_2^B}$	$\frac{\partial E_s[U^i(\hat{C}^B]}{\partial \pi_3^B}$
> 0	> 0	< 0	< 0	< 0	> 0

4.4 Model 4

Let us now propose a novel canonical general equilibrium model of insurance markets with an infinite temporal horizon, useful to illustrate more realistic situations in insurance and hedging markets. The economy is an infinite-period exchange economy, inhabited by 2 households which consume a perishable single commodity and receive an endowment of this commodity that depends upon the realization of the states of nature and its past value. There are two possible states of nature, whose occurrence is given by constant subjective and objective probabilities π_1^A, π_1^B, π_1, π_2^A, π_2^B, π_2. As in the previous models, the Bernoulli utility function and the time discount factor are the same for both households, namely $U(C) = \ln(C)$ and β. The endowments are a state dependent percentage of the agent's wealth at the immediately preceding node R_s^I: for household A, $w_{s1}^A = T_1^A R_s^A$, $w_{s2}^A = T_2^A R_s^A$, where T_1^A and T_2^A are positive, and analogously for household B. With this assumption we capture an important characteristic of insurance and hedging markets, namely the dynamical implications of the insurance activities. More specifically, in this model, the previous insurance operations contribute to the current wealth of the individuals, which, in turn, determines the future wealth, a set of observed features in insurance and hedging markets. In order to ensure the existence of insurance markets, let us suppose that state of nature 1 is the relatively good state for household A, state of nature 2 being the relatively good state for household B. Then $T_1^A > T_2^A$ and $T_2^B > T_1^B$. For this economy, the subjective general equilibrium is defined as follows:

Definition 6 (Subjective General Equilibrium) *Set of functions* $\hat{C}_{sl}^A(q, \pi_1^A)$, $\hat{C}_{sl}^B(q, \pi_1^B)$, $\hat{a}_{sl}^A(q, \pi_1^A)$ *and* $\hat{a}_{sl}^B(q, \pi_1^B)$, *and set of prices* $\hat{q}_{sl}$, $l = 1, 2$, *solving the household's problems*

$$\left.\begin{array}{rc} & \max_{C_s^A, a_{sl}^A} \sum_{s\in\mathcal{S}} \beta^{t(s)}\pi^A(s)\ln(C_s^A) \\ s.t. & C_s^A + q_{s1}a_{s1}^A + q_{s2}a_{s2}^A \le R_s^A, \\ & R_{s1}^A = T_1^A R_s^A + a_{s1}^A, \\ & R_{s2}^A = T_2^A R_s^A + a_{s2}^A, \\ & C_s^A \ge 0, \\ & R_0^A \quad \textit{historically given} \\ & s \in \mathcal{S}, \end{array}\right\},$$

$$\left.\begin{array}{rc} & \max_{C_s^B, a_{sl}^B} \sum_{s\in\mathcal{S}} \beta^{t(s)}\pi^B(s)\ln(C_s^B) \\ s.t. & C_s^B + q_{s1}a_{s1}^B + q_{s2}a_{s2}^B \le R_s^B, \\ & R_{s1}^B = T_1^B R_s^B + a_{s1}^B, \\ & R_{s2}^B = T_2^B R_s^B + a_{s2}^B, \\ & C_s^B \ge 0, \\ & R_0^B \quad \textit{historically given} \\ & s \in \mathcal{S}, \end{array}\right\},$$

and verifying the market clearing conditions

$$C_s^A + C_s^B = R_s^A + R_s^B, \qquad a_{sl}^A + a_{sl}^B = 0, \qquad s \in \mathcal{S}.$$

It is worth noting the meaning of the recursive law of formation for the wealth. For instance, in the case of household A, this law implies $R_{s1}^A = T_1^A R_s^A + a_{s1}^A$. Since R_{s1}^A will be the wealth endowment of household A after node $s1$ at any of the future subsequent nodes, we are taking into account that the insurance decisions, captured by a_{s1}^A –a component of R_{s1}^A– will be part of the future wealth, as happens in the actual economy.

Since uncertainty is stationary, the household's problems can be written in recursive terms as

$$v_i(R_s^i) = \max_{C_s^i, a_{sl}^A} \{U(C_s^A) + \beta \Sigma_{l=1}^2 \pi_l^A v_A(R_{sl}^A)\} \tag{4}$$

$$s.t \quad C_s^A + \sum_{l=1}^{2} q_{sl} a_{sl}^A \leq R_s^A,$$

$$R_{sl}^A = T_l^A R_s^A + a_{sl}^A, \qquad l = 1, 2,$$

where $i = A, B$.

Applying the standard methods to solve recursive problems, it is possible to find the exact solution of the household's problems, given by

$$C_s^A(R_s^A, q_{s1}, q_{s2}) = (1-\beta) R_s^A (1 + T_1^A q_{s1} + T_2^A q_{s2}),$$

$$a_{s1}^A(R_s^A, q_{s1}, q_{s2}) = \frac{R_s^A}{q_{s1}} [\beta \pi_1^A + T_1^A q_{s1}(\beta \pi_1^A - 1) + \beta \pi_1^A T_2^A q_{s2}],$$

$$a_{s2}^A(R_s^A, q_{s1}, q_{s2}) = \frac{R_s^A}{q_{s2}} [\beta \pi_2^A + T_2^A q_{s2}(\beta \pi_2^A - 1) + \beta \pi_2^A T_1^A q_{s1}],$$

$$C_s^B(R_s^B, q_{s1}, q_{s2}) = (1-\beta) R_s^B (1 + T_1^B q_{s1} + T_2^B q_{s2}),$$

$$a_{s1}^B(R_s^B, q_{s1}, q_{s2}) = \frac{R_s^B}{q_{s1}} [\beta \pi_1^B + T_1^B q_{s1}(\beta \pi_1^B - 1) + \beta \pi_1^B T_2^B q_{s2}],$$

$$a_{s2}^B(R_s^A, q_{s1}, q_{s2}) = \frac{R_s^B}{q_{s2}} [\beta \pi_2^B + T_2^B q_{s2}(\beta \pi_2^B - 1) + \beta \pi_2^B T_1^B q_{s1}].$$

Therefore, the subjective general equilibrium is defined as the functions $C_s^i(R_s^i, q_{s1}, q_{s2})$, $a_{s1}^i(R_s^i, q_{s1}, q_{s2},)$ and $a_{s2}^i(R_s^i, q_{s1}, q_{s2})$, where $i = A, B$, and the sequence of prices $\hat{q}_{s1}$ and $\hat{q}_{s2}$ verifying the above set of equations and the market clearing conditions

$$\sum_{i=1} C_s^i(R_s^i, q_{s1}, q_{s2}) \leq \sum_{i=1} R_s^i,$$

$$\sum_{i=1} a_{s1}^i(R_s^i, q_{s1}, q_{s2}) = 0,$$

$$\sum_{i=1} a_{s2}^i(R_s^i, q_{s1}, q_{s2}) = 0.$$

We will not discuss here the existence, uniqueness and stability of the competitive equilibrium or the properties of the value, policy and control functions in the household's problems. Under the standard hypotheses we have assumed for the proposed economy, the existence, uniqueness and stability of the competitive general equilibrium are ensured, as well as the increasing monotonicity and strict concavity of the household's value functions $v_i(R_s^i)$. The interested reader can consult Bewley (1972), Mas-Colell (1986) and Mas-Colell, Whinston and Green (1995) to analyze the questions concerning the competitive equilibrium, and Stokey and Lucas with Prescott (1984) and Rincón-Zapatero and Rodríquez-Palmero (2003) to examine the recursive formulation of the household's problem.

Solving the former equations defining the equilibrium in the prices $\hat{q}_{s1}$ and $\hat{q}_{s2}$ and substituting into the consumption and securities equilibrium demand functions, the subjective general equilibrium can be exactly calculated. These subjective general equilibrium solution functions are

$$\hat{C}_s^A(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) =$$

$$\frac{R_s^A\{\beta R_s^B[\pi_1^B(R_s^A + R_s^B)(T_1^A T_2^B - T_1^B T_2^A) + (T_2^A - T_2^B)(R_s^A T_1^A + R_s^B T_1^B)] + (R_s^A T_1^A + R_s^B T_1^B)(R_s^A T_2^A + R_s^B T_2^B)\}}{(R_s^A T_1^A + R_s^B T_1^B)(R_s^A T_2^A + R_s^B T_2^B) - \beta R_s^A R_s^B(\pi_1^A - \pi_1^B)(T_1^A T_2^B - T_1^B T_2^A)}$$

$$\hat{C}_s^B(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) =$$

$$\frac{R_s^B\{\beta R_s^A[\pi_1^A(R_s^A + R_s^B)(T_1^A T_2^B - T_1^B T_2^A) + (T_2^A - T_2^B)(R_s^A T_1^A + R_s^B T_1^B)] - (R_s^A T_1^A + R_s^B T_1^B)(R_s^A T_2^A + R_s^B T_2^B)\}}{\beta R_s^A R_s^B(\pi_1^A - \pi_1^B)(T_1^A T_2^B - T_1^B T_2^A) - (R_s^A T_1^A + R_s^B T_1^B)(R_s^A T_2^A + R_s^B T_2^B)}$$

$$\hat{a}_{s1}^A(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) =$$

$$\frac{R_s^A \mathbf{R}_s^B\{\beta[\pi_1^A(\pi_1^B(R_s^A + R_s^B)(T_1^A T_2^B - T_1^B T_2^A) + R_s^B T_1^B(T_2^A - T_2^B)) + \pi_1^B R_s^A T_1^A(T_2^A - T_2^B)] + (\pi_1^A T_1^B - \pi_1^B T_1^A)(R_s^A T_2^A + R_s^B T_2^B)\}}{\beta R_s^A R_s^B(\pi_1^A - \pi_1^B)(T_2^A - T_2^B) + (\pi_1^A R_s^A + \pi_1^B R_s^B)(R_s^A T_2^A + R_s^B T_2^B)}$$

$$\hat{a}_{s2}^A(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) =$$

$$\frac{1}{\beta R_s^A R_s^B(\pi_1^A - \pi_1^B)(T_2^A - T_2^B) + (\pi_1^A R_s^A + \pi_1^B R_s^B)(R_s^A T_2^A + R_s^B T_2^B)}\{$$

$$R_s^A \mathbf{R}_s^B\{\beta[\pi_1^A(\pi_1^B(R_s^A + R_s^B)(T_1^A T_2^B - T_1^B T_2^A) + R_s^A(T_1^B T_2^A - T_1^A T_2^B) + R_s^B T_1^B(T_2^A - T_2^B))+$$

$$\pi_1^B(R_s^A T_1^A(T_2^A - T_2^B) + R_s^B(T_1^B T_2^A - T_1^A T_2^B)) + (T_2^B - T_2^A)(R_s^A T_1^A + R_s^B T_1^B)] + ((1-\pi_1^B)T_2^A - (1-\pi_1^A)T_2^B)(R_s^A T_1^A + R_s^B T_1^B)\}\}$$

$$\hat{a}_{s1}^B(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) = -\hat{a}_{s1}^A(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B),$$

$$\hat{a}_{s2}^B(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B) = -\hat{a}_{s2}^A(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B).$$

Through a standard analysis of the solutions, the conditions under which the results in section 3 apply can be easily proved. For instance and concerning our first result, it is immediate to show that when the consumption solution functions $\hat{C}_s^i(R_s^A, R_s^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B)$ are positive, then they are also increasing and concave in $\pi_{sl|s}^i$. Given the complexity of the algebraic solutions, these are the unique general mathematical properties we can easily obtain for this model. However, numerically and for values of the wealth shocks T_l^i compatible with the existence of insurance markets, all the qualitative results are verified. In this respect, it is immediate to numerically obtain that the bounded rational optimal expectations general equilibrium problem

$$\max_{\pi_1^A} E[\ln(\hat{C}^A)] = \pi_1 \ln(\hat{C}_{s1}^A(R_{s1}^A, R_{s1}^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B))+$$

$$\pi_2 \ln(\hat{C}_{s2}^A(R_{s2}^A, R_{s2}^B, T_1^A, T_2^A, T_1^B, T_2^B, \pi_1^A, \pi_1^B))$$

has a unique interior solution, and that the inequality

$$\left.\frac{\partial E[\ln(\hat{C}^A)]}{\partial \pi_1^A}\right|_{\pi_1^A = \pi_1} > 0$$

holds for a feasible set of endowment values. Therefore, under a set of sufficient conditions, we can apply our first theoretical result and ensure the existence of bounded rational optimism for household A , i.e. we can ensure that $\hat{\pi}_1^A > \pi_1$. By applying the same arguments to household B, it is possible to numerically find a set of sufficient conditions ensuring the bounded rational optimism of household B, i.e. that $\hat{\pi}_2^B > \pi_2$. Again, the questions to elucidate are then, first, whether this optimistic rational departure from accurate beliefs has a limit; and second, whether the optimal expectations general equilibrium can imply the existence of bounded rational optimism for both agents. As for the former models, we provide some numerical results showing that moderate optimism is beneficial, extreme optimism is prejudicial, and that the second set of sufficient conditions holds and there exist optimal expectations equilibria implying bounded rational moderate optimism for both agents. It is worth noting that, in this case and unlike the former models, the Nash equilibrium implies higher true expected utility. The analysis of the causes of this result, probably related to the dynamic dimension of this model and to the fact that we have considered only the bounded rational optimal expectation for the next period[9], is beyond the scope of this paper and part of the agenda of future research.

Table 4. Model 4.

$\pi_1 = 0.5$; $\pi_2 = 0.5$; $R_s^A = R_s^B = 100$; $T_1^A = T_2^B = 1.25$; $T_2^A = T_1^B = 0.75$; $\beta = 0.9$.

π_1^B	$\hat{\pi}_1^A$	Moderate optimism	Extreme optimism	True expected utility and consumption of agent A, optimal optimism of agent A	True expected utility and consumption of agent A, accurate beliefs of agent A	True expected utility and consumption of agent B, optimal optimism of agent A
0.25	0.648	[0.5,0.648]	(0.648,1]	4.58835605 99.6004044	4.5555832 95.1791386	4.59655371 100.399596
0.5	0.719	[0.5,0.719]	(0.719,1]	4.65428598 105.051763	4.57632516 100	4.55312716 94.9482374
0.7	0.766	[0.5,0.766]	(0.766,1]	4.75072191 116.728259	4.63671967 111.395129	4.40401462 83.2717412

Nash Equilibrium.

$\pi_1^{A\star}$ (optimistic)	$\pi_2^{B\star}$ (optimistic)	$\pi_2^{A\star}$ (optimistic)	$\pi_1^{B\star}$ (optimistic)	True Expected Utility and Consumption, Nash equilibrium	True Expected Utility and Consumption, accurate beliefs
0.674	0.674	0.326	0.326	4.60011248 100	4.5763251 100

5 Conclusions and Future Research

In this paper we present a canonical model of insurance markets in which agents are bounded rational, and that allows the main theoretical and empirical results concerning the implications, accuracy, and formation mechanisms of agents' beliefs to be reconciled. As is well known, these questions are at the core of a central puzzle in today's Economics. On the one hand, the main theoretical result concerning the relevance of accurate beliefs –the so called *market selection hypothesis*– asserts that agents with accurate beliefs are selected by the market

[9]In principle, in this dynamic problem, there exists one different bounded rational optimal expectation for each period.

over those with incorrect beliefs, since they earn higher expected wealth. This *market selection hypothesis* is nevertheless contradicted by empirical evidence, and there is substantial evidence revealing that agents habitually overestimate the probability of good outcomes, consistently and continuously maintain an optimistic bias in their assessment of probabilities, and can earn, by adopting biased beliefs, higher expected wealth than agents with correct beliefs.

The introduction of a bounded rational mechanism of belief formation can help to solve this puzzle of paramount importance in current economic theory. In this respect, taking a canonical model of insurance and hedging markets with stationary uncertainty as our starting point, we show that when agents choose as subjective probabilities those maximizing their one-period-ahead true expected utility, they commit, under quite general conditions, a bounded rational optimistic bias in assessing probabilities. The consideration of the true/objective one-period-ahead expected utility appears as a very logical criterion for agents from the point of view of bounded rationality. First of all, it informs on the real gains agents obtain by changing their beliefs in their real utility and wealth. In addition, the maximization of the one-period-ahead true expected utility is a problem the agents can tackle from the perspective of bounded rationality: given the stationarity of the uncertainty, it can be considered as a measurable and identifiable objective as time passes; and it is a simple one-period problem, whose solution (carried out consciously or unconsciously) does not entail excessive calculation abilities. As we show, when agents pursue this objective, there exist very plausible conditions ensuring, as a theoretical result, the existence of a consistent and continuous bounded rational optimistic bias in assigning probabilities. Indeed, from the empirical point of view, our results are in consonance with the observed behaviors: Our model implies an incentive for agents to adopt bounded rational optimistic beliefs; second, given the appearance of actual benefits associated to optimism, the model explains the continuous and consistent overestimation of the probabilities assigned to the good states/outcomes; and third, since bounded rational optimistic agents obtain higher expected wealth than agents with accurate beliefs, our model is compatible with the evolutionary survival of agents with biased beliefs.

This research, at a preliminary stage, implies some direct and obvious conclusions and opens up several interesting lines of investigation. As a first conclusion, we can clearly see the great potential that the introduction of bounded rational behaviors in standard general equilibrium models of insurance markets has in studying the role that sentiments play in insurance and hedging activities and in elucidating how expectations and beliefs are originated in an economy, both questions of great importance in current economic theory. Indeed, using canonical models of insurance markets as the starting point, our simple analyses have shown that the introduction of a bounded rational criterion of expectation formation leads to a departure from accurate beliefs, and they have also revealed suggestive intuitions to interpret the mechanisms of belief formation in insurance and hedging markets. As a second conclusion, the huge formal difficulties in reaching conclusive results is also clear, even for the simple canonical models we have considered.

From the theoretical point of view, the future lines of research involve a double mission. First, to go more deeply into the concept of *Bounded Rational Optimal Expectations General Equilibrium* by considering, jointly with our criterion of maximization of the one-period-ahead true expected utility, different plausible alternative rules of expectation formation. These would clarify the implications of the different criteria, helping to ascertain which of them are consistent with the observed persistent optimistic bias of agents in assessing probabilities and which of them can therefore be the ones actually adopted. As a second important theoretical line of investigation, it will be necessary to obtain new mathematical results allowing the formal analysis of the models to be developed, especially in the fields of dynamic optimization and game theory.

From the empirical point of view, future research requires the analysis of the explanatory capability of the different proposed models, both from the qualitative and quantitative perspective, paying special attention to the concordance with the observed data on optimism and pessimism.

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