Evolution of quantum observables: from non-commutativity to commutativity

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Abstract

A fundamental aspect of the quantum-to-classical limit is the transition from a non-commutative algebra of observables to commutative one.However, this transition is not possible if we only consider unitary evolutions. One way to describe this transition is to consider the Gamow vectors, which introduce exponential decays in the evolution. In this paper, we give two mathematical models in which this transition happens in the infinite time limit. In the first one, we consider operators acting on the space of the Gamow vectors, which represent quantum resonances. In the second one, we use an algebraic formalism from scattering theory. We construct a non-commuting algebra which commutes in the infinite time limit.

1 Introduction

Since the work of Birkhoff and von Neumann [1], it is known that the properties of a classical system are represented by subsets of the phase space, which has a Boolean lattice structure (it is an orthocomplemented and distributive lattice), while in quantum mechanics, the properties are represented by closed vector subspaces (or by their corresponding orthogonal projectors). The set of all quantum properties is an orthocomplemented lattice (as in classical physics), but a non-distributive one. This mathematical structure was called quantum logic [1].

The different mathematical structures associated to the quantum formalism –such as quantum logics, orthomodular structures and effect algebras- are usually called *quantum structures* [2]. While there are different approaches for the study of the quantum-classical transition, the

*All authors have contributed equally to this work approaches based in quantum structures focuses almost exclusively in a static perspective. But, in the classical limit process, the quantum structures associated to a given quantum system should become those of a classical one. This implies the necessity of including a dynamical perspective in the study quantum structures.

There are several approaches to the study of the classical limit [5]. Among them, one can find the decoherence approach, based on the action of the environment. In previous papers we have discussed an alternative approach, based on the evolution of the algebra of observables. A non-commutative algebra of observables evolves into a commutative one, when time tends to infinity. However, this is not possible if the time evolution is unitary. In order to achieve a transition from a non-commutative algebra to commutative one, we need to consider more general time evolutions. The approach based in non-standard evolutions has been applied in the context of self-induced decoherence [3, 4], enviroment induced decoherence [5, 6, 7], and for quantum maps [8]. Also, we have studied its formal aspects and its algebraic properties [8, 9], using the rigged Hilbert space (RHS) formalism. In this work we will elaborate further in this framework.

The theory of quantum resonances and quantum unstable states –which can be suitably described using the RHS– illustrate several relevant features of quantum mechanics that cannot be captured using the standard approach based on the Hilbert space. However this theory is not fully developed yet, and its consequences should be the goal of further study.

A comprehensible mathematical model for quantum resonances is the Friedrichs model, in which one or several bound states interact with a system with unbounded continuous spectrum. This can be looked as an interaction between one or several atoms and an external field. As a consequence of the interaction, the bound states become unstable and decay. For large values of time, only the continuous spectrum remains.

At the same time, the Friedrichs model show us that the theory of resonances may be looked from two different points of view. In one side, resonances may be viewed as *open systems* in which the decay is the result of a system, the atom, with an external bath, the field. One the other hand, we may focus our interest in the study of the resonance by itself by considering it as a *closed system* and analyzing its behavior.

It is true that studies advocate to one or the other point of view [18, 21, 20]. In the present paper, we are considering a system of resonances as a closed system. Then, resonance states are the result of the sum of two contributions. One is the Gamow state, which decays exponentially. The other is the background, which accounts for the deviations of the exponential decay for very short and very large values of time. These deviations are difficult to be observed. Indeed, for most observations, the region in which the resonance decays exponentially covers almost any time interval. For this reason, the Gamow state could be considered as a good representation of a resonance state.

It is well known that Gamow states cannot be normalized in the sense of the scalar product in the ordinary Hilbert space. In order to properly define them and establish their properties, one needs to extend the Hilbert space to rigged Hilbert spaces [21, 38]. In this formalism, we see in a rigorous manner, that the unitary hamiltonian evolution can be extended to a purely exponential decay. Gamow vectors could be looked as dissipative structures that vanish at long values of time.

For a system with N distinct resonances, we may construct a vector space spanned by their Gamow vectors, the space of resonance states. Although on such space we could construct a legitimate scalar product, it lacks a proper physical meaning. Some studies concerning physical properties of resonances [28, 51, 60, 61], suggest that it is more convenient the introduction of a pseudo-scalar metrics on this space (Krein space). Then, we also may define observables as operators on this space and an operator giving an exponential time decay for Gamow states. This operator is produced by exponentiating a sort of truncated Hamiltonian and is not unitary in the sense that its product times its formal adjoint cannot be the unity. Within this framework, we show that time evolution of observables produces a decoherence phenomenon: observables commute after very long times.

Gamow states can be also formulated as functionals over an algebra of observables. In this sense, Gamow states are neither pure states nor mixtures, but structures similar to the van Hove states which have been defined for quantum systems far from equilibrium (see [48] and references therein). We have shown that also in this context, the time evolution of observables drives to decoherence in the above sense.

In [8], we have given an example based in the Gamow formalism for resonances. The space of states is formed by the Gamow vectors representing resonance states. The algebra of operators is given by the linear mappings on the space of this resonance states. In order to get a suitable non-unitary time evolution, it is necessary to isolate the resonances from the environment, as described in [8]. This is summarized in Section 2 of this work. It is important to recall that resonances represent quantum unstable states [10]. We shall use these two terms without distinction.

In this paper, we give a further generalization of the non-unitary evolution presented in [8]. This is done by appealing to a formalism for quantum observables and states that was originally developed so as to include van Hove states, originally conceived to be applied in statistical mechanics of systems far from equilibrium. In this formalism, observables belong to a non-commutative algebra and states are represented by positive functionals over the algebra of observables. Then, we discuss the scattering process given by a Hamiltonian pair $\{H_0, H = H_0 + V\}$, where the scattering is produced by a potential V. This gives place to two isomorphic algebras which can be put in connection with the *in* and *out* states of the scattering process. Although resonances may appear in this model and Gamow states are properly defined, we do not rely in the presence of resonances this time. Under suitable conditions, *in observables* and *out observables* commute at the limit $t \longmapsto -\infty$ and $t \longmapsto +\infty$, respectively.

The paper is organized as follows: In Section 2, we discuss a model based in the construction of the Gamow spaces with Krein pseudo-metrics and we show that in the limit $t \longmapsto \infty$ all operators commute. In Section 3, we describe the construction of in and out algebras and how this algebras become commutative at the limits of large negative and positive times. Finally, we present our conclusions.

2 Quantum resonances

There are several definitions of quantum resonances, not all of them equivalent [11, 12, 13], based either on physical or mathematical considerations. From the physical point of view, one may assume that resonances come from resonant scattering: an incident particle stays a long time on an interaction region and then scapes. Thus, from the physical point of view resonances can be detected [12] either by long Wigner times [14], by a sharp bump in the cross section, or a sudden change in the phase shift of about $\pi/2$.

The resonance scattering requires of the existence of two dynamics, a *free* dynamics governed by a Hamiltonian H_0 and a *perturbed* dynamics governed by a total Hamiltonian $H = H_0 + V$, where V is the potential giving by the perturbation which is responsible for the resonance behavior.

From the mathematical point of view, there are two definitions of resonances, not always equivalent (although in many models they are equivalent). One of them is based on the study of the poles appearing in the resolvent functions associated to the Hamiltonians, and the other is based on the S-matrix formalism.

For the former, we define the following pair of complex functions for $\psi \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the system under consideration [15]:

$$F_0(z) := \langle\psi|(H_0 - z)|\psi\rangle\,, \qquad F(z) := \langle\psi|(H - z)|\psi\rangle\,. \tag{1}$$

Then, assume that there is a dense subspace $\mathcal{D}$ such that for any $\psi \in \mathcal{D}$ these functions are meromorphic having the positive semi-axis $\mathbb{R}^+$ as a branch cut and that they are analytically continuable through the cut. Then, if $F(z)$ has a pole in the analytic continuation at $z_R = E_R - i\Gamma/2$ and $F_0(z)$ is analytic at this point, we say that the Hamiltonian pair $\{H_0, H\}$ has a resonance pole at z_R. Its real part E_R is the resonance energy and Γ is the inverse of the mean life. Note that the complex conjugate $z_R^* = E_R + i\gamma/2$ of z_R has the same property.

The second definition of resonance assumes that the S-matrix has an analytic continuation either as a function of the momentum p or of the energy E [12, 16]. In this case, which is the one we will consider in this paper, resonances are characterized by poles of the analytic continuation of $S(E)$ through the cut $\mathbb{R}^+$. These poles appear as complex conjugate pairs, $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$, both representing the same resonance. In many relevant models, such as the Friedrichs model [17, 18, 19], poles of $S(E)$ and poles of $F(z)$ coincide. However, there are models for which poles of $S(E)$ do not correspond to bumps in the cross section and vice-versa [11].

Pure states are represented by unit vectors in the Hilbert space $\mathcal{H}$. Now, the question is to assign a vector state $|\psi\rangle \in \mathcal{H}$ to a resonance state. As it can be observed in several cases [20, 21], its non-decay probability should follow an exponential law:

$$\mathcal{P}(t) = |\langle\psi|e^{-itH}|\psi\rangle|^2 \propto e^{-\Gamma t/2}\,, \qquad t \geq 0\,. \tag{2}$$

But not all models predict this behavior. Although for decaying states, $|\psi\rangle \in \mathcal{H}$, $\mathcal{P}(t)$ is approximately exponential for all times, there are large deviations from this exponential law for very short and very large values of time [11]. These deviations are due to the semi-boundedness of the Hamiltonian. In fact, pure exponential decays exist if and only if the state $|\psi\rangle$ obeys a Breit Wigner energy distribution [12], something that is not possible when the spectrum of H is semi-bounded [11, 22, 23]. However, recent experiments suggest that, in some cases, there are indeed deviations from the exponential law [24, 25, 26]. For example, for short times, the probability stands approximatelly constant and decays slowly. For extremely large times a decay following a t^{-1} law is observed. This regime, known as Khalflin effect, is very difficult to observe, although it has been experimentally detected [24, 27]. Nevertheless, for most of real life experiments the observational fact is that the non-decay probability is exponential, up to a reasonable degree of accuracy and for times of observation that are not too small to detect an effect or too large so that it just remains a quite small sample of undecayed products. Thus, for most experiments and observational times the decay rule is a good approximation. If $|\psi\rangle \in \mathcal{H}$ is a decaying state, it admits the following decomposition:

$$|\psi\rangle = |\psi^D\rangle + |\psi^B\rangle \,, \tag{3}$$

where $|\psi^D\rangle$ decays exponentially for $t \geq 0$ and $|\psi^B\rangle$ represents the interaction with the environment, which is assumed to be responsible for all deviations of the pure exponential law. As $|\psi^D\rangle$ represents the "intrinsic" decay, it is often taken as the vector state for the resonance state. It is called the *decaying Gamow vector*. The requirement for the decaying Gamow vector, $|\psi^D\rangle$, to decay exponentially has motivated its definition proposed by Nakanishi [28] as the eigenvector of the Hamiltonian with eigenvalue $z_R = E_R - i\Gamma/2$:

$$H|\psi^D\rangle = z_R\,|\psi^D\rangle \,. \tag{4}$$

As a consequence of this definition, one formally derives the exponential decay with time $t \geq 0$:

$$e^{-itH}\,|\psi^D\rangle = e^{-itE_R}\,e^{-t\Gamma/2}\,|\psi^D\rangle \,, \qquad t \geq 0 \,. \tag{5}$$

This definition for the decaying Gamow vector has an apparent difficulty: If the Hamiltonian H is self-adjoint in $\mathcal{H}$, it only admits real eigenvalues *with eigenvectors in* $\mathcal{H}$. However, it is possible to obtain complex eigenvalues if we define the eigenvalue equation in a larger space containing the Hilbert space as a subspace. This can be performed in the context of *rigged Hilbert spaces* (RHS). A RHS is a triplet of spaces [29, 30, 31, 32, 33, 34]

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{6}$$

where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. The dense subspace Φ has a topology finer than the topology inhereted from $\mathcal{H}$, so that the canonical injection $i : \Phi \longmapsto \mathcal{H}$, $i(\psi) = \psi$, $\forall\, \psi \in \Phi$, is continuous. The space $\Phi^\times$ is the antidual, i.e., the space of continuous antilinear functionals (mappings from Φ into the field of complex numbers $\mathbb{C}$) on Φ. The action of $F \in \Phi^\times$ on $\phi \in \Phi$ is here denoted as $\langle\phi|F\rangle$, so as to preserve that Dirac bra-ket notation. The antilinearity of F means that for all $\alpha, \beta \in \mathbb{C}$ and for all $\phi, \psi \in \Phi$, we have $\langle\alpha\phi + \beta\psi|F\rangle = \alpha^*\langle\phi|F\rangle + \beta^*\langle\psi|F\rangle$, where the star means complex conjugation. We endow $\Phi^\times$ with the weak topology with respect to the dual pair $\{\Phi, \Phi^\times\}$ [35, 36]. Each $\psi \in \mathcal{H}$ produces a unique $F_\psi \in \Phi^\times$ by $\langle\phi|F_\psi\rangle := \langle\phi|\psi\rangle$, where $\langle\phi|\psi\rangle$ is the scalar product on $\mathcal{H}$.

Rigged Hilbert spaces have been used in order to implement the Dirac formalism of Quantum Mechanics [20, 30, 31, 32, 33, 34], for which the Hilbert space was not sufficient. Later, it was realized that Gamow vectors acquire full mathematical meaning in the context of RHS with an appropriate implementation [21, 37, 38]. Also, RHS serve as a unifying framework for Lie group representation, special functions, discrete and continuous basis, with methods that include signal theory [39, 40, 41].

Since expressions like (4) and (5) are defined in the extension $\Phi^\times$ of the Hilbert space, we need a mechanism to extend the Hamiltonian and the evolution operator to $\Phi^\times$. This mechanism is the *duality formula*. It goes as follows: Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ be a RHS and A an operator (for us an operator is always linear and densely defined) on $\mathcal{H}$ verifying the following properties: i.) The adjoint of A, $A^\dagger$, preserves Φ, i.e., for any $\varphi \in \Phi$, $A^\dagger\varphi \in \Phi$ (alternatively, we say that $A^\dagger\,\Phi \subset \Phi$); ii.) The adjoint $A^\dagger$ is continuous on Φ with respect to the topology that we have previously defined on Φ. In this case, A admits a unique continuous extension to $\Phi^\times$ given by the following duality formula:

$$\langle A^\dagger\,\varphi|F\rangle = \langle\varphi|AF\rangle\,, \qquad \forall\,\varphi \in \Phi\,, \quad \forall\,F \in \Phi^\times\,. \tag{7}$$

Note that AF defines a unique element of $\Phi^\times$. If A were Hermitian, (7) would have an obvious form. Furthermore, If A is self-adjoint, there always exists a RHS fulfilling the above conditions with respect to A [30]. In the case under discussion, being given the total Hamiltonian $H = H_0 + V$, we may always construct a RHS, $\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$, such that i.) $H\Phi_+ \subset \Phi_+$; ii.) H is continuous on Φ_+; H may be extended to $\Phi_+^\times$ with continuity; iv.) the eigenvalue expression $H|\psi^D\rangle = z_R\,|\psi^D\rangle$ makes full sense in $\Phi_+^\times$.

In addition, $e^{itH}\,\Phi_+ \subset \Phi_+$ with continuity, at least for $t \geq 0$, so that its adjoint e^{-itH} may be extended to $\Phi_+^\times$ by continuity, at least for $t \geq 0$. This implies that (5) makes full sense on $\Phi_+^\times$. As proven in [21, 38], the space Φ_+ admits a representation in terms of Hardy functions on the lower half of the complex plane.

We have already mentioned that resonance poles appear in complex conjugate pairs of the scattering matrix written in the energy representation. In addition to the pole located at $z_R = E_R - i\Gamma/2$, there exists the pole at $z_R^* = E_R + i\Gamma/2$. Then, there is another RHS $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ and a vector $|\psi^G\rangle \in \Phi_-^\times$ with $|\psi^G\rangle \notin \mathcal{H}$, such that

$$H|\psi^G\rangle = z_R^*\,|\psi^G\rangle\,, \tag{8}$$

$$e^{-itH}\,|\psi^G\rangle = e^{-itE_R}\,e^{t\Gamma/2}\,|\psi^G\rangle\,, \qquad t \leq 0\,. \tag{9}$$

The space Φ_- admits a representation with Hardy functions on the upper half of the complex plane. The vector $|\psi^G\rangle$ is often called the *growing Gamow vector*. The decaying and the growing Gamow vectors are related via the time reversal operator T, which may be well defined on the RHS [42], and verifies the following relations:

$$T\,\Phi_\pm = \Phi_\mp\,, \quad T\,\Phi_\pm^\times = \Phi_\mp^\times\,, \quad T|\psi^D\rangle = |\psi^G\rangle\,, \quad T|\psi^G\rangle = |\psi^D\rangle\,. \tag{10}$$

Then, for each resonance there are two state vectors, the growing and the decaying Gamow vectors. Both are equally suitable for a state vector and are just time reversal of each other.

Finally, we wish to remark that we shall in principle assume that the resonance poles are simple. They may be multiple, and in that case z_R and z_R^* must have the same multiplicity [43]. Models with double pole resonances have been constructed [43, 44].

2.1 From non-commutativity to commutativity

Along this subsection, we summarize a first approach to the transition from a non-commutative algebra of observables to a commutative one, taking the limit as $t \longmapsto \infty$. This cannot be achieved using a unitary evolution, so that we need to use a formalism in which time evolution is not unitary. Moreover, we have to make use of an structure different from the Hilbert space. In this work, for simplicity, we will restrict to models with a finite number of resonance poles.

Thus, we assume that our system has N resonances, with the resonance poles given by $z_1, z_2, \ldots, z_N$ and their respective complex conjugates. The corresponding decaying Gamow vectors are

$$|\psi_1^D\rangle, |\psi_2^D\rangle, \ldots, |\psi_N^D\rangle\,, \qquad H|\psi_i^D\rangle = z_i\,|\psi_i^D\rangle\,, \quad i = 1, 2, \ldots, N\,, \tag{11}$$

and the corresponding growing Gamow vectors are

$$|\psi_1^G\rangle, |\psi_2^G\rangle, \ldots, |\psi_N^G\rangle\,, \qquad H|\psi_i^G\rangle = z_i^*\,|\psi_i^G\rangle\,, \quad i = 1, 2, \ldots, N\,. \tag{12}$$

Moreover, there are two generalized spectral decompositions of the Hamiltonian [8, 45]:

$$H = \sum_{i=1}^{N} z_i\,|\psi_i^D\rangle\langle\psi_i^G| + \mathrm{BGR}\,, \tag{13}$$

and

$$H^\dagger = \sum_{i=1}^{N} z_i^*\,|\psi_i^G\rangle\langle\psi_i^D| + \mathrm{BGR}^*\,. \tag{14}$$

The term BGR includes whatever does not depend on the Gamow vectors. This term is always non-vanishing. Observe that decompositions (13) and (14) are the formal adjoint of each other. If Φ and Ψ are two topological vector spaces, we denote by $\mathcal{L}(\Phi, \Psi)$ the space of all continuous linear operators from Φ into Ψ. Using this terminology and notation, one finds that [45]

$$H \in \mathcal{L}(\Phi_-, \Phi_+^\times)\,, \qquad H^\dagger \in \mathcal{L}(\Phi_+, \Phi_-^\times)\,. \tag{15}$$

Observe that H is an operator on Φ_-. The action of H on a vector of Φ_- gives a vector in $\Phi_+^\times$. A similar comment is in order about $H^\dagger$ as a linear mapping from Φ_+ into $\Phi_-^\times$. Since the description of resonances may be given in terms of decaying and growing Gamow vectors, it results that the distinction between H and $H^\dagger$ is purely conventional and these two operators represent two similar *non-hermitian* decomposition of the Hamiltonian which are formally time reversal of each other. However, the action of (13) or (14) on the Gamow vectors has no sense as Gamow vectors do not belong neither to Φ_- nor to Φ_+. We need to give a meaning to such actions.

We are interested in what concerns to resonances and, in consequence, we drop the background terms. Then, both H and $H^\dagger$ are non-hermitian operators depending only on resonance poles and Gamow vectors, although still verify (15). These *truncated* versions of H and $H^\dagger$ are as in (13) and (14), but without the BGR term.

Let us consider the $2N$ dimensional space $\mathcal{H}^G$, spanned by the following vectors:

$$\{|\psi_1^D\rangle\,,|\psi_1^G\rangle\,,|\psi_2^D\rangle\,,|\psi_2^G\rangle\,,\dots\,,|\psi_N^D\rangle\,,|\psi_N^G\rangle\}\,. \tag{16}$$

Then, we define a pseudometric on $\mathcal{H}^G$ as the bilinear form given in the basis (16) by the following matrix:

$$A := \left(\begin{array}{ccccccc} 0 & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & 1 & \dots & \dots & \dots \\ \dots & \dots & 1 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & 0 \end{array}\right)\,. \tag{17}$$

All entries replaced by dots are equal to zero. Then, the psuedo-scalar product of two vectors $|\psi\rangle, |\varphi\rangle \in \mathcal{H}^G$ is defined as

$$(\psi|\varphi) := \langle\psi|A|\varphi\rangle\,, \tag{18}$$

so that

$$(\psi_i^D|\psi_j^D) = (\psi_i^G|\psi_j^G) = 0\,, \quad (\psi_i^D|\psi_j^G) = (\psi_i^G|\psi_j^D) = \delta_{ij}\,. \tag{19}$$

This pseudometric is important in this context in order to define the operations that are relevant for the construction of an evolution operator. Assume that we define a standard scalar product on $\mathcal{H}^G$. The simplest one implies that (16) is an orthonormal basis. For simplicity, we may assume first the presence of only one resonance. Then,

$$H^2 = z_R^2|\psi^D\rangle\langle\psi^G|\psi^D\rangle\langle\psi^G| = 0. \tag{20}$$

Therefore, to avoid this kind of problems maintaining some simplicity in the operations is why we introduce this pseudometric. From (18), we note that if B is a matrix such that $A = B^2$, one has that $B|\psi_i^D\rangle \equiv |\psi_i^D)$ and $\langle\psi_i^G|B \equiv (\psi_i^G|$, $i = 1, 2, \dots, N$. The matrix B is not uniquely defined, although a reasonable choice is given by replacing the 2×2 dimensional non-vanishing boxes in (17) by

$$B = (-i)^{1/2}\left(\begin{array}{cc} i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & i\sqrt{2}/2 \end{array}\right)\,. \tag{21}$$

With some natural amendments in the formalism, $|\psi^D)$ and $|\psi^G)$ will represent the Gamow vectors. Then, we have to replace (13), without the BGR term, by

$$H = \sum_{i=1}^{N} z_i \, B|\psi_i^D\rangle\langle\psi_i^G|B = \sum_{i=1}^{N} z_i \, |\psi_i^D)(\psi_i^G| \,, \tag{22}$$

so that

$$H|\psi_j^D) = \sum_{i=1}^{N} z_i \, |\psi_i^D)(\psi_i^G|\psi_j^D) = \sum_{i=1}^{N} z_i \, |\psi_i^D) \, \delta_{ij} = z_j \, |\psi_j^D) \,, \tag{23}$$

$$H^2 = B \left[\sum_{i=1}^{N} z_i^2 \, |\psi_i^D\rangle\langle\psi_i^G| \right] B = \sum_{i=1}^{N} z_i^2 \, |\psi_i^D)(\psi_i^G| \,. \tag{24}$$

We obtain in the same way H^n, so that we arraive to the following formal expression:

$$e^{-itH} = \sum_{j=1}^{N} e^{-itz_j} \, |\psi_j^D)(\psi_j^G| \,. \tag{25}$$

If we replace B by $C := B^\dagger$, we have $|\psi_i^G) \equiv C|\psi_i^G\rangle$ and $(\psi_i^D|C \equiv \langle\psi^D|$. Then, $H^\dagger$ becomes:

$$H^\dagger = C \left[\sum_{i=1}^{N} z_i^* \, |\psi_i^G\rangle\langle\psi_i^D| \right] C = \sum_{i=1}^{N} z_i^* \, |\psi_i^G)(\psi_i^D| \,, \tag{26}$$

so that

$$H^\dagger \, |\psi_i^G) = z_i^* \, |\psi_i^G) \,, \qquad e^{-itH^\dagger} = \sum_{j=1}^{N} e^{-itz_j^*} \, |\psi_j^G)(\psi_j^D| \,. \tag{27}$$

In addition, we have the following results:

$$\begin{aligned}
&H|\psi_j^G) = 0 \,, \qquad H^\dagger|\psi_j^D) = 0 \,, \quad j = 1, 2, \ldots, N \\
&e^{-itH}|\psi_j^D) = e^{-itz_j} \, |\psi_j^D) \,, \qquad e^{-itH^\dagger}|\psi_j^G) = e^{-itz_j^*} \, |\psi_j^G) \,, \\
&e^{-itH}|\psi_j^G) = 0 \,, \qquad e^{-itH^\dagger}|\psi_j^D) = 0 \,.
\end{aligned} \tag{28}$$

Therefore, both evolution operators e^{-itH} and $e^{-itH^\dagger}$ act non-trivially on a half of the space $\mathcal{H}^G$ and vanish on the other half. However, it is desirable that meaningful operators on resonance states act on all Gamow states. Therefore, we need an obvious extension of the above formalism to all $\mathcal{H}^G$. To such end, we define the Hamiltonian as

$$H = \sum_{i=1}^{N} z_i \, |\psi_i^D)(\psi_i^G| + z_i^* \, |\psi_i^G)(\psi_i^D| \,. \tag{29}$$

With this definition, H is formally Hermitian. Using the pseudometric, we obtain the following properties:

$$H|\psi_i^D) = z_i\,|\psi_i^D)\,, \qquad H|\psi_i^G) = z_i^*\,|\psi_i^G)\,, \tag{30}$$

$$H^n = \sum_{i=1}^N z_i^n\,|\psi_i^D)(\psi_i^G| + (z_i^*)^n\,|\psi_i^G)(\psi_i^D|\,, \tag{31}$$

so that formally:

$$U(t) := e^{-itH} = \sum_{j=1}^N e^{-itz_j}\,|\psi_j^D)(\psi_j^G| + e^{-itz_j^*}\,|\psi_j^G)(\psi^D|\,. \tag{32}$$

The identity operator admits the following decomposition:

$$I = \sum_{i=1}^N \{\,|\psi_i^D)(\psi_i^G| + |\psi_i^G)(\psi_i^D|\,\}\,. \tag{33}$$

Note that from (32), we have that

$$U(-t) = e^{itH} = \sum_{j=1}^N e^{itz_j}\,|\psi_j^D)(\psi_j^G| + e^{itz_j^*}\,|\psi_j^G)(\psi^D|\,. \tag{34}$$

Hence, using once again the pseudometric relations (19), we find that $U(t)U(-t) = U(-t)U(t) = I$, so that $U(-t) = U^{-1}(t)$. So far, everything sounds like ordinary quantum mechanics. If we proceed further, the time evolution of an arbitrary linear operator $O(t=0)$ should be given by $O(t) = U(-t)\,O\,U(t)$. However, this produces a contribution in $O(t)$ that grows exponentially, which cannot be cancelled out with any other term [8]. This suggests that (32) may not be a good definition for the time evolution operator in our case. For this reason, we propose the following definition for the time evolution operator:

$$U(t) = \sum_{j=1}^N e^{-itz_j}\,|\psi^D)(\psi^G| + e^{itz_j^*}\,|\psi^G)(\psi^D|\,, \tag{35}$$

which is formally Hermitian. This means that the formal adjoint is given by $U^\dagger(t) = U(t)$. In this case, the relation $U^\dagger(t)U(t) = I$ is not longer correct. Instead we have

$$U(t)U^\dagger(t) = U^2(t) = e^{-t\Gamma}\,I\,. \tag{36}$$

This is not the usual quantum relation, although the exponential decay is consistent with the exponential decay of Gamow states. Observe that (35) is already time asymmetric. For the time evolution of observables, one should choose $O(t) = U^\dagger(t)\,O\,U(t)$, which is consistent with the usual Heisenberg picture.

Then, let us take two arbitrary operators O_1 and O_2 on $\mathcal{H}^G$, let them evolve with time and construct their commutator at time t, which gives [8]

$$[O_1(t), O_2(t)] = \sum_{i=1}^{N} e^{-t\Gamma_i} \{\alpha_i(t) \, |\psi_i^D)(\psi_i^G| + \beta_i(t) \, |\psi_i^G)(\psi_i^D|\} \,, \tag{37}$$

where $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, 2, \ldots, N$, are oscillating functions of time. In this way, we obtain a time evolution for which the commutator of two given observables decays as a sum of exponential terms.

2.2 Time reversal operation

It is interesting to discuss the previous formalism under the time reversal operation. We have previously established that if T is the time reversal operator, $T|\psi^D\rangle = |\psi^G\rangle$ and $T|\psi^G\rangle = |\psi^D\rangle$. Then, one may expect that the same operations result when we use $|\psi^D)$ and $|\psi^G)$ instead. However, this is not true, unless we use a correct definition for the time reversal operator suitable for this case. In fact, there are different definitions of the time reversal operator (see for example, [42, 46, 47]). Although these definitions were, in principle, related with some projective representations of the Poincaré group extended with time inversion and parity, we shall see that one of these two dimensional choices for T is suitable for our discussion.

For simplicity, we assume that the number of resonances is $N = 1$. The extension to an arbitrary number of resonances is straightforward. Now, we adopt the following notation for the basis of $\mathcal{H}^G$ as $|\psi^D\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\psi^G\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let us choose as time reversal operator [42, 46, 47] the following matrix, with C the complex conjugation operation:

$$T := \begin{pmatrix} 0 & \mathcal{C} \\ \mathcal{C} & 0 \end{pmatrix} . \tag{38}$$

Then,

$$T|\psi^D) = TB \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{C} \\ \mathcal{C} & 0 \end{pmatrix} (-i)^{1/2} \begin{pmatrix} i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & i\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= i^{1/2} \begin{pmatrix} \sqrt{2}/2 \\ -i\sqrt{2}/2 \end{pmatrix} = i^{1/2} \begin{pmatrix} -i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -i\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B^\dagger |\psi^G\rangle = |\psi^G) \,. \tag{39}$$

Analogously,

$$T|\psi^G) = |\psi^D) \,. \tag{40}$$

However, the Hamiltonian (29) is not time reversal invariant. It may be shown by a simple calculation. Let $|\psi) = \sum_{i=1}^{N} a_i \, B|\psi_i^D\rangle + \sum_{i=1}^{N} b_i \, B^\dagger \, |\psi^G\rangle$ be an arbitrary vector in $\mathcal{H}^G$, where a_i and b_i $(i = 1, \ldots, N)$ are complex numbers. We need to compare

$$(\psi|H|\psi) = \sum_{i=1}^{N} z_i (\psi|\psi_i^D)(\psi_i^G|\psi) + \sum_{i=1}^{N} z_i^* (\psi|\psi_i^G)(\psi_i^D|\psi)\,, \tag{41}$$

with

$$(\psi|THT|\psi) = \sum_{i=1}^{N} z_i^* (\psi|T|\psi_i^D)(\psi_i^G|T|\psi) + \sum_{i=1}^{N} z_i (\psi|T|\psi_i^G)(\psi_i^D|T|\psi)\,. \tag{42}$$

This calculation is rather straightforward and we sketch it in Appendix. The conclusion is that (41) and (42) do not coincide and therefore, the model is not time reversal invariant.

3 Algebraic formalism

In this section, we will use an alternative formalism for the description of observables and states of the non-relativistic quantum mechanics, which was originally proposed in order to include the van Hove states that appear in systems far from thermodynamic equilibrium [48]. This formalism was extended to further purposes [49, 50]. In particular, it has been useful to accommodate Gamow states for resonances as functionals over some type of algebras. The presentation of quantum states as functionals on algebras generated by observables is a common feature. Thus, we have shown that Gamow states for quantum unstable systems are as valid as any other type of states.

Our aim is to use this formalism in order to describe the transition of the algebra of observables from non-commutativity to commutativity. For our discussion, we will follow the formalism described in [51]. But there the purpose was different, the aim was to include Gamow states as functionals on certain algebras of observables. Note that in [51], we have focused our attention in the behavior of states. Now, we will focus on the time evolution of the observables, so that we shall use a type of Heisenberg time evolution. We have discussed the motivation for the formalism given in the present section in [51]; therefore, we shall not insist on the motivation and the details in this work.

It is important to remark that our approach differs from those which are based on state decoherence induced by the environment (such as Decoherent Histories [52, 53], Quantum Darwinism [54] or Consistent Histories [55, 56]). Our approach uses a perspective which is based in the algebra of observables –considered in the Heisenberg picture– and relies on instabilities based on resonances, which are described appealing to Gamov vectors.

Resonances are typically produced by a Hamiltonian pair $\{H_0, H = H_0 + V\}$, and we assume that it has the simplest properties and works in the energy representation. In particular, the spectrum of the *free* Hamiltonian H_0 is purely absolutely continuous, simple and given by $\mathbb{R}^+ \equiv [0, \infty)$. Let $|E\rangle$ be the eigenvector of H_0 for the particular value of the energy $E \in \mathbb{R}^+$. The meaning of such eigenvectors has been largely discussed as functionals on certain rigged Hilbert space. They are also valid for the expansion of H_0 in terms of generalized projections [57, 58] as

$$H_0 = \int_0^\infty dE\, E\, |E\rangle\langle E|\,. \tag{43}$$

An operator O is said to be compatible with H_0 if it has the form

$$O = \int_0^\infty dE\, O(E)\, |E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE\, O(E,E')\, |E\rangle\langle E'|\,, \tag{44}$$

where $O(E)$ and $O(E,E')$ are given by well behaved functions, which always can be multiplied by each other. From this point of view, they could be considered as test functions. In general $O(E,E') \neq O(E',E)$. One of the properties of the kets $|E\rangle$ is that

$$\langle E|E'\rangle = \delta(E-E')\,. \tag{45}$$

A proper choice of the space of test functions shows that the set of operators which are compatible with H_0 is a non-commutative algebra with identity, $\mathcal{A}_0$. The proof is straightforward. The identity is given by

$$I = \int_0^\infty dE\, |E\rangle\langle E|\,. \tag{46}$$

Next, we consider the scattering produced by the Hamiltonian pair $\{H_0, H\}$. Some extra assumptions are needed, which we choose with criteria of simplicity and generality. We assume asymptotic completeness and the existence of the Møller wave operators

$$\Omega_\pm := \lim_{t\to\pm} e^{itH}\, e^{-itH_0}\, \varphi =: \varphi^\pm\,, \tag{47}$$

with φ a state that evolves freely. Thus, the Møller wave operators relate states that evolve freely, according to the hamiltonian H_0, with states $\varphi^\pm$, that evolve under the total Hamiltonian $H = H_0 + V$, where V is a potential. Note that $\lim_{t\to\pm}(e^{-itH_0}\,\varphi - e^{-itH}\,\varphi^\pm) = 0\,.$

We define the following states:

$$|E^\pm\rangle = \Omega_\pm\, |E\rangle\,, \tag{48}$$

which are eigenvectors of H, with eigenvalue $E \in \mathbb{R}^+$, i.e., $H|E^\pm\rangle = E\,|E^\pm\rangle$. States given in equation (48) have a proper meaning [51]. Since (48) implies that $\langle E|\Omega_\pm^\dagger = \langle E^\pm|$, from the following definition

$$O^\pm := \Omega_\pm\, O\, \Omega_\pm^\dagger\,, \tag{49}$$

we have that

$$O^\pm = \int_0^\infty dE\, O(E)\, |E^\pm\rangle\langle E^\pm| + \int_0^\infty dE \int_0^\infty dE'\, O(E,E')\, |E^\pm\rangle\langle E'^\pm|\,. \tag{50}$$

We say that an operator O is *compatible with* H if it has the form (50) with either sign.

Taking into account that

$$\langle E^{\pm}|w^{\pm}\rangle = \langle E|\,\Omega_{\pm}^{\dagger}\,\Omega_{\pm}\,|w\rangle = \langle E|w\rangle = \delta(E-w)\,, \tag{51}$$

we conclude that the operators of the type O^{+} and O^{-} in (48) form algebras $\mathcal{A}_{+}$ and $\mathcal{A}_{-}$, respectively. These algebras have the following identities,

$$I_{\pm} = \int_0^\infty dE\,|E^{\pm}\rangle\langle E^{\pm}|\,. \tag{52}$$

The relation between the unperturbed algebra $\mathcal{A}_0$ and $\mathcal{A}_{\pm}$ is the following

$$\mathcal{A}_{\pm} = \Omega_{\pm}\,\mathcal{A}_0\,\Omega_{\pm}^{\dagger}\,. \tag{53}$$

We also introduce the notation $|E^{\pm})$ for $|E^{\pm}\rangle\langle E^{\pm}|$ and $|E^{\pm}E'^{\pm})$ for $|E^{\pm}\rangle\langle E'^{\pm}|$, so we can rewrite (50) as

$$|O^{\pm}) = \int_0^\infty dE\,O(E)\,|E^{\pm}) + \int_0^\infty dE\int_0^\infty dE'\,O(E,E')\,|E^{\pm}E'^{\pm})\,. \tag{54}$$

Observe that, using this notation and the usual definition of the dual space, we have that $(E^{\pm}|O^{\pm}) = O(E)$ and that $(E^{\pm}E'^{\pm}|O^{\pm}) = O(E,E')$. As a simple remark, $|O^{\pm})$ is an observable if and only if $O(E) = O^{*}(E)$ and $O(E,E') = O^{*}(E',E)$, where the star denotes complex conjugation [51].

Now, we are going to consider the time evolution in the Heisenberg picture. Recall that $H|E^{\pm}\rangle = E\,|E^{\pm}\rangle$, which implies that

$$e^{itH_0}\,|E^{\pm}\rangle\langle E'^{\pm}|\,e^{-itH_0} = e^{it(E-E')}\,|E^{\pm}\rangle\langle E'^{\pm}|\,. \tag{55}$$

Expression (55) is useful in order to determine the time evolution of the elements of the algebras $\mathcal{A}_{\pm}$ in the Heisenberg picture

$$\begin{aligned} O^{\pm}(t) :&= e^{itH_0}\,O^{\pm}\,e^{-itH_0} = \\ &= \int_0^\infty dE\,O(E)\,|E^{\pm}\rangle\langle E^{\pm}| + \int_0^\infty dE\int_0^\infty dE'\,O(E,E')\,e^{it(E-E')}\,|E^{\pm}\rangle\langle E'^{\pm}| = \\ &= \int_0^\infty dE\,O(E)\,|E^{\pm}) + \int_0^\infty dE\int_0^\infty dE'\,O(E,E')\,e^{it(E-E')}\,|E^{\pm}E'^{\pm})\,. \end{aligned} \tag{56}$$

We are going to analyze the limits $t \longmapsto \pm\infty$ of $O^{\pm}(t)$, respectively. This requires the concept of functional on the algebras $\mathcal{A}_{\pm}$. Using the above notation with round kets and bras, each functional on $\mathcal{A}_{\pm}$ can be formally written as

$$(\rho^{\pm}| = \int_0^\infty dE\,\rho(E)\,(E^{\pm}| + \int_0^\infty dE\int_0^\infty dE'\,\rho(E,E')\,(E^{\pm}E'^{\pm}|\,, \tag{57}$$

where $\rho(E)$ and $\rho(E,E')$ are functions or generalized functions acting on the space of test functions $O(E)$ and $O(E,E')$, respectively. Thus, we obtain that:

$$(E^{\pm}|w^{\pm}) = \delta(E-w)\,, \qquad (E^{\pm}|w^{\pm}E'^{\pm}) = 0\,,$$
$$(E^{\pm}E'^{\pm}|w^{\pm}w'^{\pm}) = \delta(E-w)\delta(E'-w')\,. \tag{58}$$

The action of $(\rho^{\pm}|$ on $|O^{\pm}) \in \mathcal{A}_{\pm}$ is given by

$$(\rho^{\pm}|O^{\pm}) = \int_0^{\infty} dE\,\rho(E)\,O(E) + \int_0^{\infty} dE \int_0^{\infty} dE'\,\rho(E,E')\,O(E,E')\,, \tag{59}$$

so that

$$(\rho^{\pm}|O^{\pm}(t)) = \int_0^{\infty} dE\,\rho(E)\,O(E) + \int_0^{\infty} dE \int_0^{\infty} dE'\,e^{it(E-E')}\,\rho(E,E')\,O(E,E')\,. \tag{60}$$

Often, the product $\rho(E,E')\,O(E,E')$ is an integrable function. In this case, the Riemann-Lebesgue theorem shows that the last term of (60) vanishes as $t \longmapsto \pm\infty$. Therefore, we may say that

$$\lim_{t\to\pm\infty} |O^{\pm}(t)) = \int_0^{\infty} dE\,O(E)\,|E^{\pm})\,, \tag{61}$$

in a weak sense. This is a mathematical fact that has been called *self-induced decoherence* in the literature [49, 50]. What is interesting from our point of view is that if $O^{\pm}$ and $U^{\pm}$ are two operators in the algebra $\mathcal{A}_{\pm}$, one can obtain the following result

$$\lim_{t\to\pm\infty} [O^{\pm}(t), U^{\pm}(t)] = 0\,. \tag{62}$$

Note that in general $[O^{\pm}(t), U^{\pm}(t)] \neq 0$ for all finite times. A similar mathematical result has also been studied by Kiefer and Polarski [59], and Ramírez and Reboiro [60, 61], using alternative approaches.

Therefore, we have obtain the same result found in the previous section, a quantum-to-classical transition for very large values of time. Nevertheless, there are some fundamental differences between the discussion on Section 2 and the presentation in this section. The most important one is that we do not make use of resonance states here. In addition, we do not mix in and out states in the same framework. Quite the contrary, the algebras $\mathcal{A}_-$ and $\mathcal{A}_+$, which refers to in and out observables are treated separately. In spite of this terminology of in and out observables, the algebras $\mathcal{A}_{\pm}$ are just time reversal of each other as demonstrated in [51].

4 Conclusions

Non-commutativity of observables is one of the characteristic features of quantum mechanics and it is related to the impossibility of realizing simultaneous measurements of incompatible observables. On the contrary, commutativity of observables is related with a classical behavior, in the sense that simultaneous measurements of compatible observables are always possible. The transition from a non-commutative algebra of observables to commutative one is a fundamental aspect of the quantum-to-classical limit. However, if we only consider unitary evolutions, it is not possible to describe this transition. In order to describe the quantum-to-classical transition, we need to consider more general time evolutions. One way to do that is to consider the Gamow vectors, which represent resonances including growing and decaying states, and they are linked with exponential decays in the evolution.

In this paper, we gave a further generalization of the non-unitary evolution presented in previous papers. We discussed two quantum models in which the transitions from the non-commutativity to commutativity happens when time goes to infinite. In the first one, the involved operators act on a space spanned by the Gamow states, endowed with a Krein pseudo-metric. Taking into account that resonances decay as time tends to infinity, we have to use a proper definition of time evolution on vectors of the Gamow space compatible with this fact. Then, all observables in this model commute when time goes to infinite.

In the second one, we use an algebraic formalism from scattering theory. We construct in and out algebras of non-commuting operators, that commute in the limits $t \longmapsto \pm\infty$ in a weak sense. These algebras has been constructed in order to include certain states representing situations far from equilibrium. It has also been used for describing Gamow states as functionals over these two algebras and for a formulation of the decoherence.

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Appendix

Let us show that formulas (41) and (42) are not equivalent. For simplicity, we shall assume that there is only one resonance so that dim $\mathcal{H}^G = 2$. The vector $|\psi)$ being arbitrary is a linear combination of $|\psi^D\rangle$ and $|\psi^G\rangle$, so that it may be written as a column vector as $\begin{pmatrix} a \\ b \end{pmatrix}$ with a and b complex. Then,

$$(\psi^D|\psi) = (1,0)B^\dagger \begin{pmatrix} a \\ b \end{pmatrix} = (1,0)\, i^{1/2} \begin{pmatrix} -i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -i\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= i^{1/2} \left[\frac{\sqrt{2}}{2}\, b - i\, \frac{\sqrt{2}}{2}\, a \right] . \tag{63}$$

Similar calculations yield (we have written $\sqrt{-1} = -i$):

$$(\psi^G|\psi) = (\psi^D|\psi)\,, \quad (\psi|\psi^D) = (\psi|\psi^G) = i^{1/2} \left[\frac{\sqrt{2}}{2}\, b^* - i\, \frac{\sqrt{2}}{2}\, a^* \right] . \tag{64}$$

This gives (41). To obtain (42), we need the following calculation:

$$(\psi^D|T|\psi) = (1,0)\, i^{1/2} \begin{pmatrix} -i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -i\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= i^{1/2} \left(-i\,\sqrt{2}/2, \sqrt{2}/2 \right) \begin{pmatrix} b^* \\ a^* \end{pmatrix} = i^{1/2} \left[\frac{\sqrt{2}}{2}\, a^* - i\, \frac{\sqrt{2}}{2}\, b^* \right] . \tag{65}$$

Taken $\sqrt{-1} = -i$, we easily find that $(\psi^D|T|\psi) = (\psi^G|T|\psi)$. From the other terms, we write

$$(\psi|T|\psi^D) = (a^*, b^*) \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} (-i)^{1/2} \begin{pmatrix} i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & i\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= i^{1/2}(a^*, b^*) \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 \\ -i\sqrt{2}/2 \end{pmatrix} = i^{1/2} \left[\frac{\sqrt{2}}{2}\, a^* - i\, \frac{\sqrt{2}}{2}\, b^* \right] . \tag{66}$$

Similarly, we obtain that

$$(\psi|T|\psi^G) = (\psi^G|T|\psi) = (\psi^D|T|\psi) = (\psi|T|\psi^D)\,. \tag{67}$$

The obvious conclusion is that (41) and (42) do not coincide. The first and third identities in (67) are not a surprise due to the properties of the time reversal operator T.

5 Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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