UNIQUENESS OF LIMIT CYCLES FOR QUADRATIC VECTOR FIELDS

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Abstract. This article deals with the study of the number of limit cycles surrounding a critical point of a quadratic planar vector field, which, in normal form, can be written as $x' = a_1x - y - a_3x^2 + (2a_2 + a_5)xy + a_6y^2$, $y' = x + a_1y + a_2x^2 + (2a_3 + a_4)xy - a_2y^2$. In particular, we study the semi-varieties defined in terms of the parameters $a_1, a_2, \dots, a_6$ where some classical criteria for the associated Abel equation apply. The proofs will combine classical ideas with tools from computational algebraic geometry.

1. Introduction and main results

The number of periodic solutions of a quadratic polynomial planar system is an open problem and the first non-trivial case of the second part of Hilbert's *XVI*-th problem.

It is known that if a quadratic system has a limit cycle, i.e., a periodic solution that is isolated in the set of periodic solutions of the system, then it must surround a focus of the system. In particular, if one takes the focus to be at the origin, then the system can be written in the form (see [5])

$$\begin{aligned} x' &= a_1x - y - a_3x^2 + (2a_2 + a_5)xy + a_6y^2, \\ y' &= x + a_1y + a_2x^2 + (2a_3 + a_4)xy - a_2y^2. \end{aligned} \tag{1.1}$$

One way to study the periodic solutions of (1.1) is to analyse the 2π-periodic positive solutions of the polar equation

$$\frac{dr}{d\theta} = \frac{a_1r + f(\theta)r^2}{1 + g(\theta)r}, \tag{1.2}$$

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where f and g are the cubic homogeneous trigonometric polynomials defined by

$$\begin{aligned} f(\theta) &= -a_3\cos^3\theta + (3a_2+a_5)\cos^2\theta\sin\theta \\ &\quad + (2a_3+a_4+a_6)\cos\theta\sin^2\theta - a_2\sin^3\theta, \\ g(\theta) &= a_2\cos^3\theta + (3a_3+a_4)\cos^2\theta\sin\theta \\ &\quad - (3a_2+a_5)\cos\theta\sin^2\theta - a_6\sin^3\theta, \end{aligned}$$

or of the Cherkas-equivalent Abel differential equation (see [8])

$$\rho' = A(\theta)\rho^3 + B(\theta)\rho^2 + a_1\rho, \tag{1.3}$$

where

$$A(\theta) = g(\theta)(a_1 g(\theta) - f(\theta)), \quad B(\theta) = f(\theta) - 2a_1 g(\theta) - g'(\theta).$$

There are several results that establish upper bounds for the number of limit cycles of (1.3). The best known ones impose the condition that one of the functions A or B has definite sign, see [14, 15, 19, 21, 23], where a 2π-periodic function $F(\theta)$ has definite sign if $F(\theta) \geq 0$ for all $\theta \in [0, 2\pi]$ or $F(\theta) \leq 0$ for all $\theta \in [0, 2\pi]$.

In the particular case of Equation (1.3), the criteria in [15, 23] give the following result.

Theorem 1.1 ([15, 23])**.** *If A or B has definite sign, then Abel equation* (1.3) *has at most one positive limit cycle.*

In [9] the quadratic systems for which the above criteria applies are described taking into account their number of critical points and the directions θ in which $g(\theta) = 0$.

To establish our main results, which determine the semi-varieties in the space of parameters where the above criteria apply, and, as a consequence, to obtain that at most one limit cycle surrounds the origin of (1.1), we shall need the following notation.

The study of whether A has definite sign can, by the change of variable $x = \tan(\theta)$ (see Section 4), be reduced to the study of the common roots of the polynomials $p_1(x)$ and $p_3(x) := a_1 p_1(x) - p_2(x)$, where

$$\begin{aligned} p_1(x) &= a_2 + (3a_3+a_4)x - (3a_2+a_5)x^2 - a_6x^3, \\ p_2(x) &= -a_3 + (3a_2+a_5)x + (2a_3+a_4+a_6)x^2 - a_2x^3. \end{aligned}$$

Let us denote by D_1, D_3, D_1', D_3' the discriminants of the polynomials p_1, p_3, p_1', p_3', respectively. If $\mathrm{res}(p_1, p_3)$ denotes the resultant of p_1 and p_3 with respect to x, then it factorizes as

$$\mathrm{res}(p_1, p_3) = R_1 R_2,$$

where

$$\begin{aligned} R_1 &= (4a_2+a_5)^2 + (3a_3+a_4+a_6)^2, \\ R_2 &= a_3a_6l_0^2 + a_2l_0l_1l_2 + a_2^2(l_1+l_2)(l_1+l_3), \end{aligned}$$

with $l_0 = 2a_3 + a_4$, $l_1 = 2a_2 + a_5$, $l_2 = a_3 + a_6$ and $l_3 = a_3 - a_6$.

Let us write

$$R_{113} = \mathrm{res}(p_1', p_3), \quad R_{133} = \mathrm{res}(p_1, p_3').$$

If r_1 (resp. r_3) denotes the remainder of the polynomial division of p_1 by p_1' (resp. p_3 by p_3'), we shall write

$$\bar{R}_{113} = \mathrm{res}(r_1, p_3), \quad \bar{R}_{133} = \mathrm{res}(p_1, r_3).$$

Note that $D_1, D_3, D_1', \ldots, \bar{R}_{113}, \bar{R}_{133}$, are defined "for the generic case", i.e., they are obtained as expressions on $a_1, \ldots, a_6$ without imposing any condition. Some of the expressions are not included in the paper as they are gruesome.

The first result determines the quadratic systems such that $A(\theta)$ has definite sign.

Theorem A. *The coefficient A has definite sign (and, in consequence,* (1.1) *has at most one limit cycle surrounding the origin) if and only if one of the following conditions holds:*

(1) p_1 or p_3 is identically null, or, equivalently, one of the following conditions holds:
  (a) $a_6 = a_5 = 3a_3 + a_4 = a_2 = 0$,
  (b) $a_1a_6 - a_2 = a_1a_5 - a_3 + a_4 + a_6 = a_1(3a_3 + a_4) - 3a_2 - a_5 = a_1a_2 + a_3 = 0$.

(2) p_1 has degree one, p_3 has degree three (i.e., $a_6 = 3a_2 + a_5 = 0$ and $(3a_3 + a_4)a_2 \neq 0$), $R_2 = 0$, and $a_2^2 \leq 4a_3^2 + 4a_1a_2a_3$.

(3) p_3 has degree one, p_1 has degree three (i.e., $a_2 - a_1a_6 = 2a_3 + a_4 + a_1(3a_2 + a_5) + a_6 = 0$ and $a_6\left(3a_2 - a_1(3a_3 + a_4) + a_5\right) \neq 0$), and one of the following conditions holds:
  (a) $R_2 = 0$, $D_1 \leq 0$, $R_{113} \neq 0$,
  (b) $4a_4 - 9a_6 = 4a_3 + 5a_6 = 9a_2 + a_5 = 9a_1a_6 + a_5 = 8a_1^2 - 1 = 0$.

(4) p_1, p_3 have degree two (i.e., $a_2 = a_6 = 0$ and $a_5(a_3 - a_1a_5) \neq 0$), $3a_3 + a_4 = 0$, and $4a_3^2 - 4a_1a_3a_5 \geq a_5^2$.

(5) p_1, p_3 have degree three (i.e., $a_6(a_2 - a_1a_6) \neq 0$), $R_2 = 0$, and one of the following conditions holds:
  (a) $D_1 < 0$, $D_3 < 0$, $(a_3 - a_6)\left(a_2^2 + (a_4 + 2a_3)^2\right) \neq 0$,
  (b) $D_1 = 0$, $D_3 < 0$, $D_1'R_{113} \neq 0$,
  (c) $D_1 = D_1' = 0$, $D_3 < 0$,
  (d) $D_3 = 0$, $D_1 < 0$, $D_3'R_{133} \neq 0$,
  (e) $D_3 = D_3' = 0$, $D_1 < 0$,
  (f) $D_1 = D_3 = 0$, $D_1'D_3'\bar{R}_{113}\bar{R}_{133} \neq 0$,
  (g) $D_1 = D_1' = D_3 = 0$, $\bar{R}_{133} \neq 0$,
  (h) $D_1 = D_3 = D_3' = 0$, $\bar{R}_{113} \neq 0$.

Remark 1.2. The codimension of the semi-varieties defined by the conditions of Theorem A are the following (Proposition 4.10):

- $5a)$ has codimension one.
- $5b), 5d)$ have codimension two.
- $2), 3a), 4)$ have codimension three.
- $1a), 1b)$ have codimension four.
- $3b)$ has codimension five.
- $5f)$ has codimension two or three.
- $5c), 5e), 5g), 5h)$ have codimension of at least two.

Note that in case $3b)$ the equations already imply $a_2 - a_1a_6 = 2a_3 + a_4 + a_1(3a_2a_5) + a_6 = 0$.

Next, we determine quadratic systems such that $B(t)$ has definite sign.

Theorem B. *The coefficient B has definite sign (indeed, it is identically null) if and only if the parameters $a_1, \ldots, a_6$ belong to any of the two codimension-four regular varieties defined by the equations*

$$a_4 + 4a_6 = 4a_3 + a_4 = 4a_2 + a_5 = a_1 = 0, \tag{1.4}$$

or

$$a_6 = 3a_3 + a_4 = 4a_2 + a_5 = 3a_1a_5 + 2a_4 = 0. \tag{1.5}$$

Moreover, (1.1) *has at most one limit cycle surrounding the origin.*

Remark 1.3. The conditions (1.4), (1.5) in Theorem B imply that B is identically null. Therefore, (1.3) reduces to a Bernoulli equation, and it is possible to obtain the exact number of limit cycles surrounding the origin (zero or one).

The rest of the paper is organized as follows. Section 2 contains some known results on the number of limit cycles of Abel equations. Section 3 describes the algebraic geometry tools that will be required for the proofs of the main results. Section 4 contains the proofs of Theorems A and B. Finally, in Appendix A we include the SINGULAR code for the proofs of Section 4.

2. Abel equations with at most one non-trivial limit cycle

In this section we collect known results about the number of limit cycles of the Abel equation (1.3) that we will use subsequently.

Proposition 2.1 ([23, 15])**.** *Assume $A(\theta)$ has definite sign. Then Equation* (1.3) *has at most one positive limit cycle.*

Proof. From [23], we have that (1.3) has at most three limit cycles. Moreover, notice that $\rho = 0$ is always a periodic solution of (1.3). Since $A(\theta+\pi) = A(\theta)$ and $B(\theta + \pi) = -B(\theta)$, we have that $\rho(\theta)$ is a solution of (1.3) if and only if $-\rho(\theta + \pi)$ also is. Thus the number of limit cycles is the same in regions $\rho > 0$ and $\rho < 0$, and consequently Equation (1.3) has at most one positive limit cycle. □

Proposition 2.2. *Assume $A(\theta)$ to be identically null. Then Equation* (1.3) *has no limit cycle.*

Proof. When $A(\theta) \equiv 0$, Equation (1.3) is the Ricatti equation $\rho' = B(\theta)\rho^2 + a_1\rho$. Since $\int_0^{2\pi} B(t)\,dt = 0$, when $a_1 = 0$ it is a centre and if $a_1 \neq 0$ it has no limit cycle. □

Proposition 2.3. *If $B(\theta)$ has definite sign, it is identically null. Moreover, equation*(1.3) *has at most one positive limit cycle.*

Proof. Since $B(\theta + \pi) = -B(\theta)$, if $B(\theta)$ has definite sign, it is necessarily identically null. Then (1.3) is the Bernoulli equation $\rho' = A(t)\rho^3 + a_1\rho$ which has at most one positive limit cycle. □

Remark 2.4. The criterion $\alpha A + \beta B$ has definite sign for some $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$, used in [1, 16] to obtain upper bounds for the number of limit cycles in Abel equations is not relevant in this context since if $\alpha A + \beta B$ has definite sign then, by the change of variables $t \to \pi + t$, $\alpha A - \beta B$ has the same definite sign. Therefore $2\alpha A = (\alpha A + \beta B) + (\alpha A - \beta B)$ has definite sign, and consequently A has definite sign if $\alpha \neq 0$ and $B(t) \equiv 0$ otherwise.

3. Algebraic geometry tools

In this section, we summarize the computational algebraic geometry results to be used subsequently. In all cases, we will include references to the SINGULAR ([11]) commands necessary to perform the corresponding computation. Those readers interested in considering computational algebraic geometry techniques in more depth are encouraged to consult [10] for an introduction, or [4] for a fuller development. Furthermore, readers familiar with differential equations will enjoy [24] which includes a comprehensive introduction to the basic generalities of computational algebraic geometry in its first chapter.

Let us consider a system of polynomial equations in n variables $x_1, \ldots, x_n$ with coefficients in a field $\mathbb{k}$,

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ f_s(x_1, \ldots, x_n) &= 0. \end{aligned} \tag{3.6}$$

Clearly, $(a_1, \ldots, a_n) \in \mathbb{k}^n$ is a solution of (3.6) if and only if

$$\sum_{i=1}^{s} g_i(a_1, \ldots, a_n) f_i(a_1, \ldots, a_n) = 0$$

for every g_i in the ring $\mathbb{k}[x_1, \ldots, x_n]$ of polynomials in n variables with coefficients in $\mathbb{k}$. Thus, the set of solutions of (3.6) in $\mathbb{k}^n$ matches the set of zeros in $\mathbb{k}^n$ of the ideal $\langle f_1, \ldots, f_s\rangle$ of $\mathbb{k}[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_s$. The set of zeros of $I = \langle f_1, \ldots, f_s\rangle$ in $\mathbb{k}^n$ is called the (affine) variety of I in $\mathbb{k}^n$. It is denoted $\mathcal{V}_{\mathbb{k}}(I)$, or simply $\mathcal{V}(I)$ when no confusion is possible.

Here, it is convenient to recall that all the ideals of $\mathbb{k}[x_1, \ldots, x_n]$ are finitely generated by the Hilbert Basis Theorem (see [4, Theorem 1.3.5]). Therefore, to study a system of polynomial equations is the same as to study the ideal generated by the polynomials of the system, and vice versa.

Furthermore, since $f(a_1, \ldots, a_n) = 0$ if and only $f^r(a_1, \ldots, a_n) = 0$ for every positive integer r, one has that $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in \mathbb{k}[x_1, \ldots, x_n] \ \mid f^r \in I, \text{ for some } r \in \mathbb{Z}_+\}$$

is the radical of I.

This ideal-variety approach has two immediate advantages. On the one hand, the varieties in $\mathbb{k}^n$ of the ideals of $\mathbb{k}[x_1, \ldots, x_n]$ form the closed sets of a topology on $\mathbb{k}^n$ called the Zariski topology of $\mathbb{k}^n$ (see [4, Lemma A.2.4]). And on the other, there exists of a kind of factorization theory for ideals of $\mathbb{k}[x_1, \ldots, x_n]$ in which the intersection of ideals plays the role of the product: the so-called primary decomposition theory that we shall outline in the following.

Observe that because of the well-known property

$$\mathcal{V}(J_1 \cap J_2) = \mathcal{V}(J_1) \cup \mathcal{V}(J_2),$$

for $J_i,\ i = 1, 2$, ideals of $\mathbb{k}[x_1, \ldots, x_n]$ (see [4, Lemma A.2.3, part (2)]), a decomposition of the ideal defined by the polynomials in (3.6) as an intersection of “simpler ideals” will mean splitting the system 3.6 into several easier-to-solve systems, hopefully!

Depending on the purpose, some systems of generators of a polynomial ideal are better than others. For example, minimal systems of generators (i.e., systems of generators such that no generator is an algebraic combination of the others) are preferred for a concise description of the variety. But Gröbner bases, which are far from being minimal in the above sense, are special systems of generators with good computational properties. Given a system of generators of an ideal I of $\mathbb{k}[x_1, \ldots, x_n]$, one can compute a minimal system of generators or a Gröbner basis of I by using the SINGULAR commands `mres(I,1)[1]` or `std(I)`, respectively.

The original aim of the Gröbner bases methods was to compute the remainder of a polynomial under division by a polynomial ideal, something that can be done with the command `reduce` in SINGULAR. Nowadays, Gröbner bases are used for more sophisticated tasks. Computing the dimension of a variety or eliminating variables are just two classic examples.

Given a system of generators of an ideal I of $\mathbb{k}[x_1, \ldots, x_n]$, the problem of the computation of the dimension of $\mathcal{V}(I)$ (equivalently, the Krull dimension of $\mathbb{k}[x_1, \ldots, x_n]/I$) may be reduced to a pure combinatorial problem after the computation of one (any) Gröbner basis of I (see [10, Chapter 9]). The SINGULAR command `dim(std(I))` will compute the dimension of $\mathcal{V}_{\mathbb{C}}(I)$ for us. The precise notion of dimension will be defined at the end of this section. On other hand, the problem of the elimination of a variable, say x_n, from the ideal I, consists of determining a system of generators of $I \cap \mathbb{k}[x_1, \ldots, x_{n-1}]$.

This can be easily computed from a Gröbner basis of I with respect to a suitable well-ordering of the monomials in $\mathbb{k}[x_1,\ldots,x_n]$. Geometrically, the elimination of variables has the following meaning:

Proposition 3.1. *Let $\mathbb{k}$ be algebraically closed, and let I be an ideal of $\mathbb{k}[x_1,\ldots,x_n]$. If $\pi : \mathbb{k}^n \to \mathbb{k}^{n-1}$ is the projection map that sends $(a_1,\ldots,a_n)$ to $(a_1,\ldots,a_{n-1})$ then the Zariski closure of $\pi(\mathcal{V}(I))$ in $\mathbb{k}^{n-1}$ is equal to $\mathcal{V}(I\cap\mathbb{k}[x_1,\ldots,x_{n-1}])$.*

Proof. See [10, Theorem 3, Section 3.2]. □

The elimination of variables is computed in SINGULAR with the command `eliminate`.

Let us now briefly summarize the primary decomposition process for ideals of $\mathbb{k}[x_1,\ldots,x_n]$. To do so, we shall first introduce the quotient operation and its most elementary properties.

Definition 3.2. Let I and J be ideals of $\mathbb{k}[x_1,\ldots,x_n]$. The quotient of I and J is the ideal $(I : J)$ of $\mathbb{k}[x_1,\ldots,x_n]$ defined as follows:

$$(I : J) = \{g \in \mathbb{k}[x_1,\ldots,x_n] \;\mid\; gf \subseteq I, \text{ for every } f \in J\}.$$

It is not difficult to see that $I \subseteq (I : J)$ and $\big((I : J) : J\big) = (I : J^2)$. Then, we have a chain of ideals $I \subseteq (I : J) \subseteq \ldots \subseteq (I : J^r) \subseteq \ldots$ that necessarily stabilizes by the Noetherian property of $\mathbb{k}[x_1,\ldots,x_n]$. If N is the smallest integer for which the above chain stabilizes, then the ideal $(I : J^N)$ is called the saturation of I by J and is usually denoted $(I : J^\infty)$.

Both quotient and saturation can be computed using the SINGULAR commands `quotient` and `sat`, respectively (the latter from the `elim` library).

Remark 3.3. Observe that an elementary necessary and sufficient condition for $J \subseteq I$ is $(I : J) = \langle 1\rangle$. Moreover, one has that $f \in \sqrt{I}$ if and only if $(I : \langle f\rangle^\infty) = \langle 1\rangle$. So, the radical membership problem can be computationally solved by computing the saturation of I by $\langle f\rangle$.

Geometrically, when $\mathbb{k}$ is algebraically closed, the quotient and the saturation of I by J have the same behaviour which is nothing but the Zariski closure of the difference of varieties. In particular, the following holds:

$$\mathcal{V}(I : J) = \overline{\mathcal{V}(I)\setminus\mathcal{V}(J)} = \overline{\mathcal{V}(I)\setminus\mathcal{V}(J^r)} = \mathcal{V}(I : J^r),$$

for every positive integer r (see [10, Theorem 7, section 4.4]).

The next result represents a first step for the decomposition of an ideal of $\mathbb{k}[x_1,\ldots,x_n]$:

Lemma 3.4. (Splitting tool). *Let I be an ideal of $\mathbb{k}[x_1,\ldots,x_n]$, and let $g \in \mathbb{k}[x_1,\ldots,x_n]$. If N is the smallest integer such that $(I : \langle g\rangle^\infty) = (I : \langle g^N\rangle)$, then*

$$I = \big(I : \langle g\rangle^\infty\big) \cap \big(I + \langle g^N\rangle\big).$$

Proof. See [4, Lemma 3.3.6]. □

When $\mathbb{k}$ is algebraically closed, an immediate consequence of the splitting tool is the formula

$$\mathcal{V}(I) = \mathcal{V}(I : f^\infty) \cup \mathcal{V}(I + \langle f \rangle) = \overline{\mathcal{V}(I) \setminus \mathcal{V}(f)} \cup \big(\mathcal{V}(I) \cap \mathcal{V}(f)\big)$$

where the varieties in the union on the right-hand side can be carefully interpreted as the solutions of the system associated with I by imposing the conditions $f(x_1, \ldots, x_n)$ different from or equal to zero, respectively.

At this point, we are in a position to clarify what "simpler ideals" means in the context of primary decomposition theory.

Definition 3.5. An ideal P of $\mathbb{k}[x_1, \ldots, x_n]$ is said to be prime if $fg \in P$ and $g \notin P$ implies $f \in P$. An ideal Q of $\mathbb{k}[x_1, \ldots, x_n]$ is said to be primary if $fg \in Q$ and $g \notin Q$ implies $f \in \sqrt{Q}$.

Notice that every *prime ideal P is primary*: indeed, if P is prime, $\sqrt{P} = P$. Moreover, one can easily check that the *radical of a primary ideal is prime*. Here, it is important to emphasize that, if $\mathbb{k}$ is algebraically closed, then P is a prime ideal of $\mathbb{k}[x_1, \ldots, x_n]$ if and only if $\mathcal{V}(P)$ is Zariski irreducible (see [10, Corollary 4, Section 4.5]). So, in this case, the variety of a primary ideal is a Zariski irreducible subset of $\mathbb{k}^n$.

Theorem 3.6. *Let I be an ideal of $\mathbb{k}[x_1, \ldots, x_n]$. If $I \neq \langle 1 \rangle$, there exists a decomposition of I as the intersection of finitely many primary ideals.*

Proof. If I is primary, there is nothing to prove. Otherwise, there exists $g \notin \sqrt{I}$ such that $(I : g^\infty) \supsetneq I$. Thus, by Lemma 3.4, I decomposes as $(I : g^\infty) \cap (I + \langle g^N \rangle)$. Both ideals strictly contain I. If they are primary, we are done. Otherwise, we can repeat the same argument with $(I : g^\infty)$ and $(I + \langle g^N \rangle)$, and so on and so forth. In so far as this process cannot continue indefinitely because of the Noetherian property of $\mathbb{k}[x_1, \ldots, x_n]$, our claim follows. $\square$

A decomposition of I into primary ideals, $I = Q_1 \cap \ldots \cap Q_r$, is called a primary decomposition of I. Since $\sqrt{I} = \sqrt{Q_1} \cap \ldots \cap \sqrt{Q_s}$, by removing redundancies if necessary, we obtain finitely many prime ideals, $P_1, \ldots, P_t$, not contained one in another, such that

$$\mathcal{V}(I) = \mathcal{V}(P_1) \cup \ldots \cup \mathcal{V}(P_t).$$

Therefore, when $\mathbb{k}$ is algebraically closed, a primary decomposition of an ideal I yields a decomposition of $\mathcal{V}(I)$ into Zariski irreducible varieties. In general, the prime ideals defining these varieties do not depend on the decomposition, and are called minimal associated primes of I ([4, Theorem 4.1.5]).

Remark 3.7. Let $\{P_1, \ldots, P_t\}$ be the set of minimal associated primes of an ideal I of $\mathbb{k}[x_1, \ldots, x_n]$. If P' is a prime ideal such that $I \subseteq P' \subseteq P_j$ for some j, then $P' = P_j$. Indeed, it suffices to note that $\sqrt{I} = P_1 \cap \ldots \cap P_t \subseteq P' \subseteq P_j$ implies $P_i \subseteq P' \subseteq P_j$ for some i, and that necessarily $i = j$. Therefore, the minimal associated primes of I are the "smallest" prime ideals containing I.

In conclusion, there exists a computational method to write the set of solutions of a system of polynomial equations in several variables as the union of the solution of finitely many systems. Moreover, if $\mathbb{k}$ is algebraically closed, the varieties associated with those systems are Zariski irreducible.

The minimal associated primes of an ideal of $\mathbb{k}[x_1, \ldots, x_n]$ can be computed by using the SINGULAR command `minAssGTZ` (library `primary`).

We end this section by defining the notion of dimension of an algebraic variety.

Definition 3.8. Let I be an ideal of $\mathbb{k}[x_1, \ldots, x_n]$. The dimension of I, $\dim(I)$, is the supremum of the lengths of all chains of prime ideals in $\mathbb{k}[x_1, \ldots, x_n]/I$.

Equivalently, $\dim(I)$ is supremum of the lengths of all chains of prime ideals in $\mathbb{k}[x_1, \ldots, x_n]$ containing I (because of the well-known correspondence between ideals of the quotient A/I and ideals of A containing I). Observe that

$$\dim(I) = \max\big\{ \dim(P) \mid P \text{ is a minimal associated prime of } I \big\}$$

by Remark 3.7.

This notion of dimension does not depend on the base field $\mathbb{k}$ in the sense that if $\mathbb{k} \hookrightarrow \mathbb{K}$ is an extension of $\mathbb{k}$, then the dimension of I is the same regardless of whether I is an ideal of $\mathbb{k}[x_1, \ldots, x_n]$ or an ideal of $\mathbb{K}[x_1, \ldots, x_n]$ (see [4, Theorem 3.5.1]).

Since the dimension of $\mathcal{V}(I)$ is the supremum of the lengths of the chains of its closed irreducible sets, when $\mathbb{k}$ is algebraically closed, the dimension of $\mathcal{V}(I)$ is the maximum of $\dim(P)$ where P is any minimal associated prime of I.

The next result is a particular version of the General Jacobian criterion (see [4, Theorem 5.7.1]).

Theorem 3.9. *Let $I = \langle f_1, \ldots, f_m \rangle \subset \mathbb{k}[x_1, \ldots, x_n]$ be an ideal and P a minimal associated prime of I. If $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{V}(P) \subseteq \mathbb{k}^n$, then*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right) \leq n - \dim(P), \tag{3.7}$$

and $\mathbf{a} = (a_1, \ldots, a_n)$ is a regular point of $\mathcal{V}(I)$ if and only if the equality holds.

Proof. This theorem is nothing but [4, Theorem 5.7.1], from taking into account that $n - \dim(P)$ is the height of P, $\mathrm{ht}(P)$, by [4, Theorem 3.5.1(4)] and the definition of regular point given in [4, Definition A.8.7]. □

The left hand side in (3.7) can be computed in SINGULAR with the following command `rank(reduce(jacob(I),std(m_a)))`, where $\mathfrak{m}_{\mathbf{a}}$ is the maximal ideal associated with $\mathbf{a}$, i.e., $\mathfrak{m}_{\mathbf{a}} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

4. Proof of the main results

In this section, we shall prove Theorem A and Theorem B.

A first consideration is that the functions A and B are homogeneous trigonometric polynomials of degrees 6 and 3, respectively. Since $\sin(\theta) = -\sin(\theta+\pi)$ and $\cos(\theta) = -\cos(\theta+\pi)$, then for all $\theta \in (-\pi/2, \pi/2]$

$$A(\theta) = A(\theta+\pi), \quad B(\theta) = -B(\theta+\pi).$$

In particular, B has definite sign if and only if $B(\theta) \equiv 0$ for all $\theta \in (-\pi/2, \pi/2]$, and A has definite sign if and only if $A(\theta) \geq 0$ for all $\theta \in (-\pi/2, \pi/2]$, or $A(\theta) \leq 0$ for all $\theta \in (-\pi/2, \pi/2]$.

By the changes of variables $x = \tan(\theta)$, we obtain that A has definite sign if and only if the rational function

$$A(\operatorname{atan}(x)) = \frac{p_1(x)\Big(a_1 p_1(x) - p_2(x)\Big)}{(1+x^2)^3}$$

has definite sign, where (we recall)

$$\begin{aligned} p_1(x) &= a_2 + (3a_3 + a_4)x - (3a_2 + a_5)x^2 - a_6 x^3, \\ p_2(x) &= -a_3 + (3a_2 + a_5)x + (2a_3 + a_4 + a_6)x^2 - a_2 x^3, \end{aligned}$$

or equivalently, that $p_1(x)\Big(a_1 p_1(x) - p_2(x)\Big)$ has definite sign.

Analogously, by the change of variable $x = \tan(\theta)$, B is identically null if and only if

$$B(\operatorname{atan}(x)) = \frac{q(x)}{(1+x^2)^{3/2}} \equiv 0, \quad \text{for all } x \in \mathbb{R},$$

where

$$\begin{aligned} q(x) = &-\Big(2a_1a_2 + 4a_3 + a_4\Big) + \Big(12a_2 + 3a_5 - a_1(6a_3 + 2a_4)\Big)x \\ &+ \Big(8a_3 + 3a_4 + 4a_6 + a_1(6a_2 + 2a_5)\Big)x^2 - \Big(4a_2 + a_5 - 2a_1a_6\Big)x^3. \end{aligned}$$

Again, that is equivalent to $q(x) \equiv 0$.

4.1. **Proof of Theorem A.** We divide the proof of Theorem A into several propositions. A first comment is that if $p_1(x) \equiv 0$ or $p_3(x) := a_1 p_1(x) - p_2(x) \equiv 0$ then $A(\theta) \equiv 0$. In Proposition 4.1 we characterize when one of the polynomials p_1, p_3 is identically null. Next, we distinguish cases in terms of the minimum of the degrees of p_1 and p_3. When this minimum is zero, Theorem A is proved in Proposition 4.2; when it is one, in Proposition 4.3; when it is two, in Proposition 4.4; and when it is three, in Proposition 4.9.

Proposition 4.1. *The polynomial p_1p_3 is identically null if and only if*

$$a_6 = a_5 = 3a_3 + a_4 = a_2 = 0, \tag{4.8}$$

or

$$\begin{aligned} a_1a_6 - a_2 = a_1a_5 - a_3 + a_4 + a_6 = 0,\\ a_1(3a_3 + a_4) - 3a_2 - a_5 = a_1a_2 + a_3 = 0. \end{aligned} \tag{4.9}$$

Proof. If suffices to consider the ideals generated by the coefficients of the polynomials, and then, for each of these ideals, compute a minimal system of generators. (See Appendix A). □

In the following, we assume that neither of p_1, p_3 is identically null. In consequence, p_1p_3 has definite sign if and only if the odd-multiplicity real roots of p_1 and p_3 coincide. We shall distinguish several cases depending on the minimum degree of p_1 and p_3.

If the minimum degree of p_1 and p_3 is zero (and neither of p_1, p_2, p_3 is identically null), then A does not have definite sign.

Proposition 4.2. *If p_1 and p_3 are not identically null and p_1 or p_3 is constant, then the odd-multiplicity real roots of p_1 and p_3 do not coincide.*

Proof. Assume p_1 is constant, i.e., $p_1(x) = a_2$. If the odd-multiplicity roots of p_1, p_3 coincide, then p_3 has even degree. Hence $a_2 = 0$, in contradiction with p_1 not being null.

Conversely, if p_3 is constant and not null, then $p_3(x) = a_1a_2+a_3$. Arguing as above, p_1 has an even degree, so $a_6 = 0$. Moreover, since $p_3(x)$ is constant, $a_2 = 0$ in particular, and

$$p_1(x) = x(3a_3 + a_4 - a_5x).$$

I.e., $x = 0$ is a root of p_1. If it is a simple root, it should be a root of p_3, in contradiction with p_3 being constant, so that $3a_3 + a_4 = 0$. But in this case,

$$p_3(x) = a_3 - a_5x + (-2a_3 - a_4 - a_1a_5)\, x^2.$$

In particular, $a_5 = 0$, so $p_1(x) \equiv 0$, and with this contradiction we conclude the proof. □

Next, we consider that one of p_1, p_3 has degree one, and the other has an equal or greater degree.

Proposition 4.3. *Assume that the minimum of the degrees of p_1 and p_3 is one. Then the odd-multiplicity real-roots of p_1, p_3 coincide if and only if*

$$a_6 = 3a_2 + a_5 = R_2 = 0,\ a_2^2 \le 4a_3^2 + 4a_1a_2a_3,\ a_2(3a_3 + a_4) \neq 0, \tag{4.10}$$

or

$$\begin{aligned} a_2 - a_1a_6 = 2a_3 + a_4 + a_1(3a_2 + a_5) + a_6 = R_2 = 0,\\ D_1 \le 0,\ a_6\,(3a_2 - a_1(3a_3 + a_4) + a_5)\, R_{113} \neq 0. \end{aligned} \tag{4.11}$$

or

$$4a_4 - 9a_6 = 4a_3 + 5a_6 = 9a_2 + a_5 = 9a_1a_6 + a_5 = 8a_1^2 - 1 = 0,\ a_6 \neq 0. \tag{4.12}$$

Proof. Assume the odd-multiplicity real-roots of p_1, p_3 coincide. Then the possible degrees of p_1, p_3 are one or three. The polynomials p_1 and p_3 can not be simultaneously linear, since in this case $p_1(x) = a_3 x$ and $p_3(x) = a_3 + a_1 a_3 x$.

CASE 1. Assume that p_1 has degree one and p_3 has degree three. Then $a_6 = 0$, $3a_2 + a_5 = 0$, $3a_3 + a_4 \neq 0$, and $a_2 \neq 0$. Moreover,

$$\begin{aligned} p_1(x) &= a_2 + (3a_3 + a_4)x, \\ p_3(x) &= a_1 a_2 + a_3 + a_1(3a_3 + a_4)x - (2a_3 + a_4)x^2 + a_2 x^3. \end{aligned}$$

Assume that the odd-multiplicity roots of p_1 and p_3 coincide. The root of p_1 is $x_1 = -a_2/(3a_3 + a_4)$. Then

$$p_3(x_1) = -\frac{(-a_2^2 + a_3(3a_3 + a_4))(a_2^2 + (3a_3 + a_4)^2)}{(3a_3 + a_4)^3} = 0.$$

Hence

$$0 = a_2^2 - a_3(3a_3 + a_4) = \frac{R_2}{a_2^2}.$$

In consequence $a_3 \neq 0$. Replacing a_4 by $\frac{a_2^2 - 3a_3^2}{a_3}$, we obtain

$$p_1(x) = \frac{a_2(a_3 + a_2 x)}{a_3}, \quad p_3(x) = \frac{(a_3 + a_2 x)(a_1 a_2 + a_3 - a_2 x + a_3 x^2)}{a_3}.$$

The odd-multiplicity roots of p_1 and p_3 coincide if and only if $a_1 a_2 + a_3 - a_2 x + a_3 x^2$ has no simple roots, i.e.,

$$a_2^2 - 4a_1 a_2 a_3 - 4a_3^2 \leq 0.$$

The converse is obvious.

CASE 2. Assume that p_3 has degree one and p_1 has degree three, or equivalently

$$\begin{aligned} &a_2 = a_1 a_6,\ 2a_3 + a_4 + a_1(3a_2 + a_5) + a_6 = 0, \\ &a_6 \neq 0,\ 3a_2 - a_1(3a_3 + a_4) + a_5 \neq 0. \end{aligned}$$

Assume that the odd-multiplicity real roots of p_1 and p_3 coincide. From $a_2 = a_1 a_6$, $a_4 = -2a_3 - a_1 a_5 - a_6 - 3a_1^2 a_6$, we obtain

$$p_3(x) = a_3 + a_1^2 a_6 - (a_5 + a_1(-a_3 + a_1 a_5 + 4a_6 + 3a_1^2 a_6))x.$$

where $a_5 + a_1(-a_3 + a_1 a_5 + 4a_6 + 3a_1^2 a_6) \neq 0$. Therefore, p_3 has the unique root

$$x_0 = \frac{a_3 + a_1^2 a_6}{a_5 + a_1\left(-a_3 + a_1 a_5 + 4a_6 + 3a_1^2 a_6\right)}.$$

As p_1 and p_3 have the same odd-multiplicity real roots, x_0 must be a root of p_1. Substituting, one has

$$p_1(x_0) = \frac{-R_2\left((a_5 + 4a_1 a_6)^2 + (a_3 - a_1 a_5 - 3a_1^2 a_6)^2\right)}{a_6^2\left(a_5 + a_1\left(-a_3 + a_1 a_5 + 4a_6 + 3a_1^2 a_6\right)\right)^3}.$$

Since $a_5 - a_1(a_3 - a_1a_5 - 4a_6 - 3a_1^2a_6) \neq 0$, then $(a_5+4a_1a_6)^2 + (a_3 - a_1a_5 - 3a_1^2a_6)^2 > 0$. Therefore x_0 is a root of p_1 if and only if $R_2 = 0$.

If $D_1 < 0$, we shall prove that $R_{113} \neq 0$, so that (4.11) holds. Assume by contradiction that $R_{113} = 0$. Consider the ideal generated by $D_1 + x^2$ (which implies $D_1 < 0$ if $x \neq 0$), $a_2 - a_1a_6$, $2a_3 + a_4 + a_1(3a_2 + a_5) + a_6$, and R_{113}. This ideal has three associated primes (see Appendix A - the computations take some time in this case). The first one contains the polynomial x, so that it corresponds to $D_1 = 0$. The second contains the polynomial $3a_2 - a_1(3a_3 + a_4) + a_5$. The third contains $1 + a_1^2$ so that it has no real points. Therefore, the variety of the ideal is contained in $3a_2 - a_1(3a_3 + a_4) + a_5 = 0$. But $3a_2 - a_1(3a_3 + a_4) + a_5 \neq 0$ by hypothesis. This contradiction proves that $R_{113} \neq 0$.

If $D_1 = 0$, the multiplicity of x_0 as a root of p_1 must be one or three. The multiplicity is two or more if and only if $p_1'(x_0) = 0$, but

$$R_{113} = 9a_6^2\left(a_5 + a_1\left(-a_3 + a_1a_5 + 4a_6 + 3a_1^2a_6\right)\right)^2 p_1'(x_0).$$

I.e., the multiplicity is one if and only if $R_{113} \neq 0$. Finally, if the multiplicity is three, then $p_1(x) = -a_6(x-a)^3$ for a certain a. We consider the ideal generated by R_2, $a_2 - a_1a_6$, $2a_3 + a_4 + a_1(3a_2 + a_5) + a_6$, and the coefficients of $p_1(x) + a_6(x-a)^3$. Eliminating a, and computing the minimal associated primes, we obtain three ideals. The first one contains $1 + a_1^2$ so that it has no real points in its variety. The second contains the polynomial a_6, and, since by hypothesis $a_6 \neq 0$, it has no real points in its variety. The third is

$$\begin{aligned}
&8a_5^2 - 81a_6^2 = 4a_4 - 9a_6 = 9a_1a_6 + a_5 = 8a_1a_5 + 9a_6 = 8a_1^2 - 1 = 0,\\
&3a_1^2a_6 + a_1a_5 + 2a_3 + a_4 + a_6 = -a_1a_6 + a_2 = 0.
\end{aligned}$$

Computing a minimal system of generators, we obtain

$$4a_4 - 9a_6 = 4a_3 + 5a_6 = 9a_2 + a_5 = 9a_1a_6 + a_5 = 8a_1^2 - 1 = 0. \tag{4.13}$$

To conclude, note that if (4.13) holds then

$$p_1(x) = -a_6\left(x \pm \frac{1}{\sqrt{2}}\right)^3, \quad p_3(x) = \frac{9\sqrt{2}}{8}a_6\left(x \pm \frac{1}{\sqrt{2}}\right).$$

□

Now, we consider that either p_1 or p_3 has degree two (and the other degree is two or more).

Proposition 4.4. *Assume that the minimum of the degrees of p_1 and p_3 is two. Then the odd-multiplicity real roots of p_1, p_3 coincide if and only if*

$$a_2 = a_6 = 3a_3 + a_4 = 0,\ 4a_3^2 - 4a_1a_3a_5 \geq a_5^2 > 0,\ a_3 - a_1a_5 \neq 0. \tag{4.14}$$

Proof. Assume that the minimum of the degrees of p_1 and p_3 is two and the real odd-multiplicity roots of p_1, p_3 coincide. Note that this implies that they are both of degree two. I.e., $a_6 = a_2 = 0$, $a_5 \neq 0$, $2a_3 + a_4 + a_1a_5 \neq 0$, and

$$p_1(x) = (3a_3 + a_4)x - a_5x^2,$$
$$p_3(x) = a_3 + (a_1(3a_3 + a_4) - a_5)x - (2a_3 + a_4 + a_1a_5)x^2.$$

The roots of p_1 are then $x_1 = 0$ and $x_2 = (3a_3 + a_4)/a_5$.

Assume that $3a_3 + a_4 \neq 0$. As the simple real roots of p_1 must be roots of p_3, we have that $p_3(0) = 0$ which implies $a_3 = 0$. Moreover, evaluating p_3 at x_2, we obtain

$$p_3\left(\frac{3a_3 + a_4}{a_5}\right) = -\frac{a_4(a_4^2 + a_5^2)}{a_5^2} = 0.$$

I.e., $a_4 = 0$. But this is contradictory with $3a_3 + a_4 \neq 0$.

If $3a_3 + a_4 = 0$ then $p_1(x) = -a_5x^2$ has no odd-multiplicity real roots. The discriminant of p_3, replacing a_4 by $-3a_3$, is

$$\operatorname{disc}(p_3) = -4a_3^2 + 4a_1a_3a_5 + a_5^2,$$

so that p_3 has no simple real roots if and only if $4a_3^2 - 4a_1a_3a_5 - a_5^2 \geq 0$. Finally, note that if $3a_3 + a_4 = 0$ then the condition $2a_3 + a_4 + a_1a_5 \neq 0$ is equivalent to $a_3 - a_1a_5 \neq 0$.

Conversely, assume that (4.14) holds. Then $p_1(x) = -a_5x^2$ and $p_3(x) = a_3 - a_5x + (a_3 - a_1a_5)x^2$. Since $\operatorname{disc}(p_3) < 0$, both p_1 and p_3 have no odd real roots. □

In the remainder of this subsection, we shall consider that both p_1 and p_3 have degree three. In this case, the number of real odd-multiplicity roots is given by the discriminant, being three if the discriminant is strictly positive and one if the discriminant is negative. Note that if $D_1 \leq 0$ and $D_3 > 0$, or $D_1 > 0$ and $D_3 \leq 0$, then the odd-multiplicity roots of p_1, p_3 do not coincide since one has three simple roots and the other has one root with odd-multiplicity. Consequently, we only need to consider the cases $D_1, D_3 > 0$ or $D_1, D_3 \leq 0$.

Firstly, we consider the case when p_1, p_3 have three simple roots, for which we prove that the real roots can not coincide. The following result is a little more general since we do not impose the condition that the real roots be simple. It will be used in proving other cases.

Proposition 4.5. *If p_1, p_3 have three real roots then the roots do not coincide (with multiplicity).*

Proof. The polynomials p_1, p_3 have three real roots if and only if $a_6, a_2 - a_1a_6 \neq 0$ and their discriminants are positive.

The three real roots of p_1, p_3 coincide (with multiplicity) if and only if there exists $\lambda \in \mathbb{R}$ such that $p_1(x) = \lambda p_3(x)$. Equating the coefficients of the leading term, one obtains

$$\lambda = \frac{a_6}{-a_2 + a_1a_6}.$$

Replacing λ in the rest of the equations yields the system (we have multiplied by $a_2 - a_1a_6 \neq 0$)

$$a_2^2+a_3a_6 = a_2(3a_3+a_4-3a_6)-a_5a_6 = a_2(3a_2+a_5)+a_6(2a_3+a_4+a6) = 0.$$

Solving this, one obtains (note that it is a staggered solution)

$$a_3 = \frac{-a_2^2}{a_6}, \quad a_5 = \frac{a_2(a_4a_6+3a_2-3a_2^2)}{a_6^2}, \quad a_4 = \frac{3a_2^2-a_6^2}{a_6}.$$

Substituting in D_1 gives $D_1 = -4(a_2^2+a_6^2)^2 < 0$, in contradiction with p_1 having three real roots. □

Recall that $\operatorname{res}(p_1, p_3)$ factorizes as the product of two polynomials, R_1, R_2. We shall prove that if p_1, p_3 have a real root in common then R_2 must vanish.

Lemma 4.6. *Assume $a_2 - a_1a_6, a_6 \neq 0$. p_1, p_3 have a real root in common if and only if $a = (a_1, \ldots a_6) \in \mathcal{V}(R_2)$.*

Proof. If p_1, p_3 have a real root in common, then $\operatorname{res}(p_1, p_3) = R_1R_2 = 0$. Hence $R_1 = 0$ or $R_2 = 0$. Assume that $R_1 = (4a_2+a_5)^2+(3a_3+a_4+a_6)^2 = 0$, i.e., $a_5 = -4a_2$ and $a_6 = -3a_3 - a_4$. Then

$$\begin{aligned} p_1(x) &= (a_2 + (3a_3+a_4)x)\,(1+x^2),\\ p_3(x) &= (a_1a_2 + a_3 + (a_2 + 3a_1a_3 + a_1a_4)x)\,(1+x^2). \end{aligned}$$

Therefore, p_1, p_3 have a real root in common if and only if $a_2^2 - 3a_3^2 - a_3a_4 = 0$. Since $R_1 = (4a_2+a_5)^2 + (3a_3+a_4+a_6)^2 = 0$ then

$$R_2 = (a_2^2 - 3a_3^2 - a_3a_4)\left(4a_2^2 + (2a_3+a_4)^2\right).$$

Thus, the real root coincide if and only if $R_2 = 0$. □

Next, we study the singular points of the variety defined by R_2. We shall show that they are the intersection of the variety with the hyperplane $a_3 = a_6$. Moreover, in the intersection, the odd-multiplicity real roots of p_1 and p_3 do not coincide.

Lemma 4.7. *The point $a = (a_1, a_2, \ldots, a_6) \in \mathcal{V}(R_2)$ is singular if and only if $a_3 = a_6$ or $a_2 = 2a_3 + a_4 = 0$.*

Moreover, if $a \in \mathcal{V}(R_2)$ is singular, then the real odd-multiplicity roots of p_1, p_3 do not coincide.

Proof. The variety of singular points of $\mathcal{V}(R_2)$ is defined by $\mathcal{V}(\langle R_2, \nabla R_2\rangle)$. It has two minimal associated prime ideals (see the SINGULAR code in Appendix A),

$$\langle 2a_2^2 + a_2a_5 + a_4a_6 + 2a_6^2, a_3 - a_6\rangle \quad \text{and} \quad \langle 2a_3 + a_4, a_2\rangle. \tag{4.15}$$

If $a_3 = a_6$, then $R_2 = (2a_2^2 + a_2a_5 + a_4a_6 + 2a_6^2)^2$. Hence,

$$R_2 = 0, a_3 = a_6 \quad \text{if and only if} \quad R_2 = 0, \nabla R_2 = 0.$$

Let $a \in \mathcal{V}(\langle R_2, a_3 - a_6\rangle)$. Then, parametrizing the variety by a_1, a_2, a_3, a_4, we obtain

$$p_1(x) = \frac{(a_3x + a_2)(a_2 + (2a_3 + a_4)x - a_2x^2)}{a_2},$$
$$p_3(x) = \frac{(a_2 + (2a_3 + a_4)x - a_2x^2)(a_3 - a_2x + a_1(a_2 + a_3x))}{a_2}.$$

I.e., p_1, p_3 have three real roots (as the quadratic factor has positive discriminant), and by Proposition 4.5 they do not coincide.

Finally, let $a \in \mathcal{V}(\langle a_2, 2a_3 + a_4\rangle)$. Then

$$p_1(x) = x(a_3 - a_5x - a_6x^2), \quad p_3(x) = (1 + a_1x)(a_3 - a_5x - a_6x^2).$$

Since $x = 0$ has different parity as root of p_1 than it does as root of p_3, they do not have the same odd-multiplicity real roots. $\square$

The next proposition considers the case of p_1 and p_3 having a unique simple real solution.

Proposition 4.8. *Assume p_1 and p_3 have one simple root and two complex conjugate roots. Then p_1 and p_3 have the same odd-multiplicity real root if and only if*

$$\begin{aligned} &R_2 = 0,\ D_1 < 0,\ D_3 < 0, \\ &a_6 \neq 0,\ a_2 - a_1a_6 \neq 0,\ a_3 \neq a_6,\ a_2^2 + (a_4 + 2a_3)^2 \neq 0. \end{aligned} \tag{4.16}$$

Proof. If p_1 and p_3 have the same real root then $R = 0$ and, by Lemma 4.6, $R_2 = 0$. Moreover, applying Lemma 4.7, $a_3 \neq a_6$, and either $a_2 \neq 0$ or $a_4 + 2a_3 \neq 0$.

Conversely, suppose $R_2 = 0$, $a_3 \neq a_6$, and either $a_2 \neq 0$ or $a_4 + 2a_3 \neq 0$. We have to prove that the real root of p_1 coincides with that of p_3.

Assume on the contrary that these real roots do not coincide. In that case, the complex conjugate roots of p_1 and p_3 must coincide. Then there exist some $a, a', b, d \in \mathbb{R}$ such that

$$\begin{aligned} p_1(x) &= -a_6(x - a)\big((x - b)^2 + d^2\big), \\ p_3(x) &= (-a_1a_6 + a_2)(x - a')\big((x - b)^2 + d^2\big). \end{aligned} \tag{4.17}$$

Equating the coefficients, eliminating the variables a, a', b, d, and computing the minimal associated prime ideals (see Appendix A), we obtain the ideals in (4.15) (which do not satisfy that $a_3 \neq a_6$, and either $a_2 \neq 0$ or $a_4 + 2a_3 \neq 0$), and an ideal such that one of its generators is R_1. By Lemma 4.6, we conclude. $\square$

The last case is p_1, p_3 of degree three with a unique odd-multiplicity real root, and possible double roots.

Proposition 4.9. *Assume p_1, p_3 have degree three (i.e., $a_6(a_2 - a_1a_6) \neq 0$) and one of them has a root of multiplicity two or more. Then p_1 and p_3*

have the same odd-multiplicity real root if and only if $R_2 = 0$ *and one of the following statements holds:*

$$D_1 = 0,\ D_3 < 0,\ D_1' \neq 0,\ R_{113} \neq 0, \tag{4.18}$$

$$D_1 = D_1' = 0,\ D_3 < 0, \tag{4.19}$$

$$D_3 = 0,\ D_1 < 0,\ D_3' \neq 0,\ R_{133} \neq 0, \tag{4.20}$$

$$D_3 = D_3' = 0,\ D_1 < 0, \tag{4.21}$$

$$D_1 = D_1' = D_3 = 0,\ R_{133} \neq 0, \tag{4.22}$$

$$D_1 = D_3 = D_3' = 0,\ R_{113} \neq 0, \tag{4.23}$$

$$D_1 = D_3 = 0,\ D_1' \neq 0,\ D_3' \neq 0,\ \bar{R}_{113} \neq 0,\ \bar{R}_{133} \neq 0, \tag{4.24}$$

Proof. Since p_1, p_3 have degree three, then $a_6 \neq 0$, $a_2 - a_1 a_6 \neq 0$. By Lemma 4.6, p_1, p_3 have a real root in common if and only if $R_2 = 0$. In the following, we shall assume this to be the case.

Assume that p_1 has a root x_1 of multiplicity two or more, and that p_3 has a simple real root, x_3, and two complex conjugate roots, i.e., $D_1 = 0$, $D_3 < 0$. The multiplicity of x_1 is two if and only if $D_1' \neq 0$, and is three if and only if $D_1' = 0$. In the former case of $D_1' \neq 0$, p_1 has a simple root $\bar{x}_1 \neq x_1$. Therefore p_1, p_3 have the same odd-multiplicity real roots if and only if $x_3 = \bar{x}_1$. As $R_2 = 0$, then either $x_3 = \bar{x}_1$ or $x_1 = \bar{x}_1$. Moreover, x_1 is a root of p_1', while $\bar{x}_1$ is not, so that $x_3 = \bar{x}_1$ if and only if $R_{113} \neq 0$. In the latter case of $D_1' = 0$, x_1 is the unique real root of p_1 with multiplicity three, and, as $R_2 = 0$, $x_1 = x_3$, so that the odd-multiplicity real roots of p_1, p_3 coincide.

Assume that p_3 has a root of multiplicity two or more, and p_1 has a simple real root and two complex conjugate roots. Arguing analogously, we obtain that the odd-multiplicity real roots of p_1, p_3 coincide if and only if (4.20) or (4.21) hold.

Assume that p_1 and p_3 have a root of multiplicity two or more, i.e., $D_1 = D_3 = 0$. Firstly, by Proposition 4.5, if both p_1 and p_3 have a root of multiplicity three, then it can not be common.

If $D_1' = 0$, then p_1 has a triple root. As $R_2 = 0$, this root coincides with one of the roots of p_3. If $R_{133} \neq 0$, then it coincides with a simple root of p_3, and in any other case ($R_{133} = 0$ and $D_3' \neq 0$), it coincides with the double root of p_3.

Analogously, if $D_3' = 0$, then the triple root of p_3 coincides with the odd-multiplicity real root of p_1 if and only if $R_{113} \neq 0$.

If $D_1', D_3' \neq 0$, then p_1 and p_3 have a root of multiplicity two and a simple root. In this case, the greatest common divisor of p_1 and p_1' is r_1, a degree-one polynomial, so that $\bar{R}_{113}$ is zero if and only if the double root of p_1 is a root of p_3. Analogously, $\bar{R}_{133}$ is zero if and only if the double root of p_3 is a root of p_1. By Proposition 4.5, if p_1, p_3 have a double root in common, then their simple root is distinct. So p_1, p_3 have the same simple root if and only if $\bar{R}_{113} \neq 0$ and $\bar{R}_{133} \neq 0$. □

Finally, we compute examples of points for some of the semi-varieties and their dimensions.

Proposition 4.10. *The codimensions of the semi-varieties defined by conditions of Theorem A are the following:*

- $5a)$ *has codimension one.*
- $5b), 5d)$ *have codimension two.*
- $2), 3a), 4), 5f)$ *have codimension three.*
- $1a), 1b)$ *have codimension four.*
- $3b)$ *has codimension five.*
- $5f)$ *has codimension two or three.*
- $5c), 5e), 5g), 5h)$ *have codimension of at least two.*

Proof. In Table 1 we give one point in each of the semi-varieties, such that if the definition of the semi-variety contains inequalities then the inequalities hold strictly.

In the same table, we include the codimension of the tangent space of the semi-variety at that point, c_p. To obtain it, we compute the rank of the Jacobian matrix of the equations (equalities) defining the semi-variety at that point. If the rank is maximum (the point is regular), then it coincides with the codimension of the variety at that point. (We set it to * if the point is not singular.)

Finally, c_I denotes the (Krull) codimension of the defining ideal I of the smallest variety cointaining the corresponding semi-variety (i.e., considering the ideal generated only by the polynomials of the equalities). In symbols, $c_I = \operatorname{codim}(\mathcal{V}_{\mathbb{C}}(I)) := n - \dim(I)$, where n is the number of indeterminates in the base ring (see Appendix A). By Theorem 3.9 $c_p \leq c_P$, where P is a minimal prime of I vanishing at p and the equality holds if the point is regular. Therefore, since the dimension of I is the maximum of the dimensions of its associated prime ideals, if c denotes the (real) codimension of the variety, then $c_p \geq c \geq c_I$ at the regular points.

□

4.2. **Proof of Theorem B.** The trigonometric polynomial $B(\theta)$ has definite sign if and only if $q(x) \equiv 0$. I.e., the parameters belong to the variety defined by the ideal obtained by equating the coefficients of $q(x)$ to zero:

$$
\begin{aligned}
2a_1a_2 + 4a_3 + a_4 &= 0, \\
12a_2 + 3a_5 - a_1(6a_3 + 2a_4) &= 0, \\
8a_3 + 3a_4 + 4a_6 + a_1(6a_2 + 2a_5)&, \\
4a_2 + a_5 - 2a_1a_6 &= 0.
\end{aligned}
$$

Case	Point	c_p	c_I
1a)	$a_1 = 1,\ a_2 = 0,\ a_3 = 1,\ a_4 = -3,\ a_5 = 0,\ a_6 = 0.$	4	4
1b)	$a_1 = 1,\ a_2 = 1,\ a_3 = -1,\ a_4 = 2,\ a_5 = -4,\ a_6 = 1.$	4	4
2)	$a_1 = -1,\ a_2 = \sqrt{14},\ a_3 = -2,$ $a_4 = -1,\ a_5 = -3\sqrt{14},\ a_6 = 0.$	3	3
3a)	$a_1 = -1,\ a_2 = (201 + 2\sqrt{1509})/58,$ $a_3 = (-33 + 4\sqrt{1509})/58,\ a_4 = -1,$ $a_5 = -16,\ a_6 = (-201 - 2\sqrt{1509})/58)$	3	3
3b)	$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1,\ a_5 = -2,\ a_6 = -1.$	5	5
4)	$a_1 = 1,\ a_2 = 0,\ a_3 = 1/3,\ a_4 = -1,\ a_5 = -1,\ a_6 = 0.$	3	3
5a)	$a_1 = 0,\ a_2 = 1,\ a_3 = -15/16,$ $a_4 = -53/16,\ a_5 = (-941 - 31\sqrt{7913})/512,\ a_6 = 1.$	1	1
5b)	$a_1 = 0,\ a_2 = (4096 - 7\sqrt{1726})/16384,\ a_3 = 0,$ $a_4 = -(58339673 + 28672\sqrt{1726})/94666752,$ $a_5 = -1,\ a_6 = -2889/16384.$	2	2
5c)	$a_1 = 1,\ a_2 = 4,\ a_3 = -12,\ a_4 = 30,\ a_5 = -15,\ a_6 = 1/2$	*	2
5d)	$a_1 = 0,\ a_2 = \sqrt{185}/32,\ a_3 = 0,$ $a_4 = -1,\ a_5 = -3\sqrt{185}/32,\ a_6 = -5/32$	2	2
5e)	$a_1 = 0,\ a_2 = 2\sqrt{2},\ a_3 = -1,\ a_4 = 0,\ a_5 = -9\sqrt{2},\ a_6 = 8$	*	2
5f)	$a_1 = 0,\ a_2 = 2/3,\ a_3 = 0,\ a_4 = -1,\ a_5 = -2,\ a_6 = -1/3$	3	2
5g)	$a_1 = 0,\ a_2 = 1,\ a_3 = -9/2,\ a_4 = 15/2,\ a_5 = -15,\ a_6 = 8$	*	2
5h)	$a_1 = 0,\ a_2 = 1,\ a_3 = -8,\ a_4 = 35/2,\ a_5 = -15,\ a_6 = 9/2$	*	2

Table 1. Codimensions of the semi-varieties.

Computing the minimal associated prime ideals and a minimal set of generators (see Appendix A), we obtain three minimal ideals. But the first one contains the polynomial a_1^2+4, so that the associated variety is empty. The other two prime ideals obtained are

$$\langle a_1, 4a_2 + a_5, a_3 - a_6, a_4 + 4a_6\rangle,$$

and

$$\langle a_6, 3a_3 + a_4, 4a_2 + a_5, 3a_1a_5 + 2a_4\rangle.$$

Appendix A. SINGULAR codes

```
// Proposition 4.1;
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x), dp;
poly p1 = -a6*x^3 - 3*a2*x^2 - a5*x^2 + 3*a3*x + a4*x + a2;
poly p2 = -a2*x^3 + 2*a3*x^2 + a4*x^2 + a6*x^2 + 3*a2*x + a5*x - a3;
poly p3 = a1*p1-p2;
ideal i1 = coeffs(p1,x);
ideal i3 = coeffs(p3,x);
mres(i1,1)[1];
```

```
mres(i3,1)[1];
```

```
// Proposition 4.3 Case 2;
// D1<0 implies R113!=0;
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x), dp;
poly p1 = -a6*x^3-(3*a2+a5)*x^2+(3*a3+a4)*x+a2;
poly p2 = -a2*x^3+(2*a3+a4+a6)*x^2+(3*a2+a5)*x-a3;
poly p3 = a1*p1-p2;
ideal R = resultant(p1,p2,x);
poly R2 = minAssGTZ(R)[1][1];
poly dp1 = diff(p1,x);
ideal j1 = coeffs(p3,x)[4,1],coeffs(p3,x)[3,1], resultant(dp1,p3,x);
ideal j = R2, resultant(dp1,p1,x)+x^2, j1;
j = sat(j,a6)[1];
list l = minAssGTZ(j); // Takes some time
reduce(x,std(l[1]));
reduce(coeffs(p3,x)[2,1],std(l[2]));
reduce(1+a1^2,std(l[3]));
```

```
// Proposition 4.3 Case 2;
// p1 with a root of multiplicity three;
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x,a), dp;
poly p1 = -a6*x^3 - 3*a2*x^2 - a5*x^2 + 3*a3*x + a4*x + a2;
poly p2 = -a2*x^3 + 2*a3*x^2 + a4*x^2 + a6*x^2 + 3*a2*x + a5*x - a3;
poly p3 = a1*p1-p2;
poly R2 = minAssGTZ(resultant(p1,p2,x))[1][1];
poly p1d = p1 + a6*(x-a)^3;
ideal i3 = coeffs(p3,x);
poly p3l = i3[3]*x+i3[4];
ideal i1d = coeffs(p1d,x);
ideal i3d = coeffs(p3l,x);
ideal i13 = i1d,i3d,R2;
ideal ie=eliminate(i13,a);
list J=minAssGTZ(ie);
mres(J[3],1)[1];
```

```
// Lemma 4.7;
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x), dp;
poly p1 = -a6*x^3 - 3*a2*x^2 - a5*x^2 + 3*a3*x + a4*x + a2;
poly p2 = -a2*x^3 + 2*a3*x^2 + a4*x^2 + a6*x^2 + 3*a2*x + a5*x - a3;
poly R2 = minAssGTZ(resultant(p1,p2,x))[1][1];
ideal sR2 = R2,jacob(R2);
minAssGTZ(sR2);
```

```
// Proposition 4.8;
```

```
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x,a,ap,b,d), dp;
poly p1 = -a6*x^3 - 3*a2*x^2 - a5*x^2 + 3*a3*x + a4*x + a2;
poly p2 = -a2*x^3 + 2*a3*x^2 + a4*x^2 + a6*x^2 + 3*a2*x + a5*x - a3;
poly p3 = a1*p1-p2;
poly R2 = minAssGTZ(resultant(p1,p2,x))[1][1];
poly p1d = p1 + a6*(x-a)*((x-b)^2-d^2);
poly p3d = p3 + (a1*a6-a2)*(x-ap)*((x-b)^2-d^2);
ideal i1d = coeffs(p1d,x);
ideal i3d = coeffs(p3d,x);
ideal i13 = i1d,i3d,R2;
ideal ie=eliminate(i13,a*ap*b*d);
list J=minAssGTZ(ie);

// Proposition 4.10;
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6,x), dp;
poly p1 = -a6*x^3 - 3*a2*x^2 - a5*x^2 + 3*a3*x + a4*x + a2;
poly p2 = -a2*x^3 + 2*a3*x^2 + a4*x^2 + a6*x^2 + 3*a2*x + a5*x - a3;
poly p3 = a1*p1-p2;
poly dp1 = diff(p1,x);
poly dp3 = diff(p3,x);
poly ddp1 = diff(dp1,x);
poly ddp3 = diff(dp3,x);
poly D1 = resultant(p1,dp1,x);
poly D1p = resultant(p1,ddp1,x);
poly D3 = resultant(p3,dp3,x);
poly D3p = resultant(p3,ddp3,x);
poly R2 = minAssGTZ(resultant(p1,p2,x))[1][1];
ideal i1a = a6,a5,3*a3+a4,a2;
ideal i1b = a1*a6-a2,a1*a5-a3+a4+a6,a1*(3*a3+a4)-3*a2-a5,a1*a2+a3;
ideal i2  = a6,3*a2+a5,R2;
ideal i3a = a2-a1*a6,2*a3+a4+a1*(3*a2+a5)+a6,R2;
ideal i3b = 4*a4-9*a6,4*a3+5*a6,9*a2+a5,9*a1*a6+a5,8*a1^2-1;
ideal i4  = a2,a6,3*a3+a4;
ideal i5a = R2;
ideal i5b = R2,D1;
ideal i5c = R2,D1,D1p;
ideal i5d = R2,D3;
ideal i5e = R2,D3,D3p;
ideal i5f = R2,D1,D3;
ideal i5g = R2,D1,D1p,D3;
ideal i5h = R2,D1,D3p,D1;
nvars(basering) - dim(std(i1a));
nvars(basering) - dim(std(i1b));
```

```
nvars(basering) - dim(std(i2));
nvars(basering) - dim(std(i3a));
nvars(basering) - dim(std(i3b));
nvars(basering) - dim(std(i4));
nvars(basering) - dim(std(i5a));
nvars(basering) - dim(std(i5b));
nvars(basering) - dim(std(i5c));
nvars(basering) - dim(std(i5d));
nvars(basering) - dim(std(i5e)); // Takes some time;
nvars(basering) - dim(std(i5f));
nvars(basering) - dim(std(i5g));
nvars(basering) - dim(std(i5h)); // Takes some time;
// Theorem B
LIB "primdec.lib";
ring r = 0, (a1,a2,a3,a4,a5,a6), dp;
poly c0 = 2*a1*a2 + 4*a3 + a4;
poly c1 = 12*a2 + 3*a5 - a1*(6*a3 + 2*a4);
poly c2 = 8*a3 + 3*a4 + 4*a6 + a1*(6*a2+2*a5);
poly c3 = 4*a2 + a5 - 2*a1*a6;
ideal iB = c0,c1,c2,c3;
list LB = minAssGTZ(iB);
mres(LB[1],1)[1];
mres(LB[2],1)[1];
mres(LB[3],1)[1];
```


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