Optimal pension funding for sophisticated managers*

Ricardo Josa-Fombellida§†, Jorge Navas♯

§*Dpto. de Estadística e Investigación Operativa and IMUVa. Universidad de Valladolid.*

♯*Dpto. de Matemàtica Econòmica, Financiera i Actuarial and BEAT. Universitat de Barcelona.*

July 25, 2019

Abstract

We consider the optimal management of a defined benefit stochastic pension plan where the participants have different rates of time preference. The fund manager invests in a portfolio with risky assets and one riskless asset. Its main objective is to select the amortization rate and the investment strategy minimizing both the contribution rate risk and the solvency risk. The problem is formulated as an optimal control problem with non-constant rate of discount and is solved analytically by means of the dynamic programming approach and the technical interest rate is selected in order to keep stable the fund evolution within prescribed targets. A numerical illustration shows a comparative of the stability of the fund assets and the rate of contribution for a convex combination of exponential functions as discount function and for the constant discount case.

Keywords: pension funding; risk management; optimal portfolio; dynamic programming; non-constant discount; optimization in Financial Mathematics

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*Both authors gratefully acknowledge financial support from Consejería de Educación de la Junta de Castilla y León (Spain) under project VA056A09. The first author is also supported by Ministerio de Economía y Competitividad (Spain) under projects ECO2014-56384-P and ECO2017-86261-P, and Consejería de Educación de la Junta de Castilla y León (Spain) under project VA148G18, and the second author under projects ECO2013-48248P and ECO2017-82227P.

†Corresponding author. Paseo de Belén 7, 47011 Valladolid, Spain. E-mail: ricar@eio.uva.es

1 Introduction

The increase in life expectancy and the periods of financial crisis have stressed the importance of alternative pension plans in order to complement public protection at retirement. As a consequence, the study of the management of pension plans is an important subject in the economic field, and also in the financial field because the fund managers use the financial markets to invest the fund assets of the pension plan.

This paper studies the optimal management of a pension plan of aggregated type using the dynamic programming approach. The analysis of the optimal management of pension plans from the dynamic optimization approach has appeared in the literature in discrete time, see Berkelaar and Kouwenberg (2003), Chang *et al.* (2003), Cox *et al.* (2013), Haberman and Sung (1994, 2005) and Haberman and Vigna (2002), and in continuous time, see Battocchio *et al* (2007), Cairns (2000), Deelstra *et al* (2003), Haberman *et al.* (2000), Menoncin and Vigna (2017) and Josa-Fombellida and Rincón-Zapatero (2001, 2004, 2008ab, 2010, 2018).

We focus our attention in plans of defined benefit type where the benefits are fixed in advance by the sponsor and contributions are designed to amortize the fund according to a previously chosen actuarial scheme. The plan sponsor can built an investment portfolio of the fund. The main aim of the plan manager is the minimization of both the solvency risk and the contribution rate risk. This objective is generally accepted since the papers Haberman and Sung (1994) and Josa-Fombellida and Rincón-Zapatero (2001). These risk concepts are defined as quadratic deviations of fund wealth and amortization rates with respect to liabilities and normal cost, respectively. The solvency risk is related with the security of the pension fund in attaining the comprised liabilities and the contribution risk with its stability, see Haberman *et al.* (2000).

In this paper we study the optimal management of a defined benefit pension plan with stochastic benefits correlated with the financial market, composed by n risky assets and one riskless asset, where the aim of the sponsor is the minimization of the contribution and the solvency risks along an infinite horizon, in order to allow a long term valuation, as in Josa-Fombellida and Rincón-Zapatero (2004). We provide an extension of that paper, supposing that the discount function is not necessarily of exponential type with the aim of capturing the diversity in the temporal preferences of the participants in the plan. Collective temporal decisions for agents with different rates of time preferences lead to time inconsistent aggregate temporal preferences (Jackson and Yariv (2015)). As a consequence, standard optimization techniques (Pontryagin Maximum Principle (PMP) or Hamilton-Jacobi-Bellman (HJB)) fail in characterizing time consistent optimal policies. As preferences change with time, as long as the decision maker goes trough the time horizon, they differ depending on the instant t at which solutions are obtained, so a t'-agent, in general, will not find optimal the solutions computed by the t-agent, for any t and t' in the time horizon.

Karp (2007) introduced the analysis of dynamic optimization problems in continuous time setting with non-constant rate of discount, deriving in infinite time horizon a modified Hamilton-Jacobi-Belman (HJB) equation. Later Marín-Solano and Navas (2009) extended the approach to the finite horizon case and study the application to a non-renewable resource problem with non-constant discount. The methodology for stochastic control problems with non-constant discount in a finite time horizon is developed in Marín-Solano and Navas (2010). The classical optimal consumption and portfolio problem by Merton (1971) with non-constant discount is detailed studied for logarithmic, power and exponential utility functions in this last paper.

Stochastic control problems with non-constant rate of discount have recent interest in the literature of economics, finance and insurance. Also in pension funding. Thus Liang *et al.* (2014) considers time-consistent strategies in a mean-variance optimization problem but in a defined benefit pension scheme starting from the model in Josa-Fombellida and Rincón-Zapatero (2008b) for strategies pre-commitment. Li *et al.* (2016) derives the time-consistent investment strategy under the mean-variance criterion for a defined contribution pension plan with stochastic salary and where the risky asset is a CEV process. Zhao *et al* (2016a) analyses a defined benefit pension plan model with non-constant discount where the aim is the minimization of the solvency and the contribution rate risks but in a finite horizon, and the manager invests the fund in a portfolio with one risky asset and one risk free asset. The model follows Josa-Fombellida and Rincón-Zapatero (2001), where the benefit is constant. Zhao *et al.* (2016b) considers a consumption-investment problem for a member of a defined contribution pension plan with non-constant time preferences, with power and logarithmic utility functions and with the exponential discounting, the mixture of exponential discounting and the hyperbolic discounting.

The main findings of the paper are that the rate of discount function intervenes in the optimal strategies and in the optimal fund evolution and it is possible to select the technical rate of interest in order that the optimal contribution does not depend on the parameters of the benefit process and it has the form of a spread method of funding, providing the stability of the plan at the long-term. We find that the speed of convergence of the optimal fund to the actuarial liability is inversely related with the degree of impatience in the collective of participants in the plan.

The paper is organized as follows. Section 2 defines the elements of the pension scheme and describes the financial market where the fund operates. We consider the fund is invested in a portfolio with several risky assets and one riskless asset. The management of the defined benefit plan is formulated as a stochastic optimal control problem with non-constant discount where the objective is to minimize on a infinite horizon the contribution rate risk and the solvency risk. In Section 3 the optimal strategies of contribution and investment, and the optimal fund are obtained with dynamic programming techniques. Some properties of the optimal solutions are found. A particular case where the technical rate of interest is selected to lead to a spread method of funding is analyzed. Section 4 serves as a numerical illustration of previous results. Finally, Section 5 establishes some conclusions. All proofs are developed in Appendix A.

2 The pension model

Consider a pension plan of aggregated type where, at every instant of time, active participants coexist with retired participants. The plan is of defined benefit type, that is to say, the benefits paid to the participants at the age of retirement are fixed in advance by the manager. The benefit is modeled by a stochastic process correlated with the financial market.

The main elements intervening in the pension plan are the following. We denote $F(t)$ the value of the fund assets at time t and $C(t)$ the contribution rate made by the sponsor at time t to the funding process in order to accrue the benefit at the moment of retirement. The risk-free market interest rate is the constant r. The technical rate of valuation δ is the constant used for the valuation of the liabilities. This valuation is made using the distribution function M on $[a, d]$, that is, $100M(x)\%$ is the percentage of the value of the future benefits accumulated until age $x \in [a, d]$, where a is the common age of entrance in the fund and d is the common age of retirement.

2.1 The actuarial functions

The main functions of the pension plan are described next. $P(t)$ denotes the benefit promised to the participants at time t. Though it is related with the salary at the moment of retirement, we will not consider the salary process into the model, as in Josa-Fombellida and Rincón-Zapatero (2008a). Its ideal value is the normal cost at time t for all participants, denoted by $NC(t)$. The ideal value of the fund is the actuarial liability at time t, denoted by $AL(t)$, that is, the total liabilities of the sponsor. The unfunded actuarial liability at time t (equal to $AL(t) - F(t)$) is denoted by $UAL(t)$, and the supplementary cost at time t (the difference $C(t) - NC(t)$) by $SC(t)$. We suppose that these actuarial functions are stochastic processes.

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on Ω and $\mathscr{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is a complete and right continuous filtration generated by the $(n+1)$-dimensional standard Brownian motion $(w_0, w_1, \ldots, w_n)$, that is to say, $\mathscr{F}_t = \sigma\{w_0(s), w_1(s), \ldots, w_n(s);\ 0 \leq s \leq t\}$. The stochastic actuarial liability and the stochastic normal cost are defined as in Josa-Fombellida and Rincón-Zapatero (2004):

$$AL(t) = \int_a^d e^{-\delta(d-x)}\, M(x)\mathbb{E}\left(P(t+d-x)\,|\,\mathscr{F}_t\right)\, dx,$$

$$NC(t) = \int_a^d e^{-\delta(d-x)}\, M'(x)\mathbb{E}\left(P(t+d-x)\,|\,\mathscr{F}_t\right)\, dx.$$

for every $t \geq 0$, where $\mathbb{E}(\cdot|\mathscr{F}_t)$ denotes conditional expectation with respect to the filtration $\mathscr{F}_t$. The actuarial liability $AL(t)$ is the total expected value of the promised benefits accumulated according to the distribution function M, and the normal cost $NC(t)$ is the total expected value of the promised benefits accumulated according to the density function M', both discounted at the constant rate δ.

In order to get analytical tractability, we assume that the benefit P is given by a geometric Brownian motion, as in Josa-Fombellida and Rincón-Zapatero (2004). Thus the expected benefit grows exponentially which is coherent because the benefit depends on the salary and the population pension plan size.

Assumption 1 *The benefit P satisfies the stochastic differential equation (SDE therefore)*

$$dP(t) = \mu P(t)\,dt + \eta P(t)\,dB(t), \quad t \geq 0,$$

where B is a standard Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$, and where $\mu \in \mathbb{R}$ and $\eta \in \mathbb{R}_+$. The initial condition $P(0) = P_0$ is a random variable that represents the initial liabilities.

Under Assumption 1 the actuarial functions satisfy $AL\,(t) = \psi_{AL} P(t)$ and $NC\,(t) = \psi_{NC} P(t)$, where $\psi_{AL} = \int_a^d e^{(\mu-\delta)(d-x)} M(x)\,dx$ and $\psi_{NC} = \int_a^d e^{(\mu-\delta)(d-x)} M'(x)\,dx$, and they are linked by the identity

$$(\delta - \mu) AL\,(t) + NC\,(t) - P(t) = 0, \tag{1}$$

for every $t \geq 0$. See Proposition 2.1 in Josa-Fombellida and Rincón-Zapatero (2004).

2.2 The financial market and the fund wealth

In this section we describe the financial market where the fund is invested. The plan sponsor manages the fund by means of a portfolio formed by n risky assets $S^1, \ldots, S^n$, which are geometric Brownian motions correlated with the benefit process, and a riskless asset S^0, as proposed Merton (1971), that is, whose evolutions are given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{2}$$

$$dS^i(t) = S^i(t)\Big(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dw_j(t)\Big), \quad S^i(0) = s_i, \quad i = 1, \ldots, n. \tag{3}$$

We have denoted by $r > 0$ the short risk-free rate of interest, $b_i > 0$ the mean rate of return of the risky asset S^i, and $\sigma_{ij} > 0$ the uncertainty parameters. We assume that $b_i > r$, for each $i = 1, ..., n$, so the sponsor has incentives to invest with risk. The matrix (σ_{ij}) is denoted by σ and the Sharpe ratio or market price of risk for this portfolio, $\sigma^{-1}(b - r\bar{1})$, by θ, where $b = (b_1, \ldots, b_n)^\top$ and $\bar{1}$ is a (column) vector of 1's. We suppose that the symmetric matrix $\Sigma = \sigma\sigma^\top$ is positive definite. There exists correlation $q_i \in [-1, 1]$ between B and w_i, for $i = 1, \ldots, n$. Thus the financial market influences the evolution of liability P. As a consequence, B is expressed in terms of $(w_0, \ldots, w_n)$ as $B(t) = \sqrt{1 - q^\top q}\, w_0(t) + q^\top w(t)$, where $w = (w_1 \ldots, w_n)^\top$ and $q = (q_1, \ldots, q_n)^\top$.

By Assumption 1 and using $AL = \psi_{AL} P$, the actuarial liability satisfies the SDE

$$dAL\,(t) = \mu AL\,(t)\,dt + \eta AL\,(t)\,dB(t),$$

with the initial condition $AL\,(0) = AL_0 = \psi_{AL} P_0$. And in terms of the Brownian motion generating $\mathscr{F}$, AL is determined by

$$dAL\,(t) = \mu AL\,(t)\,dt + \eta AL\,(t)\,\sqrt{1 - q^\top q}\,dw_0(t) + \eta AL\,(t)\,q^\top dw(t), \tag{4}$$

with $AL\,(0) = AL_0$. Thus the benefit P and actuarial liability AL depend on the financial market. By the same argument, the normal cost satisfies the same SDE (4) but with initial condition $NC\,(0) = NC_0 = \psi_{NC} P_0$.

The manager builds a portfolio based on the financial market and designs an amortization scheme varying with time. The amount of fund invested in time t in the risky asset S^i is denoted by $\pi_i(t)$, $i = 1, \ldots, n$. The remainder, $F(t) - \sum_{i=1}^n \pi_i(t)$, is invested in the bond. Borrowing and shortselling are allowed. A negative value of π_i means that the sponsor sells a part of his risky asset S^i short while, if π_i is larger than F, he or she then gets into debt to purchase the corresponding stock, borrowing at the riskless interest rate r. π denotes $(\pi_1, \ldots, \pi_n)^\top$. We suppose the contribution rate $\{C(t) : t \geq 0\}$ and the investment strategy $\{\pi(t) : t \geq 0\}$ are control processes adapted to filtration $\{\mathscr{F}_t\}_{t \geq 0}$, $\mathscr{F}_t$-measurables, markovian and stationary, satisfying

$$\mathbb{E}_{F_0, AL_0} \int_0^T SC^2(t)dt < \infty, \tag{5}$$

and

$$\mathbb{E}_{F_0, AL_0} \int_0^T \pi^\top(t)\pi(t)dt < \infty, \tag{6}$$

for every $T < \infty$. In the above, $\mathbb{E}_{F_0,AL_0}$ denotes conditional expectation with respect to the initial conditions (F_0, AL_0).

The dynamic fund evolution under the investment policy π is:

$$dF(t) = \sum_{i=1}^{n} \pi_i(t) \frac{dS^i(t)}{S^i(t)} + \left(F(t) - \sum_{i=1}^{n} \pi_i(t) \right) \frac{dS^0(t)}{S^0(t)} + (C(t) - P(t))\, dt. \tag{7}$$

By substituting (2) and (3) in (7), we obtain the SDE that determines the fund evolution,

$$dF(t) = \Big(rF(t) + \pi^\top(t)(b - r\bar{1}) + C(t) - P(t) \Big) dt + \pi^\top(t)\sigma\, dw(t), \tag{8}$$

with initial condition $F(0) = F_0 > 0$.

2.3 The optimization problem

The manager wishes to minimize a convex combination of the contribution rate risk and the solvency risk in a infinite horizon, but discounted at a non-constant rate of discount. Following Josa-Fombellida and Rincón-Zapatero (2004), at every instant of time t, we define the solvency risk as the quadratic deviation of the fund assets $F(t)$ with respect to its ideal value $AL(t)$, instead of its expected value $\mathbb{E}_{F_0} F(t)$, that is to say $\mathbb{E}_{F_0,AL_0} UAL(t)^2$. Analogously, we define the contribution rate risk as $\mathbb{E}_{F_0,AL_0} SC(t)^2$. Thus, the objective functional to be minimized over the class of admissible controls $\mathcal{A}_{F_0,AL_0}$, is given by

$$J((F_0, AL_0); (SC, \pi)) = \mathbb{E}_{F_0,AL_0} \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv} \left(\beta SC^2(s) + (1-\beta)(AL(s) - F(s))^2 \right) ds. \tag{9}$$

Note that we choose $SC = C - NC$ as the control variable instead of C, leading to an equivalent control problem. Here, $\mathcal{A}_{F_0,AL_0}$ is the set of Markovian processes (SC, π), adapted to the filter $\{\mathscr{F}_t\}_{t \geq 0}$ where C satisfies (5), π satisfies (6), and where F and AL satisfy (8) and (4), respectively. The parameter β, $0 < \beta \leq 1$, is a weighting factor reflecting the relative importance for the manager of the two different types of risks.

We consider a positive non-constant impatience rate or discount factor $\tilde{\rho}(t)$ for the manager. A high rate of impatience implies that the promotor is more concerned with the present than with the distant future, in the sense of being more impatient in the short-run decisions compared with similar decisions in the long-run. The rate of discount $\tilde{\rho}$ is a positive decreasing and bounded function in $[0,\infty)$. Note that the function $\theta(t) = e^{-\int_0^t \tilde{\rho}(s)ds}$ satisfies $\tilde{\rho}(t) = -\dot{\theta}(t)/\theta(t)$. Denote $\rho = \lim_{t\to\infty} \tilde{\rho}(t)$ and assume $\tilde{\rho}(t) \geq \rho$, for all $t \geq 0$. In the literature, we can find several functional specifications for general discounts functions with non-constant instantaneous rates of time preference, being one of them a convex linear combination of exponential functions. An economic motivation for this particular discount function is the following: consider the case that there exist N different participants in the plan that exhibit constant but different rates of time preference, and consider that in this collective there exist m $(m \leq M)$ different profiles or subgroups in terms of their time preference, which can be motivated by several factors (age, personal wealth, genre, life expectancy, etc.). If each profile has a rate of time preference of ρ_i, $i = 1, ..., m$, and with $\rho_1 < \ldots < \rho_m$, we can define:

$$\theta(t) = \sum_{i=1}^{m} \mu_i e^{-\rho_i t}$$

where μ_i, $0 < \mu_i < 1$, $i = 1, ..., m$, $\sum_{i=1}^{m} \mu_i = 1$, represents the weight of the corresponding subgroup in the whole collective. Then, the discount function $\theta(t)$ has a decreasing instantaneous rate of time preference $\tilde{\rho}(t) = -\dot{\theta}(t)/\theta(t)$:

$$\tilde{\rho}(t) = \frac{\mu_1 \rho_1 e^{-\rho_1 t} + \ldots + \mu_m \rho_m e^{-\rho_m t}}{\mu_1 e^{-\rho_1 t} + \ldots + \mu_m e^{-\rho_m t}}$$

where $d\tilde{\rho}(t)/dt < 0$ and $\lim_{t\to\infty} \theta(t) = \rho_1$.

The dynamic programming approach will be used to solve the problem. The value function is defined as

$$\widehat{V}(F, AL) = \min_{(SC,\pi)\in\mathcal{A}_{F,AL}} \big\{ J((F, AL); (SC, \pi)) \; : \; \text{s.t. } (4), (8) \big\}.$$

Since the problem is autonomous and the horizon unbounded, we may suppose that $\widehat{V}$ is time independent. The value function so defined is non-negative and strictly convex. The connection between value functions and optimal feedback controls in stochastic control theory with non-constant discount is accomplished by a modified HJB. In order to obtain a sophisticated solution, Marín-Solano and Navas (2010) analyse the finite horizon case, that is easily translated to the unbounded case as follows. The modified HJB equation is

$$-\rho V - K + \min_{SC,\pi}\Big\{\beta SC^2 + (1-\beta)(F-AL)^2 + \big(rF + \pi^\top(b - r\bar{1}) + SC + NC - P\big)V_F$$
$$+\mu AL\, V_{AL} + \frac{1}{2}\pi^\top\Sigma^{-1}\pi V_{FF} + \frac{1}{2}\eta^2 AL^2 V_{AL,AL} + \eta\pi^\top\sigma q AL\, V_{F,AL}\Big\} = 0, \quad (10)$$

where

$$K(F,AL) = \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)\mathbb{E}_{F,AL}\left\{\beta SC^2(s) + (1-\beta)(AL(s) - F(s))^2\right\} ds, \quad (11)$$

and where $\mathbb{E}_{F,AL}$ denotes the conditional expectation to $F(0) = F$ and $AL(0) = AL$.

Remark 2.1 *In the definition of the $K(F, AL)$ function, control variables SC and π are non-local terms when solving (10) leading to functional equations in the solution procedure. In some cases it is possible to simplify the process working with the total differential of K with respect to time (see for instance Remark 2 in Karp (2007)). For instance, if the discount function is a linear combination of two exponential discount functions, $\theta(\tau) = \lambda e^{-\rho_1\tau} + (1-\lambda)e^{-\rho_2\tau}$, with $\lambda \in [0,1]$ and $0 < \rho_1 < \rho_2$, by differentiating (11) we obtain*

$$\rho_2 K = (\rho_2 - \rho_1)\left(\beta SC^2(s) + (1-\beta)(AL(s) - F(s))^2\right) + \big(rF + \pi^\top(b - r\bar{1}) + SC + NC - P\big)K_F$$
$$+ \mu AL\, K_{AL} + \frac{1}{2}\pi^\top\Sigma^{-1}\pi K_{FF} + \frac{1}{2}\eta^2 AL^2 K_{AL,AL} + \eta\pi^\top\sigma q AL\, K_{F,AL}, \quad (12)$$

so, the time consistent solution can be characterized as the solution of the system (10), (12). *This property will be used in the numerical illustration in Section 4.*

3 The optimal strategies

In this section we show how the sponsor may select the rate contribution and the proportion of fund assets put into the risky assets. We analyze some properties of these optimal strategies and study the optimal fund evolution. We have the following result.

Theorem 3.1 *Suppose that Assumption 1 holds and the inequalities*

$$2\mu + \eta^2 < \rho, \quad (13)$$

$$2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta < \rho, \quad (14)$$

are satisfied. The optimal rate of contribution and the optimal investment in the risky assets are given by

$$C^* = NC - \frac{\alpha_{FF}}{\beta}F - \frac{\alpha_{F,AL}}{2\beta}AL\,, \tag{15}$$

$$\pi^* = -\Sigma^{-1}(b - r\bar{1})F - \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\Sigma^{-1}(b - r\bar{1}) + \eta\sigma^{-\top}q\right)AL\,, \tag{16}$$

respectively, where α_{FF} is a positive solution of the equation

$$-\frac{\alpha_{FF}^2}{\beta} + \left(-\rho + 2r - \theta^\top\theta\right)\alpha_{FF} + (1-\beta) - \kappa_{FF} = 0, \tag{17}$$

where

$$\kappa_{FF} = \left(\frac{\alpha_{FF}^2}{\beta} + 1 - \beta\right)\int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)e^{\left(2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta\right)s}ds, \tag{18}$$

and $\alpha_{F,AL}$ is the unique solution of the equation

$$-\frac{\alpha_{FF}}{\beta}\alpha_{F,AL} + \left(-\rho + r - \theta^\top\theta - \eta q^\top\theta + \mu\right)\alpha_{F,AL} + 2(\mu - \delta)\alpha_{FF} - 2(1-\beta) - \kappa_{FAL} = 0, \tag{19}$$

where

$$\begin{aligned}\kappa_{FAL} = {} & \frac{\left(\frac{\alpha_{FF}^2}{\beta} + 1 - \beta\right)\left(\frac{\alpha_{F,AL}}{\beta} + 2(\delta - \mu)\right)}{-r + \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta}\int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)e^{\left(2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta\right)s}ds \\ & + \left(\frac{\alpha_{FF}}{\beta}\alpha_{F,AL} - 2(1-\beta) - \frac{\left(\frac{\alpha_{FF}^2}{\beta} + 1 - \beta\right)\left(\frac{\alpha_{F,AL}}{\beta} + 2(\delta - \mu)\right)}{-r + \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta}\right) \\ & \cdot \int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)e^{\left(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta\right)s}ds. \end{aligned} \tag{20}$$

The optimal fund is the solution of the system (4)*,*

$$\begin{aligned} dF(t) = {} & \left(\left(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta}\right)F(t) - \left(\frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\theta^\top\theta + \eta q^\top\theta + \frac{\alpha_{FF}}{\beta}\right) + \delta - \mu\right)AL\,(t)\right)dt \\ & - \left(\theta^\top F(t) + \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\theta^\top + \eta q^\top\right)AL\,(t)\right)dw(t),. \end{aligned} \tag{21}$$

with $F(0) = F_0$, $AL\,(0) = AL_0$.

The optimal strategies C^*, π^* are linear functions of the fund assets F and the actuarial liability AL, and depend on the parameters of the financial market and the benefit process, and also, through $\alpha_{F,AL}$, depend on the rate of discount $\tilde{\rho}$, the technical rate of interest δ and the benefit drift parameter μ.

The optimal investment decisions π^*, (16), are composed by two terms. The first is proportional to F, with coefficient proportional to θ, but the second is proportional to AL and depends on the rate of discount, and the parameters containing the correlation between benefit and risky assets. The constant of proportionality in the first term, $\Sigma^{-1}(b - r\bar{1})$, is the so called optimal-growth portfolio strategy, that appears in the Merton model where a CRRA utility of consumption is maximized.

An interesting consequence is that there exists a linear relationship between the optimal supplementary cost and the optimal investment strategy,

$$\pi^* = \Sigma^{-1}(b - r\bar{1})\frac{\beta}{\alpha_{FF}} SC^* - \eta\sigma^{-\top} q \frac{\alpha_{F,AL}}{2\alpha_{FF}} AL\,,$$

thus for each unit of additional amortization with respect to the normal cost the manager must invest $\Sigma^{-1}(b - r\bar{1})\frac{\beta}{\alpha_{FF}}$ units in the risky assets, plus an additional quantity of $\eta\sigma^{-\top} q \frac{-\alpha_{F,AL}}{2\alpha_{FF}} AL$ units. Both quantities depend on the rate of discount through α_{FF} and α_{FAL}.

Remark 3.1 The manager must borrow money at rate r to invest in the risky asset S^i, that is to say $\pi_i^* > F^*$, when the level of the fund is below $\lambda_i AL$, where the constant λ_i is defined as

$$\lambda_i = \frac{\bar{e}_i \Sigma^{-1}(b - r\bar{1}) + \eta \bar{e}_i \sigma^{-\top} \bar{q}}{1 + \bar{e}_i \Sigma^{-1}(b - r\bar{1})} \frac{\alpha_{F,AL}}{2\alpha_{FF}}, \qquad \bar{e}_i = (0, \ldots, \overset{i)}{1}, 0, \ldots, 0),$$

for all $i = 1, 2, \ldots, n$, and he/she need to short sell asset, that is to say $\pi_i^* < 0$, when the fund is above the value $\lambda_i' AL$, where

$$\lambda_i' = \frac{\bar{e}_i \Sigma^{-1}(b - r\bar{1}) + \eta \bar{e}_i \sigma^{-\top} \bar{q}}{\bar{e}_i \Sigma^{-1}(b - r\bar{1})} \frac{\alpha_{F,AL}}{2\alpha_{FF}}.$$

Thus the manager does not need short-selling neither borrowing, $0 \le \pi_i^* \le F^*$, when the fund F^* is between $\lambda_i AL$ and $\lambda_i' AL$.

Remark 3.2 The model can be observed in some particular cases. The model analyzed in Josa-Fombellida and Rincón-Zapatero (2004) is the same but with $\tilde{\rho}(t) = \rho$, $\forall t \in [0, \infty)$, where $\kappa_{FF} = \kappa_{FAL} = 0$.

In order to maintain in the long term the fund assets near the actuarial liability and the rate of contribution near the normal cost, we give a valuation of the technical rate of actualization δ, consisting in a spread method of fund amortization, as in Josa-Fombellida and Rincón-Zapatero (2004). These spread methods, widely used in pension funding (see Owadally and Haberman (1999)), assume that the supplementary cost *SC* is proportional to the unfunded actuarial liability *UAL*.

From (15), in order to achieve a spread method the identity $\alpha_{F,AL} = -2\alpha_{FF}$ must be satisfied. Substituting in (19) we obtain

$$-\frac{\alpha_{FF}^2}{\beta} + \left(-\rho + r - \theta^\top\theta - \eta q^\top\theta + \delta\right)\alpha_{FF} + 1 - \beta - \frac{\frac{\alpha_{FF}}{\beta} + \mu - \delta}{\frac{\alpha_{FF}}{\beta} + \mu - r - \eta q^\top\theta}\kappa_{FF}$$
$$+ \frac{\left(\frac{\alpha_{FF}^2}{\beta} + 1 - \beta\right)\left(r + \eta q^\top\theta - \delta\right)}{\frac{\alpha_{FF}}{\beta} + \mu - r - \eta q^\top\theta}\int_0^\infty e^{-\int_0^s \tilde{\rho}(v)dv}(\tilde{\rho}(s) - \rho)e^{\left(r - \frac{\alpha_{FF}}{\beta} - \theta^\top\theta + \mu - \eta q^\top\theta\right)s}ds = 0, \quad (22)$$

and comparing (22) with (17), we obtain that the technical interest rate must coincide with the rate of return of the bond modified to get rid of the sources of uncertainty. Specifically we assume that the valuation of liabilities δ is r plus the product of the Sharpe ratio of the portfolio and a term depending on the parameters containing the correlations and the diffusion parameter of the benefit process, as in Josa-Fombellida and Rincón-Zapatero (2004).

Assumption 2 *The technical rate of actualization is* $\delta = r + \eta q^\top\theta$.

Note that δ does not depend on parameter μ associated to the benefit P. If there is no correlation between the benefit and the financial market, then δ is the risk-free rate of interest. Note that the existence of a non-constant discount rate does not influence in the selection of the technical rate of interest.

Besides this valuation it provides, this selection of δ will allow us to simplify the explicit solution of the problem in the following result.

Corollary 3.1 *Suppose that Assumptions 1 and 2 hold, and the inequalities* (13), (14), *are satisfied. The optimal rate of contribution and the optimal investment in the risky assets are* *given by*

$$C^* = NC + \frac{\alpha_{FF}}{\beta} UAL,$$

$$\pi^* = \Sigma^{-1}(b - r\bar{1})\, UAL + \eta\sigma^{-\top} q AL\,, \tag{23}$$

respectively, where α_{FF} *is a positive solution of the equation* (17). *The optimal fund is the solution of the system* (4),

$$dF^*(t) = \Big(\Big(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta}\Big)F^*(t) - \Big(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} - \mu\Big) AL\,(t)\Big)\, dt$$
$$+ \Big(- \theta^\top F^*(t) + \big(\theta^\top + \eta q^\top\big) AL\,(t)\Big)\, dw(t), \tag{24}$$

with $F(0) = F_0$, $AL\,(0) = AL_{\,0}$.

The supplementary cost SC^* is proportional to the unfunded actuarial liability UAL, with constant of proportionality depending on the rate of discount. The optimal investment decisions π^*, (23), are the same than in Josa-Fombellida and Rincón-Zapatero (2004), thus it does not depend on the rate of discount. They are composed by two terms. The first is again proportional to UAL, but the second is a correction term, depending on the risk parameters of the model and AL. This second term is zero when there is no uncertainty in the benefits, as in Josa-Fombellida and Rincón-Zapatero (2001), and when there is no correlation between benefit and risky asset.

We obtain that the optimal rate of contribution C^* and the optimal investment π^* do not depend on μ. However, from (24), all parameters of the benefit process influence linearly in the optimal fund evolution. Also we observe that the manager takes a greater risk when the wealth of the fund is far below the actuarial liability than when it is closer. The optimal rate of contribution C^* and the fund evolution F^* depend on the rate of discount function $\tilde{\rho}$ through α_{FF}.

Next proposition shows that this selection of δ allows the fulfillment in the long term of one objective of the pension plan manager, that is the maintenance of the fund F^* and the contribution C^* close to their ideal values AL and NC, respectively.

Proposition 3.1 *Suppose that Assumptions 1, 2 and the inequalities* (13), (14) *and*

$$\alpha_{FF} > \beta\big(r - \theta^\top \theta\big), \tag{25}$$

are satisfied. Then the expected unfunded actuarial liability and the expected supplementary cost converge in the long term to zero, that is to say,

$$\lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} UAL^*(t) = \lim_{t\to\infty} \mathbb{E}_{F_0,AL_0} SC^*(t) = 0.$$

When the pension plan is underfunded, the total expected supplementary cost is

$$\overline{SC} = \int_0^\infty \mathbb{E}_{F_0,AL_0} SC^*(t)dt = \frac{\alpha_{FF}/\beta}{\alpha_{FF}/\beta + \theta^\top\theta - r} UAL_0.$$

Notice that the fulfilment of (25) is necessary to guarantee a finite total expected supplementary cost. However, the total expected contribution rate is infinite because the normal cost NC is a geometric Brownian motion (GBM) with positive drift μ. We can obtain the accumulated expected contribution rate along the interval $[0, T]$ using that UAL^* is a GBM (see proof of Proposition 3.1.

4 A numerical illustration

Next, we numerically illustrate the dynamic behaviour of the optimal fund and the optimal policies (contribution rate and investment polices) by conducting some simulations for a specific example. We consider that benefits have mean return $\mu = 0.03$ and standard deviation $\eta = 0.1$. We assume that the distribution function M is uniform in $[a, d]$, that is to say, $M(u) = \frac{u-a}{d-a}$, for $a \leq u \leq d$. Without loss of generality , we consider that there is one risky asset with mean $\mu = 0.09$ and standard deviation $\sigma = 0.2$ (this implies a Sharpe ratio of $\theta = 0.3$), which has a correlation coefficient with benefits of $q = 0.5$, while the risk free rate of interest is equal to $r = 0.03$. Initial values for the actuarial liability and the fund are, respectively, $AL_0 = 1000$ and $F_0 = 800$, so we consider that at $t = 0$ the actuarial liability is not totally covered by the fund. For the objective function we take as weight parameter $\beta = 0.5$, i.e., the manager gives equal value to the contribution rate risk and the solvency risk to be minimized. Finally, we select the technical rate of interest leading a spread method of funding ($\delta = 0.045$), and as a discount function for the manager in the objective function a linear combination between two exponential functions, i.e., $\theta(s-t) = \lambda e^{-0.08(s-t)} + (1-\lambda)e^{-0.3(s-t)}$, which will allow us to obtain parameters of the value function as described in Remark 2.1.

These parameters satisfy the transversality condition (13) and the convergence condition (25), assuring the stability of the pension plan, that we are going to check below.

Figure 1 shows the expected fund actuarial liability $\mathbb{E}AL\,(= \mathbb{E}_{F_0,AL_0}AL\,)$ and the expected fund assets $\mathbb{E}F$ for $\lambda = 0.5$ (baseline case: F), $\lambda = 0.9$ ($F1$) and $\lambda = 0.1$ ($F2$) for a total of 1000 realizations over a time horizon of 20 years and with a step of 1/12. In the baseline case we consider two kinds of participants (patient participants with $\rho_1 = 0.08$ and impatient participants with $\rho_2 = 0.3$) with equal weight (i.e., $\lambda = 0.5$), while the case of $\lambda = 0.9$ corresponds to a situations where patient participants are majority or $\lambda = 0.1$ where impatient agents are majority.

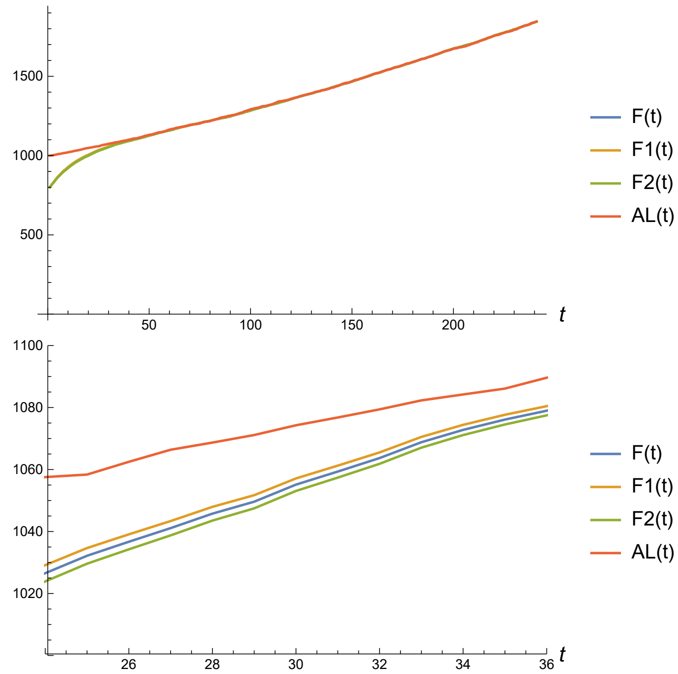

Figure 1: Expected actuarial liability and expected fund assets dynamics for different values of λ

$\delta = 0.045$	$\lambda = 1$	$\lambda = 0.9$	$\lambda = 0.5$	$\lambda = 0.1$	$\lambda = 0$
α_{FF}	0.473256	0.468554	0.449354	0.429394	0.424261
$\alpha_{F,AL}$	-0.946511	-0.937108	-0.898707	-0.858788	-0.848521
$\delta = 0.06$	$\lambda = 1$	$\lambda = 0.9$	$\lambda = 0.5$	$\lambda = 0.1$	$\lambda = 0$
α_{FF}	0.473256	0.468554	0.449354	0.429394	0.424261
$\alpha_{F,AL}$	-0.959761	-0.950119	-0.910724	-0.869735	-0.859185

Table 1: α_{FF} and $\alpha_{F,AL}$ for different values of λ under the spread method $\delta = 0.045$ and general case $\lambda = 0.06$

Despite the patient or impatient majority in the whole group, we can observe that the fund dynamics are very similar for all three cases and converge to the actuarial liability. More specifically, observing with more detail the same graph for subperiods ranging from 24 to 36 (third year of the time horizon simulated), it can be seen that $\mathbb{E}F1(t) > \mathbb{E}F(t) > \mathbb{E}F2(t)$, for all t, i.e., in the case of more patient participants the value of the fund is the higher one, and it decreases as impatience increases ($\lambda = 0.9 \to \lambda = 0.5 \to \lambda = 0.1$). This has, as a consequence, that the unfunded actuarial liability $\mathbb{E}\,UAL$ is inversely related with λ.

From Corollary 3.1 we have that contributions C^* are proportional to the unfunded actuarial liability and the value of α_{FF}, while investments π^* are proportional to UAL and AL. We next compute, at Table 1, some values of α_{FF} and $\alpha_{F,AL}$ for the case of applying a spread method ($\delta = 0.045$) where it holds that $\alpha_{F,AL} = -2\alpha_{FF}$ and for the general case where α_{FF} and $\alpha_{F,AL}$ are obtained as the solution of (17) and (19), where we have chosen as the technical rate of interest $\delta = 0.06$.

From Table 1, we can observe that α_{FF} decreases with λ. As a consequence, for a given value of UAL, contributions will be higher in the case of having a majority of patient participants. Moreover, as in our model we have that $UAL(0)$ and $AL(0)$, initial investments will be equal for our three studied cases ($\lambda = 0.9 \to \lambda = 0.5 \to \lambda = 0.1$). For these reasons, at the beginning of the time horizon $\mathbb{E}F1(t)$ will overcome $\mathbb{E}F(t)$ and $\mathbb{E}F2(t)$, and the corresponding unfunded actuarial liability will become the lowest one. While this last fact holds, investments will be the lowest ones for $\lambda = 0.9$, while contributions will depend on the cross effect between α_{FF} and the *UAL*. We next focus on the investment strategies.

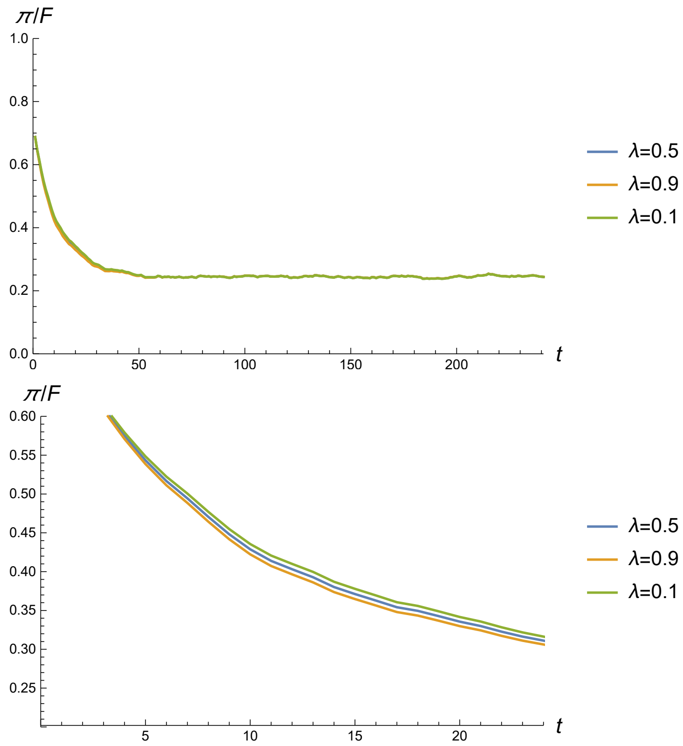

Figure 2: Expected relative investments dynamics for different values of λ

Figure 2 show the time evolution of the expected optimal investment relative to fund size, $\mathbb{E}(\pi^*(t)/F(t))$. We first mention that borrowing and shortselling are not necessary. In our three cases the global behaviour is similar, starting at the highest level and stabilizing in the long run. We can also see that relative investments are higher for more impatient participants, i.e., the higher the level of impatience, the larger the fraction of the fund invested in risky assets, not only for initial periods as explained above, but also for the whole simulated time horizon, what is related to the fact that in this case the fund $F3$ has the lowest value and, consequently, the associated *UAL* is the highest.

Looking now at the contributions C^*, as mentioned before, its value will depend not only on α_{FF} but also on *UAL*. In Figure 3, we first note that initially they start at a high level from which

$\delta = 0.045$	$\lambda = 1$	$\lambda = 0.9$	$\lambda = 0.5$	$\lambda = 0.1$	$\lambda = 0$
$\overline{SC}$	188.078	187.965	187.483	186.939	186.792
$\mathbb{E}_{F_0,AL_0}F(60)$	1159.57	1157.03	1156.73	1156.42	1153.62

Table 2: Expected accumulated contribution and expected value of the fund at fifth year

decrease up to converge the normal cost NC. Focusing on our three studied cases, we can observe that in the case of having more patient participants (large λ) initial contributions are higher, and the expected value of the fund increases faster. This has however, as a consequence, that the unfunded actuarial liability becomes smaller. As the optimal contributions are proportional to UAL and α_{FF}, it is not clear how contributions will be depending on the value of λ. This is what we observe in Figure 3 when looking at subperiods 0 to 12 (first year) and 24 to 48 (third year). In the first case, we have that initial contributions effectively are higher for $\lambda = 0.9$, but after some moment they reverse as a consequence that despite of having the largest α_{FF}, this cannot be compensated by the lower UAL. Table 2 collects accumulated contributions $\overline{SC}$ and the expected value of the fund at the end of the fifth year $\mathbb{E}_{F_0,AL_0}F(60)$, which are negatively related to participants' impatience.

5 Conclusions

We have analyzed by means of dynamic programming techniques the management of an aggregated defined benefit pension plan where the rate of discount is non-constant and the benefit is stochastic. The objective is to determine the contribution rate and the investment strategy minimizing both the contribution and the solvency risk in a infinite time horizon. We have found that the rate of discount function intervenes in the optimal strategies and in the optimal fund evolution. It is possible to select the technical rate of interest such that the optimal investment does not depend on the rate of discount (only depends on the diffusion parameter of the benefit and the financial market) and the optimal contribution depends rate of discount but does not depend on the parameters of the benefit process, getting a spread amortization and the plan stability in the long term.

A numerical illustration show how, when the discount function is a convex combination between two exponential functions, in the long term, the expected values of the optimal fund and the optimal contribution are closed, respectively, to the expected values of the actuarial liability and to the normal cost. In the case of the investment strategies, we have observed that the more impatient is the majority of participants, the higher are the investments in risky assets. However, there is not a constant relationship between the contributions and the degree of impatience, changing their ordering at some intermediate point in the planning horizon. Despite of this, more patient participants will lead to a higher value of the fund, and consequently, a lower unfunded actuarial liability.

Further research can be guide to include Poisson jumps in the financial market and to consider a regime switching model.

A Appendix

Proof of Theorem 3.1.

If there is a smooth solution V of the equation (10), strictly convex, then the minimizers values of the contribution rate and the investment rates are given by

$$\widehat{SC}(V_F) = -\frac{V_F}{2\beta}, \tag{26}$$

$$\widehat{\pi}(V_F, V_{FF}, V_{F,AL}) = -\Sigma^{-1}(b - r\overline{1})\frac{V_F}{V_{FF}} - \eta AL\,\sigma^{-\top} q\frac{V_{F,AL}}{V_{FF}}, \tag{27}$$

respectively.

From (26) and (27), the structure of the HJB equation obtained once we have substituted these values for SC and π in (10), suggests a quadratic homogeneous solution

$$V(F, AL) = \alpha_{FF}F^2 + \alpha_{F,AL}F\,AL + \alpha_{AL,AL}AL^2.$$

Imposing this solution in (26) and (27), we obtain

$$\widehat{SC} = -\frac{\alpha_{FF}}{\beta}F - \frac{\alpha_{F,AL}}{2\beta}AL\,,$$

$$\widehat{\pi} = -\Sigma^{-1}(b - r\bar{1})F - \frac{\alpha_{F,AL}}{2\alpha_{FF}}\left(\Sigma^{-1}(b - r\bar{1}) + \eta\sigma^{-\top}q\right)AL\,.$$

Now we are going to obtain the explicit form of $K(F, AL)$ in (11). Taking into account that process AL satisfies the SDE (4), after substitution in (8) of expressions for $\widehat{SC}$ and $\widehat{\pi}$, process F satisfies the SDE (21). Following Arnold (1974), we can apply the Itô's formula to the processes F^2, FAL and AL^2. Taking expected values, the functions defined by $\phi(t) = \mathbb{E}_{F_0,AL_0}F^2(t)$, $\psi(t) = \mathbb{E}_{F_0,AL_0}(FAL)(t)$ and $\xi(t) = \mathbb{E}_{F_0,AL_0}AL^2(t)$ satisfy the linear differential equations

$$\phi'(t) = \Big(2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta\Big)\phi(t) - \Big(\frac{\alpha_{F,AL}}{\beta} + 2(\delta-\mu)\Big)\psi(t) + \frac{\alpha_{F,AL}^2}{4\alpha_{FF}^2}\Big(\theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q\Big)\xi(t),$$

$$\psi'(t) = \Big(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta\Big)\psi(t) - \Big(\frac{\alpha_{F,AL}}{2\alpha_{FF}}(\theta^\top\theta + 2\eta q^\top\theta + \frac{\alpha_{FF}}{\beta} + \eta^2 q^\top q) + \delta - \mu\Big)\xi(t),$$

$$\xi'(t) = (2\mu + \eta^2)\xi(t),$$

with initial conditions $\phi(0) = F^2$, $\psi(0) = FAL$ and $\xi(0) = AL^2$, respectively. Thus the explicit expressions for these functions are:

$$\xi(t) = AL^2 e^{(2\mu+\eta^2)t},$$

$$\psi(t) = \left(FAL - \frac{b}{2\mu+\eta^2-a}AL^2\right)e^{at} + \frac{b}{2\mu+\eta^2-a}AL^2 e^{(2\mu+\eta^2)t},$$

$$\phi(t) = \left(F^2 - \frac{d}{a-c}FAL + \frac{bd}{(2\mu+\eta^2-a)(a-c)}AL^2 - \frac{1}{2\mu+\eta^2-c}\left(\frac{bd}{2\mu+\eta^2-a} + m\right)AL^2\right)e^{ct}$$
$$+ \left(\frac{d}{a-c}FAL - \frac{bd}{(2\mu+\eta^2-a)(a-c)}AL^2\right)e^{at} + \left(\frac{bd}{2\mu+\eta^2-a} + m\right)\frac{AL^2}{2\mu+\eta^2-c}e^{(2\mu+\eta^2)t}$$

where $a = r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta$, $b = -\frac{\alpha_{FAL}}{2\alpha_{FF}}\left(\theta^\top\theta + 2\eta q^\top\theta + \frac{\alpha_{FF}}{\beta} + \eta^2 q^\top q\right) - \delta + \mu$, $c = 2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta$, $d = -\frac{\alpha_{FAL}}{\beta} - 2(\delta-\mu)$ and $m = \left(\theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q\right)\frac{\alpha_{FAL}^2}{4\alpha_{FF}^2}$. Substituting the minimizers,

$$
\begin{aligned}
&\mathbb{E}_{F,AL}\left\{\beta SC^2(s)+(1-\beta)(AL\,(s)-F(s))^2\right\}\\
&=\mathbb{E}_{F,AL}\left\{\frac{\alpha_{FF}^2}{\beta}\left(F(s)+\frac{\alpha_{FAL}}{2\alpha_{FF}}AL\,(s)\right)^2+(1-\beta)(AL\,(s)-F(s))^2\right\}\\
&=\frac{\alpha_{FF}^2}{\beta}\left(\phi(s)+\frac{\alpha_{FAL}}{\alpha_{FF}}\psi(s)+\frac{\alpha_{FAL}^2}{4\alpha_{FF}^2}\xi(s)\right)+(1-\beta)\left(\phi(s)-2\psi(s)+\xi(s)\right),
\end{aligned}
$$

and then $K(F,AL)=\kappa_{FF}F^2+\kappa_{FAL}FAL+\kappa_{AL,AL}AL^2$, where κ_{FF} is given by (18), $\kappa_{F,AL}$ is given by (20) and $\kappa_{AL,AL}$ is another constant what does not need to be determined. Note that the improper integral that appears in (18) is well defined by condition (14) y because the rate of discount function $\tilde{\rho}$ is time-decreasing. Analogously with the improper integral in (20), by (31); see below.

Substituting the minimizers in (10) and using (1), the following set of three equations for the coefficients is obtained: (17), (19) and

$$
\begin{aligned}
(-\rho+2\mu-\eta^2)\alpha_{AL,AL}-\left(\theta^\top\theta+\eta^2q^\top q+2\eta q^\top\theta\right)\frac{\alpha_{F,AL}^2}{4\alpha_{FF}}&\\
-(\delta-\mu)\alpha_{F,AL}-\frac{\alpha_{F,AL}^2}{4\beta}+1-\beta-\kappa_{AL,AL}=0.&\qquad(28)
\end{aligned}
$$

In order to prove that the solution of (10) is the value function and C^* and π^*, given by (15) and (16), respectively, are the optimal strategies of the stochastic control problem, it is sufficient to check that the transversality condition

$$
\lim_{t\to\infty}e^{-\rho t}\mathbb{E}_{F_0,AL_0}V(F^*(t),AL\,(t))=\lim_{t\to\infty}e^{-\rho t}\left(\alpha_{FF}\phi(t)+\alpha_{F,AL}\psi(t)+\alpha_{AL,AL}\xi(t)\right)=0
$$

holds, where AL and F^* satisfy, respectively, (4) and (21). By our previous calculations,

$$
\xi(t)=AL_0^2e^{(2\mu+\eta^2)t},
$$

hence $\lim_{t\to\infty}e^{-\rho t}\xi(t)=0$ if and only if (13) holds. On the other hand,

$$
\psi(t)=(F_0-a_1AL_0)AL_0e^{at}+a_1\xi(t),
$$

where $a_1 = \frac{b}{2\mu+\eta^2-a}$ is a constant depending on parameters of model. Then $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0,AL_0}\psi(t) = 0$, if and only if, both (13) and the inequality

$$r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta < \rho \tag{29}$$

simultaneously hold. On the other hand,

$$\phi(t) = \left(F_0^2 + a_2 AL_0^2 - a_3 F_0 AL_0\right)e^{ct} + a_2\psi(t) - a_3\xi(t),$$

where $a_2 = \frac{d}{a-c}$ and $a_3 = a_1 a_2 - \frac{ba_1+m}{2\mu+\eta^2-c}$ are constants. Hence $\lim_{t\to\infty} e^{-\rho t}\phi(t) = 0$ if and only if (13), (29) and

$$2r - 2\frac{\alpha_{FF}}{\beta} - \theta^\top\theta < \rho \tag{30}$$

hold. Now we prove that the conditions (13) and (30) imply (29). Observe that

$$0 \le \left(\theta^\top + \eta q^\top\right)(\theta + \eta q) = \theta^\top\theta + 2\eta q^\top\theta + \eta^2 q^\top q \le \theta^\top\theta + 2\eta q^\top\theta + \eta^2,$$

implies

$$-\eta q^\top\theta \le \frac{1}{2}\theta^\top\theta + \frac{1}{2}\eta^2.$$

We have used $q^\top q \le 1$. Thus

$$r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu - \eta q^\top\theta \le r - \frac{1}{2}\theta^\top\theta - \frac{\alpha_{FF}}{\beta} + \mu + \frac{1}{2}\eta^2 < \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{31}$$

Since V is a homogeneous quadratic polynomial in F and AL, $e^{-\rho t}\mathbb{E}_{F_0,AL_0}V(F^*(t), AL(t))$ converges to 0 when t goes to ∞. Finally, applying the analogous theorem to the verification Theorem 8.1, chapter 3, in Fleming and Soner (1993), we conclude that V is the value function and C^*, given by (15), and π^*, given by (16), are the optimal controls. □

Proof of Proposition 3.1. From (21), using $\alpha_{F,AL} = -2\alpha_{FF}$ and Assumption 2 we obtain (24) and then

$$dUAL^*(t) = \left(r - \theta^\top\theta - \frac{\alpha_{FF}}{\beta}\right)UAL^*(t)dt + \eta\sqrt{1 - q^\top q}AL(t)\,dw_0(t) - \theta^\top UAL^*(t)dw(t).$$

Thus

$$\mathbb{E}_{F_0,AL_0}UAL^*(t) = \mathbb{E}_{F_0,AL_0}AL(t) - \mathbb{E}_{F_0,AL_0}F^*(t) = (AL_0 - F_0)e^{\left(r-\theta^\top\theta-\frac{\alpha_{FF}}{\beta}\right)t},$$

converges to zero when t goes to ∞, by (25). Analogously

$$\mathbb{E}_{F_0,AL_0} SC^*(t) = \frac{\alpha_{FF}}{\beta} \mathbb{E}_{F_0,AL_0} UAL^*(t),$$

converges to zero when t goes to ∞.

On the other hand, by Corollary 3.1,

$$\begin{aligned}\overline{SC} &= \int_0^\infty \mathbb{E}_{F_0,AL_0} SC(t)dt = \frac{\alpha_{FF}}{\beta} \int_0^\infty \mathbb{E}_{F_0,AL_0} UAL(t)dt \\ &= \frac{\alpha_{FF}}{\beta} UAL_0 \int_0^\infty e^{(r-\theta^\top\theta-\alpha_{FF}/\beta)t} dt = \frac{\alpha_{FF}/\beta}{\alpha_{FF}/\beta + \theta^\top\theta - r} UAL_0 > 0,\end{aligned}$$

by (25), and because the plan is underfunded, $UAL_0 = AL_0 - F_0 > 0$. □

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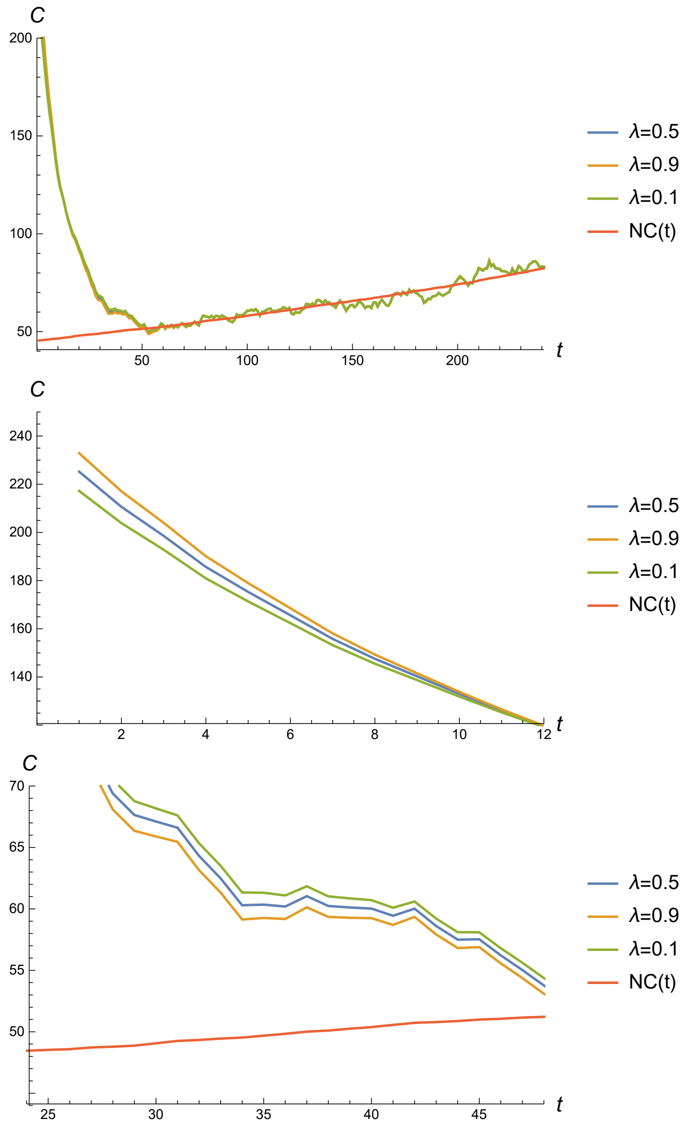

Figure 3: Expected contributions for different values of λ and expected normal cost dynamics