A 2D FM II model-based FF H_∞ SOF Control

Ismail ER-RACHID[1], Hassnae MERZOUKI[1], Taha ZOULAGH[2], Fernando TADEO[3], Khalid CHIKH[1],

***Abstract*— In this paper, the problem of $\mathbf{H_\infty}$ Static Output Feedback (SOF) Controller Design with finite frequency (FF) specification for Two-Dimensional (2D) Discrete Systems in Fornasini-Marchesini (FM) second model is investigated with the use of the Generalized Kalman Yakobovich Popov (GKYP) lemma. New condition guaranteeing the finite frequency SOF H_∞ controller design for 2D discrete systems is derived, in terms of Linear Matrix Inequalities (LMIs). This study reduces the conservatism of the existing entire-frequency (EF) design. In the end, same examples are provided to show the effectiveness of the derived results.**

Keywords: *Finite Frequency (FF) domain, static output feedback (SOF) controller design, linear matrix inequality (LMI), 2-D discrete systems, H_∞ performance, Fornasini-Marchesini (FM) second model.*

I. Introduction

Two-dimensional (2-D) systems have attracted much attention during the past decades [26], [1], [27]. However, due to the structural and analytical complexity, many 2-D control problems lack analytical solutions. Owing to the LMI approach emerged in the 1990s, the LMI-based design provides a valuable alternative to solve the 2-D control problems. In recent years, increasing LMI-based results on 2-D systems have been reported in the literature, including stability analysis [22], stabilization [27], filtering [1], control design [20], etc. On the other hand, the static-output-feedback (SOF) control problem is one of the most important open problems in control theory even for 1-D systems, and simpler to implement than dynamic output feedback ones. The design of an SOF controller with a desired H_∞ performance has received considerable attention from the control community over the past several decades [2], and it has not been fully investigated. The 2-D H_∞ SOF control problem can be represented as a BMI problem. Note that all the existing SOF control design problems for 2D FM second model are considered in the entire frequency range. However, in practical situations and design specifications are usually given in a certain frequency domains of relevance, where it is required that SOF control design problems should be designed in finite frequency field [8]. Authors of [8] considered the H_∞ design properties in finite frequency fields, and provided exact LMIs techniques with the aid of the generalized Kalman-Yakubovich-Popov (GKYP) lemma. On the basis of [8], analysis and design of finite frequency have attracted wide attention. To list same literature papers, The finite frequency H_∞ control for 2-D discrete systems, in Fornasini-Marchesini in[5]. The GKYP combined with the frequency-partitioning approach to stability analysis, were obtained in [22] for 2-D discrete system, and in [11] for 2-D continuous system. However, no result address the issue of SOF H_∞ control of 2D discrete systems in finite frequency domain.

Motivated by the previous discussions, a new approach is studied. Then, an equivalent strict LMI condition is given to guarantee the existence of controllers. The main feature of the proposed technique is the single-step design procedure for SOF H_∞ controller design problem, which reduces the drawback induced by using iterative algorithm. By virtue of the GKYP lemma, we propose a design to SOF H_∞ control problem of 2D discrete systems in FM second model, with finite frequency specifications, by solving a set of strict LMIs, whose purpose is to overcome the conservativeness of the entire-frequency results. In the end, many illustrative examples are included in order to show the advantages of the proposed approach.

Notation: Throughout this note, we use the following notations: $\mathbb{R}^n$ denotes the n$-$dimentional Euclidean space. $*$ is used for the blocks induced by symmetry. I is the identity matrix with appropriate dimensions. A^T represents the transpose matrix of A. $P{>}0$ means that P is real symmetric and positive definite, and $sym(M)$ is defined as $sym(M)$=M+M^T.

II. H_∞ SOF control of 2-D FM second model

A. problem formulation and basic Results

Consider the following 2-D discrete system described by the FM second model

$$\begin{aligned} x(i+1,j+1) &= A_1x(i,j+1)+A_2x(i+1,j) \\ &\quad +B_{11}w(i,j+1)+B_{12}w(i+1,j) \\ &\quad +B_{21}u(i,j+1)+B_{22}u(i+1,j) \\ z(i,j) &= C_1x(i,j)+D_{11}w(i,j)+D_{12}u(i,j) \\ y(i,j) &= C_2x(i,j)+D_{21}w(i,j)+D_{22}u(i,j) \end{aligned} \tag{1}$$

where $x(i,j)\in R^n$ is the system state; $w(i,j)\in R^q$ is the exogenous disturbance input with bounded energy, i.e.,

[1] LaSTI, National School of Applied Sciences (ENSA) Khouribga, Sultan Moulay Slimane University, Morocco. i.errachid@usms.ma, ORCID: https://orcid.org/0000-0002-0287-0101, hassnae.merzouki@gmail.com, genielectrique@gmail.com

[2]Electrical engineering departement, santiago university, chile. zoulagh.taha@gmail.com,

[3]Departamento de Ingenieria de Sistemas y Automatica, Universidad de Valladolid, 47005 Valladolid, Spain. Fernando.Tadeo@uva.es

$w(i,j)$ belongs to L2; $u(i,j) \in R^r$ is the control input; $z(i,j) \in R^p$ is the controlled output and $y(i,j) \in R^l$ is the measured output. $A_k, B_{1k}, B_{2k}, C_k, D_{1k}$, and D_{2k}, where $k = 1,2$, are constant matrices with appropriate dimensions. Without loss of generality, we assume $D22 = 0$. The SOF control law is defined as:

$$u(i,j) = Ky(i,j) \quad (2)$$

where $F \in R^{r\times l}$ is a gain matrix to be determined. Applying the SOF controller (2) to System (1) yields the following closed-loop system

$$\begin{aligned} x(i+1,j+1) =& A_{c1}x(i,j+1) + A_{c2}x(i+1,j) \\ &+ B_{c1}w(i,j+1) + B_{c2}w(i+1,j) \\ z(i,j) =& C_c x(i,j) + D_c w(i,j) \end{aligned} \quad (3)$$

where
$A_{c1} = A_1 + B_{21}KC_2,\ B_{c1} = B_{11} + B_2KD_{21}$,
$A_{c2} = A_2 + B_{22}KC_2,\ B_{c2} = B_{12} + B_2KD_{21}$,
$C_c = C_1 + D_{12}KC_2,\ D_c = D_{11} + D_{12}KD_{21}$.
The transfer function is given by

$$\begin{aligned} H(z_1,z_2) =& C_c(z_1z_2I - z_2A_{c1} - z_1A_{c2})^{-1}(z_2B_{c1} + z_1B_{c2}) \\ &+ D_c \end{aligned} \quad (4)$$

with $z_1 = e^{j\theta_1}$, $z_2 = e^{j\theta_2}$ and $(\theta_1,\theta_2) \in \Omega$ where $\Omega = [\theta_{m1},\theta_{M1}] \times [\theta_{m2},\theta_{M2}]$

Lemma 1 *[7] Given a symmetric matrix $\Sigma \in \mathbb{R}^{p\times p}$ and two matrices X, Z of column dimension p, there exists a matrix Y such that the LMI*

$$\Sigma + symX^TYZ < 0 \quad (5)$$

holds if and only if the following two projection inequalities with respect to Y are satisfied:

$$X^{\perp T}\Sigma X^{\perp} < 0, \quad Z^{\perp T}\Sigma Z^{\perp} < 0. \quad (6)$$

where $X^{\perp}$ and $Z^{\perp}$ are arbitrary matrices whose columns form a basis of the null spaces of X and Z, respectively.

Lemma 2 *[10] For matrices T, Q, U, and W with appropriate dimensions and scalar β. Inequality*

$$T + QW + W^TQ^T < 0 \quad (7)$$

is fulfilled if the following condition holds:

$$\begin{bmatrix} T & * \\ \beta Q^T + UW & -\beta U - \beta U^T \end{bmatrix} < 0 \quad (8)$$

Lemma 3 *[5] Consider the FM model in (3) and suppose that $det(z_1z_2I - z_2A_{c1} - z_1A_{c2}) \neq 0$ for all $|z_1| > 1$, $|z_2| > 1$ with $z_1,z_2 \in \mathbb{C}$. Given scalars $\gamma > 0$, and $\theta_{Mk},\theta_{mk} \in [-\pi,\pi]$, $k = 1,2$, satisfying $\theta_{Mk} < \theta_{mk}$, if there exist symmetric matrices $Q_k > 0$, $\bar{P}_k > 0$, P_k, $k = 1,2$, M, and a general matrix N such that*

$$\begin{bmatrix} \mathfrak{A}_\mathfrak{c} & \mathfrak{B}_\mathfrak{c} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P & Q\Lambda \\ \Lambda^*Q^T & W \end{bmatrix} \begin{bmatrix} \mathfrak{A}_\mathfrak{c} & \mathfrak{B}_\mathfrak{c} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{C}_\mathfrak{c}{}^T\mathfrak{C}_\mathfrak{c} & \mathfrak{C}_\mathfrak{c}{}^T\mathfrak{D}_\mathfrak{c} + N_d \\ \mathfrak{D}_\mathfrak{c}{}^T\mathfrak{C}_\mathfrak{c} + N_d^T & M_d - \gamma^2 I + \mathfrak{D}_\mathfrak{c}{}^T\mathfrak{D}_\mathfrak{c} \end{bmatrix} < 0 \quad (9)$$

$$\begin{bmatrix} \mathfrak{A}_\mathfrak{c} \\ I \end{bmatrix}^T \begin{bmatrix} \bar{P} & 0 \\ 0 & -\tilde{P} \end{bmatrix} \begin{bmatrix} \mathfrak{A}_\mathfrak{c} \\ I \end{bmatrix} < 0 \quad (10)$$

hold, where $P = P_1 + P_2$, $\bar{P} = \bar{P}_1 + \bar{P}_2$, $\tilde{P} = diag\{\bar{P}_1, \bar{P}_2\}$
$Q = [Q_1 \quad Q_2]$, $\Lambda = diag\{e^{\theta_1^c}I, e^{\theta_2^c}I\}$,
$W = diag\{-P_1 - 2cos(\theta_1^a)Q_1, -P_2 - 2cos(\theta_2^a)Q_2\}$,
$\Delta = diag\{-P_1 - 2cos(\bar{\theta}_1)Q_1, -P_2 - 2cos(\bar{\theta}_2)Q_2\}$,
$\theta_k^c = (\theta_{Mk} + \theta_{mk})/2$, $\theta_k^a = (\theta_{Mk} - \theta_{mk})/2$, $k = 1,2$,
$\mathfrak{A}_\mathfrak{c} = \begin{bmatrix} A_{c1} & A_{c2} \end{bmatrix}$, $\mathfrak{B}_\mathfrak{c} = \begin{bmatrix} B_{c1} & B_{c2} \end{bmatrix}$
$\mathfrak{C}_\mathfrak{c} = diag\{C_c, C_c\}$, $\mathfrak{D}_\mathfrak{c} = diag\{D_c, D_c\}$,
$N_d = diag\{N, -N\}$, $M_d = diag\{M, -M\}$
then

$$||H||_\infty^\Omega = \sup_{(\theta_1,\theta_2)\in\Omega} \sigma_{max}[H(e^{j\theta_1}, e^{j\theta_2})] < \gamma \quad (11)$$

is satisfied, with Ω is defined in (4).

B. finite frequency SOF H_∞ controller analysis for 2-D FM second model

we are now in a position to present a new approach of the finite frequency H_∞ SOF controller design for 2D discrete systems in FM model.

Theorem 1 *The closed-loop system (3) is asymptotically stable with an H_∞ performance $\gamma > 0$ if there exist general matrices F, G and N_d, symmetric matrices $Q_k > 0$, $\bar{P}_k > 0$, P_k, $k = 1,2$, and M_d, such that the following conditions are satisfied:*

$$\begin{bmatrix} P_1 + P_2 - F - F^T & \Lambda Q + F\mathfrak{A}_\mathfrak{c} & F\mathfrak{B}_\mathfrak{c} & 0 \\ * & W & N_d & \mathfrak{C}_\mathfrak{c}{}^TG^T \\ * & * & M_d - \gamma^2 I & \mathfrak{D}_\mathfrak{c}{}^TG^T \\ * & * & * & I - G - G^T \end{bmatrix} < 0 \quad (12)$$

$$\begin{bmatrix} \bar{P}_1 + \bar{P}_2 - F - F^T & F\mathfrak{A}_\mathfrak{c} \\ * & -diag\{\bar{P}_1, \bar{P}_2\} \end{bmatrix} < 0 \quad (13)$$

Proof 1 *We can verify that (13) is equivalent to,*

$$\begin{bmatrix} \bar{P} & 0 \\ * & -\bar{P} \end{bmatrix} + sym(\begin{bmatrix} F \\ 0 \end{bmatrix} \begin{bmatrix} -I & \mathfrak{A}_\mathfrak{c} \end{bmatrix}) < 0 \quad (14)$$

By Lemma 1 with

$$\Sigma = \begin{bmatrix} \bar{P} & 0 \\ * & -\bar{P} \end{bmatrix},\ X = I,\ Y = \begin{bmatrix} F \\ 0 \end{bmatrix},\ Z = \begin{bmatrix} -I & \mathfrak{A}_\mathfrak{c} \end{bmatrix}$$

the inequality (14) can guarantee

$$\begin{bmatrix} A_c^T & I \end{bmatrix} \begin{bmatrix} \bar{P} & 0 \\ * & -\bar{P} \end{bmatrix} \begin{bmatrix} A_c \\ I \end{bmatrix} < 0 \quad (15)$$

this implies that (10) *holds.*
Let

$$\Sigma = \begin{bmatrix} P & \Lambda Q & 0 \\ Q\Lambda^* & -WQ-P+\mathfrak{C}_\mathfrak{c}{}^T\mathfrak{C}_\mathfrak{c} & \mathfrak{C}_\mathfrak{c}{}^T\mathfrak{D}_\mathfrak{c} \\ 0 & \mathfrak{D}_\mathfrak{c}{}^T\mathfrak{C}_\mathfrak{c} & -\gamma^2 I+\mathfrak{D}_\mathfrak{c}{}^T\mathfrak{D}_\mathfrak{c} \end{bmatrix},$$

$X = I, \quad Y = \begin{bmatrix} F^T & 0 & 0 \end{bmatrix}^T, \quad Z = \begin{bmatrix} -I & \mathfrak{A}_\mathfrak{c} & \mathfrak{B}_\mathfrak{c} \end{bmatrix}$

By the Schur complement, (12) *is equivalent to*

$$\Sigma + sym(X^TYZ) < 0 \tag{16}$$

Choosing $Z^\perp = \begin{bmatrix} \mathfrak{A}_\mathfrak{c} & \mathfrak{B}_\mathfrak{c} \\ I & 0 \\ 0 & I \end{bmatrix}$ *and applying Lemma 1, we obtain from* (16) *that* (9) *holds. The proof is completed.*

C. finite frequency SOF H_∞ controller design for 2-D FM second model

Theorem 2 *The closed-loop system* (3) *is asymptotically stable with an H_∞ performance $\gamma > 0$ if there exist general matrices F, G, U, V and N, symmetric matrices $Q_k > 0$, $\bar{P}_k > 0$, P_k, $k = 1,2$, and M, such that the following conditions are satisfied*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 & \Psi_{18} & \Psi_{19} \\ * & \Psi_{22} & 0 & N & 0 & \Psi_{26} & 0 & \Psi_{28} & 0 \\ * & * & \Psi_{33} & 0 & -N & 0 & \Psi_{37} & 0 & \Psi_{39} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 & \Psi_{48} & 0 \\ * & * & * & * & \Psi_{55} & 0 & \Psi_{57} & 0 & \Psi_{59} \\ * & * & * & * & * & \Psi_{66} & 0 & \Psi_{68} & 0 \\ * & * & * & * & * & * & \Psi_{77} & \Psi_{78} & 0 \\ * & * & * & * & * & * & * & \Psi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Psi_{99} \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} \bar{\Psi}_{11} & \bar{\Psi}_{12} & \bar{\Psi}_{13} & \bar{\Psi}_{14} & \bar{\Psi}_{15} \\ * & -\bar{P}_1 & 0 & C_2^TV^T & 0 \\ * & * & -\bar{P}_2 & 0 & C_2^TV^T \\ * & * & * & -\beta(U+U^T) & 0 \\ * & * & * & * & -\beta(U+U^T) \end{bmatrix} < 0 \tag{18}$$

$\bar{\Psi}_{11} = \bar{P}_1 + \bar{P}_2 - F - F^T$
$\bar{\Psi}_{12} = FA_1 + B_{21}VC_2$
$\bar{\Psi}_{13} = FA_2 + B_{22}VC_2$
$\bar{\Psi}_{14} = \beta(FB_{21} - B_{21}U)$
$\bar{\Psi}_{15} = \beta(FB_{22} - B_{22}U)$

$\Psi_{11} = P_1 + P_2 - F - F^T,$
$\Psi_{12} = Q_1e^{j\theta_c} + FA_1 + B_{21}VC_2,$
$\Psi_{13} = Q_2e^{j\theta_c} + FA_2 + B_{22}VC_2,$
$\Psi_{14} = FB_{11} + B_{21}VD_{21},$
$\Psi_{15} = FB_{12} + B_{22}VD_{21},$
$\Psi_{18} = \beta(FB_{21} - B_{21}U),$
$\Psi_{19} = \beta(FB_{22} - B_{22}U),$
$\Psi_{22} = -2cos(\theta_a)Q_1 - P_1,$
$\Psi_{26} = \Psi_{37} = C_1^TG^T + C_2^TV^TD_{12}^T,$
$\Psi_{28} = \Psi_{39} = C_2^TV^T$
$\Psi_{33} = -2cos(\theta_a)Q_2 - P_2,$
$\Psi_{44} = M - \gamma^2 I$
$\Psi_{46} = \Psi_{57} = D_{11}^TG^T + D_{21}^TV^TD_{12}^T,$
$\Psi_{48} = \Psi_{59} = D_{21}^TV^T$
$\Psi_{55} = -M - \gamma^2 I$
$\Psi_{66} = \Psi_{77} = I - G - G^T,$
$\Psi_{68} = \Psi_{78} = \beta(GD_{12} - D_{12}U),$
$\Psi_{88} = \Psi_{99} = -\beta(U + U^T),$

Moreover, the gain of the H_∞ SOF controller is given by $K = U^{-1}V$.

Proof 2 *Suppose that inequality* (17) *holds, it guarantees $-\beta U - \beta U^T < 0$, which implies U is nonsingular.*
we can verify that (18) *is equivalent to,*

$$\begin{bmatrix} \bar{P}_1+\bar{P}_2-F-F^T & FA_1+B_{21}VC_2 & FA_2+B_{22}VC_2 \\ * & -\bar{P}_1 & 0 \\ * & * & -\bar{P}_2 \end{bmatrix}$$
$$+sym\left\{ \begin{bmatrix} FB_{21}-B_{21}U & FB_{22}-B_{22}U \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} 0 & VC_2 & 0 \\ 0 & 0 & VC_2 \end{bmatrix} \right\} \tag{19}$$

By Lemma 2 with
$W = \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} 0 & VC_2 & 0 \\ 0 & 0 & VC_2 \end{bmatrix},$
$Q = \begin{bmatrix} FB_{21}-B_{21}U & FB_{22}-B_{22}U \\ 0 & 0 \\ 0 & c \end{bmatrix}$*, and* $T = \begin{bmatrix} \bar{P}_1+\bar{P}_2-F-F^T & FA_1+B_{21}VC_2 & FA_2+B_{22}VC_2 \\ * & -\bar{P}_1 & 0 \\ * & * & -\bar{P}_2 \end{bmatrix}$
and defining $K = U^{-1}V$, we can guarantee that (19) *is equivalent to*

$$\begin{bmatrix} \bar{P}_1+\bar{P}_2-F-F^T & FA_1+FB_{21}KC_2 & FA_2+FB_{22}KC_2 \\ * & -\bar{P}_1 & 0 \\ * & * & -\bar{P}_2 \end{bmatrix} < 0$$

this replies that (18) *is equivalent to* (13).
Let $W = \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} 0 & VC_2 & 0 & VD_{21} & 0 & 0 & 0 \\ 0 & 0 & VC_2 & 0 & VD_{21} & 0 & 0 \end{bmatrix},$

$$Q = \begin{bmatrix} FB_{21}-B_{21}U & FB_{22}-B_{22}U \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ GD_{12}-D_{12}U & 0 \\ 0 & GD_{12}-D_{12}U \end{bmatrix}, \text{and}$$

$$T = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 \\ * & \Psi_{22} & 0 & N & 0 & \Psi_{26} & 0 \\ * & * & \Psi_{33} & 0 & -N & 0 & \Psi_{37} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 \\ * & * & * & * & \Psi_{55} & 0 & \Psi_{57} \\ * & * & * & * & * & \Psi_{66} & 0 \\ * & * & * & * & * & * & \Psi_{77} \end{bmatrix},$$

applying Lemma 2 the inequality in (17) *leads to*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 \\ * & \Psi_{22} & 0 & N & 0 & \Psi_{26} & 0 \\ * & * & \Psi_{33} & 0 & -N & 0 & \Psi_{37} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 \\ * & * & * & * & \Psi_{55} & 0 & \Psi_{57} \\ * & * & * & * & * & \Psi_{66} & 0 \\ * & * & * & * & * & * & \Psi_{77} \end{bmatrix}$$

$$+ sym \left\{ \begin{bmatrix} FB_{21}-B_{21}U & FB_{22}-B_{22}U \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ GD_{12}-D_{12}U & 0 \\ 0 & GD_{12}-D_{12}U \end{bmatrix} \right. \quad (20)$$

$$\times \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix}$$

$$\left. \begin{bmatrix} 0 & VC_2 & 0 & VD_{21} & 0 & 0 & 0 \\ 0 & 0 & VC_2 & 0 & VD_{21} & 0 & 0 \end{bmatrix} \right\}$$

By defining $K = U^{-1}V$, we can verify that (20) is equivalent to

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 \\ * & \Psi_{22} & 0 & N & 0 & \Psi_{26} & 0 \\ * & * & \Psi_{33} & 0 & -N & 0 & \Psi_{37} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & 0 \\ * & * & * & * & \Psi_{55} & 0 & \Psi_{57} \\ * & * & * & * & * & \Psi_{66} & 0 \\ * & * & * & * & * & * & \Psi_{77} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & 0 & 0 \\ * & 0 & 0 & 0 & 0 & \psi_{26} & 0 \\ * & * & 0 & 0 & 0 & 0 & \psi_{37} \\ * & * & * & 0 & 0 & \psi_{46} & 0 \\ * & * & * & * & 0 & 0 & \psi_{57} \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix} < 0 \quad (21)$$

$\psi_{12} = FB_{21}KC_2 - B_{21}NC_2$
$\psi_{13} = FB_{22}KC_2 - B_{22}NC_2$
$\psi_{14} = FB_{21}KD_{21} - B_{21}VD_{21}$
$\psi_{15} = FB_{22}KD_{21} - B_{22}VD_{21}$
$\psi_{26} = \psi_{37} = C_2^T K^T D_{12}^T G^T - C_2^T V^T D_{12}^T$
$\psi_{46} = \psi_{57} = D_{21}^T K^T D_{12}^T G^T - D_{21}^T V^T D_{12}^T$
From (21), the condition (12) is obtained. The proof is completed.

III. Illustrative examples

In this section, we provide some examples to illustrate the application of the proposed method in this paper.

Example 1 *[3] Consider the H_∞ SOF control problem of the FM second model (1) with*

$$A_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0.4 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 1 \\ -0.5 & 0 & 0 \end{bmatrix},$$

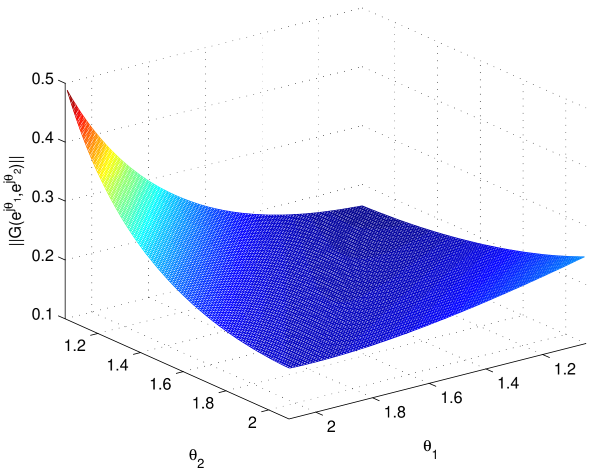

Fig. 1. Frequency response of transfer function in $[\frac{\pi}{3}, \frac{2\pi}{3}] \times [\frac{\pi}{3}, \frac{2\pi}{3}]$

TABLE I

γ_{min}'s values and controller gains. Example 1.

methods	γ_{min}	K	
[3]	1.9975	1.0332	1.0232
Theorem 2	0.5010	0.8121	0.5415

$$B_{11} = \begin{bmatrix} -0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.5 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, C_1 = \begin{bmatrix} -0.3 & 0.4 & -0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -0.5 & -1 & 1 \\ 0.6 & 0.1 & -1 \end{bmatrix}, \quad D_{11} = 0.5, \quad D_{12} = 0,$$

$$D_{21} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

Choosing $\beta = 5.5$, we obtain the results in Table I, which shows the values of γ_{min} and the controller gain matrices obtained with the full frequency approach existing in [3], and the finite frequency approach designed by Theorem 2 in this paper, with $[\theta_{m1}, \theta_{M1}] \times [\theta_{m2}, \theta_{M2}] = [\frac{\pi}{3}, \frac{2\pi}{3}] \times [\frac{\pi}{3}, \frac{2\pi}{3}]$. We can see that the proposed method provides the best results for this example. Figure 1 shows the frequency response of transfer function. It's obvious that this amplitude is less than the prescribed values of γ_{min}. The effectiveness of the designed approach is demonstrated.

Example 2 *[2] Consider the H_∞ SOF control problem of the FM second model (1) with*

$$A_1 = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0.6 & 0.1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 1 \\ 0.06 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 0.02 \\ 0.04 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.05 & 0.1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 10 \end{bmatrix}, \; D_{11} = 0.3, \; D_{12} = 0.1, \; D_{21} = 0.$$

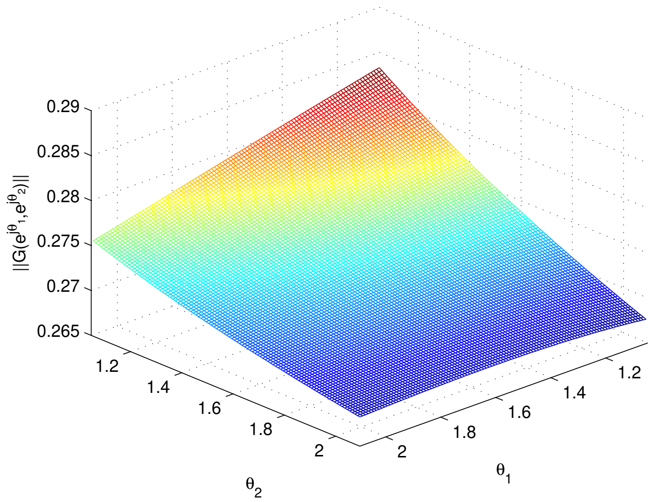

Fig. 2. Frequency response of transfer function in $[\frac{\pi}{3}, \frac{2\pi}{3}] \times [\frac{\pi}{3}, \frac{2\pi}{3}]$

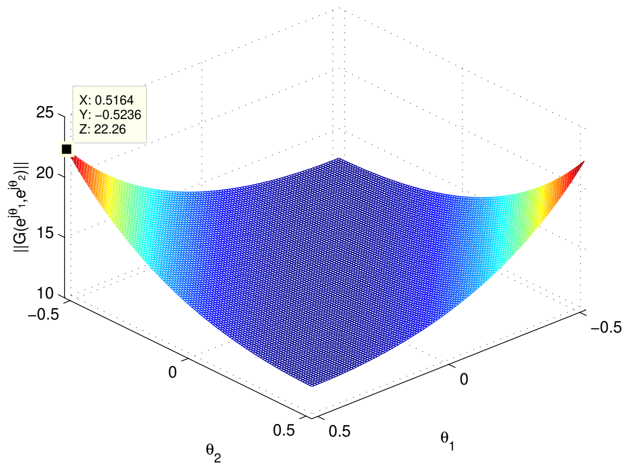

Fig. 3. Frequency response of transfer function in $[\frac{-\pi}{6}, \frac{\pi}{6}] \times [\frac{-\pi}{6}, \frac{\pi}{6}]$

TABLE II

γ_{min}'S VALUES AND CONTROLLER GAINS. EXAMPLE 2.

	[2]		Theorem 2	
ε	γ_{min}	K	γ_{min}	K
$\varepsilon = 0.6$	0.3536	−0.1238	0.3004	−0.0142
$\varepsilon = 0.8$	0.3606	−0.1222	0.3006	−0.0129
$\varepsilon = 1$	0.3883	−0.1098	0.3006	−0.0109
$\varepsilon = 1.2$	0.5356	−0.0859	0.3007	−0.0094
$\varepsilon = 1.4$	10.9041	−0.0353	0.3007	−0.0097

For different values of ε, choosing $\beta = 1$, we obtain the results in Table 2, which shows the values of γ_{min} and the controller gain matrices obtained with the full frequency approach existing in [2], and the finite frequency approach designed by Theorem 2 in this paper, with $[\theta_{m1}, \theta_{M1}] \times [\theta_{m2}, \theta_{M2}] = [\frac{\pi}{3}, \frac{2\pi}{3}] \times [\frac{\pi}{3}, \frac{2\pi}{3}]$. We can see that the proposed method provides the best results for this example. Figure 2 shows the frequency response of transfer function. It's obvious that this amplitude is less than the prescribed values of γ_{min}.

Example 3 *[4] Consider the H_∞ SOF control problem of the FM second model (1) with*

$$A_1 = \begin{bmatrix} -0.2 & 0.1 & -0.1 & 0.1 \\ -0.2 & 0.1 & 0.2 & 0.6 \\ 0.2 & -0.4 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.1 & -0.4 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.1 & 0.6 & -0.2 & 0.2 \\ 0.1 & 0.2 & 0.5 & 0 \\ -0.5 & 0.1 & -0.2 & 0.1 \\ 0.5 & 0.4 & 0.2 & 0.5 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 2.3 & 1.6 & 0.8 & 0.8 \\ 0.4 & -1.7 & 0.3 & -0.9 \\ -1.4 & 0.1 & -1.2 & -0.6 \\ 0.2 & -0.8 & 0.6 & 1.4 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} -0.3 & 1.4 & 1.2 & 0.2 \\ 0.9 & -0.4 & 1.8 & 0.2 \\ 1.5 & 0.9 & -1.7 & 1.2 \\ 0.9 & 1.5 & 0.8 & 0 \end{bmatrix},$$

TABLE III

γ_{min}'S VALUES AND CONTROLLER GAINS. EXAMPLE 3.

methods	γ_{min}	K
[4]	34.6658	not given
Theorem 2	22.3265	$\begin{bmatrix} -0.1505 & 0.1264 \\ 0.0282 & -0.0234 \\ -0.2203 & 0.0918 \\ -0.1268 & 0.0159 \end{bmatrix}$

$$B_{21} = \begin{bmatrix} 0.6 & 0 & 0.5 & -2.5 \\ -0.9 & -0.5 & -1.2 & -0.3 \\ 0.2 & 1.0 & 0 & 0.6 \\ 0.6 & 1.6 & -0.1 & 0.7 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} -2.0 & 0.7 & 0.8 & -0.4 \\ -0.3 & -1.0 & -0.2 & 0.6 \\ 0.8 & 0.9 & 0 & 0.1 \\ -0.6 & 1.4 & -0.6 & 0.2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.9 & 1.0 & -0.1 & -1.2 \\ 0.6 & 0.1 & 1.9 & 0.2 \\ 0.2 & -1.3 & -0.2 & -0.4 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.8 & -0.1 & -0.3 & -1.4 \\ 1.2 & 1.0 & -0.5 & -0.5 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 1.4 & -0.6 & 0 & -0.1 \\ 0.3 & -0.5 & 0.2 & -1.6 \\ -0.7 & 0.9 & 0.2 & 1.4 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} -1.6 & 0.7 & -0.2 & 0.8 \\ -2.6 & 0 & 0.4 & -0.8 \\ -0.4 & -0.3 & 0.4 & -0.9 \end{bmatrix},$$

Choosing $\beta = 3.3$, we obtain the results in Table III, which shows the values of γ_{min} obtained with the full frequency approach existing in [4], and γ_{min} and the controller gain matrices obtained with the finite frequency approach, designed by Theorem 2 in this paper, with $[\theta_{m1}, \theta_{M1}] \times [\theta_{m2}, \theta_{M2}] = [\frac{-\pi}{6}, \frac{\pi}{6}] \times [\frac{-\pi}{6}, \frac{\pi}{6}]$. We can see that the proposed method provides the best results for this example.

Figure 3 shows the frequency response of transfer function. It's obvious that this amplitude is less than the prescribed value of γ_{min}. The effectiveness of the designed approach is demonstrated.

IV. Conclusions

This paper has investigated the problem of finite frequency H_∞ static output feedback controller desing for 2-D discrete systems in Fornasini-Marchesini FM second model. Using the GKYP lemma, new condition for the existence of SOF control is proposed in order to overcome the drawback induced by the previous studies. Controller gains can be obtained by solving a set of strict LMIs. Finally, numerical examples are proposed to illustrate the effectiveness of the results.

Acknowledgement: Partly Funded by Conserjería de Educación of Junta de Castilla y León and EU-FEDER (CLU-2017-09, VA232P18, UIC 225) and Agencia Estatal de Investigacion (PID2020-112871RB-C21 / AEI / 10.13039/501100011033).

References

[1] C. Du and L. Xie, H_∞ Control and Filtering of Two-Dimensional Systems, Springer-Verlag, Berlin Heidelberg; 2002.

[2] Feng, Zhi-Yong, Li Xu, and Yoshihisa Anazawa. Sufficient LMI conditions for H8 static output feedback control of 2-D systems. Control Automation Robotics & Vision (ICARCV), 2010 11th International Conference on. IEEE, 2010.

[3] Feng, Zhi-Yong, et al. H_∞ Static Output Feedback Control of 2-D Discrete Systems in FM Second Model. Asian Journal of Control 14.6 (2012): 1505-1513.

[4] Feng, Zhi-Yong, and Li Xu. Optimization of coordinate transformation matrix for H_∞ static-output-feedback control of 2-D discrete systems in FM second model. Multidimensional Systems and Signal Processing (2017): 1-11.

[5] Li, Xianwei, Huijun Gao, and Changhong Wang. Generalized Kalman-Yakubovich-Popov lemma for 2-D FM LSS model. IEEE Transactions on Automatic Control 57.12 (2012): 3090-3103.

[6] Feng, Zhi-Yong, and Li Xu. H_∞static output feedback controller design for two-dimensional discrete systems in Roesser model. Computer-Aided Control System Design (CACSD), 2010 IEEE International Symposium on. IEEE, 2010.

[7] P. Gahinet, P. Apkarian, A linear matrix inequality approach to H_∞ control, International Journal of Robust and Nonlinear Control 4 421-448, (1994).

[8] T. Iwasaki, S. Hara, Generalized KYP lemma: Unified frequency domain inequalities with design applications, IEEE Trans. Autom. Control, vol. 50, no. 1, pp. 41-59, (Jan. 2005).

[9] Feng, Zhi-Yong, et al. "Optimization of Coordinate Transformation Matrix for H_∞ Static-Output-Feedback Control of Linear Discrete-Time Systems." Asian Journal of Control 17.2 (2015): 604-614.

[10] Chang, X. H., Zhang, L., and Park, J. H. (2015). Robust static output feedback H_∞ control for uncertain fuzzy systems. Fuzzy Sets and Systems, 273, 87-104.

[11] Er rachid, I., & Hmamed, A. (2017). Stability and Robust Stabilization of 2-D Continuous Systems in Roesser Model Based on GKYP Lemma. International Journal of Power Electronics and Drive Systems (IJPEDS), 8(3), 990-1001.

[12] Er rachid, I., & Hmamed, A. (2017). Stability of 2-D Continuous Systems in Roesser Model Based on KYP Lemma. IEEE Conferences, 14th International Multi-Conference on Systems, Signals & Devices (SSD), 554-559. DOI: 10.1109/SSD.2017.8166968.

[13] Er Rachid, I., Chaibi, R., Tissir, E.H., Hmamed, A. and TADEO, F., 2019, October. Finite Frequency H_∞ Static Output Feedback Control for 2-D Systems. In 2019 8th International Conference on Systems and Control (ICSC) (pp. 340-345). IEEE.

[14] Er-Rachid I, Chaibi R, Tissir EH, Hmamed A. Robust finite frequency H static output feedback control for uncertain fuzzy systems via a sum-of-squares approach. Optim Control Appl Meth. 2021;1{20. https://doi.org/10.1002/oca.2785.

[15] Er Rachid, I., Chaibi, R. and Hmamed, A., 2019. Stability of 2-D Continuous TS Fuzzy Systems Based on KYP Lemma. Systems, Automation, and Control, 9, p.267.

[16] Er Rachid, I., Chaibi, R., Tissir, E.H. and Hmamed, A., 2018. Observer-based H_∞ control with finite frequency specifications for discrete-time T-S fuzzy systems. International Journal of systems science, 49(16), pp.3307-3319.

[17] Er Rachid, I., Chaibi, R., El Haiek, B., Tissir, E.H. and Hmamed, A., 2018, May. Robust observer-based controller design for uncertain discrete-time systems in finite frequency domain. In 2018 6th International Conference on Multimedia Computing and Systems (ICMCS) (pp. 1-6). IEEE.

[18] Chaibi, R., Er Rachid, I., Tissir, E.H. and Hmamed, A., 2019. Finite-frequency static output feedback H_∞ control of continuous-time T-S fuzzy systems. Journal of Circuits, Systems and Computers, 28(02), p.1950023.

[19] Chaibi, R., Er Rachid, I., Tissir, E.H. and Hmamed, A., 2019. Finite frequency H_∞ control for 2-D TS fuzzy FM II model with stochastic perturbation. International Journal of Systems Science, 50(15), pp.2785-2801.

[20] Z. Duan, Z. Xiang. Finite frequency H_∞ control of 2-D continuous systems in Roesser model, Multidim Syst. Sign. Process, (2016).

[21] R. Yang ,L. Xie, and C. Zhang, Generalized two−dimensional Kalman − Yakubovich − Popov lemma for discrete Roesser model. IEEE Transactions on Circuits and Systems I: Regular Papers, 55(10), 3223−3233, (2008).

[22] X. Li, J. Lam, H. Gao, Y. Gu, A frequency-partitioning approach to stability analysis of two-dimensional discrete systems. Multidimensional Systems and Signal Processing, 26(1), 67−93, (2013).

[23] Li, Xianwei, and Huijun Gao. "Robust finite frequency H_∞ filtering for uncertain 2-D Roesser systems." Automatica 48.6 (2012): 1163-1170.

[24] Li, Xianwei, and Huijun Gao. "Robust finite frequency H_∞ filtering for uncertain 2-D systems: The FM model case." Automatica 49.8 (2013): 2446-2452.

[25] Xie, L., Du, C., Soh, Y. C., & Zhang, C. (2002). H_∞ and robust control of 2-D systems in FM second model. Multidimensional Systems and Signal Processing, 13, 265.287.

[26] Fornasini, Ettore, and Giovanni Marchesini. State-space realization theory of two-dimensional filters. IEEE Transactions on Automatic Control 21.4 (1976): 484-492.

[27] Du, Chunling, Lihua Xie, and Cishen Zhang. H_∞ control and robust stabilization of two-dimensional systems in Roesser models. Automatica 37.2 (2001): 205-211.

[28] El Ghaoui, Laurent, Francois Oustry, and Mustapha AitRami. "A cone complementarity linearization algorithm for static output-feedback and related problems." IEEE transactions on automatic control 42.8 (1997): 1171-1176.