Generalized Pitchfork Bifurcations in D-Concave Nonautonomous Scalar Ordinary Differential Equations

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Abstract
The global bifurcation diagrams for two different one-parametric perturbations ($+\lambda x$ and $+\lambda x^2$) of a dissipative scalar nonautonomous ordinary differential equation $x' = f(t, x)$ are described assuming that 0 is a constant solution, that f is recurrent in t, and that its first derivative with respect to x is a strictly concave function. The use of the skewproduct formalism allows us to identify bifurcations with changes in the number of minimal sets and in the shape of the global attractor. In the case of perturbation $+\lambda x$, a so-called *generalized pitchfork bifurcation* may arise, with the particularity of lack of an analogue in autonomous dynamics. This new bifurcation pattern is extensively investigated in this work.

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1 Introduction

The interest that the description of nonautonomous bifurcation patterns arouses in the scientific community has increased significantly in recent years, as evidenced by the works [1–3, 5, 11–13, 15, 17, 20, 21, 24, 25, 27, 29–32], and references therein. This paper constitutes an extension of the work initiated in [11], were we describe several possibilities for the global bifurcation diagrams of certain types of one-parametric variations of a coercive equation. We make use of the skewproduct formalism, which allows us to understand bifurcations as changes in the number of minimal sets and in the shape of the global attractor, which of course give rise to substantial changes in the global dynamics.

Let us briefly describe the skewproduct formalism. Standard boundedness and regularity conditions ensure that the hull Ω of a continuous map $f_0 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as the closure of the set of time-shifts $\{f_0 \cdot t \colon t \in \mathbb{R}\}$ in a suitable topology of $C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, is a compact metric space, and that the map $\sigma \colon \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega \cdot t$ (where, as in the case of f_0, $(\omega \cdot t)(s, x) = \omega(t + s, x)$) defines a global continuous flow. The continuous function $f(\omega, x) = \omega(0, x)$ provides the family of equations

$$x' = f(\omega \cdot t, x), \qquad \omega \in \Omega, \tag{1.1}$$

which includes $x' = f_0(t, x)$: it corresponds to $\omega = f_0 \in \Omega$. When, in addition, f_0 satisfies some properties of recurrence in time, the flow (Ω, σ) is minimal, which means that Ω is the hull of any of its elements. If $u(t, \omega, x)$ is the solution of $(1.1)_\omega$ with $u(0, \omega, x) = x$, then $\tau \colon \mathcal{U} \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$, $(t, \omega, x) \mapsto (\omega \cdot t, u(t, \omega, x))$ defines a (possibly local) flow on $\Omega \times \mathbb{R}$ of skewproduct type: it projects over the flow (Ω, σ). If f_0 is coercive with respect to x uniformly in $t \in \mathbb{R}$, so is f uniformly in $\omega \in \Omega$, and this ensures the existence of the global attractor and of at least one minimal compact τ-invariant subset of $\Omega \times \mathbb{R}$. In the simplest nonautonomous cases, the minimal subsets are (hyperbolic or nonhyperbolic) graphs of continuous functions, and thus they play the role performed by the critical points of an autonomous equation; but there are cases in which both the shape of a minimal set and the dynamics on it are extremely complex, without autonomous analogues, and therefore impossible bifurcation scenarios for a autonomous equation can appear in the nonautonomous setting.

So, we take as starting points a (global) continuous minimal flow (Ω, σ) and a continuous map $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$, assume that f is coercive in x uniformly on Ω, and define the dissipative flow τ. Throughout this paper, we also assume that the derivatives f_x and f_{xx} globally exist and are jointly continuous on $\Omega \times \mathbb{R}$, as well as the fundamental property of strict concavity of f_x with respect to x: d-concavity. Not all these conditions are in force to obtain the results of [11], but, for simplicity, we also assume them all to summarize part of the properties there proved.

The first goal in [11] is to describe the possibilities for the μ-bifurcation diagram of the one-parametric family $x' = f(\omega \cdot t, x) + \mu$, with global attractor $\mathcal{A}_\mu$. In particular, it is proved that, if there exist three minimal sets for a value $\mu_0 \in \mathbb{R}$ of the parameter, then: $\mathcal{A}_\mu$ contains three (hyperbolic) minimal sets if and only if μ belongs to a nondegenerate interval (μ_-, μ_+); the two upper (resp. lower) minimal sets collide on a residual invariant subset of Ω when $\mu \downarrow \mu_-$ (resp. $\mu \uparrow \mu_+$); and $\mathcal{A}_\mu$ reduces to the (hyperbolic) graph of a continuous map on Ω if $\mu \notin [\mu_-, \mu_+]$. That is, the bifurcation diagram presents at μ_- and μ_+ two *local saddle-node bifurcation points* of minimal sets and two points of discontinuity of $\mathcal{A}_\mu$: is the nonautonomous analogue of the bifurcation diagram of $x' = -x^3 + x + \mu$.

A second type of perturbation is considered in [11], namely $x' = f(\omega \cdot t, x) + \lambda x$, with global attractor $\mathcal{A}_\lambda$, under the additional assumption $f(\cdot, 0) \equiv 0$. Now, $\mathcal{M}_0 = \Omega \times \{0\}$ is a

minimal set for all λ, and its hyperbolicity properties are determined by the Sacker and Sell spectrum $[-\lambda_+, -\lambda_-]$ of the map $\omega \mapsto f_x(\omega, 0)$. Two possible global bifurcation diagrams are described, and some conditions ensuring their occurrence are given. The first one is the *classical pitchfork bifurcation diagram*, with unique bifurcation point λ_+: $\mathcal{M}_0$ is the unique minimal set for $\lambda \leq \lambda_+$, and two more (hyperbolic) minimal sets occur for $\lambda > \lambda_+$, which collide with $\mathcal{M}_0$ as $\lambda \downarrow \lambda_+$. An autonomous analogue is the diagram of $x' = -x^3 + \lambda x$. The second one is the *local saddle-node and transcritical bifurcation diagram*, with a local saddle node bifurcation of minimal sets at a point $\lambda_0 < \lambda_-$ and a so-called *generalized transcritical bifurcation* of minimal sets around $\mathcal{M}_0$. We will describe this diagram in detail in the next pages, pointing out now the most remarkable fact: $\mathcal{M}_0$ collides with another (hyperbolic) minimal set as $\lambda \uparrow \lambda_-$ and as $\lambda \downarrow \lambda_+$, and it is the unique minimal set lying on a band $\Omega \times [-\rho, \rho]$ for a $\rho > 0$ if $\lambda \in [\lambda_-, \lambda_+]$. This local transcritical bifurcation becomes classical if $\lambda_- = \lambda_+$, being $x' = -x^3 + 2x^2 + \lambda x$ an autonomous example of this situation.

This analysis of the family $x' = f(\omega{\cdot}t, x) + \lambda x$ initiated in [11] is now completed: the main goal in this paper is to describe *all* the possibilities for its bifurcation diagram. Besides the two described ones, only a third situation may arise: a *generalized pitchfork bifurcation diagram*, just possible when $\lambda_- < \lambda_+$. It is characterized by the existence of two bifurcation points, $\lambda_0 \in [\lambda_-, \lambda_+)$ and λ_+: $\mathcal{M}_0$ is the unique minimal set for $\lambda < \lambda_0$, there are two of them for $\lambda \in (\lambda_0, \lambda_+]$, and there are three for $\lambda > \lambda_+$. The lack of an autonomous analogue raises a nontrivial question: does this bifurcation diagram correspond to some actual family? We also answer it, explaining how to construct nonautonomous patterns fitting at each one of the described possibilities. Furthermore, we prove that, given $\lambda_- < \lambda_+$, any $\lambda_0 \leq \lambda_+$ is the first bifurcation point of a suitable family $x' = g_{\lambda_0}(\omega{\cdot}t, x) + \lambda x$ with Sacker and Sell spectrum of $(g_{\lambda_0})_x(\cdot, 0)$ given by $[-\lambda_+, -\lambda_-]$, and that the three possible diagrams actually occur: they correspond to $\lambda_0 < \lambda_-$, $\lambda_0 = \lambda_+$ and $\lambda_0 \in [\lambda_-, \lambda_+)$. As a tool to prove of this last result, we analyze the bifurcation possibilities for a new one-parametric family, namely $x' = f(\omega{\cdot}t, x) + \xi x^2$. In order not to lengthen this introduction too much, we omit here the (self-interesting) description of the bifurcation possibilities for this case, and refer the reader to Sect. 6.

These are the main results of this paper, which presents more detailed descriptions in some particular cases. Its contents are organized in five sections. Section 2 contains the basic notions and properties required to start with the analysis. Section 3 is devoted to the description of the three mentioned possibilities for the bifurcation diagrams of $x' = f(\omega{\cdot}t, x) + \lambda x$. In Sect. 4, we focus on the case of a cubic polynomial $f(\omega, x) = -a_3(\omega)x^3 + a_2(\omega)x^2 + a_1(\omega)x$ with strictly positive a_3, and show how some suitable properties of the coefficients a_1, a_2 and a_3 and some factible relations among them either preclude or ensure each one of the three different bifurcation diagrams. Section 5 extends these results to more general functions $f(\omega, x) = (-a_3(\omega) + h(\omega, x))x^3 + a_2(\omega)x^2 + a_1(\omega)x$, describing in this way other patterns fitting each one of the possibilities. And Sect. 6 begins with the description of the possibilities for the bifurcation diagrams of $x' = f(\omega{\cdot}t, x) + \xi x^2$ to conclude with the consequence mentioned at the end of the previous paragraph.

2 Preliminaries

Throughout the paper, the map $\sigma: \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma_t(\omega) = \omega{\cdot}t$ defines a global continuous flow on a compact metric space Ω, and we assume that the flow (Ω, σ) is minimal, that is, that every σ-orbit is dense in Ω. This paper will be focused on describing the bifurcation

diagrams of simple parametric variations of the family

$$x' = f(\omega{\cdot}t, x), \quad \omega \in \Omega, \tag{2.1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be jointly continuous, f_x and f_{xx} are supposed to exist and to be jointly continuous (which we represent as $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$), and $f(\omega, 0) = 0$ for all $\omega \in \Omega$ (that is, $x \equiv 0$ solves the equation). If only f and f_x are assumed to exist and to be jointly continuous, then we shall say that $f \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$. Additional coercivity and concavity properties will be assumed throughout the paper. In Sect. 4, we focus on the case in which $f(\omega, x)$ is a cubic polynomial in the state variable x with strictly negative cubic coefficient.

We develop our bifurcation theory through the skewproduct formalism: as explained in the Introduction, our bifurcation analysis studies the variations on the global attractors and on the number and structure of minimal sets for the corresponding parametric family of skewproduct flows. In the next subsections, we summarize the most basic concepts and some basic results required in the formulations and proofs of our results. The interested reader can find in Sect. 2 of [11] more details on these matters, as well as a suitable list of references.

2.1 Scalar Skewproduct Flow

We define the *local skewproduct flow* on $\Omega \times \mathbb{R}$ induced by (2.1) as

$$\tau : \mathcal{U} \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}, \quad (t, \omega, x_0) \mapsto (\omega{\cdot}t, u(t, \omega, x_0)), \tag{2.2}$$

where $\mathcal{I}_{\omega, x_0} \to \mathbb{R}$, $t \mapsto u(t, \omega, x_0)$ is the maximal solution of (2.1) with initial value $u(0, \omega, x_0) = x_0$, and $\mathcal{U} = \bigcup_{(\omega, x_0) \in \Omega \times \mathbb{R}} (\mathcal{I}_{\omega, x_0} \times \{(\omega, x_0)\})$. The flow τ projects on (Ω, σ), which is called its *base flow*. It is known that u satisfies the *cocycle property* $u(t + s, \omega, x_0) = u(t, \omega{\cdot}s, u(s, \omega, x_0))$ whenever the right-hand term is defined; and, clearly, the flow τ is fiber-monotone: if $x_1 < x_2$, then $u(t, \omega, x_1) < u(t, \omega, x_2)$ for any t in the common interval of definition of both solutions.

2.2 Functions of Bounded Primitive

Throughout this paper, the space of continuous functions from Ω to $\mathbb{R}$ will be represented by $C(\Omega)$, the subspace of functions $a \in C(\Omega)$ such that $\int_{\Omega} a(\omega) \, dm = 0$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ will be represented by $C_0(\Omega)$, the subspace of functions $a \in C(\Omega)$ such that the map $t \mapsto a_\omega(t) = a(\omega{\cdot}t)$ is continuously differentiable on $\mathbb{R}$ will be represented by $C^1(\Omega)$ (in this case we shall represent $a'(\omega) = a'_\omega(0)$), and the subspace of functions $a \in C(\Omega)$ with continuous primitive, that is, such that there exists $b \in C^1(\Omega)$ with $b' = a$, will be represented by $CP(\Omega)$.

It is frequent to refer to a function $a \in CP(\Omega)$ as "with bounded primitive". Let us explain briefly the reason. Recall that (Ω, σ) is minimal. Then, $a \in CP(\Omega)$ if and only if there exists $\omega_0 \in \Omega$ such that the map $a_0 : \mathbb{R} \to \mathbb{R}$, $t \mapsto a(\omega_0{\cdot}t)$ has a bounded primitive $b_0(t) = \int_0^t a_0(s) \, ds$, in which case this happens for all $\omega \in \Omega$ (see e.g. Lemma 2.7 of [16] or Proposition A.1 of [19]).

It follows from Birkoff's Ergodic Theorem that $CP(\Omega) \subseteq C_0(\Omega)$. It is well known that $CP(\Omega) = C_0(\Omega)$ if (Ω, σ) is periodic, and Lemma 5.1 of [7] ensures that $CP(\Omega)$ is a dense subset of $C_0(\Omega)$ of first category in $C_0(\Omega)$ if (Ω, σ) is a minimal aperiodic flow. In addition, $C^1(\Omega)$ is dense in $C(\Omega)$ (see Sect. 2 of [34]).

2.3 Equilibria and Semiequilibria

Let $\beta : \Omega \to \mathbb{R}$ be such that $u(t, \omega, \beta(\omega))$ is defined for all $\omega \in \Omega$ and $t \geq 0$ (resp. $t \leq 0$). We shall say that β is a τ-*equilibrium* if $\beta(\omega{\cdot}t) = u(t, \omega, \beta(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, a τ-*subequilibrium* (resp. *time-reversed τ-subequilibrium*) if $\beta(\omega{\cdot}t) \leq u(t, \omega, \beta(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$ (resp. $t \leq 0$), and a τ-*superequilibrium* (resp. *time-reversed τ-superequilibrium*) if $\beta(\omega{\cdot}t) \geq u(t, \omega, \beta(\omega))$ for all $\omega \in \Omega$ and all $t \geq 0$ (resp. $t \leq 0$). Superequilibria and subequilibria are generally denominated *semiequilibria*. We will frequently omit the reference to τ if there is no risk of confusion.

Given a Borel measure m on Ω, we shall say that $\beta : \Omega \to \mathbb{R}$ is *m-measurable* if it is measurable with respect to the m-completion of the Borel σ-algebra, and we shall say that $\beta : \Omega \to \mathbb{R}$ is C^1 *along the base orbits* if $t \mapsto \beta_\omega(t) = \beta(\omega{\cdot}t)$ is C^1 on $\mathbb{R}$ for all $\omega \in \Omega$, in which case we represent $\beta'(\omega) = \beta'_\omega(0)$. Note that any τ-equilibrium is C^1 along the base orbits. We shall say that $\beta : \Omega \to \mathbb{R}$ is a *semicontinuous equilibrium* (resp. *semiequilibrium*) if it is an equilibrium (resp. semiequilibrium) and a bounded semicontinuous map. A *copy of the base for the flow τ* is the graph of a continuous τ-equilibrium.

We shall say that a τ-superequilibrium (resp. time-reversed τ-superequilibrium) $\beta : \Omega \to \mathbb{R}$ is *strong* if there exists $s_* > 0$ (resp. $s_* < 0$) such that $\beta(\omega{\cdot}s_*) > u(s_*, \omega, \beta(\omega))$ for all $\omega \in \Omega$, and we shall say that a τ-subequilibrium (resp. time-reversed τ-subequilibrium) is *strong* if there exists $s_* > 0$ (resp. $s_* < 0$) such that $\beta(\omega{\cdot}s_*) < u(s_*, \omega, \beta(\omega))$ for all $\omega \in \Omega$. Since (Ω, σ) is minimal, Proposition 4.3 of [23] and its analogue in the time-reversed cases show that, if β is a semicontinuous strong τ-superequilibrium (resp. time-reversed τ-superequilibrium), then there exist $e_0 > 0$ and $s^* > 0$ (resp. $s^* < 0$) such that $\beta(\omega) \geq u(s^*, \omega{\cdot}(-s^*), \beta(\omega{\cdot}(-s^*))) + e_0$ for all $\omega \in \Omega$, and if β is a semicontinuous strong τ-subequilibrium (resp. time-reversed τ-subequilibrium), then there exist $e_0 > 0$ and $s^* > 0$ (resp. $s^* < 0$) such that $\beta(\omega) \leq u(s^*, \omega{\cdot}(-s^*), \beta(\omega{\cdot}(-s^*))) - e_0$ for all $\omega \in \Omega$.

Let $\beta : \Omega \to \mathbb{R}$ be C^1 along the base orbits. The map β shall be said to be a *global upper* (resp. *lower*) *solution* of (2.1) if $\beta'(\omega) \geq f(\omega, \beta(\omega))$ (resp. $\beta'(\omega) \leq f(\omega, \beta(\omega))$) for every $\omega \in \Omega$, and to be *strict* if the previous inequalities are strict for all $\omega \in \Omega$. Some comparison arguments prove the following facts (see Sects. 3 and 4 of [23]): if every forward τ-semiorbit is globally defined, then β is a τ-superequilibrium (resp. τ-subequilibrium) if and only if is a global upper (resp. lower) solution of (2.1), and it is strong as superequilibrium (resp. subequilibrium) if it is strict as global upper (resp. lower) solution. Analogously, if every backward τ-semiorbit is globally defined, then β is a time-reversed τ-subequilibrium (resp. time-reversed τ-superequilibrium) if and only if it is a global upper (resp. lower) solution of (2.1), and it is strong as time-reversed subequilibrium (resp. time-reversed superequilibrium) if it is strict as global upper (resp. lower) solution.

2.4 Minimal Sets, Coercivity and Global Attractor

A set $\mathcal{K} \subseteq \Omega \times \mathbb{R}$ is τ-*invariant* if it is composed by globally defined τ-orbits, and it is *minimal* if it is compact, τ-invariant and it does not contain properly any other compact τ-invariant set. Let us recall some properties of compact τ-invariant sets and minimal sets for the local skewproduct flow $(\Omega \times \mathbb{R}, \tau)$ over a minimal base (Ω, σ). Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a compact τ-invariant set. Since (Ω, σ) is minimal, $\mathcal{K}$ projects onto Ω, that is, the continuous map $\pi : \mathcal{K} \to \Omega, (\omega, x) \mapsto \omega$ is surjective. In addition,

$$\mathcal{K} \subseteq \bigcup_{\omega \in \Omega} \left(\{\omega\} \times [\alpha_\mathcal{K}(\omega), \beta_\mathcal{K}(\omega)] \right),$$

where $\alpha_{\mathcal{K}}(\omega) = \inf\{x \in \mathbb{R}: (\omega, x) \in \mathcal{K}\}$ and $\beta_{\mathcal{K}}(\omega) = \sup\{x \in \mathbb{R}: (\omega, x) \in \mathcal{K}\}$ are, respectively, lower and upper semicontinuous τ-equilibria whose graphs are contained in $\mathcal{K}$. In particular, the residual sets of their continuity points are σ-invariant. They will be called the *lower and upper delimiter equilibria* of $\mathcal{K}$. The compact τ-invariant set $\mathcal{K}$ is said to be *pinched* if there exists $\omega \in \Omega$ such that the section $(\mathcal{K})_\omega = \{x: (\omega, x) \in \mathcal{K}\}$ is a singleton. A τ-minimal set $\mathcal{M} \subset \Omega \times \mathbb{R}$ is said to be *hyperbolic attractive* (resp. *repulsive*) if it is uniformly exponentially asymptotically stable at ∞ (resp. $-\infty$). Otherwise, it is said to be *nonhyperbolic*.

A function $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be *coercive* ((Co) for short) if

$$\lim_{|x| \to \infty} \frac{f(\omega, x)}{x} = -\infty$$

uniformly on Ω. A stronger definition of coercivity will be needed in part of Sect. 5 and in Sect. 6: $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be *2-coercive* ((Co)$_2$ for short) if

$$\lim_{x \to \pm\infty} \frac{f(\omega, x)}{x^2} = \mp\infty,$$

uniformly on Ω. It is clear that, if f is (Co)$_2$, then it is (Co). The arguments leading to Theorem 16 of [8] (see also Sect. 1.2 of [9]) ensure that, if $f \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ is (Co), then the flow τ is globally forward defined and admits a global attractor. That is, a compact τ-invariant set $\mathcal{A}$ which satisfies $\lim_{t \to \infty} \operatorname{dist}(\tau_t(\mathcal{C}), \mathcal{A}) = 0$ for every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$, where $\tau_t(\mathcal{C}) = \{(\omega{\cdot}t, u(t, \omega, x_0)): (\omega, x_0) \in \mathcal{C}\}$ and

$$\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(\omega_1, x_1) \in \mathcal{C}_1} \left(\inf_{(\omega_2, x_2) \in \mathcal{C}_2} \left(\operatorname{dist}_{\Omega \times \mathbb{R}}((\omega_1, x_1), (\omega_2, x_2)) \right) \right).$$

In addition, the attractor takes the form $\mathcal{A} = \bigcup_{\omega \in \Omega}(\{\omega\} \times [\alpha_{\mathcal{A}}(\omega), \beta_{\mathcal{A}}(\omega)])$ and is composed by the union of all the globally defined and bounded τ-orbits. And, as proved in Theorem 5.1(iii) of [11], any global (strict) lower solution κ satisfies $\kappa \leq \beta_{\mathcal{A}}$ ($\kappa < \beta_{\mathcal{A}}$) and a (strict) upper solution κ satisfies $\kappa \geq \alpha_{\mathcal{A}}$ ($\kappa > \alpha_{\mathcal{A}}$). Besides, if ω_1 (resp. ω_2) is a continuity point of $\alpha_{\mathcal{A}}$ (resp. $\beta_{\mathcal{A}}$), then the set $\mathcal{M}^l = \operatorname{cl}_{\Omega \times \mathbb{R}}\{(\omega_1{\cdot}t, \alpha_{\mathcal{A}}(\omega_1{\cdot}t)): t \in \mathbb{R}\}$ (resp. $\mathcal{M}^u = \operatorname{cl}_{\Omega \times \mathbb{R}}\{(\omega_2{\cdot}t, \beta_{\mathcal{A}}(\omega_2{\cdot}t)): t \in \mathbb{R}\}$) is the lower (resp. upper) τ-minimal set, and its sections reduce to the points $\alpha_{\mathcal{A}}(\omega)$ (resp. $\beta_{\mathcal{A}}(\omega)$) at all the continuity points ω of $\alpha_{\mathcal{A}}$ (resp. $\beta_{\mathcal{A}}$): see Theorem 3.3 of [6]. Moreover, it is easy to check by contradiction that, if $\mathcal{M}^l$ (resp. $\mathcal{M}^u$) is hyperbolic, then it is attractive and it coincides with the graph of $\alpha_{\mathcal{A}}$ (resp. $\beta_{\mathcal{A}}$), which therefore is a continuous map.

2.5 Measures and Lyapunov Exponents

We shall say that a Borel measure m on Ω is *normalized* if $m(\Omega) = 1$, that it is *σ-invariant* if $m(\sigma_t(\mathcal{B})) = m(\mathcal{B})$ for every $t \in \mathbb{R}$ and every Borel subset $\mathcal{B} \subseteq \Omega$, and that it is *$\sigma$-ergodic* if it is normalized, σ-invariant and $m(\mathcal{B}) \in \{0, 1\}$ for every σ-invariant subset $\mathcal{B} \subseteq \Omega$. $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ and $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ are, respectively, the nonempty sets of normalized σ-invariant and σ-ergodic Borel measures on Ω. The flow (Ω, σ) is said to be *uniquely ergodic* if $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ reduces to just one element m, in which case m is ergodic; and it is said to be *finitely ergodic* if $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ is a finite set.

The *Lyapunov exponent* of $a \in C(\Omega)$ with respect to $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ is

$$\gamma_a(\Omega, m) = \int_\Omega a(\omega) \, dm.$$

The family of scalar linear differential equations $x' = a(\omega \cdot t)\, x$ has *exponential dichotomy* over Ω if there exist $k \geq 1$ and $\delta > 0$ such that either

$$\exp \int_0^t a(\omega \cdot s)\, ds \leq k e^{-\delta t} \text{ whenever } \omega \in \Omega \text{ and } t \geq 0$$

or

$$\exp \int_0^t a(\omega \cdot s)\, ds \leq k e^{\delta t} \text{ whenever } \omega \in \Omega \text{ and } t \leq 0\,.$$

The set of $\lambda \in \mathbb{R}$ such that the family $x' = (a(\omega \cdot t) - \lambda)\, x$ does not have exponential dichotomy over Ω is called the *Sacker and Sell spectrum of* $a \in C(\Omega)$, and represented by $\mathrm{sp}(a)$. Recall that Ω is connected, since (Ω, σ) is minimal. The arguments in [18] and [33] show the existence of $m^l, m^u \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ such that $\mathrm{sp}(a) = \big[\gamma_a(\Omega, m^l), \gamma_a(\Omega, m^u)\big]$, and also that $\int_\Omega a(\omega)\, dm \in \mathrm{sp}(a)$ for any $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$. We shall say that a has *band spectrum* if $\mathrm{sp}(a)$ is a nondegenerate interval and that a has *point spectrum* if $\mathrm{sp}(a)$ reduces to a point. As seen in Sect. 2.2, $\mathrm{sp}(a) = \{0\}$ if $a \in C_0(\Omega)$.

On the other hand, assume that $f \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$, where f is the function on the right hand side of (2.1). The *Lyapunov exponent* of a compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ with respect to $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K}, \tau)$ is

$$\gamma_{f_x}(\mathcal{K}, \nu) = \int_{\mathcal{K}} f_x(\omega, x)\, d\nu\,.$$

We will frequently omit the subscript f_x if no confusion may arise. We will refer to the Sacker and Sell spectrum of $f_x \colon \mathcal{K} \to \mathbb{R}$ as the *Sacker and Sell spectrum of* f_x *on a compact τ-invariant set* $\mathcal{K} \subset \Omega \times \mathbb{R}$. Since (Ω, σ) is a minimal flow, a τ-minimal set $\mathcal{M} \subset \Omega \times \mathbb{R}$ is nonhyperbolic if and only if 0 belongs to the Sacker and Sell spectrum of f_x on $\mathcal{M}$. Moreover, Proposition 2.8 of [6] proves that $\mathcal{M}$ is an attractive (resp. repulsive) hyperbolic copy of the base if and only if all its Lyapunov exponents are strictly negative (resp. positive).

Theorems 1.8.4 of [4] and 4.1 of [14] provide a fundamental characterization of the set $\mathfrak{M}_{\mathrm{erg}}(\mathcal{K}, \tau)$ given by the τ-ergodic measures concentrated on a compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$: for any $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K}, \tau)$, there exists an m-measurable τ-equilibrium $\beta \colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{K}$ such that, for every continuous function $g \colon \Omega \times \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathcal{K}} g(\omega, x)\, d\nu = \int_\Omega g(\omega, \beta(\omega))\, dm\,, \tag{2.3}$$

where $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ is the ergodic measure on which ν projects, given by $m(A) = \nu((A \times \mathbb{R}) \cap \mathcal{K})$. In particular, the Lyapunov exponent on $\mathcal{K}$ for (2.1) with respect to any τ-ergodic measure projecting onto m is given by an integral of the form $\int_\Omega f_x(\omega, \beta(\omega))\, dm$. The converse also holds: any m-measurable τ-equilibrium $\beta \colon \Omega \to \mathbb{R}$ with graph in $\mathcal{K}$ defines $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K}, \tau)$ projecting onto m by (2.3). Note that β_1 and β_2 define the same measure if and only if they coincide m-a.e.

2.6 Strict D-Concavity

We shall say that $f \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ is *d-concave* ((DC) for short) if its derivative f_x is concave on $\mathbb{R}$ for all $\omega \in \Omega$. With the purpose of measuring the degree of strictness of the concavity of f_x, the standardized ϵ-modules of d-concavity of f on a compact interval J were introduced in [11], and several subsets of strictly d-concave functions of $C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ were defined in terms of these modules. In this paper, we will only be interested in the set

(SDC)$_*$ of *strictly d-concave functions with respect to every measure* (see Definition 3.8 of [11]). Proposition 3.9 of [11] gives a characterization of this set of functions which will be sufficient for the purposes of this paper: $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ is (SDC)$_*$ if and only if $m(\{\omega \in \Omega \colon f_{xx}(\omega, \cdot) \text{ is strictly decreasing on } J\}) > 0$ for every compact interval J and every $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. In particular, it can be easily checked that any polynomial of the form $p(\omega, x) = -a_3(\omega)x^3 + a_2(\omega)x^2 + a_1(\omega)x + a_0(\omega)$, where the coefficients are continuous and Ω and a_3 is nonnegative and nonzero, is (SDC)$_*$, since $p_{xx}(\omega, \cdot)$ is strictly decreasing on $\mathbb{R}$ for every ω on an open subset of Ω (recall that the minimality of (Ω, σ) ensures that every open set has positive m-measure for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$).

Assume that the function f of (2.1) is (SDC)$_*$. Following the methods of [26] and [35], Theorems 4.1 and 4.2 of [11] state relevant dynamical properties of the local skewproduct flow τ in terms of the previous properties. Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a compact τ-invariant set. Then, there exist at most three distinct τ-invariant measures of $\mathfrak{M}_{\mathrm{erg}}(\mathcal{K}, \tau)$ which project onto m. Moreover, if there exist three such measures ν_1, ν_2 and ν_3 projecting onto m, and they are respectively given by the m-measurable equilibria β_1, β_2 and β_3 (see (2.3)) with $\beta_1(\omega) < \beta_2(\omega) < \beta_3(\omega)$ for m-a.e. $\omega \in \Omega$, then $\gamma_{f_x}(\mathcal{K}, \nu_1) < 0$, $\gamma_{f_x}(\mathcal{K}, \nu_2) > 0$ and $\gamma_{f_x}(\mathcal{K}, \nu_3) < 0$. In addition, $\mathcal{K}$ contains at most three disjoint compact τ-invariant sets, and if it contains exactly three, then they are hyperbolic copies of the base: attractive the upper and lower ones, and repulsive the middle one. These properties will be often combined with those established in Proposition 5.3 of [11]: if f is coercive and either if there exists a repulsive hyperbolic τ-minimal set or if there exist two hyperbolic τ-minimal sets, then there exist three τ-minimal sets.

3 Generalized Pitchfork Bifurcation Patterns

Let (Ω, σ) be a minimal flow, and let $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ be (Co) and (SDC)$_*$, and satisfy $f(\omega, 0) = 0$ for every $\omega \in \Omega$. The description of the global bifurcation diagram for the family

$$x' = f(\omega{\cdot}t, x) + \lambda x, \quad \omega \in \Omega \tag{3.1}$$

with respect the real parameter λ was initiated in Sect. 6 of [11] and is completed in Theorem 3.1. We denote by $\mathcal{I}_{\omega,x_0}^{\lambda} \to \mathbb{R}$, $t \mapsto u_\lambda(t, \omega, x_0)$ the maximal solution of (3.1)$_\lambda$ with initial value $u_\lambda(0, \omega, x_0) = x_0$, and by τ_λ the corresponding local skewproduct flow induced by (3.1)$_\lambda$; i.e., $\tau_\lambda(t, \omega, x_0) = (\omega{\cdot}t, u_\lambda(t, \omega, x_0))$. The coercivity property ensures the existence of the global attractor $\mathcal{A}_\lambda$ for all $\lambda \in \mathbb{R}$. Theorems 5.1 and 6.2 of [11] describe the structure of $\mathcal{A}_\lambda$ and its variation with respect to λ: it can be written as

$$\mathcal{A}_\lambda = \bigcup_{\omega \in \Omega} \left(\{\omega\} \times [\alpha_\lambda(\omega), \beta_\lambda(\omega)] \right),$$

where: $\alpha_\lambda \colon \Omega \to \mathbb{R}$ and $\beta_\lambda \colon \Omega \to \mathbb{R}$ are lower and upper semicontinuous τ_λ-equilibria; $\alpha_\lambda(\omega) \leq 0 \leq \beta_\lambda(\omega)$ for all $\omega \in \Omega$; the maps $\lambda \mapsto \beta_\lambda(\omega)$ and $\lambda \mapsto \alpha_\lambda(\omega)$ are respectively nondecreasing and nonincreasing on $\mathbb{R}$ for all $\omega \in \Omega$ and both are right-continuous; $\lim_{\lambda \to \infty} \alpha_\lambda(\omega) = -\infty$ and $\lim_{\lambda \to \infty} \beta_\lambda(\omega) = \infty$ uniformly on Ω; and there exists $\lambda_* \in \mathbb{R}$ such that $\mathcal{A}_\lambda = \Omega \times \{0\}$ and it is hyperbolic attractive for every $\lambda < \lambda_*$. We represent by $\mathcal{M}_\lambda^l$ and $\mathcal{M}_\lambda^u$ the lower and upper τ_λ-minimal sets, and recall that any of them is hyperbolic if and only it is hyperbolic attractive, in which case it coincides with the graph of the corresponding delimiter equilibrium of $\mathcal{A}_\lambda$ (see Sect. 2.4).
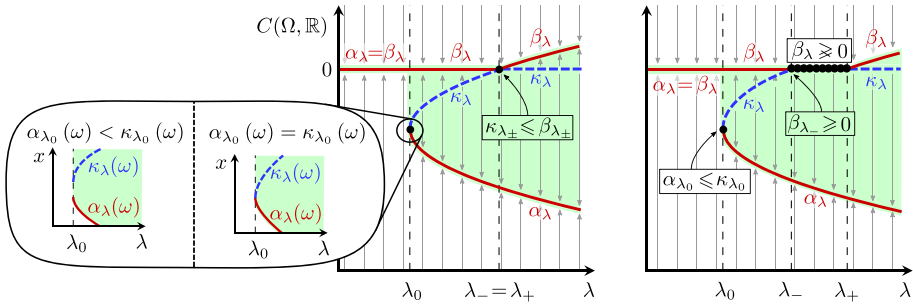
Fig. 1 Local saddle-node and transcritical bifurcation diagrams described in Theorem 3.1(i). The left one corresponds to $\lambda_- = \lambda_+$ and the right one to $\lambda_- < \lambda_+$. The solid red curves represent the families of attractive hyperbolic solutions of the λ-parametric family (3.1): α_λ for $\lambda \neq \lambda_0$ and β_λ for $\lambda \notin [\lambda_-, \lambda_+]$. The blue curve represents the family of repulsive hyperbolic solutions of (3.1): κ_λ for $\lambda \in (\mu_-, \lambda_-) \cup (\lambda_+, \infty)$. In the first diagram, a large black point with abscissa λ_0 represents the complex possibilities which arise for the collision of α_λ and κ_λ as $\lambda \downarrow \lambda_0$, which is partly explained in the left zoom: the limit maps α_{λ_0} and κ_{λ_0} are not necessarily continuous, but lower and upper semicontinuous; they take the same value for a residual invariant set of points ω; but this residual set may (not necessarily) coexist with a nonempty invariant set $\Omega_0 \subsetneq \Omega$ at whose points $\alpha_{\lambda_0}(\omega) < \kappa_{\lambda_0}(\omega)$; and the set Ω_0 can have measure 0 or 1 for an ergodic measure on Ω. The situation is analogous for all $\lambda = \lambda_\pm$ with the maps $\kappa_{\lambda_\pm}$ and $\beta_{\lambda_\pm}$ given by the limits of κ_λ as $\lambda \uparrow \lambda_-(= \lambda_+)$ and of β_λ as $\lambda \downarrow \lambda_+(= \lambda_-)$, and represented by "$\kappa_{\lambda_\pm} \leq \beta_{\lambda_\pm}$" and the large black point. The hyperbolic minimal sets are given by the graphs of the curves $\alpha_\lambda, \kappa_\lambda$ and β_λ whenever they are hyperbolic. A nonhyperbolic minimal set $\mathcal{M}^l_{\lambda_0}$ exists for λ_0, lying in the region delimited by the graphs of α_{λ_0} and κ_{λ_0}, and with a possibly highly complex shape. The situation is, again, analogous for $\lambda_-(= \lambda_+)$, and no more minimal sets exist for any λ. The green-shadowed area represents the global attractor $\mathcal{A}_\lambda$, and the light grey arrows just try to depict the attracting and repelling properties of $\alpha_\lambda, \kappa_\lambda$ and β_λ. In the second diagram, besides the features already described for the first one, $\mathcal{M}_0$ is a nonhyperbolic minimal set for $\lambda \in [\lambda_-, \lambda_+]$, and β_λ is not identically zero for $\lambda \in (\lambda_-, \lambda_+]$, as deduced from the existence of $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $\int_\Omega (f_x(\omega, 0) + \lambda) \, dm > 0$ and Proposition 5.2 of [11], which ensures that $\int_\Omega (f_x(\omega, \beta_\lambda(\omega)) + \lambda) \, dm \leq 0$. (We will use "large black points" and analogous inequalities in the remaining figures to depict similar situations, as well as red and blue "hyperbolic" curves, green-shadowing, and grey arrows) (Color figure online)

Let $\{\kappa^1_\lambda : \lambda \in (\lambda_1, \lambda_2)\}$ and $\{\kappa^2_\lambda : \lambda \in (\lambda_1, \lambda_2)\}$ be two families of τ_λ-equilbria. We will say that κ^1_λ and κ^2_λ *collide* (or that κ^1_λ collides with κ^2_λ) as $\lambda \downarrow \lambda_1$ or $\lambda \uparrow \lambda_2$ on a residual set $\mathcal{R} \subseteq \Omega$ if the limit of the difference $\kappa^1_\lambda(\omega) - \kappa^2_\lambda(\omega)$ vanishes for all $\omega \in \mathcal{R}$. The same terminology will be used for the different parametric families of flows appearing throughout the paper.

Theorem 3.1 describes the possibilities for the global bifurcation diagram, which are depicted in Figs. 1 and 2. Its proof is given after a required technical result, Lemma 3.2.

Theorem 3.1 *Let $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ be (Co) and (SDC)$_*$, and let $[-\lambda_+, -\lambda_-]$ be the Sacker and Sell spectrum of f_x on $\mathcal{M}_0 = \Omega \times \{0\}$, with $\lambda_- \leq \lambda_+$. Then, $\mathcal{M}_0$ is hyperbolic attractive (resp. repulsive) if $\lambda < \lambda_-$ (resp. $\lambda > \lambda_+$) and nonhyperbolic if $\lambda \in [\lambda_-, \lambda_+]$; τ_λ admits three different hyperbolic minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0 < \mathcal{M}^u_\lambda$ for $\lambda > \lambda_+$, where $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ are hyperbolic attractive and given by the graphs of α_λ and β_λ respectively; and either α_λ or β_λ (or both of them) collide with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$.*

Assume that this is the case for β_λ. Then, one of the following situations holds:

(i) *(Local saddle-node and transcritical bifurcations). There exists $\lambda_0 < \lambda_-$ such that: $\mathcal{A}_\lambda = \mathcal{M}_0$ for $\lambda < \lambda_0$; τ_{λ_0} admits exactly two different minimal sets $\mathcal{M}^l_{\lambda_0} < \mathcal{M}_0$, with $\mathcal{M}^l_{\lambda_0}$ nonhyperbolic; τ_λ admits three hyperbolic minimal sets $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}_0$ for $\lambda \in (\lambda_0, \lambda_-)$, where $\mathcal{M}^l_\lambda$ is hyperbolic attractive and given by the graph of α_λ, and $\mathcal{N}_\lambda$ is hyperbolic repulsive and given by the graph of a continuous map $\kappa_\lambda : \Omega \to \mathbb{R}$ which*
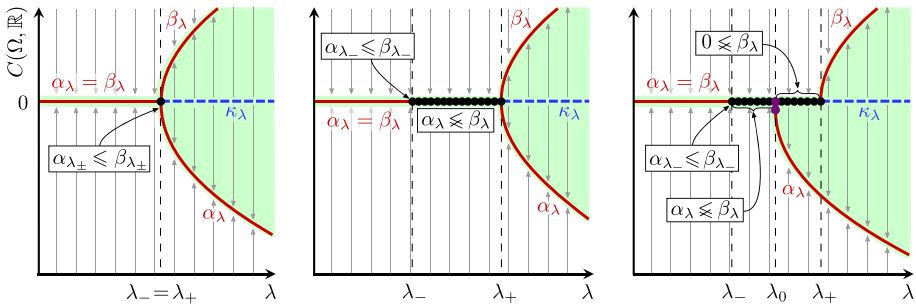
Fig. 2 The classical pitchfork bifurcation (left and center) and generalized pitchfork bifurcation (right) diagrams described in Theorem 3.1(ii) and (iii). In the last case, the possible existence of one or two nonhyperbolic minimal sets at λ_0 is depicted by a solid-filled purple eight, and it will be proved in a forthcoming paper. See Fig. 1 to understand the meaning of the remaining elements

increases strictly as λ increases in (λ_0, λ_-), and which collides with α_λ (resp. with 0) on a residual σ-invariant set as $\lambda \downarrow \lambda_0$ (resp. $\lambda \uparrow \lambda_-$); and τ_λ admits exactly two minimal sets $\mathcal{M}_\lambda^l < \mathcal{M}_0$ for $\lambda \in [\lambda_-, \lambda_+]$, where $\mathcal{M}_\lambda^l$ is hyperbolic attractive and given by the graph of α_λ. In particular, λ_0, λ_- and λ_+ are the unique bifurcation points: a local saddle-node bifurcation of minimal sets occurs around $\mathcal{M}_{\lambda_0}$ at λ_0, as well as a discontinuous bifurcation of attractors; and a classical (resp. generalized) transcritical bifurcation of minimal sets arises around $\mathcal{M}_0$ at λ_- (resp. on $[\lambda_-, \lambda_+]$) if $\lambda_- = \lambda_+$ (resp. $\lambda_- < \lambda_+$).

(ii) (Classical pitchfork bifurcation). $\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda \leq \lambda_+$, and $\mathcal{A}_\lambda = \mathcal{M}_0$ if $\lambda < \lambda_-$; and both α_λ and β_λ collide with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$. A classical pitchfork bifurcation of minimal sets arises around $\mathcal{M}_0$ at λ_+.

(iii) (Generalized pitchfork bifurcation). $\lambda_- < \lambda_+$ and there exists $\lambda_0 \in [\lambda_-, \lambda_+)$ such that: $\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda < \lambda_0$ and $\mathcal{A}_\lambda = \mathcal{M}_0$ if $\lambda < \lambda_-$; and τ_λ-admits two minimal sets $\mathcal{M}_\lambda^l < \mathcal{M}_0$ for $\lambda \in (\lambda_0, \lambda_+]$, where $\mathcal{M}_\lambda^l$ is hyperbolic attractive and given by the graph of α_λ. A generalized pitchfork bifurcation of minimal sets around $\mathcal{M}_0$ on $[\lambda_-, \lambda_+]$ arises, with λ_0 and λ_+ as bifurcation points.

The possibilities for the bifurcation diagram are symmetric with respect to the horizontal axis to those described if α_λ collides with 0 as $\lambda \downarrow \lambda_+$.

Lemma 3.2 *Let* $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ *be* (Co) *and* (SDC)$_*$ *and satisfy* $f(\omega, 0) = 0$ *for all* $\omega \in \Omega$. *Assume that there exist exactly two minimal sets* $\mathcal{M}^l < \mathcal{M}_0$ *(resp.* $\mathcal{M}^u > \mathcal{M}_0$*) for the local skewproduct flow* τ *defined by the solutions* $u(t, \omega, x_0)$ *of the family* $x' = f(\omega \cdot t, x)$, *with* $\mathcal{M}^l$ *(resp.* $\mathcal{M}^u$*) hyperbolic attractive. Assume also that* $\int_\Omega f_x(\omega, 0) \, dm \neq 0$ *for an* $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. *Then, there exist a lower semicontinuous* τ-equilibrium $\kappa_1 \colon \Omega \to (-\infty, 0]$ *and an upper semicontinuous* τ-equilibrium $\kappa_2 \colon \Omega \to [0, \infty)$ *which take the value 0 on the residual* σ-invariant sets of their continuity points and such that $\int_\Omega f_x(\omega, \kappa_1(\omega)) \, dm > 0$ *and* $\int_\Omega f_x(\omega, \kappa_2(\omega)) \, dm < 0$ *(resp.* $\int_\Omega f_x(\omega, \kappa_1(\omega)) \, dm < 0$ *and* $\int_\Omega f_x(\omega, \kappa_2(\omega)) \, dm > 0$*). Moreover,*

$$\kappa_1(\omega) = \sup\{x_0 \in \mathbb{R} \colon \lim_{t \to \infty} (u(t, \omega, x_0) - \alpha(\omega \cdot t)) = 0\} > \alpha(\omega)$$

$$(\text{resp. } \kappa_2(\omega) = \inf\{x_0 \in \mathbb{R} \colon \lim_{t \to \infty} (u(t, \omega, x_0) - \beta(\omega \cdot t)) = 0\} < \beta(\omega)),$$

where the τ-equilibrium α *(resp.* β*) is the lower (resp. upper) delimiter of the global attractor, with graph* $\mathcal{M}^l$ *(resp.* $\mathcal{M}^u$*).*

Proof Proposition 5.3(ii) of [11] ensures the nonhyperbolicity of $\mathcal{M}_0$. We reason in the case $\mathcal{M}^l < \mathcal{M}_0$. Theorem 5.12 of [11] ensures that the ξ-bifurcation diagram for $x' = f(\omega{\cdot}t, x) + \xi$ is described by Theorem 5.10 of [11], with 0 as left local saddle-node bifurcation point, at which the upper (nonhyperbolic) minimal set is $\mathcal{M}_0$. As explained in the proof of Theorem 5.10 of [11], $\mathcal{M}_0$ is contained on a compact τ-invariant pinched set $\bigcup_{\omega \in \Omega}\left(\{\omega\} \times [\kappa_1(\omega), \kappa_2(\omega)]\right)$, where: $\kappa_1 \colon \Omega \to (-\infty, 0]$ (called κ in [11]) is a lower semicontinuous τ-equilibrium; $\kappa_2 \colon \Omega \to (-\infty, 0]$ (called β in [11]) is the upper delimiter of the global attractor; and both of them coincide (and hence take the value 0) on the residual set of their common continuity points (and hence at all their continuity points: see Proposition 2.4 of [11]). Since $\int_\Omega f_x(\omega, 0)\, dm \neq 0$, Proposition 5.11(i) of [11] ensures that $\int_\Omega f_x(\omega, \kappa_1(\omega))\, dm > 0$ and $\int_\Omega f_x(\omega, \kappa_2(\omega))\, dm < 0$.

Let us check the last assertion, also in the case $\mathcal{M}^l < \mathcal{M}_0$. Recall that the hyperbolic attractiveness of $\mathcal{M}^l$ ensures that it coincides with the graph of α, which is continuous, and that every global strict upper solution is strictly greater than α (see Sect. 2.4). Since κ_1 is a τ-equilibrium, $\kappa_1(\omega) \geq \sup\{x_0 \in \mathbb{R} \colon \lim_{t \to \infty}(u(t, \omega, x_0) - \alpha(\omega{\cdot}t)) = 0\}$ for all $\omega \in \Omega$. We fix ω_0 to check that $\lim_{t \to \infty}(u(t, \omega_0, x_0) - \alpha(\omega_0{\cdot}t)) = 0$ for any $x_0 < \kappa_1(\omega_0)$. The ξ-bifurcation diagram for $x' = f(\omega{\cdot}t, x) + \xi$ described by Theorem 5.10 of [11] ensures the existence of $\xi_+ > 0$ and strictly negative continuous equilibria $\kappa_\xi \colon \Omega \to \mathbb{R}$ of $x' = f(\omega{\cdot}t, x) + \xi$ for $\xi \in (0, \xi_+)$ which decrease strictly as ξ increases and such that $\kappa_1(\omega) = \lim_{\xi \downarrow 0} \kappa_\xi(\omega)$ for all $\omega \in \Omega$. Hence, for any $x_0 < \kappa_1(\omega_0)$, there exists $\xi_0 > 0$ such that $x_0 < \kappa_{\xi_0}(\omega_0)$. Since κ_{ξ_0} is a strict upper solution of $x' = f(\omega{\cdot}t, x)$ and hence a strong τ-superequilibrium, the $\boldsymbol{\omega}$-limit for τ of (ω_0, x_0) is strictly below the graph of κ_{ξ_0} (see Sect. 2.3), so this $\boldsymbol{\omega}$-limit contains $\mathcal{M}^l$, which is the unique τ-minimal set strictly below it. Therefore, there exists a sequence $\{t_n\} \uparrow \infty$ such that $\lim_{n \to \infty}(u(t_n, \omega_0, x_0) - \alpha(\omega_0{\cdot}t_n)) = 0$, and the hyperbolic attractiveness of $\mathcal{M}^l$ ensures that $\lim_{t \to \infty}(u(t, \omega_0, x_0) - \alpha(\omega_0{\cdot}t)) = 0$, as asserted.

The proof is analogous in the other case. $\qquad\square$

Proof of Theorem 3.1 The Sacker and Sell spectrum of $f_x + \lambda$ on $\mathcal{M}_0$ is $[-\lambda_+ + \lambda, -\lambda_- + \lambda]$, which ensures the stated hyperbolicity properties of $\mathcal{M}_0$ (see Sect. 2.5). As in Theorem 6.3 of [11], we define

$$\mu_- = \inf\{\lambda \colon \forall\, \xi > \lambda \text{ the graph of } \alpha_\xi \text{ is the hyperbolic minimal set } \mathcal{M}^l_\xi < \mathcal{M}_0\},$$

$$\mu_+ = \inf\{\lambda \colon \forall\, \xi > \lambda \text{ the graph of } \beta_\xi \text{ is the hyperbolic minimal set } \mathcal{M}^u_\xi > \mathcal{M}_0\},$$

which, as proved there, belong to $(-\infty, \lambda_+]$. This property guarantees the stated structure of the τ_λ-minimal sets for $\lambda > \lambda_+$, since there exist at most three τ_λ-minimal sets (see Sect. 2.6). Theorem 6.3(ii) of [11] also ensures that at least one of these two parameters μ_-, μ_+ coincides with λ_+, which proves the stated collision properties for α_λ or for β_λ as $\lambda \downarrow \lambda_+$. As in the statement, we assume that this is the case for β_λ, i.e., that $\mu_+ = \lambda_+$. Then, since $\lambda \mapsto \beta_\lambda(\omega)$ is nondecreasing for all $\omega \in \Omega$ and the intersection of two residual sets is also residual, $\mathcal{M}_0$ is the upper minimal set for all $\lambda \leq \lambda_+$. If also $\mu_- = \lambda_+$, then Theorem 6.3(iii) of [11] ensures that the bifurcation diagram is that of (ii). If $\mu_- < \lambda_-$, then Theorem 6.4 of [11] shows that the diagram is that of (i), with $\lambda_0 = \mu_-$. The remaining case is, hence, $\mu_- \in [\lambda_-, \lambda_+)$. We will check that, in this case, the situation is that of (iii), which will complete the proof.

Let us call $\lambda_0 = \mu_-$. Notice that, if $\lambda \in (\lambda_0, \lambda_+]$, then there exist only two τ_λ-minimal sets, as otherwise the nonhyperbolicity of $\mathcal{M}_0$ would be contradicted (see Sect. 2.6); and, as explained before, $\mathcal{M}^l_\lambda$ is hyperbolic attractive (and given by the graph of α_λ). Consequently, it only remains to prove that $\alpha_\lambda(\omega) = 0$ on the residual σ-invariant set of its continuity points for $\lambda < \lambda_0$. This will ensure that $\mathcal{M}_0$ is the unique τ_λ-minimal set for $\lambda < \lambda_0$, and

the hyperbolic attractiveness of $\mathcal{M}_0$ for $\lambda < \lambda_-$ will ensure that $\mathcal{A}_\lambda = \mathcal{M}_0$ for $\lambda < \lambda_-$ (see Theorem 3.4 of [6]). Recall also that α_λ vanishes at all its continuity points if it vanishes at one of them (see e.g. Proposition 2.5 of [11]).

First, let us assume that $\mathcal{M}^l_{\lambda_0} = \mathcal{M}_0$, which means that $\alpha_{\lambda_0}(\omega) = 0$ on the residual set of its continuity points (see Sect. 2.4). Therefore, the same happens with α_λ if $\lambda < \lambda_0$, since $\alpha_{\lambda_0} \le \alpha_\lambda \le 0$ and the intersection of two residual sets is also residual. This proves the result in this case.

So, we assume $\mathcal{M}^l_{\lambda_0} < \mathcal{M}_0$, which in particular ensures that $\alpha_{\lambda_0}(\omega) < 0$ for all $\omega \in \Omega$. Let us take $\lambda > \lambda_0$. Lemma 3.2 shows that the upper delimiter of the basin of attraction of the graph of α_λ is a lower semicontinuous τ_λ-equilibrium $\kappa^1_\lambda \colon \Omega \to (-\infty, 0]$ which vanishes at its continuity points, with $\kappa^1_\lambda > \alpha_\lambda$, and which satisfies $\int_\Omega (f_x(\omega, \kappa^1_\lambda(\omega)) + \lambda)\, dm > 0$ whenever $\int_\Omega (f_x(\omega, 0) + \lambda)\, dm < 0$. In particular, since m is ergodic, $\kappa^1_\lambda < 0$ m-a.e. whenever $\int_\Omega (f_x(\omega, 0) + \lambda)\, dm < 0$. Let us check that the map $\lambda \mapsto \kappa^1_\lambda(\omega)$ is nondecreasing on (λ_0, ∞) for all $\omega \in \Omega$. It is easy to deduce from the hyperbolicity of the graph $\mathcal{M}^l_\lambda$ of α_λ that $x < \kappa^1_\lambda(\omega)$ if and only if $\mathcal{M}^l_\lambda$ is the ω-limit for τ_λ of (ω, x). We take $\lambda_0 < \lambda_1 < \lambda_2$, $\omega \in \Omega$ and $x < \kappa^1_{\lambda_1}(\omega)$. Then, $u_{\lambda_2}(t, \omega, x) < u_{\lambda_1}(t, \omega, x) < \kappa^1_{\lambda_1}(\omega{\cdot}t)$ for all $t \ge 0$, which precludes the possibility that $\mathcal{M}_0$ is contained in the ω-limit set for τ_{λ_2} of (ω, x) and hence guarantees that $x < \kappa^1_{\lambda_2}(\omega)$. This proves the assertion. As a consequence of this nondecreasing character, the map $\kappa^1_{\lambda_0}(\omega) = \lim_{\lambda \downarrow \lambda_0} \kappa^1_\lambda(\omega)$ is well defined, and it satisfies $\alpha_{\lambda_0}(\omega) \le \kappa^1_{\lambda_0}(\omega) \le \kappa^1_\lambda(\omega)$ for all $\omega \in \Omega$ if $\lambda > \lambda_0$. It is clear that $\kappa^1_{\lambda_0}$ it is an m-measurable τ_{λ_0}-equilibrium for every $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$.

Let $\beta_\mathcal{M}$ be the upper delimiter τ_{λ_0}-equilibrium of $\mathcal{M}^l_{\lambda_0}$. Our next purpose is to check that there exist points $\omega_0 \in \Omega$ with $\kappa^1_{\lambda_0}(\omega_0) \le \beta_\mathcal{M}(\omega_0)$. Since $\mathcal{M}^l_{\lambda_0}$ is nonhyperbolic, there exists $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ and an m-measurable τ_{λ_0}-equilibrium $\tilde{\kappa}$ with graph in $\mathcal{M}^l_{\lambda_0}$ such that $\int_\Omega (f_x(\omega, \tilde{\kappa}(\omega)) + \lambda_0)\, dm \ge 0$ (see Sect. 2.5). Proposition 4.4 of [11] ensures that $\int_\Omega (f_x(\omega, 0) + \lambda_0)\, dm < 0$, so $\int_\Omega (f_x(\omega, 0) + \lambda)\, dm < 0$ for $\lambda \ge \lambda_0$ close enough. Therefore, as seen before, $\kappa^1_\lambda(\omega) < 0$ m-a.e. for these values of the parameter, and hence $\kappa^1_{\lambda_0}(\omega) < 0$ m-a.e. Assume for contradiction that $\kappa^1_{\lambda_1}(\omega) > \beta_\mathcal{M}(\omega)$ for all $\omega \in \Omega$, and hence that $\kappa^1_{\lambda_1}(\omega) > \tilde{\kappa}(\omega)$ for all $\omega \in \Omega$. Then, 0, $\kappa^1_{\lambda_0}$ and $\tilde{\kappa}$ define three different τ_{λ_0}-ergodic measures by (2.3). This ensures that $\int_\Omega (f_x(\omega, \tilde{\kappa}(\omega)) + \lambda_0)\, dm < 0$ (see Sect. 2.6), which is not the case. This proves the assertion.

Finally, let us fix $\lambda < \lambda_0$ and check that $\alpha_\lambda(\omega) = 0$ at one of its continuity points, which completes the proof. Propositions 6.1 and 2.1 of [11] show the existence of $s^* > 0$ such that $\beta_\mathcal{M}(\omega) < u_\lambda(s^*, \omega{\cdot}(-s^*), \alpha_{\lambda_0}(\omega{\cdot}(-s^*))) \le \alpha_\lambda(\omega)$ for all $\omega \in \Omega$ (the last inequality follows from the flow monotonicity and the nonincreasing character of $\lambda \mapsto \alpha_\lambda(\omega)$). We take $\omega_0 \in \Omega$ with $\kappa^1_{\lambda_0}(\omega_0) \le \beta_\mathcal{M}(\omega_0)$. Then, $\lim_{\lambda \downarrow \lambda_0} \kappa^1_\lambda(\omega_0) = \kappa^1_{\lambda_0}(\omega_0) \le \beta_\mathcal{M}(\omega_0) < \alpha_\lambda(\omega_0)$. Therefore, there exists $\xi > \lambda_0$ close enough to get $\kappa^1_\xi(\omega_0) < \alpha_\lambda(\omega_0)$. Consequently, since $\xi > \lambda$ and hence the nonpositive semicontinuous map κ^1_ξ is a global lower solution for (3.1)$_\lambda$, $\kappa^1_\xi(\omega_0{\cdot}t) \le u_\lambda(t, \omega_0, \kappa^1_\xi(\omega_0)) < u_\lambda(t, \omega_0, \alpha_\lambda(\omega_0)) = \alpha_\lambda(\omega_0{\cdot}t)$ for all $t > 0$. Let ω_1 be a common continuity point of κ^1_ξ and α_λ. As (Ω, σ) is minimal, there exists a sequence $\{t_n\} \uparrow \infty$ such that $\omega_0{\cdot}t_n \to \omega_1$ as $n \to \infty$. Since κ^1_ξ vanishes at ω_1, $0 = \kappa^1_\xi(\omega_1) = \lim_{n \to \infty} \kappa^1_\xi(\omega_0{\cdot}t_n) \le \lim_{n \to \infty} \alpha_\lambda(\omega_0{\cdot}t_n) = \alpha_\lambda(\omega_1) \le 0$. That is, $\alpha_\lambda(\omega_1) = 0$, and the proof is complete. $\qquad\square$

The first and second diagrams in Fig. 1 (resp. in Fig. 2) depict the two cases of local saddle-node and transcritical bifurcations (resp. classical pitchfork bifurcation) described in detail in Theorem 3.1 when the Sacker and Sell spectrum is a point (the first one) and a

band (the second one). The third diagram in Fig. 2 depicts the case of a generalized pitchfork bifurcation, which requires the Sacker and Sell spectrum to be a band (and hence never happens in the autonomous case, which is uniquely ergodic). Notice that we have examples of one, two, and three bifurcation points. In particular, the band case of local saddle-node and transcritical bifurcation diagram exhibits three of these values.

We complete the section by mentioning that there are simple autonomous examples of classical pitchfork bifurcation (as $x' = -x^3 + \lambda x$, with $\lambda_{\pm} = 0$ as bifurcation point) and local saddle-node and transcritical bifurcation (as $x' = -x^3 \pm 2x^2 + \lambda x$, with $\lambda_0 = -1$ as local saddle-node bifurcation point and $\lambda_{\pm} = 0$ as local classical transcritical bifurcation point; the two signs of the second-order term correspond to the two possible sings for the curve κ_λ of nonhyperbolic critical points). We will go deeper in this matter in Sects. 4, 5 and 6, where we will show that all the possibilities realize for suitable families (3.1).

4 Criteria for Cubic Polynomial Equations

Let us consider families of cubic polynomial ordinary differential equations

$$x' = -a_3(\omega{\cdot}t)x^3 + a_2(\omega{\cdot}t)x^2 + (a_1(\omega{\cdot}t) + \lambda)x, \quad \omega \in \Omega, \tag{4.1}$$

where $a_i \in C(\Omega)$ for $i \in \{1, 2, 3\}$, a_3 is strictly positive and $\lambda \in \mathbb{R}$. It is clear that the function $f(\omega, x) = -a_3(\omega)x^3 + a_2(\omega)x^2 + a_1(\omega)x$ is (Co). In addition, f is $(SDC)_*$ (see Sect. 2.6). Then, Theorem 3.1 describes the three possible λ-bifurcation diagrams for (4.1). Our first goal in this section, achieved in Sects. 4.1 and 4.2, is to describe conditions on the coefficients a_i determining the specific diagram. The last subsection is devoted to explain how to get actual patterns satisfying the previously established conditions.

As in Sect. 3, $\tau_\lambda(t, \omega, x_0) = (\omega{\cdot}t, u_\lambda(t, \omega, x_0))$ repesents the local skewproduct flow induced by (4.1)$_\lambda$ on $\Omega \times \mathbb{R}$. Notice that the Sacker and Sell spectrum of f_x on $\mathcal{M}_0 = \Omega \times \{0\}$ (which is τ_λ-minimal for all $\lambda \in \mathbb{R}$) coincides with $\mathrm{sp}(a_1)$.

4.1 The Case of a_1 with Continuous Primitive

Throughout this subsection, we assume that $a_1 \in CP(\Omega)$. Since $CP(\Omega) \subseteq C_0(\Omega)$ (see Sect. 2.2), the Sacker and Sell spectrum of a_1 is $\mathrm{sp}(a_1) = \{0\}$. Hence, the bifurcation diagram of (4.1) fits in (i) or (ii) of Theorem 3.1, and our objective is to give criteria ensuring each one of these two possibilities. The relevant fact in terms of which the criteria will be constructed is that the number of τ_0-minimal sets distinguishes the type of bifurcation: there is either one τ_0-minimal set in (ii) or two τ_0-minimal sets in (i).

Proposition 4.2 provides a simple classification for (4.1) in this case. It is based on the previous bifurcation analysis of

$$x' = -a_3(\omega{\cdot}t)x^3 + (a_2(\omega{\cdot}t) + \xi)x^2, \quad \omega \in \Omega, \tag{4.2}$$

made in Proposition 4.1. These two results extend Proposition 6.6 and Corollary 6.7 of [11] to the case of strictly positive a_3 (instead of $a_3 \equiv 1$), since the case of $a_1 \equiv 0$ is trivially covered by Proposition 4.2 with $b \equiv 0$. We will call $\check{\tau}_\xi$ the local skewproduct flow induced by (4.2)$_\xi$ on $\Omega \times \mathbb{R}$.

Proposition 4.1 *The $\check{\tau}_\xi$-minimal set $\mathcal{M}_0 = \Omega \times \{0\}$ is nonhyperbolic for all $\xi \in \mathbb{R}$. In addition, if $\mathrm{sp}(a_2) = [-\xi_+, -\xi_-]$, with $\xi_- \leq \xi_+$, then*

(i) $\check{\tau}_\xi$ admits exactly two minimal sets $\mathcal{M}_\xi^l < \mathcal{M}_0$ respectively given by the graphs of α_ξ and 0 for $\xi < \xi_-$, where $\mathcal{M}_\xi^l$ is hyperbolic attractive; and α_ξ collides with 0 on a residual σ-invariant set as $\xi \uparrow \xi_-$;

(ii) $\mathcal{M}_0$ is the unique $\check{\tau}_\xi$-minimal set for $\xi \in [\xi_-, \xi_+]$;

(iii) $\check{\tau}_\xi$ admits exactly two minimal sets $\mathcal{M}_0 < \mathcal{M}_\xi^u$ respectively given by the graphs of 0 and β_ξ for $\xi > \xi_+$, where $\mathcal{M}_\xi^u$ is hyperbolic attractive; and β_ξ collides with 0 on a residual σ-invariant set as $\xi \downarrow \xi_+$.

Proof The proof follows step by step that of Proposition 6.6 of [11]. The only remarkable differences arise in checking the existence of a bounded solution of $(4.2)_\xi$ for $\xi > \xi_+$ which is bounded away from 0 for $t > 0$. Let us explain these differences. Take $\alpha \in (0, \xi - \xi_+)$ with $\alpha < \sqrt{2r}$ for $r = \|a_3\| = \max_{\omega \in \Omega} a_3(\omega) > 0$. Let us check that the solution $w(t)$ of $w' = -(a_2(\omega \cdot t) + \xi) + a_3(\omega \cdot t)/w$ with $w(0) = \alpha < 2r/\alpha$ is bounded on $[0, \infty)$. We define $t_1 = \sup\{t > 0 : w(s) \le 2r/\alpha + lt_\alpha \text{ for all } s \in [0, t]\}$, where $l = r/\alpha + \|a_2 + \xi\|$ and $t_\alpha > 0$ satisfies $\int_0^{t_\alpha}(a_2(\omega \cdot s) + \xi)\, ds \ge \alpha t_\alpha$ for all $\omega \in \Omega$. The existence of such t_α is proved in the proof of Proposition 6.6 of [11]. We assume for contradiction that $t_1 < \infty$ and define $t_0 = \inf\{t < t_1 : w(s) \ge 2r/\alpha \text{ for all } s \in [t, t_1]\}$. Then, $t_0 < t_1 - t_\alpha$: otherwise

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1}\left(-(a_2(\omega_0 \cdot s) + \xi) + \frac{a_3(\omega_0 \cdot s)}{w(s)}\right) ds < \frac{2r}{\alpha} + lt_\alpha,$$

which is not the case. In particular, $w(t) \ge 2r/\alpha$ for $t \in [t_1 - t_\alpha, t_1]$, and hence

$$w(t_1) = w(t_1 - t_\alpha) - \int_0^{t_\alpha}(a_2(\omega_0 \cdot (t_1 - t_\alpha) \cdot s) + \xi)\, ds + \int_{t_1 - t_\alpha}^{t_1} \frac{a_3(\omega_0 \cdot s)}{w(s)}\, ds$$

$$\le w(t_1 - t_\alpha) - \alpha t_\alpha + \frac{\alpha}{2} t_\alpha < w(t_1 - t_\alpha),$$

which contradicts the definition of t_1. $\qquad\square$

Proposition 4.2 *Let b be a continuous primitive of a_1. Then,*

(i) $\mathrm{sp}(e^b a_2) \subset (0, \infty)$ *if and only if* (4.1) *exhibits the local saddle-node and classical transcritical bifurcations of minimal sets described in Theorem* 3.1(i), *with α_λ colliding with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$. In particular, this situation holds if $0 \not\equiv a_2 \ge 0$.*

(ii) $\mathrm{sp}(e^b a_2) \subset (-\infty, 0)$ *if and only if* (4.1) *exhibits the local saddle-node and classical transcritical bifurcations of minimal sets described in Theorem* 3.1(i), *with β_λ colliding with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$. In particular, this situation holds if $0 \not\equiv a_2 \le 0$.*

(iii) $0 \in \mathrm{sp}(e^b a_2)$ *if and only if* (4.1) *exhibits the classical pitchfork bifurcation of minimal sets described in Theorem* 3.1(ii).

Proof The family of changes of variable $y(t) = e^{-b(\omega \cdot t)} x(t)$ takes (4.1) to

$$\begin{aligned} y' &= -e^{2b(\omega \cdot t)} y^3 + e^{b(\omega \cdot t)} a_2(\omega \cdot t) y^2 + (a_1(\omega \cdot t) - b'(\omega \cdot t) + \lambda) y \\ &= -e^{2b(\omega \cdot t)} y^3 + e^{b(\omega \cdot t)} a_2(\omega \cdot t) y^2 + \lambda y. \end{aligned} \tag{4.3}$$

The possibilities for (4.1) follow from here, since $\mathcal{N}$ is a minimal set for $(4.3)_\lambda$ if and only if $\mathcal{M} = \{(\omega, e^{b(\omega)} x) : (\omega, x) \in \mathcal{N}\}$ is minimal for $(4.1)_\lambda$. The rest of the proof adapts that of Corollary 6.7 of [11]. Theorem 3.1 applied to $f(\omega, y) = -e^{2b(\omega \cdot t)} y^3 + e^{b(\omega \cdot t)} a_2(\omega \cdot t) y^2$ shows that $\lambda_- = \lambda_+ = 0$ is a bifurcation point for (4.3), and that the corresponding bifurcation diagram is that described in its points (i) or (ii). According to Proposition 4.1, the

flow induced by $y' = f(\omega{\cdot}t, y)$, that is, $(4.3)_0$, admits just one minimal set if and only if $0 \in \mathrm{sp}(e^b a_2)$; two minimal sets, being $\mathcal{M}_0 = \Omega \times \{0\}$ the lower one, if and only if $\mathrm{sp}(e^b a_2) \subset (0, \infty)$; and two minimal sets, being $\mathcal{M}_0$ the upper one, if and only if $\mathrm{sp}(e^b a_2) \subset (-\infty, 0)$. As said before, this determines the global bifurcation diagram. The last assertions in (i) and (ii) are trivial. □

As a consequence of the previous result, given any strictly positive a_3 and any changing-sign a_2, we are able to construct a_1 with bounded primitive in such a way that (4.1) exhibits the classical pitchfork bifurcation of minimal sets described in Theorem 3.1(ii).

Proposition 4.3 *Let $a_2 \in C(\Omega)$ change sign. Then, there exists $a_1 \in CP(\Omega)$ such that (4.1) exhibits the classical pitchfork bifurcation described in Theorem 3.1(ii).*

Proof Let $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ be arbitrarily fixed. We define the nonempty open sets $\Gamma_1 = \{\omega \in \Omega : a_2(\omega) > 0\}$ and $\Gamma_2 = \{\omega \in \Omega : a_2(\omega) < 0\}$. As (Ω, σ) is minimal, $\mathrm{supp}(m) = \Omega$, so $m(\Gamma_1) > 0$ and $m(\Gamma_2) > 0$. A suitable application of Urysohn's Lemma provides nonnegative and not identically zero continuous functions $c_1, c_2 \colon \Omega \to \mathbb{R}$ such that $\mathrm{supp}(c_1) \subseteq \Gamma_1$ and $\mathrm{supp}(c_2) \subseteq \Gamma_2$. Then, $\int_\Omega c_1(\omega)a_2(\omega)\,dm > 0$ and $\int_\Omega c_2(\omega)a_2(\omega)\,dm < 0$, so there exists $\epsilon > 0$ such that $\int_\Omega (c_1(\omega) + \epsilon)a_2(\omega)\,dm > 0$ and $\int_\Omega (c_2(\omega) + \epsilon)a_2(\omega)\,dm < 0$. The density of $C^1(\Omega)$ on $C(\Omega)$ and the strict positiveness of $c_1 + \epsilon$ and $c_2 + \epsilon$ ensure the existence of strictly positive functions $\hat{c}_1, \hat{c}_2 \in C^1(\Omega)$ such that $\int_\Omega \hat{c}_1(\omega)a_2(\omega)\,dm > 0$ and $\int_\Omega \hat{c}_2(\omega)a_2(\omega)\,dm < 0$. Therefore, there exists $s \in (0, 1)$ such that $\int_\Omega (s\hat{c}_1(\omega) + (1 - s)\hat{c}_2(\omega))a_2(\omega)\,dm = 0$. Since $s\hat{c}_1(\omega) + (1 - s)\hat{c}_2(\omega)$ is strictly positive, $b(\omega) = \log(s\hat{c}_1(\omega) + (1 - s)\hat{c}_2(\omega))$ is well defined and $b \in C^1(\Omega)$. So, $\int_\Omega e^{b(\omega)}a_2(\omega)\,dm = 0$, and hence $0 \in \mathrm{sp}(e^b a_2)$ (see Sect. 2.5). We take $a_1 = b'$ and apply Proposition 4.2 to complete the proof. □

4.2 The Case of Sign-Preserving a_2

In what follows, we will describe some conditions ensuring which one of the bifurcation possibilities of minimal sets described in Theorem 3.1 holds for (4.1). Our starting point are the functions a_1, a_3, with a_3 strictly positive. Let $\mathrm{sp}(a_1) = [-\lambda_+, -\lambda_-]$, with $\lambda_- \leq \lambda_+$, be the Sacker and Sell spectrum of a_1, let $k_1 < k_2$ be such that $k_1 \leq a_1(\omega) \leq k_2$ for all $\omega \in \Omega$, and let $0 < r_1 \leq r_2$ be such that $r_1 \leq a_3(\omega) \leq r_2$ for all $\omega \in \Omega$. Since $\mathrm{sp}(a_1) \subseteq [k_1, k_2]$ (see Sect. 2.5), $\lambda_- + k_1 \leq \lambda_+ + k_1 \leq 0 \leq \lambda_- + k_2 \leq \lambda_+ + k_2$, and the first and last inequalities are strict if a_1 has band spectrum. Our goal is to relate the "size" of a_2 to these six constants in order to get the different diagrams. A remarkable fact is that, in the cases studied this subsection, a_2 never changes sign, in contrast with the situation of Proposition 4.3 and those analyzed at the end of Sect. 6.

Now, we recall and complete the statement of Proposition 6.5 of [11] when applied to our current model (4.1), which gives a sufficient criterium for the classical pitchfork bifurcation.

Proposition 4.4

(i) (A criterium ensuring classical pitchfork bifurcation). *If $a_2(\omega) = 0$ for all $\omega \in \Omega$, then (4.1) exhibits the classical pitchfork bifurcation of minimal sets described in Theorem 3.1(ii).*

(ii) *If $a_2(\omega) \geq 0$ (resp. $a_2(\omega) \leq 0$) for all $\omega \in \Omega$, then α_λ (resp. β_λ) takes the value 0 at its continuity points for all $\lambda \leq \lambda_+$.*

Proof Proposition 6.5 of [11] ensures (i) and (ii) for $\lambda < \lambda_+$. To check (ii) for λ_+, it suffices to observe that all the alternatives of Theorem 3.1 in which α_λ (resp. β_λ) takes the value 0 at its continuity points for all $\lambda < \lambda_+$, also α_{λ_+} (resp. β_{λ_+}) takes the value 0 at its continuity points. $\square$

The main results of this subsection are stated in Propositions 4.7, 4.8 and 4.9, whose proofs use the next technical results. The first one shows that one of the conditions required in Proposition 4.7 always holds if a_1 has band spectrum (in which case a_1 is not a constant function).

Lemma 4.5 *Let $a \in C(\Omega)$. Then, the next three assertions are equivalent: a is nonconstant; $\min_{\omega \in \Omega} a(\omega) < \inf \mathrm{sp}(a)$; $\max_{\omega \in \Omega} a(\omega) > \sup \mathrm{sp}(a)$.*

Proof Clearly a is nonconstant if $\min_{\omega \in \Omega} a(\omega) < \inf \mathrm{sp}(a)$. Now, we assume that a is nonconstant and take $0 < \epsilon < \max_{\omega \in \Omega} a(\omega) - \min_{\omega \in \Omega} a(\omega)$. We define the nonempty open set $U_1 = \{\omega \in \Omega : a(\omega) > \min_{\omega \in \Omega} a(\omega) + \epsilon\}$, and note that $m(U_1) > 0$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$, since Ω is minimal. Let $m_1 \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ satisfy $\inf \mathrm{sp}(a) = \int_\Omega a(\omega)\, dm_1$ (see Sect. 2.5). Then,

$$\inf \mathrm{sp}(a) = \int_\Omega a(\omega)\, dm_1 = \int_{\Omega \setminus U_1} a(\omega)\, dm_1 + \int_{U_1} a(\omega)\, dm_1$$
$$\geq m_1(\Omega \setminus U_1) \min_{\omega \in \Omega} a(\omega) + m_1(U_1)(\min_{\omega \in \Omega} a(\omega) + \epsilon) > \min_{\omega \in \Omega} a(\omega)\,.$$

This completes the proof of the equivalence of the two first assertions. To check the equivalence of the first and the third ones, we work with $U_2 = \{\omega \in \Omega : a(\omega) < \max_{\omega \in \Omega} a(\omega) - \epsilon\}$. $\square$

Lemma 4.6 *Let $\lambda + k_1 < 0$.*

(i) *If $a_2(\omega) > 2\sqrt{r_2(-\lambda - k_1)}$ for all $\omega \in \Omega$, then $\rho_1 = \sqrt{(-\lambda - k_1)/r_2}$ is a global strict lower solution of $(4.1)_\lambda$. Consequently, τ_λ admits a minimal set $\mathcal{M}_\lambda^u > \mathcal{M}_0$.*
(ii) *If $a_2(\omega) < -2\sqrt{r_2(-\lambda - k_1)}$ for all $\omega \in \Omega$, then $-\rho_1$ is a global strict upper solution of $(4.1)_\lambda$. Consequently, τ_λ admits a minimal set $\mathcal{M}_\lambda^l < \mathcal{M}_0$.*

Proof Let us prove (i). We define $g(\rho) = r_2\rho - (\lambda + k_1)/\rho$. Then, $g(\rho_1) = 2\sqrt{r_2(-\lambda - k_1)}$, and

$$-a_3(\omega)\rho_1^3 + a_2(\omega)\rho_1^2 + (a_1(\omega) + \lambda)\rho_1 \geq \rho_1^2 \left(a_2(\omega) - \left(r_2\rho_1 - \frac{\lambda + k_1}{\rho_1} \right) \right)$$
$$= \rho_1^2 (a_2(\omega) - g(\rho_1)) > 0$$

for all $\omega \in \Omega$, which proves the first assertion. In turn, this property ensures that $\rho_1 < \beta_\lambda$ (see Sect. 2.4), which proves the second assertion. The proof of (ii) is analogous. $\square$

Proposition 4.7

(i) (A criterium ensuring saddle-node and transcritical bifurcations). *If $k_1 < -\lambda_+$ and*

$$a_2(\omega) > 2\sqrt{r_2(-\lambda_- - k_1)} \quad (\text{resp. } a_2(\omega) < -2\sqrt{r_2(-\lambda_- - k_1)})$$

for all $\omega \in \Omega$, then (4.1) exhibits the local saddle-node and transcritical bifurcations of minimal sets described in Theorem 3.1(i), with α_λ (resp. β_λ) colliding with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$.

(ii) (A criterium precluding classical pitchfork bifurcation). *If $k_1 < -\lambda_+$ and*

$$a_2(\omega) > 2\sqrt{r_2(-\lambda_+ - k_1)} \quad (resp. \ a_2(\omega) < -2\sqrt{r_2(-\lambda_+ - k_1)})$$

for all $\omega \in \Omega$, then (4.1) *does not exhibit the classical pitchfork bifurcation of minimal sets described in Theorem* 3.1(ii).

Proof

(i) Note that if $k_1 < -\lambda_+$, then $-\lambda_- - k_1 \geq -\lambda_+ - k_1 > 0$. There exists $\delta > 0$ such that $a_2(\omega) > 2\sqrt{r_2(-\lambda_- + \delta - k_1)}$ (resp. $a_2(\omega) < -2\sqrt{r_2(-\lambda_- + \delta - k_1)}$) for all $\omega \in \Omega$. Hence, Lemma 4.6 ensures the existence of a $\tau_{\lambda_- - \delta}$-minimal set $\mathcal{M}^u_{\lambda_- - \delta} > \mathcal{M}_0$ (resp. $\mathcal{M}^l_{\lambda_- - \delta} < \mathcal{M}_0$), and this situation only arises in the stated case of Theorem 3.1(i).

(ii) The existence of a τ_{λ_+}-minimal set $\mathcal{M}^u_{\lambda_+} > \mathcal{M}_0$ (resp. $\mathcal{M}^l_{\lambda_+} < \mathcal{M}_0$), ensured by Lemma 4.6, precludes the situation of Theorem 3.1(ii). $\qquad\square$

Recall that $\lambda_+ < k_2$ if a_1 has band spectrum: see Lemma 4.5. The following two results refer to the case that a_1 has band spectrum: $\lambda_- < \lambda_+$. (This is ensured in Proposition 4.9 by its condition (4.4)).

Proposition 4.8 (A criterium ensuring pitchfork bifurcation) *If $\lambda_- < \lambda_+$ and*

$$0 \leq a_2(\omega) < \frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} \quad \left(resp. \ -\frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} < a_2(\omega) \leq 0\right)$$

for all $\omega \in \Omega$, then (4.1) *does not exhibit the saddle-node and transcritical bifurcations of minimal sets described in Theorem* 3.1(i).

Proof We reason in the case of positive a_2. Let $\hat{\beta}_\lambda$ be the upper delimiter equilibrium of the global attractor of the skewproduct flow induced by $x' = -a_3(\omega{\cdot}t)x^3 + (a_1(\omega{\cdot}t) + \lambda)x$, so that $\hat{\beta}_\lambda \colon \Omega \to \mathbb{R}$ is a strictly positive continuous map if $\lambda > \lambda_+$ (see Proposition 4.4(i)). Note that, if $\lambda + k_2 \geq 0$ and $\rho > \sqrt{(\lambda + k_2)/r_1}$, then

$$-a_3(\omega)\rho^3 + (a_1(\omega) + \lambda)\rho \leq -r_1\rho^3 + (\lambda + k_2)\rho < 0\,;$$

that is, $\omega \mapsto \rho$ is a global strict upper solution of $x' = -a_3(\omega{\cdot}t)x^3 + (a_1(\omega{\cdot}t) + \lambda)x$. Hence, if $\lambda > \lambda_+$, for any $\omega \in \Omega$, then the ω-limit set of (ω, ρ) contains a minimal set $\mathcal{M}^u$ which is below of the graph of ρ (see Sect. 2.3), and which cannot be $\mathcal{M}_0$, since $\mathcal{M}_0$ is repulsive for τ_λ. Hence, $\mathcal{M}^u$ is the graph of $\hat{\beta}_\lambda$, which ensures that $\hat{\beta}_\lambda(\omega) \leq \rho$ for all $\omega \in \Omega$. It follows easily that $\hat{\beta}_\lambda(\omega) \leq \sqrt{(\lambda + k_2)/r_1}$ for all $\omega \in \Omega$.

We take $\delta > 0$ such that, if $\lambda \in [\lambda_+, \lambda_+ + \delta]$, then $a_2(\omega) \leq \sqrt{r_1}\,(\lambda_+ - \lambda_-)/\sqrt{\lambda + k_2}$ for all $\omega \in \Omega$. Then, if $\lambda \in (\lambda_+, \lambda_+ + \delta]$,

$$a_2(\omega)\hat{\beta}_\lambda(\omega) \leq a_2(\omega)\sqrt{(\lambda + k_2)/r_1} \leq \lambda_+ - \lambda_-$$

for all $\omega \in \Omega$, and hence

$$\hat{\beta}'_\lambda(\omega) > -a_3(\omega)\hat{\beta}^3_\lambda(\omega) + (\lambda_+ - \lambda_-)\hat{\beta}_\lambda(\omega) + (a_1(\omega) + \lambda_-)\hat{\beta}_\lambda(\omega)$$
$$\geq -a_3(\omega)\hat{\beta}^3_\lambda(\omega) + a_2(\omega)\hat{\beta}^2_\lambda(\omega) + (a_1(\omega) + \lambda_-)\hat{\beta}_\lambda(\omega)$$

for all $\omega \in \Omega$: $\hat{\beta}_\lambda$ is a global strict upper τ_{λ_-}-solution, and, in particular, a continuous strong τ_{λ_-}-superequilibrium. In addition, Proposition 4.4(i) ensures that $\hat{\beta}_{\lambda_+} = \lim_{\lambda \downarrow \lambda_+} \hat{\beta}_\lambda$ collides with 0 on a residual set of points $\mathcal{R}_1$.

Let us assume for contradiction that the situation for (4.1) is that described in Theorem 3.1(i). It follows from Proposition 4.4(ii) that α_λ collides with 0 on a residual set of points as $\lambda \downarrow \lambda_+$, so Theorem 3.1(i) ensures that β_{λ_-} is a continuous strictly positive τ_{λ_-}-equilibrium whose graph is the attractive hyperbolic τ_{λ_-}-minimal set $\mathcal{M}^u_{\lambda_-}$. Since the Sacker and Sell spectrum of $\mathcal{M}_0$ for τ_{λ_-} is the nondegenerate interval $[-\lambda_+ + \lambda_-, 0]$, there exists a measure m such that $\int_\Omega (f_x(\omega, 0) + \lambda_-) \, dm < 0$ (see Sect. 2.5), and hence Lemma 3.2 ensures that the lower delimiter $\kappa^2_{\lambda_-}$ of the basin of attraction of β_{λ_-} vanishes on its residual set of continuity points $\mathcal{R}_2$. Let us take $\omega_0 \in \mathcal{R}_1 \cap \mathcal{R}_2$, a sequence $\{t_n\} \uparrow \infty$ such that $\lim_{n\to\infty} \omega_0 \cdot t_n = \omega_0$ and $\lambda \in (\lambda_+, \lambda_+ + \delta]$ such that $\kappa^2_{\lambda_-}(\omega_0) = 0 < \hat{\beta}_\lambda(\omega_0) < \beta_{\lambda_-}(\omega_0)$, which guarantees that $\lim_{n\to\infty}(u_{\lambda_-}(t_n, \omega_0, \hat{\beta}_\lambda(\omega_0)) - \beta_{\lambda_-}(\omega_0 \cdot t_n)) = 0$. Since $\hat{\beta}_\lambda$ is a continuous strong τ_{λ_-}-superequilibrium, there exist $t_0 > 0$ and $e_0 > 0$ such that $u_{\lambda_-}(t, \omega_0, \hat{\beta}_\lambda(\omega_0)) + e \leq \hat{\beta}_\lambda(\omega_0 \cdot t)$ for all $t \geq t_0$, and hence $(u_{\lambda_-}(t_n, \omega_0, \hat{\beta}_\lambda(\omega_0)) - \beta_{\lambda_-}(\omega_0 \cdot t_n)) + \beta_{\lambda_-}(\omega_0 \cdot t_n) + e \leq \hat{\beta}_\lambda(\omega_0 \cdot t_n)$ for n large enough. Taking limits as $n \to \infty$ we obtain $\beta_{\lambda_-}(\omega_0) + e \leq \hat{\beta}_\lambda(\omega_0)$, a contradiction. This completes the proof. $\qquad\square$

Proposition 4.9 (A criterium ensuring generalized pitchfork bifurcation) *If*

$$r_1(\lambda_+ - \lambda_-)^2 + 4\,r_2(\lambda_+ + k_1)(\lambda_+ + k_2) > 0 \,, \tag{4.4}$$

and

$$2\sqrt{r_2(-\lambda_+ - k_1)} < a_2(\omega) < \frac{(\lambda_+ - \lambda_-)\sqrt{r_1}}{\sqrt{\lambda_+ + k_2}}$$
$$\left(resp. \quad -\frac{(\lambda_+ - \lambda_-)\sqrt{r_1}}{\sqrt{\lambda_+ + k_2}} < a_2(\omega) < -2\sqrt{r_2(-\lambda_+ - k_1)} \right) \tag{4.5}$$

for all $\omega \in \Omega$, then (4.1)$_\lambda$ exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.1(iii), with α_λ (resp. β_λ) colliding with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$.

Proof Condition (4.4) ensures that a_1 has band spectrum and that the intervals in which a_2 can take values are nondegenerate. Propositions 4.8 and 4.7(ii) respectively preclude situations (i) and (ii) of Theorem 3.1, and Proposition 4.4(ii) ensures the stated collision property for α_λ (resp. for β_λ). $\qquad\square$

4.3 Cases of Generalized Pitchfork Bifurcation

As said after Theorem 3.1, there are autonomous cases presenting either the local saddle-node and classical transcritical bifurcations or the classical pitchfork bifurcation of minimal sets described in cases (i) and (ii) of that theorem. These two possibilities are also the unique ones in nonautonomous examples when a_1 has point spectrum, and we have classified them if $a_1 \in CP(\Omega)$ in Sect. 4.1, where in addition we have shown simple ways to construct examples fitting in each one of these two situations. In Sect. 4.2, we have observed that cases (i) (with a local nonclassical transcritical bifurcation) and (ii) of Theorem 3.1 can also occur when a_1 has band spectrum. In addition, Propositions 4.4(i) and 4.7(i) provide simple ways to construct such examples by choosing a suitable a_2 once fixed a_1 and a_3. In the same line, Proposition 4.9 establishes conditions ensuring the generalized pitchfork case of Theorem 3.1(iii). But the existence of polynomials satisfying these last conditions is not so obvious.

Therefore, our next objective is to develop systematic ways of constructing third degree polynomials giving rise to families (4.1) for which the bifurcation diagram is that of Theorem 3.1(iii). Hence, all the situations described in that theorem actually realize.

Lemma 4.10 *Let $m_1, \ldots, m_n$ be different elements of $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $n \geq 1$, and let $0 < \epsilon < 1$ be fixed. For every $i \in \{1, \ldots, n\}$, there exists a continuous function $c_i : \Omega \to [0, 1]$ with $\min_{\omega \in \Omega} c_i(\omega) = 0$ and $\max_{\omega \in \Omega} c_i(\omega) = 1$ such that, if $i, j \in \{1, \ldots, n\}$ and $j \neq i$, then $c_i \cdot c_j \equiv 0$ and*

$$1 - \epsilon < \int_\Omega c_i(\omega)\, dm_i \leq 1\,, \qquad 0 \leq \int_\Omega c_i(\omega)\, dm_j < \epsilon\,. \tag{4.6}$$

Proof Let $\epsilon > 0$ be fixed. As explained in Remark 1.10 of [15] and Sect. 6 of Chapter II of [22], there exist disjoint σ-invariant sets $\Omega_1, \ldots, \Omega_n \subseteq \Omega$ such that $m_i(\Omega_j) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$, where δ_{ij} is the Kronecker delta: just take Ω_i as the so-called *ergodic component* of m_i. We take compact sets $K_i \subseteq \Omega_i$ such that $m_i(K_i) > 1 - \epsilon$ for $i \in \{1, \ldots, n\}$. Let $d = \min_{1 \leq i < j \leq n} d(K_i, K_j)$ and let U_i be an open set such that $K_i \subset U_i \subseteq B(K_i, d/3)$, $m_i(U_i \setminus K_i) < \epsilon$ and $m_j(U_i) = m_j(U_i \setminus K_i) < \epsilon$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. The choice of d ensures that $U_1, \ldots, U_n$ are pairwise disjoint. Finally, Urysohn's Lemma provides continuous functions $c_i : \Omega \to [0, 1]$ with $c_i(\omega) = 1$ for all $\omega \in K_i$ and $c_i(\omega) = 0$ for all $\omega \notin U_i$, for $i \in \{1, \ldots, n\}$. All the statements follow easily. $\qquad \square$

Proposition 4.11 *Let $m_1, \ldots, m_n$ be different elements of $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $n \geq 2$. Take $r \geq 1$ and $\epsilon > 0$ with*

$$\epsilon < \epsilon_1 = \frac{n + 2r(n-1) - 2\sqrt{r(n-1)\,(r(n-1)+n)}}{n^2}\,,$$

(so that $0 < \epsilon < 1/n$). Let $c_1 \ldots, c_n : \Omega \to \mathbb{R}$ be the functions given by Lemma 4.10 for $m_1, \ldots, m_n$ and ϵ. Take constants $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ with $\alpha_1 < 0$ and $\alpha_n > 0$, and define $a_1 = \sum_{i=1}^n \alpha_i c_i$, so that $\alpha_1 \leq a_1(\omega) \leq \alpha_2$ for all $\omega \in \Omega$. Then, a_1 has band spectrum $\mathrm{sp}(a_1) = [-\lambda_+, -\lambda_-] \subset (\alpha_1, \alpha_2)$ and

$$(\lambda_+ - \lambda_-)^2 + 4r(\lambda_+ + \alpha_1)(\lambda_+ + \alpha_n) > 0\,. \tag{4.7}$$

Consequently, if a_3 takes values in $[r_1, r_2]$ for $r_1 > 0$ and $r_2 = rr_1$, and if a_2 satisfies (4.5) for $k_1 = \alpha_1$ and $k_2 = \alpha_n$, then (4.1) exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.1(iii).

Proof It is easy to check that $0 < \epsilon_1 < 1/n$. In addition, according to Lemma 4.10,

$$\int_\Omega a_1(\omega)\, dm_1 = \alpha_1 \int_\Omega c_1(\omega)\, dm_1 + \sum_{i \geq 2} \alpha_i \int_\Omega c_i(\omega)\, dm_1 < \alpha_1(1 - \epsilon) + (n-1)\alpha_n \epsilon\,,$$

$$\int_\Omega a_1(\omega)\, dm_n = \alpha_n \int_\Omega c_n(\omega)\, dm_n + \sum_{i \leq n-1} \alpha_i \int_\Omega c_i(\omega)\, dm_n > \alpha_n(1 - \epsilon) + (n-1)\alpha_1 \epsilon\,.$$

These inequalities ensure that $\alpha_1 \leq -\lambda_+ < \alpha_1(1 - \epsilon) + (n-1)\alpha_n \epsilon$ and $\alpha_n(1 - \epsilon) + (n-1)\alpha_1 \epsilon < -\lambda_- \leq \alpha_n$. Then, $\lambda_+ - \lambda_- > (1 - n\epsilon)(\alpha_n - \alpha_1) > 0$ (which shows the nondegeneracy of $\mathrm{sp}(a_1)$), $0 > \lambda_+ + \alpha_1 > \epsilon(\alpha_1 - (n-1)\alpha_n)$ and $0 < \lambda_+ + \alpha_n \leq \alpha_n - \alpha_1$, which in turn yield

$$(\lambda_+ - \lambda_-)^2 + 4r(\lambda_+ + \alpha_1)(\lambda_+ + \alpha_n)$$
$$> (\alpha_n - \alpha_1)^2(1 - n\epsilon)^2 + 4r\epsilon(\alpha_n - \alpha_1)(\alpha_1 - (n-1)\alpha_n)\,.$$

So, it is enough to check that the right-hand side is strictly positive; that is,

$$\alpha_n((1 - n\epsilon)^2 - 4\epsilon r(n-1)) > \alpha_1((1 - n\epsilon)^2 - 4\epsilon r)\,.$$

Since $(1 - n\epsilon)^2 - 4\epsilon r(n - 1) \leq (1 - n\epsilon)^2 - 4\epsilon r$ and $\alpha_1 < 0 < \alpha_n$, it suffices to check that $(1 - n\epsilon)^2 - 4r\epsilon(n - 1) > 0$. That is, $n^2\epsilon^2 - 2(n + 2r(n - 1))\epsilon + 1 > 0$. And this follows from $\epsilon < \epsilon_1$, since ϵ_1 is the lowest root of the polynomial.

Since $\alpha_1 = \min_{\omega \in \Omega} a_1(\omega)$ and $\alpha_n = \max_{\omega \in \Omega} a_1(\omega)$, Lemma 4.5 ensures that $[-\lambda_+, -\lambda_-] \subset (\alpha_1, \alpha_2)$. Finally, note that all the hypotheses of Proposition 4.9 are fulfilled with $k_1 = \alpha_1$ and $k_2 = \alpha_n$. This proves the last assertion. $\qquad\square$

Note that every function a_1 constructed by the procedure of Proposition 4.11 takes positive and negative values. But this is not a real restriction to get a generalized pitchfork bifurcation diagram, since that corresponding to $a_1 + \mu$ for any constant $\mu \in \mathbb{R}$ is of the same type.

Proposition 4.11 shows that the occurrence of families (4.1) with generalized pitchfork bifurcation diagram only requires the existence of two different ergodic measures. The functions a_1 constructed as there indicated are intended to satisfy (4.4); that is, their extremal Lyapunov exponents are near its maximum and minimum. But in fact this is not a necessary condition for a function a_1 to be the first order coefficient of a polynomial giving rise to a generalized pitchfork bifurcation. Theorem 4.14 proves this assertion in the case of a finitely ergodic base flow. Its proof in based on Proposition 4.12 and Corollary 4.13.

Proposition 4.12 *Assume that $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma) = \{m_1, \ldots, m_n\}$ with $n \geq 1$. There exists $\epsilon_2 > 0$ such that, if $0 < \epsilon \leq \epsilon_2$ and $c_1, \ldots, c_n \colon \Omega \to \mathbb{R}$ are the functions constructed in Lemma 4.10 for $m_1, \ldots, m_n$ and ϵ, then*

$$C(\Omega) = \langle c_1, \ldots, c_n \rangle \oplus C_0(\Omega)$$

as topological sum of vector spaces, where $C(\Omega)$ is endowed with the uniform topology, given by $\|a\| = \max_{\omega \in \Omega} |a(\omega)|$. In particular, the Sacker and Sell spectrum of $a \in C(\Omega)$ coincides with that of its projection onto $\langle c_1, \ldots, c_n \rangle$.

Proof Let $M_{n \times n}(\mathbb{R})$ be the linear space of $n \times n$ real matrices, which we endow with the norm $\|C\|_\infty = \max_{1 \leq i, j \leq n} |c_{ij}|$, where $C = \{c_{ij}\}_{1 \leq i, j \leq n}$. The set of regular $n \times n$ real matrices $GL_n(\mathbb{R})$ is an open subset of $M_{n \times n}(\mathbb{R})$, and the identity matrix I belongs to $GL_n(\mathbb{R})$. Hence, there exists $\epsilon_2 \in (0, 1)$ such that, if $\|C - I\|_\infty \leq \epsilon_2$, then C is regular. Therefore, if $\epsilon \in (0, \epsilon_2]$, then the corresponding functions $c_1, \ldots, c_n$ of Lemma 4.10 provide a regular matrix

$$C = \begin{pmatrix} \int_\Omega c_1(\omega)\, dm_1 & \ldots & \int_\Omega c_n(\omega)\, dm_1 \\ \vdots & \ddots & \vdots \\ \int_\Omega c_1(\omega)\, dm_n & \ldots & \int_\Omega c_n(\omega)\, dm_n \end{pmatrix}.$$

Let us consider the continuous linear functionals $T_i \colon C(\Omega) \to \mathbb{R}$, $f \mapsto \int_\Omega f(\omega)\, dm_i$ for $i \in \{1, \ldots, n\}$, and note that $\mathrm{Ker}(T_i)$ has codimension 1. Therefore, the codimension of the set $C_0(\Omega)$, which coincides with $\bigcap_{i \in \{1, \ldots, n\}} \mathrm{Ker}(T_i)$, is at most n. In addition, the linear space $\langle c_1, \ldots, c_n \rangle$ has dimension n, since the supports of $c_1, \ldots, c_n$ are pairwise disjoint. Let us check that $\langle c_1, \ldots, c_n \rangle \cap C_0(\Omega) = \{0\}$: if $c = \sum_{i=1}^n \alpha_i c_i \in C_0(\Omega)$, then $0 = \int_\Omega c(\omega)\, dm_j = \sum_{i=1}^n \alpha_i \int_\Omega c_i(\omega)\, dm_j$ for every $j \in \{1, \ldots, n\}$. These n equations provide a homogeneous linear system for $\alpha_1, \ldots, \alpha_n$ with regular coefficient matrix C; so $\alpha_1 = \cdots = \alpha_n = 0$ and hence $c = 0$. Consequently, $C(\Omega)$ is the algebraic direct sum of $\langle c_1, \ldots, c_n \rangle$ and $C_0(\Omega)$. We will check that the projections of $C(\Omega)$ onto each one of the subspaces are continuous, which will complete the proof of the first assertion. Given $a \in C(\Omega)$, its projection $P_{\langle c_1, \ldots, c_n \rangle} a =$

$\sum_{i=1}^{n} \alpha_i c_i$ onto $\langle c_1, \ldots, c_n \rangle$ is given by

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = C^{-1} \begin{pmatrix} \int_\Omega a(\omega) \, dm_1 \\ \vdots \\ \int_\Omega a(\omega) \, dm_n \end{pmatrix}.$$

Therefore, $\|\alpha_i c_i\| = |\alpha_i| \leq n\|C^{-1}\|_\infty \|a\|$ for every $i \in \{1, \ldots, n\}$, and hence $\|P_{\langle c_1, \ldots, c_n \rangle} a\| = \|\sum_{i=1}^{n} \alpha_i c_i\| \leq n^2 \|C^{-1}\|_\infty \|a\|$. So, $P_{\langle c_1, \ldots, c_n \rangle} \colon C(\Omega) \to \langle c_1, \ldots, c_n \rangle$ is continuous. Finally, as $P_{C_0(\Omega)} a = a - P_{\langle c_1, \ldots, c_n \rangle} a$, also the projection $P_{C_0(\Omega)}$ is continuous, as asserted. The second assertion is an easy consequence of the first one. □

Corollary 4.13 *Assume that $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma) = \{m_1, \ldots, m_n\}$ with $n \geq 2$, and take $r \geq 1$. Let $a_1 \in C(\Omega)$ have Sacker and Sell spectrum $sp(a_1) = [-\lambda_+, -\lambda_-]$ with $\lambda_- < 0 < \lambda_+$. Then, there exist $\tilde{a}_1 \in C(\Omega)$ with Sacker and Sell spectrum $sp(\tilde{a}_1) = [-\lambda_+, -\lambda_-]$ and $a_1 - \tilde{a}_1 \in C_0(\Omega)$, and $k_1, k_2 \in \mathbb{R}$ such that $k_1 \leq \tilde{a}_1(\omega) \leq k_2$ for all $\omega \in \Omega$ and*

$$(\lambda_+ - \lambda_-)^2 + 4\, r(\lambda_+ + k_1)(\lambda_+ + k_2) > 0. \tag{4.8}$$

Proof We take $\epsilon < \min(\epsilon_1, \epsilon_2)$, with ϵ_1 and ϵ_2 respectively provided by Propositions 4.11 and 4.12. Let $c_1, \ldots, c_n$ be the functions given by Lemma 4.10 for $m_1, \ldots, m_n$ and ϵ. Proposition 4.12 provides $\alpha_1, \cdots, \alpha_n$ such that the map $\tilde{a}_1 = \alpha_1 c_1 + \cdots + \alpha_n c_n$ satisfies $sp(\tilde{a}_1) = sp(a_1)$. Hence, since $\lambda_- < 0 < \lambda_+$, $\tilde{a}_1$ takes positive and negative values, and therefore there exist i_1 with $\alpha_{i_1} < 0$ and i_2 with $\alpha_{i_2} > 0$. The result follows from Proposition 4.12 by reordering the measures $m_1, \ldots, m_n$. □

Theorem 4.14 *Assume that $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma) = \{m_1, \ldots, m_n\}$ with $n \geq 2$. Let $a_1 \in C(\Omega)$ have Sacker and Sell spectrum $sp(a_1) = [-\lambda_+, -\lambda_-]$ with $\lambda_- < \lambda_+$. Then, there exist strictly positive functions $a_2, a_3 \in C(\Omega)$ such that (4.1) exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.1(iii).*

Proof We take any strictly positive $\tilde{a}_3 \in C(\Omega)$ and $0 < r_1 \leq r_2$ with $r_1 \leq \tilde{a}_3(\omega) \leq r_2$ for all $\omega \in \Omega$, and call $r = r_2/r_1$. There is no loss of generality in assuming that $\lambda_- < 0 < \lambda_+$, since the bifurcation diagrams for a_1 and $a_1 + \mu$ coincide for any $\mu \in \mathbb{R}$. We associate $\tilde{a}_1$ to a_1 and r by Corollary 4.13. Note that there exists $\delta_0 > 0$ such that, if $|\lambda_+ - \mu_+| < \delta_0$ and $|\lambda_- - \mu_-| < \delta_0$, then $\mu_- < \mu_+$ and (4.8) holds for μ_+, μ_- instead of λ_+, λ_- and $k_1 + \delta_0, k_2 - \delta_0$ instead of k_1, k_2. That is, if $c \in C(\Omega)$ satisfies $|c(\omega)| < \delta_0$ for all $\omega \in \Omega$, then $sp(\tilde{a}_1 + c) = [-\mu_+, -\mu_-]$ with $\mu_- < \mu_+$, $k_1 - \delta_0 \leq \tilde{a}_1(\omega) + c(\omega) \leq k_2 + \delta_0$ for all $\omega \in \Omega$, and

$$(\mu_+ - \mu_-)^2 + 4\, r(\mu_+ + k_1 - \delta_0)(\mu_+ + k_2 + \delta_0) > 0.$$

Since $CP(\Omega)$ is dense in $C_0(\Omega)$ (see Sect. 2.2), there exists $b \in C^1(\Omega)$ such that $\max_{\omega \in \Omega} |a_1(\omega) - \tilde{a}_1(\omega) - b'(\omega)| < \delta_0$. We take $c = a_1 - \tilde{a}_1 - b'$ and call $sp(\tilde{a}_1 + c) = [-\mu_+, -\mu_-]$. Now, in order to apply Proposition 4.9, we take $\tilde{a}_2 \in C(\Omega)$ satisfying $2\sqrt{r_2(\mu_+ - k_1 - \delta_0)} < \tilde{a}_2(\omega) < \sqrt{r_1}(\mu_+ - \mu_-)/(\sqrt{(\mu_+ + k_2 - \delta_0)})$ for all $\omega \in \Omega$. Hence, the parametric family

$$x' = -\tilde{a}_3(\omega{\cdot}t)x^3 + \tilde{a}_2(\omega{\cdot}t)x^2 + (\tilde{a}_1(\omega{\cdot}t) + c(\omega{\cdot}t) + \lambda)x, \quad \omega \in \Omega \tag{4.9}$$

presents a generalized pitchfork bifurcation of minimal sets. As explained in the proof of Proposition 4.2, the family of changes of variables $y(t) = e^{b(\omega{\cdot}t)}x(t)$ takes (4.9) to

$$y' = -e^{-2b(\omega{\cdot}t)}\tilde{a}_3(\omega{\cdot}t)y^3 + e^{-b(\omega{\cdot}t)}\tilde{a}_2(\omega{\cdot}t)y^2 + (\tilde{a}_1(\omega{\cdot}t) + c(\omega{\cdot}t) + b'(\omega{\cdot}t) + \lambda)y$$

$$= -e^{-2b(\omega{\cdot}t)}\tilde{a}_3(\omega{\cdot}t)y^3 + e^{-b(\omega{\cdot}t)}\tilde{a}_2(\omega{\cdot}t)y^2 + (a_1(\omega{\cdot}t) + \lambda)y$$

without changing the global structure of the bifurcation diagram. That is, the strictly positive functions $a_3 = e^{-2b}\tilde{a}_3$ and $a_2 = e^{-b}\tilde{a}_2$ fulfill the statement. □

5 Criteria in a More General Framework

The ideas of Sects. 4.2 and 4.3 can be used to construct examples of all the three possible types of bifurcation diagrams described in Theorem 3.1 for families of differential equations of a more general type. Let us consider

$$x' = (-a_3(\omega{\cdot}t) + h(\omega{\cdot}t, x))x^3 + a_2(\omega{\cdot}t)x^2 + (a_1(\omega{\cdot}t) + \lambda)x, \quad \omega \in \Omega, \qquad (5.1)$$

where $a_i \in C(\Omega)$ for $i \in \{1, 2, 3\}$, a_3 is strictly positive, $h \in C^{0,2}(\Omega{\times}\mathbb{R}, \mathbb{R})$, and $h(\omega, 0) = 0$ for all $\omega \in \Omega$. Throughout the section, we will represent the Sacker and Sell spectrum of a_1 as $\mathrm{sp}(a_1) = [-\lambda_+, -\lambda_-]$, with $\lambda_- \leq \lambda_+$. In addition, $k_1 \leq k_2$ represent real constants satisfying $k_1 \leq a_1(\omega) \leq k_2$, and $0 < r_1 \leq r_2$ represent real constants satisfying $r_1 \leq a_3(\omega) \leq r_2$ for all $\omega \in \Omega$.

In the line of part of the results of Sect. 4, we fix a_1, a_3 and h satisfying the mentioned hypotheses as well as a fundamental extra condition which relates the behavior of h for small values of x to the properties of a_1 and a_3, and such that the function

$$f(\omega, x) = (-a_3(\omega) + h(\omega, x))x^3 + a_2(\omega)x^2 + a_1(\omega)x \qquad (5.2)$$

is (Co) and (SDC)$_*$. The goal is to describe conditions on a_2 determining each one of the possible bifurcation cases described in Theorem 3.1 for (5.1). The function a_2 will be sign-preserving under all these conditions. Note the $\mathrm{sp}(a_1)$ is the Sacker and Sell spectrum of f_x on $\mathcal{M}_0$.

Proposition 5.1 *Assume that the function f given by* (5.2) *is* (Co) *and* (SDC)$_*$, *and that*

(H) *there exist $\rho_0 > 0$ and $0 < \epsilon_0 < r_1$ such that $|h(\omega, x)| \leq \epsilon_0$ for all $\omega \in \Omega$ if $|x| \leq \rho_0$, $\sqrt{(\lambda_+ + k_2)/(r_1 - \epsilon_0)} < \rho_0$, and $\sqrt{(-\lambda_- - k_1)/(r_2 + \epsilon_0)} < \rho_0$.*

We call $s_1 = r_1 - \epsilon_0$ and $s_0 = r_2 + \epsilon_0$. Then,

(i) *if $a_2(\omega) = 0$ for all $\omega \in \Omega$, then* (5.1) *exhibits the classical pitchfork bifurcation of minimal sets described in Theorem 3.1(ii).*

(ii) *If $k_1 < -\lambda_+$ and*

$$a_2(\omega) > 2\sqrt{s_2(-\lambda_- - k_1)} \quad (resp.\ a_2(\omega) < -2\sqrt{s_2(-\lambda_- - k_1)})$$

for all $\omega \in \Omega$, then (5.1) *exhibits the local saddle-node and transcritical bifurcations of minimal sets described in Theorem 3.1(i), with α_λ (resp. β_λ) colliding with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$.*

(iii) *If $k_1 < -\lambda_+$ and*

$$a_2(\omega) > 2\sqrt{s_2(-\lambda_+ - k_1)} \quad (resp.\ a_2(\omega) < -2\sqrt{s_2(-\lambda_+ - k_1)})$$

for all $\omega \in \Omega$, then (5.1) *does not exhibit the classical pitchfork bifurcation of minimal sets described in Theorem 3.1(ii).*

(iv) *If $0 \leq a_2(\omega) < (\lambda_+ - \lambda_-)\sqrt{s_1/(\lambda_+ + k_2)}$ (resp. $-(\lambda_+ - \lambda_-)\sqrt{s_1/(\lambda_+ + k_2)} < a_2(\omega) \leq 0$) for all $\omega \in \Omega$, then* (5.1) *does not exhibit the local saddle-node and transcritical bifurcations of minimal sets described in Theorem 3.1(i).*

(v) *If*

$$s_1(\lambda_+ - \lambda_-)^2 + 4 s_2(\lambda_+ + k_1)(\lambda_+ + k_2) > 0,$$

  and

$$2\sqrt{s_2(-\lambda_+ - k_1)} < a_2(\omega) < \frac{(\lambda_+ - \lambda_-)\sqrt{s_1}}{\sqrt{\lambda_+ + k_2}}$$

$$\left(resp. \quad -\frac{(\lambda_+ - \lambda_-)\sqrt{s_1}}{\sqrt{\lambda_+ + k_2}} < a_2(\omega) < -2\sqrt{s_2(-\lambda_+ - k_1)}\right)$$

for all $\omega \in \Omega$, then (5.1) exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.1(iii).

Proof (i) This property is proved in Proposition 6.5(ii) of [11].

(ii)-(iii) We define $\rho_\pm = \sqrt{(-\lambda_\pm - k_1)/s_2}$ and observe that $\rho_+ \leq \rho_- < \rho_0$. Hence, $a_3(\omega) + h(\omega, \rho_\pm) \leq r_2 + \epsilon_0 = s_2$. This allows us to check the assertions of Lemma 4.6 for $\lambda = \lambda_+$ and $\lambda = \lambda_- - \delta$ (for some small $\delta > 0$ for which $\sqrt{(-(\lambda_- - \delta) - k_1)/s_2} < \rho_0$) and with r_2 replaced by s_2, just by repeating the same proof. Once this is done, the arguments of Proposition 4.7(i) and (ii) prove assertions (ii) and (iii).

(iv) The proof is analogous to that of Proposition 4.8. We sketch it in the case of positive a_2 and refer to Proposition 4.8 for the details. Condition (H) and the hypothesis in (iv) allow us to take $\delta > 0$ such that, if $\lambda \in [\lambda_+, \lambda_+ + \delta]$, then $\sqrt{(\lambda + k_2)/(r_1 - \epsilon_0)} < \rho_0$ and $a_2(\omega) \leq (\lambda_+ - \lambda_-)\sqrt{s_1/(\lambda + k_2)}$ for all $\omega \in \Omega$. We fix one of these values of λ and take ρ satisfying $\sqrt{(\lambda + k_2)/s_1} < \rho \leq \rho_0$ in order to deduce from the bound $a_3(\omega) + h(\omega, \rho) \geq r_1 - \epsilon_0 = s_1$ (ensured by (H)) that ρ is a strict global upper solution of $x' = (-a_3(\omega \cdot t) + h(\omega \cdot t, x))x^3 + (a_1(\omega \cdot t) + \lambda)x$. Let $\hat{\beta}_\lambda$ be the upper delimiter equilibrium of the global attractor of the corresponding skew-product flow. The next step is checking that $\hat{\beta}_\lambda(\omega) \leq \sqrt{(\lambda + k_2)/s_1}$ for all $\omega \in \Omega$ if $\lambda \in (\lambda_+, \lambda_+ + \delta]$. Once this is done, we can deduce that $\hat{\beta}_\lambda$ is also a strong τ_{λ_-}-superequilibrium and that the limit $\hat{\beta}_{\lambda_+} = \lim_{\lambda \downarrow \lambda_+} \hat{\beta}_\lambda$ takes the value 0 on a residual set of points. Finally, we assume that (5.1) is in case (i) of Theorem 3.1 and use the previous properties to get a contradiction.

(v) As in Proposition 4.9, this follows from (iii), (iv) and Theorem 3.1. □

We complete this short section by analyzing ways to guarantee the initial hypotheses of Proposition 5.1: f is (Co) and (SDC)$_*$, and (H) holds. The first objectives are achieved, at least, in the following situations:

Proposition 5.2 *Assume that $(-a_3(\omega) + h(\omega, x))x^3$ is (Co)$_2$ and (SDC)$_*$. Then, the function f defined by (5.2) is (Co)$_2$ and (SDC)$_*$. Moreover, if $(-r + h(\omega, x))x^3$ is (Co) and (DC) for some $r < r_1$ (or $r \leq r_1$ if a_3 is not constant), then $(-a_3(\omega) + h(\omega, x))x^3$ is (Co)$_2$ and (SDC)$_*$.*

Proof Consider $f(\omega, x) = ((-a_3(\omega) + h(\omega, x))x^3) + (a_2(\omega)x^2 + a_1(\omega)x)$ and recall that any quadratic polynomial is (DC); that the sum of a (SDC)$_*$ function and a (DC) one is (SDC)$_*$ (see Sect. 3 of [11]); and that the sum of a (Co)$_2$ function and a second degree polynomial is (Co)$_2$.

Let us check the second assertion. Since $(r - a_3(\omega))x^3$ is (SDC)$_*$ (see Sect. 2.6) and $(-r + h(\omega, x))x^3$ is (DC), the sum is (SDC)$_*$. In addition, the (Co) property of $(-r + h(\omega, x))x^3$ means that $\lim_{|x| \to \infty}(-r + h(\omega, x))x^2 = -\infty$ uniformly on Ω, and hence there exists $\rho > 0$ such that $h(\omega, x) < r$ for all $\omega \in \Omega$ and $x \geq \rho$. Consequently,

$$\limsup_{x \to \infty}(-a_3(\omega) + h(\omega, x))x \leq \limsup_{x \to \infty}(-a_3(\omega) + r)x = -\infty$$

for all $\omega \in \Omega$, and hence $\lim_{x\to\infty}(-a_3(\omega)+h(\omega,x))x = -\infty$ for all $\omega \in \Omega$. The compactness of Ω ensures that the limit is uniform. Analogously, $\lim_{x\to-\infty}(-a_3(\omega)+h(\omega,x))x = \infty$ uniformly on Ω. Hence, $(-a_3(\omega) + h(\omega,x))x^3$ is $(Co)_2$. $\qquad\square$

Regarding (H), notice that the included inequality $\sqrt{(-\lambda_- - k_1)/(r_2 + \epsilon_0)} < \rho_0$ is fulfilled by taking a large enough upper bound r_2 for a_3 (although the smaller r_2 is, the less restrictive the conditions in points (ii), (iii) and (v) of Proposition 5.1 are). The following results indicate three ways to get the rest of the conditions in (H). Recall that h is always assumed to belong to $C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ and to satisfy $h(\omega, 0) = 0$ for all $\omega \in \Omega$. And recall also the meaning of $\lambda_-, \lambda_+, k_1, k_2, r_1$ and r_2.

Proposition 5.3

(i) *Assume that a_3 and h are fixed, take $0 < \epsilon_0 < r_1$ and $\rho_0 > 0$ such that $|h(\omega, x)| \le \epsilon_0$ for all $\omega \in \Omega$ if $|x| \le \rho_0$, and call $s_1 = r_1 - \epsilon_0$. Let $k_1 < 0 < k_2$ satisfy $k_2 - k_1 < \rho_0^2 s_1$ and $a_1 \in C(\Omega)$ satisfy $k_1 \le a_1(\omega) \le k_2$ for all $\omega \in \Omega$, choose the upper bound r_2 for a_3 large enough to get $\sqrt{(k_2 - k_1)/(r_2 + \epsilon_0)} < \rho_0$, and call $s_2 = r_2 + \epsilon_0$. Then, (H) is fulfilled. If, in addition, (Ω, σ) is not uniquely ergodic, then a_1 can be chosen to get $\lambda_- < \lambda_+$ and $s_1(\lambda_+ - \lambda_-)^2 + 4 s_2(\lambda_+ + k_1)(\lambda_+ + k_2) > 0$. Consequently, if the function f given by (5.2) is (Co) and (SDC)$_*$, suitable choices of a_1 and a_2 provide the three possible bifurcation diagrams for (5.1).*

(ii) *Assume that a_1 and h are fixed and take $\epsilon_0 > 0$ and $\rho_0 > 0$ such that $|h(\omega, x)| \le \epsilon_0$ for all $\omega \in \Omega$ if $|x| \le \rho_0$. If $r_1 > (\lambda_+ + k_2)/\rho_0^2 + \epsilon_0$ and $a_3 \in C(\Omega)$ satisfies $r_1 \le a_3(\omega)$ for all $\omega \in \Omega$, then (H) is fulfilled for large enough r_2.*

(iii) *Assume that a_1 and a_3 are fixed, and take ρ_0 and m with $0 < \rho_0 < \sqrt{(\lambda_+ + k_2)/r_1}$ and $0 < m < r_1/\rho_0 - (\lambda_+ + k_2)/\rho_0^3$. If $h \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies $h(\omega, 0) = 0$ and $|h_x(\omega, x)| \le m$ for all $\omega \in \Omega$ if $|x| \le \rho_0$, then (H) is fulfilled with $\epsilon_0 = \rho_0 m$ and a large enough r_2.*

Proof

(i) Notice that $\sqrt{(-\lambda_- - k_1)/(r_2 + \epsilon_0)} \le \sqrt{(k_2 - k_1)/s_2} < \rho_0$ and also $\sqrt{(\lambda_+ + k_2)/(r_1 - \epsilon_0)} \le \sqrt{(k_2 - k_1)/s_1} < \rho_0$. This proves the first assertion.

To prove the second one, we apply Proposition 4.11 with $\alpha_1 = k_1$ and $\alpha_n = k_2$. It shows that a_1 has band spectrum $[-\lambda_+, -\lambda_-]$ and that $(\lambda_+ - \lambda_-)^2 + 4s(\lambda_+ + k_1)(\lambda_+ + k_2) > 0$. Lemma 4.5 shows that $\mathrm{sp}(a_1) \subset (k_1, k_2)$. The last assertion in (i) follows from Proposition 5.1.

(ii) If $r_1 > (\lambda_+ + k_2)/\rho_0^2 + \epsilon_0$, then $\sqrt{(\lambda_+ + k_2)/(r_1 - \epsilon_0)} < \rho_0$.

(iii) It is clear that $0 < \epsilon_0 < r_1$, and easy to check that $\sqrt{(\lambda_+ + k_2)/(r_1 - \epsilon_0)} < \rho_0$. A large enough s_2 ensures $\sqrt{(-\lambda_- - k_1)/(r_2 + \epsilon_0)} < \rho_0$. The equality $h(\omega, x) = \int_0^1 x h_x(\omega, sx)\,ds$ yields $|h(\omega, x)| \le \rho_0 m = \epsilon_0$ for $|x| \le \rho_0$. $\qquad\square$

6 A Second Bifurcation Problem

The ideas and methods developed in [11] and in the previous sections of this paper allow us to classify and describe all the possibilities for the bifurcation diagram of a problem different from that analyzed in Sects. 3, 4 and 5, namely

$$x' = f(\omega{\cdot}t, x) + \mu x^2, \quad \omega \in \Omega. \tag{6.1}$$

Besides its own interest, this analysis allows us to go deeper in the construction of patterns for the three bifurcation possibilities described in Theorem 3.1, as explained at the end of this section.

As before, f is assumed to be $C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ and $(Co)_2$ (see Sect. 2.4) with $f(\omega, 0) = 0$ for all $\omega \in \Omega$, and $\mu \in \mathbb{R}$ is the bifurcation parameter. We expect not to generate risk of confusion while denoting by $\tau_\mu : \mathcal{U}_\mu \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ the local skewproduct flow $(t, \omega, x_0) \mapsto (\omega \cdot t, u_\mu(t, \omega, x_0))$ defined from the maximal solutions $\mathcal{I}_{\omega, x_0}^\mu \to \mathbb{R}$, $t \mapsto u_\mu(t, \omega, x_0)$ of $(6.1)_\mu$. Note that the set $\mathcal{M}_0 = \Omega \times \{0\}$ is τ_μ-minimal for all $\mu \in \mathbb{R}$, and that the Sacker and Sell spectrum of $f_x + 2\mu x$ on $\mathcal{M}_0$ (see Sect. 2.5) is independent of μ. We represent it by $\mathrm{sp}(f_x(\cdot, 0))$.

Note that $f + \mu x^2$ is $(Co)_2$ if f is (unlike what happens with the (Co) condition).

We will denote by $\mathcal{A}_\mu$ the corresponding global attractor. As recalled in Sect. 2.4, $\mathcal{A}_\mu = \bigcup_{\omega \in \Omega} (\{\omega\} \times [\alpha_\mu(\omega), \beta_\mu(\omega)])$, where α_μ and β_μ are (lower and upper) semicontinuous τ_μ-equilibria. The following proposition analyzes the behavior of these maps as μ varies.

Proposition 6.1 *Let $f \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ be $(Co)_2$. Then,*

(i) *for any $\omega \in \Omega$, the maps $\mu \mapsto \beta_\mu(\omega)$ and $\mu \mapsto \alpha_\mu(\omega)$ are nondecreasing on $\mathbb{R}$ and they are, respectively, right- and left-continuous. If β_{μ_0} (resp. α_{μ_0}) is strictly positive (resp. strictly negative) for some $\mu_0 \in \mathbb{R}$, then $\beta_\mu(\omega) < \beta_{\mu_0}(\omega)$ (resp. $\alpha_\mu(\omega) < \alpha_{\mu_0}(\omega)$) for all $\mu < \mu_0$ and $\omega \in \Omega$, and $\beta_{\mu_0}(\omega) < \beta_\mu(\omega)$ (resp. $\alpha_{\mu_0}(\omega) < \alpha_\mu(\omega)$) for all $\mu > \mu_0$ and $\omega \in \Omega$.*

(ii) $\lim_{\mu \to -\infty} \alpha_\mu(\omega) = -\infty$ *and* $\lim_{\mu \to \infty} \beta_\mu(\omega) = \infty$ *uniformly on Ω.*

Proof

(i) Let $\xi < \mu$. Since $\beta'_\xi(\omega) = f(\omega, \beta_\xi(\omega)) + \xi \beta_\xi^2(\omega) \leq f(\omega, \beta_\xi(\omega)) + \mu \beta_\xi^2(\omega)$, β_ξ is a global lower solution of $(6.1)_\mu$. Consequently, $\beta_\xi(\omega) \leq \beta_\mu(\omega)$ for all $\omega \in \Omega$ (see Sect. 2.4). Moreover, if β_{μ_0} is strictly positive, the previous inequalities are strict. The case of α_μ follows analogously. To show the one-side continuity, we proceed as in the proof of Theorem 5.5(i) of [11].

(ii) Let $\rho > 0$ be fixed. There exists $\mu_\rho > 0$ such that $f(\omega, \rho) + \mu_\rho \rho^2 \geq 0$ for all $\omega \in \Omega$, and $f(\omega, -\rho) - \mu_\rho(-\rho)^2 < 0$ for all $\omega \in \Omega$, and hence $\omega \mapsto \rho$ is a global lower τ_μ-solution and $-\rho$ is a global upper $\tau_{-\mu}$-solution for $\mu \geq \mu_\rho$. This ensures that $\rho \leq \beta_\mu(\omega)$ for all $\omega \in \Omega$ and $-\rho \geq \alpha_{-\mu}(\omega)$ for all $\omega \in \Omega$ if $\mu \geq \mu_\rho$ (see again Sect. 2.4), as we wanted to prove.

□

The proof of Theorem 6.3, which describes the possible bifurcation diagrams for (6.1) (depicted in Figs. 3 and 4), requires the next technical result, similar to Proposition 4.4 of [11].

Proposition 6.2 *Let $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ be (DC), let us fix $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$, $\lambda_0 > 0$ and $v < \mu$ (resp. $\mu < v$), let $\kappa_\mu^1 : \Omega \to \mathbb{R}$ be an m-measurable τ_μ-equilibrium, and let $\kappa_v^2 : \Omega \to \mathbb{R}$ be an m-measurable equilibrium of $x' = f(\omega \cdot t, x) - \lambda_0 x + v x^2$ such that $0 < \kappa_v^2(\omega) < \kappa_\mu^1(\omega)$ (resp. $\kappa_\mu^1(\omega) < \kappa_v^2(\omega) < 0$) for m-a.e. $\omega \in \Omega$. Then,*

$$\int_\Omega \left(f_x(\omega, \kappa_\mu^1(\omega)) + 2\mu \kappa_\mu^1(\omega) \right) dm + \int_\Omega \left(f_x(\omega, \kappa_v^2(\omega)) - \lambda_0 + 2v \kappa_v^2(\omega) \right) dm < 0 \,.$$

Proof We define the m-a.e. positive (resp. negative) function $k(\omega) = \kappa_\mu^1(\omega) - \kappa_\nu^2(\omega)$ and $F(\omega, y) = \int_0^1 f_x(\omega, sy + \kappa_\nu^2(\omega)) \, ds$. Then, it is not hard to check that

$$\frac{k'(\omega \cdot t)}{k(\omega \cdot t)} = F(\omega \cdot t, k(\omega \cdot t)) + (\mu - \nu)\frac{(\kappa_\mu^1(\omega \cdot t))^2}{k(\omega \cdot t)} + \nu(\kappa_\mu^1(\omega \cdot t) + \kappa_\nu^2(\omega \cdot t)) + \lambda_0 \frac{\kappa_\nu^2(\omega \cdot t)}{k(\omega \cdot t)}.$$

Notice that κ_μ^1, κ_ν^2 and k are bounded (see Sect. 2.4), that $k'(\omega)/k(\omega) - F(\omega, k(\omega)) - \nu(\kappa_\mu^1(\omega) + \kappa_\nu^2(\omega))$ is strictly positive for m-a.e. $\omega \in \Omega$, and that $F(\omega, k(\omega))$ and $\nu(\kappa_\mu^1(\omega) + \kappa_\nu^2(\omega))$ are in $L^1(\Omega)$. Then, Birkoff's Ergodic Theorem (see Theorem 1 in Sect. 2 of Chapter 1 of [10] and Proposition 1.4 of [15]) yields

$$\begin{aligned}
0 = \int_\Omega F(\omega, k(\omega)) \, dm + (\mu - \nu) \int_\Omega \frac{(\kappa_\mu^1(\omega))^2}{k(\omega)} \, dm \\
+ \nu \int_\Omega (\kappa_\mu^1(\omega) + \kappa_\nu^2(\omega)) \, dm + \lambda_0 \int_\Omega \frac{\kappa_\nu^2(\omega)}{k(\omega)} \, dm.
\end{aligned} \tag{6.2}$$

The d-concavity of f and Taylor's Theorem ensure that $F(\omega, k(\omega)) - F(\omega, 0) \geq k(\omega) F_y(\omega, k(\omega))$ for all $\omega \in \Omega$. Now, we derive $y F(\omega, y) = f(\omega, y + \kappa_\nu^2(\omega)) - f(\omega, \kappa_\nu^2(\omega))$ with respect to y and evaluate at $y = k(\omega)$, obtaining $F(\omega, k(\omega)) + k(\omega) F_y(\omega, k(\omega)) = f_x(\omega, \kappa_\mu^1(\omega))$. Since $F(\omega, 0) = f_x(\omega, \kappa_\nu^2(\omega))$, the last equality and the previous inequality yield $f_x(\omega, \kappa_\mu^1(\omega)) + f_x(\omega, \kappa_\nu^2(\omega)) \leq 2F(\omega, k(\omega))$. In turn, this inequality and (6.2) yield

$$\begin{aligned}
\int_\Omega (f_x(\omega, \kappa_\mu^1(\omega)) + 2\mu\kappa_\mu^1(\omega)) \, dm + \int_\Omega (f_x(\omega, \kappa_\nu^2(\omega)) - \lambda_0 + 2\nu\kappa_\nu^2(\omega)) \, dm \\
\leq 2 \int_\Omega F(\omega, k(\omega)) \, dm + 2 \int_\Omega (\mu\kappa_\mu^1(\omega) + \nu\kappa_\nu^2(\omega)) \, dm - \lambda_0 \\
= -2(\mu - \nu) \int_\Omega \frac{\kappa_\mu^1(\omega)\kappa_\nu^2(\omega)}{k(\omega)} \, dm - \lambda_0 \int_\Omega \frac{\kappa_\mu^1(\omega) + \kappa_\nu^2(\omega)}{k(\omega)} \, dm < 0,
\end{aligned}$$

which proves the statement in any of the two stated situations. □

Theorem 6.3 *Let $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ be* (Co)$_2$ *and* (SDC)$_*$.

(i) *(No bifurcation). If* $\mathrm{sp}(f_x(\cdot, 0)) \subset (0, \infty)$, *then for all* $\mu \in \mathbb{R}$ *there exist three hyperbolic τ_μ-minimal sets* $\mathcal{M}_\mu^l < \mathcal{M}_0 = \Omega \times \{0\} < \mathcal{M}_\mu^u$, *where* $\mathcal{M}_\mu^l$ *and* $\mathcal{M}_\mu^u$ *are respectively attractive hyperbolic copies of the base and given by the graphs of* α_μ *and* β_μ, $\mathcal{M}_0$ *is a repulsive hyperbolic copy of the base, and* $\lim_{\mu \to \infty} \alpha_\mu(\omega) = \lim_{\mu \to -\infty} \beta_\mu(\omega) = 0$ *uniformly on* Ω.

(ii) *(Two local saddle-node bifurcations). If* $\mathrm{sp}(f_x(\cdot, 0)) \subset (-\infty, 0)$, *then* $\mathcal{M}_0$ *is an attractive hyperbolic τ_μ-copy of the base for all* $\mu \in \mathbb{R}$ *and there exist* $\mu_1 < \mu_2$ *such that: for all* $\mu < \mu_1$ *(resp.* $\mu > \mu_2$), *there exist three hyperbolic τ_μ-copies of the base* $\mathcal{M}_\mu^l < \mathcal{N}_\mu < \mathcal{M}_0$ *(resp.* $\mathcal{M}_0 < \mathcal{N}_\mu < \mathcal{M}_\mu^u$) *which are hyperbolic copies of the base, given by the graphs of* $\alpha_\mu < \kappa_\mu < 0$ *(resp.* $0 < \kappa_\mu < \beta_\mu$), *where* $\mathcal{M}_\mu^l$ *(resp.* $\mathcal{M}_\mu^u$) *is hyperbolic attractive and* $\mathcal{N}_\mu$ *is hyperbolic repulsive, and* $\mu \mapsto \kappa_\mu$ *is strictly decreasing on* $(-\infty, \mu_1)$ *(resp.* (μ_2, ∞)); *the graphs of* α_μ *and* κ_μ *(resp.* κ_μ *and* β_μ) *collide on a residual σ-invariant set as* $\mu \uparrow \mu_1$ *(resp.* $\mu \downarrow \mu_2$), *giving rise to a nonhyperbolic minimal set* $\mathcal{M}_{\mu_1}^l$ *(resp.* $\mathcal{M}_{\mu_2}^u$); *for* $\mu \in (\mu_1, \mu_2)$, $\mathcal{A}_\mu = \mathcal{M}_0$; *and* $\lim_{\mu \to \pm\infty} \kappa_\mu(\omega) = 0$ *uniformly on* Ω.

(iii) *(Weak generalized transcritical bifurcation). If* $0 \in \mathrm{sp}(f_x(\cdot, 0))$, *then* $\mathcal{M}_0$ *is a nonhyperbolic τ_μ-copy of the base for all* $\mu \in \mathbb{R}$. *In addition, there exist* $\mu_1 \leq \mu_2$ *such that: for*

all $\mu < \mu_1$ (resp. $\mu > \mu_2$) there exist exactly two τ_μ-minimal sets $\mathcal{M}^l_\mu < \mathcal{M}_0$ (resp. $\mathcal{M}_0 < \mathcal{M}^u_\mu$), where $\mathcal{M}^l_\mu$ (resp. $\mathcal{M}^u_\mu$) is an attractive hyperbolic copy of the base given by the graph of α_μ (resp. β_μ); and if $\mathcal{M}_0$ is the unique τ_{μ_1}-minimal set (resp. τ_{μ_2}-minimal set), then α_μ and 0 (resp. 0 and β_μ) collide on a residual σ-invariant set as $\mu \uparrow \mu_1$ (resp. $\mu \downarrow \mu_2$). In addition, if $0 \neq \inf \mathrm{sp}(f_x(\cdot, 0))$, then $\mu_1 < \mu_2$ and $\mathcal{M}_0$ is the unique τ_μ-minimal set for any $\mu \in (\mu_1, \mu_2)$; and if $0 = \inf \mathrm{sp}(f_x(\cdot, 0))$, then $\mathcal{M}_0$ is the unique τ_μ-minimal set for all $\mu \in [\mu_1, \mu_2]$.

Proof We call $\mathcal{M}^l_\mu$ and $\mathcal{M}^u_\mu$ the lower and upper τ_μ-minimal sets, defined as in Sect. 3, and recall that they are attractive if they are hyperbolic, in which case they respectively coincide with the graphs of the continuous maps α_μ and β_μ.

(i) Since every Lyapunov exponent of $\mathcal{M}_0$ is strictly positive for any $\mu \in \mathbb{R}$ (see Sect. 2.5), $\mathcal{M}_0$ is a repulsive hyperbolic τ_μ-minimal set for every $\mu \in \mathbb{R}$. Consequently, there exist three different hyperbolic τ_μ-minimal sets $\mathcal{M}^l_\mu < \mathcal{M}_0 < \mathcal{M}^u_\mu$ (see Sect. 2.6), with $\mathcal{M}^l_\mu$ and $\mathcal{M}^u_\mu$ given respectively by the graphs of $\alpha_\mu < \beta_\mu$, which are continuous. The hyperbolic continuation of minimal sets (see Theorem 3.8 of [28]) guarantees the continuity of the maps $\mathbb{R} \to C(\Omega)$, $\mu \mapsto \beta_\mu$ and $\mathbb{R} \to C(\Omega)$, $\mu \mapsto \alpha_\mu$ in the uniform topology.

   To check that $\lim_{\mu \to -\infty} \beta_\mu(\omega) = 0$ uniformly on Ω, we take any $\epsilon > 0$, use coercivity to find $\rho > \epsilon > 0$ such that $f(\omega, x) \leq 0$ for all $x \geq \rho$ and $\omega \in \Omega$, and choose $\mu_\epsilon < 0$ such that $f(\omega, x) + \mu_\epsilon x^2 \leq 0$ for all $x \in [\epsilon, \rho]$ and $\omega \in \Omega$. Then, $f(\omega, x) + \mu_\epsilon x^2 \leq 0$ for all $x \geq \epsilon$ and $\omega \in \Omega$. According to Theorem 5.1(i) of [11], $\beta_\mu(\omega) \leq \epsilon$ for all $\omega \in \Omega$ if $\mu \leq \mu_\epsilon$, which proves the assertion. The argument is analogous for α_μ.

(ii) Since every Lyapunov exponent of $\mathcal{M}_0$ is strictly negative, $\mathcal{M}_0$ is always an attractive hyperbolic τ_μ-minimal set for all $\mu \in \mathbb{R}$. We fix $\rho > 0$ and look for μ^+_ρ such that $f(\omega, \rho) + \mu^+_\rho \rho^2 > 0$: if $\mu \geq \mu^+_\rho$, then ρ is a global strict lower solution of $(6.1)_\mu$, and hence its graph is strictly below (resp. strictly above) a minimal set $\mathcal{M}^u_\mu$ (resp. $\mathcal{N}_\mu$) contained in the ω-limit (resp. α-limit) of any point (ω, ρ) (see Sect. 2.3). The attractive hyperbolicity of $\mathcal{M}_0$ precludes $\mathcal{N}_\mu = \mathcal{M}_0$. That is, there exist three distinct τ_μ-minimal sets, so they are hyperbolic copies of the base, with $\mathcal{M}^u_\mu$ attractive (and given by the graph of β_μ) and $\mathcal{N}_\mu$ repulsive. A similar argument works for $-\rho$ and $\mu \leq \mu^-_\rho$ if $f(\omega, -\rho) + \mu^-_\rho \rho^2 < 0$, providing three different hyperbolic τ_μ-copies of the base $\mathcal{M}^l_\mu < \mathcal{N}_\mu < \mathcal{M}_0$, with $\mathcal{M}^l_\mu$ attractive (and given by the graph of α_μ) and $\mathcal{N}_\mu$ repulsive. Let κ_μ be τ_μ-equilibrium whose graph is $\mathcal{N}_\mu$, both for $\mu \geq \mu^+_\rho$ and $\mu \leq \mu^-_\rho$. Since the initially fixed $\rho > 0$ is as small as desired, $\lim_{\mu \to \pm \infty} \kappa_\mu(\omega) = 0$ uniformly on Ω.

   Let us define $\mathcal{I}_1 = \{\mu \in \mathbb{R} : \mathcal{M}^l_\xi \text{ is hyperbolic and } \mathcal{M}^l_\xi < \mathcal{M}_0 \text{ for all } \xi < \mu\}$ and $\mathcal{I}_2 = \{\mu \in \mathbb{R} : \mathcal{M}^u_\xi \text{ is hyperbolic and } \mathcal{M}^u_\xi > \mathcal{M}_0 \text{ for all } \xi > \mu\}$, and observe that $\mu^-_\rho \in \mathcal{I}_1$ and $\mu^+_\rho \in \mathcal{I}_2$. We also define $\mu_1 = \sup \mathcal{I}_1$ and $\mu_2 = \inf \mathcal{I}_2$ and note that $\mu_1 \notin \mathcal{I}_1$ and $\mu_2 \notin \mathcal{I}_2$: $\mathcal{I}_1 = (-\infty, \mu_1)$ and $\mathcal{I}_2 = (\mu_2, \infty)$. The existence of at most three τ_μ-minimal sets for any μ ensures $\mu_1 \leq \mu_2$, and hence they are finite. If $\mu \in \mathcal{I}_1$ (resp. $\mu \in \mathcal{I}_2$) then $\mathcal{M}^l_\mu$ (resp. $\mathcal{M}^u_\mu$) is attractive and coincides with the graph of α_μ (resp. β_μ), which is continuous, and there exists a repulsive hyperbolic copy of the base $\mathcal{N}_\mu$ for τ_μ with $\mathcal{M}^l_\xi < \mathcal{N}_\mu < \mathcal{M}_0$ (resp. $\mathcal{M}_0 < \mathcal{N}_\mu < \mathcal{M}^u_\xi$): see Sect. 2.6. Let κ_μ be the continuous τ_μ-equilibrium whose graph is $\mathcal{N}_\mu$ for $\mu \in \mathcal{I}_1 \cup \mathcal{I}_2$. As in (i), the hyperbolic continuation of minimal sets guarantees the continuity with respect to the uniform topology of the maps $\mu \mapsto \alpha_\mu$, κ_μ on $\mathcal{I}_1$ and $\mu \mapsto \kappa_\mu$, β_μ on $\mathcal{I}_2$. In addition, if $\xi_1 < \xi_2 < \mu_1$ or $\mu_2 < \xi_1 < \xi_2$, then κ_{ξ_1} is a strong time-reversed τ_{ξ_2}-superequilibrium, and hence the α-limit set for τ_{ξ_2} of a point $(\omega, \kappa_{\xi_1}(\omega))$ contains a τ_{ξ_2}-minimal set which

is strictly below the graph $\mathcal{N}_{\xi_1}$ of κ_{ξ_1}, and which (see again Sect. 2.3), as above, must coincide with $\mathcal{N}_{\xi_2}$. That is, $\mu \mapsto \kappa_\mu(\omega)$ is strictly decreasing on $\mathcal{I}_1$ and $\mathcal{I}_2$ for all $\omega \in \Omega$.

The maps $\alpha_{\mu_1}(\omega) = \lim_{\mu \uparrow \mu_1} \alpha_\mu(\omega)$ and $\kappa_{\mu_1}(\omega) = \lim_{\mu \uparrow \mu_1} \kappa_\mu(\omega)$ satisfy $\alpha_{\mu_1} \le \kappa_{\mu_1} < 0$ and respectively define lower and upper semicontinuous τ_{μ_1}-equilibria which must coincide on the residual set of its common continuity points: otherwise they would define two different τ_{μ_1}-minimal sets $\mathcal{M}^l_{\mu_1} < \mathcal{N}_{\mu_1} < \mathcal{M}_0$ (see Proposition 2.3 of [11]), so $\mathcal{M}^l_{\mu_1}$ would be hyperbolic attractive and $\mu_1 \in \mathcal{I}_1$, which is not the case. Therefore, μ_1 is a local saddle-node bifurcation point of minimal sets and a point of discontinuity of the global attractor. In addition, the unique τ_{μ_1}-minimal set $\mathcal{M}^l_{\mu_1}$ that they define is nonhyperbolic, which also follows from $\mu_1 \notin \mathcal{I}_1$. Similar arguments apply to μ_2. In addition, notice that $\mu_1 < \mu_2$, as otherwise there would exist three τ_{μ_1}-minimal sets, contradicting the nonhyperbolicity of $\mathcal{M}^l_{\mu_1}$.

Let $\mu \in (\mu_1, \mu_2)$, and let ω_0 be a common continuity point of $\alpha_\mu, \alpha_{\mu_1}$ and κ_{μ_1}, so that $\alpha_{\mu_1}(\omega_0) = \kappa_{\mu_1}(\omega_0)$. Since κ_{μ_1} is a strong τ_μ-subequilibrium and $\alpha_{\mu_1} \le \alpha_\mu$, there exists $t_0 > 0$ such that $\kappa_{\mu_1}(\omega_0 \cdot t_0) < u_\mu(t_0, \omega_0, \alpha_{\mu_1}(\omega_0)) \le \alpha_\mu(\omega_0 \cdot t_0)$. Let us call $\omega_1 = \omega_0 \cdot t_0$ and assume for contradiction that $\alpha_\mu(\omega_0) < 0$ or, equivalently, $\alpha_\mu(\omega_1) < 0$. The previous property combined with $\lim_{\xi \to -\infty} \kappa_\xi(\omega_1) = 0$ and the continuity of $\xi \mapsto \kappa_\xi(\omega_1)$ on $\mathcal{I}_1$ provides $\alpha_\mu(\omega_1) = \kappa_{\xi_1}(\omega_1)$ for a $\xi_1 \in \mathcal{I}_1$. Since κ_{ξ_1} is a strong (and continuous) τ_μ-subequilibrium, there exists $s_1 > 0$ and $e_1 > 0$ such that $\alpha_\mu(\omega_1 \cdot t) > \kappa_{\xi_1}(\omega_1 \cdot t) + e_1$ for all $t \ge s_1$ (see Sect. 2.3), and we obtain the contradiction $\alpha_\mu(\omega_1) \ge \kappa_{\xi_1}(\omega_1) + e_1$ by taking $\{t_n\} \uparrow \infty$ with $\omega_1 = \lim_{n \to \infty} \omega_1 \cdot t_n$. This means that $\mathcal{M}^l_\mu = \mathcal{M}_0$, and a similar argument shows $\mathcal{M}^u_\mu = \mathcal{M}_0$. Consequently, $\mathcal{M}_0$ is the unique τ_μ minimal set for $\mu \in (\mu_1, \mu_2)$ and, since it is hyperbolic attractive, Theorem 3.4 of [11] proves that $\mathcal{A}_\mu = \mathcal{M}_0$.

(iii) In this case, $\mathcal{M}_0$ is nonhyperbolic for all $\mu \in \mathbb{R}$ (see Sect. 2.5), and this fact precludes the existence of three τ_μ-minimal sets (see Sect. 2.6). Let us check that $\mathcal{M}^u_\mu$ is an attractive hyperbolic copy of the base if μ is large enough. To this end, we choose $\lambda_0 > 0$ such that the Sacker and Sell spectrum of $f_x(\omega, 0) - \lambda_0$ is contained in $(-\infty, 0)$. Consequently, the bifurcation diagram of minimal sets of

$$x' = f(\omega \cdot t, x) - \lambda_0 x + \nu x^2 \qquad (6.3)$$

with respect to ν is that described in (ii). Let ν_2 be the upper bifurcation point, fix $\nu > \nu_2$, and let $\hat{\kappa}_\nu$ and $\hat{\beta}_\nu$ be the equilibria giving rise to the repulsive and attractive hyperbolic copies of the base for (6.3)$_\nu$. Then, if $\mu > \nu$,

$$f(\omega, \hat{\beta}_\nu(\omega)) + \mu \hat{\beta}^2_\nu(\omega) > f(\omega, \hat{\beta}_\nu(\omega)) - \lambda_0 \hat{\beta}_\nu(\omega) + \nu \hat{\beta}^2_\nu(\omega),$$

which means that $\hat{\beta}_\nu$ is a global strict lower τ_μ-solution and hence that $\hat{\kappa}_\nu(\omega) < \hat{\beta}_\nu(\omega) < \beta_\mu(\omega)$ for all $\omega \in \Omega$ (see Sect. 2.4). The definition of $\mathcal{M}^u_\mu$ ensures that it is above the graph of $\hat{\beta}_\nu$, and hence $\hat{\kappa}_\nu(\omega) < \gamma_\mu(\omega)$ for any τ_μ-equilibrium $\gamma_\mu \colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{M}^u_\mu$. Since $\hat{\kappa}_\nu$ defines a repulsive hyperbolic copy of the base of (6.3)$_\nu$, Proposition 6.2 ensures that $\int_\Omega (f_x(\omega, \gamma_\mu(\omega)) + 2\mu \gamma_\mu(\omega)) \, dm < 0$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$. This means that $\mathcal{M}^u_\mu$ is an attractive hyperbolic minimal set for all $\mu > \nu_2$. An analogous argument shows that $\mathcal{M}^l_\mu$ is an attractive hyperbolic minimal set if $-\mu$ is large enough.

Now, we define $\mathcal{I}_1, \mathcal{I}_2, \mu_1$ and μ_2 as in (ii). Note that $\mu_1 \le \mu_2$: otherwise there would be three τ_{μ_1}-minimal sets. The definition of these two values of the parameter shows the first assertion in (iii), concerning $\mu \notin [\mu_1, \mu_2]$. In addition, the definition on $\mathcal{M}^l_\mu$ (resp. $\mathcal{M}^u_\mu$) shows that, if $\mathcal{M}_0$ is the unique τ_{μ_1}-minimal set (resp. τ_{μ_2}-minimal set), then α_μ and 0 (resp. 0 and β_μ) collide on a residual σ-invariant set as $\mu \uparrow \mu_1$ (resp. $\mu \downarrow \mu_2$).

Let us prove that $\mathcal{M}_0$ is the unique τ_μ-minimal set if $\mu \in [\mu_1, \mu_2]$ in the case $\inf \operatorname{sp}(f_x(\cdot, 0)) = 0$, which means that $\int_\Omega f_x(\omega, 0) \, dm \geq 0$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. In this case, any τ_μ-minimal set different from $\mathcal{M}_0$ is hyperbolic attractive for any $\mu \in \mathbb{R}$: Proposition 4.4 of [11] ensures that all the Lyapunov exponents of such a minimal set are strictly negative, from where the assertion follows. Hence, $\mathcal{M}_{\mu_1}^l$ and $\mathcal{M}_{\mu_2}^u$ coincide with $\mathcal{M}_0$ to not contradict the definition of μ_1 and μ_2. The monotonicity properties of α_μ and β_μ established in Proposition 6.1 ensure that $\mathcal{M}_\mu^l = \mathcal{M}_0$ for all $\mu \geq \mu_1$ and $\mathcal{M}_\mu^u = \mathcal{M}_0$ for all $\mu \leq \mu_1$, and this proves the assertion.

In the rest of the proof, we work in the remaining case, $\inf \operatorname{sp}(f_x(\cdot, 0)) < 0$, which implies the existence of a measure $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $\int_\Omega f_x(\omega, 0) \, dm < 0$.

Let us prove that $\mu_1 < \mu_2$. We fix $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $\int_\Omega f_x(\omega, 0) \, dm < 0$. For $\mu > \mu_2$, Lemma 3.2 ensures that $\int_\Omega (f_x(\omega, \kappa_\mu^2(\omega)) + 2\mu \kappa_\mu^2(\omega)) \, dm > 0$, where κ_μ^2 is the m-measurable τ_μ-equilibrium providing the lower delimiter of the basin of attraction of the graph of β_μ, which satisfies $0 \leq \kappa_\mu^2 < \beta_\mu$. Reasoning as in Theorem 3.1, we check that $\mu \mapsto \kappa_\mu^2(\omega)$ is nonincreasing on $\mathcal{I}_2$ for all $\omega \in \Omega$. Hence, there exists the limit $\kappa_{\mu_2}^2 = \lim_{\mu \downarrow \mu_2} \kappa_\mu^2 \geq 0$. Lebesgue's Convergence Theorem ensures that $\int_\Omega (f_x(\omega, \kappa_{\mu_2}^2(\omega)) + 2\mu \kappa_{\mu_2}^2(\omega)) \, dm \geq 0$. In particular, $\kappa_{\mu_2}^2$ is different from 0 with respect to m. A symmetric procedure performed for $\mu < \mu_1$ (now defining κ_μ^1 as the upper delimiter of the basin of attraction of the graph of α_μ, so that $\mu \mapsto \kappa_\mu^1$ is nonincreasing on $\mathcal{I}_1$) shows the existence of a τ_{μ_1}-equilibrium $\kappa_{\mu_1}^1 \leq 0$ with is different from 0 with respect to m. Finally, we assume for contradiction that $\mu_1 = \mu_2$, observe that $\kappa_{\mu_1}^1 \leq 0 \leq \kappa_{\mu_2}^2$ define three τ_{μ_1}-equilibria which are different with respect to m, and conclude that $\int_\Omega f_x(\omega, 0) \, dm > 0$ (see Sect. 2.6), which is not the case. This contradiction proves the assertion.

It remains to check that $\mathcal{M}_\mu^l = \mathcal{M}_0$ for $\mu > \mu_1$ and $\mathcal{M}_\mu^u = \mathcal{M}_0$ for $\mu < \mu_2$, which ensures that $\mathcal{M}_0$ is the unique τ_μ-minimal set for $\mu \in (\mu_1, \mu_2)$. We begin by assuming that $\mathcal{M}_{\mu_2}^u = \mathcal{M}_0$, which means that β_{μ_2} vanishes on the residual set of its continuity points. Proposition 6.1(i) ensures that, if $\mu < \mu_2$, then $0 \leq \beta_\mu(\omega) \leq \beta_{\mu_2}(\omega) = 0$ at a common continuity point of both maps, and this ensures that $\mathcal{M}_\mu^u = \mathcal{M}_0$, as asserted. Now, let us work in the case $\mathcal{M}_{\mu_2}^u > \mathcal{M}_0$. The argument adapts to this situation that of the two last paragraphs of the proof of Theorem 3.1, as we sketch in what follows. First, for $\mu > \mu_2$, we consider the τ_μ-equilibrium κ_μ^2 of the previous paragraph, which satisfies $\kappa_\mu^2(\omega) > 0$ m-a.e. if $\int_\Omega f_x(\omega, 0) \, dm < 0$ for an $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. Second, we combine this property with the nonhyperbolicity of $\mathcal{M}_{\mu_2}^u$ to deduce that there are points in the graph of $\kappa_{\mu_2}^2 = \lim_{\mu \downarrow \mu_2} \kappa_\mu^2$ which are between the delimiter equilibria of $\mathcal{M}_{\mu_2}^u$. And third, we deduce from this fact that $\beta_\mu(\omega) = 0$ on its residual set of continuity points if $\mu < \mu_2$, which proves that $\mathcal{M}_\mu^u = \mathcal{M}_0$ also in this case. The argument is analogous for $\mathcal{M}_\mu^l$ and $\mu > \mu_1$, and the proof is complete. $\qquad \square$

Autonomous cases $x' = f(x) + \lambda x^2$ fitting the possibilities described in Theorem 6.3 are very easy to find, since they just depend on the sign of $f'(0)$. For example, $x' = -x^3 + x + \lambda x^2$ for the first one, $x' = -x^3 - x + \lambda x^2$ for (ii) for the second one, and $x' = -x^3 + \lambda x^2$ for the third one. Note also that the model analyzed in Proposition 4.1 fits in the situation of Theorem 6.3(iii), and that in that case we can determine the values of μ_1 and μ_2.

The two first diagrams described in Theorem 6.3 (the absence of bifurcation points and the occurrence of two local saddle-node ones) are depicted in Figs. 3, and 4 shows several cases of weak generalized transcritical bifurcations.

We close this paper by using the information just obtained to go deeper in the analysis of the bifurcation possibilities for our initial problem (3.1), i.e., $x' = f(\omega{\cdot}t, x) + \lambda x$. More precisely,
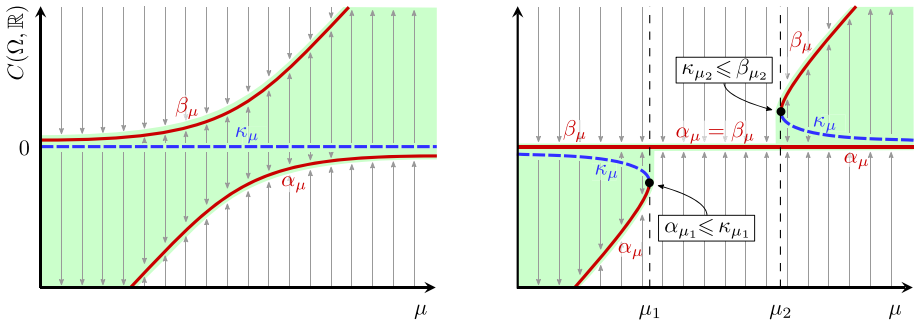
Fig. 3 The no bifurcation (left) and two local saddle-node bifurcation (right) diagrams described in Theorem 6.3(i) and (ii). The meaning of the different elements is explained in Fig. 1
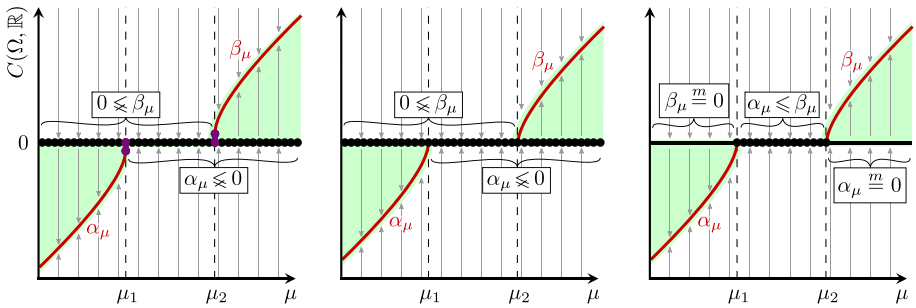
Fig. 4 Some possibilities for the weak generalized transcritical bifurcation diagrams described in Theorem 6.3(iii). See Figs. 1 and 2 to understand the meaning of the different elements. For the first two diagrams, $\mathrm{sp}(f_x(\cdot, 0))$ is a nondegenerate interval which contains 0. The first one corresponds to $\inf \mathrm{sp}(f_x(\cdot, 0)) < 0 < \sup \mathrm{sp}(f_x(\cdot, 0))$ (which yields $\mu_1 < \mu_2$), and the second one to $\inf \mathrm{sp}(f_x(\cdot, 0)) = 0$ with $\mu_1 < \mu_2$ (the diagram for $\mu_1 = \mu_2$ is obtained by deleting the vertical strip given by (μ_1, μ_2)). In these diagrams, the existence of $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ such that $\int_{\Omega} f_x(\omega, 0) \, dm > 0$ and Proposition 5.2 of [11] (see the caption of Fig. 1) yield α_μ and β_μ not identically 0 for all μ. (If $\sup \mathrm{sp}(f_x(\cdot, 0)) = 0$, we can just say $\alpha_\mu \le 0$ if $\mu \ge \mu_1$ and $0 \le \beta_\mu$ if $\mu \ge \mu_2$.) The third diagram corresponds to $\mathrm{sp}(f_x(\cdot, 0)) = \{0\}$ with $\mu_1 < \mu_2$ (again, erasing the vertical strip given by (μ_1, μ_2) provides the diagram for $\mu_1 = \mu_2$ in this point spectrum case). For this case, Theorem 4.1 of [11] ensures the existence of a set Ω_0 with $m(\Omega_0) = 1$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ such that $\alpha_\mu(\omega) = 0$ and $\beta_\mu(\omega) = 0$ for $\omega \in \Omega_0$ if $\mu > \mu_2$ and $\mu < \mu_1$, respectively, which we represent as $\alpha_\mu \overset{m}{=} 0$ and $\beta_\mu \overset{m}{=} 0$

Corollary 6.5 shows that for any $\lambda_0 \le -\inf \mathrm{sp}(f_x(\cdot, 0))$ there exists a suitable $\mu_0 \in \mathbb{R}$ such that λ_0 is the lower bifurcation point of the modified family $x' = f(\omega{\cdot}t, x) + \mu_0 x^2 + \lambda x$, and that the three different possibilities of Theorem 3.1 correspond to $\lambda_0 < -\sup \mathrm{sp}(f_x(\cdot, 0))$, $\lambda_0 = -\inf \mathrm{sp}(f_x(\cdot, 0))$ and $-\sup \mathrm{sp}(f_x(\cdot, 0)) \le \lambda_0 < -\inf \mathrm{sp}(f_x(\cdot, 0))$. To this end, we consider the two-parametric bifurcation problem of minimal sets

$$x' = f(\omega{\cdot}t, x) + \mu x^2 + \lambda x \,, \tag{6.4}$$

where $f \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ is $(\mathrm{Co})_2$ and $(\mathrm{SDC})_*$ and $f(\omega, 0) = 0$ for all $\omega \in \Omega$. Let $\mathcal{A}_{\lambda,\mu}$ be the global attractor of the local skewproduct flow $\tau_{\lambda,\mu}$ induced on $\Omega \times \mathbb{R}$ by $(6.4)_{\lambda,\mu}$, with delimiter equilibria $\alpha_{\lambda,\mu}(\omega) = \inf(\mathcal{A}_{\lambda,\mu})_\omega$ and $\beta_{\lambda,\mu}(\omega) = \sup(\mathcal{A}_{\lambda,\mu})_\omega$, and let $\mathrm{sp}(f_x(\cdot, 0)) = [-\lambda_+, -\lambda_-]$. We define

$$\hat{\mu} \colon (-\infty, \lambda_+] \to \mathbb{R}, \ \lambda \mapsto \inf\{\mu \in \mathbb{R} \colon \text{the graph of } \beta_{\lambda,\xi} \text{ defines a}$$
$$\text{hyperbolic minimal set } \mathcal{M}^u_{\lambda,\xi} \neq \mathcal{M}_0 \text{ for all } \xi > \mu\},$$
$$\hat{\lambda} \colon \mathbb{R} \to (-\infty, \lambda_+], \ \mu \mapsto \inf\{\lambda \in \mathbb{R} \colon \text{the graph of } \beta_{\xi,\mu} \text{ defines a}$$
$$\text{hyperbolic minimal set } \mathcal{M}^u_{\xi,\mu} \neq \mathcal{M}_0 \text{ for all } \xi > \lambda\}. \tag{6.5}$$

Notice that, for any fixed $\lambda_1 \leq \lambda_+$, $\mathrm{sp}(f_x(\cdot, 0) + \lambda_1) = [\lambda_1 - \lambda_+, \lambda_1 - \lambda_-] \not\subset (0, \infty)$, which ensures that the bifurcation diagram for the μ-parametric family $(6.4)_{\lambda_1}$ is not that of Theorem 6.3(i) and hence that $\hat{\mu}(\lambda_1)$ is well defined. In addition, Theorem 3.1 ensures that $\hat{\lambda}$ is well defined and satisfies $\hat{\lambda}(\mu_1) \leq \lambda_+$ in the three bifurcation cases which may arise for the λ-parametric family $(6.4)_{\mu_1}$, for any $\mu_1 \in \mathbb{R}$.

Proposition 6.4 *Let $\hat{\lambda}$ and $\hat{\mu}$ be the maps defined in (6.5). Then,*

(i) $\hat{\lambda} \circ \hat{\mu} = \mathrm{Id}_{(-\infty, \lambda_+]}$, *and consequently, $\hat{\lambda}$ is onto.*

(ii) $\hat{\lambda}$ *is nonincreasing, and consequently, $\hat{\lambda}$ is continuous.*

Proof

(i) We fix $\lambda_0 \in (-\infty, \lambda_+]$ and call $\mu_0 = \hat{\mu}(\lambda_0)$ and $\overline{\lambda}_0 = \hat{\lambda}(\mu_0)$. The goal is to check that $\overline{\lambda}_0 = \lambda_0$. The definition of μ_0 ensures that the graph of β_{λ_0,μ_0} does not define a hyperbolic τ_{λ_0,μ_0}-minimal set distinct from $\mathcal{M}_0$, so $\lambda_0 \leq \overline{\lambda}_0$. By contradiction, we assume that $\lambda_0 < \overline{\lambda}_0$, and fix $\lambda \in (\lambda_0, \overline{\lambda}_0)$. Observe that μ_0 is the upper bifurcation point of the diagram described by Theorem 6.3(ii) or (iii) for the μ-family of $(6.4)_{\lambda_0}$, and that, for any $\epsilon > 0$, the upper delimiter $\beta_{\lambda_0,\mu_0+\epsilon}$ of $\mathcal{A}_{\lambda_0,\mu_0+\epsilon}$ is continuous and strictly positive. Then, if $\epsilon \in (0, 1)$ satisfies $\epsilon < (\lambda - \lambda_0)/\sup_{\omega \in \Omega} \beta_{\lambda_0,\mu_0+1}(\omega)$, we have $\lambda_0 - \lambda + \epsilon \beta_{\lambda_0,\mu_0+\epsilon}(\omega) < 0$ for all $\omega \in \Omega$, and hence

$$\begin{aligned}\beta'_{\lambda_0,\mu_0+\epsilon}(\omega) &= f(\omega, \beta_{\lambda_0,\mu_0+\epsilon}(\omega)) + \lambda_0 \beta_{\lambda_0,\mu_0+\epsilon}(\omega) + (\mu_0 + \epsilon)\beta^2_{\lambda_0,\mu_0+\epsilon}(\omega) \\ &= f(\omega, \beta_{\lambda_0,\mu_0+\epsilon}(\omega)) + \lambda \beta_{\lambda_0,\mu_0+\epsilon}(\omega) + \mu_0 \beta^2_{\lambda_0,\mu_0+\epsilon}(\omega) \\ &\quad + \beta_{\lambda_0,\mu_0+\epsilon}(\omega)(\lambda_0 - \lambda + \epsilon \beta_{\lambda_0,\mu_0+\epsilon}(\omega)) \\ &< f(\omega, \beta_{\lambda_0,\mu_0+\epsilon}(\omega)) + \lambda \beta_{\lambda_0,\mu_0+\epsilon}(\omega) + \mu_0 \beta^2_{\lambda_0,\mu_0+\epsilon}(\omega).\end{aligned}$$

That is, $\beta_{\lambda_0,\mu_0+\epsilon}$ is a global strict lower solution for $(6.4)_{\lambda,\mu_0}$. This ensures that $0 < \beta_{\lambda_0,\mu_0+\epsilon} < \beta_{\lambda,\mu_0}$ (see Sect. 2.4), which in particular implies the existence of a strictly positive τ_{λ,μ_0}-minimal set. But this is not possible: Theorem 3.1 and the definition of $\hat{\lambda}$ show that there is no τ_{λ,μ_0}-minimal set above $\mathcal{M}_0$ if $\lambda < \overline{\lambda}_0$. This contradiction shows that $\overline{\lambda}_0 = \lambda_0$.

(ii) By contradiction, we assume that there exist $\mu_1 < \mu_2$ such that $\lambda_1 = \hat{\lambda}(\mu_1) < \hat{\lambda}(\mu_2) = \lambda_2$. We take $\lambda \in (\lambda_1, \lambda_2)$. As $\lambda > \lambda_1$, Theorem 3.1 and the definition of λ_1 ensure that the upper delimiter β_{λ,μ_1} of $\mathcal{A}_{\lambda,\mu_1}$ is continuous and strictly positive. Notice that

$$\begin{aligned}\beta'_{\lambda,\mu_1}(\omega) &= f(\omega, \beta_{\lambda,\mu_1}(\omega)) + \lambda \beta_{\lambda,\mu_1}(\omega) + \mu_2 \beta^2_{\lambda,\mu_1}(\omega) + (\mu_1 - \mu_2)\beta^2_{\lambda,\mu_1}(\omega) \\ &< f(\omega, \beta_{\lambda,\mu_1}(\omega)) + \lambda \beta_{\lambda,\mu_1}(\omega) + \mu_2 \beta^2_{\lambda,\mu_1}(\omega),\end{aligned}$$

so β_{λ,μ_1} is a global strict lower solution for $(6.4)_{\lambda,\mu_2}$. Therefore, $0 < \beta_{\lambda,\mu_1} < \beta_{\lambda,\mu_2}$ (see Sect. 2.4), which in particular implies the existence of a strictly positive τ_{λ,μ_2}-minimal

set. But the definition of λ_2 ensures that $\mathcal{M}_0$ is the unique τ_{λ,μ_2}-minimal set, since $\lambda < \lambda_2$. Hence, $\hat{\lambda}$ is nonincreasing. Finally, a nondecreasing and onto function defined from an interval to an interval is always continuous.

$\square$

Corollary 6.5 *Let* $\mathrm{sp}(f_x(\cdot, 0)) = [-\lambda_+, -\lambda_-]$ *and* $\lambda_0 \leq \lambda_+$. *The* λ-*parametric family*

$$x' = f(\omega{\cdot}t, x) + \hat{\mu}(\lambda_0)x^2 + \lambda x \tag{6.6}$$

exhibits

- *the local saddle-node and transcritical bifurcations of Theorem* 3.1(i) *if* $\lambda_0 < \lambda_-$, *with* λ_0 *as local saddle-node bifurcation point and* α_λ *colliding with* 0 *as* $\lambda \downarrow \lambda_+$;
- *the classical pitchfork bifurcation of Theorem* 3.1(ii) *if* $\lambda_0 = \lambda_+$, *with unique bifurcation point* λ_0;
- *and the generalized pitchfork bifurcation of Theorem* 3.1(iii) *if* $\lambda_0 \in [\lambda_-, \lambda_+)$, *with* λ_0 *as lower bifurcation point and* α_λ *colliding with* 0 *as* $\lambda \downarrow \lambda_+$.

Proof Proposition 6.4(i) ensures that $\hat{\lambda}(\hat{\mu}(\lambda_0)) = \lambda_0$. That is, $\lambda_0 = \inf\{\lambda \in \mathbb{R} \colon \text{ the graph of } \beta_{\xi,\hat{\mu}(\lambda_0)} \text{ defines a hyperbolic minimal set } \mathcal{M}^u_{\xi,\hat{\mu}(\lambda_0)} \neq \Omega \times \{0\} \text{ for all } \xi > \lambda\}$. The conclusions follow from the description of the three cases made in Theorem 3.1 applied to (6.6). $\square$

By taking the lower delimiters of the global attractors instead of the upper ones in (6.5), we get a result analogous to Corollary 6.5, with β_λ colliding with 0 at the upper bifurcations points.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

References

1. Alonso, A.I., Obaya, R.: The structure of the bounded trajectories set of a scalar convex differential equation. Proc. R. Soc. Edinb. A **133**(2), 237–263 (2003)
2. Anagnostopoulou, V., Jäger, T.: Nonautonomous saddle-node bifurcations: random and deterministic forcing. J. Differ. Equ. **253**(2), 379–399 (2012)
3. Anagnostopoulou, V., Jäger, T., Keller, G.: A model for the nonautonomous Hopf bifurcation. Nonlinearity **28**(7), 2587–2616 (2015)
4. Arnold, L.: Random Dynamical Systems. Springer, Berlin (1998)

5. Braaksma, B., Broer, H., Huitema, G.: Toward a quasi-periodic bifurcation theory. Memoirs Am. Math. Soc. **83**(421), 83–175 (1990)
6. Campos, J., Núñez, C., Obaya, R.: Uniform stability and chaotic dynamics in nonhomogeneous linear dissipative scalar ordinary differential equations. Preprint (2021)
7. Campos, J., Obaya, R., Tarallo, M.: Favard theory for the adjoint equation and Fredholm alternative. J. Differ. Equ. **262**(2), 749–802 (2017)
8. Caraballo, T., Langa, J., Obaya, R.: Pullback, forward and chaotic dynamics in 1D non-autonomous linear-dissipative equations. Nonlinearity **30**(1), 274–299 (2017)
9. Carvalho, A., Langa, J., Robinson, J.: Attractors for infinite-dimensional non-autonomous dynamical systems. Appl. Math. Sci. **182**, Springer, New York (2013)
10. Cornfeld, I.P., Fomin, S.V., Sinai, Ya..G..: Ergodic Theory. Springer, New York (1982)
11. Dueñas, J., Núñez, C., Obaya, R.: Bifurcation theory of attractors and minimal sets in d-concave nonautonomous scalar ordinary differential equations. J. Differ. Equ. **361**(15), 138–182 (2023)
12. Fabbri, R., Johnson, R.: A nonautonomous saddle-node bifurcation pattern. Stoch. Dyn. **4**(3), 335–350 (2004)
13. Fuhrmann, G.: Non-smooth saddle-node bifurcations III: strange attractors in continuous time. J. Differ. Equ. **261**(3), 2109–2140 (2016)
14. Furstenberg, H.: Strict ergodicity and transformation of the torus. Am. J. Math. **83**, 573–601 (1961)
15. Johnson, R., Obaya, R., Novo, S., Núñez, C., Fabbri, R.: Nonautonomous linear Hamiltonian systems: oscillation, spectral theory and control. Dev. Math. **36**, Springer (2016)
16. Johnson, R.: Minimal functions with unbounded integral. Israel J. Math. **31**(2), 133–141 (1978)
17. Johnson, R., Mantellini, F.: A nonautonomous transcritical bifurcation problem with an application to quasi-periodic bubbles. Discrete Contin. Dyn. Syst. **9**(1), 209–224 (2003)
18. Johnson, R., Palmer, K., Sell, G.R.: Ergodic properties of linear dynamical systems. SIAM J. Math. Anal. **18**(1), 1–33 (1987)
19. Jorba, À., Núñez, C., Obaya, R., Tatjer, J.C.: Old and new results on strange nonchaotic attractors. Int. J. Bifurc. Chaos **17**(11), 3895–3928 (2007)
20. Kloeden, P.: Pitchfork and transcritical bifurcations in systems with homogeneous nonlinearities and an almost periodic time coefficient. Commun. Pure Appl. Anal. **3**(2), 161–173 (2004)
21. Langa, J.A., Robinson, J.C., Suárez, A.: Bifurcations in non-autonomous scalar equations. J. Differ. Equ. **221**(1), 1–35 (2006)
22. Mañé, R.: Ergodic theory and Differentiable Dynamics. Springer, Berlin (1987)
23. Novo, S., Núñez, C., Obaya, R.: Almost automorphic and almost periodic dynamics for quasimonotone non-autonomous functional differential equations. J. Dyn. Differ. Equ. **17**(3), 589–619 (2005)
24. Núñez, C., Obaya, R.: Li–Yorke chaos in nonautonomous Hopf bifurcation patterns-I. Nonlinearity **32**(10), 3940–3980
25. Núñez, C., Obaya, R.: A non-autonomous bifurcation theory for deterministic scalar differential equations. Discrete Contin. Dyn. Syst. Ser. B **9**(3–4), 701–730 (2008)
26. Pliss, V.A.: Nonlocal Problems of the Theory of Oscillations. Harry Herman Academic Press, New York (1966)
27. Pötzsche, C.: Nonautonomous bifurcation of bounded solutions I: a Lyapunov–Schmidt approach. Discrete Contin. Dyn. Syst. Ser. B **14**(2), 739–776 (2010)
28. Pötzsche, C.: Nonautonomous continuation of bounded solutions. Commun. Pure Appl. Anal. **10**(3), 937–961 (2011)
29. Pötzsche, C.: Nonautonomous bifurcation of bounded solutions II: A shovel-bifurcation pattern. Discrete Contin. Dyn. Syst. **31**(3), 941–973 (2011)
30. Rasmussen, M.: Attractivity and Bifurcation for Nonautonomous Dynamical Systems. Lecture Notes in Mathematics, vol. 1907. Springer, Berlin (2007)
31. Rasmussen, M.: Nonautonomous bifurcation patterns for one-dimensional differential equations. J. Differ. Equ. **234**(1), 267–288 (2007)
32. Remo, F., Fuhrmann, G., Jäger, T.: On the effect of forcing of fold bifurcations and early-warning signals in population dynamics. Nonlinearity **35**, 6485 (2022)
33. Sacker, R.J., Sell, G.R.: A spectral theory for linear differential systems. J. Differ. Equ. **27**, 320–358 (1978)
34. Schwartzman, S.: Asymptotic cycles. Ann. Math. **66**(2), 270–284 (1957)
35. Tineo, A.: A result of Ambrosetti–Prodi type for first-order ODEs with cubic non-linearities. I, II. Ann. Mat. Pura Appl. (4) **182**(2), 113–128, 129–141 (2003)