GENERATION OF AXISYMMETRIC MAGNETOSONIC SHOCK WAVES: APPLICATIONS TO STATIC EQUILIBRIA AND ACCRETION DISKS

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Abstract

Magnetosonic waves traveling in a MHD equilibrium may evolve into shocks. We develop a criterion for the creation of fast shocks in the equatorial plane of axisymmetric equilibria and analyze the influence of the most important parameters. The results are applied to Grad-Shafranov equilibria and accretion disks.

PACS numbers: 02.30.Jr, 52.35.Bj, 52.35.Tc
Keywords: Shock waves, fast magnetosonic waves, MHD equilibria, accretion disks

1 Introduction

This paper deals with the creation of magnetosonic shock waves in axisymmetric MHD equilibria. While the theoretical results hold for all cases, the practical application of them requires a knowledge of the particular situation we are dealing with. We will concentrate on two important instances: static equilibria described by the Grad-Shafranov equation, which play such a significant part in the analysis of stability of confined plasmas; and accretion rings near astrophysical objects ranging from planets to stars to black holes. In the last instance the full relativistic approach would be necessary, but as soon as we leave the immediate vicinity of the object the Newtonian equations are extremely precise. Since each of these topics has its own rationale, we introduce them in turn.

Perhaps the most successful endeavor in classical Magnetohydrodynamics was the proof of the existence of one and two-dimensional static equilibria, as well as the analysis of their linear stability. Among the many references we may select the classical review [1], and the excellent modern exposition [2]. One of the aims of this theory is to identify perturbation frequencies which lead to exponential growth and therefore show the instability of the equilibrium. As befits a linear theory, this does not depend on the initial size of the perturbation, provided it is small enough for the linearized MHD equations to be considered valid. Nonlinear stability depends more heavily on computational algorithms, but there is an important instance where we can go a long way by analytical methods. This concerns the propagation of MHD waves into an unperturbed state, a subject forming part of the study of nonlinear hyperbolic systems [3, 4]. Both sound and MHD waves have been studied in simple special configurations from a long time ago [5–8], but never including the case of axisymmetric equilibria.

Accretion disks form one of the most conspicuous astrophysical structures, present e.g. in galactic nuclei, young and dwarf stars, black holes, binary systems and wherever jets are observed. Among the plethora of general monographs and review articles dealing with this topic, [9–11] may be commended by their breadth and clarity. Although perhaps the main object of theoretical interest today is the presence and consequences of the magnetorotational instability [12–14], the existence of shocks is also relevant [15]. In fact, one of its effects is plasma heating, which remains one of the most relevant topics in the study of accretion disks [16].

There are several analytic approaches to the problem [17, 18] of shock formation, but all lead to a differential equation satisfied along the rays by the jump of the time derivative of the solution at the wavefront separating the perturbed and the unperturbed states. This equation is of Riccati type and may lead to a blow up of the solution in a finite time, a fact which is interpreted as the formation of a discontinuity in both velocity and magnetic field consistent with the formation of a shock wave. A MHD shock wave yields several undesirable effects such as the creation of surface currents in the plasma, which does not bode well for the preservation of the equilibrium; and shock waves in accretion diks are equally disrupting. Being a nonlinear feature, the size of the initial perturbation is a key parameter governing the future evolution of the system. While most classical papers assume that in the state where the wave propagates all the quantities are constant, this simplifying hypothesis does not hold for nontrivial static equilibria, both in the static case an in accretion disks. Fortunately the general case is also included in modern treatises on nonlinear waves, such as [19, 20].

Let us end the introduction by admitting some weaknesses. Since the only comprehensive theory of nonlinear waves and shocks involve quasilinear hyperbolic systems, we must use the ideal MHD system, thus ignoring both resistive and turbulent dissipation. Also the geometry is constrained, not only because of axisymmetry but also because of several North-South symmetries we will impose; and finally, to descend to precise predictions, we must consider certain simple configurations of the main quantities: low beta and poloidal field for static settings, self-similar behavior for accretion disks. On the positive side, we will obtain rigorous criteria guaranteeing that a certain perturbation located at a fixed radius will eventually evolve into a fast magnetosonic shock, and we will be able to pinpoint the exact location where this will occur.

2 General results

Since the main results on nonlinear waves propagating into an equilibrium are not so well known, we will recall them briefly. Let

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{3} A_j(t, \mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{C}(t, \mathbf{x}, \mathbf{u}) = \mathbf{0}, \tag{1}$$

be a quasilinear hyperbolic system, where all functions are assumed smooth enough. Let $\mathbf{u}_0$ be a known stationary solution of the system (1). Let $\tau(t, \mathbf{x})$ be a solution of the eikonal equation

$$\det \left(\sum_{j=1}^{3} A_j(t, \mathbf{x}, \mathbf{u}_0) \frac{\partial \tau}{\partial x_j} + \frac{\partial \tau}{\partial t} I \right) = 0. \tag{2}$$

Assume that $\partial\tau/\partial t$ is an eigenvalue of order one of the matrix

$$\sum_{j=1}^{3} A_j(t, \mathbf{x}, \mathbf{u}_0) \frac{\partial \tau}{\partial x_j}, \tag{3}$$

and let $\mathbf{R}$ be a right eigenvector of this matrix, $\mathbf{L}$ a left eigenvector. Let $\Omega(t) : \tau(t, \mathbf{x}) = \text{const.}$, be a level surface of τ (a characteristic surface) separating two states, one of which is precisely $\mathbf{u}_0$. Equation (2) corresponds to the characteristic surfaces of the system. Assume that the variable solution propagates into the state $\mathbf{u} = \mathbf{u}_0$, where the normal vector to Ω points. This means that

$$\frac{\partial \tau}{\partial t} < 0, \quad \mathbf{n} = \frac{\nabla \tau}{|\nabla \tau|}, \tag{4}$$

and the velocity of Ω is

$$c = -\frac{\partial \tau / \partial t}{|\nabla \tau|}. \tag{5}$$

We also assume that $\mathbf{u}$ is continuous at both sides of Ω, but perhaps not its derivatives. In fact, let $[\ \ ]$ denote the jump at Ω, i.e. the magnitude at the positive side of Ω, $\mathbf{u} = \mathbf{u}_0$, minus the one on the negative side. Then

$$\left[\frac{\partial \mathbf{u}}{\partial t} \right] = w_0 \mathbf{R}, \tag{6}$$

for a scalar $w_0(t, \mathbf{x})$ whose evolution along the rays satisfies a certain Riccati equation which we will detail later. Eigenvectors are determined up to a multiplicative constant, so that w_0 depends on our particular choosing; nevertheless, the left hand side of (6) is independent of it, as well as the time when shock waves occur. Rays are the bicharacteristic curves of system (1), i.e. solutions of the following equations: if $c = c(\mathbf{n}, \mathbf{x}, t)$ is the velocity (5), a ray $t \to \mathbf{x}(t)$

satisfies

$$\frac{dx_i}{dt} = cn_i + (\delta_{ij} - n_i n_j)\frac{\partial c}{\partial n_j},$$
$$\frac{dn_i}{dt} = (n_i n_j - \delta_{ij})\frac{\partial c}{\partial x_j}. \tag{7}$$

It is known that in general rays are not orthogonal to wavefronts: this is guaranteed only if the speed of propagation c does not depends on $\mathbf{n}$. Since all the quantities are evaluated at $\mathbf{u} = \mathbf{u}_0$, this is an ordinary differential equation in $(t, \mathbf{x})$ [5]. For any matrix or vector $B(t, \mathbf{x}, \mathbf{u})$ we use the following notation:

$$(\nabla_u B) \cdot \mathbf{R} = \sum_i R_i \frac{\partial B}{\partial u_i}, \tag{8}$$

all of them evaluated at $\mathbf{u}_0$. Let

$$q_0 = \mathbf{L} \cdot \left(\sum_{j=1}^{3} \frac{n_j}{c} (\nabla_u A_j) \cdot \mathbf{R} \right) \cdot \mathbf{R}, \tag{9}$$

$$p_0 = \mathbf{L} \cdot \left(\sum_{j=1}^{3} A_j \frac{\partial \mathbf{R}}{\partial x_j} + \sum_{j=1}^{3} (\nabla_u A_j \cdot \mathbf{R}) \cdot \left(\frac{\partial \mathbf{u}_0}{\partial x_j} \right) \right) + \mathbf{L} \cdot (\nabla_u \mathbf{C}) \cdot \mathbf{R}, \tag{10}$$

The expression in (9) depends only on the values of the vectors $\mathbf{L}$ and $\mathbf{R}$ at a single point, which means that once chosen a fixed normal vector $\mathbf{n}$, $\mathbf{L}$ and $\mathbf{R}$ are left and right eigenvectors of $A_n = \sum n_j A_j$. Then

$$q_0 = \mathbf{L} \cdot \left(\frac{1}{c} (\nabla_u A_n) \cdot \mathbf{R} \right) \cdot \mathbf{R}. \tag{11}$$

This value depends only on the normal, and not on the geometry of the wavefront. The same may be said of the product $\mathbf{L} \cdot \mathbf{R}$. Things are different for (10); this term involves derivatives of the quantities which must be found along a wavefront, so we need some information on the local geometry of the surface. If we parametrize the surface Ω in the form $x_i = x_i(y_1, y_2, t)$, and $(g_{\alpha\beta})$ is the metric tensor of the surface, we have

$$p_0 = \mathbf{L} \cdot \left(A_n \frac{\partial \mathbf{R}}{\partial n} + \sum_{j,\alpha,\beta} A_j g^{\alpha\beta} \frac{\partial x_j}{\partial y_\beta} \frac{\partial \mathbf{R}}{\partial y_\alpha} \right)$$
$$+ \mathbf{L} \cdot \left([\nabla_u A_n \cdot \mathbf{R}] \frac{\partial \mathbf{u}_0}{\partial n} + \sum_{j,\alpha,\beta} [\nabla_u A_j \cdot \mathbf{R}] g^{\alpha\beta} \frac{\partial x_j}{\partial y_\beta} \frac{\partial \mathbf{u}_0}{\partial y_\alpha} \right) + \mathbf{L} \cdot (\nabla_u \mathbf{C}) \cdot \mathbf{R}. \tag{12}$$

The main result is as follows: let $t \to \mathbf{x}(t)$ represent a ray associated to the phase τ, $w(t) = w_0(t, \mathbf{x}(t))$, $p = (\mathbf{L} \cdot \mathbf{R})^{-1} p_0$, $q = (\mathbf{L} \cdot \mathbf{R})^{-1} q_0$. Then w satisfies

$$\frac{dw}{dt} + pw + qw^2 = 0. \tag{13}$$

Since this Riccati equation lacks an independent term, it may be immediately reduced to a linear one:

$$\frac{d}{dt}\left(\frac{1}{w}\right) - \frac{p}{w} - q = 0, \tag{14}$$

whose solution is

$$\frac{1}{w(t)} = \frac{1}{w(t_0)} \exp\left(\int_{t_0}^{t} p(s)\, ds\right)$$
$$+ \exp\left(\int_{t_0}^{t} p(s)\, ds\right) \int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} p(r)\, dr\right) q(s)\, ds. \tag{15}$$

Therefore, if there exists t_1 such that

$$\frac{1}{w(t_0)} = -\int_{t_0}^{t_1} \exp\left(-\int_{0}^{s} p(r)\, dr\right) q(s)\, ds, \tag{16}$$

then $w(t)$ tends to ∞ when $t \to t_1$, which, provided that $\mathbf{R}$ does not tend to zero, means that the jump of the differential of $\mathbf{u}$ tends to infinity. Hence $\mathbf{u}$ undergoes a jump and becomes discontinuous, i.e. a shock appears. The interval of time where the integral is evaluated depends on the problem; it may be limited by the physical characteristics of the process under study (as in our case, by the dimensions of the device or the accretion disk where the equilibrium exists) or by the possible formation of caustics and loss of regularity of the wavefront. On the other hand, if $w(t_0) > 0$ and the integral in (16) tends to ∞, then $w(t) \to 0$ and the wavefront becomes a mild discontinuity.

3 Propagation in axisymmetric equilibria

Obviously our first aim is to state the equations of ideal MHD in cylindrical coordinates (z, r, ϕ). The main magnitudes are velocity, magnetic field and two of three thermodynamic quantities: density ρ, entropy S and pressure P, related by a state equation. The entropy uncouples from the rest and may be ignored. We will follow ([21], p. 16) (notice that there are a few misprints in this text) and use the density as primary variable. In order to simplify the result, will assume from the beginning axisymmetry ($\partial/\partial\phi = 0$). Let us denote the velocity and magnetic field by

$$\mathbf{v} = v_z\hat{z} + v_r\hat{r} + v_\phi\hat{\phi} \tag{17}$$

$$\mathbf{B} = B_z\hat{z} + B_r\hat{r} + B_\phi\hat{\phi}. \tag{18}$$

Let I_7 denote the 7×7 identity matrix. The main equations are

$$\frac{\partial}{\partial t}\begin{bmatrix} v_z \\ v_r \\ v_\phi \\ B_z \\ B_r \\ B_\phi \\ \rho \end{bmatrix} + (v_z I_7 + \begin{bmatrix} 0 & 0 & 0 & 0 & B_r/\rho & B_\phi/\rho & P_\rho/\rho \\ 0 & 0 & 0 & 0 & -B_z/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_z/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_r & -B_z & 0 & 0 & 0 & 0 & 0 \\ B_\phi & 0 & -B_z & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix})\frac{\partial}{\partial z}\begin{bmatrix} v_z \\ v_r \\ v_\phi \\ B_z \\ B_r \\ B_\phi \\ \rho \end{bmatrix}$$

$$+(v_r I_7 + \begin{bmatrix} 0 & 0 & 0 & -B_r/\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & B_z/\rho & 0 & B_\phi/\rho & P_\rho/\rho \\ 0 & 0 & 0 & 0 & 0 & -B_r/\rho & 0 \\ -B_r & B_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_\phi & -B_r & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix})\frac{\partial}{\partial r}\begin{bmatrix} v_z \\ v_r \\ v_\phi \\ B_z \\ B_r \\ B_\phi \\ \rho \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ -v_\phi^2/r + B_\phi^2/\rho r \\ v_r v_\phi/r - B_r B_\phi/\rho r \\ B_z v_r/r \\ B_r v_r/r \\ \rho v_\phi/r \\ \rho v_r/r \end{bmatrix} + \begin{bmatrix} \frac{GM}{(r^2+z^2)^{3/2}}z \\ \frac{GM}{(r^2+z^2)^{3/2}}r \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{19}$$

We will abbreviate this to

$$\frac{\partial\mathbf{u}}{\partial t} + A_z\frac{\partial\mathbf{u}}{\partial z} + A_r\frac{\partial\mathbf{u}}{\partial r} + \mathbf{C} = \mathbf{0}. \tag{20}$$

Without axisymmetry a further matrix A_ϕ would appear. Although later we will write it formally its specific value will not be necessary. Also the last vector, representing the gravitational force (G is the gravitational constant and M the mass of the central object) applies only to accretions disks, and disappears for laboratory plasmas.

We will set a number of conditions on the velocity and the field which will be different in each case, but all have in common that the propagation velocity c is not only axisymmetric, but even with respect to the variable z. Thus, if the initial value $\tau(0)$ is axisymmetric, wavefronts will intersect the equator $\Pi : z = 0$ as circumferences and their normal vectors will be radial. We will limit our study to fast magnetosonic shock waves occurring first in the plane Π. The choosing of fast waves is logical: because they have the largest velocity, they will be the ones to extend into an untouched equilibrium state; moreover, they are the only ones which may develop across the equilibrium magnetic field, which will be needed in our geometry. While one could imagine wavefront configurations creating shocks above and below Π, usually compression is maximized at the plane and one expects shocks beginning at the plane of symmetry. Thus the normal matrix A_n coincides with A_r. The eigenvalue associated to the fast magnetosonic wave is given by $v_n \pm \mu$, where v_n is the velocity of the fluid normal to the wavefront, and

$$2\mu^2 = c^2 + \sqrt{c^4 - \frac{4B_r^2 P_\rho}{\rho}}, \tag{21}$$

where c is the total velocity, sum of the sound and Alfvén ones:

$$c^2 = P_\rho + \frac{B^2}{\rho}. \tag{22}$$

Provided $\mu^2 \neq B_r^2/\rho$ (which never happens unless $P_\rho = B_z = B_\phi = 0$), the right and left eigenvectors turn out to be

$$\mathbf{R} = \left(-\frac{B_z B_r}{\rho\mu}, \mu - \frac{B_r^2}{\rho\mu}, -\frac{B_r B_\phi}{\rho\mu}, B_z, 0, B_\phi, \frac{\rho}{\mu}\left(\mu - \frac{B_r^2}{\mu}\right)\right), \tag{23}$$

$$\mathbf{L} = \left(-\frac{B_z B_r}{\mu}, \rho\left(\mu - \frac{B_r^2}{\rho\mu}\right), -\frac{B_r B_\phi}{\mu}, B_z, 0, B_\phi, \frac{P_\rho}{\mu}\left(\mu - \frac{B_r^2}{\mu}\right)\right), \tag{24}$$

up to multiplication by a real constant. We defer the calculations in this section to the appendix; for typographical convenience we write vectors as rows instead of columns. We will see that in all our examples, in the plane Π we have $B_r = 0$; thus those vectors simplify to

$$\mathbf{R} = (0, c, 0, B_z, 0, B_\phi, \rho), \tag{25}$$

$$\mathbf{L} = (0, \rho c, 0, B_z, 0, B_\phi, P_\rho). \tag{26}$$

To find $\partial \mathbf{R}/\partial r$ at Π we may use (25), but for $\partial \mathbf{R}/\partial z$ we need the full formula (23). Anyway, at Π,

$$\mathbf{L} \cdot \mathbf{R} = \rho c^2 + B_z^2 + B_\phi^2 + \rho P_\rho = 2\rho c^2. \tag{27}$$

We need now to evaluate all the terms in the expressions of q_0 and p_0. Relegating again a scheme of the calculations to the Appendix, one finds in all cases

$$q_0 = 2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}, \tag{28}$$

and

$$\mathbf{L} \cdot A_r \cdot \frac{\partial \mathbf{R}}{\partial r} = c \frac{\partial}{\partial r}\left(P + \frac{B^2}{2}\right) + \rho c^2 \frac{\partial c}{\partial r}, \tag{29}$$

$$\mathbf{L} \cdot (\nabla_u \mathbf{C}) \cdot \mathbf{R} = \frac{\rho c^3}{r}. \tag{30}$$

The state $\mathbf{u}_0$ will have a different form in each case, but they will always satisfy at the plane Π

$$\mathbf{L} \cdot (\nabla_u A_r \cdot \mathbf{R}) \cdot \frac{\partial \mathbf{u}_0}{\partial r} = c\left(\frac{1}{2}\frac{\partial B^2}{\partial r} + \rho P_{\rho\rho}\frac{\partial \rho}{\partial r}\right). \tag{31}$$

It remains to find the terms where tangential derivatives to the wavefront occur. We know nothing about the wavefront surface other than it is axisymmetric and z-symmetric. We may parametrize it by its section by the plane $\phi = 0$, a curve $z \to r(z)$, which satisfies $r(z) = r(-z)$, so that $r'(0) = 0$. The surface will be $\mathbf{x}(z, \phi) = (r(z)\cos\phi, r(z)\sin\phi, z)$. Thus

$$g^{zz} = \frac{1}{1 + r'(z)^2}, \quad g^{\phi\phi} = \frac{1}{r(z)^2}, \quad g^{z\phi} = 0. \tag{32}$$

Thus, for any axisymmetric vector field $\mathbf{F}$ and matrices $\mathcal{A}_j$,

$$\sum_{j,\alpha,\beta} \mathcal{A}_j g^{\alpha\beta} \frac{\partial x_j}{\partial y_\beta}\frac{\partial \mathbf{F}}{\partial y_\alpha} = \mathcal{A}_z \frac{\partial \mathbf{F}}{\partial z}. \tag{33}$$

This is applied to the vectors and matrices $\mathbf{F} = \mathbf{R}$ with $\mathcal{A}_j = A_j$, as well as $\mathbf{F} = \mathbf{u}_0$ with $\mathcal{A}_j = \nabla_u A_j \cdot \mathbf{R}$. Now it is easy to see (shown in the appendix) that the second term in the expression of p_0 in (12) becomes

$$\mathbf{L} \cdot \left(\sum_{j,\alpha,\beta} A_j g^{\alpha\beta} \frac{\partial x_j}{\partial y_\beta} \frac{\partial \mathbf{R}}{\partial y_\alpha} \right) = \mathbf{L} \cdot A_z \frac{\partial \mathbf{R}}{\partial z} = -\frac{B_z P_\rho}{c} \frac{\partial B_r}{\partial z}. \tag{34}$$

Finally

$$\mathbf{L} \cdot \left(\sum_{j,\alpha,\beta} [\nabla_u A_j \cdot \mathbf{R}] g^{\alpha\beta} \frac{\partial x_j}{\partial y_\beta} \frac{\partial \mathbf{u}_0}{\partial y_\alpha} \right) = \mathbf{L} \cdot [\nabla_u A_z \cdot \mathbf{R}] \frac{\partial \mathbf{u}_0}{\partial z} = -B_z c \frac{\partial B_r}{\partial z}. \tag{35}$$

Adding (29), (30), (31), (34) and (35), we obtain

$$\begin{aligned} p_0 &= c \frac{\partial}{\partial r} \left(P + \frac{B^2}{2} \right) + \rho c^2 \frac{\partial c}{\partial r} + c \left(\frac{1}{2} \frac{\partial B^2}{\partial r} + \rho P_{\rho\rho} \frac{\partial \rho}{\partial r} \right) \\ &+ \frac{\rho c^3}{r} - \frac{B_z (P_\rho + c^2)}{c} \frac{\partial B_r}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r c^3 \rho) - \frac{B_z (P_\rho + c^2)}{c} \frac{\partial B_r}{\partial z}. \end{aligned} \tag{36}$$

Thus

$$p = \frac{p_0}{2\rho c^2} = c \frac{\partial}{\partial r} \ln(\sqrt{\rho c^3 r}) - \frac{B_z (P_\rho + c^2)}{2\rho c^3} \frac{\partial B_r}{\partial z}, \tag{37}$$

$$q = \frac{2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2}. \tag{38}$$

Our next step is to find the displacement along the rays, which as we know are radial. Equations (7) simplify because by axisymmetry c depends only on r at Π; thus

$$\frac{d\mathbf{n}}{dt} = (n^2 - 1) \frac{dc}{dr} \hat{r} = \mathbf{0}, \tag{39}$$

$$\frac{d\mathbf{x}}{dt} = c \hat{r}, \tag{40}$$

which means that as expected the rays are traveled at speed c. Thus $d/dt = c d/dr$. Let us write (13) in terms of the parameter r:

$$\frac{dw}{dr} + \left[\frac{\partial}{\partial r} \ln \sqrt{\rho c^3 r} - \frac{B_z \rho (P_\rho + c^2)}{2\rho^2 c^4} \frac{\partial B_r}{\partial z} \right] w + \frac{1}{c} \left[\frac{2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2} \right] w^2 = 0. \tag{41}$$

The first factor in the exponential of the integral of p (see (15) may be found easily:

$$\exp \left(- \int_{r_0}^{r} \frac{\partial}{\partial s} \ln \sqrt{\rho c^3 s} \, ds \right) = \sqrt{\frac{\rho_0 c_0^3 r_0}{\rho c^3 r}}, \tag{42}$$

Thus equation (16) may be written in terms of the radii

$$-\frac{1}{w(r_0)}$$

$$= \int_{r_0}^{r_1} \left(\frac{\rho_0 c_0^3 r_0}{\rho c^3 r}\right)^{1/2} \left(1 + \frac{B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2}\right) \exp\left(\int_{r_0}^{s} \frac{B_z \rho (P_\rho + c^2)}{2\rho^2 c^4} \frac{\partial B_r}{\partial z}\, ds\right) \frac{1}{c}\, dr, \tag{43}$$

where the subindex 0 means that the quantities are taken at the point r_0, and

$$\left[\frac{\partial \mathbf{u}}{\partial t}\right]_{r=r_0} = w(r_0)\mathbf{R}(r_0). \tag{44}$$

Let us consider now equation (43) to see the terms contributing to the blow up of the solution. There is little to say in general for the term q in (16) given by (41), other than the obvious bound

$$q = \left(1 + \frac{B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2}\right) \frac{1}{c} \geq \frac{1}{c}. \tag{45}$$

For polytropic plasmas $P = A\rho^\gamma$ with $\gamma \leq 2$, we also have the upper bound

$$q \leq \frac{3}{2c}, \tag{46}$$

thus we cannot expect great effects from this term. The remaining factor is the only one where the magnetic field configuration outside the plane Π plays any role. Since the focusing of compressive waves in the central plane may be an important source of shocks, we will analyze this term carefully. First we relate it to the curvature of poloidal field lines. Let us take a parametrization of this field line as $\xi \to (r(\xi), z(\xi))$, where $\dot{r} = B_r$, $\dot{z} = B_z$. As we know, the curvature may be written as

$$\kappa = \frac{\dot{r}\ddot{z} - \ddot{r}\dot{z}}{(\dot{r}^2 + \dot{z}^2)^{3/2}} = \frac{B_r \dot{B}_z - \dot{B}_r B_z}{(B_z^2 + B_r^2)^{3/2}}. \tag{47}$$

At the plane Π, $B_r = 0$, so that $\kappa = -\dot{B}_r / B_z^2$. Thus

$$B_z \frac{\partial B_r}{\partial z} = \frac{dz}{d\xi} \frac{\partial B_r}{\partial z} = \dot{B}_r = -B_z^2 \kappa. \tag{48}$$

Thus the remaining factor in the integral in (43) may be written as

$$\exp\left(\int_{r_0}^{r} \frac{B_z \rho (P_\rho + c^2)}{2\rho^2 c^4} \frac{\partial B_r}{\partial z}\, ds\right) = \exp\left(-\int_{r_0}^{r} \frac{B_z^2 \rho (P_\rho + c^2)}{2\rho^2 c^4} \kappa\, ds\right). \tag{49}$$

Hence the exponent is positive whenever $\kappa < 0$, negative for $\kappa > 0$. The first case correspond to field lines concave with respect to $r = 0$, as occurs in the left hand side of the magnetic axis, provided the equilibrium configuration possesses such an axis; and the second to convex poloidal lines, as to the right of the magnetic axis. Since the exponential grows very rapidly with the exponent, concavity in the direction of the positive r-axis is an important factor in the formation of shock waves. This is very intuitive: this geometry tends to push the fluid towards the central plane, thus creating a compressive wave that may evolve into a shock. To quantify all the terms ocurring in (43), we must descend to particular cases.

4 Shock waves in specific axisymmetric equilibria

4.1 Static equilibria

The initial state for a static axisymmetric plasma has the form

$$\mathbf{u}_0 = (0, 0, 0, B_z, B_r, B_\phi, \rho), \tag{50}$$

As for the north-south symmetry, if we wish to preserve the classical picture of magnetic field lines coiling around tori, we must demand B_z even, B_r odd, B_ϕ even, ρ even. This implies that at $z = 0$,

$$\frac{\partial B_z}{\partial z} = 0, \quad B_r = 0, \quad \frac{\partial B_\phi}{\partial z} = 0. \tag{51}$$

Thus, in the plane Π (see appendix),

$$\frac{\partial \mathbf{u}_0}{\partial r} = \left(0, 0, 0, \frac{\partial B_z}{\partial r}, 0, \frac{\partial B_\phi}{\partial r}, \frac{\partial \rho}{\partial r}\right), \tag{52}$$

and

$$\frac{\partial \mathbf{u}_0}{\partial z} = \left(0, 0, 0, 0, \frac{\partial B_r}{\partial z}, 0, 0\right). \tag{53}$$

All possible static axisymmetric equilibria may be found by solving the Grad-Shafranov equation. Among them, the family of specific equilibria found by

Solov'ev is well known [2]. Even these analytic examples become impossibly complex when applied to equation (43) with full generality of parameters. Thus we take simplicity a step further by assuming a plasma of very low beta, so that the pressure may be taken as constant and the speed of sound as negligible when compared to the Alfvén speed. Also we want to keep the main characteristic of tokamaks, the presence of a magnetic axis where the magnetic field is toroidal. Thus we choose the Solov'ev parameters so as to take the flux function

$$\psi(r,z) = (r^2 - r_m^2)^2 - 4r^2z^2, \tag{54}$$

whose magnetic axis is located at $r = r_m$, $z = 0$, and take $P = P_0$, $I = 0$ (purely poloidal field). In these conditions, at the plane Π,

$$\begin{aligned} B_r = -8r^2z, \quad \frac{\partial B_r}{\partial z} = -8r^2, \quad B_z = 4(r^2 - r_m^2), \\ c = \frac{4}{\sqrt{\rho}}|r^2 - r_m^2|, \quad \rho c^2 = B^2 = 16(r^2 - r_m^2)^2. \end{aligned} \tag{55}$$

Let us study the factors in the integral of (43). We have

$$\left(\frac{\rho(r_0)c(r_0)^3 r_0}{\rho(r)c(r)^3 r}\right)^{1/2} = \frac{\rho(r)^{1/4}|r_0^2 - r_m^2|^{3/2} r_0^{1/2}}{\rho(r_0)^{1/4}|r^2 - r_m^2|^{3/2} r^{1/2}}, \tag{56}$$

Also

$$\frac{2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2} = \frac{3}{2}, \tag{57}$$

and

$$\frac{B_z \rho(P_\rho + c^2)}{2\rho^2 c^4}\frac{\partial B_r}{\partial z} = -\frac{r^2}{r^2 - r_m^2}, \tag{58}$$

so that

$$\exp\left(\int_{r_0}^{r} \frac{B_z \rho(P_\rho + c^2)}{2\rho^2 c^4}\frac{\partial B_r}{\partial z}\, ds\right) = \exp(r_0 - r)\left|\frac{r + r_m}{r - r_m}\right|^{r_m/2}\left|\frac{r_0 + r_m}{r_0 - r_m}\right|^{-r_m/2}. \tag{59}$$

Equation (43) becomes now

$$\begin{aligned} -\frac{1}{w(r_0)} &= \frac{3}{8}e^{r_0}\rho(r_0)^{-1/4}|r_0 - r_m|^{(3+r_m)/2}|r_0 + r_m|^{(3-r_m)/2} \\ &\times \int_{r_0}^{r_1} e^{-r}\rho(r)^{3/4}|r + r_m|^{(-5+r_m)/2}|r - r_m|^{(-5-r_m)/2}\, dr. \end{aligned} \tag{60}$$

Since due to the presence of the factor $|r - r_m|^{(-5-r_m)/2}$ the integral is infinite as soon as the interval $[r_0, r_1]$, includes the magnetic axis r_m, for any initial condition $w(r_0) < 0$ (if $r_0 < r_m$) or $w(r_0) > 0$ (if $r_0 > r_m$) a shock wave forms before the wavefront reaches the magnetic axis. This is logical since the fact that the magnetic field vanishes there and the waves travel at the Alfvén velocity imply that no wave can cross the magnetic axis, so successive wavefronts pile there until a discontinuity is created.

4.2 Accretion disks with non toroidal magnetic field

Axisymmetric equilibria for accretion disks must take into account the presence of the gravitational term and the flow velocity; the result is a generalized Grad-Shafranov equation where the pressure is non longer a function of the magnetic flux. It can be analytically solved if we assume that the temperature, the density or the entropy are functions of the flux. The results, when numerically integrated, show some differences between these three models [22]. In our case, however, we will only worry about the equilibrium equations at the plane Π, where many quantities vanish, and assume the existence of a general equilibrium coinciding with ours in the equator. Later we will even assume a self-similar form where all the quantities are powers of the radius r, and the toroidal velocity is Keplerian. The result cannot be valid for the whole length of an accretion disk, but we will restrict ourselves to the zone where this description holds, not too near to the central object.

Assuming the matter in an accretion disk flows towards the equator and rotates with the same direction in the northern and southern hemispheres, both reasonable assumptions, the velocity must satisfy that v_z is odd, v_r even and v_ϕ even. If the magnetic field starts with e.g. a dipole topology and then it is dragged by the flow, we must have B_z even, B_r odd and B_ϕ odd. The odd quantities vanish at $z = 0$, whereas the derivatives with respect to z of the even ones vanish there. Taking this into account, the equilibrium equations at Π may be written as

$$-B_z\frac{\partial B_r}{\partial z}+B_z\frac{\partial B_z}{\partial r}+\rho v_r\frac{\partial v_r}{\partial r}+P_\rho\frac{\partial\rho}{\partial r}-\frac{\rho v_\phi^2}{r}+\frac{\rho GM}{r^2}=0 \quad (61)$$

$$-B_z\frac{\partial B_\phi}{\partial z}+\rho v_r\frac{\partial v_\phi}{\partial r}=0 \quad (62)$$

$$v_r\frac{\partial B_z}{\partial r}+B_z\frac{\partial v_r}{\partial r}+\frac{B_z v_r}{r}=0 \quad (63)$$

$$\rho\frac{\partial v_z}{\partial z}+v_r\frac{\partial\rho}{\partial r}+\rho\frac{\partial v_r}{\partial r}+\frac{\rho v_r}{r}=0. \quad (64)$$

From (63) one deduces $rv_rB_z=$ const. In all classical accretion disks one has $|v_r|\ll|v_\phi|$, and moreover the magnetic field decreases faster than r^{-1}, so we must take $v_r=0$ at Π. Taking this to (62) we find $\partial B_\phi/\partial z=0$, and to (64) $\partial v_z/\partial z=0$. We are left with (61), which now reads

$$B_z\left(\frac{\partial B_z}{\partial r}-\frac{\partial B_r}{\partial z}\right)+\frac{\partial P}{\partial r}-\frac{\rho v_\phi^2}{r}+\frac{\rho GM}{r^2}=0. \quad (65)$$

We assume that the velocity v_ϕ must be Keplerian, and the plasma polytropic, with $\gamma=5/3$. Taking a dependence of all magnitudes in powers of the radius, [23],

$$\rho=\rho_0r^{-3/2},\quad P=P_0r^{-5/2},\quad B_z=B_{z0}r^{-5/4},\quad v_\phi=v_{\phi0}r^{-1/2}, \quad (66)$$

and taking this to (65), we find

$$\frac{\partial B_r}{\partial z}=kr^{-9/4}, \quad (67)$$

with

$$-B_{z0}\left(\frac{5B_{z0}}{4}+k\right)-\frac{5}{2}P_0-\rho_0v_{\phi0}^2+\rho_0GM=0. \quad (68)$$

Hence

$$B_z\frac{\partial B_r}{\partial z}=\lambda r^{-7/2}=\left(-\frac{5}{4}B_{z0}^2-\frac{5}{2}P_0-\rho_0v_{\phi0}^2+\rho_0GM\right)r^{-7/2}. \quad (69)$$

The expressions of $\mathbf{u}_0$ and its derivatives are shown in the appendix. Formula (43) remains valid, and its factors become

$$\left(\frac{\rho_0c_0^3r_0}{\rho c^3r}\right)^{1/2}=\frac{r}{r_0}$$

$$1+\frac{B^2+\rho^2P_{\rho\rho}}{2\rho c^2}=\frac{3B_{z0}^2+(40/9)P_0}{2B_{z0}^2+(10/3)P_0}=\kappa>0$$

$$\frac{1}{c}=\left(\frac{\rho_0}{B_{z0}^2+(5/3)P_0}\right)^{1/2}=\delta r^{-1/2}. \quad (70)$$

The factor within the exponential is

$$\frac{B^2 + 2\gamma P}{2(\rho c^2)^2} B_z \frac{\partial B_r}{\partial z} = \frac{B_{z0}^2 + (10/3)P_0}{2(B_{z0}^2 + (5/3)P_0)^2} \lambda r^{-7/2} r^{5/2} = \mu r^{-1}. \tag{71}$$

Notice that the sign of μ is the same as the one of λ. Hence its associated factor is

$$\exp\left(\int_{r_0}^{r} \mu \frac{ds}{s}\right) = \left(\frac{r}{r_0}\right)^{\mu}, \tag{72}$$

so that (43) becomes

$$-\frac{1}{w(r_0)} = \int_{r_0}^{r_1} \frac{\kappa\delta}{r_0^{\mu+1}} r^{\mu+1/2}\, dr = \frac{\kappa\delta}{r_0^{\mu+1}} \frac{1}{\mu+3/2}\left(r_1^{\mu+1/2} - r_0^{\mu+1/2}\right), \tag{73}$$

for $\mu \neq -3/2$, and

$$-\frac{1}{w(r_0)} = \kappa\delta r_0^{1/2} \ln\left(\frac{r_1}{r_0}\right), \tag{74}$$

in the improbable case that $\mu = -3/2$. Notice that for $\mu + 3/2 > 0$ and $r_1 > r_0$, necessarily $w(r_0 > 0$; the remaining possible combination of signs are equally easy. Always

$$r_1 = r_0 \left(1 - \frac{1}{w(r_0)} \frac{r_0^{1/2}(\mu+3/2)}{\kappa\delta}\right)^{1/(\mu+1/2)}, \tag{75}$$

for $\mu \neq -3/2$, and

$$r_1 = r_0 \exp\left(-\frac{1}{w(r_0)\kappa\delta r_0^{1/2}}\right), \tag{76}$$

for $\mu = -3/2$. In theory this predicts precisely the point of formation of a shock wave. Obviously r_1 may be inaccessible because there the power description of the main quantities (66) does not hold.

4.3 Cylindrical accretion rings

If we wish to have a nontrivial toroidal field at the equator, we may take equilibria with $v_r = v_z = B_r = 0$, v_ϕ, B_z, B_ϕ even functions of z [24]. The equilibrium equations reduce at the plane Π to the single equation

$$B_z \frac{\partial B_z}{\partial r} + B_\phi \frac{\partial B_\phi}{\partial r} + \frac{\partial P}{\partial r} - \frac{\rho v_\phi^2}{r} + \frac{B_\phi^2}{r} + \frac{\rho G M}{r^2} = 0. \tag{77}$$

Positing a form of the quantities as powers of r, and a Keplerian velocity, we choose

$$\rho = \rho_0 r^{-3/2}, \quad P = P_0 r^{-5/2}, \quad B_z = B_{z0} r^{-5/4}$$
$$B_\phi = B_{\phi 0} r^{-5/4}, \quad v_\phi = v_{\phi 0} r^{-1/2}, \tag{78}$$

so that (77) reduces to

$$-\frac{5}{4} B_{z0}^2 - \frac{1}{4} B_{\phi 0}^2 - \frac{5}{2} P_0 - \rho_0 v_{\phi 0}^2 + \rho_0 GM = 0. \tag{79}$$

Since $B_r = 0$, the exponential term becomes 1 in equation (43). The remaining factors are

$$\left(\frac{\rho_0 c_0^3 r_0}{\rho c^3 r}\right)^{1/2} = \frac{r}{r_0}$$
$$1 + \frac{B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2} = \frac{3B_{z0}^2 + 3B_{\phi 0}^2 + (40/9)P_0}{2B_{z0}^2 + 2B_{\phi 0}^2 + (10/3)P_0} = \kappa_1 > 0$$
$$\frac{1}{c} = \left(\frac{\rho_0}{B_{z0}^2 + B_{\phi 0}^2 + (5/3)P_0}\right)^{1/2} = \delta_1 r^{-1/2}. \tag{80}$$

Equation (43) is now

$$-\frac{1}{w(r_0)} = \int_{r_0}^{r_1} r^{1/2}\, dr = \frac{2}{3}\frac{\kappa_1 \delta_1}{r_0}\left(r_1^{3/2} - r_0^{3/2}\right). \tag{81}$$

Notice that for $r_1 > r_0$, $w(r_0) < 0$, and for $r_1 < r_0$, $w(r_0) > 0$. Thus

$$r_1 = \left(r_0^{3/2} - \frac{3r_0}{2\kappa_1 \delta_1 w(0)}\right)^{2/3}, \tag{82}$$

with the same caveats about the location of r_1 as before.

5 Conclusions

One rarely studied source of instability in axisymmetric equilibria is the possible formation of shock waves in the plasma. Since this is a nonlinear effect, it depends on an essential way on the size as well as the functional form of the initial perturbation. For the case where the solution of an hyperbolic nonlinear system moves into a stationary state, the equations describing the evolution as well as the possible degeneration of the wavefront into a shock are known. Since the quickest MHD wave as well as the one which may cross magnetic surfaces is the fast magnetosonic one, we study how this wave may start a shock into an axisymmetric equilibrium state. The result is given in the form of an integral identity relating the initial condition with the rest variables. This identity is analyzed in depth in three cases: static equilibria appropriate for the laboratory (without gravitational forcing) and described by the Grad-Shafranov equations, in particular in the case of low beta plasma; accretion rings with a nontrivial poloidal magnetic field, which are the most frequently posited, and cylindrical accretion rings. In these last two cases the main quantities are assumed to be function of a power of the radius, and the velocity to be Keplerian. In all cases a definite prediction on the location of the formation of the shock is made.

Acknowledgment

Partially supported by the Ministry of Economy and Innovation of Spain under contract MTM2012-31439.

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A Calculations for the equation coefficients

1) The calculations of the left and right eigenvectors are routine. Denoting as asserted by I_7 the 7×7 identity matrix, the derivatives of the matrices A_r and A_z are as follows:

$$\nabla_{v_z} A_r = \nabla_{v_\phi} A_r = 0, \quad \nabla_{v_r} A_r = I_7 \tag{83}$$

$$\nabla_{B_z} A_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{84}$$

$$\nabla_{B_r} A_r = \begin{bmatrix} 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{85}$$

$$\nabla_{B_\phi} A_r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{86}$$

$$\nabla_{\rho} A_r = \begin{bmatrix} 0 & 0 & 0 & B_r/\rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_z/\rho^2 & 0 & -B_\phi/\rho^2 & \partial/\partial\rho(P_\rho/\rho) \\ 0 & 0 & 0 & 0 & 0 & B_r/\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{87}$$

$$\nabla_{v_r} A_z = \nabla_{v_\phi} A_z = 0, \quad \nabla_{v_z} A_z = I_7 \tag{88}$$

$$\nabla_{B_z} A_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{89}$$

$$\nabla_{B_r} A_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{90}$$

$$\nabla_{B_\phi} A_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{91}$$

$$\nabla_\rho A_z = \begin{bmatrix} 0 & 0 & 0 & 0 & -B_r/\rho^2 & -B_\phi/\rho^2 & \partial/\partial\rho(P_\rho/\rho) \\ 0 & 0 & 0 & B_z/\rho^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_z/\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{92}$$

In all the cases under study we have $B_r = 0$ at the plane Π. Then

$$\begin{aligned} \mathbf{R} \cdot \nabla_u A_r &= cI_7 + B_z \nabla_{B_z} A_r + B_\phi \nabla_{B_\phi} A_r + \rho \nabla_\rho A_r \\ &= cI_7 + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho\partial/\partial\rho(P_\rho/\rho) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{93}$$

Thus

$$(\nabla_u A_r \cdot \mathbf{R}) \cdot \mathbf{R} = c\mathbf{R} + \left(0, \rho^2 \frac{\partial}{\partial \rho}\left(\frac{P_\rho}{\rho}\right), 0, cB_z, 0, cB_\phi, c\rho\right), \tag{94}$$

and

$$q_0 = \frac{1}{c}\mathbf{L} \cdot (\nabla_u A_r \cdot \mathbf{R}) \cdot \mathbf{R}$$
$$= 2\rho c^2 + \rho^3 \frac{\partial}{\partial \rho}\left(\frac{P_\rho}{\rho}\right) + B_z^2 + B_\phi^2 + \rho P_\rho = 2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}. \tag{95}$$

Since for all cases

$$\frac{\partial \mathbf{R}}{\partial r} = \left(0, \frac{\partial c}{\partial r}, 0, \frac{\partial B_z}{\partial r}, 0, \frac{\partial B_\phi}{\partial r}, \frac{\partial \rho}{\partial r}\right), \tag{96}$$

formula (29) is routine.

To obtain (30) we need to find

$$\nabla_u \mathbf{C} \cdot \mathbf{R} = c\frac{\partial \mathbf{C}}{\partial v_r} + B_z \frac{\partial \mathbf{C}}{\partial B_z} + B_\phi \frac{\partial \mathbf{C}}{\partial B_\phi} + \rho \frac{\partial \mathbf{C}}{\partial \rho}. \tag{97}$$

We have

$$\frac{\partial \mathbf{C}}{\partial v_r} = \left(0, 0, \frac{v_\phi}{r}, \frac{B_z}{r}, \frac{B_r}{r}, 0, \frac{\rho}{r}\right)$$
$$\frac{\partial \mathbf{C}}{\partial B_z} = \left(0, 0, 0, \frac{v_r}{r}, 0, 0, 0\right)$$
$$\frac{\partial \mathbf{C}}{\partial B_\phi} = \left(0, \frac{2B_\phi}{\rho r}, -\frac{B_r}{\rho r}, 0, 0, 0, 0\right)$$
$$\frac{\partial \mathbf{C}}{\partial \rho} = \left(0, -\frac{B_\phi^2}{\rho r}, \frac{B_r B_\phi}{\rho^2 r}, 0, 0, 0, \frac{v_r}{r}\right). \tag{98}$$

Thus, for $B_r = 0$

$$\nabla_u \mathbf{C} \cdot \mathbf{R} = \left(0, \frac{B_\phi^2}{\rho r}, 0, \frac{cB_z}{r}, 0, 0, \frac{c\rho}{r}\right) \tag{99}$$

$$\mathbf{L} \cdot (\nabla_u \mathbf{C} \cdot \mathbf{R}) = \rho c \frac{B_\phi^2}{\rho r} + \frac{cB_z^2}{r} + \frac{c\rho P_\rho}{r} = \frac{\rho c^3}{r}. \tag{100}$$

The equilibrium states have the form at Π

$$\mathbf{u}_0 = (0, 0, 0, B_z, 0, B_\phi, \rho) \qquad \text{(Case 4.1)}$$
$$\mathbf{u}_0 = (0, v_r, v_\phi, B_z, 0, 0, \rho) \qquad \text{(Case 4.2)}$$
$$\mathbf{u}_0 = (0, 0, v_\phi, B_z, 0, B_\phi, \rho) \qquad \text{(Case 4.3)}. \tag{101}$$

and satisfy

$$
\begin{aligned}
\frac{\partial \mathbf{u}_0}{\partial r} &= \left(0, 0, 0, \frac{\partial B_z}{\partial r}, 0, \frac{\partial B_\phi}{\partial r}, \frac{\partial \rho}{\partial r}\right) \qquad \text{(Case 4.1)} \\
\frac{\partial \mathbf{u}_0}{\partial r} &= \left(0, 0, \frac{\partial v_\phi}{\partial r}, \frac{\partial B_z}{\partial r}, 0, 0, \frac{\partial \rho}{\partial r}\right) \qquad \text{(Case 4.2)} \\
\frac{\partial \mathbf{u}_0}{\partial r} &= \left(0, 0, \frac{\partial v_\phi}{\partial r}, \frac{\partial B_z}{\partial r}, 0, \frac{\partial B_\phi}{\partial r}, \frac{\partial \rho}{\partial r}\right) \qquad \text{(Case 4.3).}
\end{aligned}
\tag{102}
$$

as well as

$$
\begin{aligned}
\frac{\partial \mathbf{u}_0}{\partial z} &= \left(0, 0, 0, 0, \frac{\partial B_r}{\partial z}, 0, 0\right) \qquad \text{(Case 4.1)} \\
\frac{\partial \mathbf{u}_0}{\partial z} &= \left(0, 0, 0, 0, \frac{\partial B_r}{\partial z}, 0, 0\right) \qquad \text{(Case 4.2)} \\
\frac{\partial \mathbf{u}_0}{\partial z} &= \mathbf{0} \qquad \text{(Case 4.3).}
\end{aligned}
\tag{103}
$$

The expression of $\partial \mathbf{R}/\partial r$ is already known (96). As for $\partial \mathbf{R}/\partial z$, we have the general form

$$
\frac{\partial \mathbf{R}}{\partial z} = \left(-\frac{B_z}{\rho c}, 0, -\frac{B_\phi}{\rho c}, 0, 0, 0, 0\right) \frac{\partial B_r}{\partial z}, \tag{104}
$$

which degenerates into

$$
\begin{aligned}
\frac{\partial \mathbf{R}}{\partial z} &= \left(-\frac{B_z}{\rho c}, 0, 0, 0, 0, 0, 0\right) \frac{\partial B_r}{\partial z} \qquad \text{(Case 4.2)} \\
\frac{\partial \mathbf{R}}{\partial z} &= \mathbf{0} \qquad \text{(Case 4.3).}
\end{aligned}
\tag{105}
$$

Since we have all the needed vectors and matrices, the proof of (34-36) is now a mere calculation.