Conventions in Repeated Games with Endogenous Separation

Segismundo S. Izquierdo*[a] and Luis R. Izquierdo[b]

[a] Department of Industrial Organization and BioEcoUva, Universidad de Valladolid, Spain. ORCID 0000-0002-7113-0633

[b] Department of Management Engineering, Universidad de Burgos, Spain. ORCID 0000-0003-1057-4465

October 29, 2025

Abstract

Games with endogenous separation are repeated games where players have the option to leave their current partnership and keep on playing in a newly-formed partnership. Arguably, most repeated interactions in real life fall into this category. We present a general framework to analyze equilibria in games with endogenous separation, with a special focus on social conventions, i.e., stable strategies that are resistant to direct invasion by any conceivable strategy. Our search for conventions leads to *path-protecting* strategies, which play a similar role to trigger strategies in standard (fixed-partnership) repeated games. We provide a constructive proof of existence for path-protecting strategies, and a folk theorem for neutrally stable conventions. *JEL* classification numbers: C72, C73.

Keywords: Endogenous separation; conventions; neutral stability; path-protecting strategy; voluntarily repeated games

1 Introduction

Games with endogenous separation (Rob and Yang, 2010) are repeated games where players have the option to leave their current partnership and keep on playing in a newly-formed partnership. Thus, in these games, partnerships may be broken for reasons that do not depend on the players' choices (exogenous separation), but also because players may decide to break the partnership (endogenous separation).[1]

*Correspondence to: EII, Universidad de Valladolid, Dr. Mergelina s/n, 47011 Valladolid, Spain. *e-mail*: segismundo.izquierdo@uva.es.

[1] In most models with endogenous separation, one single player's decision to leave is sufficient to break the partnership, but other alternatives have also been considered (see e.g. Kurokawa (2022) or Křivan and Cressman (2020)).

Arguably, in most social and biological interactions in real life, individuals have the option to leave and change partners; however, the option to leave is mostly absent from the theory of repeated games. Thus, it is not clear the extent to which standard results from the theory of repeated games remain valid for settings with the option to leave. To our knowledge, a standard framework for the analysis of games with endogenous separation has not been developed yet. Here we present a general framework for the analysis of symmetric two-player games with endogenous separation considering all possible strategies. We draw on the seminal work of Carmichael and MacLeod (1997) and Fujiwara-Greve and Okuno-Fujiwara (2009), who focus on the Prisoner's Dilemma, as well as on Vesely and Yang (2010) and Izquierdo et al. (2021).[2]

We pay special attention to *conventions*, which can be defined as "*behavioral regularities that serve as stable but to some degree arbitrary solutions to repeated coordination problems*" (Hawkins et al., 2019, p. 160). In our formal setting, a convention is a behavior or strategy which, if adopted by every player in a population, produces a stable equilibrium state. The key property of a convention is its stability, which must be formally defined. Considering that many reasonable processes of learning, adaptation or evolution lead to the replicator dynamics, we are particularly interested in stability under such dynamics. Two useful concepts of stability in game theory are neutral stability and evolutionary stability. In standard population games, neutral stability guarantees Lyapunov stability under the replicator dynamics, and evolutionary stability guarantees asymptotic stability (Weibull, 1995).

It is well known (Boyd and Lorberbaum, 1987) that in standard (fixed-partnership) infinitely repeated games there are no evolutionarily stable pure strategies (or mixed strategies with finite support). The same arguments can be used to show that there are no evolutionarily stable strategies in games with endogenous separation.[3] Therefore, we focus on neutral stability.

Interestingly, extending the standard definition of neutral stability to games with endogenous separation is not at all straightforward. Previous definitions of neutral stability for games with endogenous separation (Carmichael and MacLeod, 1997; Fujiwara-Greve and Okuno-Fujiwara, 2009; Izquierdo et al., 2021) can be considered unsatisfactory for different reasons (as discussed in section 3.4 and in appendix B). For instance, the definition of neutral stability for games with endogenous separation by Fujiwara-Greve and Okuno-Fujiwara (2009) does not guarantee dynamic stability under replicator dynamics.

The first challenge for games with endogenous separation consists in finding an appropriate characterization of the payoff function for a group of potential invaders in a population. The second difficulty stems from the fact that there are different standard definitions of neutral stability (Bomze and Weibull, 1995). These definitions are equivalent in the standard model of two-player games, where payoff functions are linear, but

[2] Graser et al. (2025) indicate that the option to leave leads to higher cooperation levels in the repeated Prisoner's Dilemma. Several studies focus on specific sets of strategies, such as Izquierdo et al. (2010, 2014), Zheng et al. (2017) and Li and Lessard (2021), or, in a spatial setting, Aktipis (2004, 2011) and Premo and Brown (2019). For experimental studies see Zhang et al. (2016) and the references therein.

[3] Given any equilibrium with finite support, there are other strategies that, when interacting with the equilibrium strategies, behave equivalently.

they are not equivalent when payoff functions are not linear (Bomze and Weibull, 1995), which is the case in games with endogenous separation.

In this paper we propose a simple definition of neutral stability for games with endogenous separation that is a direct adaptation of the original concept (Maynard Smith, 1982; Banerjee and Weibull, 2000) for population games, and which guarantees Lyapunov stability under the replicator dynamics. Therefore, neutrally stable strategies thus defined can be seen as social conventions in settings with the option to leave. It is important to note that our definition of neutral stability for games with endogenous separation is different from the definitions proposed by Carmichael and MacLeod (1997), Fujiwara-Greve and Okuno-Fujiwara (2009) and Izquierdo et al. (2021), which can be considered unsatisfactory (see appendix B); consequently, all the results in those references about *neutrally stable states* or conventions cannot be applied directly here.

Having found a suitable definition for a social convention, we then study necessary and sufficient conditions for strategies to be neutrally stable. It turns out that, in most games, a necessary condition for a strategy to be neutrally stable is that it never breaks up with itself (i.e., with a partner using the same strategy). We also identify a sufficient condition for neutral stability: *path-protection.* A *path-protecting* strategy never leaves a partner who mimics its behavior and, if adopted by all the players in a population, it guarantees that any player who deviates from the equilibrium path obtains a strictly lower payoff than the population's average. These conditions are shown to guarantee neutral stability. The concept of path-protecting strategy generalizes the idea behind *trust-building* strategies, which appear in previous works focused on the Prisoner's Dilemma and in other games with a similar structure (Datta, 1996; Ghosh and Ray, 1996; Carmichael and MacLeod, 1997; Kranton, 1996; Fujiwara-Greve and Okuno-Fujiwara, 2009).

Our main result is a constructive proof of existence for path-protecting strategies and a *folk theorem* for neutral stability or conventions. This result identifies behavior that can be stable in a population (conventions) and shows that in a game with endogenous separation, for large enough values of the (exogenous) continuation probability, any payoff between the pure minmax payoff and the maximum symmetric payoff of the stage game can be approached arbitrarily closely as the equilibrium payoff of some (neutrally stable) convention.

Path-protecting strategies present some similarities and some differences with the classical *trigger* strategies that, in standard repeated games, prevent deviations from an equilibrium path by playing a minmax action after a deviation. In standard repeated games, trigger strategies protect a path by the threat of punishment, but such potential punishment does not materialize in the equilibrium path. By contrast, in games with endogenous separation, a player who deviates from a convention can avoid punishment from his/her current partners by breaking up the partnership. Therefore, a convention in a population needs to ensure that players who start new partnerships bear some initial cost. In our setting, this cost can only take place through a painful *deviation-deterring* phase at the beginning of every new partnership. Furthermore, since there is no information flow between partnerships, every player must go through this initial deviation-deterring phase, so this unpleasant experience necessarily becomes part of the equilibrium path.

The paper is structured as follows. In section 2 we define games with endogenous separation derived from normal-form stage games, and we present their main elements: strategies, population states, pool states and payoff functions. In section 3 we provide definitions for Nash, evolutionarily stable and neutrally stable states in this framework. We discuss Nash states and the non-existence of evolutionarily stable strategies; we then focus on neutral stability. Having defined the payoff function for strategies and for distributions of strategies, it becomes natural to adapt a standard definition of neutral stability (Banerjee and Weibull, 2000) to repeated games with endogenous separation. We then show that neutral stability thus defined guarantees Lyapunov stability in the replicator dynamics for games with endogenous separation. Section 4 introduces *path-protecting* strategies, and shows how these strategies can be created, giving rise to monomorphic neutrally stable states, for sufficiently high exogenous continuation probabilities. Here we also provide a *folk theorem* for neutrally stable strategies or conventions in games with endogenous separation. Finally, in section 5 we present some conclusions.

The paper includes four appendices with proofs and additional results. Appendix A considers polymorphic neutrally stable states (mixtures of strategies). Here we show a strong limitation to the existence of polymorphic equilibria made up by different path-protecting strategies, and we extend the concept of path-protecting strategy to path-protecting state. Appendix B discusses some previous definitions of neutral stability that have been proposed for games with endogenous separation, and their limitations. Appendix C studies robustness against indirect invasions (van Veelen, 2012) in games with endogenous separation. Finally, appendix D contains most of the proofs.

2 Repeated Games with endogenous separation

In this section we present repeated Games with Endogenous Separation derived from normal-form stage games. For simplicity, we focus the presentation and the analysis on symmetric two-player stage games.

We consider a unit-mass population of agents who are matched in couples or *partnerships* to play a symmetric two-player normal-form stage game. The stage game $G = \{A, U\}$ is defined by an action set $A = \{a_1, ..., a_n\}$, and a payoff function $U\colon A^2 \to \mathbb{R}$, where $U(a_k, a_l)$ represents the payoff obtained by a player using action a_k whose opponent plays action a_l. Every stage game G has an associated repeated game with endogenous separation G^{Ends}, which is characterized in this section. Following Mailath and Samuelson (2006), we refer to choices in the stage game G as *actions*, reserving *strategy* for behavior in the repeated game.

2.1 Strategies in G^{Ends}

After playing a stage game G, partnerships may remain together and play the stage game again. A partnership is broken if either one of the players, according to their strategy, decides to break it (endogenous separation) or if some exogenous factor breaks the partnership, which happens with probability $(1-\delta) \in (0,1)$ after every interaction (exogenous separation). Thus, δ is the continuation probability of the partnership assuming that both players decide to stay. At the beginning of every (discrete) time period, all single players are randomly (re-)matched in partnerships, and then all players play the stage game, i.e., every player plays the stage game at every period, either in newly-formed partnerships or in older ones. We assume that there is no information flow between partnerships (Ghosh and Ray, 1996), so there are no reputation effects: single players (those who make up new partnerships) are anonymous.[4]

Considering the sequence of action profiles taken in a partnership, let the stage-t game, with $t \in \{1, 2, ...\}$, be the t^{th} time that the stage game is played in that partnership, assuming the partnership is not broken before. A strategy i for a player determines the choice that the player makes given any past history of play within a partnership. If the strategies followed by the two players in a partnership are i and j, the action profile played at stage t (assuming the partnership survives to play for the t^{th} time together), is $a_{ij}^{[t]} \equiv (a_i^{[t]}, a_j^{[t]}) \in A^2$, where $a_i^{[t]}$ is the action played by the player using strategy i (at stage t) and $a_j^{[t]}$ is the action played by the player using strategy j.

Denoting the null (empty) history by $a^{[0]}$, and taking $(A^2)^0 \equiv \{\emptyset\}$, a *history of play of length* $t \geq 0$, $a^{[0,t]} = (a^{[0]}, a^{[1]}, ..., a^{[t]}) \in (A^2)^t$, is a sequence of t action profiles.[5] The set of all possible histories of any length (including the empty history, or history of length 0) is

$$\mathcal{H} \equiv \bigcup_{t=0}^{\infty} (A^2)^t.$$

Let $\tilde{A} \equiv A \cup \{\textit{break}\}$ be the set of choices, where *break* represents the decision to break the current partnership. A strategy i for the repeated game is a mapping $i : \mathcal{H} \to \tilde{A}$, from the set of possible histories to the set of choices, that prescribes one choice $i(a^{[0,t]}) \in \tilde{A}$ for every possible history $a^{[0,t]}$, for every $t \geq 0$. As players in a new partnership are assumed to play at least once together before deciding whether to break their partnership, we require $i(\emptyset) \in A$. Let Ω be the set of strategies.

Note that:

- We assume $0 < \delta < 1$. The process for $\delta = 0$, where every partnership is exogenously broken after every stage game, would correspond to the standard framework for evolutionary population games.

- Constraining the strategy space to strategies that never choose *break* provides an evolutionary framework for standard indefinitely repeated games, where the stage game is iteratively repeated with probability δ.

[4] Fujiwara-Greve et al. (2012) consider a model where players may voluntarily provide information across partnerships in the context of the Prisoner's Dilemma.

[5] $a^{[0,t]}$ represents some sequence of t action profiles, while $a_{ij}^{[0,t]}$ represents the first t action profiles generated by strategy i when playing against strategy j, assuming they do not break up before stage t.

2.2 States and payoffs in G^{Ends}

We consider populations where the number of different strategies being played at any time is finite. Let x_i be the fraction of the population using strategy $i \in \Omega$. A (population) state $\mathbf{x}$ is a strategy distribution over Ω with finite support $\mathbb{S}(\mathbf{x}) \subset \Omega$, i.e., $\mathbf{x}$ is a function from Ω to $[0,1]$ that:

i) assigns a positive value $x_i > 0$ to each strategy i in a finite set $\mathbb{S}(\mathbf{x})$,

ii) assigns the value 0 to strategies that are not in $\mathbb{S}(\mathbf{x})$, and

iii) satisfies $\sum_{i \in \mathbb{S}(\mathbf{x})} x_i = 1$.

Let $\mathbb{D}$ be the set of distributions with finite support, and let $\mathbf{e}_i$ represent the monomorphic state at which all players use strategy i (i.e., the distribution satisfying $x_i = 1$ and $x_j = 0$ for every $j \in \Omega \setminus \{i\}$).

Consider an index $\mathcal{T}$ for periods of play of the game in the population. At every period, single players are matched and every player plays a stage game. In contrast, index t refers to repetitions of the stage game within a partnership: at period $\mathcal{T}$, after matching and before playing the stage game, every partnership has its own value for t, which, if the partnership has just been matched at that period, is set to 0 before playing the stage game and becomes 1 after playing the stage game. For any pair of strategies i and j, let their endogenous break-up period $T_{ij} \geq 1$ be the number of stages that an i-j partnership is to play together if the partnership is not broken by exogenous factors (i.e., the number of stage games they play together before one of them decides to break up). If an i-j partnership never breaks up endogenously, let $T_{ij} = \infty$.

To calculate the average payoff $F_i(\mathbf{x})$ obtained by players using strategy i when the population state is $\mathbf{x}$ (average per player in each period), we consider a stationary strategy distribution $\mathbf{p}$ in the pool of singles consistent with the population state $\mathbf{x}$. If the strategy distribution $\mathbf{p}$ in the pool of singles is stationary, then it should satisfy the following:

- Before matching, the mass of players in the pool of singles is a stationary value ϕ. The mass of single i-players in the pool is $\phi\, p_i$.

- After matching, the mass of i-players just matched to j-players, i.e., the mass of i-players in newly-formed (0-period-old) i-j partnerships, is $\phi\, p_i\, p_j$.

- For $1 \leq t \leq T_{ij}$, the mass of i-players in $(t-1)$-period-old i-j partnerships (after matching and before playing), is $\phi\, p_i\, p_j\, \delta^{t-1}$. These are the i-players that were matched in i-j partnerships $(t-1)$ periods ago and have survived exogenous (and endogenous) separation to play their t^{th} stage game in the current period $\mathcal{T}$. The total mass or fraction of i-players in the population is then

$$x_i = \phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i\, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} = \phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i\, p_j\, \frac{1-\delta^{T_{ij}}}{1-\delta}$$

and considering that $\sum_{j \in \mathbb{S}(\mathbf{x})} x_j = 1$, we have

$$x_i = \frac{p_i \sum_{j \in \mathbb{S}(\mathbf{x})} p_j (1 - \delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(\mathbf{x})} p_k \, p_j (1 - \delta^{T_{kj}})}. \tag{1}$$

Technically, in (1) we are assuming that the pool distribution has been stationary for at least as many periods as the longevity of the oldest partnership in the population.

Equation (1) defines a function $f : \mathbb{D} \to \mathbb{D}$ such that $\mathbf{x} = f(\mathbf{p})$, which provides the population state $\mathbf{x}$ corresponding to pool state $\mathbf{p}$.

- Let $a_{ij}^{[t]} = (a_i^{[t]}, a_j^{[t]}) \in A^2$ be the action profile played at the t^{th} stage of an i-j partnership, with the first action in the profile corresponding to the player using strategy i and the second action in the profile corresponding to the player using strategy j. The total payoff obtained (at each and every period $\mathcal{T}$) by the mass of i-players is

$$\phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i \, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} \, U(a_{ij}^{[t]}),$$

so, dividing by the mass of i-players, we have that the per-period per-player average payoff to an i-player is

$$\hat{F}_i(\mathbf{p}) \equiv (1 - \delta) \, \frac{\sum_{j \in \mathbb{S}(\mathbf{p})} \, p_j \, \sum_{t=1}^{T_{ij}} \delta^{t-1} \, U(a_{ij}^{[t]})}{\sum_{j \in \mathbb{S}(\mathbf{p})} \, p_j \, (1 - \delta^{T_{ij}})}, \tag{2}$$

which is defined for every $i \in \Omega$.

From (2) we have a formula for $\hat{F}_i(\mathbf{p})$ that provides the payoff to strategy i corresponding to pool state $\mathbf{p}$, and from (1) we have a formula $\mathbf{x} = f(\mathbf{p})$, that provides the population state $\mathbf{x}$ corresponding to pool state $\mathbf{p}$. In order to use existing results and concepts from the literature in population games, it would be convenient to have payoff functions F_i that provide the payoff to strategy i corresponding to population state $\mathbf{x}$, i.e., $F_i(\mathbf{x})$. Considering $\mathbf{x} = f(\mathbf{p})$ as defined in (1), it can be shown[6] that there is an inverse function f^{-1} such that $\mathbf{p} = f^{-1}(\mathbf{x})$, so we can define payoff functions F_i from population states as

$$F_i(\mathbf{x}) = \hat{F}_i(f^{-1}(\mathbf{x})). \tag{3}$$

Interestingly, for more than three strategies, $f^{-1}(\mathbf{x})$ does not admit a general closed-form algebraic expression. Our results are based on a series of properties of the payoff functions $F_i(\mathbf{x})$ that we indicate in the following section.

[6] See lemma D.2 in appendix D. A detailed proof can be found in Izquierdo et al. (2021).

Finally, for a group of players with strategy distribution $\mathbf{y} \in \mathbb{D}$ entering a population with strategy distribution $\mathbf{x}$, we define the average *payoff of* $\mathbf{y}$ *against* $\mathbf{x}$, $E(\mathbf{y}, \mathbf{x})$, as:

$$E(\mathbf{y}, \mathbf{x}) \equiv \sum_{i \in \mathbb{S}(\mathbf{y})} y_i F_i(\mathbf{x}). \tag{4}$$

We can interpret this payoff as the average payoff obtained by a very small mass of players whose strategy distribution is $\mathbf{y}$ (sometimes called *mutants* or *entrants*) when they play in a population of players whose strategy distribution is $\mathbf{x}$.

2.3 Properties of the payoff functions in G^{Ends}

The payoff functions $F_i : \mathbb{D} \to \mathbb{R}$, defined in (3) for every $i \in \Omega$, satisfy the following properties:

- At monomorphic population states (where $\mathbf{x} = \mathbf{e}_j = \mathbf{p}$) we have, from (2):

$$F_{ij} \equiv F_i(\mathbf{e}_j) = \frac{1-\delta}{1-\delta^{T_{ij}}} \sum_{t=1}^{T_{ij}} \delta^{t-1} U(a_{ij}^{[t]}). \tag{5}$$

  Note that the payoff F_{ij} to an i-player in a population of j-players is a convex combination of the stage payoffs $U(a_{ij}^{[t]})$ for $1 \leq t \leq T_{ij}$.

- It follows from (2), (3) and (5) that, for $\mathbf{p} = f^{-1}(\mathbf{x})$, we have

$$F_i(\mathbf{x}) = \hat{F}_i(\mathbf{p}) = \sum_{j \in \mathbb{S}(\mathbf{x})} \frac{p_j \left(1 - \delta^{T_{ij}}\right)}{\sum_{k \in \mathbb{S}(\mathbf{x})} p_k \left(1 - \delta^{T_{ik}}\right)} F_{ij}, \tag{6}$$

  which shows that $F_i(\mathbf{x})$ is a convex combination of the payoffs F_{ij} for $j \in \mathbb{S}(\mathbf{x})$, with (strictly) positive coefficients for the convex combination.

Let the *path* $a_{ij}^{[1,T_{ij}]} = ((a_i^{[1]}, a_j^{[1]}), (a_i^{[2]}, a_j^{[2]}), ..., (a_i^{[T_{ij}]}, a_j^{[T_{ij}]}))$ be the series of T_{ij} action profiles that strategy i generates when playing with strategy j until they decide to break up. Let the *repeated path* $h_{ij}^{[\infty]}$ be the infinite series of action profiles that corresponds to (or is generated by) one i-player in a population of j-players, with no exogenous separation and with re-matching after each endogenous separation:

$$h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]}, a_{ij}^{[1,T_{ij}]}, ...). \tag{7}$$

For a sequence of T action profiles $a^{[1,T]}$, where the t^{th} action profile in the sequence is $a^{[t]} \in A^2$, let the *normalized discounted value* $V(a^{[1,T]})$ be

$$V(a^{[1,T]}) \equiv \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} U(a^{[t]}).$$

From the previous definitions and the properties of geometric series, we have:

$$F_{ij} = V\left(a_{ij}^{[1,T_{ij}]}\right) = V\left(h_{ij}^{[\infty]}\right) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}U(h_{ij}^{[t]}), \tag{8}$$

where $h_{ij}^{[t]}$ is the t^{th} action profile in $h_{ij}^{[\infty]}$. Formula (8) shows that F_{ij} coincides with $V(h_{ij}^{[\infty]})$, which is the normalized discounted value of the infinite sequence of action profiles in the repeated path $h_{ij}^{[\infty]}$.

Note that in the framework we have presented for games with endogenous separation there is no discounting, and F_{ij} is defined as a per-period per-player average payoff (averaged over individuals whose prevalence in t-period-old partnerships is proportional to δ^t). However, the definition of repeated path in (7) allows to establish an equivalence between F_{ij} and the normalized discounted value $V(h_{ij}^{[\infty]})$. It follows from this equivalence that any two strategies j_1 and j_2 that generate the same repeated path against i-players obtain the same payoff against i-players, even if they have different break-up periods, i.e., even if they have different paths (as long as these paths, when repeated, generate the same sequence), i.e.:

$$h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} \implies F_{j_1 i} = F_{j_2 i}. \tag{9}$$

3 Equilibria in games with endogenous separation: Nash, evolutionarily stable and neutrally stable states

In this section we adapt standard definitions of Nash state, evolutionarily stable state and neutrally stable state to games with endogenous separation. For completeness, and in order to introduce the notation, we begin with the definitions for the stage game G.

3.1 Definitions for the stage game G

Here we present the main definitions and concepts for a stage game G that will be useful for the analysis of the repeated game with endogenous separation G^{Ends} derived from G.

The *best-response payoff* to action a is the best payoff that an action can obtain when playing against a, defined by

$$U^{BR}(a) \equiv \max_{a_l \in A} U(a_l, a).$$

The set of *best-response actions* to action a, $BR(a)$, is the set of actions that obtain the best-response payoff against a. If $a \in BR(a)$, i.e., if action a is a best-response to itself, we say that:

- (a, a) is a (symmetric) *Nash profile.*
- a is a *Nash action.*

The *pure minmax payoff* of G, m, is the minimum of the best-response payoffs to actions in A:

$$m \equiv \min_{a \in A} U^{BR}(a).$$

Every best-response payoff to an action is greater than or equal to m, i.e., $U^{BR}(a) \geq m \ \forall a \in A$.

A *minmax action* $\tilde{a} \in A$ is an action such that $U^{BR}(\tilde{a}) = m$. By choosing a minmax action, a player can guarantee that her opponent's payoff does not exceed m.

Let $q \in \Delta(A) \equiv \{(q_k)_{k=1}^n \in \mathbb{R}_+^n : \sum_{k=1}^n q_k = 1\}$ be a distribution over actions or *mixture* of actions. The payoff of action a against q is defined by $U_a(q) \equiv \sum_{l=1}^n U(a, a_l) q_l$. With some abuse in notation, the payoff of mixture $p \in \Delta(A)$ against $q \in \Delta(A)$ is defined by

$$U(p, q) \equiv \sum_{k=1}^{n} p_k \, U_{a_k}(q) = \sum_{k,l} p_k \, q_l \, U(a_k, a_l)$$

The best-response payoff against q is defined by

$$U^{BR}(q) \equiv \max_{p \in \Delta(A)} U(p, q) = \max_{a \in A} U_a(q).$$

The set of best-response actions to q, $BR(q)$, is the set of actions that obtain the best-response payoff against q.

A (symmetric) Nash equilibrium of G is a distribution $q \in \Delta(A)$ such that

$$U(q, q) = U^{BR}(q).$$

The (mixed) *minmax payoff* $\underline{m}$ of G is the minimum of the best-response payoffs to mixtures in $\Delta(A)$:

$$\underline{m} \equiv \min_{q \in \Delta(A)} \max_{a \in A} U_a(q).$$

Every best-response payoff (to some mixture) is greater than or equal to $\underline{m}$: $U^{BR}(q) \geq \underline{m}$, i.e., independently of q, if a is a best response to q, then the payoff of a against q is at least $\underline{m}$. It follows from the definitions that $\underline{m} \leq m$.

A distribution over actions $q \in \Delta(A)$ is *evolutionarily stable* (Maynard Smith and Price, 1973) if, for every other distribution $p \in \Delta(A)$:

$$\begin{aligned} &U(q, q) \geq U(p, q), \text{ i.e., } q \text{ is Nash, and} \\ &U(p, q) = U(q, q) \implies U(q, p) > U(p, p). \end{aligned}$$

An alternative definition of evolutionary stability only requires the condition $U(q, p) > U(p, p)$ to hold locally, i.e., in some punctured relative neighborhood of q. Evolutionary stability implies asymptotic stability under the replicator dynamics (Weibull, 1995; Sandholm, 2010).

There are several definitions of neutral stability (Maynard Smith, 1982) that are equivalent in this setting (Bomze and Weibull, 1995). Here we adopt the following one:

A distribution over actions $q \in \Delta(A)$ is *neutrally stable* if, for every distribution $p \in \Delta(A)$:

$$U(q,q) \geq U(p,q), \text{ i.e., } q \text{ is Nash, and}$$
$$U(p,q) = U(q,q) \implies U(q,p) \geq U(p,p).$$

Neutral stability requires that q is Nash and that it is robust to the introduction of (any combination of) alternative best responses to q, in the sense that q will not do worse than the average ($U(q,p) \geq U(p,p)$) when such alternative best responses are introduced. Neutral stability implies Lyapunov stability under the replicator dynamics (Thomas, 1985; Bomze and Weibull, 1995).

3.2 Nash states in G^{Ends}

A strategy j is a best response to state $\mathbf{x}$ if, when playing against $\mathbf{x}$, no other strategy (or distribution) can obtain a payoff greater than j's payoff, i.e., if and only if $F_j(\mathbf{x}) \geq F_k(\mathbf{x})$ for every $k \in \Omega$. Let $BR(\mathbf{x})$ be the set of best-response strategies to $\mathbf{x}$. A strategy distribution $\mathbf{y} \in \mathbb{D}$ is a best response to state $\mathbf{x}$ if and only if $E(\mathbf{y},\mathbf{x}) \geq E(\mathbf{z},\mathbf{x})$ for every $\mathbf{z} \in \mathbb{D}$. It follows from (4) that $\mathbf{y}$ is a best response to $\mathbf{x}$ if and only if every strategy in its support $\mathbb{S}(\mathbf{y})$ is a best response to $\mathbf{x}$.

Definition 1 (Nash equilibrium state)**.** *A state $\mathbf{x} \in \mathbb{D}$ is Nash (short for Nash equilibrium state) if $E(\mathbf{x},\mathbf{x}) \geq F_j(\mathbf{x})$ for every $j \in \Omega$. Equivalently, a state $\mathbf{x} \in \mathbb{D}$ is Nash if it is a best response to itself.*

If a monomorphic state $\mathbf{e}_i$ is Nash, we say that strategy i is a Nash strategy. Consequently, a strategy i is Nash if and only if $F_{ii} \geq F_{ji}$ for every $j \in \Omega$.

Let us now consider some implications of being a Nash strategy. The action profiles played at a monomorphic population $\mathbf{e}_i$ are always symmetric[7], i.e. in the set $\{(a,a)\}_{a \in A}$. Consequently, the payoff F_{ii} in a monomorphic population (see equation (5)) is a convex combination of the payoffs $\{U(a,a)\}_{a \in A}$ corresponding to the main diagonal of the payoff matrix of the stage game G. This implies that the maximum symmetric stage-game payoff $M \equiv \max_{a \in A} U(a,a)$ is an upper bound for F_{ii}.

If i is a Nash strategy, it cannot be beaten by any other strategy in its corresponding monomorphic population $\mathbf{e}_i$; in particular, strategy i cannot be beaten by what we call *reap-and-leave* strategies. Reap-and-leave strategies are those which, in a partnership with i, play exactly as i up to stage $T \leq T_{ii}$, at stage T adopt a best-response action to the action chosen by i, and then break the partnership. We say that such strategies reap-and-leave i at stage T.

The fact that being Nash implies robustness against reap-and-leave strategies allows us to derive simple conditions that must be satisfied by Nash strategies and Nash states in general. The next two propositions are based on robustness against strategies that

[7] In the symmetric setting that we consider, it is assumed that there is no role asymmetry (like row-player and column-player) on which players could condition their actions.

reap-and-leave i at the first stage of an i-j partnership, while the third proposition considers robustness against a strategy that reaps-and-leaves i at stage T_{ii}.

Lemma 3.1. *The first action $a^{\emptyset}$ played by a Nash strategy in G^{Ends} must satisfy*

$$U^{BR}(a^{\emptyset}) \leq M,$$

where $U^{BR}(a^{\emptyset})$ is the best-response stage payoff to action $a^{\emptyset}$ and $M = \max_{a \in A} U(a, a)$ is the maximum symmetric stage-game payoff.

To illustrate some practical applications of each result, we will consider the Prisoner's Dilemma and the Hawk-Dove game (also known as Snowdrift), with actions C and D (table 1). In the Prisoner's Dilemma, C stands for cooperate and D for defect; in the Hawk-Dove game, C corresponds to Dove and D to Hawk. In both cases, coordinating on C is more efficient than on D (i.e., the maximum diagonal stage payoff is $M = U_{CC} > U_{DD}$), and D is the minmax action. For the examples, we use the simpler notation $U_{a_k,a_l} \equiv U(a_k, a_l)$.

In the Prisoner's Dilemma ($U_{CD} < U_{DD} < U_{CC} < U_{DC}$), D is a dominant action and (D, D) is a Nash action profile. In the Hawk-Dove ($U_{DD} < U_{CD} < U_{CC} < U_{DC}$), the best-response to each action is the other action (this is an anti-coordination game). We will also consider the so-called 1-2-3 coordination game (table 1).

$$\begin{array}{c} \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{array}$$

Table 1: Left: A Prisoner's Dilemma game, with C for Cooperate and D for Defect. Middle: A Hawk-Dove game, with C for Dove and D for Hawk. Right: the 1-2-3 coordination game.

Example 1. *In the Prisoner's Dilemma, the only action that satisfies the condition in lemma 3.1 is action D. Consequently, every Nash strategy must begin a partnership by playing action D: no Nash strategy can be "nice" (Axelrod, 1984). This rules out strategies such as Tit for Tat.*

Similarly, for the Hawk-Dove, lemma 3.1 implies that every Nash strategy must begin a partnership by playing D (Hawk).

Lemma 3.2. *The minmax payoff $\underline{m}$ of a stage game G is a lower bound for the payoff at Nash states of G^{Ends}:*

$$\mathbf{x} \in \mathbb{D} \text{ is Nash} \implies E(\mathbf{x}, \mathbf{x}) \geq \underline{m}.$$

The pure minmax payoff m of a stage game G is a lower bound for the payoff F_{ii} at a Nash strategy i of G^{Ends}, and $M \equiv \max_{a \in A} U(a, a)$ is an upper bound:

$$i \in \Omega \text{ is Nash} \implies m \leq F_{ii} \leq M.$$

Example 2. *With the payoffs shown on table 1, lemma 3.2 implies that the payoffs to Nash strategies are: between* 2 *and* 3 *in the Prisoner's Dilemma; also between* 2 *and* 3 *in the Hawk-Dove; and between* 1 *and* 3 *in 1-2-3 coordination.*

Lemma 3.3. *If i is a Nash strategy with finite T_{ii}, then the action profile at the break-up stage T_{ii} of an i-i partnership is a Nash profile of the stage game G.*

Example 3. *In a Prisoner's Dilemma with endogenous separation, the action profile at the break-up stage of a Nash strategy with finite T_{ii} has to be (D,D).*

In a Hawk-Dove game, neither (C,C) nor (D,D) are Nash profiles, so in a Hawk-Dove game with endogenous separation there is no Nash strategy i with finite T_{ii}.

For the Prisoner's Dilemma, Observation 1 below strengthens the previous result.

Observation 1. *In the Prisoner's Dilemma with endogenous separation, Nash strategies with finite T_{ii} never play C in the equilibrium path.*

Observation 1 follows from considering that, in the Prisoner's Dilemma with endogenous separation, if a strategy i with finite T_{ii} ever plays the action profile (C, C) in an i-i partnership, then there is a stage T_l in $[1, T_{ii}]$ at which (C, C) is played for the last time, and a strategy j that reaps-and-leaves i at stage T_l beats i (in the sense $F_{ji} > F_{ii}$), so i cannot be Nash. Observation 1 can be extended to games G with only one symmetric Nash action profile which is the least efficient of the symmetric action profiles.

Lemma 3.4. *Let (a^N, a^N) be a Nash profile of G.*

- *Every strategy i that always chooses action a^N before breaking a partnership is a Nash strategy of G^{Ends}.*
- *Any mixture of strategies that satisfy the previous condition (for the same action a^N) is a Nash state of G^{Ends}.*

Example 4. *In a Prisoner's Dilemma with endogenous separation, any strategy i that for every history of length between* 0 *and T_{ii} (for some $T_{ii} > 0$) plays D, and breaks every partnership that gets to stage T_{ii}, is a Nash strategy (i.e., $\mathbf{e}_i$ is a monomorphic Nash state). Any mixture of such strategies is a Nash (polymorphic) state.*

In a Hawk-Dove game, neither (C,C) nor (D,D) are Nash profiles, so we cannot use lemma 3.4 to find Nash strategies for the game with endogenous separation.

After discussing Nash strategies, and given that we are mainly interested in conventions, or *stable* equilibrium strategies, we next analyze the existence of evolutionarily stable and neutrally stable strategies.

3.3 Evolutionarily stable strategies in G^{Ends}

Definition 2 (Evolutionarily stable strategy)**.** *A strategy $i \in \Omega$ is evolutionarily stable (Maynard Smith and Price, 1973) if*

$$\begin{aligned} &F_{ii} \geq F_{ji} && \text{for every } j \in \Omega, \text{ i.e., } i \text{ is Nash, and} \\ &F_i(\mathbf{y}) > E(\mathbf{y}, \mathbf{y}) && \text{for every } \mathbf{y} \in \mathbb{D} \setminus \{\mathbf{e}_i\} \text{ such that } E(\mathbf{y}, \mathbf{e}_i) = F_{ii}. \end{aligned}$$

Evolutionary stability for a Nash strategy i requires that there are no alternative best-response strategies j with $F_{ij} = F_{jj}$. The concept of *path-equivalent strategy*, defined below, will be useful to show that, in games with endogenous separation, there are no evolutionarily stable strategies. The argument extends easily to polymorphic states with finite support (which, in the standard framework, are equivalent to mixed strategies), and is basically the same argument used to show that there are no evolutionarily stable strategies in standard repeated games (Selten and Hammerstein, 1984).

Definition 3 (Path-equivalent strategy)**.** *Strategy j is path-equivalent to strategy i if*

$$a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}.$$

Considering that the action profiles in $a_{ii}^{[1,T_{ii}]}$ are symmetric, it follows that if j is path-equivalent to i, then $a_{ii}^{[1,T_{ii}]} = a_{ji}^{[1,T_{ji}]} = a_{ij}^{[1,T_{ij}]} = a_{jj}^{[1,T_{jj}]}$ and, consequently, $F_{ii} = F_{ji} = F_{ij} = F_{jj}$. If i is Nash and j is path-equivalent to i, then j is an alternative best-response to i (i.e., $F_{ji} = F_{ii}$) that satisfies $F_{ij} = F_{jj}$. By modifying the choices made by strategy i after histories $a^{[0,t]}$ that do not belong to the set of histories $\{a_{ii}^{[0,t]}\}_{t\in[0,T_{ii}]}$ generated by an i-i partnership, one can create (an infinite number of) strategies that are path-equivalent to strategy i. This proves that no strategy is evolutionarily stable in a game with endogenous separation, given that evolutionary stability does not admit the existence of any (different) path-equivalent strategy.

For completeness, in appendix C we discuss another equilibrium concept stronger than neutral stability: robustness against indirect invasions (van Veelen, 2012). We show that in many games with endogenous separation, such as those whose stage game is the Prisoner's Dilemma or the Hawk-Dove game, no strategy can be robust against indirect invasions.

3.4 Neutrally stable states in G^{Ends}

After showing that there are no evolutionarily stable strategies, in this section we define neutral stability for games with endogenous separation.

There have been several attempts to define neutral stability in games with endogenous separation, but all of them present undesirable features (see discussion in appendix B). For instance, the definition in Fujiwara-Greve and Okuno-Fujiwara (2009) (Definition B.2 in appendix B) does not guarantee Lyapunov stability in the replicator dynamics because it only requires robustness to monomorphic invasions, i.e., it does not consider invasions by groups of players using a mix of different strategies. As an example (see also Izquierdo et al. (2021)), consider the one-shot game with payoff matrix (10).

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \end{array} \tag{10}$$

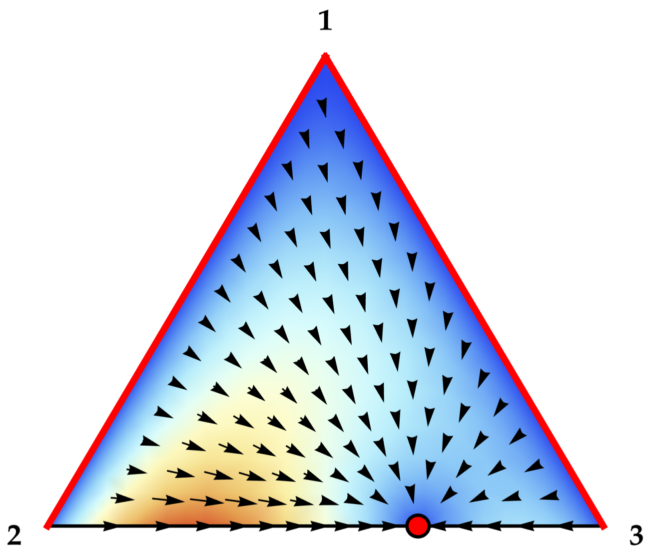

Figure 1: Phase portrait of the replicator dynamics in the game with payoff matrix (10). Rest points are shown in red: there is an isolated rest point at $(x_1, x_2, x_3) = (0, 1/3, 2/3)$ and a connected component of rest points along the edges where $x_2 = 0$ or $x_3 = 0$. Strategy 1 is neutrally stable according to Fujiwara-Greve and Okuno-Fujiwara (2009)'s definition because there are no exit routes along the edges from $\mathbf{e}_1$, but it is not neutrally stable according to the standard definition, and it is not Lyapunov stable. The background is colored according to the speed of motion: from blue (slowest) to brown (fastest). This figure has been generated with *EvoDyn-3s* (Izquierdo et al., 2018).

In this game, strategy 1 is neutrally stable according to Fujiwara-Greve and Okuno-Fujiwara (2009) because a monomorphic population of 1-players is robust to invasions by 2-players and also to invasions by 3-players (when considered separately). However, state $\mathbf{e}_1$ is not Lyapunov stable under the replicator dynamics (see figure 1). Strategy 1 is not neutrally stable according to the standard definition presented in section 3.1. The reason is that any mixture of 2-players and 3-players obtains a strictly greater payoff than 1-players at any interior state.

The definition we propose here is based on one of the standard definitions of neutral stability (Banerjee and Weibull, 2000): a state is neutrally stable if it is (i) a best response to itself, and also (ii) a weakly-better response to all its best-response states (than such states are to themselves).

Definition 4 (Neutrally stable state)**.** *A state $\mathbf{x} \in \mathbb{D}$ is neutrally stable if*

$$E(\mathbf{x},\mathbf{x}) \geq E(\mathbf{y},\mathbf{x}) \qquad \text{for every } \mathbf{y} \in \mathbb{D}, \text{ i.e., } \mathbf{x} \text{ is Nash, and}$$
$$E(\mathbf{x},\mathbf{y}) \geq E(\mathbf{y},\mathbf{y}) \qquad \text{for every } \mathbf{y} \in \mathbb{D} \text{ such that } E(\mathbf{y},\mathbf{x}) = E(\mathbf{x},\mathbf{x}).$$

A strategy i is said to be neutrally stable, or a convention, if and only if its associated monomorphic state $\mathbf{e}_i$ is neutrally stable.

Neutral stability requires strategy i to satisfy $F_i(\mathbf{y}) \geq E(\mathbf{y},\mathbf{y})$ whenever $\mathbf{y}$ is a mixture of alternative best-response strategies to $\mathbf{e}_i$. This robustness to every possible *mixture* of best response strategies is stronger than robustness against all best response strategies considered individually, as defined by the following condition:

$$F_{ij} \geq F_{jj} \text{ for every } j \in \Omega \text{ that is a best-response to } \mathbf{e}_i.$$

This latter condition is necessary but not sufficient for neutral stability. The reason is that, if j_1 and j_2 are two best-response strategies to $\mathbf{e}_i$, and $\mathbf{y}$ is a mixture of j_1 and j_2, then $E(\mathbf{y},\mathbf{y})$ depends not only on the payoffs $F_{j_1j_1}$ and $F_{j_2j_2}$ of each best-response strategy against itself, but also on the payoffs $F_{j_1j_2}$ and $F_{j_2j_1}$ for the crossed interactions.

Considering a finite set of strategies, we can define the replicator dynamics for games with endogenous separation as a direct adaptation of the standard replicator dynamics (Taylor and Jonker, 1978). Specifically, for a game with endogenous separation and a finite set of strategies $S \subset \Omega$, the replicator dynamics in S is the set of differential equations

$$\dot{x}_i = x_i[F_i(\mathbf{x}) - E(\mathbf{x},\mathbf{x})] \tag{11}$$

for $i \in S$. The replicator dynamics provides a good approximation to the (stochastic) dynamics of many reasonable evolutionary processes in large populations.[8] In our case, payoffs are calculated under the assumption of a stationary pool of singles. This means that, after any change in the composition of strategies in the population (caused, for instance, by reproduction or imitation), the pool of singles is assumed to approach its new

[8]See e.g. Weibull (1995, section 3.1.1), Sandholm (2010, examples 5.4.2-4), and Izquierdo et al. (2024, chapter V-1).

stationary distribution (i.e., the payoffs are assumed to approach their new theoretical values) before new changes in the composition of strategies in the population take place.

For the replicator dynamics with finite strategy set S, by numbering the s strategies in S, we can associate every strategy distribution with support contained in S with a point in the standard simplex $\Delta(S) \subset \mathbb{R}^s$.[9] Our next proposition shows that the definition of neutral stability that we adopt for games with endogenous separation (definition 4) guarantees Lyapunov stability in the replicator dynamics, independently of which strategies are included in S.

Proposition 1. *Let $\mathbf{x} \in \mathbb{D}$ be neutrally stable, and let S be any finite (numbered) superset of its support. Let $\hat{\mathbf{x}} \in \Delta(S)$ be the point that represents $\mathbf{x}$ in $\mathbb{R}^s$. Then $\hat{\mathbf{x}}$ is a Lyapunov stable rest point of the replicator dynamics in S.*

Proposition 1 shows that, if $\mathbf{x}$ is neutrally stable according to definition 4, then its associated point $\hat{\mathbf{x}}$ representing $\mathbf{x}$ in $\Delta(S)$ is a Lyapunov stable rest point in the replicator dynamics, considering any finite set of strategies S that includes the support of $\mathbf{x}$ (*incumbents*), and any other set of strategies (*potential invaders*), whatever those potential invaders may be.

Next we study the existence of neutrally stable strategies. Lemma 3.5 below shows a strong limitation for the stability of strategies with finite break-up period: if the symmetric action profile of G with maximum payoff $M = \max_{a\in A} U(a, a)$ is not[10] Nash, then no strategy i with finite break-up period T_{ii} can be neutrally stable. Formally, let N_M^G be the (possibly empty) set of symmetric Nash profiles of stage game G that obtain the maximum symmetric payoff $M = \max_{a\in A} U(a, a)$.

Lemma 3.5. *If i is a neutrally stable strategy (of G^{Ends}) with finite break-up period T_{ii}, the action profiles played in an i-i partnership are in N_M^G.*

As a consequence of lemma 3.5, for games with $N_M^G = \emptyset$ (such as the Prisoner's Dilemma or the Hawk-Dove) no strategy i with finite break-up period can be neutrally stable. This result shows that, for many games, no convention can display (endogenous) break-up in the equilibrium path.

After this first result in our search for conventions, we present *path-protecting* and *weakly path-protecting* strategies, which imply neutral stability and whose existence, for sufficiently large values of δ, is guaranteed for many games.

4 Path-protecting strategies

In this section we define *path-protecting* and *weakly path-protecting* strategies. Both concepts imply neutral stability. We also discuss their existence.

[9] The s strategies in S can be numbered following any order $i_1, i_2, ..., i_{s-1}, i_s$, and then we can define $\hat{x}_k = x_{i_k}$ for $k \in \{1, ..., s\}$.

[10] are not, if there are several.

Definition 5 (Path-protecting strategy)**.** *A strategy $i \in \Omega$ is path-protecting if:*

$$a_{jj}^{[1,T_{jj}]} \neq a_{ii}^{[1,T_{ii}]} \implies F_{ji} < F_{ii}.$$

In words, a strategy i is path-protecting if, when playing against i, only those strategies that are path-equivalent to i (those with $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}$) obtain the same payoff as i, while every strategy j that is not path-equivalent to i obtains a strictly lower payoff.

Note that a necessary condition for a strategy i to be path-protecting is that $T_{ii} = \infty$. The reason is that, if T_{ii} is finite, then any strategy j with $T_{jj} > T_{ii}$ whose path of play up to stage T_{ii} coincides with that of i (i.e., $a_{jj}^{[1,T_{ii}]} = a_{ii}^{[1,T_{ii}]}$) satisfies $F_{ji} = F_{ii}$.

Considering that $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,\infty]}$ if and only if $a_{ij}^{[1,T_{ij}]} = a_{ii}^{[1,\infty]}$, it is easy to see that a strategy i is path-protecting if and only if $T_{ii} = \infty$ and

$$a_{ji}^{[1,T_{ij}]} \neq a_{ii}^{[1,\infty]} \implies F_{ji} < F_{ii}.$$

This alternative characterization shows that a path-protecting strategy i "protects" the equilibrium path against strategies that, when playing with i, deviate at some point from i's choice, either by choosing a different action or by breaking the partnership.

We now define a concept weaker than path-protecting strategy, which will turn out to be sufficient to guarantee neutral stability, namely *weakly path-protecting strategy.* Before doing so, for convenience, let us recall that $h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]})^{\infty}$ is the infinite sequence of action profiles generated by strategy i in a population of j-players with no exogenous separation (7), and F_{ij} coincides with $V(h_{ij}^{[\infty]})$, the normalized discounted value of (the action profiles in) $h_{ij}^{[\infty]}$.

Definition 6 (Weakly path-protecting strategy)**.** *A strategy $i \in \Omega$ with $T_{ii} = \infty$ is weakly path-protecting if:*

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}.$$

In words, a strategy i is weakly path-protecting if

- it never breaks a partnership with a partner who takes the same actions as i does, and
- if the repeated path $h_{ji}^{[\infty]}$ that strategy j generates with i-players is different from the path $a_{ii}^{[1,\infty]}$ that i generates, then j obtains a strictly lower payoff (in a population of i-players) than i.

Note that any strategy j that at some stage of an i-j partnership adopts a different action from the action adopted by i generates a different repeated path $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$. Strategies j that, before breaking an i-j partnership at a finite stage T_{ij}, do not adopt different actions from i's, may still generate the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$, but only if $h_{ii}^{[\infty]}$ is an infinite repetition of the finite sequence of T_{ij} (symmetric) action profiles $a_{ii}^{[1,T_{ij}]} = a_{ji}^{[1,T_{ij}]}$.

For any strategy i with a path $a_{ii}^{[1,\infty]}$ that is not an infinite repetition $(a^{[1,T]})^\infty$ of some finite sequence $a^{[1,T]}$ of action profiles, being weakly path-protecting is equivalent to being path-protecting, because, in that case, the only way a strategy i can protect the repeated path $h_{ii}^{[\infty]}$ is by protecting the path $a_{ii}^{[1,\infty]}$. By contrast, strategies i with a path $a_{ii}^{[1,\infty]}$ that is an infinite repetition of some finite sequence may be weakly path-protecting, but cannot be path-protecting.

Considering equation (9), it follows from definition 5 that if strategy i is weakly path-protecting, then:

- Strategy i is Nash, because strategies with the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$ obtain the same payoff $F_{ji} = F_{ii}$ and strategies with different repeated path obtain a lower payoff $F_{ji} < F_{ii}$, so $F_{ji} \leq F_{ii}$ for every j.
- Every best-response strategy j to $\mathbf{e}_i$ must generate the same (symmetric) repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$. This implies that, if j is a best-response to $\mathbf{e}_i$, then $F_{ij} = F_{ji} = F_{ii}$. It also implies that if $\mathbf{y}$ is a mixture of best-response strategies to $\mathbf{e}_i$, then $F_i(\mathbf{y}) = F_{ii}$.

Our next result states that (weakly) path-protecting strategies are neutrally stable. Its proof shows that, if strategy i is weakly path-protecting, then any mixture $\mathbf{y}$ of best-response strategies to $\mathbf{e}_i$ must satisfy $E(\mathbf{y}, \mathbf{y}) = F_i(\mathbf{y})$. The reason is that every repeated path $h_{j_1 j_2}^{[\infty]}$ generated between any two best-response strategies (j_1 and j_2) to $\mathbf{e}_i$ must also be equal to $h_{ii}^{[\infty]}$, so if $\mathbf{y}$ is a mixture of best-response strategies to $\mathbf{e}_i$, then $E(\mathbf{y}, \mathbf{y}) = F_{ii} = F_i(\mathbf{y})$.

Proposition 2. *(Weakly) path-protecting strategies are neutrally stable.*

Weakly path-protecting strategies can be easily found if the stage game has some strict Nash profile, as our next result shows.

Lemma 4.1. *If $(\hat{a}, \hat{a})$ is a strict Nash profile of a stage game G, then any strategy of G^{Ends} that:*

- *chooses action $\hat{a}$ whenever it does not choose to break a partnership, and*
- *does not break a partnership while profile $(\hat{a}, \hat{a})$ is played*

is weakly path-protecting (and, consequently, neutrally stable).

Example 5. *In the Prisoner's Dilemma, (D, D) is a strict Nash profile. Consequently, any strategy that never plays C and never breaks up after a history of mutual defections is weakly path-protecting. For instance, the strategy "always play D and never leave", that maps every history to D, is weakly path-protecting and, consequently, neutrally stable.*

Much more generally than the case in which G has some strict Nash profile, Proposition 3 below shows that, for large enough δ, every stage game G with $M > m$ admits path-protecting strategies. Proposition 3 leads to a *folk theorem* for neutral stability which basically says that, for large enough δ, any payoff between m and M can be obtained, or approached arbitrarily closely, as the equilibrium payoff of some neutrally stable strategy.

Before stating proposition 3, let us define the average stage-payoff for a finite sequence of action profiles. Considering a sequence $\Phi = (\Phi^{[t]})_{t=1}^{T}$of T action profiles, where each $\Phi^{[t]} \in A^2$ is an action profile, let the average stage-payoff of sequence Φ be

$$\bar{U}_\Phi \equiv \frac{\sum_{t=1}^{T} U(\Phi^{[t]})}{T}.$$

The average stage payoff is specially relevant for large δ and for paths that end up repeating some sequence Φ of action profiles, because the normalized payoff of any infinite path $([...], \Phi, \Phi, \Phi, ...)$ which, after a finite number of periods, eventually repeats the finite sequence of outcomes Φ forever, converges to the average stage-payoff $\bar{U}_\Phi$ as δ goes to 1.

Proposition 3. *Let Φ be a finite sequence of symmetric action profiles with average stage payoff $\bar{U}_\Phi$ strictly greater than the pure minmax payoff. For large enough $\delta < 1$, there are path-protecting strategies whose equilibrium path, after a finite transient phase, is an infinite repetition of the sequence Φ, and whose equilibrium payoff converges to $\bar{U}_\Phi$ as $\delta \to 1$.*

Considering that $\bar{U}_\Phi$ can approximate any real payoff between m and M as much as desired, Proposition 3 has as a corollary the following folk theorem for neutrally stable strategies.

Corollary 3.1. *(Folk theorem for neutral stability). In a game with endogenous separation, for large enough values of the continuation probability δ, any payoff between the pure minmax payoff m and the maximum symmetric payoff M of the stage game can be obtained, or approximated as much as desired, as the equilibrium payoff of some neutrally stable strategy.*

The proof of proposition 3 is included in appendix D, but here we provide a sketch. The proof is constructive and considers a strategy i such that:

- It never breaks a partnership with a partner who takes the same actions as i does (i.e., $T_{ii} = \infty$).
- As soon as strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership. Because of this condition, we know that an i-j partnership will not survive if j chooses a different action from the action chosen by i. Naturally, it will not survive either if j chooses to break the partnership. The only way in which an i-j partnership can survive indefinitely is if j chooses the same initial action as i does and, for every history $a_{ii}^{[0,t]}$ corresponding to an i-i partnership, j chooses the same action as i does.

- The path $a_{ii}^{[1,\infty]}$ is made up by three phases, each one associated to one finite sequence of symmetric action profiles (Φ_m, Φ_f and Φ_p), with

$$a_{ii}^{[1,\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^\infty),$$

  where Φ_m is a repetition of a minmax action profile, Φ_f is arbitrary (but finite), Φ_p (which corresponds to the infinitely repeated pattern Φ in proposition 3) has an average stage payoff greater than the pure minmax payoff m of the stage game, and $(\Phi_p)^\infty$ represents an infinite sequence of action profiles made up by repeating the sequence Φ_p infinitely.

- The first phase in $a_{ii}^{[1,\infty]}$ is a T_m-period-long phase, $T_m \geq 1$, during which a minmax action profile $(\tilde{a}, \tilde{a})$ is played, producing the sequence

$$\Phi_m = a_{ii}^{[1,T_m]} = ((\tilde{a}, \tilde{a}), (\tilde{a}, \tilde{a}), ..., (\tilde{a}, \tilde{a})).$$

  During this minmax or *deviation-deterring* phase, the stage payoff is $U(\tilde{a}, \tilde{a}) \leq m$ and any strategy j that deviates in choice during this phase obtains a payoff $F_{ji} \leq m$.

- The second phase in the path $a_{ii}^{[1,\infty]}$ is an arbitrary finite sequence of $T_f \geq 0$ (symmetric) action profiles.

- The last phase, or *pattern-playing* phase, in $a_{ii}^{[1,\infty]}$ is an infinite repetition of a finite sequence (pattern) Φ_p of $T_p \geq 1$ symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

The proof of proposition 3 combines three intermediate results to create path-protecting strategies. These strategies are initially built to be weakly path-protecting, and then fine-tuned so the path when they play against themselves is not an infinite repetition of any finite sequence of action profiles, so they are also path-protecting.

- The first result shows that, in order to prove that the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}$ holds for every strategy j, it is enough to prove that it holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation before and up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful.

- The second result states that, for any given Φ_f and Φ_p (with $\bar{U}_{\Phi_p} > m$), the deviation-deterring phase can be chosen to be long enough to guarantee that deviations in $h_{ji}^{[\infty]}$ from $h_{ii}^{[\infty]}$ at or before $t = T_m + T_f + T_p$ lead to payoffs F_{ji} close to or below m.

- The third result states that, for sufficiently large δ, the payoff F_{ii} is close to $\bar{U}_{\Phi_p} > m$.

Combining the three results shows that, given Φ_f and Φ_p, there is a length of the deviation-deterring phase T_m such that, for large enough δ, $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$ implies $F_{ji} < F_{ii}$, so strategy i is weakly path protecting. Finally, by choosing Φ_p so that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, we make sure that strategy i is also path-protecting.

Example 6. *In a Prisoner's Dilemma or in a Hawk-dove game, the minmax profile is* DD,[11] *so:*

- *For the deviation-deterring or minmax phase, Φ_m is a T_m-long series of DD action profiles.*
- *For the pattern-playing phase, the infinitely repeated finite pattern Φ_p can be any finite sequence of DD and CC action profiles with at least one CC in the sequence, which guarantees an average stage payoff $\bar{U}_{\Phi_p} > m = U_{DD}$.*

For instance, choosing $T_m = 3$, $\Phi_f = (CC, DD)$ and $\Phi_p = (CC)$, we obtain a strategy i with path $h_{ii}^{\infty} = (DD, DD, DD \,|\, CC, DD \,|\, (CC)^{\infty})$. For the stage payoffs shown on table 1 for the Prisoner's Dilemma, the sequence of payoffs corresponding to h_{ii}^{∞} is $(2, 2, 2, 3, 2, (3)^{\infty})$, where $()^{\infty}$ represents an infinite repetition of the payoffs in brackets, so

$$F_{ii} = (1 - \delta)(2 + 2\delta + 2\delta^2 + 3\delta^3 + 2\delta^4 + 3\frac{\delta^5}{1 - \delta}) > 2.$$

The pattern $\Phi_p = (CC)$ begins to be repeated after period 6. Strategies j with $T_{ji} \leq 3$ obtain a payoff F_{ji} of at most the minmax payoff $2 < F_{ii}$. For $T_{ji} = 4$ the payoff F_{ji} is bounded by that of the series $(2, 2, 2, 4)^{\infty}$, and for $5 \leq T_{ji} \leq 6$ the payoff is bounded by that of the series $(2, 2, 2, 3, 2, 4)^{\infty}$. For $\delta > 0.71$, F_{ii} is greater than the payoffs corresponding to both series, so i is path-protecting.

Example 7. *In 1-2-3 coordination, the minmax profile is (a_1, a_1), so*

- *Φ_m is a T_m-long series of (a_1, a_1) action profiles.*
- *Φ_p can be any finite sequence of symmetric action profiles where at least one action profile is not (a_1, a_1), which guarantees $\bar{U}_{\Phi_p} > m = U(a_1, a_1)$.*

5 Conclusions

In the standard approach to repeated games, partners are tied to each other and do not have a say on whether they wish to stay together or whether they prefer to leave their current partner and meet a new one. For many real-life situations, the field of games with endogenous separation constitutes a natural and more realistic alternative.

Following some pioneers (most notably, Carmichael and MacLeod (1997)), a major step forward to study games with endogenous separation was taken by Fujiwara-Greve

[11] For compactness, here we represent action profiles (D, D) as DD.

and Okuno-Fujiwara (2009).[12] This seminal paper, while focused on the Prisoner's Dilemma, provided the first fundamental framework for the study of games with endogenous separation taking into account the whole strategy space.[13] Fujiwara-Greve and Okuno-Fujiwara's (2009) framework is based on the strategy distribution in the *pool of singles*. In contrast, in this paper we develop an approach based on the distribution of strategies in the population, or population state. This approach allows us to establish clear links and differences between games with endogenous separation and standard repeated games, including the definition of appropriate payoff functions for (a group of) potential invaders, the adaptation of static equilibrium concepts such as neutral stability (given that previous attempts to define neutral stability for games with endogenous separation did not manage to provide a satisfactory definition), and the adaptation of standard dynamics such as the replicator dynamics.

In this paper, we have also introduced the notion of *path-protecting* strategy, shown that they constitute conventions (behavior that, if adopted in a population, is resistant to invasion by small groups of players using any conceivable strategy), and provided an existence result for path-protecting strategies in games with endogenous separation: in general, for sufficiently large continuation probability, there is a large variety of path-protecting neutrally stable strategies. The concept of path-protecting strategy generalizes the idea of *trust-building* strategies that appear in previous related works for the Prisoner's Dilemma and some of its variations (Datta, 1996; Ghosh and Ray, 1996; Carmichael and MacLeod, 1997; Kranton, 1996; Fujiwara-Greve and Okuno-Fujiwara, 2009). As an additional result to our search for behavior that can constitute a convention, we obtain a folk theorem for neutrally stable states in games with endogenous separation. We also extend the concept of path-protecting strategy from strategies (monomorphic states) to mixtures of strategies in a population (polymorphic states).

Extensions of the framework of games with endogenous separation to multiplayer asymmetric games or multi-population games present additional challenges and remain an open field of research.

Acknowledgements

We thank Prof. Christoph Hauert and the mathematics department of the University of British Columbia for their feedback and support, and for hosting us during the development of this work. We also thank an anonymous reviewer for helpful comments.

[12] It is also worth noting the work of Vesely and Yang (2010), which constitutes an approach based on behavioral strategies.

[13] This framework has been used and extended in subsequent papers such as Fujiwara-Greve et al. (2012, 2015).

Declarations

Funding

Partial financial support was received from the Spanish State Research Agency (PID2024-159461NB-I00/MICIU and PID2020-118906GB-I00/MCIN, AEI/10.13039/501100011033/EU-FEDER), the Regional Government of Castilla y León and EU-FEDER program (CLU-2019-04 - BIOECOUVA Unit of Excellence of the University of Valladolid), and the Spanish Ministry of Universities (PRX22/00064 and PRX22/00065).

References

Aktipis, C. A. (2004). Know when to walk away: contingent movement and the evolution of cooperation. *Journal of Theoretical Biology*, *231*, 249–260, https://doi.org/10.1016/j.jtbi.2004.06.020.

Aktipis, C. A. (2011). Is cooperation viable in mobile organisms? simple walk away rule favors the evolution of cooperation in groups. *Evolution and Human Behavior*, *32*, 263–276, https://doi.org/10.1016/J.EVOLHUMBEHAV.2011.01.002.

Axelrod, R. (1984). *The Evolution of Cooperation*. Basic Books.

Banerjee, A. & Weibull, J. W. (2000). Neutrally stable outcomes in cheap-talk coordination games 1. *Games and Economic Behavior*, *32*, 1–24, https://doi.org/10.1006/game.1999.0756.

Bomze, I. M. & Weibull, J. W. (1995). Does neutral stability imply Lyapunov stability? *Games and Economic Behavior*, *11*, 173–192, https://doi.org/10.1006/game.1995.1048.

Boyd, R. & Lorberbaum, J. P. (1987). No pure strategy is evolutionarily stable in the repeated prisoner's dilemma game. *Nature*, *327*(6117), 58–59, https://doi.org/10.1038/327058a0.

Carmichael, H. L. & MacLeod, W. B. (1997). Gift Giving and the Evolution of Cooperation. *International Economic Review*, *38*(3), 485, https://doi.org/10.2307/2527277.

Cox, D. A., Little, J., & O'Shea, D. (2015). *Ideals, Varieties, and Algorithms*. Springer International Publishing http://link.springer.com/10.1007/978-3-319-16721-3.

Datta, S. (1996). Building trust. *STICERD - Theoretical Economics Paper Series*, *TE/1996/305*. https://sticerd.lse.ac.uk/_NEW/PUBLICATIONS/abstract/?index=1541.

Fujiwara-Greve, T. & Okuno-Fujiwara, M. (2009). Voluntarily separable repeated prisoner's dilemma. *Review of Economic Studies*, *76*(3), 993–1021, https://doi.org/10.1111/j.1467-937X.2009.00539.x.

Fujiwara-Greve, T., Okuno-Fujiwara, M., & Suzuki, N. (2012). Voluntarily separable repeated prisoner's dilemma with reference letters. *Games and Economic Behavior*, *74*, 504–516, https://doi.org/10.1016/J.GEB.2011.08.019.

Fujiwara-Greve, T., Okuno-Fujiwara, M., & Suzuki, N. (2015). Efficiency may improve when defectors exist. *Economic Theory*, *60*(3), 423–460, https://doi.org/10.1007/s00199-015-0909-4.

Ghosh, P. & Ray, D. (1996). Cooperation in community interaction without information flows. *The Review of Economic Studies*, *63*, 491, https://doi.org/10.2307/2297892.

Gordon, W. B. (1972). On the diffeomorphisms of euclidean space. *The American Mathematical Monthly*, *79*, 755, https://doi.org/10.2307/2316266 https://www.jstor.org/stable/2316266?origin=crossref.

Graser, C., Fujiwara-Greve, T., García, J., & van Veelen, M. (2025). Repeated games with partner choice. *PLOS Computational Biology*, *21*, e1012810, https://doi.org/10.1371/journal.pcbi.1012810 https://dx.plos.org/10.1371/journal.pcbi.1012810.

Hawkins, R. X., Goodman, N. D., & Goldstone, R. L. (2019). The emergence of social norms and conventions. *Trends in Cognitive Sciences*, *23*(2), 158–169, https://doi.org/https://doi.org/10.1016/j.tics.2018.11.003.

Izquierdo, L. R., Izquierdo, S. S., & Sandholm, W. H. (2018). *EvoDyn-3s*: A Mathematica computable document to analyze evolutionary dynamics in 3-strategy games. *SoftwareX*, *7*, 226–233, https://doi.org/10.1016/J.SOFTX.2018.07.006.

Izquierdo, L. R., Izquierdo, S. S., & Sandholm, W. H. (2024). *Agent-Based Evolutionary Game Dynamics*. University of Wisconsin Pressbooks https://wisc.pb.unizin.org/agent-based-evolutionary-game-dynamics.

Izquierdo, L. R., Izquierdo, S. S., & Vega-Redondo, F. (2014). Leave and let leave: A sufficient condition to explain the evolutionary emergence of cooperation. *Journal of Economic Dynamics and Control*, *46*, 91–113, https://doi.org/10.1016/j.jedc.2014.06.007.

Izquierdo, S. S., Izquierdo, L. R., & Veelen, M. V. (2021). Repeated games with endogenous separation. *Universidad de Valladolid, Mimeo*. https://uvadoc.uva.es/handle/10324/52054.

Izquierdo, S. S., Izquierdo, L. R., & Vega-Redondo, F. (2010). The option to leave: Conditional dissociation in the evolution of cooperation. *Journal of Theoretical Biology*, *267*(1), 76–84, https://doi.org/10.1016/j.jtbi.2010.07.039.

Kranton, R. E. (1996). The formation of cooperative relationships. *Journal of Law, Economics, and Organization*, *12*, 214–233, https://doi.org/10.1093/oxfordjournals.jleo.a023358.

Křivan, V. & Cressman, R. (2020). Defectors' intolerance of others promotes cooperation in the repeated public goods game with opting out. *Scientific Reports*, *10*, 19511, https://doi.org/10.1038/S41598-020-76506-3.

Kurokawa, S. (2022). Evolution of cooperation in an n-player game with opting out. *Behavioural Processes*, *203*, 104754, https://doi.org/10.1016/J.BEPROC.2022.104754.

Li, C. & Lessard, S. (2021). The effect of the opting-out strategy on conditions for selection to favor the evolution of cooperation in a finite population. *Journal of Theoretical Biology*, *510*, https://doi.org/10.1016/j.jtbi.2020.110543.

Mailath, G. J. & Samuelson, L. (2006). *Repeated Games and Reputations: Long-Run Relationships*. Oxford University Press.

Maynard Smith, J. (1982). *Evolution and the Theory of Games*. Cambridge University Press.

Maynard Smith, J. & Price, G. R. (1973). The logic of animal conflict. *Nature*, *246*(5427), 15–18, https://doi.org/10.1038/246015a0.

Premo, L. S. & Brown, J. R. (2019). The opportunity cost of walking away in the spatial iterated prisoner's dilemma. *Theoretical Population Biology*, *127*, 40–48, https://doi.org/10.1016/J.TPB.2019.03.004.

Rob, R. & Yang, H. (2010). Long-term relationships as safeguards. *Economic Theory*, *43*(2), 143–166, https://doi.org/10.1007/s00199-008-0421-1.

Sandholm, W. H. (2010). *Population games and evolutionary dynamics*. The MIT Press.

Selten, R. & Hammerstein, P. (1984). Gaps in harley's argument on evolutionarily stable learning rules and in the logic of "tit for tat". *Behavioral and Brain Sciences*, *7*(1), 115–116, https://doi.org/10.1017/S0140525X00026479.

Taylor, P. D. & Jonker, L. B. (1978). Evolutionary stable strategies and game dynamics. *Mathematical Biosciences*, *40*, 145–156, https://doi.org/10.1016/0025-5564(78)90077-9.

Thomas, B. (1985). On evolutionarily stable sets. *Journal of Mathematical Biology*, *22*(1), https://doi.org/10.1007/BF00276549.

van Veelen, M. (2012). Robustness against indirect invasions. *Games and Economic Behavior*, *74*, 382–393, https://doi.org/10.1016/j.geb.2011.05.010.

Vesely, F. & Yang, C.-L. (2010). On optimal and neutrally stable population equilibrium in voluntary partnership prisoner's dilemma games. *SSRN*, https://doi.org/10.2139/ssrn.1541684.

Vesely, F. & Yang, C.-L. (2012). Breakup, secret handshake and neutral stability in repeated prisoner's dilemma with option to leave: A note. *SSRN*, https://doi.org/10.2139/ssrn.2179126.

Weibull, J. W. (1995). *Evolutionary Game Theory*. The MIT Press.

Zhang, B.-Y., Fan, S.-J., Li, C., Zheng, X.-D., Bao, J.-Z., Cressman, R., & Tao, Y. (2016). Opting out against defection leads to stable coexistence with cooperation. *Scientific Reports*, *6*, 35902, https://doi.org/10.1038/srep35902.

Zheng, X. D., Li, C., Yu, J. R., Wang, S. C., Fan, S. J., Zhang, B. Y., & Tao, Y. (2017). A simple rule of direct reciprocity leads to the stable coexistence of cooperation and defection in the prisoner's dilemma game. *Journal of Theoretical Biology*, *420*, 12–17, https://doi.org/10.1016/J.JTBI.2017.02.036.

A Polymorphic neutrally stable states

Let us now consider polymorphic neutrally stable states, in which players in a population use different strategies (beyond those states already considered in lemma 3.4). In the standard setting of population games, polymorphic states can alternatively be interpreted as mixed strategies. In games with endogenous separation, the average payoff to a group of players with strategy distribution $\mathbf{y}$ in a population $\mathbf{x}$, equation (4), does not need to coincide with the payoff to an individual using mixed strategy $\mathbf{y}$ in a population $\mathbf{x}$ (because each of the pure strategies in the support of $\mathbf{y}$ may have different break-up periods with the strategies used in $\mathbf{x}$). In this setting, interpreting a strategy distribution as an individual's mixed strategy is not straightforward.

Looking for stable polymorphic states, the first candidate wold seem to be a mixture of path-protecting strategies. However, our next result shows that, if two path-protecting strategies i and j have different paths $a_{ii}^{[1,\infty]} \neq a_{jj}^{[1,\infty]}$, then they cannot both be in the support of a neutrally stable state. The result holds for weakly path-protecting strategies with different repeated path. Consequently, there are no neutrally stable states with more than one (weakly) path-protecting strategy, unless the different strategies are actually generating the same repeated path.

Proposition A.1. *If a neutrally stable state* $\mathbf{x}$ *has some (weakly) path-protecting strategy* i *in its support then all the repeated paths in* $\mathbf{x}$ *are equal to* $h_{ii}^{[\infty]}$.

Proposition A.1 shows that mixtures of path-protecting strategies with different paths do not satisfy definition 4 of neutral stability.

Next we present a series of definitions and a proposition that allow us to extend some of the results for monomorphic states to polymorphic states, and we conclude this section with an example of a polymorphic neutrally stable state.

Definition A.1 (Path-equivalent strategy in a set)**.** *Let S be a finite set of strategies satisfying $T_{ij} = \infty$ for every $i, j \in S$. We say that strategy k is path-equivalent in S to strategy $i \in S$ if, for every $j \in S$,*

$$T_{kj} = \infty \text{ and } a_{kj}^{[1,\infty]} = a_{ij}^{[1,\infty]}.$$

The idea here is that, with each of the strategies in S, strategy k behaves exactly as strategy i does, and there is no difference also between $a_{ii}^{[1,\infty]}$ and $a_{kk}^{[1,\infty]}$.

Definition A.2 (Path-protecting state)**.** *A population state $\mathbf{x}$ with finite support $\mathbb{S}(\mathbf{x})$ is path-protecting if:*

- *$T_{ij} = \infty$ for every $i, j \in \mathbb{S}(\mathbf{x})$, and*
- *If strategy j is not path-equivalent in $\mathbb{S}(\mathbf{x})$ to some strategy $i \in \mathbb{S}(\mathbf{x})$, then $F_j(\mathbf{x}) < E(\mathbf{x}, \mathbf{x})$.*

It follows from the definition that path-protecting states are Nash.

Definition A.3 (Internally neutrally stable state)**.** *A state $\mathbf{x}$ is internally neutrally stable if $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$ for every $i \in \mathbb{S}(\mathbf{x})$ and $E(\mathbf{x}, \mathbf{y}) \geq E(\mathbf{y}, \mathbf{y})$ for every $\mathbf{y}$ with support contained in $\mathbb{S}(\mathbf{x})$.*

This condition only considers strategies in the support of state $\mathbf{x}$, and it is clearly a necessary condition for neutral stability, which considers the whole strategy space.

Proposition A.2. *If a state is path-protecting and internally neutrally stable, then it is neutrally stable.*

A.1 Example of a bimorphic neutrally stable path-protecting equilibrium

Consider a Prisoner's Dilemma game with the payoffs shown in table 2. For the game with endogenous separation, let strategy 1 and strategy 2 be two strategies that generate the paths $a_{ij}^{[1,\infty]}$ shown in table 2, with the corresponding payoffs F_{ij} shown in table 3. Strategy 1 is such that, if an opposing strategy j generates in a j-1 partnership a history that is not coherent with either $a_{11}^{[1,\infty]}$ or $a_{21}^{[1,\infty]}$, strategy 1 breaks up the partnership. In the same way, strategy 2 breaks any j-2 partnership as soon as the history deviates from both $a_{12}^{[1,\infty]}$ and $a_{22}^{[1,\infty]}$.

	C	D
C	3	-1
D	5	0

	1	2
1	$(DD)^{T_1}\,(CC)^\infty$	$(DD)^{T_2}\,DC\,(CC)^\infty$
2	$(DD)^{T_2}\,CD\,(CC)^\infty$	$(DD)^{T_2}(CC)^\infty$

Table 2: Left: Stage game payoffs for a Prisoner's Dilemma, with C for Cooperate and D for Defect. Right: Paths $a_{ij}^{[1,\infty]}$ that strategy 1 and strategy 2 generate together, with i for the row strategy and j for the column. It is assumed that $T_1 > T_2$

	1	2
1	$\delta^{T_1}\,3$	$\delta^{T_2}[5\,(1-\delta)+3\,\delta]$
2	$\delta^{T_2}[(-1)(1-\delta)+3\,\delta]$	$\delta^{T_2}\,3$

Table 3: Payoffs F_{ij} corresponding to the paths shown in table 2.

Let us take $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$, leading to the F_{ij} payoffs shown in table 4.

	1	2
1	1.59	2.10
2	1.71	1.97

Table 4: Payoffs F_{ij} corresponding to the paths shown in table 2, for $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$.

At a population state made up by strategies 1 and 2 in proportions x_1 and x_2, considering that all paths have the same length, we have $F_1(\mathbf{x}) = x_1 F_{11} + x_2 F_{12}$ and $F_2(\mathbf{x}) = x_1 F_{21} + x_2 F_{22}$. These formulas together with the payoffs in Table 4 show that the internal or restricted game for strategies 1 and 2 has the structure of an anti-coordination game (such as a Hawk-Dove game), which presents an internally neutrally stable (in fact, internally evolutionarily stable) equilibrium $\hat{x}$ where $F_1(\hat{\mathbf{x}}) = F_2(\hat{\mathbf{x}})$, at $\hat{x}_1 = \frac{20}{37} \approx 0.54$ and $\hat{x}_2 = \frac{17}{37} \approx 0.46$, with $E(\hat{\mathbf{x}}, \hat{\mathbf{x}}) \approx 1.83$.

Let us check that $\hat{\mathbf{x}}$ is path-protecting.

- Strategies that do not get past history $(DD)^4$ when playing with strategies 1 or 2 (they break up or deviate in action before stage 5) obtain at most the minmax payoff $U_{DD} = 0 < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$.
- Strategies j that after history $(DD)^4$ play D (as strategy 1 does and strategy 2 does not) may go on generating with 1 and 2 the same paths $a_{11}^{[1,\infty]}$ and $a_{12}^{[1,\infty]}$ as strategy 1 does, may break up at stage 5 (after playing), or may deviate from $a_{11}^{[1,\infty]}$ at stage $T_{j1} > 5$ and from $a_{12}^{[1,\infty]}$ at stage $T_{j2} > 5$, obtaining a payoff (see (6), considering that the pool and population strategy distributions at $\hat{\mathbf{x}}$ are the same):

$$F_j(\hat{\mathbf{x}}) = \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j1} + \frac{\hat{\mathbf{x}}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j2}.$$

Let us focus first on F_{j2} for deviations from $a_{12}^{[1,\infty]}$ after stage 5. Applying lemma D.1, we have that if $F_{j2} < F_{12}$ for every possible deviation at $T_{j2} = 6$ (first play of the repeated pattern CC in $h_{12}^{[\infty]}$), then $F_{j2} < F_{12}$ for every finite $T_{j2} > 6$. F_{j2} for a strategy j that breaks up at $T_{j2} = 5$ or deviates at $T_{j2} = 6$ is bounded by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 5, 5)$, which is $\frac{1-\delta}{1-\delta^6}\delta^4\,(5 + 5\,\delta) = 1.33 < F_{12}$, so $F_{j2} < F_{12}$ for $T_{j2} \geq 5$.

Let us focus now on F_{j1} for break-up at stage 5 or deviations from $a_{11}^{[1,\infty]}$ after stage 5. The payoff to these strategies is bounded by 0 for $5 \leq T_{j1} \leq 6$ and by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 0, 0, 3, ..., 3, 5)$, which is $\frac{1-\delta}{1-\delta^{T_{j1}}}\left[\frac{\delta^6\,3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\,5\right]$ for $T_{j1} > 6$. Applying lemma D.1 for deviations at $T_{j1} = 7$ (first play of the repeated pattern (CC) in $h_{11}^{[\infty]}$) shows $F_{j1} < F_1(\mathbf{e}_1) < F_{12}$. We can now state the following bound:

$$F_j(\hat{\mathbf{x}}) \leq \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j1} + \frac{\hat{x}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{12}.$$

Considering that $F_{j1} < F_1(\mathbf{e}_1) < F_{12}$, and that the weight multiplying F_{12} on the previous convex combination of F_{12} and F_{j1} increases with T_{j2}, we find that, for every T_{j1}, the maximum value of the bound corresponds to $T_{j2} = \infty$ (being smaller for finite T_{j2}). Thus, bearing in mind that $F_{j1} \leq 0$ for $5 \leq T_{j1} \leq 6$, we have:

$$F_j(\hat{x}) \leq \frac{\hat{x}_2}{\hat{x}_1(1-\delta^5) + \hat{x}_2} F_{12} \approx 1.42 < E(\hat{x}, \hat{x}),\ \textit{for}\ 5 \leq T_{j1} \leq 6.$$

And, for $T_{j1} > 6$,

$$F_j(\hat{\mathbf{x}}) \leq \frac{\hat{x}_1(1-\delta)}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2}\left[\frac{\delta^6\,3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\,5\right] + \frac{\hat{x}_2\,F_{12}}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2}.$$

Note that the only variable in the previous bound is T_{j1}, with all the other terms being known numbers. By taking the derivative of this bound with respect to T_{j1} it can be checked that it is monotonic increasing with T_{j1} (for $T_{j1} > 6$), and its limit is $E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$. Consequently, any strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays D there (as strategy 1 and its path-equivalent-in-$\{1, 2\}$ strategies do) obtains a payoff $F_j(\hat{\mathbf{x}}) < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ if j is not path-equivalent to strategy 1 in the set of strategies $\{1, 2\}$.

- We now consider strategies j that after history $(DD)^4$ play C (as strategy 2 does and strategy 1 does not). Applying the same procedure that we followed before, it can be shown that any such strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays C there (as strategy 2 and its path-equivalent-in-$\{1,2\}$ strategies do), obtains a payoff $F_j(\hat{\mathbf{x}}) < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ if j is not path-equivalent to strategy 2 in the set of strategies $\{1,2\}$.

B Other approaches to neutral stability in games with endogenous separation

Here we summarize previous definitions of neutral stability for games with endogenous separation.

Definition B.1. *Carmichael and MacLeod (1997). A Nash equilibrium population state* $\mathbf{x}$ *is a neutrally stable state* NSS_{CM} *if for every* $\mathbf{y} \in \mathbb{D}$ *there exists an* $\bar{\epsilon}_{\mathbf{y}} \in (0,1)$ *such that for every* $\epsilon \in (0, \bar{\epsilon}_{\mathbf{y}})$,

$$F_i(\,(1-\epsilon)\mathbf{x} + \epsilon\mathbf{y}) \geq F_j(\,(1-\epsilon)\mathbf{x} + \epsilon\mathbf{y})$$

for all $i \in supp(\mathbf{x})$ *and* $j \in supp(\mathbf{y})$.

Definition B.2. *Fujiwara-Greve and Okuno-Fujiwara (2009). A distribution in the matching pool* $\mathbf{p}$ *is a neutrally stable pool distribution* NSS_{FO} *if for every* $j \in \Omega$ *there exists an* $\bar{\epsilon}_j \in (0,1)$ *such that for every* $\epsilon \in (0, \bar{\epsilon}_j)$ *and every* $i \in supp(\mathbf{p})$,

$$\hat{F}_i(\,(1-\epsilon)\,\mathbf{p} + \epsilon\,\mathbf{e}_j) \geq \hat{F}_j(\,(1-\epsilon)\;\mathbf{p} + \epsilon\,\mathbf{e}_j)$$

These definitions are related to a standard condition for (neutral) stability from Taylor and Jonker (1978), which can be adapted as follows (Bomze and Weibull, 1995):

Definition B.3. *Considering a finite set of strategies* S, *a state* $\mathbf{x} \in \Delta(S)$ *is neutrally stable* NSS_{TJ} *in* $\Delta(S)$ *if for every* $\mathbf{y} \in \Delta(S)$ *there is some* $\bar{\epsilon}_y \in (0,1)$ *such that*

$$F(\mathbf{x}, \epsilon\,\mathbf{y} + (1-\epsilon)\,\mathbf{x}) \geq F(\mathbf{y}, \epsilon\,\mathbf{y} + (1-\epsilon)\,\mathbf{x})$$

for all $\epsilon \in (0, \bar{\epsilon}_y)$.

On the one hand, when considering a finite set of strategies, it is easy to see that the conditions for NSS_{CM} and NSS_{FO} are not equivalent to the standard condition NSS_{TJ}. Izquierdo et al. (2021, Appendix C) present simple examples of states that are neutrally stable (NSS_{TJ}) but are not NSS_{CM} or NSS_{FO}. They also show examples of states that are NSS_{FO} but are not neutrally stable NSS_{TJ} and are instable in the replicator dynamics (such as the example we presented in section 3.4). In short:

- Definitions NSS_{CM} and NSS_{FO} are not consistent with the standard definition of neutral stability.
- NSS_{FO} does not guarantee Lyapunov stability under the replicator dynamics.

Considering behavioral strategies, Vesely and Yang (2010) provide a definition of neutral stability for games with endogenous separation that is close to NSS_{TJ}. However, if the payoff functions $F_i(\mathbf{x})$ are not linear (and this is generically the case in games with endogenous separation), then the different "standard" definitions of neutral stability, which are equivalent in the linear setting, are not equivalent any more (Bomze and Weibull (1995)), and being NSS_{TJ} does not guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$. In contrast, the condition that we use to define neutral stability does guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$.

Izquierdo et al. (2021) provide a definition of neutral stability that looks rather involved because it uses the population and pool distributions, related by the function f as defined by (1), as well as a function $\hat{E}(\mathbf{z}, \mathbf{p})$ that provides the payoff to a group of players with strategy distribution $\mathbf{z}$ entering a population with pool distribution $\mathbf{p}$.

Definition B.4. *A population-pool state* $\{\mathbf{x}, \mathbf{p}\}$ *with* $\mathbf{x} = f(\mathbf{p})$ *is neutrally stable* NNS_{IIV} *if* $\mathbf{x}$ *is a Nash equilibrium and for any finite set of strategies* $S \in \mathbb{D}$ *such that* $\mathbb{S}(\mathbf{p}) \subseteq S$ *there is a neighborhood* O_S *of* $\mathbf{p}$ *in* $\Delta(S)$ *such that* $\hat{E}(\mathbf{x}, \mathbf{y}) \geq \hat{E}(f(\mathbf{y}), \mathbf{y})$ *for every* $\mathbf{y} \in O_S$ *satisfying* $\hat{E}(f(\mathbf{y}), \mathbf{p}) = \hat{E}(\mathbf{x}, \mathbf{p})$.

It can be shown that our condition for neutral stability (definition 4), which is a global condition, involves satisfaction of the condition NNS_{IIV} (which is actually a set of local conditions).

When comparing our results with those in Carmichael and MacLeod (1997), Fujiwara-Greve and Okuno-Fujiwara (2009) or Izquierdo et al. (2021), the reader should keep in mind the different definitions of neutral stability used in each of those papers. In particular, many of the polymorphic equilibria discussed by Fujiwara-Greve and Okuno-Fujiwara (2009) do not satisfy definition 4 of neutral stability, and can be destabilized by other strategies in the replicator dynamics (see also Vesely and Yang (2012)).

C Strategies robust against indirect invasions

Here we consider the equilibrium condition of robustness against indirect invasions or RAII (van Veelen, 2012) for games with endogenous separation. It can be argued that any reasonable extension of this concept to games with endogenous separation would require at least neutral stability and that every weakly path-equivalent strategy is also neutrally stable, where j is said to be weakly path-equivalent to i if $h_{jj}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$ (the second equality is implied by the first), which implies that any mixture $\mathbf{y}$ of strategies i and j satisfy $E(\mathbf{y}, \mathbf{e}_i) = F_{ii} = F_i(\mathbf{y}) = E(\mathbf{y}, \mathbf{y})$. With these minimum requirements, our results below show that, in many cases of interest, there are no RAII strategies in games with endogenous separation. We first show that being RAII requires playing Nash action profiles of the stage game and, in most cases of interest, it requires $T_{ii} = \infty$ and a sufficiently low value of δ. For (sufficiently) large values of δ, and unless the maximum payoff of the stage game is attained at a symmetric Nash action profile, no strategy is robust against indirect invasions. The reason is that every strategy i has a path-equivalent strategy j_1 that would let a potential invader j_2 who deviates in action (from

i or j_1) at the first stage of an j_2-j_1 partnership obtain the maximum stage game payoff afterwards, in an infinite path $a_{j_2 j_1}^{[1,\infty]}$. The payoff $F_{j_2 j_1}$ to such a strategy j_2 converges to the maximum stage game payoff as $\delta \to 1$.

Proposition C.1. *A strategy $i \in \Omega$ can be robust against indirect invasions only if the action profiles played in the i-i equilibrium path are Nash profiles of the stage game.*

It follows from lemma 3.5 that, unless the maximum symmetric payoff of the stage game corresponds to a Nash profile, $T_{ii} = \infty$ is also a necessary condition for a strategy to be RAII, as it is a necessary condition for neutral stability.

Proof of proposition C.1. Suppose that the action profile $(a_i^{[t]}, a_i^{[t]}) = (a, a)$ is not Nash. Consider two strategies j and k such that:

- Strategy j is path-equivalent to i, so $F_{jj} = F_{ii}$.
- Strategy k behaves with j (or with i) like j up to stage t (i.e., $a_{kj}^{[1,t-1]} = a_{jj}^{[1,t-1]} = a_{ii}^{[1,t-1]}$ if $t > 1$) and deviates at t by playing a best response action to action a, obtaining at that stage a greater payoff than what j obtains in a j-j partnership.
- From stage t, strategies j and k do not break up and play the action profile that provides k the maximum possible payoff of the stage game.

Then $F_{kj} > F_{jj}$, so strategy j is not Nash. □

Proposition C.2. *For stage games with a single Nash action profile which does not obtain the maximum symmetric payoff, such as the Prisoner's Dilemma, no strategy in the game with endogenous separation is RAII.*

Proof of proposition C.2. The only possible candidates to be RAII are strategies with $T_{ii} = \infty$ that always play the Nash action profile at the equilibrium. But any such strategy i has a weakly path-equivalent strategy j with finite T_{jj} that always plays the Nash action profile in $a_{jj}^{[1,T_{jj}]}$, and which, by lemma 3.5, is not neutrally stable. □

Example C.1. *For the Prisoner's Dilemma, the only Nash action profile is (D, D) and it does not obtain the maximum symmetric payoff U_{CC}, so there are no RAII strategies in the game with endogenous separation. For the Hawk-Dove game, no symmetric action profile is Nash, so there are no RAII strategies in the game with endogenous separation.*

D Proofs

Proof of lemma 3.1. Let i be a Nash strategy and let $a^\emptyset = i(\emptyset)$ be the first action played by i. Let j be a strategy that plays a best-response action to $a^\emptyset$ when starting a new partnership, i.e., $j(\emptyset) \in BR(a^\emptyset)$, and then breaks the partnership. We have $F_{ji} = \max_l U(a_l, a^\emptyset)$. Considering that M is an upper bound for F_{ii}, the Nash condition $F_{ii} \geq F_{ji}$ requires $M \geq \max_l U(a_l, a^\emptyset)$ or, equivalently, $U^{BR}(a^\emptyset) \leq M$. □

Proof of lemma 3.2. State $\mathbf{x}$ has an associated pool state $\mathbf{p} = f^{-1}(\mathbf{x})$. Any strategy arriving at the pool of singles $\mathbf{p}$ to be matched faces a distribution of initial actions $q \in \Delta(A)$ (given by $q_k = \sum_{i \in \mathbb{S}(\mathbf{x}): i(\emptyset)=a_k} p_i$). Given a state $\mathbf{x}$ and its associated q, consider a strategy j that at the beginning of a partnership plays a best response action to the distribution of actions q and then breaks the partnership. The payoff $F_j(\mathbf{x})$ to such a strategy is at least $\underline{m}$. Consequently, if $\mathbf{x}$ is Nash, then $E(\mathbf{x}, \mathbf{x})$ has to be greater than or equal to $\underline{m}$. For monomorphic states, we have that F_{ji} is at least m, while M is an upper bound for F_{ii}. □

Proof of lemma 3.3. Suppose that i is a strategy with finite self-break-up period T_{ii} and the last action profile $(a_i^{[T_{ii}]}, a_i^{[T_{ii}]})$ in an i-i partnership is not a Nash profile of the stage game G. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage $T_{ii} - 1$, i.e., $j(a_{ii}^{[0,t]}) = i(a_{ii}^{[0,t]})$ for $0 \leq t < T_{ii} - 1$,
- at stage T_{ii} of an i-j partnership plays a best-response action against the action $a_i^{[T_{ii}]}$ played by i at that stage, and
- leaves i (i.e., breaks the partnership with i) after stage $T_{ij} = T_{ii}$.

Strategy j obtains the same stage payoff against i as i itself for the first $T_{ii} - 1$ stages of a partnership and a higher payoff at the last stage T_{ii}. Consequently, considering (8), $F_{ji} > F_{ii}$, so i cannot be a Nash strategy. □

Proof of observation 1. Suppose that i is a Nash strategy with finite self-break-up period T_{ii} that plays the action profile (C, C) at some stage (between stages $t = 1$ and $t = T_{ii}$) of an i-i partnership. Then we have $F_{ii} > U(D, D)$. Let t_l be the last stage at which (C, C) is played. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage $t_l - 1$, i.e., $j(a_{ii}^{[0,t]}) = i(a_{ii}^{[0,t]})$ for $0 \leq t < t_l - 1$,
- at stage t_l of an i-j partnership plays action D, obtaining a stage payoff $U(D, C) > U(C, C)$, and
- breaks the partnership with i after stage t_l.

  Using formula (8), it can be seen that $F_{ji} > F_{ii}$, so i is not a Nash strategy (contradiction). The reason, comparing the sequence of payoffs to i in the infinite series $h_{ii}^{[\infty]}$ and to j in the infinite series $h_{ji}^{[\infty]}$ is that j obtains a higher payoff at stage t_l and (if $t_l < T_{ii}$) shortens the sequence of lowest payoffs $U(D, D)$ until the next high payoffs $U(C, C)$ or $U(D, C)$.

□

Proof of lemma 3.4. With the conditions on i, the infinite series of actions that a j-player faces in a population of i-players (see 7) is $(a^N, a^N, ...)$. The best stage-payoff against

a^N is obtained by a^N, and, considering that $F_{ji} = V(h_{ji}^{\infty}) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} U(h_{ji}^{[t]})$, the best payoff against any strategy i satisfying the condition is obtained by strategies j that generate the path $h_{ji}^{[\infty]} = ((a^N, a^N), (a^N, a^N), ...)$, which obtain the payoff $F_{ji} = U(a^N, a^N)$. If i_1 and i_2 satisfy the conditions for i, we have $F_{i_1}(\mathbf{e}_{i_1}) = F_{i_1}(\mathbf{e}_{i_2}) = F_{i_2}(\mathbf{e}_{i_1}) = F_{i_2}(\mathbf{e}_{i_2}) = U(a^N, a^N)$. As $F_i(\mathbf{x})$ is a strictly convex combination of the payoffs F_{ij} for $j \in \mathbb{S}(\mathbf{x})$, we have proved the result: if $\mathbf{x}$ is a mixture of strategies satisfying the condition for i, we have $F(\mathbf{x}, \mathbf{x}) = U(a^N, a^N) \geq F_j(\mathbf{x})$ for every $j \in \Omega$.

□

Proof of proposition 1. Given any finite set of strategies S, we can number the strategies and identify distributions $\mathbf{x}$ whose support is in S with real vectors $\hat{\mathbf{x}} \in \Delta(S) \equiv \{\hat{\mathbf{x}} \in \mathbb{R}_+^{|S|} : \sum_{k=1}^{|S|} \hat{x}_k = 1\}$. The restriction of F_i to distributions with support in S can then be identified with a function $F_{i|S} : \Delta(S) \to \mathbb{R}$. By lemma D.2, $F_{i|S}$ is Lipschitz continuous in $\Delta(S)$.[14]

Given any finite set of strategies $S \subset \Omega$ and a neutrally stable state $\mathbf{x}$ with support in S, it follows from definition 4 and from the Lipschitz property of the payoff functions $F_{i|S}$ in $\Delta(S)$ that the point $\hat{\mathbf{x}} \in \Delta(S)$ associated to state $\mathbf{x}$ satisfies the conditions in Thomas (1985) [Theorem 1] to be a weakly evolutionarily stable state in $\Delta(S)$ and, consequently, $\hat{\mathbf{x}}$ is Lyapunov stable in the replicator dynamics restricted to S. □

Proof of lemma 3.5. Let $(a^M, a^M) \in N_M^G$ be one of the symmetric action profiles (there may be more than one) that attain the maximum symmetric payoff $M = \max_{a \in A} U(a, a)$.

Suppose that T_{ii} is finite and $F_{ii} < M$. This implies that $h_{ii}^{[\infty]}$ is a repetition of a pattern of length T_{ii}, and, for any fixed t_0, there is always $t > t_0$ with $U(h_{ii}^{[t]}) < M$. Consider a strategy j that when playing with i behaves like i up to period T_{ii}, but at that period does not break the partnership and turns to playing action a^M forever, without breaking the partnership. That would make play between strategy i and strategy j unfold in the same way as it does between two players that play strategy i, with $h_{ji}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$, and hence $F_{ji} = F_{ii} = F_{ij}$. For $t \leq T_{ii}$, two players that play strategy j obtain a payoff $U(h_{jj}^{[t]}) = U(h_{ii}^{[t]}) = U(h_{ij}^{[t]})$. For $t > T_{ii}$, we have $U(h_{jj}^{[t]}) = M$, while $U(h_{ij}^{[t]}) = U(h_{ii}^{[t]}) \leq M$ and, for some $t > T_{ii}$, $U(h_{ij}^{[t]}) < M$. Consequently, considering (8), $F_{jj} > F_{ij}$, so i is not neutrally stable. Up to now we have proved that if i is neutrally stable with finite T_{ii} then $F_{ii} = M$, which implies $U(h_{ii}^{[t]}) = M$ for every t. Suppose that payoff M is obtained at time t_1 by some action profile $h_{ii}^{t_1}$ which is not a Nash equilibrium of the stage game. Then a strategy j that when playing with i chooses the same action as i up to period t_1 (obtaining M at every period up to t_1 if $t_1 \geq 1$), but at period t_1 plays a best response the action taken in $h_{ii}^{t_1}$ and breaks the partnership, obtains a payoff $F_{ji} > M = F_{ii}$, which cannot happen if i is neutrally stable. □

Proof of proposition 2. Let strategy i be weakly path-protecting and let j_1 and j_2 be two alternative best responses to $\mathbf{e}_i$, i.e., $\{j_1, j_2\} \in BR(\mathbf{e}_i)$. Considering that the action

[14] A function $f : \Delta(S) \to \mathbb{R}$ is Lipschitz continuous in $\Delta(S)$ if there exists a positive real constant K such that, for all $\mathbf{x}$ and $\mathbf{y}$ in $\Delta(S)$, $|f(\mathbf{x}) - f(\mathbf{y})| \leq K||\mathbf{x} - \mathbf{y}||$.

profiles in $h_{ii}^{[\infty]}$ are symmetric, we have $h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$. In a j_1-j_2 partnership, no strategy can take an action different from the one they take when playing with i until the split-up period $T_{j_1 j_2} = \min(T_{ij_1}, T_{ij_2})$, because the generated histories up to that point are the same as in an i-i partnership and, until they break the partnership, both j_1 and j_2 take the same action as i does given the history. Consequently, $h_{j_1 j_2}^{[\infty]}$ coincides either with $h_{j_1 i}^{[\infty]} = h_{ii}^{[\infty]}$ or with $h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]}$. Then we have $h_{j_1 j_2}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$, which implies that, for $\{j_1, j_2\} \in BR(\mathbf{e}_i)$, $F_{j_1 j_2} = F_{j_1 j_1} = F_{ii} = F_{ij_1} = F_{ij_2}$. Now, if $\mathbf{y}$ is a mixture of best responses to $\mathbf{e}_i$ and j is a best response to $\mathbf{e}_i$, we have $F_j(\mathbf{y}) = F_{ii} = F_i(\mathbf{y})$ and, consequently, $E(\mathbf{y}, \mathbf{y}) = F_i(\mathbf{y})$, proving that i is neutrally stable.

□

Proof of lemma 4.1. Let i be a strategy satisfying the conditions of the proposition. It is clear that $T_{ii} = \infty$, $h_{ii}^{[\infty]} = ((\hat{a}, \hat{a}), (\hat{a}, \hat{a}), ...)$ and $F_{ii} = U(\hat{a}, \hat{a})$. Any strategy j playing with i-players generates a repeated path $h_{ji}^{[\infty]}$ in which the action taken by i is always $\hat{a}$, so, given that $(\hat{a}, \hat{a})$ is a (strict) Nash profile and that any deviation from the profile $(\hat{a}, \hat{a})$ is caused by strategy j (i always plays $\hat{a}$, so the second action in the profile is always $\hat{a}$), we have $U(h_{ji}^{[t]}) \leq U(\hat{a}, \hat{a})$ for every t. In fact, since $(\hat{a}, \hat{a})$ is strict Nash, we have $h_{ji}^{[t]} \neq (\hat{a}, \hat{a}) \implies U(h_{ji}^{[t]}) < U(\hat{a}, \hat{a})$, and, considering that F_{ji} is a strictly convex combination of the payoffs $U(h_{ji}^{[t]})$, it follows that $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < U(\hat{a}, \hat{a}) = F_{ii}$, proving that i is weakly path-protecting. □

Proof of proposition 3. Consider a strategy i such that $T_{ii} = \infty$ and

$$h_{ii}^{[\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^\infty),$$

where:

- Φ_m is a T_m-long repetition of a minmax action profile $(\tilde{a}, \tilde{a})$.
- Φ_f is a T_f-long sequence of symmetric action profiles.
- Φ_p is a T_p-long sequence of symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

As soon as another strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership.

Take Φ_f and Φ_p as fixed, and the length T_m of Φ_m as a parameter. We will show that, for large enough T_m and, then, for large enough δ,

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii},$$

i.e., strategy i is weakly path-protecting. By choosing Φ_p in a way such that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, strategy i is also path-protecting.

We will need some intermediate results. First, lemma D.1 implies that, in order to prove the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}$, it is enough to prove that this statement holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful. Consequently, it is enough to consider a finite number of possible deviating paths: those that deviate at some t not greater than $T_m + T_f + T_p$.

Second, the payoff to a strategy that deviates at $t \leq T_m$ is bounded above by the minmax payoff m (because i plays a minmax action up to stage T_m, so the stage payoff for a strategy j at every stage up to and including the deviating stage $t \leq T_m$ is bounded above by m). Let L be the maximum payoff in the stage game. Considering a repeated sequence $(m, ..., m, L, ..., L)$ of T_m payoffs m and $T_f + T_p$ payoffs L, we have that the payoff to a strategy that deviates not later than $T_m + T_f + T_p$ is bounded above[15] by

$$V_L \equiv \frac{m(1-\delta^{T_m}) + \delta^{T_m}(1-\delta^{T_f+T_p})L}{1-\delta^{T_m+T_f+T_p}},$$

and V_L is non-decreasing with δ (increasing if $L > m$).

Third, if an infinite sequence of action profiles Φ ends up repeating some finite pattern Φ_1, i.e., if $\Phi = (\Phi_0, (\Phi_1)^\infty)$ for some finite sequences Φ_0 and Φ_1, then[16]

$$\lim_{\delta \to 1} V(\Phi) = \bar{U}_{\Phi_1}.$$

This implies

$$V_L \leq \lim_{\delta \to 1} V_L = \alpha \equiv \frac{m\,T_m + L\,(T_f + T_p)}{T_m + T_f + T_p}, \tag{12}$$

with $\lim_{T_m \to \infty} \alpha = m$, and

$$\lim_{\delta \to 1} F_{ii} = \bar{U}_{\Phi_p} > m. \tag{13}$$

Choose some positive $\epsilon < \frac{\bar{U}_{\Phi_p} - m}{2}$. From (12), and considering that α approaches m as T_m grows, we can find a value for T_m such that $\alpha < m + \epsilon$, and then, fixing such T_m, we have $V_L < m + \epsilon$.

From (13), there is some $\delta_1 < 1$ such that, for $\delta > \delta_1$, $F_{ii} > \bar{U}_{\Phi_p} - \epsilon > m + \epsilon$. Consequently, for $\delta > \delta_1$,

$$V_L < F_{ii},$$

proving that strategy i is path-protecting.

□

[15] It is easy to check that, for a fixed number of m payoffs T_m, V_L is non-decreasing with the number of L payoffs (V_L is a weighted average of m and $L \geq m$, with the weight of m decreasing if the number of L payoffs increases), so, by taking a number of L values equal to $T_f + T_p$, we can be sure that V_L is an upper bound for the payoff to any strategy that deviates up to $t = T_m + T_f + T_p$.

[16] This can be shown using L'Hopital rule.

In preparation of the following result, for any finite series of action profiles Φ, let $(\Phi)^k$ represent the sequence made up by repeating k times the action profiles in Φ. Remember that $(\Phi)^\infty$ represents the infinite repetition.

Lemma D.1. *Consider two (not necessarily different) strategies j and i with $h_{ji}^\infty = (\Phi_0, (\Phi_p)^\infty)$, where Φ_0 and Φ_p are finite sequences of action profiles (and where Φ_0 may be empty). Let Φ_1 be another finite sequence of action profiles. If j_1 and j_2 are strategies such that*

$$h_{j_1 i}^\infty = (\Phi_0, \Phi_1)^\infty \text{ and}$$

$$h_{j_2 i}^\infty = (\Phi_0, (\Phi_p)^k, \Phi_1)^\infty \text{ for some } k \in \mathbb{N}$$

then

$$F_{j_1 i} < F_{ji} \iff F_{j_2 i} < F_{ji}.$$

Proof of lemma D.1. For any sequence Φ of length $T \geq 1$, let

$$V(\Phi) = \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} U(\Phi^{[t]}).$$

Let the respective lengths of Φ_0, Φ_p and Φ_1 be $T_0 \geq 0, T_p \geq 1$ and $T_1 \geq 1$. If $T_0 = 0$ let $V(\Phi_0) = 0$. Then

$$F_{ji} = (1-\delta^{T_0})V(\Phi_0) + \delta^{T_0} V(\Phi_p),$$

$$F_{j_1 i} = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+T_1}}, \text{ and}$$

$$F_{j_2 i} = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{k\,T_p})V(\Phi_p) + \delta^{T_0+k\,T_p}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+k\,T_p+T_1}}.$$

Any of the two conditions $F_{j_1 i} < F_{ji}$ or $F_{j_2 i} < F_{ji}$ can then be seen to be equivalent (rearranging and simplifying terms) to the condition

$$\delta^{T_1}(1-\delta^{T_0})V(\Phi_0) + (1-\delta^{T_1})V(\Phi_1) < (1-\delta^{T_0+T_1})V(\Phi_p).$$

□

Proof of proposition A.1. Suppose that a Nash equilibrium state $\mathbf{x}$ includes a weakly path protecting strategy i and some strategy j with $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$, then:

- $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$, because $\mathbf{x}$ is Nash and i is in its support, so $i \in BR(\mathbf{x})$, and
- $E(\mathbf{x}, \mathbf{e}_i) < F_{ii}$, because i is weakly path-protecting (so it is Nash) and $\mathbf{x}$ includes a strategy j that deviates from $h_{ii}^{[\infty]}$ when playing with i, obtaining a payoff $F_{ji} < F_{ii}$.

Consequently, $\mathbf{x}$ is not neutrally stable. □

Proof of proposition A.2. Let $Eq(\mathbf{x})$ be the set of strategies that are path-equivalent in $\mathbb{S}(\mathbf{x})$ to some strategy in $\mathbb{S}(\mathbf{x})$, and let $\bar{Eq}(\mathbf{x})$ be the complement of this set. As $\mathbf{x}$ is Nash and path-protecting, we have $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$ for $i \in Eq(\mathbf{x})$ and $F_i(\mathbf{x}) < E(\mathbf{x}, \mathbf{x})$ for $i \in \bar{Eq}(\mathbf{x})$. Consequently, any state $\mathbf{y}$ that includes strategies both in $Eq(\mathbf{x})$ (for which $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$) and in $\bar{Eq}(\mathbf{x})$ satisfies $E(\mathbf{y}, \mathbf{x}) < E(\mathbf{x}, \mathbf{x})$, and only mixtures of strategies in $Eq(\mathbf{x})$ can be (are) alternative best responses to $\mathbf{x}$. Because any strategy that is path-equivalent in $\mathbb{S}(\mathbf{x})$ to strategy $i \in \mathbb{S}(\mathbf{x})$ behaves like i does with strategies in $Eq(\mathbf{x})$, for any mixture $\mathbf{y}$ of strategies in $Eq(\mathbf{x})$ there is an "internal" state $\hat{\mathbf{y}}$ satisfying $\mathbb{S}(\hat{\mathbf{y}}) = \mathbb{S}(\mathbf{x})$ such that $E(\hat{\mathbf{y}}, \mathbf{x}) = E(\mathbf{x}, \mathbf{x})$, $E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}, \hat{\mathbf{y}})$ and $E(\mathbf{y}, \mathbf{y}) = E(\hat{\mathbf{y}}, \hat{\mathbf{y}})$. Consequently, internal neutral stability (which guarantees $E(\mathbf{x}, \hat{\mathbf{y}}) \geq E(\hat{\mathbf{y}}, \hat{\mathbf{y}})$) guarantees neutral stability ($E(\mathbf{x}, \mathbf{y}) \geq E(\mathbf{y}, \mathbf{y})$). □

Lemma D.2. *Given a finite set of strategies S, the function defined by equation (1) that relates the pool and population distributions with support in S, $f : \Delta(S) \to \Delta(S)$ such that $\mathbf{x} = f(\mathbf{p})$, is a bi-Lipschitz homeomorphism.*

A detailed proof can be found in Izquierdo et al. (2021). For completeness, we discuss here the main steps of the proof. The component functions of f are defined by

$$f_i(\mathbf{p}) = \frac{p_i \sum_{j \in \mathbb{S}(\mathbf{p})} p_j (1 - \delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(\mathbf{p})} p_k \, p_j (1 - \delta^{T_{kj}})}.$$

Let $g : \mathbb{R}^s \to \mathbb{R}^s$ be defined by $g_i(\mathbf{p}) = p_i \sum_{j \in \mathbb{S}(\mathbf{x})} p_j (1 - \delta^{T_{ij}})$, for $\mathbf{p} \in \mathbb{R}^s$. Note that $f(\mathbf{p}) = \frac{g(\mathbf{p})}{\|g(\mathbf{p})\|_1}$. The Jacobian of g is non-singular in $\mathbb{R}^s_{\geq 0} \setminus \{\mathbf{0}\}$ (in $\mathbb{R}^s_{>0}$ it is column strictly diagonally dominant). Using a theorem by Gordon (1972), it can be shown that the restriction of g to $\mathbb{R}^s_{\geq 0}$, $g|_{\mathbb{R}^s_{\geq 0}} : \mathbb{R}^s_{\geq 0} \to \mathbb{R}^s_{\geq 0}$ is globally invertible, with a continuous inverse $g_+^{-1} : \mathbb{R}^s_{\geq 0} \to \mathbb{R}^s_{\geq 0}$ which is Lipschitz continuous away from $\mathbf{0}$.

$g|_{\mathbb{R}^s_{\geq 0}}$ maps injectively rays from $\mathbf{0}$ through $\mathbf{p} \in \Delta(S)$ to rays from $\mathbf{0}$ through $g(\mathbf{p})$ (see figure 2). The extension of f to $\mathbb{R}^s_{\geq 0} \setminus \{\mathbf{0}\}$ maps (injectively) an open ray from $\mathbf{0}$ through $\mathbf{p}$ to the point in $\Delta(S)$ at the intersection with the ray from $\mathbf{0}$ through $g(\mathbf{p})$. The preimage of $\mathbf{x} = f(\mathbf{p})$ is then the unique intersection of $\Delta(S)$ with the ray from $\mathbf{0}$ through $g_+^{-1}(\mathbf{x})$, so

$$f^{-1}(\mathbf{x}) = \frac{g_+^{-1}(\mathbf{x})}{\|g_+^{-1}(\mathbf{x})\|_1}.$$

Additionally, it can be shown that f^{-1} does not admit a general algebraic expression. For a given population state $\underline{\mathbf{x}}$ with support $\mathbb{S}(\underline{\mathbf{x}})$ and a given value for δ, the system of equations

$$\underline{x}_i = \frac{p_i \sum_{j \in \mathbb{S}(\underline{\mathbf{x}})} p_j (1 - \delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(\mathbf{x})} p_k \, p_j (1 - \delta^{T_{kj}})} \tag{14}$$

with $0 \leq p_i \leq 1$ and $\sum_{i \in \mathbb{S}(\mathbf{x})} p_i = 1$, is a polynomial system in the components of $\mathbf{p}$. The solution of this system can be found using Gröbner basis (Cox et al., 2015). For

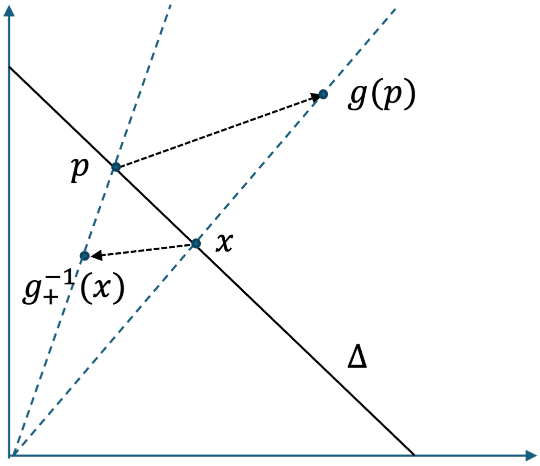

Figure 2: g maps the ray through $\mathbf{p}$ to the ray through $g(\mathbf{p})$. $\mathbf{x} = f(\mathbf{p})$ is found at the intersection of this last ray with the simplex Δ. Starting from $\mathbf{x}$, with $g_+^{-1}(\mathbf{x})$ we recover the ray through $\mathbf{p}$, which intersects Δ at $\mathbf{p}$.

four strategies (in the support of $\underline{\mathbf{x}}$), it is easy to find examples of (14) with rational coefficients (rational $\underline{x}_i$ and δ) which do not admit a solution in radicals, proving that, for more than three strategies, there is no general solution in radicals to (14), i.e., there is no general formula that allows to calculate $\mathbf{p}$ from $\mathbf{x}$, δ and (T_{ij}) using addition, subtraction, multiplication, division, and root extraction.