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TESIS DOCTORAL:

Graded Betti numbers of edge ideals

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Resumen y conclusiones

Esta tesis tiene por objeto el estudio de los números de Betti graduados correspondientes a los llamados ideales de grafos. Los ideales de grafos son ideales generados por monomios de grado 2 libres de cuadrados. Al ser ideales monomiales, son homogéneos y finitamente generados y, por tanto, admiten una resolución libre graduada minimal, que es finita y donde todos los módulos libres tiene rango finito,

$$\mathcal{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} \cdots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} I \longrightarrow 0 \,.$$

Los números de Betti graduados son invariantes homológicos del ideal que recogen la información numérica de su resolución libre graduada minimal. Más concretamente, el número de Betti graduado $\beta_{i,j}$ es el número de generadores de grado j en una base del i-ésimo módulo libre en la resolución libre graduada minimal.

Los ideales de grafos tienen una fuerte estructura combinatoria y pueden representarse de forma única mediante grafos simples, cuyos vértices están etiquetados por las variables del anillo de polinomios en el que consideramos el ideal y cuyas aristas corresponden a los generadores minimales del mismo. Estos ideales también pueden verse como ideales de Stanley-Reisner, cuyo complejo simplicial asociado es el complejo de independencia del ideal. Esto nos permite recurrir a la fórmula de Hochster para expresar los números graduados de Betti de un ideal de grafo en función de la homología reducida de su complejo de independencia.

Capítulo 1: Bagaje combinatorio y algebraico

El primer apartado de este trabajo consiste en una recopilación de la terminología relativa a la teoría de grafos y de complejos simpliciales, así como los resultados básicos sobre homología simplicial, destacando de entre estos últimos la sucesión de Mayer-Vietoris y el teorema del Nervio por su repetida utilización a lo largo de este trabajo. Se han incluido también diferentes tipos de colapsos en complejos simpliciales que preservan los grupos de homología y que tienen un potencial uso en la teoría de Stanley-Reisner a través de la teoría discreta de Morse o la propiedad de Leray.

La parte algebraica se centra en la construcción de la resolución libre graduada minimal para el caso más general de módulos graduados finitamente generados sobre un anillo de polinomios R. Sobre R consideraremos dos posibles graduaciones: la graduación estándar y la multigraduación. Revisamos la existencia y unicidad de la resolución libre graduada minimal y definimos a partir de ella los invariantes que vamos a estudiar: los números de Betti graduados, la regularidad de Castelnuovo-Mumford y la dimensión proyectiva. A continuación nos centramos en el caso de ideales monomiales. Además, asumiremos sin pérdida de generalidad, que estos ideales serán libres de cuadrados, ya que éstos se obtienen por polarización a partir de ideales monomiales y que los números de Betti graduados no son modificados por esta transformación. En este contexto, introducimos la correspondencia de Stanley-Reisner entre ideales monomiales libres de cuadrados en un anillo de polinomios con n varibles y los complejos simpliciales con n vértices. La herramienta principal que nos proporciona esta correspondencia es la llamada fórmula de Hochster, que nos permite expresar los números de Betti graduados de un ideal monomial libre de cuadrados en función de las homologías reducidas del complejo simplicial asociado.

Finalizamos este capítulo con la definición de los objetos de estudio en esta tesis, los ideales de grafos. Estos ideales están generados por monomios de grado 2 libres de cuadrados, por lo que pertenecen al ámbito de aplicación de la correspondencia de Stanley-Reisner. El complejo simplicial asociado a un ideal de grafo es el complejo de independencia de un grafo simple. En esta sección vemos la relación entre las propiedades de un grafo simple y las de su complejo de independencia y su homología reducida.

Capítulo 2: Diagramas de Betti de ideales de grafos

Este capítulo recoge resultados sobre los valores concretos de los números de Betti graduados de ciertos ideales de grafos y también sobre el conjunto de los números de Betti que son no nulos. Por ejemplo, en el teorema 2.3.3 damos una fórmula combinatoria para todos los números de Betti graduados del ideal de grafo asociado al complementario de un ciclo, mientras que en el teorema 2.1.2 demostramos que el conjunto de números de Betti graduados no nulos, en su presentación a través del llamado diagrama de Betti, tiene forma escalonada en su margen izquierda en el caso de ideales de grafos.

La forma más sencilla que puede adoptar el diagrama de Betti de un ideal de grafo consiste en una única fila. Esta propiedad equivale a que el ideal de grafo tenga regularidad 2 y es satisfecha por los ideales asociados a grafos cuyo grafo complementario es cordal. Este resultado, formulado por R Fröberg, fue mejorado por D. Eisenbud, M. Green, K. Hulek y S. Popescu en [22] precisando en qué paso de la resolución libre graduada minimal deja de ser lineal si el grafo complementario no es cordal. En este caso, se demuestra en el teorema 2.4.6 que todas las sizigias minimales no lineales que aparecen en ese primer paso tienen el mismo grado. Además, determinamos el número de Betti graduado correspondiente. Cerramos el capítulo generalizando estos resultados al caso de ideales monomiales generados en grado 2 (no necesariamente libres de cuadrados).

Capítulo 3: Diagramas de Betti ideales de grafos bipartitos

En este capítulo nos centramos en ideales de grafos bipartitos para trasladar los resultados del capítulo anterior al caso de regularidad 3. Lo hacemos siguiendo una analogía con el caso de regularidad 2:

- Empezamos determinando los grafos bipartitos minimales con regularidad mayor que 3 y damos fórmulas combinatorias para todos sus números de Betti graduados.
- Caracterizamos aquellos grafos bipartitos con regularidad igual a 3.
- En el caso de ideales de grafos bipartitos cuya regularidad es mayor que 3, determinamos el primer paso de la resolución libre graduada minimal en el que aparecen sizigias cuyo grado es al menos dos unidades mayor que el grado de cualquier sizigia no lineal en el paso anterior. Demostramos que todas esas sizigias tienen el mismo grado y determinamos ese grado y el número de tales sizigias.
- Los resultados son generalizados al caso de ideales monomiales generados en grado 2.

En la última sección de este capítulo estudiamos la relación entre el número de emparejamientos inducidos de un grafo bipartito y la regularidad de su ideal de grafo.

Apéndice: Algoritmo para grafos bipartitos conexos no isomorfos

Bajo este epígrafe detallamos un algoritmo que nos permite obtener una lista exhaustiva con un representante de cada clase de isomorfía de grafos bipartitos conexos. Aportamos el desarrollo teórico e incluimos un pseudocódigo de todas sus partes.

Conclusiones

En esta tesis hemos proporcionado fórmulas puramente combinatorias para el cálculo de los números de Betti graduados de dos familias de grafos. Este problema no sólo es difícil sino que es imposible en muchos casos debido a que éstos pueden depender de la característica del cuerpo. En particular, no es posible obtener una caracterización de los ideales de grafos en general con regularidad 3 a partir exclusivamente de las propiedades del grafo. Ni siquiera restringiéndonos a grafos bipartitos podemos conseguir una caracterización semejante para aquellos ideales de grafos bipartitos con regularidad 4, ya que en ambos casos hay ejemplos en los que dicha regularidad depende de la característica del cuerpo base.

Nuestros resultados sobre la forma del diagrama de Betti de un ideal de grafo y la determinación del primer y segundo escalón para ideales de grafos e ideales de grafos bipartitos, respectivamente, permiten obtener cotas ajustadas sobre los grados de las sizigias de toda la resolución. Además, A. Conca ha sugerido que estas cotas pueden generalizarse o servir de referencia para las cotas de las sizigias de las álgebras de Koszul.

Finalmente, este trabajo motiva el estudio de cotas para la regularidad de ideales de grafos bipartitos y la dependencia de dicha regularidad respecto de la característica del cuerpo base.

List of Symbols

Introduction

The scope of this thesis lies in the field of Combinatorial Commutative Algebra. In this dissertation, we address questions on invariants related to the minimal graded free resolution of a homogeneous ideal I in a polynomial ring R with coefficients in a field $\mathbb{K}$,

$$0 \to \oplus_j R(-j)^{\beta_{p,j}} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_{i+1}} \oplus_j R(-j)^{\beta_{i,j}} \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_1} \oplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\varphi_0} I \to 0\,.$$

Namely, we focus on the study of graded Betti numbers, $\beta_{i,j}$, and other related invariants like the Castelnuovo-Mumford regularity and the projective dimension of particular families of homogeneous ideals. The minimal graded free resolution of a homogeneous ideal can theoretically be computed by using algorithms based on Gröbner basis and implemented in software systems like CoCoA, Macaulay2 or Singular. However, in practice, these procedures can take too long or require unaffordable resources. This makes very desirable any possible combinatorial description of these invariants.

Graded Betti numbers encode all the numerical information about the minimal graded free resolution of a homogeneous ideal. From them, we can recover the Hilbert function, the Castelnuovo-Mumford regularity or the projective dimension. In particular, the regularity is one of the most important invariants of such an ideal and measures its complexity in some sense. These invariants have also important geometric interpretations (see [21]).

In this dissertation, we restrict ourselves to the case of monomial ideals. These homogeneous ideals have interesting properties that make them especially suitable for a combinatorial and computational approach. The relevance of this family can be justified by the fact that some homological invariants of general homogeneous ideals in R are equal to or bounded by the corresponding invariants of their initial ideals, which are monomial ideals. There is extensive literature about monomial ideals but it is worthing to mention the celebrated book of R.H. Villarreal [68] and the nice updated surveys [55] and [40].

The combinatorial structure behind monomial ideals can be exploited to describe in combinatorial terms the minimal free resolution for some concrete families. For instance, we can find in the paradigmatic paper by S. Eliahou and M. Kervaire [23] a description of the minimal graded free resolution of a Borel ideal in terms of the basic data of its minimal generators. This family of monomial ideals includes generic initial ideals of homogeneous ideals in a polynomial ring with coefficients in a field of characteristic zero. The minimal graded free resolution of a monomial ideal can also be described as a simplicial chain complex in several successful cases. One can label the vertices of a simplicial complexes, Δ, with the minimal generators of a monomial ideal and the faces with the least common multiple of the labels of the vertices they comprise, and then consider the chain complex of Δ with coefficients in the polynomial ring. Some examples of graded free resolutions constructed following this procedure are given by D. Taylor in [62] and G. Lyubeznik in [50]. In the former, the resolution is built from a simplex whereas it is reduced to a subcomplex in the latter. Both resolutions always contain the minimal graded free resolution of the monomial ideal, though, in most of the cases, the containment is strict. On the other hand, in [4], the construction of the simplicial chain complex is based on the so-called Scarf complex and is always contained in the minimal graded free resolution. When this chain complex is a resolution, i.e., when it is exact as a sequence, it coincides with the minimal resolution. However, it was proved in [66] that the minimal graded free resolution of a monomial ideal can not always be obtained in this way even if we consider more general structures as CW-complexes instead of simplicial complexes. To overcome this limitation, the concept of *frame* was introduced in [56].

Another approach to the study of graded Betti numbers of monomial ideals is the Stanley-Reisner correspondence, which establishes a bijection between the set of squarefree monomial ideals in $R = \mathbb{K}[x_1, \ldots, x_n]$ and the set of simplicial complexes on n vertices (see [57] and [60]). The additional condition of squarefreeness does not mean any loss of generality in the study of graded Betti numbers as they are preserved by polarization, a procedure that turns a monomial ideal into a squarefree monomial ideal in a new polynomial ring with more variables (see [28]). This trend has led to characterizations of some algebraic properties of squarefree monomial ideals, such as being Cohen-Macaulay, sequentially Cohen Macaulay or Gorenstein in terms of the topology of the simplicial complex associated (see [19] or [12]). It also provides formulas for the numerical invariants. A celebrated tool is Hochster's Formula ([42]), that expresses the graded Betti numbers of a squarefree monomial ideal in terms of the reduced homology of the corresponding simplicial complex with coefficients in the ground field. Unfortunately, this theoretical machinery is not sufficient for the effective computation of these invariants and combinatorial descriptions have been reached only for a few families. This is the approach taken in this work.

Edge ideals are squarefree monomial ideals generated in degree 2. The Stanley-Reisner complex of an edge ideal is the independence complex of a simple graph. They were first introduced by R.H. Villarreal in [67] and [59]. Many recent papers are devoted to describing algebraic properties of these ideals in terms of the combinatorial properties of the graph (see [68] and the surveys [64], [36] and [53], and the references therein) and it is still a very active research topic.

We will pay special attention to bipartite graphs. In the case of bipartite edge ideals, Cohen-Macaulayness is characterized in terms of the independence complex by the property of being pure shellable ([26]) and also in a graph-theoretical way ([39]). If we relax the algebraic condition to sequentially Cohen-Macaulayness, then the characterization is given by shellability in a non-pure sense ([65]). Though bipartite edge ideals have been studied by several authors, one only has formulas for their graded Betti number in a few particular cases like Ferrer graphs ([13]), cycles of even length, complete bipartite graphs, path graphs or star graphs ([43]).

We begin the body of this dissertation with a review of the combinatorial and algebraic terminology used later and a survey on the fundamental results regarding these concepts. We collect the basic terminology of the combinatorial structures that we associate to the homogeneous ideals we are considering in this work: graphs and simplicial complexes. Also, we recall the basic definitions and results on simplicial homology, including the Mayer-Vietoris sequence, the Nerve Theorem and several ways of collapsing a simplicial complex preserving its homotopy type.Then, we overview the existence and uniqueness (up to isomorphism) of the minimal graded free resolution of a finitely generated graded R-module and define some associated numerical invariants like graded Betti numbers, Castelnuovo-Mumford regularity and projective dimension. We go from monomial ideals to squarefree monomial ideals through polarization and describe the Stanley-Reisner corresponence. In this context, Hochster's Formula and Eagon-Reiner's version are formulated. Finally, we focus on the objects we will deal with in this thesis: edge ideals. In this case, the Stanley-Reisner complex coincides with the independence complex of a simple graph and we rewrite some properties of this simplicial complex in terms of the graph.

In the second chapter, we study the set of graded Betti numbers that we store in the so-called Betti diagram. We prove in Theorem 2.1.2 that the Betti diagram of an edge ideal has a left-justified staircase shape. The simplest Betti diagram that an edge ideal can have is when it consists of only one row. In this case, one says that the ideal has a linear resolution. Moreover, when this happens, the graded Betti numbers do not depend on the characteristic of the ground field $\mathbb{K}$. A classical characterization of edge ideals with linear resolution was given by R. Fröberg. He formulated that the edge ideal associated to a graph G has linear resolution if and only if the complement of G is chordal. This result was later refined by D. Eisenbud, M. Green, K. Hulek and S. Popescu by determining the first step in the minimal resolution where nonlinear syzygies appear for the first time if G^c is not chordal. In Theorem 2.4.6, we compute the multidegrees of those minimal syzygies and determine how many there are. This result is extended to the non-squarefree case of ideals generated by monomials of degree 2. Furthermore, we obtain in Theorem 2.3.3 a purely combinatorial description of all the graded Betti numbers of edge ideals corresponding to complementary graphs of cycles. These are the minimal graphs whose edge ideals do not have linear resolution in the sense that any graph with that property contains the complement of a cycle as an induced subgraph by the Froberg's Theorem.

In chapter 3, we single out bipartite edge ideals. We give a characterization of bipartite edge ideals having regularity 3 in Theorem 3.3.11. The first column in the Betti diagram with a nonzero entry outside the first two rows is determined and the multidegrees of all such graded Betti numbers are computed in Theorem 3.3.9. Then, we generalize our results to the non-squarefree case. Moreover, combinatorial formulas for all the graded Betti numbers of the edge ideal associated to the bipartite complement of a cycle of even length are also given in Theorem 3.2.15. These ideals are also the minimal ones having regularity greater than or equal to 4. At the end of this chapter, there is a brief study of the relation between the induced matching number of a bipartite graph and the regularity of its edge ideal.

The last chapter is devoted to conclusions and further work. We present applications of the main results in this thesis as well as propose some future research interests.

Finally, we include an appendix where we describe an algorithm to obtain an exhaustive list with a representative of all the equivalence classes defined by isomorphy of connected bipartite graphs.

Chapter 1

Combinatorial and Algebraic Background

We start collecting the basic terminology related to graphs, simplicial complexes and minimal graded free resolutions. Also, we present a survey on the fundamental results on simplicial homology and independence complexes.

1.1 Graphs

In this section we fix some terminology and notation from the graph theory. All the items addressed here can be found in [18], [37] or [10].

A *graph* G is an ordered pair of sets (V, E) such that V is nonempty and $E \subset V \times V$. The elements of V and E are called *vertices* and *edges*, respectively. The elements of an edge $e = \{u, v\} \in E$ are called the *endvertices* of e and they are said to be *adjacent* to each other and *incident* with e. Also, e is said to *cover* u and v. An edge $\{u, v\}$ is called a *loop* if $u = v$. Two graphs G and G' are said to be isomorphic if there exists a one-to-one correspondence between their vertex sets $f : V(G) \longrightarrow V(G')$ such that $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(G')$. We denote this situation by $G \cong G'$.

A graph $G = (V, E)$ is said to be

- *directed* (or a *digraph*) if the elements of E are ordered pairs. Otherwise, it is said to be *undirected*.
- *finite* if $|G| := |V|$, the *order* of G, is finite.
- *simple* if there is no loop in E.

Remark 1.1.1. In this work we consider only undirected finite graphs. We assume also that all graphs are simple except in the sections 2.5 and 3.4.

Let $G = (V, E)$ be a graph, $v \in V$ and $W \subset V$. Then, we denote by

- $N_G(v) := \{u \in V : \{u, v\} \in E\}$, the *neighborhood* of v in G.
- $N_G[v] := N_G(v) \cup \{v\}$, the *closed neighborhood* of v in G.
- $N_G(W) := \cup_{v \in W} N_G(v)$, the neighborhood of W in G.
- $N_G[W] := \cup_{v \in W} N_G[v]$, the closed neighborhood of W in G.

The *degree* (or *valency*) of a vertex v in a graph G is defined as the number of vertices adjacent to v, i.e. $|N_G(v)|$, and is denoted by $\deg_G(v)$. Two vertices $u, v \in V$ are called *twins* if $N_G(u) = N_G(v)$. If $\deg_G(v) = 0$ then v is said to be an *isolated vertex*. If $\deg_G(v) = 1$ and $e = \{u, v\} \in E(G)$ is the unique edge in G covering v, then e is said to be a *whisker at* u.

Given a graph G, a *subgraph* of G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If we fix a subset of vertices $W \subset V(G)$, the *induced subgraph* of G on W is the subgraph H with $V(H) = W$ and maximal $E(H) \subset E(G)$, i.e., all edges in G whose endvertices belong to W are in $E(H)$. We write $H \subset G$ if H is a subgraph of G, $G[W]$ for the induced subgraph on the vertices W and $H < G$ if H is an induced subgraph on some omitted vertex set.

Given two vertices u, u' of a graph G, a *path* in G from u to u' is an alternating sequence of vertices and edges of G, $v_0, e_1, v_1, e_2, v_2, \ldots, v_{l-1}, e_l, v_l$ such that $v_0 = u, v_l = u', v_i \neq v_j$ if $i, j > 0$ and $i \neq j$, and $e_i = \{v_{i-1}, v_i\}, \forall i = 1, \ldots, l$. The number of edges in the path, l, is called the *length* of the path. The vertices u and u' are *connected* if there exists a path between them. The *distance* between u and u', denoted by $d_G(u, u')$ is the minimal length of a path between them if they are connected and infinity otherwise. The maximum among the distances between two vertices of G is called the *diameter* of G and denoted by $d(G)$. G is said to be a *connected graph* if every pair of distinct vertices in G are connected. A *connected component* of G is a maximal (under inclusion of vertex sets) connected induced subgraph. We denote by comp(G) the set of connected components of G.

We can construct new graphs from a given one. Let G, G' be graphs, $v \in V(G)$ and $e \in E(G)$, we can consider the following graphs:

- $G^c := (V(G), \{\{u, v\} : u, v \in V(G), \{u, v\} \notin E(G)\})$, the *complement* of G,
- $\mathcal{L}(G) := (E(G), \{\{e, e'\} : e, e' \in E(G), |e \cap e'| = 1\})$, the *line graph* of G,

- $G \setminus v := (V(G) \setminus \{v\}, \{e \in E(G) : v \notin e\})$,
- $G \setminus e := (V(G), E(G) \setminus \{e\})$,
- $G \cup G' := (V(G) \cup V(G'), E(G) \cup E(G'))$, the *union* of G and G',
- kG, the union of k copies of G on different vertex sets.

Some particular families of graphs are listed below:

- a *cycle* C_l (of length l) is a graph isomorphic to the graph $\mathcal{C}_l$ on the vertices $[l] := \{1, \ldots, l\}$ where $l \geq 3$ and with edge set $\{\{1,2\}, \{2,3\}, \ldots, \{l-1, l\}, \{l, 1\}\}$. An induced subgraph isomorphic to a cycle is called an *induced cycle.*
- a graph is said to be *chordal* if it has no induced cycle of length $l \geq 4$.
- a *forest* is a graph such that no subgraph is a cycle.
- a *tree* is a connected forest. Equivalently, a graph G is a tree if it is a forest with $|V(G)| = |E(G)| + 1$ or if it is connected and $|V(G)| = |E(G)| + 1$.
- *the complete graph* on the set V, denoted by K_V, or K_m if $V = [m]$, is the graph with vertex set V and edge set containing all pairs of distinct vertices in V.
- a graph G is said to be *bipartite* if its vertex set can be splitted into two disjoint subsets $V(G) = A \sqcup B$ in such a way that $E(G) \subset A \times B$, i.e., every edge in G has its endpoints in different sets of the *bipartition* $\{A, B\}$.
- *the complete bipartite graph* on $V = A \sqcup B$, denoted by $K_{A,B}$, or $K_{a,b}$ if $A = [a]$ and $B = [b]$, is the bipartite graph on V with the bipartition $\{A, B\}$ and edge set $A \times B$.

Given a graph G, a subset $W \subset V(G)$ is called a *clique* if $G[W]$ is a complete graph on W. We say that a subset S of vertices (resp. edges) of G is *independent* if no edge (resp. vertex) of G covers (resp. is incident with) two elements of S. If every edge (resp. vertex) of G covers (resp. is incident with) an element in S then we say that S is a *vertex (resp. edge) cover* of G. The maximum cardinality of an independent set of vertices (resp. edges) in G is denoted by $\beta_0(G)$ (resp. $\beta_1(G)$) and the minimum cardinality of a vertex (resp. edge) cover in G is denoted by $\alpha_0(G)$ (resp. $\alpha_1(G)$). Independent sets of edges are also called *matchings.* A matching is said to be *perfect* if it is also an edge cover. An *induced matching* is a matching M in G such that $E(G[\cup_{e \in M} e]) = M$, i.e., there is no edge in G covering two endpoints of two different edges in M. The maximum cardinality of an induced matching is called *the induced matching number* of G and is denoted by $\mu(G)$.

1.2 Simplicial Complexes

Now, we fix the terminology regarding finite abstract simplicial complexes according to the algebraic point of view (we consider the empty set as a possible face of a simplicial complex).

Given a finite set V, we define a (*finite abstract*) *simplicial complex on* V as a family Δ of subsets of V such that

1. if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$;

2. for every element $v \in V$, $\{v\} \in \Delta$.

Remark 1.2.1. Some authors do not require condition 2 in the definition of simplicial complex. Also some authors exclude the emptyset from being an element of a simplicial complex.

The set V is called the *vertex set* of Δ and its elements are called the *vertices* of Δ. Elements of Δ are called *faces*. By abuse of notation, we also refer to a singleton face as a vertex. The simplicial complex consisting of all subsets of a set V is called the *simplex* on V and is denoted by Δ_V. There exists two simplicial complexes on the empty vertex set: $\Delta = \emptyset$, the *void complex*, and $\Delta = \{\emptyset\}$, the *empty complex*. Notice that if $\Delta \neq \emptyset$, then $\emptyset \in \Delta$.

Maximal faces with respect to set inclusion are called *facets* and the set of all facets of a simplicial complex Δ is denoted by $\mathcal{F}(\Delta)$. Given a set V and a family of subsets $F_1, \ldots, F_r$, we denote by $\langle F_1, \ldots, F_r\rangle$ the simplicial complex consisting of all subsets of every F_i, $i = 1, \ldots, r$. If Δ is a simplicial complex with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_s\}$, then $\Delta = \langle F_1, \ldots, F_s\rangle$.

Let Δ and Δ' be two simplicial complexes. We say that Δ and Δ' are *isomorphic* and denote by $\Delta \cong \Delta'$ if there exists a bijection $f : V(\Delta) \longrightarrow V(\Delta')$ such that $\{v_1, \ldots, v_s\} \in \Delta$ if and only if $\{f(v_1), \ldots, f(v_s)\} \in \Delta'$. We say that Δ' is a *subcomplex* of Δ if $\Delta' \subset \Delta$. A subcomplex Δ' of Δ is said to be *full* provided every face of Δ having its elements in $V(\Delta')$ also belongs to Δ'. Let $W \subset V(\Delta)$, we call the *induced subcomplex* of Δ on W to the subcomplex $\Delta[W] := \{\sigma \in \Delta : \sigma \subset W\}$. We also can denote an induced subcomplex Δ' of Δ by $\Delta' < \Delta$, omitting the vertex set on which it is induced. A subcomplex Δ' is full if and only if it is an induced subgraph on $V(\Delta')$.

Let Δ be a simplicial complex and let $\sigma \subset V(\Delta)$. We define the following subcomplexes of Δ:

- $\mathrm{del}_\Delta(\sigma) := \{\tau \in \Delta : \tau \cap \sigma = \emptyset\}$,

- $\mathrm{fdel}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \not\subset \tau\}$,
- $\mathrm{link}_\Delta(\sigma) := \{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ y } \tau \cup \sigma \in \Delta\}$,
- $\mathrm{star}_\Delta(\sigma) := \{\tau \in \Delta : \tau \cup \sigma \in \Delta\}$.

In the case $\sigma = \{v\}$, we shorten the notation to $\mathrm{del}_\Delta(v)$, $\mathrm{link}_\Delta(v)$ and $\mathrm{star}_\Delta(v)$. We notice that $\mathrm{fdel}_\Delta(\{v\}) = \mathrm{del}_\Delta(v)$. If $\sigma \notin \Delta$, then $\mathrm{fdel}_\Delta(\sigma) = \Delta$ and $\mathrm{link}_\Delta(\sigma) = \mathrm{star}_\Delta(\sigma) = \emptyset$.

The *dimension of a face* σ is the cardinality of σ as a set minus one, $\dim \sigma := |\sigma| - 1$. We adopt the convention that $\dim \emptyset = -1$. The *dimension of a simplicial complex* Δ is the maximum among the dimensions of its faces, $\dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}$. We say that Δ is *pure* if all its facets have the same dimension. We denote $\Delta^i := \{\sigma \in \Delta : \dim(\sigma) = i\}$ and $\Delta^{(i)} := \{\sigma \in \Delta : \dim(\sigma) \leq i\}$. The first set is not a simplicial complex unless $V(\Delta) = \emptyset$ and $i = -1$, whereas $\Delta^{(i)}$ is always a simplicial complex named the *i-skeleton* of Δ. We notice that the 1-skeleton of a simplicial complex is a simple graph provided $V(\Delta) \neq \emptyset$.

The connectivity in Δ is defined as the connectivity in $\Delta^{(1)}$ as a graph: two vertices of Δ are connected if they are connected in $\Delta^{(1)}$ and $\Delta[W]$ is a connected component of Δ if and only if $\Delta^{(1)}[W]$ is a connected component of $\Delta^{(1)}$. The set of connected components of Δ will be denoted by $\mathrm{comp}(\Delta)$.

Let Δ and Δ' be simplicial complexes. We can construct new simplicial complexes from them:

- the *complementary simplicial complex*, $\Delta^c := \langle V(\Delta) \setminus F : F \in \mathcal{F}(\Delta)\rangle$,
- the *Alexander dual*, $\Delta^\vee := \{\sigma \subset V(\Delta) : V(\Delta) \setminus \sigma \notin \Delta\}$,
- the *(first) barycentric subdivision*, $\mathrm{sd}(\Delta)$. It is a simplicial complex on the vertex set $\Delta \setminus \{\emptyset\}$ and a family of faces of Δ, $\{\sigma_1, \dots, \sigma_s\}$, is a face of $\mathrm{sd}(\Delta)$ if it is the emptyset or $s \geq 1$ and its elements can be arranged in an inclusion chain $\sigma_{i_1} \subset \sigma_{i_2} \subset \cdots \subset \sigma_{i_s}$.
- the *union complex* and the *intersection complex*, $\Delta \cup \Delta'$ and $\Delta \cap \Delta'$, respectively,
- assuming that $V(\Delta) \cap V(\Delta') = \emptyset$, the *join* of Δ and Δ' is the simplicial complex $\Delta * \Delta' := \{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Delta'\}$ on the vertex set $V(\Delta) \cup V(\Delta')$.
- the *wedge* of Δ and Δ' (w.r.t. $u \in V(\Delta)$ and $v \in V(\Delta')$), denoted by $\Delta \vee \Delta'$, is obtained from the union $\Delta \cup \Delta'$ by identifying the vertices u and v.

We pay special attention to two types of joins:

- a simplicial complex Δ is said to be a *cone with appex* v and is denoted by $v*\Delta'$ if there exists some simplicial complex Δ', called the *base* of the cone, such that $v \notin V(\Delta')$ and $\Delta = \{\emptyset, \{v\}\} * \Delta'$;

- Δ is said to be a *suspention* of Δ' on the vertices u, v if $\Delta = \{\emptyset, \{u\}, \{v\}\} * \Delta'$ with $u, v \notin V(\Delta')$. In this case, Δ is denoted by $\sum_u^v \Delta'$ or just $\sum \Delta'$.

Notice that $\sum_u^v \Delta' = (u * \Delta') \cup (v * \Delta')$ and that $\mathrm{star}_\Delta(v)$ is a cone with appex v for any simplicial complex Δ and any $v \in V(\Delta)$.

1.3 Simplicial Homology

We do a brief reminder on the basics of simplicial homology theory based on the references [60] and [54].

Let Δ be a simplicial complex on the vertices V. Given a nonempty subset of vertices $\{v_0, \ldots, v_d\} \subset V$, we can consider the set of all posible orderings on its elements $\{(v_{b(0)}, \ldots, v_{b(d)}) : b \text{ is a bijection from } \{0, \ldots, d\} \text{ to itself}\}$ and the equivalence relation in this set given by $(v_{b(0)}, \ldots, v_{b(d)}) \sim (v_{b'(0)}, \ldots, v_{b'(d)})$ if $b = p \circ b'$ where p is an even permutation on $\{0, \ldots, d\}$. Thus, we have two equivalence classes if $d > 0$ and only one if $d = 0$. These equivalence classes are called *orientations* and a face of Δ provided with an orientation is called an *oriented face*. An oriented face $\sigma = \{v_0, \ldots, v_d\}$ is denoted by $[v_0, \ldots, v_d]$ if we want to make explicit that the orientation corresponds to the equivalence class of $(v_0, \ldots, v_d)$, otherwise we just denote by σ the oriented face and by $-\sigma$ the same face with the opposite orientation (if σ is not a vertex). $\Delta^{[i]}$ denotes the set of oriented faces of dimension i of Δ.

A *d-chain on Δ with coefficients in G* is a function c from $\Delta^{[d]}$ to an abelian group G satisfying $c(-\sigma) = -c(\sigma)$ if $d > 0$. The set $C_d(\Delta, G)$ of all d-chains on Δ with coefficients in G is a group with the addition defined by $(c+c')(\sigma) := c(\sigma)+c'(\sigma)$, $\forall \sigma \in \Delta^{[d]}$.

The *elementary d-chain* on an oriented face σ of dimension $d > 0$ is the d-chain defined by $c(\sigma) = 1, c(-\sigma) = -1$ and $c(\tau) = 0$ for all $\tau \in \Delta^{[d]} \setminus \{\sigma\}$. By abuse of notation, we identify the elementary chain on σ with σ itself. With this notation, $\sigma + (-\sigma) = 0$ in $C_d(\Delta, G)$. Let $g \in G$ and $\sigma \in C_d(\Delta, G)$, then we denote by $g\sigma$ the d-chain defined by $c(\sigma) = g, c(-\sigma) = -g$ and $c(\tau) = 0, \forall \tau \neq \sigma$. Notice that $g(-\sigma) = -g\sigma$. Let c be an arbitrary d-chain on Δ. Since $c(-\sigma) = -c(\sigma)$, then c is determined by its values in just one of the two possible orientations for each face of dimension $d > 0$. Thus, if we fix an orientation for each $\sigma \in \Delta^d$, we can write uniquely $c = \sum_{\sigma \in \Delta^d} c(\sigma)\sigma$. Therefore, $C_d(\Delta, G)$ is the direct sum of subgroups isomorphic to G, one for each unoriented face of dimension d in Δ, $C_d(\Delta, G) \cong \bigoplus_{\sigma \in \Delta^d} G$.

In the case of $C_0(\Delta, G)$, the elementary 0-chain c_v on a vertex $v \in V(\Delta)$ is defined by $c_v(v) = 1$ and $c_v(u) = 0$ if $u \neq v$. The set $\{c_v : v \in V(\Delta)\}$ form a natural basis of $C_0(\Delta, G)$. The (-1)-chains are functions from $\{\emptyset\}$ to G, so $C_{-1}(\Delta, G) \cong G$.

Given an elementary d-chain $\sigma = [v_0, \ldots, v_d]$, the *boundary operator* is defined as

$$\partial_d(\sigma) = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_d], \text{ if } d > 0, \text{ and}$$

$$\partial_0(c_v) = 1\,.$$

These applications are well defined and extend uniquely to homomorphisms

$$\partial_d : C_d(\Delta, G) \longrightarrow C_{d-1}(\Delta, G),\ \forall d > 0, \text{ and}$$

$$\partial_0 : C_0(\Delta, G) \longrightarrow G$$

and satisfy $\partial_d \circ \partial_{d+1} = 0$, i.e., $\mathrm{Im}(\partial_{d+1}) \subset \mathrm{Ker}(\partial_d),\ \forall d \geq 0$.

We consider the two following exact sequences of groups and homomorphisms:

$$\mathcal{C}(\Delta) : 0 \longrightarrow C_d(\Delta, G) \longrightarrow \cdots \longrightarrow C_i(\Delta, G) \xrightarrow{\partial_i} C_{i-1}(\Delta, G) \longrightarrow \cdots \xrightarrow{\partial_1} C_0(\Delta, G) \longrightarrow 0$$

and

$$\tilde{\mathcal{C}}(\Delta) : 0 \to C_d(\Delta, G) \to \cdots \longrightarrow C_i(\Delta, G) \xrightarrow{\partial_i} C_{i-1}(\Delta, G) \longrightarrow \cdots \xrightarrow{\partial_1} C_0(\Delta, G) \xrightarrow{\partial_0} G \to 0$$

where $d := \dim(\Delta)$. The first one is called the *simplicial chain complex* (over G) of Δ and $\tilde{\mathcal{C}}(\Delta)$ is the *augmented simplicial chain complex*, which includes $C_{-1}(\Delta, G)$.

The quotient group $\mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i+1})$ from the chain complex $\mathcal{C}(\Delta)$, considering $\partial_0 = 0$, is called the *i-th homology group* of Δ with coefficients in G and is denoted by $H_i(\Delta, G)$, $\forall i \geq 0$, whereas if the quotient is taken from $\tilde{\mathcal{C}}(\Delta)$, it is denoted by $\tilde{H}_i(\Delta, G)$ and is referred to as the *i-th reduced homology group*. We can consider also $\tilde{H}_{-1}(\Delta, G) := G/\mathrm{Im}(\partial_0)$, which is the trivial group for every simplicial complex but the empty complex. For $\Delta = \{\emptyset\}$, $\tilde{H}_{-1}(\Delta, G) \cong G$ and this is its only nontrivial homology group.

Notice that $H_i(\Delta, G) = \tilde{H}_i(\Delta, G), \forall i > 0$. For 0-chains, $\{c_v : v \in V(\Delta)\}$ form a basis of $C_0(\Delta, G)$ and if two vertices $v, v' \in V(\Delta)$ are connected by a path $v = u_0, e_1, u_1, \cdots, u_{l-1}, e_l, u_l = v'$ with $e_i = \{u_{i-1}, u_i\} \in \Delta^1$, then $\partial_1(e_1 + \cdots + e_l) = v' - v$, so they are in the same class of equivalence in $H_0(\Delta, G)$. Moreover, if they are not connected, they belong to different equivalence classes (see [54, Theorem 7.1] for details). We take the class of one vertex in each connected component of Δ, let us say $B = \{[v_1], \ldots, [v_{|\mathrm{comp}(\Delta)|}]\}$. Then, B is a basis of the free group $H_0(\Delta, G)$. Similar reasoning for the reduced homology case shows that there exists a basis of $\tilde{H}_0(\Delta, G)$ consisting of the classes of the 0-chains $v_i - v_1$ with $i \neq 1$ if Δ is not conected. Otherwise, any element in $\mathrm{Ker}(\partial_0)$ belongs to the equivalence class of 0 if Δ. Both groups are then related by the equality ([54, Theorem 7.2])

$$H_0(\Delta, G) = \tilde{H}_0(\Delta, G) \oplus G\,.$$

If G is a commutative ring with unit, then $\tilde{H}_d(\Delta, G)$ also is and if G is a field, $\tilde{H}_d(\Delta, G)$ is a vector space.

Remark 1.3.1. Hereafter, we use the short notations $H_d(\Delta)$ and $\tilde{H}_d(\Delta)$ instead of $\tilde{H}_d(\Delta, G)$ and $\tilde{H}_d(\Delta, G)$ when G is a field. We also denote the trivial group $\{0\}$ just by 0 in the context of homology groups.

We include some definitions regarding the vanishing of reduced homology groups. A simplicial complex is said to be

- *acyclic* (over G) if $\tilde{H}_i(\Delta, G) = 0,\ \forall i \geq -1$,
- *k-acyclic* (over G) if $\tilde{H}_i(\Delta, G) = 0,\ \forall i, -1 \leq i \leq k$,
- *k-connected* (over G) if $\tilde{H}_i(\Delta, G) = 0,\ \forall i, -1 \leq i \leq k$ and it is simply connected if $k \geq 1$ (this is the characterization given by Hurewicz's Theorem, see [72]),
- *k-Leray* (over G) if $\tilde{H}_i(\Delta[W], G) = 0, \forall i \geq k, \forall W \subset V(\Delta)$.

Example 1.3.2. Basic examples of acyclic simplicial complexes are simplices and cones (see [54, Theorem 8.2]). The void complex is acyclic while the empty complex is not.

1.4 Homological Tools

Computing homology groups directly from definition is an unaffordable task for non-trivial examples so we will need some tecniques for computing homological groups in practice. Namely, we focus on the Mayer-Vietoris Sequence, the Nerve Theorem and collasibility, all of them based on relating the homology groups we are interested in with others simpler.

The first way we present consists in putting the homology groups of our simplicial complex in an exact long sequence together with the homology groups of some proper subcomplexes.

Theorem 1.4.1 (Mayer-Vietoris Sequence; see e.g., [54, Theorem 25.1])**.** *Let Δ be a simplicial complex, Δ_1 and Δ_2 be subcomplexes of Δ such that $\Delta = \Delta_1 \cup \Delta_2$ and denote $\Delta_0 := \Delta_1 \cap \Delta_2$. Then, there is a long exact sequence*

$$\cdots \longrightarrow H_i(\Delta_0) \longrightarrow \begin{matrix} H_i(\Delta_1) \\ \oplus \\ H_i(\Delta_2) \end{matrix} \longrightarrow H_i(\Delta) \longrightarrow H_{i-1}(\Delta_0) \longrightarrow \cdots$$

which is called the Mayer-Vietoris sequence. Provided $\Delta_0 \neq \{\emptyset\}$, there is an analogous long exact sequence for reduced homology groups.

Example 1.4.2. Let Δ be a simplicial complex and $v \in V(\Delta)$. Then,

$$\Delta = \mathrm{del}_\Delta(v) \cup star_\Delta(v) \text{ and } del_\Delta(v) \cap star_\Delta(v) = link_\Delta(v) \tag{1.1}$$

and hence we can apply Theorem 1.4.1 taking into account that $\mathrm{star}_\Delta(v)$ is acyclic. We obtain the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow H_i(\mathrm{link}_\Delta(v)) \longrightarrow H_i(\mathrm{del}_\Delta(v)) \longrightarrow H_i(\Delta) \longrightarrow H_{i-1}(\mathrm{link}_\Delta(v)) \longrightarrow \cdots \\ \cdots \longrightarrow H_0(\mathrm{link}_\Delta(v)) \longrightarrow H_0(\mathrm{del}_\Delta(v)) \longrightarrow H_0(\Delta) \longrightarrow 0. \end{aligned} \tag{1.2}$$

and the analogous sequence for reduced homology groups is also exact if $\mathrm{link}_\Delta(v) \neq \{\emptyset\}$.

Corollary 1.4.3. *Let Δ be a simplicial complex such that $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are acyclic subcomplexes of Δ and $\Delta_1 \cap \Delta_2 \neq \{\emptyset\}$. Then*

$$\tilde{H}_i(\Delta) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2), \forall i > 0 \ .$$

If we change the direction of every arrow in the sequence, it keeps being a long exact sequence (see [41, Lemma 2.1]).

Corollary 1.4.4. *If $\tilde{H}_i(\mathrm{link}_\Delta(v)) = \tilde{H}_{i-1}(\mathrm{link}_\Delta(v)) = 0$ for some $i \geq 0$, then*

$$\tilde{H}_i(\Delta) \cong \tilde{H}_i(\mathrm{del}_\Delta(v)) \, .$$

Likewise, if $\tilde{H}_i(\mathrm{del}_\Delta(v)) = \tilde{H}_{i-1}(\mathrm{del}_\Delta(v)) = 0$ for some $i > 0$, then

$$\tilde{H}_i(\Delta) \cong \tilde{H}_{i-1}(\mathrm{link}_\Delta(v)) \, .$$

Example 1.4.5. Let us consider $\sum_u^v \Delta$, the suspention of a simplicial complex Δ on the vertices $u, v \notin V(\Delta)$. We assume that $V(\Delta) \neq \emptyset$. We notice that $\mathrm{link}_{\sum_u^v \Delta}(v) = \Delta$, $\mathrm{star}_{\sum_u^v \Delta}(v) = v * \Delta$ and $\mathrm{del}_{\sum_u^v \Delta}(v) = u * \Delta$, so, taking into account that cones are acyclic and that $\text{link}_{\sum_u^v \Delta}(v) \neq \emptyset$, one has the following long exact sequence of reduced homologies,

$$\cdots \longrightarrow \tilde{H}_i(\Delta) \longrightarrow 0 \longrightarrow \tilde{H}_i(\sum_u^v \Delta) \longrightarrow \tilde{H}_{i-1}(\Delta) \longrightarrow 0 \longrightarrow \tilde{H}_{i-1}(\sum_u^v \Delta) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tilde{H}_0(\Delta) \longrightarrow 0 \longrightarrow \tilde{H}_0(\sum_u^v \Delta) \longrightarrow 0.$$

Hence, $\tilde{H}_0(\sum_u^v \Delta)$ is the trivial group and

$$\tilde{H}_i(\sum_u^v \Delta) \cong \tilde{H}_{i-1}(\Delta), \forall i > 0\,.$$

A second way to compute the homology groups of a simplicial complex Δ is looking for isomorphisms between them and the homology groups of new simplicial complexes constructed from Δ, like the nerve of a decomposition, the Alexander dual or the barycentric subdivision.

Given a finite family of sets $\mathcal{A} = \{A_i\}_{i\in\mathcal{I}}$ ($\mathcal{I}$ finite set), we correspond to $\mathcal{A}$ the simplicial complex on the vertex set $\mathcal{I}$ whose faces are subsets $\sigma \subset \mathcal{I}$ such that $\bigcap_{i\in\sigma} A_i \neq \emptyset$. This simplicial complex is called the *nerve* of $\mathcal{A}$ and is denoted by $\mathcal{N}(\mathcal{A})$. Those simplicial complexes that can be obtained as the nerve of a finite family of convex sets in $\mathbb{R}^d$ are called *d- representable*. Nerves of coverings were introduced in [1] and the first homotopical versions of the Nerve Theorem for topological spaces seems to be in [49], [11] and [70], however, it is usually atributed to Borsuk.

We formulate here a homological version of the Nerve Theorem for simplicial complexes, though, there exist several variations (see [51]).

Theorem 1.4.6 (Nerve Theorem)**.** *Let Δ be a simplicial complex and $\mathcal{A} = \{\Delta_j\}_{j\in\mathcal{I}}$ a finite family of subcomplexes such that*

- $\Delta = \bigcup_{j\in\mathcal{I}} \Delta_j$, *and*
- $\forall S \subset \mathcal{I}$, $\bigcap_{j\in S} \Delta_j$ *is the empty set or acyclic.*

Then, $\tilde{H}_i(\Delta) \cong \tilde{H}_i(\mathcal{N}(\mathcal{A}))$, $\forall i \geq 0$.

An interesting refinement is the following version:

Theorem 1.4.7 ([38, Lemma 7])**.** *Let Δ be a simplicial complex, $k \geq 0$ and $\mathcal{A} = \{\Delta_j\}_{j\in\mathcal{I}}$ a finite family of subcomplexes such that*

- $\Delta = \bigcup_{j\in\mathcal{I}} \Delta_j$, *and*
- $\forall S \subset \mathcal{I}$, $\bigcap_{j\in S} \Delta_j$ *is the empty set or* $\tilde{H}_i(\bigcap_{j\in S} \Delta_j) = 0,\ \forall i \geq k - |S|$.

Then, $\tilde{H}_i(\Delta) \cong \tilde{H}_i(\mathcal{N}(\mathcal{A})),\ \forall i \geq k$.

There are other versions similar to Theorem 1.4.7, for example [9, Lemma1.2] or [6, Theorem 6], regarding k-conectivity. See [8] or [7] for a survey.

The homology groups of a simplicial complex are also related to the homology groups of its Alexander dual $\Delta^\vee$ under certain conditions.

Theorem 1.4.8 ([44, Theorem 3.4])**.** *Let Δ be a simplicial complex on the nonempty vertex set V and $\mathbb{K}$ be a field or $\mathbb{Z}$. Assuming that $\tilde{H}_i(\Delta)$ is $\mathbb{K}$-free for every i, then*

$$\tilde{H}_i(\Delta) \cong \tilde{H}_{|V|-i-3}(\Delta^\vee),\ \forall i \geq 0\,.$$

It is also well known (see [44, Lemma 4.24] for a particular proof using discrete Morse theory) that the geometric realizations of a simplicial complex and its barycentric subdivision are homeomorphic, so, in particular, we have

$$\tilde{H}_i(\Delta) \cong \tilde{H}_i(\mathrm{sd}(\Delta)),\ \forall i \geq 0\,.$$

Finally, we collect some recursive procedures to modify a simplicial complex Δ keeping the same homology groups up to isomorphism.

A vertex $v \in V(\Delta)$ is said to be *dominated* by another vertex u in Δ if $\mathrm{link}_\Delta(v)$ is a cone with appex u, in other words, if every facet containing v also contains u. A simplicial complex with no dominated vertex is called a *minimal (or taut) complex*. A simplicial complex Δ is minimal if and only if $\mathcal{N}(\Delta)$ is minimal and also if and only if $\mathcal{N}(\mathcal{N}(\Delta)) \cong \Delta$ (see [3],[33]).

Let $\sigma \in \Delta$ and $\tau \in \mathcal{F}(\Delta)$, then we say that σ is a *free face of* τ if τ is the unique facet of Δ containing σ. In this case, (σ, τ) is said to be a *free pair* of Δ.

Let σ be a face of Δ, then the subcomplex $\mathrm{fdel}_\Delta(\sigma)$ has the same homology groups (in fact, some stronger properties) when it is obtained as the result of one of the following processes:

- an *elementary collapse* of Δ: when (σ, τ) is a free pair of Δ with $\dim \tau = \dim \sigma + 1$. In this case, $\mathrm{fdel}_\Delta(\sigma) = \Delta \setminus \{\sigma, \tau\}$ and σ is said to be a *collapsible face* of Δ. The reduction from Δ to $\mathrm{fdel}_\Delta(\sigma)$ is denoted by $\Delta \searrow \mathrm{fdel}_\Delta(\sigma)$. We say that Δ *collapses* to another simplicial complex Δ' if there exists a sequence of elementary collapses starting from Δ and ending in Δ'. In that case, it is denoted by $\Delta \searrow \Delta'$. When $\Delta \searrow \emptyset$, we say that Δ is *collapsible* (see [72]).

- an *elementary d-collapse* of Δ: when σ is a free face of some facet of Δ and $\dim \sigma < d$. In this case, σ is said to be a *d-collapsible face* of Δ. The reduction from Δ to $\mathrm{fdel}_\Delta(\sigma)$ is denoted by $\Delta \searrow_d \mathrm{fdel}_\Delta(\sigma)$. We say that Δ *d-collapses* to Δ' if there exists a sequence of elementary d-collapses from Δ to Δ'. In that case, it is denoted by $\Delta \searrow_d \Delta'$. When $\Delta \searrow_d \emptyset$ we say that Δ is *d-collapsible* (see [69]).

- an *elementary strong collapse* of Δ: when $\sigma = \{v\}$ and v is a dominated vertex of Δ. The reduction from Δ to $\mathrm{fdel}_\Delta(\sigma)$ is denoted by $\Delta \searrow\searrow \mathrm{fdel}_\Delta(v)$. If there exists a sequence of elementary strong collapses from Δ to Δ', we say that Δ *strong collapses* to Δ' and we denote it by $\Delta \searrow\searrow \Delta'$. Δ is said to be *strong collapsible* if $\Delta \searrow\searrow \emptyset$ (see [3]).

- an *elementary strong d-collapse* of Δ: when $\sigma = \{v\}$ and $\mathrm{link}_\Delta(v)$ is $(d-1)$-collapsible. The reduction from Δ to $\mathrm{fdel}_\Delta(\sigma)$ is denoted by $\Delta \searrow\searrow_d \mathrm{fdel}_\Delta(v)$. If there exists a sequence of elementary strong d-collapses from Δ to Δ', we say that Δ *strong d-collapses* to Δ' and we denoted $\Delta \searrow\searrow_d \Delta'$. Δ is said to be *strong d-collapsible* if $\Delta \searrow\searrow_d \emptyset$ (see [61]).

Therefore, if Δ (d, strong, d-strong) collapses to Δ', then $\tilde{H}_i(\Delta) \cong \tilde{H}_i(\Delta')$ for all i.

A justification of the adjetive "strong" can be found in the next result.

Lemma 1.4.9 (Fold Lemma; see, e.g., [25, Lemma 3.2])**.** *If v is a dominated vertex in Δ, then $\Delta \searrow \mathrm{del}_\Delta(v)$.*

A *core* of Δ is defined as a minimal complex Δ' such that $\Delta \searrow\searrow \Delta'$. Every simplicial complex has a core, which is unique up to isomorphism ([3, Theorem 2.11]).

The next properties in this section will make d-collapsability specially interesting in combination with Hochster's Formula (Theorem 1.6.3). If Δ is d-collapsible, then

- Δ collapses to a simplicial complex of dimension strictly smaller than d,
- $\Delta[W]$ is d-collapsible for any subset $W \subset V(\Delta)$,
- Δ is d-Leray.

On the other hand, if Δ is d-representable, then Δ is d-collapsible. These results are proved in [69].

Theorem 1.4.10 ([45, Theorem 1.2])**.** *Let $\Delta_1, \ldots, \Delta_s$ simplicial complexes on the same vertex set. We denote by $L_{\mathbb{K}}(\Delta)$ the minimal integer k such that Δ is k-Leray over $\mathbb{K}$. Then,*

- $L_{\mathbb{K}}(\cap_{i=1}^s \Delta_i) \leq \sum_{i=1}^s L_{\mathbb{K}}(\Delta_i)$,
- $L_{\mathbb{K}}(\cup_{i=1}^s \Delta_i) \leq \sum_{i=1}^s L_{\mathbb{K}}(\Delta_i) + s - 1$.

1.5 Graded Betti Numbers

This section is devoted to the algebraic background on minimal graded free resolutions of finitely generated graded modules with special interest in the homological invariants that appear therein like Betti numbers, (Castelnuovo-Mumford) regularity or projective dimension. We suggest [14], [47], [20] and [12] for deeper details.

Let A be a commutative ring and $(\Gamma,+)$ be a commutative monoid with $+$ satisfying the cancellation law. A Γ-*grading for* A is a decomposition of A as a direct sum of additive subgroups, $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, such that $A_{\gamma_1} \cdot A_{\gamma_2} \subset A_{\gamma_1+\gamma_2}$, $\forall \gamma_1, \gamma_2 \in \Gamma$. Each summand A_γ is called the *homogeneous component* of A of degree γ and any element in A_γ is said to be *homogeneous of degree* γ. If A is Γ-graded, then A_0 is a subring of A and A_γ is an A_0-module for every $\gamma \in \Gamma$.

Let A be a Γ-graded ring and M be an A-module. A Γ-*grading for* M is a decomposition of M as a direct sum of additive subgroups, $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$, such that $A_{\gamma_1} \cdot M_{\gamma_2} \subset M_{\gamma_1+\gamma_2}$, $\forall \gamma_1, \gamma_2 \in \Gamma$. Each summand M_γ is called the *homogeneous component* of M of degree γ and any element in M_γ is said to be *homogeneous of degree* γ. If M is Γ-graded, then M_γ is an A_0-module for every $\gamma \in \Gamma$.

Let A be a Γ-graded ring, $I \subset A$ be an ideal, M be a Γ-graded A-module and $N \subset M$ be an A-submodule. Any element $m \in M$ can be written uniquely as $m = \sum_{\gamma\in\Gamma} m_\gamma$ with $m_\gamma \in M_\gamma$. We say that $\{m_\gamma : \gamma \in \Gamma\}$ are the *homogeneous components* of m. We say that N is a Γ-*graded submodule* of M if $N = \bigoplus_{\gamma\in\Gamma}(N \cap M_\gamma)$. The ideal I is said to be a *homogeneous ideal* if it is Γ-graded as an A-module.

Proposition 1.5.1 ([47, Proposition 1.7.10])**.** *Let M be a Γ-graded module and $N \subset M$ be a submodule. Then the following are equivalent:*

- *N is a Γ-graded submodule,*
- *for every $n \in N$, if $n = \sum_{\gamma\in\Gamma} n_\gamma$ with $n_\gamma \in M_\gamma$, then $n_\gamma \in N$, $\forall \gamma \in \Gamma$,*
- *there exists a generating system of N consisting of homogeneous elements.*

Henceforth, R denotes the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by $\mathbf{x}^\alpha$ the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$. We also use the notation $|\alpha| := \sum_{i=1}^n \alpha_i$. Thus, every polynomial $f \in R$ can be written uniquely as $f = \sum_{\alpha\in\mathbb{N}^n} k_\alpha \mathbf{x}^\alpha$ with $k_\alpha \in \mathbb{K}$.

We consider two different gradings for a polynomial ring R:

- *the standard grading*,
$$R = \bigoplus_{d\in\mathbb{N}} R_d$$
where $R_d := \{f = \sum_{\alpha\in\mathbb{N}^n} k_\alpha \mathbf{x}^\alpha \in R \ : \ |\alpha| = d\}$ is the $\mathbb{K}$-vector space generated by the monomials $\mathbf{x}^\alpha$ with $|\alpha| = d$. A polynomial f in a summand R_d is said to be *homogeneous of degree* d and we denote it by $\deg(f) = d$.

- *the standard multigrading* (or *fine grading*),
$$R = \bigoplus_{\alpha\in\mathbb{N}^n} R_\alpha$$
where $R_\alpha := \{f = k_\alpha \mathbf{x}^\alpha \in R\}$ is the $\mathbb{K}$-vector space generated by the monomial $\mathbf{x}^\alpha$. A polynomial f in a summand R_α is said to be *homogeneous of degree* α and we denote it by $\mathrm{mdeg}(f) = \alpha$.

Both gradings can be extended to the groups $\mathbb{Z}$ and $\mathbb{Z}^n$, respectively, by defining $R_d = 0$ if $d < 0$ and $R_\alpha = 0$ if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_i < 0$ for some $i \in [n]$. Let Γ denote any of the groups $\mathbb{Z}$ or $\mathbb{Z}^n$ and $d \in \Gamma$, then we define the grading on R *shifted* (or *twisted*) *by* d, which is denoted by $R(d)$, as

$$R(d) := \bigoplus_{\gamma\in\Gamma} R(d)_\gamma$$

where $R(d)_\gamma := R_{d+\gamma}$.

Let M be a finitely generated graded R-module, with the standard grading or multigrading over R. A *graded free resolution* of M is a sequence $\mathcal{F}$ of R-modules and morphisms

$$\mathcal{F} : \cdots \longrightarrow F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \xrightarrow{\varphi_{-1}} 0$$

satisfying the following properties:

- F_i is a graded free R-module, $\forall i \geq 0$,

- $\mathcal{F}$ is *exact*, i.e., $\mathrm{Im}(\varphi_i) = \mathrm{Ker}(\varphi_{i-1}), \forall i \geq 0$. $\mathrm{Ker}(\varphi_{i-1})$ is a finitely generated graded submodule of F_{i-1} and is called the *i-th syzygy module* of M.

- φ_i is *homogeneous*, i.e., for every degree d, $\varphi_i((F_i)_d) \subset (F_{i-1})_d, \forall i > 0$, and $\varphi_0((F_0)_d) \subset M_d$.

We say that the resolution $\mathcal{F}$ is *finite* whenever there exists an index l such that $F_i = 0, \forall i > l$. The *length of the resolution* is the minimal index l with that property.

Theorem 1.5.2 (Graded Hilbert's Syzygy Theorem)**.** *Every finitely generated graded R-module has a finite graded free resolution with length smaller than or equal to n.*

A graded free resolution of a graded R-module M

$$\mathcal{F} : \cdots \longrightarrow F_l \xrightarrow{\varphi_l} F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

is said to be *minimal* if $\varphi_i(F_i) \subset (x_1, \ldots, x_n)F_{i-1}$, $\forall i > 0$, i.e., no nonzero entry in the matrices representing the morphisms φ_i belongs to $\mathbb{K}$. Using the graded version of Nakayama's Lemma, it can be proved (see [14, (3.10)Proposition]) that the condition of minimality is also equivalent to asking that φ_i sends a basis of F_i to a minimal generating system of $\mathrm{Im}(\varphi_i)$, $\forall i > 0$.

Given a graded free resolution of a finitely generated graded R-module M, whose existence is assured by theorem 1.5.2, it can always be reduced to a minimal one following the procedure shown in the proof of [14, (3.15)Theorem].

Theorem 1.5.3. *Every finitely generated graded R-module has a minimal graded free resolution of length $l \leq n$.*

Moreover, that minimal graded free resolution is unique up to isomorphism ([14, (3.13)Theorem]), in the sense that if $\mathcal{F}$ and $\mathcal{G}$ are two minimal graded free resolutions of M, then there exist homogeneous isomorphisms $\alpha_i : F_i \longrightarrow G_i$ such that every cell in the diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & F_0 & \xrightarrow{\varphi_0} & M \\
\big\downarrow & & \alpha_1 \big\downarrow & & \alpha_0 \big\downarrow & & \big\downarrow \mathrm{Id}_M \\
\cdots & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 & \xrightarrow{\psi_0} & M
\end{array}$$

commutes. So, from now on, we refer to a minimal graded free resolution as *the* minimal graded free resolution or just the minimal resolution, for short.

Let us consider now the standard grading over R and let $\mathcal{F}$ be the minimal resolution of M,

$$\mathcal{F} : 0 \longrightarrow F_l \longrightarrow \cdots \longrightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\varphi_0} M \longrightarrow 0\,.$$

Since M is a finitely generated R-module, all the free R-modules in the resolution have finite rank, so they are isomorphic to finite direct sums of copies of R with the suitable shifted grading: let $\{e_{i,1}, \ldots, e_{i,r_i}\}$ be a basis of F_i and $d_{i,1}, \ldots, d_{i,r_i}$ the corresponding degrees, then

$$F_i \cong \bigoplus_{k=1}^{r_i} R(-d_{i,k}),\ \forall i \geq 0\,.$$

We can collect in that expression those copies of R with the same shifting and denote the number of summands with an exponent, $R(-j)^{\beta_{i,j}} := \bigoplus_{k/d_{i,k}=j} R(-d_{i,k})$. Hence, $F_i \cong \bigoplus_j R(-j)^{\beta_{i,j}}$ and

$$\mathcal{F} : 0 \to \oplus_j R(-j)^{\beta_{p,j}} \to \cdots \to \oplus_j R(-j)^{\beta_{i,j}} \to \cdots \to \oplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\varphi_0} M \to 0 \ .$$

With the above notation, we define the *graded Betti numbers* of M, that are denoted by $\beta_{i,j}(M)$, as the exponents $\beta_{i,j}$, that correspond to the number of elements in the basis of F_i of degree j and the (*Castelnuovo-Mumford*) *regularity* of M as $\text{reg}(I) := \max_{i,j}\{d_{i,j} - i\}$.

These definitions depend on the bases choosen for each free module in the minimal resolution. However, the following result shows that the Betti numbers and the regularity only depend on M.

Proposition 1.5.4 ([21, Proposition 1.7]). *Let*

$$\mathcal{F} : 0 \to \oplus_j R(-j)^{\beta_{p,j}} \to \cdots \to \oplus_j R(-j)^{\beta_{i,j}} \to \cdots \to \oplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\varphi_0} M \to 0$$

be the minimal resolution of M. Then, any minimal set of homogeneous generators of F_i contains precisely $\dim_{\mathbb{K}} \text{Tor}_i^R(\mathbb{K}, M)_j$ *generators of degree j.*

It can be also deduced from this claim that $\text{pd}(M)$, the *proyective dimension* of M, i.e., the minimal length among the finite projective resolutions of M, is equal to the length of the minimal resolution.

The set of the graded Betti numbers of a finitely generated graded R-module M is usually arranged in a table whose columns correspond to steps in the minimal resolution and are labeled from 0 to the projective dimension of M. The labels of the rows run from the minimal degree of a homogeneous generator of M up to the regularity of M. The graded Betti numbers of M, $\beta_{i,j}$, are placed in the column i but in the row $j - i$ instead of j because of the following fact: since $\varphi_i(F_i) \subset (x_1, \ldots, x_n)F_{i-1}$, $\forall i > 0$, if there exits J such that $\beta_{i,j} = 0, \forall j < J$, for some i, then $\beta_{i+1,j} = 0, \forall j < J + 1$ ([21, Proposition 1.9]). On this way, all nonzero Betti numbers are displayed in this compact table,

	0	$\ldots$	i	$\ldots$	$p = \text{pd}$
$j_{\min}$	$\beta_{0,j_{\min}}$	$\ldots$	$\beta_{i,i+j_{\min}}$	$\ldots$	$\beta_{p,p+j_{\min}}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
j	$\beta_{0,j}$	$\ldots$	$\beta_{i,i+j}$	$\ldots$	$\beta_{p,p+j}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$r = \text{reg}$	$\beta_{0,r}$	$\ldots$	$\beta_{i,i+r}$	$\ldots$	$\beta_{p,p+r}$

which is called *the Betti diagram of* M.

We would like to highlight that the regularity and the projective dimension can be read off the Betti diagram and expressed in terms of the graded Betti numbers:

$$\mathrm{pd}(M) = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

and

$$\mathrm{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0 \text{ for some } i\}.$$

Remark 1.5.5. If we are considering R provided with the multigrading, the multigraded Betti numbers are defined in the same way as graded Betti numbers but considering multidegrees. The latter can be recovered from the former as

$$\beta_{i,j} = \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|=j}} \beta_{i,\alpha},\ \forall i, j\ .$$

1.6 Stanley-Reisner Ideals and Hochster's Formula

From now on, we restrict ourselves to the homogeneous ideals in R with respect to the standard multigrading: monomial ideals. These ideals have interesting properties that make them especially suitable for a combinatorial approach.

A *monomial ideal* $I \subset R$ is an ideal for which there exists a generating system consisting of monomials, i.e., $I = (\mathbf{x}^\beta : \beta \in B)$ for some $B \subset \mathbb{N}^n$.

Monomial ideals have nice properties from the combinatorial and computational points of view (see [15]). Let $I = (\mathbf{x}^\beta : \beta \in B)$ be a monomial ideal, then the following properties hold:

- let $\mathbf{x}^\alpha$ be a monomial in R, then $\mathbf{x}^\alpha \in I \Leftrightarrow \exists \beta \in B$ such that $\mathbf{x}^\beta | \mathbf{x}^\alpha$,
- let $f = \sum_{\alpha\in A} k_\alpha \mathbf{x}^\alpha \in R$, then $f \in I \Leftrightarrow \mathbf{x}^\alpha \in I$, $\forall \alpha \in A$,
- the set of monomials in I form a $\mathbb{K}$-basis of I,
- (Dickson's Lemma) there exists a finite set $A \subset B$ such that $I = (\mathbf{x}^\alpha : \alpha \in A)$.

We say that a generating system $\{\mathbf{x}^\alpha : \alpha \in A\}$ of a monomial ideal I is *minimal* if $\mathbf{x}^\alpha \nmid \mathbf{x}^\beta, \forall \alpha, \beta \in A$, $\alpha \neq \beta$.

Theorem 1.6.1. *Every monomial ideal* $I \subset R$ *has a unique minimal generating system, denoted by* $\mathcal{G}(I)$*, which is necessarily finite.*

A monomial $\mathbf{x}^\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is said to be *squarefree* if $\alpha_i \in \{0,1\}$ for all $i \in [n]$. If every monomial in $\mathcal{G}(I)$ is squarefree, then we say that I is a *squarefree monomial ideal*. Squarefree monomial ideals correspond to radical monomial ideals.

Construction 1.6.2 (Polarization)**.** Given a monomial $m = \mathbf{x}^\alpha$ in $R = \mathbb{K}[x_1, \dots, x_n]$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we define the *polarization of* m as the squarefree monomial

$$m_{pol} := \prod_{i=1}^{n} \prod_{j=1}^{\alpha_i} x_{i,j} \in \mathbb{K}[x_{1,1}, \dots, x_{1,\alpha_1}, x_{2,1}, \dots, x_{n,\alpha_n}].$$

Let I be a monomial ideal with $\mathcal{G}(I) = \{m_1, \dots, m_s\}$, then we call the *polarization of* I to the squarefree monomial ideal generated by the polarizations of the minimal generators of I,

$$I_{pol} := ((m_1)_{pol}, \dots, (m_s)_{pol}) \in R_{pol} := \mathbb{K}[x_{1,1}, \dots, x_{1,t_1}, x_{2,1}, \dots, x_{n,t_n}]\,,$$

where $t_i := \max\{e : \exists j \in [s] \,/\, x_i^e | m_j\}$.

The interest of this construction is that the polarization of a monomial ideal has similar properties to the original monomial ideal. Namely, they have the same graded Betti numbers and, as a consequence, they have the same regularity and projective dimension:

$$\beta_i, j(I) = \beta_{i,j}(I_{pol}),\ \forall (i,j);\ \mathrm{reg}(I) = \mathrm{reg}(I_{pol});\ \mathrm{pd}(I) = \mathrm{pd}(I_{pol}). \tag{1.3}$$

See [40, Corollary 1.6.3] for a proof of 1.3 and other properties of polarizations.

Therefore, the study of Betti numbers of monomial ideals can be reduced to the case of squarefree monomial ideals, which are in correspondence with simplicial complexes, as we describe below.

Let I be a squarefree monomial ideal in R. We associate to I the following simplicial complexes (see [27]):

- *the facet complex*, whose facets correspond to the minimal generators of I:

$$\Delta_{\mathcal{F}}(I) := \langle \{i_i, \dots, i_l\} : x_{i_1} \cdots x_{i_l} \in \mathcal{G}(I) \rangle\ ;$$

- *the Stanley-Reisner complex*, whose faces correspond to squarefree monomials not belonging to I:

$$\Delta_{\mathcal{N}}(I) := \{\{i_i, \dots, i_l\} : x_{i_1} \cdots x_{i_l} \notin I\}\ .$$

Conversely, given a simplicial complex Δ on the vertices $\{1, \ldots, n\}$, we define two squarefree monomial ideals associated to Δ:

- *the facet ideal*, generated by squarefree monomials corresponding to facets:

$$I(\Delta) := (x_{i_1} \cdots x_{i_s} : \{i_1, \ldots, i_s\} \in \mathcal{F}(\Delta)) \ ;$$

- *the Stanley-Reisner ideal*, generated by squarefree monomials corresponding to non-faces:

$$I_\Delta := (x_{i_1} \cdots x_{i_s} : \{i_1, \ldots, i_s\} \notin \Delta) \ .$$

These applications satisfy $\Delta_{\mathcal{F}}(I(\Delta)) = \Delta$, $I(\Delta_{\mathcal{F}}(I)) = I$, $\Delta_{\mathcal{N}}(I_\Delta) = \Delta$ and $I_{\Delta_{\mathcal{N}}(I)} = I$, so we have two bijections between simplicial complexes and squarefree monomial ideals:

$$\{\text{Squarefree monomial ideals in } R\} \longleftrightarrow \{\text{Simplicial complexes on } [n]\}$$

$$\begin{array}{ccc} I & \longmapsto & \Delta_{\mathcal{F}}(I) \\ I(\Delta) & \longleftarrow & \Delta \end{array}$$

$$\begin{array}{ccc} I & \longmapsto & \Delta_{\mathcal{N}}(I) \\ I_\Delta & \longleftarrow & \Delta \end{array}$$

The second one is called the Stanley-Reisner correspondence and it is close related to the first one through Alexander duality:

$$I(\Delta) = I_{(\Delta^c)^\vee} \text{ and } \Delta_{\mathcal{F}(I)} = ((\Delta_{\mathcal{N}}(I))^c)^\vee \ .$$

The quotient ring $\mathbb{K}[x_1, \ldots, x_n]/I_\Delta$ is called *the Stanley-Reisner ring* and it is usually denoted by $\mathbb{K}[\Delta]$.

Now, we recall a celebrated way to relate invariants from the minimal resolution of a Stanley-Reisner ideal and the simplicial homology of the corresponding simplicial complex:

Theorem 1.6.3 (Hochster's Formula [42])**.** *Let I be a squarefree monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with the standard multigrading over $\mathbb{N}^n$ and let*

$\Delta = \Delta_{\mathcal{N}}(I)$ be the Stanley-Reisner complex of I on $[n]$. Let $\alpha \in \mathbb{N}^n$ and $i \geq 0$, then $\beta_{i,\alpha}(I) = 0$ if $\mathbf{x}^\alpha$ is not squarefree. Otherwise,

$$\beta_{i,\alpha}(I) = \dim_{\mathbb{K}} \tilde{H}_{|\alpha|-i-2}(\Delta[W])\,,$$

where $W := \{j \in [n] : \alpha_j = 1\}$. In particular, for the standard grading over $\mathbb{N}$, we have

$$\beta_{i,j}(I) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_{\mathbb{K}} \tilde{H}_{j-i-2}(\Delta[W])\,.$$

As a direct consequence, if $\Delta' < \Delta$, then $\beta_{i,j}(I_{\Delta'}) \leq \beta_{i,j}(I_\Delta)$ for all i, j. In particular, $\operatorname{reg}(I_{\Delta'}) \leq \operatorname{reg}(I_\Delta)$. Also, this formula allows us to relate the regularity of I_Δ with the k-Lerayness of Δ:

$$\operatorname{reg}(I_\Delta) = L_{\mathbb{K}}(\Delta) + 1\,,$$

and one can rewrite Theorem 1.4.10 in terms of regularities. The following result is the case $s = 2$.

Theorem 1.6.4 ([45, Theorem 1.4])**.** *Let Δ, Δ' simplicial complexes on the same vertex set. Then,*

- $\operatorname{reg}(I_\Delta + I_{\Delta'}) = \operatorname{reg}(I_{\Delta\cap\Delta'}) \leq \operatorname{reg}(I_\Delta) + \operatorname{reg}(I_{\Delta'}) - 1,$
- $\operatorname{reg}(I_\Delta \cap I_{\Delta'}) = \operatorname{reg}(I_{\Delta\cup\Delta'}) \leq \operatorname{reg}(I_\Delta) + \operatorname{reg}(I_{\Delta'}).$

Hochster's Formula was later rewritten in terms of links of the Alexander dual of Δ taking into account the following facts:

- $\operatorname{link}_{\Delta^\vee}(\sigma) \cong \Delta[V(\Delta) \setminus \sigma]^\vee$ as simplicial complexes on the vertex set $V(\Delta) \setminus \sigma$;
- $\tilde{H}_i(\Delta) \cong \tilde{H}^{|V|-3-i}(\Delta^\vee)$ (see [12, Lemma 5.5.3]);
- $\tilde{H}_i(\Delta) \cong \tilde{H}^i(\Delta)$ since $\mathbb{K}$ is a field and they are finite-dimensional vector spaces (see [54, Theorem 53.5]).

Proposition 1.6.5 ([19, Proposition 1])**.** *Let I be a squarefree monomial ideal and Δ the Stanley-Reisner complex associated to I. Then,*

$$\beta_{i,j}(I) = \sum_{\substack{\sigma \in \Delta^\vee \\ |\sigma| = |V(\Delta)| - j}} \dim_{\mathbb{K}} \tilde{H}_{i-3}(\operatorname{link}_{\Delta^\vee}(\sigma))\,.$$

1.7 Edge Ideals and Independence Complexes

Finally, we focus on the algebraic and combinatorial objects that this work is concerned with: edge ideals and independence complexes associated to graphs.

Given a graph G on the vertices $[n]$, we define *the edge ideal* associated to G as the squarefree monomial ideal $I(G) = (x_i x_j : \{i,j\} \in E(G)) \subset R = \mathbb{K}[x_1, \ldots, x_n]$. Also we can associate to G two simplicial complexes:

- *the clique complex*, Δ_G, whose faces are the cliques of G, and
- *the independence complex*, $\Delta(G)$, whose faces are the independent sets of vertices of G.

They are directly related to each other by the identity $\Delta(G) = \Delta_{G^c}$ and both of them are simplicial complexes belonging the family of so-called flag complexes.

A *flag* complex is a simplicial complex Δ verifying that if $\sigma \subset V(\Delta)$ and every pair of elements in σ is a face of Δ, then $\sigma \in \Delta$. In particular, a flag complex containing all pairs of vertices is necessarily a simplex. A simplicial complex Δ is flag if and only if Δ is the clique complex of the graph $\Delta^{(1)}$, so the set of independence complexes, clique conplexes and flag complexes are the same.

Notice that a subset $\sigma \subset V(G)$ is not in $\Delta(G)$ if and only if exists a pair of elements $i, j \in \sigma$ such that $\{i,j\} \in E(G)$. Thus, a monomial $m = x_{i_1} \cdots x_{i_s} \in R$ belongs to the Stanley-Reisner ideal associated to the independence complex of G if and only if there are two indices i_p, i_q such that $\{i_p, i_q\} \in E(G)$, i.e., m is divisible by a product $x_i x_j$ with $\{i,j\} \in E(G)$, which are the generators of $I(G)$. So

$$I(G) = I_{\Delta(G)} .$$

Hence, we can reformulate Hochster's Formula when applied to edge ideals:

Proposition 1.7.1 ([58, Proposition 1.2])**.** *Let G be a graph. Then, for all $i, j \geq 0$,*

$$\beta_{i,j}(I(G)) = \beta_{i,j}(I_{\Delta(G)}) = \sum_{\substack{W \subset V(G) \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}(\Delta(G)[W]) .$$

Some properties of the independence complex of a graph G can be reformulated in terms of G.

- $\Delta(G)[W] = \Delta(G[W])$, $\forall v \in V(G)$.

- $\Delta(G)[W] \cap \Delta(G)[W'] = \Delta(G[W \cap W'])$, $\forall W, W' \subset (V(G))$.
- $\mathrm{del}_{\Delta(G)}(v) = \Delta(G \setminus v)$, $\forall v \in V(G)$.
- $\mathrm{star}_{\Delta(G)}(v) = \Delta(G[V(G) \setminus N(v)])$, $\forall v \in V(G)$.
- $\mathrm{link}_{\Delta(G)}(v) = \Delta(G[V(G) \setminus N[v]])$, $\forall v \in V(G)$.
- $\Delta(G) = \mathrm{star}_{\Delta(G)}(v) \cup (\bigcup_{u \in N_G(v)} \mathrm{star}_{\Delta(G)}(u))$, $\forall v \in V(G)$.
- If $e = \{u, v\} \in E(G)$, then $\Delta(G) = \mathrm{del}_{\Delta(G)}(u) \cup \mathrm{del}_{\Delta(G)}(v)$.
- If v is an isolated vertex in G, then $\Delta(G)$ is a cone with appex v.
- A vertex v in $\Delta(G)$ is dominated by $u \neq v$ if $N_G(v) \subset N_G(u)$, or equivalently, if $N_{G^c}[u] \subset N_{G^c}[v]$ ($u \in \mathrm{link}_\Delta(V)$ implies u and v are not adjacent in G).

And, in some cases, it is known how transformations on the graph G affect to the independence complex:

- If G' is obtained from G by adding a whisker at the vertex v, then
$$\tilde{H}_i(\Delta(G')) \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta(G)}(v)),\ \forall i > 0\,.$$
- If G' is obtained from G by *adding an ear* at the edge $\{u, v\} \in E(G)$, i.e., $G' = (V(G) \sqcup \{z\}, E(G) \cup \{\{u, z\}, \{v, z\}\})$, then
$$\tilde{H}_i(\Delta(G')) \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta(G)}(u)) \oplus \tilde{H}_{i-1}(\mathrm{link}_{\Delta(G)}(v)),\ \forall i > 0\,.$$
- ([52, Claim 3.1]) If $e = \{u, v\} \in E(G)$ and we denote by $G' := G[V(G) \setminus N_G[e]]$, then there exists a long exact sequence
$$\begin{aligned} \cdots \longrightarrow \tilde{H}_{i-1}(\Delta(G')) \longrightarrow \tilde{H}_i(\Delta(G)) \longrightarrow \tilde{H}_i(\Delta(G \setminus e)) \longrightarrow \\ \longrightarrow \tilde{H}_{i-2}(\Delta(G')) \longrightarrow \tilde{H}_{i-1}(\Delta(G)) \longrightarrow \tilde{H}_{i-1}(\Delta \setminus e) \longrightarrow \cdots \end{aligned} \quad (1.4)$$
- Likewise, if $\{u, v\} \notin E(G)$ and we denote by $G_1 := (V(G), E(G) \cup \{\{u, v\}\})$ and $G' := G[V(G) \setminus N_G[\{u, v\}]]$, then there exists a long exact sequence
$$\begin{aligned} \cdots \longrightarrow \tilde{H}_{i-1}(\Delta(G')) \longrightarrow \tilde{H}_i(\Delta(G_1)) \longrightarrow \tilde{H}_i(\Delta(G)) \longrightarrow \\ \longrightarrow \tilde{H}_{i-2}(\Delta(G')) \longrightarrow \tilde{H}_{i-1}(\Delta(G)) \longrightarrow \tilde{H}_{i-1}(\Delta \setminus e) \longrightarrow \cdots \end{aligned} \quad (1.5)$$
since $G_1[V(G) \setminus N_{G_1}[\{u, v\}]] = G[V(G) \setminus N_G[\{u, v\}]]$.

- ([24, Lemma 2.5]) Let $v \in V(G)$ such that $G[N_G(v)]$ is a complete graph, then

$$\Delta(G) \cong \bigvee_{u \in N_G(v)} \sum \mathrm{link}_{\Delta(G)}(u)\,.$$

- ([16, Theorem 6]) Let $G_2 := (V(G) \cup E(G), \{\{v, e\} \in V(G) \cup E(G) : v \in e\})$, then

$$\Delta(G_2) = \sum \Delta(G)^{\vee}\,.$$

Example 1.7.2. We can see mK_2 as the graph obtained by adding a whisker to $G = (m-1)K_2 \cup \{\{u\}, \emptyset\}$ at the vertex u. So

$$\begin{aligned} \tilde{H}_i(\Delta(mK_2)) &\cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta(mK_2)}(u)) \cong \tilde{H}_{i-1}(\Delta((m-1)K_2)) \cong \\ &\cdots \\ &\cong \tilde{H}_{i-m+1}(\Delta(K_2)) \cong \begin{cases} \mathbb{K} \text{ if } i-m+1=0 \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

We can also obtain the same result using the Example 1.4.5.

Chapter 2

Betti Diagrams of Edge Ideals

In this chapter we study graded Betti numbers of edge ideals looking for particular values but also paying attention to the shape of the whole set of nonzero entries in the Betti diagram. In particular, we provide formulas for all the graded Betti numbers of the edge ideal associated to the complement of a cycle, and prove that the Betti diagram has a left-justified staircase shape. Fröberg's characterization of edge ideals with regularity 2 will be refined by determining the multidegree of the first nonlinear syzygies and how many there are. This improvement is also extended to the non-squarefree case. Most of the results obtained in this section are published in [29].

2.1 Betti Diagram Shape

Monomial ideals can have Betti diagrams with a great variety of shapes in general. Nevertheless, the Betti diagram of an edge ideal satisfies specific conditions. It follows from the definitions of edge ideal and Betti diagram that the first row is labeled by 2 and that the only nonzero entry in the first column is $\beta_{0,2}$, which is nothing else but the number of minimal generators of the edge ideal, or equivalently, the number of edges in the associated graph. We present below some more subtle properties.

Lemma 2.1.1 ([46, Lemma 2.2])**.** *Let $I(G)$ be an edge ideal. Then, $\beta_{i,j}(I(G)) = 0$ if $j > 2(i+1)$ and*

$$\beta_{i,2(i+1)}(I(G)) = |\{W \subset V(G) : G[W] \cong (i+1)K_2\}| \,.$$

Thus, all entries under the main diagonal passing through $\beta_{0,2}$ are zeros and the entries in that diagonal are determined by the number of induced matchings in the graphs. In particular,

$$\beta_{i,2(i+1)}(I(G)) = 0 \text{ if } i \geq \mu(G) \,.$$

We highlight three consequences of this result:

- the graded Betti numbers in the main diagonal, $\beta_{i,2(i+1)}(I(G))$, do not depend on the characteristic of the ground field $\mathbb{K}$,

- the regularity is bounded below:

$$\operatorname{reg}(I(G)) \geq \mu(G) + 1 \ ,$$

- the Betti diagram has a left-justified staircase shape up to the row $\mu(G)+1$ with steps of height one and length one.

	0	1	2	...	$\mu(G)-1$	...
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	...	...	...
3	-	$\beta_{1,4}$	$\beta_{2,5}$	...	...	...
4	-	-	$\beta_{2,6}$	...	...	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	...	...
$\mu(G)+1$	-	-	-	-	$\beta_{\mu(G)-1,2\mu(G)}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The next result states that the whole diagram has a left-justified staircase shape with steps of height one and that the length is strictly greater than one in rows r with $\mu(G)+1 < r \leq \operatorname{reg}(I(G))$.

Theorem 2.1.2. *The following two properties hold for any edge ideal $I = I(G)$.*

1. *If $\beta_{i,j}(I) = \beta_{i,j+1}(I) = 0$ with $i \geq 0$, then $\beta_{i+1,j+2}(I) = 0$.*

2. *Let $d > 2$. If $\beta_{i,i+d} = 0$, $\forall i \leq i_0$ with $i_0 \geq d-1$, then $\beta_{i,i+d+1} = 0$, $\forall i \leq i_0 + 2$.*

Proof. *1*. Let us suppose that $\beta_{i+1,j+2}(I) \neq 0$. Then, by Hochster's Formula (Theorem 1.6.3), there exists $W \subset V(\Delta)$ such that $|W| = j+2$ and $\dim_{\mathbb{K}} \tilde{H}_{j-i-1}(\Delta(G)[W]) > 0$. We denote by $\Delta := \Delta(G)[W]$. As Δ is a flag complex, there exist $u, v \in W$ such that $\{u,v\} \notin \Delta$ (otherwise, Δ would be a simplex and hence acyclic). Then, we have the following decomposition of Δ,

$$\Delta = \operatorname{del}_\Delta(u) \cup \operatorname{del}_\Delta(v)$$

with

$$\operatorname{del}_\Delta(u) \cap \operatorname{del}_\Delta(v) = \operatorname{del}_\Delta(\{u,v\})$$

and the corresponding Mayer-Vietoris sequence for the reduced homology vector spaces (note that $\mathrm{del}\Delta(\{u,v\}) \neq \emptyset$ since $|W| > 2$):

$$\cdots \longrightarrow \tilde{H}_{j-i-1}(\mathrm{del}_\Delta(\{u,v\})) \longrightarrow \begin{matrix} \tilde{H}_{j-i-1}(\mathrm{del}_\Delta(u)) \\ \oplus \\ \tilde{H}_{j-i-1}(\mathrm{del}_\Delta(v)) \end{matrix} \longrightarrow \tilde{H}_{j-i-1}(\Delta) \longrightarrow$$
$$\longrightarrow \tilde{H}_{j-i-2}(\mathrm{del}_\Delta(\{u,v\})) \longrightarrow \cdots .$$

Invoking Hochster's Formula again, we have

- $\tilde{H}_{j-i-1}(\mathrm{del}_\Delta(u)) = \tilde{H}_{j-i-1}(\mathrm{del}_\Delta(v)) = 0$ since $\beta_{i,j+1}(I) = 0$, and
- $\tilde{H}_{j-i-2}(\mathrm{del}_\Delta(\{u,v\})) = 0$ since $\beta_{i,j}(I) = 0$.

Hence, $\tilde{H}_{j-i-1}(\Delta[W]) = 0$ and we obtain a contradiction.

2. Let us suppose that $\beta_{i,i+d+1} \neq 0$ with $i > i_0 + 2$. Let $W \subset V(\Delta)$ such that $|W| = j + 2$ and $\dim_{\mathbb{K}} \tilde{H}_{d-1}(\Delta(G)[W]) > 0$ and denote $\Delta := \Delta(G)[W]$.

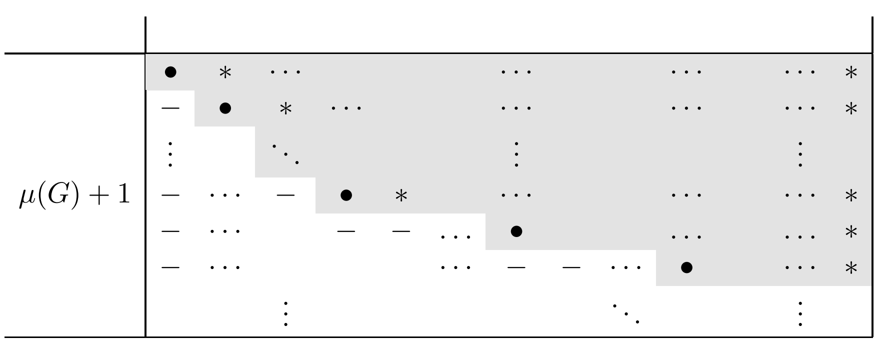
Let us consider the Mayer-Vietoris sequence (1.2) in Example 1.4.2 for an arbitrary vertex $v \in W$,

$$\cdots \longrightarrow \tilde{H}_{d-1}(\mathrm{del}_\Delta(v)) \longrightarrow \tilde{H}_{d-1}(\Delta) \longrightarrow \tilde{H}_{d-2}(\mathrm{link}_\Delta(v)) \longrightarrow \cdots .$$

Notice that $|V(\mathrm{link}_\Delta(v))| = |W| - |N_{G[W]}[v]|$. If $|N_{G[W]}[v]| = 1$, then Δ is acyclic because it is a cone or the appex v. On the other hand, $|N_{G[W]}[v]| = 2$, $\forall v \in W$ is equivalent to $G[W] \cong \frac{|W|}{2} K_2$ and, by Example 1.7.2, $\tilde{H}_{d-1}(\Delta(mK_2)) \neq 0$ only if $m = d$, but in that case, $2d = 2m = |W| = i + d + 1$, so $d - 1 = i > i_0 + 2$, that is a contradiction with $d - 1 \leq i_0$. Therefore, there must exists a vertex $v \in W$ with $|N_{G[W]}[v]| > 2$. Let $v \in W$ such a vertex, then

- $\tilde{H}_{j_0-i_0-3}(\mathrm{link}_\Delta(v)) = 0$ since $\beta_{i,i+j_0-i_0-1} = 0$, $\forall i \leq i_0 - 2$ and
- $\tilde{H}_{j_0-i_0-2}(\mathrm{del}_\Delta(v)) = 0$ since $\beta_{i_0,j_0} = 0$.

Then, $\tilde{H}_{j_0-i_0-2}(\Delta) = 0$, which is a contradiction with the condition defining Δ. □



$\bullet$ = nonzero entry ; $-$= zero entry ; $*$ = entry that may be zero or not.

There are also some partial results on the length of the steps in [71].

In particular, we can deduce from Theorem 2.1.2 *1*. the well known fact that $\beta_{i_0,i_0+2}(I(G)) = 0$ with $i_0 \geq 0$ implies $\beta_{i,i+2}(I(G)) = 0$, $\forall i \geq i_0$. It is not known, however, whether the following property holds for edge ideals:

$$\beta(I(G))_{i_0,i_0+d} \neq 0 \text{ and } \beta(I(G))_{i_0+1,i_0+1+d} = 0 \text{ implies } \beta_{i,i+d}(I(G)) = 0,\ \forall i \geq i_0. \tag{2.1}$$

Theorem 2.1.2.*1*. is equivalent to the following lemma if (2.1) is true, and a refinement otherwise. This reformulation was suggested by A. Conca.

Lemma 2.1.3 ([29, Lemma 5.3]). *Let denote* $i_d := \min_{1\leq i\leq p}\{i : \ \beta_{i,i+d} \neq 0\}$*,* $2 \leq d \leq \operatorname{reg}(I(G))$*, and assume that* $\operatorname{reg}(I(G) \geq 3$*. Then there exists* $i < i_d$ *such that* $\beta_{i,i+d-1} \neq 0$ *for all* d*,* $2 < d \leq \operatorname{reg}(I(G))$*.*

2.2 Linear Resolutions and Linear Strands

The simplest form for the Betti diagram of a nontrivial homogeneous ideal I is the case in which it consists of only one row. In that case, there exists an integer d such that $\beta_{i,j}(I) = 0$ if $j \neq i + d$. The entries in the column 0 of the Betti diagram, $\beta_{0,j}(I)$, correspond to the number of minimal generators of I of degree j and hence it is necessary that all minimal generators have the same degree in order to have a one-row Betti diagram. In addition to that, we need all the minimal generators of any free module F_i in the minimal resolution to have the same degree, which must be one unit bigger than the degree of all minimal generators of the previous free module F_{i-1}. If these two conditions hold for a homogeneous ideal I, we say that I has a *d-linear resolution.*

Let us assume now that I is an edge ideal. In this case, the following are equivalent.

- The Betti diagram of I consists of a unique row.
- I has a 2-linear resolution.
- $\operatorname{reg}(I) = 2$.

Using Hochster's Formula for computing the Betti numbers located in the linear strand requires to compute the dimension of the reduced homology vector spaces $\tilde{H}_0(\Delta(G)[W])$ for every subset W with $i + 2$ vertices. The advantage of this case lies in the facts that $\dim_{\mathbb{K}} \tilde{H}_0(\Delta, \mathbb{K}) = |\operatorname{comp}(\Delta)| - 1$, independently on whichever the field $\mathbb{K}$ is, and that $|\operatorname{comp}(\Delta(G)[W])| = |\operatorname{comp}(G^c[W])|$. So we can get purely combinatorial description for those graded Betti numbers.

Proposition 2.2.1 ([58, Proposition 2.1]). *Let G be a graph. Then, for all $i \geq 0$,*

$$\beta_{i,i+2}(I(G)) = \sum_{\substack{W \subset V(G) \\ |W| = i+2}} \left(|\mathrm{comp}(G^c[W])| - 1\right).$$

Unfortunately, the problem of counting connected components of subgraphs is not easy in general. A complete description of these graded Betti numbers exclusively in terms of the numerical data of the graph has only been achieved under some additional conditions.

Proposition 2.2.2 ([58, Proposition 2.4]). *Let G be a simple graph with no induced cycle of length 4. Then, for all $i \geq 0$,*

$$\beta_{i,i+2}(I(G)) = \sum_{v \in V(G)} \binom{\deg(v)}{i+1} - k_{i+2}(G),$$

where $k_{i+2}(G)$ is the number of cliques of size $i+2$ in G.

This result enables us to describe the Betti numbers located on the linear strand of the Betti diagrams of some families of graphs. The following examples are taken from [58].

Example 2.2.3. If G is a forest, then it has no cycle and $k_i(G) = 0$ if $i > 2$. So, $\beta_{0,2}(I(G)) = |E(G)|$ and

$$\beta_{i,i+2}(I(G)) = \sum_{v \in V_G} \binom{\deg(v)}{i+1}, \forall i > 0.$$

Example 2.2.4. Let us consider the complete graph on n vertices, K_n. Every cycle of length 4 in K_n has a chord, so none of them are induced. Moreover, every subset of $i+2$ vertices is a clique in K_n, so $k_{i+2}(K_n) = \binom{n}{i+2}$. Hence,

$$\begin{aligned}
\beta_{i,i+2}(I(K_n)) &= \sum_{v \in V_G} \binom{\deg(v)}{i+1} - k_{i+2}(K_n) \\
&= n\binom{n-1}{i+1} - \binom{n-1}{i+2} \\
&= (i+2)\binom{n}{i+2} - \binom{n-1}{i+2} \\
&= (i+1)\binom{n}{i+2}.
\end{aligned}$$

Example 2.2.5. In the case of complete bipartite graphs, $K_{m,n}$, we can not use the formula in Proposition 2.2.2 since they contain induced cycles of length 4 if $m, n \geq 2$. However, we can directly apply 2.2.1 as the number of connected components in $K^c_{m,n}[W]$ is 1 or 2, depending on whether W is contained in one set of the bipartition or not, respectively. Thus, we can write

$$\beta_{i,i+2}(I(K_{m,n})) = \binom{m+n}{i+2} - \binom{m}{i+2} - \binom{n}{i+2}.$$

We will see in section 2.4 that the formulas in 2.2.4 and 2.2.5 give a combinatorial description for all the nonzero graded Betti numbers of the edge ideals associated to those graphs since they have linear resolutions.

2.3 Complements of Cycles

We describe the entire Betti diagram of edge ideals associated to graphs in a particular family that will play a main role in Theorem 2.4.1. This family, the complementary graphs of cycles of length greather than or equal to 4, is the simplest one whose corresponding edge ideals do not have a 2-linear resolution in the sense that any graph whose edge ideal does not have a linear resolution contains an element of this family as an induced subgraph, as we will see in section 2.4.

We prove a couple of technical lemmas previously to give the combinatorial description of the graded Betti numbers in Theorem 2.3.3.

Lemma 2.3.1. *Let i, k, n be integers such that $0 < k \leq i < n$ and let C be a cycle of length n. Then, the number of induced subgraphs of C with i vertices and k connected components is $\frac{n}{k}\binom{i-1}{k-1}\binom{n-i-1}{k-1}$.*

Proof. Let us assume that $V = V(C) = [n]$. Given a subset S of V, the induced subgraph $C[S]$ can be represented by a vector w_S of length n whose l-th entry is 1 if $l \in S$ and 0 otherwise. Henceforth, we identify C_S with the vector w_S.

We use this identification to compute the number of induced subgraphs of C with i vertices and k connected components. Indeed, the number i of vertices in $G[S]$ is the number of nonzero entries in w_S and the number k of connected components of $G[S]$ is the number of runs of nonzero entries in w_S if the fist or the last entry in w_S is 0. In order to avoid distinguishing among cases when the vector w_S starts/ends with 1/0, we make an easy observation. Consider the set W of vectors w of length n with entries 0 or 1, whose first entry is 1 and last entry is 0, with i nonzero entries and k runs of nonzero entries (hence k runs of zero entries as well). To each w in W, we can correspond n subgraphs of C with i vertices and k connected components by assigning to the first entry of w_S one of the entries in w and preserving nearness. Conversely, each induced subgraph w_S of C with i vertices and k connected components can come from k vectors w in W depending on which of the connected components of w_S is the first run of w. This implies that the number of induced subgraphs of C with i vertices and k components is $\frac{n}{k} \times |W|$.

$$W = \left\{ \begin{array}{c} w \in \{0,1\}^n : w_1 = 1,\ w_n = 0 \\ \sum_{j=1}^n w_j = 1 \text{ and } k \text{ runs of 1's} \end{array} \right\} \qquad \{w_S : |G[S]| = i,\ |\text{comp}(G[S])| = k\}$$

$$(1, w_2, \ldots, w_{n-1}, 0) \longrightarrow \left\{ \begin{array}{c} \{1, w_2, \ldots, w_{n-1}, 0\} \\ \{w_2, \ldots, w_{n-1}, 0, 1\} \\ \vdots \\ \{0, 1, w_2, \ldots, w_{n-1}\} \end{array} \right.$$

$$\left. \begin{array}{c} \{w_i, w_{i+1}, \ldots, w_n, w_1, \ldots, w_{i-1}\} \\ \text{with} \\ w_i = 1 \text{ and } w_{i-1} = 0 \text{ if } i > 1 \\ w_1 = 1 \text{ and } w_n = 0 \text{ if } i = 1 \end{array} \right\} \longleftarrow \{w_1, \ldots, w_n\}$$

For computing $|W|$, we notice that a vector in W is uniquely determined by the length of each run (of 0's and of 1's) in the vector, so there is a bijection between W and the set L of vectors $(l_1^1, l_1^0, l_2^1, \ldots, l_k^1, l_k^0)$ with $l_j^0 > 0, l_j^1 > 0, \sum_{j=1}^k l_j^1 = i$ and $\sum_{j=1}^k l_j^0 = n - i$. Finally, choosing a vector in L is equivalent to choosing $k-1$ places to break a run of i elements into k pieces and $k-1$ places to break a run of $n-i$ elements into k pieces. Therefore, $|W| = \binom{i-1}{k-1}\binom{n-i-1}{k-1}$. □

Lemma 2.3.2. *For any two integers a and m such that $1 \le a < m$, one has the following identity:*

$$\sum_{k=1}^{a} \frac{k}{k+1} \binom{m-a}{k} \binom{a}{k} = \frac{a}{m-a+1} \binom{m}{a+1} .$$

Proof. Let F and g be the two polynomials in $\mathbb{Q}[X]$ defined as follows:

$F := (1+X)^a = \sum_{k=0}^a \binom{a}{k} X^k$, and

$g := (1+X)^{m-a} = \sum_{k=0}^{m-a} \binom{m-a}{k} X^k$.

Set $f := F'$ and $G := \int_0^x g(u)du$. Then

$$f = a(1+X)^{a-1} = \sum_{k=1}^{a} k \binom{a}{k} X^{k-1} = \sum_{k=1}^{a} k \binom{a}{k} X^{a-k}$$

where the last equality follows from the fact that X^{k-1} and X^{a-k} have the same coefficients in the polynomial $\sum_{k=1}^{a} k\binom{a}{k}X^{k-1}$ since $k\binom{a}{k} = k\frac{a!}{k!(a-k)!} = a\binom{a-1}{k-1}$ and $(a-k+1)\binom{a}{a-k+1} = (a-k+1)\frac{a!}{(a-k+1)!(k-1)!} = a\binom{a-1}{k-1}$. On the other hand,

$$G = \frac{(1+X)^{m-a+1}-1}{m-a+1} = \sum_{k=0}^{m-a} \frac{1}{k+1}\binom{m-a}{k}X^{k+1}.$$

Expressing the polynomial fG in two different ways, one gets that

$$\frac{a((1+X)^m-(1+X)^{a-1})}{m-a+1} = (\sum_{k=1}^{a} k\binom{a}{k}X^{a-k})(\sum_{k=0}^{m-a}\frac{1}{k+1}\binom{m-a}{k}X^{k+1})$$

and the desired identity now follows determining the coefficient of X^{a+1} in both sides of this equality. □

Theorem 2.3.3. *Let $n \geq 4$ and $I = I(C_n^c)$ be the edge ideal of the complementary graph of a cycle of length n. The minimal resolution of I is*

$$0 \longrightarrow R(-n) \longrightarrow R(-n+2)^{\beta_{n-4}} \longrightarrow \cdots \longrightarrow R(-2)^{\beta_0} \longrightarrow I \longrightarrow 0$$

where $\beta_i := n\frac{i+1}{n-i-2}\binom{n-2}{i+2}$ for all i, $0 \leq i \leq n-4$.

Proof. Let us assume $V = V(C_n^c) = [n]$ and denote $G = C_n^c$ and $\Delta = \Delta(C_n^c)$. We use Hochster's Formula

$$\beta_{i,j}(I) = \sum_{S\subseteq V\,;\ |S|=j} \dim_K \tilde{H}_{j-i-2}(\Delta[S]) \ , \quad \forall i,j \geq 0$$

to justify that, since Δ is a simplicial complex of dimension 1 (i.e. a graph) isomorphic to the cycle of length n, the only subcomplex of Δ with nontrivial homology in degree > 0 is Δ itself, whose reduced homology is $\mathbb{K}$ and is concentrated in degree 1. Thus, $\beta_{n-3,n} = 1$ and $\beta_{i,j} = 0$ for any other i,j such that $j > i+2$.

Now, we determine the linear strand of the Betti diagram using the formula given in Proposition 2.2.1:

$$\beta_{i,i+2} = \sum_{S\subseteq V\,;\ |S|=i+2} (|\mathrm{comp}(G[S]^c)| - 1) \ , \quad \forall i \geq 0\,. \tag{2.2}$$

Subgraphs with one component have no contribution in (2.2) so $\beta_{i,i+2} = 0$ if $i+2 \geq n-1$. As the number k of components of $G[S]^c$ satisfies $1 \leq k \leq i+2$ for all $S \subseteq V$ with $|S| = i+2$, one has that for all $i \geq 0$ such that $i+2 < n-1$,

$$\beta_{i,i+2} = \sum_{k=2}^{i+2}(\sum_{\substack{S\subseteq V\,;\\ |S|=i+2 \text{ and}\\ |\mathrm{comp}(G[S]^c)|=k}} (k-1)) = \sum_{k=2}^{i+2}(k-1)\,N(i+2,k) \tag{2.3}$$

where $N(i+2,k)$ is the number of induced subgraphs of $G[S]^c$ with $i+2$ vertices and k components.

Applying Lemma 2.3.1 in (2.3), one gets that, for all $i < n-3$,

$$\beta_{i,i+2} = \sum_{k=2}^{i+2}(k-1)\frac{n}{k}\binom{i+1}{k-1}\binom{n-i-3}{k-1} = n\sum_{k=1}^{i+1}\frac{k}{k+1}\binom{i+1}{k}\binom{n-i-3}{k}$$

and we are done applying the lemma 2.3.2 to $m = n-2$ and $a = i+1$. □

Remark 2.3.4. Using the Auslander-Buchsbaum formula, the above result implies that R/I is Cohen-Macaulay and then a Gorenstein ring of dimension two because its Cohen-Macaulay type is 1. The well-know symmetry of the graded Betti numbers when R/I is Gorenstein can be observed checking easily in our formula that $\beta_{n-4-i} = \beta_i$ for all i, $0 \leq i \leq n-4$.

We also point out that for this family, not only the graded Betti numbers in the linear strand are independent of the characteristic on the ground field $\mathbb{K}$ but also the entire minimal resolution, in particular, the projective dimension $\mathrm{pd}(I(C_n^c)) = n-3$ and the regularity $\mathrm{reg}(I(C_n^c)) = 3$.

Example 2.3.5. The ideal $I = (x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_8, x_3x_5, x_3x_6, x_3x_7, x_3x_8, x_4x_6, x_4x_7, x_4x_8, x_5x_7, x_5x_8, x_6x_8) \subset \mathbb{K}[x_1,\ldots,x_8]$ is the edge ideal of $G = (C_8)^c$. Then its Betti numbers can be computed using the formula in Theorem 2.3.3.

	0	1	2	3	4	5
2	20	64	90	64	20	-
3	-	-	-	-	-	1

2.4 Fröberg's Theorem

In [31], a characterization of edge ideals having 2-linear resolution was given in terms of the graph.

Theorem 2.4.1 (Fröberg)**.** *An edge ideal $I(G)$ has a linear resolution if and only if the graph G^c is chordal.*

Example 2.4.2. The edge ideals of complete graphs and complete bipartite graphs have 2-linear resolutions since K_n^c has no edge and $K_{m,n}^c$ consists of the disjoint union of K_m and K_n, that does not contain any (induced) cycle of length strictly greater than 3. Therefore, their Betti diagrams consist of a single row whose entries can be computed using the formulas in examples 2.2.4 and 2.2.5.

Example 2.4.3. The condition $C_l \not\leq G^c$ is equivalent to $C_l^c \not\leq G$ for any $l > 3$, so $I(C_l^c)$ does not have a 2-linear resolution as we saw in Theorem 2.3.3.

Fröberg's theorem was recovered in [22], where, in addition, the authors determined the first step in the resolution where nonlinear syzygies appear for the first time if the ideal does not have a linear resolution. In terms of the Betti diagram, they pointed out the first column with a nonzero entry outside the linear strand.

Theorem 2.4.4 ([22, Theorem 2.1])**.** *Let G be a graph such that $I(G)$ does not have a 2-linear resolution and r be the smallest integer $l \geq 4$ such that G^c has an induced cycle of length l. Then*

- $\beta_{i,j} = 0$ *if* $i < r-3$ *and* $j > i+2$;
- $\exists j \geq r$ *such that* $\beta_{r-3,j} \neq 0$.

	0	1	$\cdots$	$r-4$	$r-3$	$r-2$	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\cdots$	$\beta_{r-4,r-2}$	$\beta_{r-3,r-1}$	$\beta_{r-2,r}$	$\cdots$
3	-	-	$\cdots$	-	$*$	$\beta_{r-2,r+1}$	$\cdots$
4	-	-	$\cdots$	-	$*$	$\beta_{r-2,r+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$*$	$\vdots$	$\ddots$

We will now recover this result and improve it computing all the nonlinear syzygies at step $r-3$. Let us first recall a result about restricting minimal resolutions to certain multidegrees.

Let us consider R with its mulgrading over $\mathbb{Z}^n$. Let $\mathcal{F}$ be the minimal multigraded resolution of a monomial ideal I and $\alpha \in \mathbb{N}^n$ be a multidegree. Then, we denote by $\mathcal{F}_\alpha$ the subcomplex of $\mathcal{F}$ generated by the homogeneous minimal generators in $\mathcal{F}$ whose multidegrees are componentwise smaller than or iqual to α.

Theorem 2.4.5 ([32, Theorem 2.1])**.** *Let $I = (m_1, \ldots, m_s) \subset R$ be a monomial ideal, $\alpha \in \mathbb{N}^n$ and I_α the ideal generated by $\{m_i : m_i | \mathbf{x}^\alpha\}$. Then, $\mathcal{F}_\alpha$ is a minimal multigraded free resolution of the monomial ideal I_α.*

Let G be a graph such that $I(G)$ does not have a linear resolution. This means that there exists $\alpha \in \mathbb{N}^n$ such that $\beta_{i,\alpha}(I(G)) \neq 0$ for some $i > 0$ with $|\alpha| > i+2$. By Hochster's theorem, we know that $\alpha_j \in \{0,1\}$. Thus, a minimal generator of degree α in the i-th step of the minimal resolution $\mathcal{F}$ of $I(G)$ is also a minimal generator in $\mathcal{F}_\alpha$, the minimal resolution of $I(G)_\alpha$ by Theorem 2.4.5. In particular, $I(G)_\alpha$ does not have a 2-linear resolution. The ideal $I(G)_\alpha$ is generated by those minimal generators $x_k x_l$ such that $\alpha_k = \alpha_l = 1$, so $I(G)_\alpha = I(G[W])$ where $W := \{v_i \in V(G) : \alpha_i = 1\}$.

Fröberg's theorem assures that $G[W]^c$ has an induced cycle of length greater than or equal to 4. This induced cycle is necessarily also an induced cycle of G^c of length l, $4 \leq l \leq |\alpha|$.

On the other hand, if a graph G satisfies that $G^c[S]$ is a cycle for some $S \subset V(G)$ with $|S| > 3$, we take $\alpha_S = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i = 1$ if $v_i \in S$ and $\alpha_i = 0$ otherwise. Then, $I_\alpha = I(C^c_{|S|})$ and $\mathcal{F}_\alpha$ is the resolution described in Theorem 2.3.3, that must be contained in the minimal resolution $\mathcal{F}$ de $I(G)$. In particular, $\beta_{|S|-3,\alpha}(I(G)) = \beta_{|S|-3,\alpha}(I(G)_\alpha) = 1$.

Therefore, if an edge ideal $I(G)$ does not have a 2-linear resolution and we denote by i_0 the minimal i such that there exists $\alpha \in \{0,1\}^n$ with $\beta_{i,\alpha}(I(G)) \neq 0$ and $|\alpha| > i+2$, then we know that $|\alpha| = i_0 + 3$ by Theorem 2.1.2 and that there exists $W \subset V(G)$ such that $G^c[W]$ is a cycle of length $l \leq i_0 + 3$. If $l < i_0 + 3$, then, by the previous paragraph, $\beta_{|S|-3,\alpha_S}(I(G)) = 1$ for some $S \subset V(G)$ with $|S| = l < i_0 + 3$, which is a condradiction with the minimality of i_0. As a conclusion, $l = i_0 + 3$. Moreover, for each $W \subset V(G)$ with that property we have $\beta_{i_0,\alpha}(I(G)) = 1$. Thus,

$$\beta_{i_0,i_0+3} = |\{W \subset V(G) : G[W] \text{ is an induced cycle of length } i_0 + 3\}| \, .$$

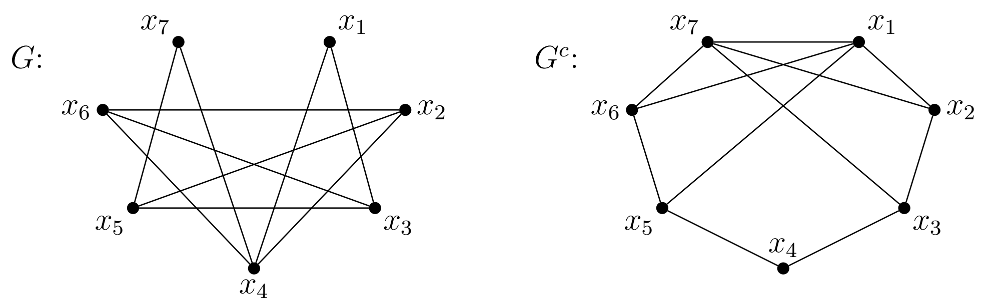
We collect all these facts in the following theorem.

Theorem 2.4.6. *Let $I = I(G)$ be an edge ideal with $reg(I) > 2$ and r be the minimal length of an induced cycle in G^c. Then*

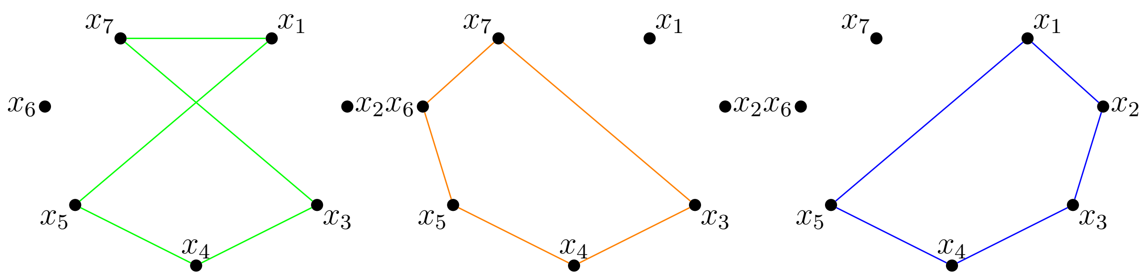
- *$\beta_{i,j} = 0$ if $i < r - 3$ and $j > i + 2$;*
- *$\beta_{r-3,j} = 0$ if $j > r$;*
- *$\beta_{r-3,r} = |\{$induced cycles in G^c of length $r\}|$. More precisely, if we consider the multigrading on R and let $\alpha \in \mathbb{N}^n$, then $\beta_{r-3,\alpha}(I(G)) = 1$ if $\alpha \in \{0,1\}^n$ and $G[\{v_i \in V(G) : \alpha_i = 1\}]^c$ is a cycle of length r and $\beta_{r-3,\alpha}(I(G)) = 0$ otherwise.*

	0	1	$\cdots$	r-4	r-3	r-2	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\cdots$	$\beta_{r-4,r-2}$	$\beta_{r-3,r-1}$	$\beta_{r-2,r}$	$\cdots$
3	-	-	$\cdots$	-	$\beta_{r-3,r}$	$\beta_{r-2,r+1}$	$\cdots$
4	-	-	$\cdots$	-	-	$\beta_{r-2,r+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example 2.4.7. Let $I = (x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_6, x_4x_7, x_5x_7)$ be the edge ideal of the graph G drawn below together with its complement.



Examining the complementary graph, we realize that there are no induced cycle of length 4 but there are 3 induced cycles of length 5.



Thus, the resolution is linear only up to the second step, F_2, which has 3 minimal generators of degree 5. If we compute the minimal resolution using any software system for Commutative Algebra like CoCoA, Macaulay2 or Singular, we obtain

$$0 \to R(-7) \to \begin{array}{c} R(-5) \\ \oplus \\ R(-6)^4 \end{array} \to \begin{array}{c} R(-4)^{11} \\ \oplus \\ R(-5)^3 \end{array} \to R(-3)^{19} \to R(-2)^{10} \to I \to 0$$

and the Betti diagram is

	0	1	2	3	4
2	10	19	11	1	-
3	-	-	3	4	1

2.5 Non-squarefree Case

In this section we omit the condition of squarefreeness for edge ideals and extend the results in section 2.4 using polarization. We get a correspondence between this class of ideals and the set of finite undirected graphs (now, loops are allowed).

Let I be an ideal in R generated by monomials of degree 2, $I = (m_1, \ldots, m_s)$. We assume without loss of generality that $m_1 = x_1^2, \ldots, m_l = x_l^2$, $m_{l+1} = x_{i_{l+1}} x_{j_{l+1}}, \ldots,$ $m_s = x_{i_s} x_{j_s}$ with $0 \leq l \leq s$ and $x_{i_k} \neq x_{j_k}$, $\forall k = l+1, \ldots, s$.

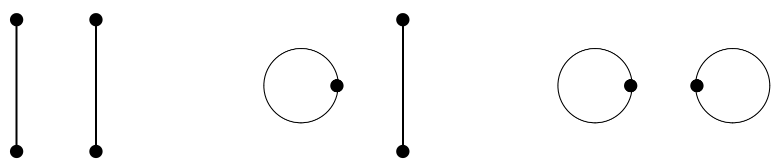
We introduced some specific terminology for this section and section 3.4. We define

- $I_{sqf} := (m_{l+1}, \ldots, m_s) \subset R$, and
- $I_{pol} := (x_1y_1, \ldots, x_ly_l, m_{l+1}, \ldots, m_s) \subset R^* := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_l]$,

and denote by

- G the (not necessarily simple) graph associated to I,
- G' the simple graph associated to I_{sqf},
- G^* the simple graph (on the vertices $\{x_1, \ldots, x_n, y_1, \ldots, y_l\}$) associated to I_{pol},
- $G^c := (G')^c$.

In this context, we say that two edges $e_1, e_2 \in E(G)$ are *totally disjoint* provided $\{u, v\} \notin E(G)$ if $u \in e_1$ and $v \in e_2$. Two totally disjoint edges correspond to one of the following configurations:



The graph G^* associated to I_{pol} is obtained from G by replacing every loop at a vertex v_i by a whisker at $v_i, i \leq l$, with a new vertex y_i of degree 1. Thus, totally disjoint edges in G correspond to induced subgraphs in G^* isomorphic to $2K_2$. Applying Lemma 2.1.1 to I_{pol}, we have that

$$\beta_{1,4}(I) = |\{\text{pairs of totally disjoint edges in } G\}|.$$

Therefore, given an ideal I generated by monomials of degree two, the following are equivalent:

1. $\beta_{1,4}(I) = 0$
2. G has no pair of totally disjoint edges.
3. ([22, Proposition 2.3 (a)]) $\beta_{1,4}(I_{sqf}) = 0$, if there are loops in G at v_i and v_j then $\{v_i, v_j\} \in E(G)$, and if there is a loop in G at v_i and $\{v_j, v_k\} \in E(G)$, then $\{v_i, v_j\} \in E(G)$ or $\{v_i, v_k\} \in E(G)$.

4. ([5, Lemma 4.28]) $d(\mathcal{L}(G)) \leq 2$.

We prove now that any induced cycle of length t in $(G^*)^c$ is also an induced cycle in G^c if $t > 4$.

Lemma 2.5.1. *Let H be a simple graph and $v \in V(H)$ with $\deg_H(v) = 1$. Then, v does not belong to any induced cycle of H^c of length $t \geq 5$.*

Proof. Let u be the neighbor of v in H. Consider $W \subset V(H)$ with $|W| = t$ and $v \in W$. Then,

$$\deg_{H[W]^c}(v) = \begin{cases} t-2 & \text{if } u \in W, \\ t-1 & \text{otherwise.} \end{cases}$$

Since $t-1$ and $t-2$ are strictly bigger that 2 if $t \geq 5$ and the degree of any vertex in a cycle is 2, the result follows. □

Thus, $(G^*)^c$ is chordal if and only if G^c has no induced cycle of length $l > 5$ and there are no pair of totally disjoint edges in G.

Theorem 2.5.2. *Let I be an ideal in R generated by monomials of degree 2 and G, G^c as above. Then I has a 2-linear resolution if and only if G has no totally disjoint edges and G^c has no induced cycle of length greater than or equal to 5.*

Proof.

$$\begin{aligned} \operatorname{reg}(I) = 2 &\Leftrightarrow \operatorname{reg}(I_{pol}) = 2 && \text{(by (1.3))} \\ &\Leftrightarrow (G^*)^c \text{ is chordal} && \text{(by Fröberg's theorem)} \\ &\Leftrightarrow \begin{cases} G \text{ has no totally disjoint edges} \\ G^c \text{ has no induced cycle of length } l \geq 5 \end{cases} && \text{(by Lemma 2.5.1)} \end{aligned}$$

□

In case that the regularity is strictly greater than 2, Lemma 2.5.1 implies the following result.

Theorem 2.5.3. *Assume that $\operatorname{reg}(I) > 2$. If G contains two totally dijoint edges, then*

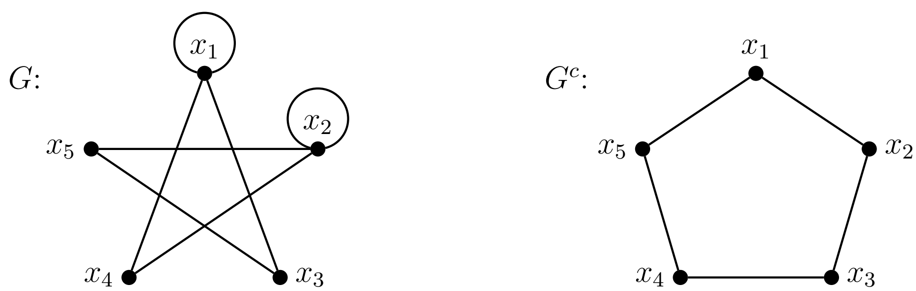
- $\beta_{1,4}(I) = |\{$*pairs of disjoint edges in* $G\}|$;
- $\beta_{1,j}(I) = 0$ *for all* $j > 4$;
- *Considering the* $\mathbb{N}^n$*-multigrading on* R, *for all* $\alpha \in \mathbb{N}^n$ *such that* $|\alpha| = 4$, *one has*

$$\beta_{1,\alpha}(I) = \begin{cases} 1 & \text{if } G[\{v_i \in V(G) : \alpha_i = 1\}] \text{ is a pair of totally disjoint edges} \\ & \text{and } \alpha \in \{0,1\}^n, \\ 0 & \text{otherwise}. \end{cases}$$

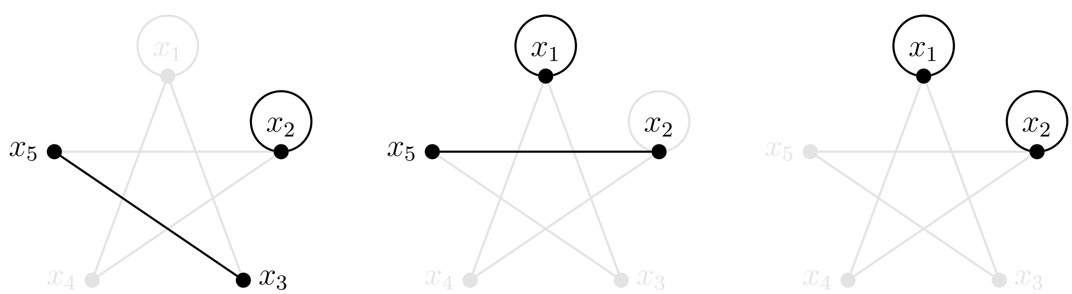
Otherwise, G^c contains a cycle of length greater than or equal to 5. Let t be the minimal length of such a cycle. Then,

- $\beta_{t-3,t}(I) = |\{W \subset V(G) : G^c[W] \cong C_t\}|$;
- $\beta_{t-3,j}(I) = 0$ *for all* $j > t$;
- *Considering the $\mathbb{N}^n$-multigrading on R, for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = t$, one has* $\beta_{t-3,\alpha}(I) = \begin{cases} 1 & \text{if } \alpha \in \{0,1\}^n \text{ and } G^c[\{v_i \in V(G) : \alpha_i = 1\}] \cong C_t \ , \\ 0 & \text{otherwise} \ . \end{cases}$

Example 2.5.4. We consider the ideal $I = (x_1^2, x_2^2, x_1x_3, x_3x_5, x_5x_2, x_2x_4, x_4x_1) \subset \mathbb{K}[x_1, \ldots, x_5]$. The complement G^c of G has no induced cycle of length 4.



However, there three pairs of disjoint edges in G.



Then $\beta_{1,4} = 3$, as we can see in its Betti diagram:

	0	1	2	3	4
2	7	9	2	-	-
3	-	3	8	5	1

Chapter 3

Betti Diagrams of Bipartite Edge Ideals

In this chapter, we deal with edge ideals associated to bipartite graphs, that we will call *bipartite edge ideals*. Several properties of this family of edge ideals have been studied like being Cohen-Macaulay or sequentially Cohen-Macaulay, having linear resolution and computing or bounding the regularity, the depth or the arithmetical rank. For example, we can find a graph theoretical characterization for Cohen-Macaulayness in [39], a topological one for sequentially Cohen-Macaulayness in [65], partial results determining the regularity, depth and arithmetical rank in [48], another partial result for the value of the regularity in [63], a proof of the characteristic dependence of homological invariants in [17]. Bipartite edge ideals having linear resolution are classified in [13].

We present here a characterization of bipartite edge ideals with regularity 3 as well as a combinatorial description of all the graded Betti numbers of a particular family, the edge ideals corresponding to bipartite complements of cycles of length $2s \geq 6$. Moreover, we study the relation between the regularity of a bipartite edge ideal and the induced matching number of the associated bipartite graph. We define a transformation on bipartite graphs that increases these invariants in a controlled way under certain conditions.

3.1 Bipartite Graphs and Biadjacency Matrices

We begin by recalling some basic properties of bipartite graphs. The reader can find the proofs and other properties in [37] or any other graph theory manual. The following classical characterization is due to by König:

Theorem 3.1.1. *A graph G is bipartite if and only if all cycles in G have even length.*

For any connected simple graph G with at least 2 vertices, the equalities $\alpha_0(G) + \beta_0(G) = |V(G)| = \alpha_1(G) + \beta_1(G)$ are satisfied (see [37, Theorem 10.1]). In the bipartite case, we have another relation between these numerical invariants.

Theorem 3.1.2 (König). *If G is a bipartite graph, then $\beta_1(G) = \alpha_0(G)$.*

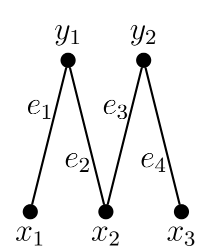
Another peculiarity of bipartite graphs comes from the matrix representation of graphs. We say that a matrix $M = (a_{i,j})$ is *binary* if $a_{i,j} \in \{0, 1\}, \forall i, j$, and we denote by $\mathcal{M}_{m\times n}(\{0, 1\})$ the set of all binary matrix with m rows and n columns. Given two matrices M, N, we denote by $M \sim N$ if one can be obtained from the other by permuting rows and columns.

Definition 3.1.3. Given a graph $G = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_s\})$, we can consider several matrices associated to G:

- the *adjacency matrix*, $M_0(G) = (b_{i,j}) \in \mathcal{M}_{n\times n}(\{0, 1\})$ where $b_{i,j} = 1$ if $\{v_i, v_j\} \in E(G)$ and 0 otherwise;
- the *incidence matrix*, $M_1(G) = (c_{i,j}) \in \mathcal{M}_{n\times s}(\{0, 1\})$ where $b_{i,j} = 1$ if $v_i \in e_j$ and 0 otherwise;
- for a bipartite graph G with $V(G) = \{x_1, \ldots, x_m\} \sqcup \{y_1, \ldots, y_n\}$, the *biadjacency matrix* $M(G) = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0, 1\})$ is defined by $a_{i,j} = 1$ if $\{x_i, y_j\} \in E(G)$ and $a_{i,j} = 0$ otherwise.

Remark 3.1.4. Notice that any matrix in $\mathcal{M}_{m\times n}(\{0, 1\})$ is the biadjacency matrix of a bipartite graph whereas it needs to be symmetric and with all entries in the main diagonal equal to 0 in order to be an adjacency matrix. Also, incidence matrices require that all the columns have exactly two entries equal to 1.

Example 3.1.5. Let G be the bipartite graph drawn below:



Then, the corresponding matrices defined above are:

$$M_0(G) = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & y_1 & y_2 \\ \hline x_1 & 0 & 0 & 0 & 1 & 0 \\ x_2 & 0 & 0 & 0 & 1 & 1 \\ x_3 & 0 & 0 & 0 & 0 & 1 \\ y_1 & 1 & 1 & 0 & 0 & 0 \\ y_2 & 0 & 1 & 1 & 0 & 0 \end{array} \qquad M_1(G) = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline x_1 & 1 & 0 & 0 & 0 \\ x_2 & 0 & 1 & 1 & 0 \\ x_3 & 0 & 0 & 0 & 1 \\ y_1 & 1 & 1 & 0 & 0 \\ y_2 & 0 & 0 & 1 & 1 \end{array} \qquad M(G) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1 & 0 \\ x_2 & 1 & 1 \\ x_3 & 0 & 1 \end{array}$$

We use biadjacency matrices to describe some properties of G and $\Delta(G)$. For that, we introduce some new notation. Let Δ be the independence complex of a bipartite graph G and M its biadjacency matrix. Then, we denote $M(\Delta) := M$, $G(M) := G$, and $\Delta(M) := \Delta$. If $W \subset V(G) = X \sqcup Y$ with $W \cap X \neq \emptyset$ and $W \cap Y \neq \emptyset$, then we denote $M[W] := M(G[W])$, the submatrix of M corresponding to the rows and columns labeled by vertices in W.

Let G be a bipartite graph with bipartition $V(G) = X \sqcup Y$. Then, we have the following immediate properties:

- $\sigma \in \Delta(G)$ if and only if $\sigma \subset X, \sigma \subset Y$ or $M(G)[\sigma]$ is a null matrix;
- the facets of $\Delta(G)$ are X, Y and maximal null submatrices of $M(G)$;
- $mK_2 < G$ if and only if exists $W \subset V(G)$ such that $M(G)[W] \sim \mathrm{Id}_m$, where Id_m is the identity matrix of size m.
- $\mu(G) = \max\{n : \exists W \subset V(G) \,/\, M(G)[W] \sim \mathrm{Id}_n\}$;
- if $e = \{i_0, j_0\} \in E(G)$, then $M(G \setminus e) = (a'_{i,j})$ where $a'_{i,j} = a_{i,j}$ if $(i,j) \neq (i_0, j_0)$ and $a'_{i_0,j_0} = 0$.

In particular, if $y_{j_0} \in Y$ (analogous identities for $x_{i_0} \in X$), $e \in E(G)$ and G has no isolated vertex, then

- $\mathrm{link}_{\Delta(G)}(y_{j_0}) = \Delta(M[W])$, where $W = \{x_i : a_{i,j_0} = 0\} \cup (Y \setminus \{y_{j_0}\})$, if $Y \neq \{y_{j_0}\}$. Otherwise, $\mathrm{link}_{\Delta(G)}(y_{j_0}) = \Delta_{\{x_i : a_{i,j_0} = 0\}}$.
- $\mathrm{star}_{\Delta(G)}(y_{j_0}) = \Delta(M[W])$, where $W = \{x_i : a_{i,j_0} = 0\} \cup Y$.
- $\mathrm{del}_{\Delta(G)}(y_{j_0}) = \Delta(M[W])$, where $W = X \cup (Y \setminus \{y_{j_0}\})$, if $Y \neq \{y_{j_0}\}$. Otherwise, $\mathrm{del}_{\Delta(G)}(y_{j_0}) = \Delta_X$.
- $\mathrm{fdel}_{\Delta}(G)(e) = \Delta(M(G \setminus e))$.

Example 3.1.6. Let G be the bipartite graph corresponding to the biadjacency matrix

$$M(G) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Then, we have

- $M(\mathrm{link}_{\Delta(G)}(y_1)) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$

- $M(\mathrm{star}_{\Delta(G)}(y_1)) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$

- $M(\mathrm{del}_{\Delta(G)}(y_1)) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccc} y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$

- $M(\mathrm{fdel}_{\Delta(G)}(\{x_1, y_1\})) = \begin{pmatrix} 1 \to 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$

Some of the results in section 1.7 on the reduced homology of the independence complex of a graph can be rewritten in terms of biadjacency matrices when the graph is bipartite.

Lemma 3.1.7. *Let* $(a_{i,j}) \in \mathcal{M}_{m\times n}(\{0,1\})$*. Then,*

1.
$$\tilde{H}_i(\Delta(\begin{pmatrix} 0 & \dots & 0 \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix})) = 0,\ \forall i \geq 0\,.$$

2.
$$\tilde{H}_i(\Delta(\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{m,2} & \dots & a_{m,n} \end{pmatrix})) \cong \tilde{H}_{i-1}(\Delta(\begin{pmatrix} a_{2,2} & \dots & a_{2,n} \\ \vdots & & \vdots \\ a_{m,2} & \dots & a_{m,n} \end{pmatrix})),\ \forall i \geq 0\,.$$

3. *If* $m > 1$,
$$\tilde{H}_i(\Delta(\begin{pmatrix} 1 & \dots & 1 \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix})) \cong \tilde{H}_i(\Delta(\begin{pmatrix} a_{2,1} & \dots & a_{2,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix})),\ \forall i \geq 0\,.$$

4.
$$\tilde{H}_i(\Delta(\begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & a_{2,2} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & \dots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix})) \cong \tilde{H}_i(\Delta(\begin{pmatrix} 0 & a_{2,2} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & \dots & a_{3,n} \\ \vdots & \vdots & & \vdots \\ a_{m,2} & a_{m,2} & \dots & a_{m,n} \end{pmatrix})),\ \forall i \geq 0\,.$$

5.
$$\tilde{H}_i(\Delta(\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & * & \dots & * & 1 & \dots & 1 \\ a_{3,1} & \dots & & & \dots & & & \dots & a_{3,n} \\ \vdots & & & & & & & & \vdots \\ a_{m,1} & \dots & & & \dots & & & \dots & a_{m,n} \end{pmatrix})) \cong$$
$$\cong \tilde{H}_i(\Delta(\begin{pmatrix} 0 & \dots & 0 & * & \dots & * & 1 & \dots & 1 \\ a_{3,1} & \dots & & & \dots & & & \dots & a_{3,n} \\ \vdots & & & & & & & & \vdots \\ a_{m,1} & \dots & & & \dots & & & \dots & a_{m,n} \end{pmatrix})),\ \forall i \geq 0\,.$$

Proof. 1. The vertex x_1 is isolated in G and hence $\Delta(G)$ is a cone with apex x_1, so it is acyclic.

2. As $a_{1,1} = 1$, $\{x_1, y_1\} \in E(G)$, so $\{x_1, y_1\} \not\subset \sigma, \forall \sigma \in \Delta(G)$. Therefore, we can write $\Delta(G) = \mathrm{del}_{\Delta(G)}(x_1) \cup \mathrm{del}_{\Delta}(G)(y_1)$ and $\mathrm{del}_{\Delta(G)}(x_1) \cap \mathrm{del}_{\Delta(G)}(y_1) = \mathrm{del}_{\Delta(G)}(\{x_1, y_1\})$. We apply 1. to $\mathrm{del}_{\Delta(G)}(x_1)$ and $\mathrm{del}_{\Delta(G)}(x_1)$ and we have $\tilde{H}_i(\mathrm{del}_{\Delta(G)}(y_1)) = \tilde{H}_i(\mathrm{del}_{\Delta(G)}(x_1)) = 0$. Thus, the result follows from the Mayer-Vietoris sequence

$$\cdots \to \begin{array}{c}\tilde{H}_i(\mathrm{del}_{\Delta(G)}(x_1))\\ \oplus \\ \tilde{H}_i(\mathrm{del}_{\Delta(G)}(y_1))\end{array} \to \tilde{H}_i(\Delta(G)) \to \tilde{H}_{i-1}(\mathrm{del}_{\Delta(G)}(\{x_1, y_1\})) \to$$

$$\to \begin{array}{c}\tilde{H}_{i-1}(\mathrm{del}_{\Delta(G)}(x_1))\\ \oplus \\ \tilde{H}_{i-1}(\mathrm{del}_{\Delta(G)}(y_1))\end{array} \to \cdots .$$

3. and 4. are particular cases of 5.

5. $\mathrm{link}_{\Delta(G)}(x_1)$ is acyclic by 1., so we are done by Corollary 1.4.4.

□

Remark 3.1.8. If M is the biadjacency matrix of a bipartite graph G and we permute rows or columns or we transpose M, then the new matrix is the biadjacency matrix of a graph isomorphic to G and hence the previous result is still valid if the rows and columns with fixed entries are other than the first ones or if the role of rows is played by columns, and viceversa.

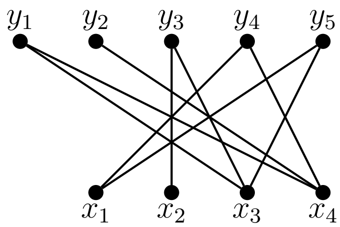
3.2 Bipartite Complement of an Even Cycle

In this section, we focus on the family of bipartite complements of cycles of even length, C_{2s}^{bc}, with $s \geq 3$. We first determine the induced matching number of C_{2s}^{bc} and then we compute all graded Betti numbers of its associated edge ideal.

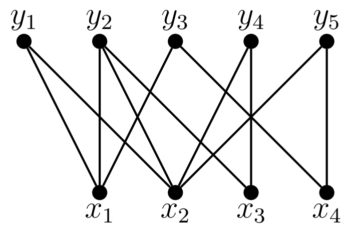
Definition 3.2.1. Given a bipartite graph $G = (X \sqcup Y, E)$, we define the *bipartite complement* of G, that is denoted by G^{bc}, as the bipartite graph on the same bipartition, $V(G^{bc}) = X \sqcup Y$, with edge set $E(G^{bc}) := \{e \in X \times Y : e \notin E(G)\}$.

Example 3.2.2. Let G be the bipartite graph on the vertex set $V = \{x_1, x_2, x_3, x_4\} \sqcup \{y_1, y_2, y_3, y_4, y_5\}$ with edges $E(G) = \{\{x_1, y_4\}, \{x_1, y_5\}, \{x_2, y_3\}, \{x_3, y_1\}, \{x_3, y_3\}, \{x_3, y_5\}, \{x_4, y_1\}, \{x_4, y_2\}, \{x_4, y_4\}\}$. We draw below G and G^{bc}.

G :



G^{bc} :

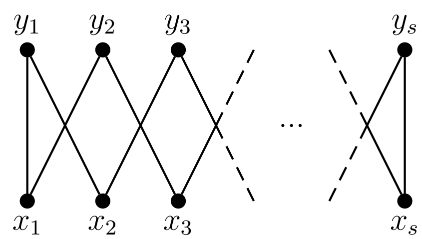


We consider a cycle of even length, C_{2s}, with $s \geq 3$. We adopt the labeling

$$V := V(C_{2s}) = X \sqcup Y = \{x_1, \ldots, x_s\} \cup \{y_1, \ldots, y_s\}, \text{ and}$$

$$E(C_{2s}) = \{\{x_1, y_1\}, \{x_1, y_2\}, \{x_2, y_1\}, \{x_2, y_3\}, \{x_3, y_2\}, \ldots, \{x_s, y_{s-1}\}, \{x_s, y_s\}\}.$$

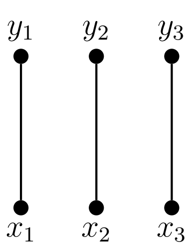
We draw C_{2s} with this labeling.



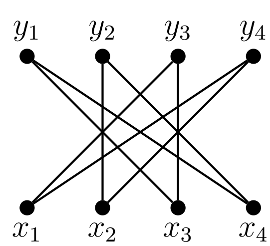
Notice that C_{2s} is a bipartite graph, so we can consider its bipartite complement, whose biadjacency matrix has the form

$$M(C_{2s}^{bc}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & & 1 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 1 & & 1 & 0 & 1 & 0 & 1 \\ 1 & \ldots & 1 & 1 & 0 & 1 & 0 \\ 1 & \ldots & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

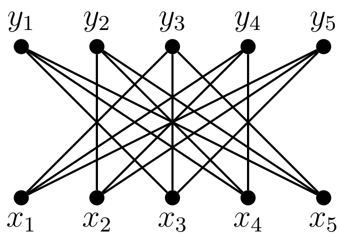
Example 3.2.3. For $s = 3, 4, 5, 6$, we have



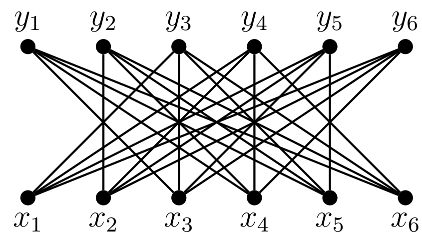
$C_6^{bc} \cong 3K_2$



$C_8^{bc} \cong C_8$



C_{10}^{bc}



C_{12}^{bc}

Lemma 3.2.4. *$\mu(C_6^{bc}) = 3$ and $\mu(C_{2s}^{bc}) = 2$ if $s > 3$.*

Proof. The first equality is deduced from the fact that $C_6^{bc} = 3K_2$. Let us now assume that $s > 3$. We have $3K_2 \not< C_{2s}^{bc}$ as $C_6 \not< C_{2s}$, so $\mu(C_{2s}^{bc}) < 3$. Moreover, $\mu(C_{2s}^{bc}) \geq 2$ since $C_{2s}^{bc}[\{x_1, x_3, y_1, y_2\}] \cong 2K_2$. □

All graded Betti numbers of $I(C_{2s}^{bc})$ can be described in a combinatorial way, as we will see in Theorem 3.2.15. This implies that they do not depend on the characteristic of the field $\mathbb{K}$. We will obtain the combinatorial formulas for the graded Betti numbers using Hochster's Formula. In order to use that formula, we need to compute the reduced homology of induced subcomplexes of $\Delta(C_{2s}^{bc})$. We start with the case of $\Delta(C_{2s}^{bc})$.

Proposition 3.2.5. $\tilde{H}_i(\Delta(C_{2s}^{bc})) = \tilde{H}_{i-1}(\Delta(\mathrm{Id}_2)) = \begin{cases} \mathbb{K} & \text{if } i = 2, \\ 0 & \text{otherwise} . \end{cases}$

Proof. We first prove that $\mathrm{del}_{\Delta(C_{2s}^{bc})}(x_1)$ is acyclic. $\mathrm{del}_{\Delta(C_{2s}^{bc})}(x_1)$ is the independence complex of $C_{2s}^{bc} \setminus x_1$ and its biadjacency matrix has the form

$$\begin{pmatrix} 0 & 1 & 0 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \\ 1 & \dots & 1 & 0 & 1 & 0 \\ 1 & \dots & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Applying Lemma 3.1.7.*4*, we have that $\tilde{H}_i(\mathrm{del}_{\Delta(C_{2s}^{bc})}(x_1)) \cong \tilde{H}_i(\mathrm{del}_{\Delta(C_{2s}^{bc})}(\{x_1, y_1\}))$. We can repeat this process to remove x_2, y_2, x_3, ... until the only remaining vertices are x_{n-1}, y_{n-1}, x_n, y_n. Then, $\tilde{H}_i(\mathrm{del}_{\Delta(C_{2s}^{bc})}(x_1)) \cong \tilde{H}_i(\Delta(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}))$, which is acyclic by Lemma 3.1.7.*1*.

Now, by Corollary 1.4.4, $\tilde{H}_i(\Delta(C_{2s}^{bc})) \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta(C_{2s}^{bc})}(x_1))$, $\forall i > 0$. After applying, if $s > 3$, Lemma 3.1.7.3, we get $\tilde{H}_{i-1}(\mathrm{link}_{\Delta(C_{2s}^{bc})}(x_1)) \cong \tilde{H}_{i-1}(\Delta(\mathrm{Id}_2))$. Hence, we conclude using Example 1.7.2. □

Hochster's Formula assures that $\beta_{i,j}(I(C_{2s}^{bc})) = 0$ if $j > |V(C_{2s}^{bc})| = 2s$. For the case $j = 2s$, Proposition 3.2.5 implies directly the following result.

Corollary 3.2.6. *Let $s \geq 3$. Then, $\beta_{2s-4,2s}(I(C_{2s}^{bc})) = 1$ and $\beta_{i,2s}(I(C_{2s}^{bc})) = 0$ if $i \neq 2s-4$.*

Now, we take $W \subsetneq V(C_{2s}^{bc}) = X \sqcup Y$. We set $W_X := W \cap X$ and $W_Y := W \cap Y$ and we assume that $W_X \neq \emptyset \neq W_Y$ since otherwise $\Delta(C_{2s}^{bc}[W])$ is a simplex and hence acyclic.

Let denote $M := M(G[W]) = (a_{i,j})$. Notice that every column and row has at most two entries equal to 0 and that some row or column does not reach that bound.

Let $w \in W$ be a vertex such that the corresponding column or row in M has no zero as an entry. Then $\tilde{H}_i(\Delta(G[W]) \cong \tilde{H}_i(\mathrm{del}_{\Delta(G[W])}(w))$ by Lemma 3.1.7*3*. The elimination of this vertex does not affect the number of zeros in the remaining columns or rows. We can remove all vertices corresponding to columns or rows with no zero as an entry. If there is no vertex remaining, then $a_{i,j} = 1, \forall i,j$. Otherwise, we can reduce the study to $M' := M(G[W'])$ where $W' := \{v \in W :$ not all entries in the corresponding column or row in M are 1's$\}$.

Now we focus on columns and rows of M' with exactly one zero as an entry and we pay attention to the number of connected components of the graph $C_{2s}[W']$, whose biadjacency matrix is $(\mathbf{1}_{m\times n} - M)[W']$. Let x be such a row (likewise for a column). $C_{2s}[W']$ has no isolated vertex since we have remove them in the previous part. We proceed according to these two possibilities:

1. if the only zero in the row x is also the unique zero in the corresponding column, then we keep x.

2. if there exists another zero in its column, we remove the vertex x and, by the remark 3.1.7.4, the new simplicial complex has the same reduced homology. The number of connected components in the graph $C_{2s}[W']$ does not change (the vertex we remove has degree 1 in $C_{2s}[W']$) and the number of zeros only varies in its column, which now has only one zero.

We repeat this step for any remaining vertex (after the previous steps) with only one zero in the corresponding column or row and denote by W'' the set of remaining vertices and $M'' := M(C_{2s}^{bc}[W''])$. Finally, we have that every column and row in M'' has exactly one entry equal to 0. Thus, after relabeling vertices if necessary, $M'' = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \in \mathcal{M}_{k\times k}(\{0,1\})$ where $k := |W''/2| = |comp(C_{2s}[W'])|$, and

$$\tilde{H}_i(\Delta(M)) \cong \tilde{H}_i(\Delta(M'')),\ \forall i \geq 0.$$

The biadjacency matrix of the bipartite graph $(kK_2)^{bc}$ has the form (up to permuting rows and columns)

$$M((kK_2)^{bc}) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \dots & y_k \end{array} \\ \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \end{array}.$$

We denote $\Delta^k := \Delta((kK_2)^{bc}) = \Delta(M'')$.

Lemma 3.2.7. $\dim_{\mathbb{K}}(\tilde{H}_i(\Delta^k)) = \begin{cases} k-1 & \textit{if } i = 1, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. For $i = 0$, Δ^k is connected since $\Delta^k[X] = \Delta_X$ and $\Delta^k[Y] = \Delta_Y$ are connected and they are connected each other by any edge $\{x_j, y_j\}, 1 \leq j \leq k$. Hence, $\dim_{\mathbb{K}}(\tilde{H}_0(\Delta^k)) = |comp(\Delta^k)| - 1 = 0$.

Let us assume $i \geq 1$. We notice that $\mathcal{F}(\Delta^k) = \{X, Y, \{x_1, y_1\}, \dots, \{x_k, y_k\}\}$ and consider the subcomplexes of Δ^k generated for every single facet, $A_x := \Delta_X, A_y := \Delta_Y, A_{z_1} := \langle\{x_1, y_1\}\rangle, \dots, A_{z_k} := \langle\{x_k, y_k\}\rangle$ and the family $\mathcal{A}; = \{A_i : i \in \mathcal{I}\}$ indexed by $\mathcal{I} := \{x, y, z_1, \dots, z_k\}$. Then, the nerve of $\mathcal{A}$ is the simplicial complex on the vertex set $\mathcal{I}$ whose $2k$ facets are $\{x, z_i\}$ and $\{y, z_i\}$ for every $i = 1, \dots, k$. Therefore, $N(\mathcal{A})$ is a suspention on the vertices x and y of the 0-dimensional simplicial complex $\langle\{x_1\}, \dots, \{x_k\}\rangle$. Thus, by the Nerve Theorem, $\dim_{\mathbb{K}}(\tilde{H}_i(\Delta^k)) = \dim_{\mathbb{K}}(\tilde{H}_i(\mathcal{N}(\mathcal{A}))) = \dim_{\mathbb{K}}(\tilde{H}_{i-1}(\langle\{x_1\}, \dots, \{x_k\}\rangle))) = \begin{cases} k-1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$ □

Collecting the previous information, we have the following result.

Proposition 3.2.8. *Let $W \subsetneq V(C_{2s}^{bc}) = X \sqcup Y$, we have the following trichotomy:*

- *If $W \cap X = \emptyset$ or $W \cap Y = \emptyset$ then $\dim_{\mathbb{K}}(\tilde{H}_i(\Delta(C_{2s}^{bc}[W])))) = 0,\ \forall i \geq 0$.*
- *If all entries in $M(C_{2s}^{bc})[W]$ are 1, then*

$$\dim_{\mathbb{K}}(\tilde{H}_i(\Delta(G[W]))) = \begin{cases} 1 & \textit{if } i = 0, \\ 0 & \textit{otherwise} . \end{cases}$$

- *Otherwise,*

$$\dim_{\mathbb{K}}(\tilde{H}_i(\Delta(G[W]))) = \begin{cases} k-1 & \textit{if } i = 1, \\ 0 & \textit{otherwise} . \end{cases}$$

where k is the number of connected components of $C_{2s}[W]$ that are not isolated vertices.

We can now start the description of all the graded Betti numbers of $I(C_{2s}^{bc})$ by the last row in the Betti diagram.

Proposition 3.2.9. *Let $s \geq 3$, then*

- $\beta_{i,i+4}(I(C_{2s}^{bc})) = \begin{cases} 1 & \text{if } i = 2s-4 \\ 0 & \text{otherwise} \end{cases}$
- $\beta_{i,j}(I(C_{2s}^{bc})) = 0$ *if* $j > i+4$,

As a consequence, $\operatorname{reg}(I(C_{2s}^{bc})) = 4$.

Proof. The equality $\beta_{2s-4,2s}(I(C_{2s}^{bc})) = 1$ was established in Corollary 3.2.6. Applying Hochster's Formula for $i < 2s-4$ and $j = i+4$, we get

$$\beta_{i,i+4} = \sum_{\substack{W \subset V(C_{2s}^{bc}) \\ |W| = i+4}} \dim_{\mathbb{K}}(\tilde{H}_2(\Delta(C_{2s}^{bc}[W])))\,.$$

As $|W| = i+4 < 2s$, then $W \subsetneq V(C_{2s}^{bc})$ and we can apply Proposition 3.2.8, that in particular, states that $\tilde{H}_i(\Delta(C_{2s}^{bc}[W])) = 0$ if $i \geq 2$.

The second part, $\beta_{i,j} = 0$ if $j > i+4$, can be deduced directly from the first part using Theorem 2.1.2 and the fact that $\beta_{i,j} = 0$ if $j > |V(C_{2s}^{bc})| = 2s$. □

In order to complete the description of the Betti diagram of $I(C_{2s}^{bc})$, we have to determine the entries on the first two rows, i.e., $\beta_{i,j}$ for $i+2 \leq j \leq i+3$.

We focus first on the linear strand. According to Hochster's Formula, Corollary 3.2.6 and Proposition 3.2.8, we needs to determine all the proper subsets W of $V(C_{2s}^{bc}) = X \sqcup Y$ such that $W_X := W \cap X \neq \emptyset$, $W_Y := W \cap Y \neq \emptyset$ and $(C_{2s}^{bc})^c[W]$ is not connected.

Let $s \geq 3$ and $W \subset V(C_{2s}^{bc})$. Notice that $(C_{2s}^{bc})^c = C_{2s} \cup K_X \cup K_Y$. Since $(C_{2s}^{bc})^c[W_X]$ and $(C_{2s}^{bc})^c[W_Y]$ are connected, if one of them is empty or they are connected to each other then $(C_{2s}^{bc})^c[W]$ is connected. Otherwise, $(C_{2s}^{bc})^c[W_X]$ and $(C_{2s}^{bc})^c[W_Y]$ are its connected components. Thus, $\beta_{i,i+2}(I(C_{2s}^{bc}))$ is the number of non-connected induced subgraphs of C_{2s}^{bc} with $i+2$ vertices. $(C_{2s}^{bc})^c[W]$ is not connected if and only if the two following conditions hold:

- $W_X \neq \emptyset$, $W_Y \neq \emptyset$,
- $N_{C_{2s}}(W_X) \cap W_Y = \emptyset$.

Let us denote by C_X the cycle on the vertex set X with edge set $E(C_X) = \{\{x_1,x_2\},\{x_1,x_3\},\{x_2,x_4\},\{x_3,x_5\},\ldots,\{x_{s-2},x_s\},\{x_{s-1},x_s\}\}$. Notice that

$$\{x_i,x_j\} \in E(C_X) \Leftrightarrow N_{C_{2s}}(x_i) \cap N_{C_{2s}}(x_j) \neq \emptyset\ .$$

Lemma 3.2.10. *Assume that $(C_{2s}^{bc})^c[W]$ is not connected.*

1. *There exists $x \in W_X$ such that $N_{C_{2s}}(x) \not\subset N_{C_{2s}}(W_X \backslash \{x\})$.*

2. *$|N_{C_{2s}}(W_X)| = |W_X| + |\mathrm{comp}(C_X[W_X])|$.*

Proof. If $x \in W_X$ and $N_{C_{2s}}(x) \subset N_{C_{2s}}(W_X \backslash \{x\})$, then $N_{C_X}(x) \subset W_X$. Thus, if $N_{C_{2s}}(x) \subset N_{C_{2s}}(W_X \backslash \{x\})$ for all $x \in W_X$, then $W_X = X$. In that case, $N_{C_{2s}}(W_X) = N_{C_{2s}}(X) = Y$ and $W_Y \neq \emptyset$ implies $N_{C_{2s}}(W_X) \cap W_Y \neq \emptyset$. Then, $(C_{2s}^{bc})^c[W]$ is connected and *1*. follows.

We prove *2*. by induction on $r := |W_X|$. If $W_X = \{x\}$ then $|N_{C_{2s}}(\{x\})| = 2$, $|\{x\}| = 1$, $|\mathrm{comp}((C_X)[\{x\}])| = 1$ and the statement holds. Consider now W such that $|W_X| = r > 1$ and assume that the statement holds for subsets W' such that $|W'_X| = r - 1$. By *1*., we know that there exists $x_0 \in W_X$ such that $N_{C_{2s}}(x_0) \not\subset N_{C_{2s}}(W_X \backslash \{x_0\})$. There are two possibilities:

- if x_0 is adjacent in C_X to some $x \in W_X \backslash \{x_0\}$, then $|\mathrm{comp}((C_X)[W_X])| = |\mathrm{comp}(C_X[W_X \backslash \{x_0\}])|$ and $N_{C_{2s}}(x_0) \cap N_{C_{2s}}(W_X \backslash \{x_0\}) \neq \emptyset$. Thus, $|N_{C_{2s}}(W_X)| = |N_{C_{2s}}(W_X \backslash \{x_0\})| + 1$;

- otherwise, we have that $|\mathrm{comp}(C_X[W_X])| = |\mathrm{comp}(C_X[W_X \backslash \{x_0\}])| + 1$ and $N_{C_{2s}}(x_0) \cap N_{C_{2s}}(W_X \backslash \{x_0\}) = \emptyset$. Therefore, $|N_{C_{2s}}(W_X)| = |N_{C_{2s}}(W_X \backslash \{x_0\})| + 2$.

In both cases, applying our inductive hypothesis, we get that

$$|N_{C_{2s}}(W_X)| = |W_X \backslash \{x_0\})| + |\mathrm{comp}(C_X[W_X])| + 1 = |W_X| + |\mathrm{comp}(C_X[W_X])|\,.$$

□

Proposition 3.2.11. 1. *For all $j \geq s$, $\beta_{j-2,j}(I(C_{2s}^{bc})) = 0$.*

2. *For $j = 2, \ldots, s-1$,*

$$\beta_{j-2,j}(I(C_{2s}^{bc})) = \sum_{k=1}^{j-1} \sum_{c=1}^{k} \frac{s}{c} \binom{k-1}{c-1} \binom{s-k-1}{c-1} \binom{s-k-c}{j-k}\,.$$

Proof. $\beta_{i,i+2}(I(C_{2s}^{bc}))$ is the number of induced subgraphs $(C_{2s}^{bc})^c[W]$ on $i+2$ vertices that are non connected. If $(C_{2s}^{bc})^c[W]$ is not connected then $|W_X| > 0$, $|W_Y| > 0$ and $|N_{C_{2s}}(W_X)| + |W_Y| \leq |Y| = s$. Thus, $|W_Y| \leq s - |N_{C_{2s}}(W_X)| < s - |W_X|$ by Lemma 3.2.10 *2*. and hence $|W| = |W_X| + |W_Y| < s$. Therefore, if $j = |W| \geq s$, $(C_{2s}^{bc})^c[W]$ is connected and $\beta_{j-2,j}(I(C_{2s}^{bc})) = 0$.

Now, for j with $2 \leq j \leq s-1$, we have to count how many subsets W of $X \sqcup Y$ with $|W| = j$ satisfy that $(C_{2s}^{bc})^c[W]$ is not connected. For each choice of W_X with k

elements ($1 \leq k \leq j-1$ in order to have $W_X \neq \emptyset$ and $W_Y \neq \emptyset$), we must choose $j-k$ elements from $Y \setminus N_{C_{2s}}(W_X)$ for W_Y, so there are $\binom{s-|N_{C_{2s}}(W_X)|}{j-k} = \binom{s-k-|\mathrm{comp}(C_X[W_X])|}{j-k}$ possible choices by Lemma 3.2.10 *2*. If we fix the number of connected components of $C_X[W_X]$ and denote it by c $1 \leq c \leq k$, then there are $\frac{s}{c}\binom{k-1}{c-1}\binom{s-k-1}{c-1}$ possible subsets W_X with $|W_X| = k$ and $|\mathrm{comp}(C_X[W_X])| = c$ by Lemma 2.3.1, and the result follows. □

Corollary 3.2.12. *The first and the last nonzero entries on the linear strand of the Betti diagram of $I(C_{2s}^{bc})$ coincide, i.e., $\beta_{s-3,s-1}(I(C_{2s}^{bc})) = \beta_{0,2}(I(C_{2s}^{bc}))$.*

Proof. For $j = s-1$ one has that $\binom{s-k-c}{j-k} \neq 0$ if and only if $c = 1$. In this case $\binom{k-1}{c-1} = \binom{s-k-1}{c-1} = \binom{s-k-c}{j-k} = 1$, and hence $\beta_{s-3,s-1}(I(C_{2s}^{bc})) = \sum_{k=1}^{s-2} s = s(s-2) = |E(C_{2s}^{bc})| = \beta_{0,2}(I(C_{2s}^{bc}))$. □

The description of the Betti diagram of $I(C_{2s}^{bc})$ will be complete once we give the Betti numbers located on the second row. This is our next result.

Proposition 3.2.13. *1. For all $j \geq 2s-1$, $\beta_{j-3,j}(I(C_{2s}^{bc})) = 0$.*

2. For $j = 4, \ldots, 2s-2$,

$$\beta_{j-3,j}(I(C_{2s}^{bc})) = \sum_{m=2}^{\lfloor j/2 \rfloor}(m-1)\sum_{a=0}^{j-2m}\frac{2s}{m}\binom{j-m-a-1}{m-1}\binom{2s-j-1}{m-1}\binom{2s-j-m}{a}.$$

Proof. By Proposition 3.2.8, $\tilde{H}_1(\Delta(C_{2s}^{bc}[W]))$ will contribute to Hochster's Formula for $\beta_{j-3,j}(I(C_{2s}^{bc}))$ if and only if W is a proper subset of V with $|W| = j \geq 4$ such that $C_{2s}[W]$ has at least 2 connected components that are not isolated vertices. More precisely, if we denote by $w(j,m)$ the number of proper subsets W of V with $|W| = j$ and such that $C_{2s}[W]$ has $m \geq 2$ connected components that are not isolated vertices, then

$$\beta_{j-3,j}(I(C_{2s}^{bc})) = \sum_{m=2}^{\lfloor \frac{j}{2} \rfloor}(m-1)w(j,m)\ . \tag{3.1}$$

In particular, since for any subset W of V with $2s-1$ elements, one has that $C_{2s}[W]$ is connected, *1.* follows.

Now, for $j \leq 2s-2$, we denote by $W(j,m,a)$ the set of proper subsets W of V with $|W| = j$ and such that $C_{2s}[W]$ has a isolated vertices and $m \geq 2$ connected components that are not isolated vertices. Then, $w(j,m) = \sum_{a=0}^{j-2m} w(j,m,a)$ where $w(j,m,a) = |W(j,m,a)|$, and we are reduced to prove that, for all possible j, m, a,

$$w(j,m,a) = \frac{2s}{m}\binom{j-m-a-1}{m-1}\binom{2s-j-1}{m-1}\binom{2s-j-m}{a}.$$

For determining the value of $w(j, m, a)$, we construct the set H of binary vectors of length $2s$ with the condition that the first element is 1, the last element is 0 and there are j 1's arranged in m runs of length greater than 1 and a runs of length 1.

We start with the set of binary vectors of length $2s - m - a$ whose first element is 1, whose last element is 0 and with $j - m - a$ 1's arranged in m runs. In the proof of [29, Lemma 3.3] is shown that this set has $\binom{j-m-a-1}{m-1}\binom{s-j-1}{m-1}$ elements. To each of those vectors, we add one 1 at the end of each run of 1's and a 1's between two 0's, one 1 each time. In each run of 0's of length l there are $l - 1$ places where we can insert a 1 between two 0's, so we generate $\binom{s-j-m}{a}$ vectors from every starting vector. Then, H has $\binom{s-j-m}{a}\binom{j-m-a-1}{m-1}\binom{s-j-1}{m-1}$ elements.

Now, to each element of $h \in H$ we can correspond $2s$ elements $\{v_i \in V : h_i = 1\}, \{v_i \in V : h_{i+1} = 1\}, \ldots, \{v_i \in V : h_{i+2s-1} = 1\}$ of $W(j, m, a)$. However, an element $W = \{v_{i_1}, \ldots, v_{i_j}\} \in W(j, m, a)$ is represented by m vectors in H: $\{h \in H : h_{i_k - c} = 1 \forall k = 1, \ldots, j\}$ where c is the smallest index in a connected component of $C_{2s}[W]$, Therefore, $w(j, m, a) = \frac{2s}{m}|H|$ and we are done. □

Corollary 3.2.14. *The first and the last nonzero entries on the second row of the Betti diagram of $I(C_{2s}^{bc})$ coincide, i.e., $\beta_{2s-5,2s-2}(I(C_{2s}^{bc})) = \beta_{1,4}(I(C_{2s}^{bc}))$.*

Proof. For $j = 2s - 2$, $\binom{2s-j-1}{m-1} \neq 0$ if and only if $m = 2$, and then $\binom{2s-j-m}{a} \neq 0$ if and only if $a = 0$, and hence $\beta_{2s-5,2s-2}(I(C_{2s}^{bc})) = \frac{2s}{2}\binom{2s-5}{1} = s(2s - 5)$. On the other hand, $\beta_{1,4}(I(C_{2s}^{bc})) = \frac{2s}{2}\binom{1}{1}\binom{2s-5}{1}\binom{2s-6}{0}$ and we are done. □

Now, we can describes completely the Betti diagram of $I(C_{2s}^{bc})$.

Theorem 3.2.15. *The edge ideal $I := I(C_{2s}^{bc})$ associated to the bipartite complement of acycle of length $t := 2s \geq 6$ has the following Betti diagram*

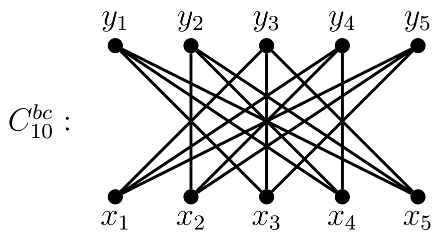
	0	1	...	$s-3$	...	$t-5$	$t-4$
2	$\beta_{0,2}$	$\beta_{1,3}$	...	$\beta_{s-3,s-1}$			
3		$\beta_{1,4}$	...	...	...	$\beta_{t-5,t-2}$	
4							1

where the nonzero entries are located inside the shadowed area and its values are

- $\beta_{j-2,j} = \sum_{k=1}^{j-1}\sum_{c=1}^{k}\frac{s}{c}\binom{k-1}{c-1}\binom{s-k-1}{c-1}\binom{s-k-c}{j-k}$, $j = 2, \ldots, s-1$;
- $\beta_{j-3,j} = \sum_{m=2}^{\lfloor j/2 \rfloor}(t\frac{m-1}{m}\binom{t-j-1}{m-1}\sum_{a=0}^{j-2m}\binom{j-m-a-1}{m-1}\binom{t-j-m}{a})$, $j = 4, \ldots, t-2$;
- $\beta_{t-4,t} = 1$.

In particular, $\operatorname{reg}(I) = 4$ *and* $\operatorname{pd}(I) = t - 4$.

Example 3.2.16. Let $I = (x_1y_3, x_1y_4, x_1y_5, x_2y_2, x_2y_4, x_2y_5, x_3y_1, x_3y_3, x_3y_5, x_4y_1, x_4y_2, x_4y_4, x_5y_1, x_5y_2, x_5y_3)$. I is the edge ideal associated to the bipartite compement of the cycle of length 10.



Then, the Betti diagram of $I = I(C_{10}^{bc})$ is

	0	1	2	3	4	5	6
2	15	30	15	-	-	-	-
3	-	25	100	140	90	25	-
4	-	-	-	-	-	-	1

3.3 Regularity 3

We begin this section recalling some results from [17] that we will need later.

Construction 3.3.1 ([17, Construction 4.4.])**.** Let Γ be a simplicial complex on $X = \{x_1, \dots, x_n\}$. Denote the number of facets of Γ by m. Let $G_j, 1 \leq j \leq m$ be such that for all $1 \leq j \leq m$, $X \backslash G_j$ is a face of Γ and such that every facet of Γ is of the form $X \backslash G_j$ for some j. Let $y_1, \dots, y_m$ be new vertices. Let Δ_X be the $(n-1)$-simplex on $x_1, \dots, x_n$. Define

$$\Delta' = \{\sigma \cup \tau : \sigma \in \Gamma, \tau \subset \{y_j : \sigma \subset (X \backslash G_j)\}\} \text{ and } \Delta = \Delta' \cup \Delta_X.$$

Let I be the Stanley-Reisner ideal of Δ, in the ring $R = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$. Let I_Γ denote the extension of the Stanley-Reisner ideal of Γ from the ring $\mathbb{K}[x_1, \dots, x_n]$ to R.

Theorem 3.3.2 ([17, Theorem 4.7])**.** *Let Γ and Δ as in Construction 3.3.1 Then for all $i \geq 0$, $\tilde{H}_{i+1}(\Delta, \mathbb{Z}) \cong \tilde{H}_i(\Gamma, \mathbb{Z})$.*

We will use Construccion 3.3.1 in the inverse way, i.e., given Δ, the independence complex of a bipartite graph, we look for a simplicial complex such that Γ with $\tilde{H}_{i-1}(\Gamma) \cong \tilde{H}_i(\Delta)$, $\forall i \geq 0$.

Let G be a bipartite graph with bipartition $V(G) = X \sqcup Y$ and $W \subset V(G)$. We use the notation $W_X = W \cap X$ and $W_Y = W \cap Y$.

Definition 3.3.3. We say that a subset of vertices $W \subset V(G)$ is *relevant* if no pair of vertices $u, v \in W$ satisfies $N_{G[W]}(u) \subset N_{G[W]}(v)$, or equivalently, if $\Delta(G)$ is a minimal complex.

Remark 3.3.4. If $W \subset V(G)$ is not relevant, then there exists $v \in W$ such that $\tilde{H}_i(\Delta(G[W]) \cong \tilde{H}_i(\mathrm{del}_{\Delta(G[W])}(v))$, by Lemma 3.1.7.5.

We define the set

$$\Gamma_G := \{\sigma \subset N_{G^{bc}}(y) : y \in Y\}$$

which is a simplicial complex on $X \setminus \{$ isolated vertices of G^{bc} in $X\}$.

Lemma 3.3.5. *Let G be a bipartite graph and $W \subset V(G)$ be a relevant subset of vertices. The simplicial complex obtained in Construction 3.3.1 from $\Gamma_{G[W]}$ is $\Delta(G)[W]$.*

Proof. Since W is relevant, no isolated vertex in $G[W]$ belongs to W_X, $\mathcal{F}(\Gamma_{G[W]}) = \{N_{G[W]^{bc}}(y) : y \in W_Y\}$ and $|F(\Gamma_G[W])| = |W_Y|$. Then, we define

$$\Delta' = \{\sigma \cup \tau : \sigma \in \Gamma_{G[W]}, \tau \subset \{y \in W_Y : \sigma \subset N_{G[W]^{bc}}(y)\}\}, \text{ and}$$

$$\Delta = \Delta' \cup \Delta_{W_X}$$

as in Construction 3.3.1.

Let $\sigma \subset W$. If $\sigma \subset W_X$, then $\sigma \in \Delta_{W_X}$ and $\sigma \in \Delta(G)[W]$. On the other hand,

$$\begin{aligned}
\sigma \in \Delta' \setminus \Delta_{W_X} &\Leftrightarrow \sigma_X \in \Gamma_{G[W]}, \sigma_Y \neq \emptyset \text{ and } \sigma_X \subset N_{G[W]^{bc}}(y)\,, \forall y \in \sigma_Y \\
&\Leftrightarrow \sigma_Y \neq \emptyset \text{ and } \sigma_X \subset N_{G[W]^{bc}}(y)\,, \forall y \in \sigma_Y \\
&\Leftrightarrow \sigma_Y \neq \emptyset \text{ and } \{x, y\} \notin E(G[W])\,, \forall x \in \sigma_X\,, \forall y \in \sigma_Y \\
&\Leftrightarrow \sigma \not\subset W_X \text{ and } \sigma \in \Delta(G)[W] \\
&\Leftrightarrow \sigma \in \Delta(G)[W] \setminus \Delta_{W_X}.
\end{aligned}$$

Thus, $\sigma \in \Delta \Leftrightarrow \sigma \in \Delta(G)[W]$. ☐

Lemma 3.3.6. *Let* $I := I_{\Gamma_{G[W]}} = (m_1, \ldots, m_s)$ *be the Stanley-Reisner ideal associated to the simplicial complex* $\Gamma_{G[W]}$.

- *If* W *is relevant then* $\deg(m_i) > 1, \forall i \in [s]$.
- $\max\{\deg(m_i) : i \in [s]\} \leq \mu(G[W])$.

Proof. Minimal generators of I correspond to minimal non-faces of $\Gamma_{G[W]}$ and their degrees are the cardinals of the corresponding non-face. For proving the first claim we only have to notice that if W is relevant then every vertex v in W_X is not isolated in $G[W]$ and hence it is also a vertex in $\Gamma_{G[W]}$. Then, non-faces can not be singletons.

If $m = x_{i_1} \cdots x_{i_d}$ is a minimal generator of I, then $x_{i_k} \in W_X$, $\frac{m}{x_{i_k}} \notin I$, $\forall k \in [d]$ and $\{x_{i_1}, \ldots, x_{i_d}\} \not\subset N_{G^{bc}[W]}(y)$, $\forall y \in W_Y$. Hence, $\forall k \in [d]$, $\{x_{i_l} : l \neq k\} \subset N_{G^{bc}[W]}(y(k))$ for some $y(k) \in W_Y$. Then, $x_{i_k} \notin N_{G^{bc}[W]}(y(k))$, or equivalently, $x_{i_k} \in N_{G[W]}(y(k))$ and $x_{i_l} \notin N_{G[W]}(y(k))$ if $l \neq k$. Therefore, $G[\{\{x_{i_k}, y(k)\} : k \in [d]\}] \cong dK_2$ and hence $d \leq \mu(G[W])$. □

In the sequel, we will assume without loss of generality that all bipartite graph whose graded Betti numbers we want to study are connected. For non-connected graphs, the graded Betti numbers of their edge ideals can be recovered from the graded Betti numbers of the edge ideals associated to their connected components (see [34, Remark 2.6] or [43, Corollary 2.2]).

Definition 3.3.7. A *Ferrers graph* is a bipartite graph G whose vertices can be relabeled on the bipartition $V(G) = \{x_1, \ldots, x_m\} \sqcup \{y_1, \ldots, y_n\}$ in such a way that $\{x_1, y_n\} \in E(G), \{x_m, y_1\} \in E(G)$ and if $\{x_i, y_j\} \in E(G)$ then $\{x_k, y_l\} \in E(G)$ for all $1 \leq k \leq i$ and $1 \leq l \leq j$.

The name of these graphs comes from the fact that their biadjacency matrices can be seen as Ferrers tableaux (up to permuting rows and columns). This means that Id_2 can not be a submatrix of the biadjacency matrix of a Ferrers graph, or equivalently, that the complement of a Ferrers graph has no induced cycle of length 4. Furthermore, the complement of a bipartite graph in general can not have an induced cycle of length $l > 4$ since otherwise at least three vertices would belong to one of the sets in the bipartition, so they would form a triangle. Therefore, Ferrers graphs are those bipartite graphs that have no induced cycle of length greater than or equal to 4 in their complementary graphs, that is the characterization given in 2.4.1 for graphs whose edge ideal has 2-linear resolution. Recall that this condition is equivalent to having regularity 2.

Theorem 3.3.8 ([13, Theorem 4.2])**.** *Let G be a bipartite graph without isolated vertices. Then, its edge ideal has a 2-linear free resolution if and only if G is a Ferrers graph.*

Now, we give a necessary condition for bipartite edge ideals to have regularity strictly greater than 3. Moreover, if the condition is satisfied, we can determine the first step in the minimal resolution where there is at least one syzygy contributing to a graded Betti number located outside the first two rows of the Betti diagram and compute the corresponding multigraded Betti numbers.

Theorem 3.3.9. *Let $I = I(G)$ be a bipartite edge ideal with $\operatorname{reg}(I) > 3$. Then, there exists an induced cycle in G^{bc} of length $2s$ for some $s \geq 3$. Moreover, if we set $t := min\{2s : s \geq 3 \text{ and } C_{2s} < G^{bc}\}$, then*

- *$\beta_{i,j} = 0$ if $i < t-4$ and $j > i+3$;*
- *if $t \geq 8$, $\beta_{i,j} = 0$ if $i \geq t-4$ and $j > t + \lfloor \frac{3(i-t+4)}{2} \rfloor$;*
- *$\beta_{t-4,t} = |\{\text{induced cycles of length } t \text{ in } G^{bc}\}|$;*
- *Considering the $\mathbb{N}^{m+n}$-multigrading on R, if $\alpha \in \mathbb{N}^{m+n}$ with $|\alpha| = t$ and we denote by $W := \{v_i \in V(G) : \alpha_i = 1\}$, then*

$$\beta_{t-4,\alpha} = \begin{cases} 1 & \text{if } \alpha \in \{0,1\}^{m+n} \text{ and } G^{bc}[W] \cong C_t \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Remark 3.3.10. If the following table is the Betti diagram corresponding to I then the first item tells us that all entries in the light gray area are zero and the second item tells that the same happens in the dark gray area (assuming $t \geq 8$).

	0	1	2	$\cdots$	$t-5$	$t-4$	$t-3$	$t-2$	$t-1$	$\cdots$	p
2	$\bullet$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$
3	$-$	$\bullet$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$
4	$-$	$-$	$-$	$\cdots$	$-$	$\beta_{t-4,t}$	$*$	$*$	$*$	$\cdots$	$*$
5	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$*$	$*$	$\cdots$	$*$
6	$-$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$-$	$-$	$\cdots$	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

$\bullet$ = nonzero entry ; $-$ = zero entry ; $*$ = entry that may be zero or not.

Proof. According to Theorem 2.1.2, if $\operatorname{reg}(I(G)) > 3$ then there exists i such that $\beta_{i,i+4}(I) \neq 0$. Let denote by i_4 the smallest index with this property. The same theorem allows us to reduce the proof of the first two points to prove that $i_4 = t-4$.

By Lemma 2.1.1, $i_4 \geq 2$ and if $i_4 = 2$ then $\beta_{2,6}(I)$ is the number of induced subgraphs of G isomorphic to $3K_2$, so we only have to notice that $(3K_2)^{bc} \cong C_6$. Then, $t = 6$ and $i_4 = t - 4$. Moreover, by Theorem 2.4.5, $\beta_{t-4,\alpha} = 1$ if and only if $\{v_i \in V(G) : \alpha_i = 1\} = V(3K_2)$, as desired.

For the case $i_4 \geq 3$, we use Hochster's Formula. $\beta_{i_4,i_4+4}(I) \neq 0$ implies that there exists $W \subset V(G)$ such that

$$|W| = i_4 + 4 \text{ and } \dim_{\mathbb{K}}(\tilde{H}_2(\Delta(G)[W]))) > 0\,. \tag{3.2}$$

We notice that W is a relevant subset of vertices because of the minimality of i_4. Our goal is to see that subsets $W \subset V(G)$ satisfying property (3.2) are exactly the subsets satisfying

$$G[W] \cong C^{bc}_{i+4}\,. \tag{3.3}$$

Let us consider $\Gamma := \Gamma_{G[W]} = \{\sigma \subset N_{G[W]^{bc}}(y) : y \in W_Y\}$ and $\Delta(\Gamma)$, the simplicial complex defined in Construction 3.3.1 for Γ. Applying Lemma 3.3.5 and Theorem 3.3.2, we have

$$\dim_{\mathbb{K}}(\tilde{H}_1(\Gamma)) = \dim_{\mathbb{K}}(\tilde{H}_2(\Delta(G)[W])) > 0\,.$$

Moreover, $\dim_{\mathbb{K}}(\tilde{H}_1(\Gamma[X'])) = 0$, $\forall X' \subsetneq W_X$ since $\Delta(\Gamma)[X'] \cong \Delta(G)[W']$, where $W' = X' \sqcup W_Y$, and the minimality of the size of W implies that $\dim_{\mathbb{K}}(\tilde{H}_1(\Gamma[X'])) = \dim_{\mathbb{K}}(\tilde{H}_2(\Delta(G)[W'])) = 0$. The fact that W is relevant also implies that Γ is a flag complex by Lemma 3.3.6. Therefore, we have $\Gamma = \Delta(G^*)$ for some graph G^* and Theorem 2.4.6 assure that $(G^*)^c$ has an induced cycle of length greater than or equal to 4. The same theorem together with Theorem 2.4.5 imply that no proper induced subgraph of $(G^*)^c$ contain any induced cycle with that length. Thus, $G^* \cong C^c_{|V(\Gamma)|}$. As $\Gamma = \Delta(C^c_{|V(\Gamma)|}) = C_{|V(\Gamma)|}$, we have $|N_{G[W]^{bc}}(y)| = 2$, $\forall y \in W_Y$. Since W is relevant, $\forall u, v \in W, u \neq v$, it holds $N_{G[W]^{bc}}(u) \not\subset N_{G[W]^{bc}}(v)$ and hence $|N_{G[W]^{bc}}(u)| > 1, \forall u \in W_X$. From the fact that $\sum_{u \in W_X} \deg_{G[W]^{bc}}(u) = \sum_{v \in W_Y} \deg_{G[W]^{bc}}(v)$, we also have that $|N_{G[W]^{bc}}(u)| \leq 2, \forall u \in W_X$. Therefore, $|N_{G[W]^{bc}}(y)| = 2$, $\forall y \in W$. Moreover, $G[W]^{bc}$ is connected because Γ is. Thus $G[W] \cong C^{bc}_{|W|}$ and we are done. ☐

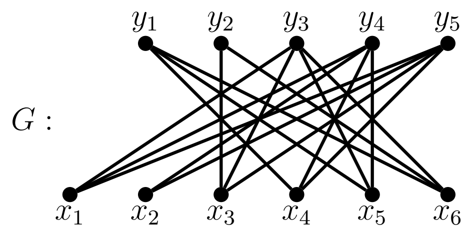
Finally, we give a combinatorial characterization of bipartite edge ideals having regularity 3.

Theorem 3.3.11. *Let $I = I(G)$ be a bipartite edge ideal. Then I has regularity 3 if and only if G^c has some induced cycle of length 4 and G^{bc} has no induced cycle of length greater than or equal to 6.*

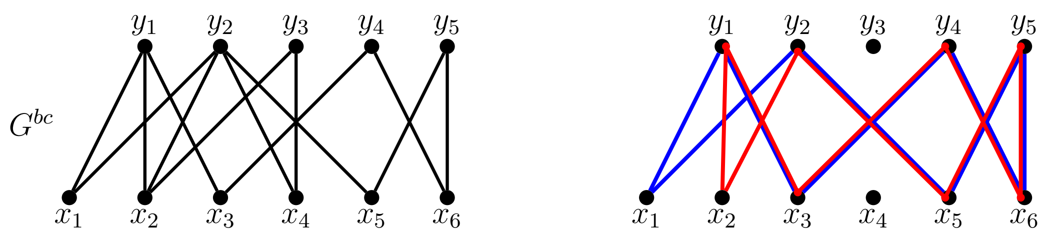
Proof. It only remains to prove that if G^{bc} has an induced cycle of length greater than or equal to 6 then I has regularity strictly greater than 3.

Let us assume that there exists $W \subset V(G)$ such that $G[W]^{bc} \cong C_t$ with $t \geq 6$. Then, $\beta_{i,j}(I) \geq \beta_{i,j}(I(G[W]))$ by Theorem 2.4.5. Hence, by 3.2.15, $\beta_{t-4,t}(I) \geq \beta_{t-4,t}(I(C^{bc}_t)) > 0$ and $\operatorname{reg}(I) \geq \operatorname{reg}(I(C^{bc}_t)) = 4$. ☐

Example 3.3.12. Let $I = (x_1y_3, x_1y_4, x_1y_5, x_2y_4, x_2y_5, x_3y_2, x_3y_3, x_3y_5, x_4y_1, x_4y_4, x_4y_5, x_5y_1, x_5y_3, x_5y_4, x_6y_1, x_6y_2, x_6y_3)$ the bipartite edge ideal associated to the graph



We look at its bipartite complementary graph searching for induced cycles of legnth greater than or equal to 6 and one can check that there are exactly two induced cycles of length 6.



And the Betti diagram of I is

	0	1	2	3	4	5	6	7
2	17	38	26	4	-	-	-	-
3	-	32	136	219	173	69	11	-
4	-	-	-	-	2	5	4	1

Remark 3.3.13. One can find in [46] several examples of edge ideals whose regularity is 3 or 4 depending on the characteristic of the field $\mathbb{K}$. This shows that a result analogous to Theorem 3.3.11 is not possible for edge ideals in general. That is why we restricted ourselves to bipartite edge ideals. But observe that even for bipartite edge ideals, it is hopeless to try a purely combinatorial characterization of bipartite edge ideals with a given regularity $r \geq 4$ in terms of the graph, as the following example shows.

Example 3.3.14 ([17, Example 4.8]). Let I be the bipartite edge ideal

$$(x_1y_1, x_2y_1, x_3y_1, x_7y_1, x_9y_1, x_1y_2, x_2y_2, x_4y_2, x_6y_2, x_{10}y_2, x_1y_3, x_3y_3, x_5y_3, x_6y_3, x_8y_3, x_2y_4, x_4y_4, x_5y_4, x_7y_4, x_8y_4, x_3y_5, x_4y_5, x_5y_5, x_9y_5, x_{10}y_5, x_6y_6, x_7y_6, x_8y_6, x_9y_6, x_{10}y_6)$$

obtained from Construction 3.3.1 applied to the the triangulation on six vertices of the projective plane, whose Stanley-Reisner ideal is known to have graded Betti numbers that depend on the characteristic of the ground field (see [57, Remark 3]).

Then, we have to possible Betti diagrams

	0	1	2	3	4	5	6	7	8	9	10	11
2	30	90	85	30	6	-	-	-	-	-	-	-
3	-	135	810	1875	2240	1470	500	75	-	-	-	-
4	-	-	90	740	2640	5270	6492	5166	2705	910	180	16

Betti diagram of $I \subset \mathbb{Q}[x_1, \ldots, x_{10}, y_1, \ldots, y_6]$

	0	1	2	3	4	5	6	7	8	9	10	11	12
2	30	90	85	30	6	-	-	-	-	-	-	-	-
3	-	135	810	1875	2240	1470	500	75	-	-	-	-	-
4	-	-	90	740	2640	5270	6492	5166	2705	910	180	16	1
5	-	-	-	-	-	-	-	-	-	-	-	1	-

Betti diagram of $I \subset \mathbb{Z}_2[x_1, \ldots, x_{10}, y_1, \ldots, y_6]$

3.4 Non-squarefree Case

We use the same notation as in section 2.5:

- G the (not necessarily simple) bipartite graph associated to I,
- G' the simple bipartite graph associated to I_{sqf}, and
- G^* the simple bipartite graph associated to I_{pol} where loops are replaced by whiskers.
- $G^{bc} := (G')^{bc}$.

We can also bring the characterization of bipartite edge ideals having regularity 3 to the non-squarefree case.

Theorem 3.4.1. *Let $I \subset R = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be an ideal generated by monomials $x_{i_k} y_{i_l}$ or squares of variables, then I has regularity 3 if and only if*

- *G has two totally disjoint edges or $C_l < G^c$ with $l \geq 5$,*
- *G does not have three edges pairwise totally disjoint, and*
- *G^{bc} has no induced cycle of length ≥ 8.*

Proof.

$$\begin{aligned} \operatorname{reg}(I) = 3 \quad &\Leftrightarrow \quad \operatorname{reg}(I_{pol}) = 3 \\ &\Leftrightarrow \quad \begin{cases} (G^*)^c \text{ has any induced cycle of length 4, and} \\ (G^*)^{bc} \text{ has no induced cycle of length} \geq 6 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} G \text{ has two totally disjoint edges or } C_l < G^c \text{ with } l \geq 5, \\ G \text{ does not have three edges pairwise totally disjoint, and} \\ G^{bc} \text{ has no induced cycle of length} \geq 8. \end{cases} \end{aligned}$$

□

Theorem 3.4.2. *Assume* $\operatorname{reg}(I) > 3$. *If G contains three pairwise totally dijoint edges, then*

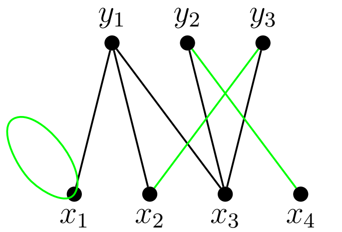
- *$\beta_{i,j} = 0$ if $i \leq 1$ and $j > i + 3$;*
- *$\beta_{2,6}$ is the number of induced subgraphs of G isomorphic to three pairwise totally disjoint edges;*
- *$\beta_{2,j} = 0$ for all $j > 6$;*
- *Considering the $\mathbb{N}^{m+n}$-multigrading on R, for all $\alpha \in \mathbb{N}^{m+n}$ such that $|\alpha| = 6$ we have $\beta_{2,\alpha} = \begin{cases} 1 & \text{if } G[\{x_i : \alpha_i = 1\}] \text{ consits of three totally disjoint edges;} \\ 0 & \text{otherwise .} \end{cases}$*

Otherwise, if we denote $t := min\{2s : s \geq 4 \text{ and } C_{2s} < G^{bc}\}$, then

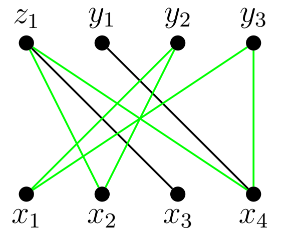
- *$\beta_{i,j} = 0$ if $i < t - 4$ and $j > i + 3$;*
- *if $t \geq 8$, $\beta_{i,j} = 0$ if $i \geq t - 4$ and $j > t + \lfloor \frac{3(i-t+4)}{2} \rfloor$;*
- *$\beta_{t-4,t} = |\{\text{induced cycles of length } t \text{ in } G^{bc}\}|$;*
- *Considering the $\mathbb{N}^{m+n}$-multigrading on R, if $\alpha \in \mathbb{N}^{m+n}$ with $|\alpha| = t$ and we denote by $W := \{v_i \in V(G) : \alpha_i = 1\}$, then*

$$\beta_{t-4,\alpha} = \begin{cases} 1 & \text{if } \alpha \in \{0,1\}^{m+n} \text{ and } G^{bc}[W] \cong C_t \,, \\ 0 & \text{otherwise .} \end{cases}$$

Example 3.4.3. Let consider the squarefree monomial ideal generated in degree two $I = (x_1^2, x_1x_5, x_2x_5, x_2x_7, x_3x_5, x_3x_6, x_3x_7, x_4x_6)$. If we look at its Betti diagram, $\beta_{2,6} = 1$. This is not because of the fact that there is an induced cycle of length 6 in G^{bc} but for the triple of pairwise disjoint edges in G.



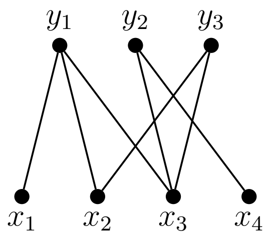
three pairwise disjoint edges in G



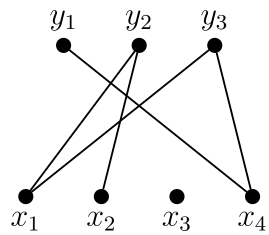
induced cycle of length 6 in $(G^*)^{bc}$

	0	1	2	3	4
2	8	10	3	-	-
3	-	7	12	5	-
4	-	-	1	2	1

Betti diagram of I



no triple of pairwise disjoint edges in G'



no induced cycle in $G^{bc} = (G')^{bc}$

	0	1	2	3
2	7	9	3	-
3	-	3	5	2

Betti diagram of I_{sqf}

3.5 Regularity and the Induced Matching Number

It seems that there exists a close relation between the regularity of an edge ideal and the induced matching number of the associated graph. It was noticed after Lemma 2.1.1 that the regularity of an edge ideal is bounded below by $\mu(G)+1$. The equality holds for some families of graphs, including particular subfamilies of bipartite graphs:

- forests ([74]),
- chordal graphs ([35]),
- weakly chordal graphs ([73]),
- unmixed bipartite graphs ([48]),
- sequentially Cohen-Macaulay bipartite graphs ([63]),
- Ferrer graphs ([13]).

Let us see a method to modify these invariants:

Construction 3.5.1. Given $M \in \mathcal{M}_{m\times n}(\{0,1\})$, we consider the following biadjacency matrices:

$$MM := \left(\begin{array}{c|c} M & M \end{array}\right) \qquad \overline{MM} := \left(\begin{array}{c|c} \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \\ \hline M & M \end{array}\right)$$

$$\underline{MM} := \left(\begin{array}{c|c} M & M \\ \hline \mathbf{0}_{1\times n} & \mathbf{1}_{1\times n} \end{array}\right) \qquad \overline{\underline{MM}} := \left(\begin{array}{c|c} \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \\ \hline M & M \\ \hline \mathbf{0}_{1\times n} & \mathbf{1}_{1\times n} \end{array}\right)$$

The matrix $\overline{\underline{MM}}$ is called the *setting* of M. If M is labeled by $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_n\}$, then we label its rows and columns as follows:

$$\begin{array}{c} \\ x_0 \\ \hline x_1 \\ \vdots \\ x_m \\ \hline x_{m+1} \end{array} \begin{array}{c} y_1 \ \cdots \ y_n \ \big| \ z_1 \ \cdots \ z_n \\ \left(\begin{array}{c|c} \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \\ \hline & \\ M & M \\ & \\ \hline \mathbf{0}_{1\times n} & \mathbf{1}_{1\times n} \end{array}\right) \end{array}$$

Proposition 3.5.2. *Let $M = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0,1\})$ with at least one 1 and one 0 in a row and denote $\mu := \mu(G(M))$. Then*

- $\mu(G(MM)) = \mu$,
- $\mu(G(\overline{MM})) = \mu(G(\underline{MM})) = \mu(G(\overline{\underline{MM}}))$,
- $\mu(G(\underline{MM})) = \begin{cases} \mu + 1 & \textit{if } \exists W \subset V(G(M)) \ / \ M[W] = \left(\begin{array}{c|c} \mathbf{0}_{\mu\times 1} & \mathrm{Id}_\mu \end{array}\right); \\ \mu & \textit{otherwise}. \end{cases}$

Proof. From the definition of MM, it is clear that $\mu(G(MM)) = \mu$. The equality $\mu(G(\overline{MM})) = \mu(G(\underline{MM}))$ follows from the fact that $\overline{MM} \sim \underline{MM}$. Suppose that $\mu(G(\underline{MM})) \neq \mu(G(\overline{\underline{MM}}))$. As there exists $W = \{x_{i_1}, y_{i_1}, y_{i_2}\}$ with $M[W] \sim \left(\begin{array}{cc} 1 & 0 \end{array}\right)$, then $\underline{MM}[\{x_{i_1}, x_{m+1}, y_{i_1}, z_{i_2}\}] \sim \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) = \mathrm{Id}_2$ and hence $\mu(G(\underline{MM})) \geq 2$. Therefore $\mu(G(\overline{\underline{MM}})) > \mu(G(\underline{MM})) \geq 2$. Let $W' \subset V(G(\overline{\underline{MM}}))$ with $G(\overline{\underline{MM}})[W'] \cong$

$\mu(G(\underline{\overline{MM}}))K_2$. Since $\mu(G(\underline{\overline{MM}})) \gneq \mu(G(\underline{MM})) = \mu(G(\overline{MM}))$, x_0 and x_{m+1} must be in W'. However, $\{x_0, x_{m+1}\} \subset W'$ with $\underline{\overline{MM}}[W'] \sim \mathrm{Id}_{\mu(G(\underline{\overline{MM}}))}$ for $\mu(G(\underline{\overline{MM}})) > 2$ is a contradiction since there are $|W' \cap Y|$ nonzero entries in the row x_0 and $|W' \cap Z|$ nonzero entries in the row x_{m+1} of the matrix $\underline{\overline{MM}}[W']$ and $|W' \cap Y| + |W' \cap Z| = \mu(G(\underline{\overline{MM}})) > 2$.

Let us assume now that $\mu(G(\underline{\overline{MM}})) = \mu(G(\overline{MM})) > \mu$. If $G(\overline{MM})[W] \cong \mu(G(\underline{\overline{MM}}))K_2$, then $x_0 \in W$. Hence, $MM[W \setminus \{x_0\}] \sim \left(\begin{array}{c|c} \mathbf{0}_{\mu\times 1} & \mathrm{Id}_{\mu(G(\overline{MM}))-1} \end{array} \right)$ and $\mu(G(\underline{\overline{MM}})) = \mu(G(\overline{MM})) \leq \mu(G(MM)) + 1 = \mu + 1$. Therefore, $\mu(G(\underline{\overline{MM}})) = \mu + 1$ and $M[W'] \sim \left(\begin{array}{c|c} \mathbf{0}_{\mu\times 1} & \mathrm{Id}_{\mu} \end{array} \right)$, where $W' := W_X \cup \{y_i : y_i \in W \text{ or } z_i \in W\}$.

Conversely, if there exists $W' = \{y_{i_0}, x_{i_1}, \ldots, x_{i_\mu}, y_{i_1}, \ldots, y_{i_\mu}\}$ such that $M[W'] \sim \left(\begin{array}{c|c} \mathbf{0}_{\mu\times 1} & \mathrm{Id}_{\mu} \end{array} \right)$, then $G(\overline{MM})[W] \sim \mathrm{Id}_{\mu+1}$ for $W := \{x_0, y_{i_0}, x_{i_1}, \ldots, x_{i_\mu}, z_{i_1}, \ldots, z_{i_\mu}\}$. Hence, $\mu(G(\underline{\overline{MM}})) = \mu(G(\overline{MM})) \geq \mu + 1$ and we are in the previous case, so $\mu(G(\underline{\overline{MM}})) = \mu + 1$. □

Proposition 3.5.3. $\tilde{H}_i(\Delta(\underline{\overline{MM}})) \cong \tilde{H}_{i-1}(\Delta(M))$, $\forall i \geq 0$.

Proof. We consider the Mayer-Vietoris sequence corresponding to the decomposition $\Delta(\underline{\overline{MM}}) = \mathrm{star}_{\Delta(\underline{\overline{MM}})}(x_0) \cup \mathrm{del}_{\Delta(\underline{\overline{MM}})}(x_0)$,

$$\begin{array}{l} \cdots \longrightarrow \tilde{H}_i(\mathrm{del}_{\Delta(\underline{\overline{MM}})}(x_0)) \longrightarrow \tilde{H}_i(\Delta(\underline{\overline{MM}})) \longrightarrow \tilde{H}_{i-1}(\mathrm{link}_{\Delta(\underline{\overline{MM}})}(x_0)) \longrightarrow \\ \longrightarrow \tilde{H}_{i-1}(\mathrm{del}_{\Delta(\underline{\overline{MM}})}(x_0)) \longrightarrow \cdots . \end{array}$$

By Lemma 3.1.7.*3*, $\tilde{H}_{i-1}(\mathrm{link}_{\Delta(\underline{\overline{MM}})}(x_0)) \cong \tilde{H}_{i-1}(\Delta(\underline{\overline{MM}}[X \cup \{x_{m+1}\} \cup Z\}])) \cong \tilde{H}_{i-1}(\Delta(\underline{\overline{MM}}[X \cup Z\}])) \cong \tilde{H}_{i-1}(\Delta(M))$.

It only remains to prove that $\tilde{H}_i(\mathrm{del}_{\Delta(\underline{\overline{MM}})}(x_0)) = 0$, $\forall i \geq 0$. We can apply Lemma 3.1.7.*5* to every column z_j since the positions of its null entries are positions of null entries in the column y_j. Then, $\tilde{H}_i(\mathrm{del}_{\Delta(\underline{\overline{MM}})}(x_0)) \cong \tilde{H}_i(\Delta(\underline{\overline{MM}}[X \cup \{x_{m+1}\} \cup Y]))$ and the claim follows applying Lemma 3.1.7.*1* to row x_{m+1}. □

Theorem 3.5.4. $\mathrm{reg}(I(G(\underline{\overline{MM}}))) = \mathrm{reg}(I(G(M))) + 1$.

Proof. Let $r := \mathrm{reg}(I(G(M)))$. By Hochster's Formula, there exists $W \subset V(G(M))$ such that $\tilde{H}_{r-2}(\Delta(M)[W]) \neq 0$. We take $\Delta(\underline{\overline{M[W]M[W]}}) < \Delta(\underline{\overline{MM}})$. Then, $\tilde{H}_{r-1}(\Delta(\underline{\overline{M[W]M[W]}})) \neq 0$, by Proposition 3.5.3. Therefore, $\mathrm{reg}(I(G(\underline{\overline{MM}}))) \geq r+1$.

Conversely, if $r' := \mathrm{reg}(I(G(\underline{\overline{MM}})))$, then there exists $W \subset V(G(\underline{\overline{MM}}))$ such that $\tilde{H}_{r'-2}(\Delta(\underline{\overline{MM}})[W]) \neq 0$. We may assume that $|W|$ is minimal for that property. We have necessarily that $W \not\subset V(G(M))$ since $\mathrm{reg}(I(G(M))) < r'$.

If $x_0 \in W$, then the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(\mathrm{del}_{\Delta(\overline{MM})[W]}(x_0)) \to \tilde{H}_i(\Delta(\overline{MM})[W]) \to \tilde{H}_{i-1}(\mathrm{link}_{\Delta(\overline{MM})[W]}(x_0)) \to \cdots$$

implies that $\tilde{H}_{i-1}(\mathrm{link}_{\Delta(\overline{MM})[W]}(x_0)) \neq 0$ as $\tilde{H}_i(\mathrm{del}_{\Delta(\overline{MM})[W]}(x_0)) = 0$ because of the minimality of $|W|$. If $x_{m+1} \in W$, we can delete it by Lemma 3.1.7.*3*. Then, we get $\tilde{H}_{i-1}(\Delta[W \cap (X \cup Y)]) \cong \tilde{H}_{i-1}(\Delta(\overline{MM})[W \cap (X \cup Z)]) \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta(\overline{MM})[W]}(x_0)) \neq 0$. Thus, $\mathrm{reg}(I(G(M))) \geq r' - 1$. Likewise if $x_{m+1} \in W$.

Finally, if $W \subset X \cup Y \cup Z$, then $\tilde{H}_i(\Delta(\overline{MM})[W]) \cong \tilde{H}_i(\Delta(\overline{MM})[W \cap (X \cup Y)])$ applying Lemma 3.1.7.*5* to the columns $z_j \in W$. Thus, $\mathrm{reg}(I(G(M))) \geq r' - 1$. □

An interesting example of the previous construction is the case $M = \mathrm{Id}_n$ with $n \geq 2$. Let us denote by $M_n \in \mathcal{M}_{n+2,2n}(\{0,1\})$ the setting of Id_n

$$M_n := \left(\begin{array}{c|c} \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \\ \hline \mathrm{Id}_n & \mathrm{Id}_n \\ \hline \mathbf{0}_{1\times n} & \mathbf{1}_{1\times n} \end{array}\right) = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}$$

and by G_n the bipatite graph on the vertex set $V(G_n) = \{x_0, x_1, \dots, x_n, x_{n+1}\} \cup \{y_1, \dots, y_n, z_1, \dots, z_n\} =: X \sqcup (Y \sqcup Z)$ whose biadjacency matrix is M_n.

Proposition 3.5.5. *$\mu(G_n) = n$.*

Proof. We have immediately that $\mu(G_n) \geq n$ from the fact that Id_n is a submatrix of M_n. Suppose that exists $W \subset V(G_n)$ with $|W| > 2n$ such that $M_n[W] \sim \mathrm{Id}_{\frac{|W|}{2}}$. Then, $|W \cap (Y \cup Z)| > n$. However, if $x_i \in W$ $\forall i \in [n]$, then $\nexists i \in [n]$ with $y_i, z_i \in W$ and hence $|W \cap (Y \cup Z)| \leq n$. Otherwise, $\{x_0, x_{n+1}\} \subset W$. Then $|W \cap Y| = 1$ and $|W \cap Z| = 1$, that implies $|W| = 4 \leq 2n$ and we get a contradiction. □

Proposition 3.5.6. *$\beta_{2n,3n+2}(I(G_n)) = 1$ and $\beta_{i,i+n+2}(I(G_n)) = 0$, $\forall i \neq 2n$.*

Proof. By Hochster's Formula, $\beta_{2n,3n+2}(I(G_n)) = \dim_{\mathbb{K}} \tilde{H}_n(\Delta(G_n))$ and, by proposition 3.5.3, $\tilde{H}_n(\Delta(G_n)) = \tilde{H}_{n-1}(\Delta(\mathrm{Id}_n))$. We can conclude $\beta_{2n,3n+2}(I(G_n)) = 1$, according to Example 1.7.2.

Let $W \subsetneq V(G_n)$, we claim that $\tilde{H}_n(\Delta(G_n)[W]) = 0$.

If $\{x_0, x_{m+1}\} \cap W = \emptyset$, then $\tilde{H}_n(\Delta(G_n)[W]) \cong \tilde{H}_n(\Delta(\mathrm{Id}_n)[W']) = \tilde{H}_n(\Delta(nk_2)[W'])$ where $W' := W \cap (X \cup Y)$, by Lemma 3.1.7.*5*. The only two possibilities are a cone or $\Delta(\mathrm{Id}_m)$ with $m \leq n$. In both cases $\tilde{H}_n(\Delta(\mathrm{Id}_n)[W']) = \{0\}$.

If $x_0 \in W$ but $x_{m+1} \notin W$ (likewise if $x_{n+1} \in W$ and $x_0 \notin W$), then we can remove every column $y_i \in W$ such that $z_i \in W$. If there is no y_i remaning, then we get a cone and we are done. Otherwise,

- if $x_i \in W$, then we apply Lemma 3.1.7.*5* to x_0 and we are in the first case.

- if $x_i \notin W$, then the first entry in the column y_i is 1 and the rest are 0. Therefore, we can remove all other remaining y_j applying Lemma 3.1.7.*5*, and we apply Lemma 3.1.7.*2* in the end. After that, we obtain again $\tilde{H}_n(\Delta(G_n)[W]) \cong \tilde{H}_{n-1}(\Delta(\mathrm{Id}_n)[W'])$ for some $m < n$ (since $z_i \notin W$) and $W' \subset W$ at is a cone or $\Delta(\mathrm{Id}_s)$ with $s < n$. In both cases $\tilde{H}_{n-1}(\Delta(\mathrm{Id}_m)[W']) = 0$.

Finally, if $\{x_0, x_{n+1}\} \subset W$, then there exist $i \in [n]$ with $\{x_i, y_i, z_i\} \not\subset W$. We denote by $W_i := \{x_i, y_i, z_i\} \cap W$. Then, $\exists i \in [n] : \emptyset \neq W_i \subsetneq \{x_i, y_i, z_i\}$ (if $W_i = \emptyset$, then $\Delta(M_n)[W] \cong \Delta(M_m)$ with $m < n$ and hence $\tilde{H}_n(\Delta_W) = 0$). We fix such an index i and we consider the following three cases:

- if $W_i = \{x_i\}$, then $\Delta(M_n)[W]$ is a cone with appex x_i;

- if $W_i = \{x_i, y_i\}$, (or $W_i = \{x_i, z_i\}$) then $link_{\Delta_W}(x_0)$ is a cone with appex x_i and hence $\tilde{H}_i(\Delta(M_n)[W]) \cong \tilde{H}_i(del_{\Delta(M_n)[W]}(x_0))$ by Corollary 1.4.4. This situation was considered above.

- if $y_i \in W_i$ and $x_i \notin W_i$ (likewise for $z_i \in W_i$ and $x_i \notin W_i$), then $link_{\Delta(M_n([W]))}(y_j)$ is a cone with appex y_i for any $y_j \in (W \cap Y)\backslash\{y_i\}$ since the column y_i has a one in the first entry and 0 in the rest. Therefore, we can remove every $y_j \neq y_i$ applying Lemma 3.1.7.*5*. After that, we apply Lemma 3.1.7.*2* to x_0 and y_i. Then, $\tilde{H}_n(\Delta(M_n)[W]) \cong \tilde{H}_{n-1}(\Delta(M_n)[W']) = \tilde{H}_{n-1}(\Delta(M_n)[W' \setminus \{x_{n+1}\}])$ (all entries in the row x_{n+1} of $M_n[W']$ are 1) where $W' := (W \cap (X \cup Z)) \cup \{x_{m+1}\}$. So, $\tilde{H}_n(\Delta(M_n)[W]) = \tilde{H}_{n-1}(\Delta(\mathrm{Id}_m)[W^*])$ for some $m < n$ (since $z_i \notin W$) and $W^* \subset \{x_1, \ldots, x_n\} \cup Z$, that is a cone or $\Delta(\mathrm{Id}_s)$ with $s < n$. In both cases $\tilde{H}_{n-1}(\Delta(\mathrm{Id}_m)[W^*]) = 0$.

Therefore $\beta_{i,i+n+2}(I(G_n)) = 0$, $\forall i < 2n$. $\beta_{i,i+n+2}(I(G_n)) = 0$, $\forall i > 2n$ by Hochster's Formula. □

Applying Theorem 2.1.2 and realizing that $\beta_{i,j}(I(G_n)) = 0$, $\forall j > |V(G_n)| = 3n+2$, we have that $\beta_{i,j}(I(G_n)) = 0$, $\forall j > i + n + 2$. Then,

Corollary 3.5.7. $\mathrm{reg}(I(G_n)) = n + 2$.

Chapter 4

Conclusions and Further Work

It is a hard problem to give closed formulas for the graded Betti numbers of families of edge ideals apart from the basic ones. In this dissertation, we have provided combinatorial formulas for all the graded Betti numbers of edge ideals associated to two particular families of graphs: the complementary graphs of cycles of length greater than or equal to 4, C_l^c with $l \geq 4$, and the bipartite complementary graphs of cycles of even length greater than or equal to 6, C_{2s}^{bc} with $s \geq 3$ (Theorems 2.3.3 and 3.2.15). These two families are the minimal edge ideals with regularity strictly greater than 2 and the minimal bipartite edge ideals with regularity strictly greater than 3, respectively, in the sense that

$$\mathrm{reg}(I(G)) > 2 \Rightarrow \exists\, l \geq 4 \,/\, C_l^c < G$$

and, when G is bipartite,

$$\mathrm{reg}(I(G)) > 3 \Rightarrow \exists\, s \geq 3 \,/\, C_{2s}^{bc} < G.$$

Let denote by u_i the maximal degree of an i-th syzygy of I. From the definition of regularity, it is clear that these degrees are bounded by $\mathrm{reg}(I) + i$. This bound is eventually reached, though, it is not sharp in general. The knowledge of some of the graded Betti numbers allows us to get better bounds for u_i.

When we deal with monomial ideals generated in degree 2, Taylor's resolution provides the bound $u_i \leq 2(i+1)$, which is thight up to $i = \mu(G) - 1$ by Lemma 2.1.1. However, with the notation $i_d := \min\limits_{1 \leq i \leq p} \{i : \beta_{i,i+d} \neq 0\}$, $2 \leq d \leq \mathrm{reg}(I(G))$, introduced in Lemma 2.1.3, we can thighten the conjunction of those bounds by using Theorem 2.1.2 in the case of edge ideals:

- $u_i = i + 2$, if $i < \mu(G)$,

- $u_i \leq \min_{i_{\mu(G)-1} \leq i_d \leq i} \{i_d + d + \lfloor \frac{3(i-i_d)}{2} \rfloor\}$ if $\mu(G) \leq i < i_{\mathrm{reg}(I(G))}$,
- $u_i \leq \mathrm{reg}(I(G)) + i$, if $i \geq i_{\mathrm{reg}(I(G))}$.

Therefore, the more values i_d we know, the thighter the bound in the middle part is. Our contribution is a combinatorial characterization of i_3 for edge ideals I with $\mathrm{reg}(I) > 2$ and i_4 for bipartite edge ideals I with $\mathrm{reg}(I) > 3$ (Theorems 2.4.6 and 3.3.9).

Aldo Conca brought to our attention that similar bounds are still valid in the more general context of Koszul algebras (see [2]). Edge ideals are Koszul algebras ([30]) and the behaviour of their graded Betti numbers suggest the same behaviour in the general case of Koszul algebras.

Edge ideals with regularity 2 are characterized by Fröberg's Theorem. In the bipartite case, they correspond to the family of Ferrer graphs. Furthermore, one of the main results in this dissertation is the characterization of bipartite edge ideals with regularity 3 given in Theorem 3.3.11. Unfortunately, we have seen examples where the regularity of an edge ideal (resp. bipartite edge ideal) is 3 or 4 (resp. 4 or 5) depending on the characteristic of the ground field, see Remark3.3.13. Therefore, it is hopeless to find a purely combinatorial characterization in terms of the defining monomials of edge ideals with regularity strictly greater than 2, or 3 for bipartite edge ideals. A natural question arising from that is

Question 4.0.8. Under what conditions is the regularity of an edge ideal independ on the characteristic of the ground field?

In spite of the characteristic dependence, there is hope for a very sharp bound on the regularity of a bipartite edge ideal $I(G)$. We saw in section 3.5 that

$$\mathrm{reg}(I(G)) \geq \mu(G) + 1$$

and mentioned some families of graphs for which the bound becomes an equality. Many of these families are bipartite graphs under additional conditions. However, there are examples of bipartite graphs G such that $\mathrm{reg}(I(G)) > \mu(G) + 1$ like C_{2s}^{bc} and G_n (Theorem 3.2.15, Corollary 3.5.7), path graphs and cycles of length $6s + 2$ ([43]). In all these cases, one has the identity $\mathrm{reg}(I(G)) = \mu(G) + 2$. After heavy computer-assisted computations based on the algorithm in Appendix A, we have not found any bipartite graph such that $\mathrm{reg}(I(G)) > \mu(G) + 2$, and hence we are interested in proving or finding some counterexample to

Conjecture 4.0.9. Let G be a bipartite graph, then

$$\mathrm{reg}(I(G)) \leq \mu(G) + 2\,.$$

Appendix A

An algorithm for constructing all non-isomorphic bipartite graphs

We deal with bipartite graphs represented by its biadjacency matrices. The problem we address in this appendix is related to isomorphisms between connected bipartite graphs. In terms of biadjacency matrices, two bipartite graphs on the same vertex set $V = A \sqcup B$ are isomorphic if and only if the biadjacency matrix of one of those graphs can be obtained from the biadjacency matrix of the other one by permuting rows and columns and, in the case $|A| = |B|$, transposing. We consider the equivalence relation given for those operations on the set of biadjacency matrices. Then, our aim is to design an algorithm for constructing a representative element for each equivalence class. We will define also an order in that set of matrices and we will choose the greatest one for each equivalence class.

A.1 Terminology

First of all, we introduce some terminology specific for this context, not related to other possible meanings in the literature.

We denote the set of permutations on $[n]$ by $\mathcal{S}_n$ and the permutation matrix corresponding to a permutation σ by P_σ.

Let denote by $\mathcal{M}_{m\times n}(\{0,1\})$ the set of all matrices with m rows and n columns and whose entries are 0 or 1. Let $M = (a_{i,j}), M' = (a'_{i,j}) \in \mathcal{M}_{m\times n}(\{0,1\})$, then

- We say that M and M' are *equivalent*, and denote $M \sim M'$, if M can be obtained from M' after permuting rows and columns. If $n = m$, transposition is also allowd. This is an equivalence relation.

- M is said to be *lexicographically greater* than M' if

  $$(a'_{1,1},\dots,a'_{1,n},a'_{2,1},\dots,a'_{m,1},\dots,a'_{m,n}) \leq_{lex} (a_{1,1},\dots,a_{1,n},a_{2,1},\dots,a_{m,1},\dots,a_{m,n}) \;.$$

- M is called a *lexicographic representative* if $M' \leq_{lex} M$, $\forall M' \sim M$. There is a unique lexicographic representative in each equivalence class.

- We say that M is *row-normalized* (resp. *column-normalized*) if it is lexicographically greater that any other matrix we can get permuting rows (resp. columns) in M. In other words, M is row-normalized (resp. column-normalized) if $MP_\sigma \leq_{\text{lex}} M$, $\forall \sigma \in \mathcal{S}_m$ (resp. $P_\gamma M \leq_{\text{lex}} M$, $\forall \gamma \in \mathcal{S}_n$). M is said to be *normalized* if it is row-normalized and column-normalized. If M is a lexicographic representative, then it is normalized.

- We say that $r \in [n]$ is a *break for the row* $f \in [m]$ if either $r = 1$ or $r > 1$ and $\exists k,\ 1 \leq k < f$, such that $a_{k,r-1} = 1$ and $a_{k,r} = 0$. We notice that if j is a break for i then it is also a break for every $i' \geq i$.

A.2 Theoretical Results

Lemma A.2.1. *A matrix $M = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0,1\})$ is (P1) column-normalized if and only if it satisfies the property (P2): if $a_{i,j} = 1$, then either $a_{i,j-1} = 1$ or j is a break for the row i.*

Proof. We prove $\neg P1 \Leftrightarrow \neg P2$.

$\neg P2 \Rightarrow \neg P1$) Let us assume that there exist i_1, j such that $a_{i_1,j-1} = 0$, $a_{i_1,j} = 1$ and j is not a break for i_1. We fix such a j, and denote by i_0 the minimum i satisfying those conditions. Hence, $a_{i,j-1} = 0$ implies $a_{i,j} = 0$, $\forall i < i_0$, and $a_{i,j-1} = 1$ implies $a_{i,j} = 1$.

After swapping columns $j-1$ and j in M, we obtain a new matrix $M' = (a'_{i,j})$ such that

i) $(a'_{1,1},\dots,a'_{i_0,j-2}) = (a_{1,1},\dots,a_{i_0,j-2})$ if $j > 2$, or

ii) $(a'_{1,1},\dots,a'_{i_0-1,n}) = (a_{1,1},\dots,a_{i_0-1,n})$ if $j = 2$.

Therefore, we have $M <_{\text{lex}} M'$ since $a'_{i_0,j-1} = a_{i_0,j} = 1 > 0 = a_{i_0,j-1}$.

$\neg P1 \Rightarrow \neg P2$) Let assume now that there exists a permutation $\gamma \in \mathcal{S}_n$ such that $M <_{\text{lex}} MP_\gamma = M'$. Let set $f := \min\{i \in [m] : \exists j \in [n] \text{ with } a_{i,j} \neq a'_{i,j}\}$, and $k = \min\{j \in [n] : a_{f,k} \neq a'_{f,k}\}$. There exists $c > k$ with $a_{f,c} = 1$ since $a_{f,j} = 0$, $\forall j > k$, implies $|\{j \in [n] : a_{f,j} = 0| \neq |\{j \in [n] : a'_{f,j} = 0\}|$. Then, there exists $j_0, k \leq j_0 < c$, such that $a_{f,j_0} = 0$ and $a_{f,j_0+1} = 1$ and $a_{l,j_0} = a_{l,j_0+1}$, $\forall l < f$ (i.e., j_0 is not a break for f). ∎

Corollary A.2.2. *Let M be a column-normalized matrix with n columns, and let p, q be two consecutive breaks for the same row r or p be the greatest break for r and $q = n + 1$. Then*

$$\exists 0 \leq l \leq q - p \;/ a_{r,p+t} = \begin{cases} 1 & \text{if } 0 \leq t < l; \\ 0 & \text{if } l \leq t < q - p. \end{cases} \tag{A.1}$$

This result allows us to represent each row i in a column-normalized matrix $M = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0, 1\})$ by a pair of vectors of the same length:

- $R_i = \{r_{i,1}, \ldots, r_{i,t_i}\}$, whose entries are the breaks for that row; and
- $L_i = \{l_{i,1}, \ldots, l_{i,t_i}\}$, where $l_{i,j}$ is the length of the run of ones starting at $r_{i,j}$, that correspond to the index l in (A.1).

Thus, every column-normalized matrix $M \in \mathcal{M}_{m\times n}(\{0, 1\})$ is determined by the vectors $R(M) := \{R_1, \ldots, R_m\}$ and $L(M) := \{L_1, \ldots, L_m\}$.

Remark A.2.3. Notice that

- $1 = r_{i,1} \leq r_{i,2} \leq \cdots \leq r_{i,t_i} \leq n$,
- $t_1 \leq t_2 \leq \cdots \leq t_m$, and
- $\forall i, j, \ \forall k > i, \ \exists j'/r_{i,j} = r_{k,j'}$.

Lemma A.2.4. *Let $M = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0, 1\})$ be column-normalized with breaks vector $R(M) = \{R_1, \ldots, R_m\}$. We fix a row $f > 1$ and two columns k_1, k_2 with $1 \leq k_1 < k_2 \leq n$. Then,*

- *if there exists a break for f, $r_{f,j}$, with $k_1 < r_{f,j} \leq k_2$, then there exists $i < f$ such that $a_{i,k_1} = 1, a_{i,k_2} = 0$ and $a_{i',k_1} = a_{i',k_2}, \ \forall i' < i$;*
- *if there exists no such a break, then $a_{i',k_1} = a_{i',k_2}, \ \forall i' < f$.*

Proof. Let us assume first that there is no break $r_{f,j} \in R_f$ with $k_1 < r_{f,j} \leq k_2$ and let $i < f$. If $a_{i,k_2} = 0$, then $a_{i,k_1} = 0$ (otherwise there is a break between k_1 and k_2 for the row $i + 1$ and, as a consequence, for every row greater that or equal to $i + 1$, including f). Likewise, if $a_{i,k_2} = 1$ then $a_{i,k_1} = 1$ by Lemma A.2.1.

Let us suppose now that there exists $r_{f,j} \in R_f$ with $k_1 < r_{f,j} \leq k_2$. We use induction on f. For $f = 2$, if there exists $r_{2,j}$ with $1 \leq k_1 < r_{2,j}$, then $a_{1,j'} = 1, \ \forall j' < r_{2,j}$ and $a_{1,j'} = 0, \ \forall j' \geq r_{2,j}$. In particular, $a_{1,k_1} = 1$ and $a_{1,k_2} = 0$. We assume that the claim is true for $1 < f < m$. Let suppose that there exists $J \subset [t_{f+1}]$ with $J \neq \emptyset$ such that $\forall j \in J$, $1 \leq k_1 < r_{f+1,j} \leq k_2 \leq n$. If $r_{f+1,j} \in R_f$ for $j \in J$, then

$\exists i < f < f+1 \ / \ a_{i,k_1} = 1, a_{i,k_2} = 0$ and $a_{i',k_1} = a_{i',k_2}$, $\forall i' < i$. If $r_{f+1,j} \notin R_f$, $\forall j \in J$ (so $|J| = 1$), then $a_{f,r_{f+1,j-1}} = 1$ and $a_{f,r_{f+1,j}} = 0$. By Lemma A.2.1, $a_{f,k_1} = 1$ and $a_{f,k_2} = 0$. Moreover, $a_{i',k_1} = a_{i',k_2}$, $\forall i' < f < f+1$ because of the first part in this proof. □

Given a column-normalized matrix $M = (a_{i,j})$, $i \in [m], X \subset [n]$ and a permutation $\sigma \in \mathcal{S}_m$, we set

$$B^{\sigma}_{i,0}(X) := \{c \in X \ / \ a_{\sigma^{-1}(i),c} = 0\} \ , \ B^{\sigma}_{i,1}(X) := \{c \in X \ / \ a_{\sigma^{-1}(i),c} = 1\} \ ,$$

and, if we are given also a vector $\underline{b} \in \{0,1\}^i$,

$$B^{\sigma}_{\underline{b}} := B^{\sigma}_{i,b_i} \circ B^{\sigma}_{i-1,b_{i-1}} \circ \cdots \circ B^{\sigma}_{1,b_1}([n]) \ .$$

We establish a partial order among the last kind of sets. Let $\underline{b}, \underline{b}' \in \{0,1\}^i$, then

$$B^{\sigma}_{\underline{b}} < B^{\sigma}_{\underline{b}'} \overset{def}{\Leftrightarrow} (b_1, \ldots, b_i) <_{lex} (b'_1, \ldots, b'_i) \ .$$

and we denote by $B^{\sigma}_{\langle i,j \rangle}$ the j-th largest non-empty set $B^{\sigma}_{\underline{b}}$ with $\underline{b} \in \{0,1\}^i$.

We also denote by $X^{\sigma}_{1,1} := B^{\sigma}_{1,1}([n])$ and, if $i > 1$, $X^{\sigma}_{i,j} := B^{\sigma}_{i,1}(B^{\sigma}_{\langle i-1,j \rangle})$.

Lemma A.2.5. *Let $M = (a_{i,j}) \in \mathcal{M}_{m \times n}(\{0,1\})$, $\sigma \in \mathcal{S}_m$ and $\gamma \in \mathcal{S}_n$. Then, the matrix $M' = (a'_{i,j}) = P_\sigma M P_\gamma$ is column-normalized if and only if γ holds*

$$\forall i \in [m], \forall \underline{b} \in \{0,1\}^{i-1}, \quad x \in B^{\sigma}_{\underline{b}|\{1\}} \text{ and } y \in B^{\sigma}_{\underline{b}|\{0\}} \Rightarrow \gamma(x) < \gamma(y). \tag{A.2}$$

Proof. $\Rightarrow$) Let M' be column-normalized and $x \in B^{\sigma}_{\underline{b}|\{1\}}$ and $y \in B^{\sigma}_{\underline{b}|\{0\}}$ with $\underline{b} \in \{0,1\}^{i-1}$ and $i \in [m]$. Then, $a'_{i,\gamma(x)} = 1, a'_{i,\gamma(y)} = 0$ and $a'_{l,\gamma(x)} = a'_{l,\gamma(y)}$, $\forall l < i$. By Lemma A.2.4, there is no break for i between $\gamma(x)$ and $\gamma(y)$. Thus, if $\gamma(x) > \gamma(y)$, we have that $a'_{i,\gamma(x)-1} = 1, a'_{i,\gamma(x)-2} = 1, \ldots, a'_{i,\gamma(y)} = 1$ by Lemma A.2.2. Hence, $\gamma(x) < \gamma(y)$.

$\Leftarrow$) Let us prove now that if $a'_{i,j} = 1$ with $j \notin R'_i$, then $a'_{i,j-1} = 1$. After that, we will be done by Lemma A.2.1.

Let us suppose $\exists (i,j) \ / \ a'_{i,j-1} = 0, a'_{i,j} = 1$ and $j \notin R'_i$. We consider

$$i_0 := \min\{i \in [m] / \exists j \in [n] \text{ with } a'_{i,j-1} = 0, a'_{i,j} = 1 \text{ and } j \notin R'_i\}, \text{ and}$$

$$j_0 := \min\{j \in [n] \text{ with } a'_{i_0,j-1} = 0, a'_{i_0,j} = 1 \text{ and } j \notin R'_{i_0}\}\} \ .$$

We have $a'_{i,j_0-1} = a'_{i,j_0}$, $\forall i < i_0$ by definition of i_0, since $j \notin R'_{i_0}$. Thus, $\gamma^{-1}(j_0 - 1) \in B^{\sigma}_{\underline{b}|\{0\}}$ and $\gamma^{-1}(j_0) \in B^{\sigma}_{\underline{b}|\{1\}}$ for $\underline{b} \in \{0,1\}^{i_0-1}$ with $b_i = a'_{i,j_0-1} = a'_{i,j_0}$. Therefore, $\gamma(\gamma(j_0)) < \gamma(\gamma(j_0 - 1))$, which is a contradiction. □

A.3 Representative Test

Proposition A.3.1. *A normalized matrix is a lexicographic representative if and only if there exist $\sigma \in \mathcal{S}_m$, $f \in [m]$ and $k \in [t_f]$ such that*

- $l_{i,j} = |X^{\sigma}_{i,j}|$, $\forall i < f$ *and, if* $i = f$, $\forall j < k$;
- $l_{f,k} < |X^{\sigma}_{f,k}|$.

Proof. $\Leftarrow$) Let us suppose that there exist such σ, f and k. We consider $\gamma \in \mathcal{S}_n$ satisfying (A.2) and $M' = (a'_{i,j}) = P_\sigma M P_\gamma$. We denote by $R' := R(M') = \{\{r'_{1,1}\}, \dots, \{r'_{m,1}, \dots, r'_{m,t_m}\}\}$ and $L' := L(M') = \{\{l_{1,1}\}, \dots, \{l_{m,1}, \dots, l_{m,t_m}\}\}$.

We will see that $X^{\sigma}_{i.j} = \{\gamma^{-1}(c) : c \in [r'_{i,j}, r'_{i,j+1}) \text{and} a'_{i,c} = 1\}$ $(r'_{i,t_i+1} = n+1)$.

By definition, $X^{\sigma}_{1,1} = \{\gamma^{-1}(c) : c \in [n]$ and $a'_{1,c} = 1\}$ and, for $i > 1$, we have to check $[r'_{i,j}, r'_{i,j+1}) = \gamma(B^{\sigma}_{\langle i-1,j\rangle})$.

We first prove that $\gamma(B^{\sigma}_{\underline{b}})$ is a closed interval. If $i = 2$, then $\gamma(B^{\sigma}_1) = [1, |B^{\sigma}_1|]$ and $\gamma(B^{\sigma}_0) = [|B^{\sigma}_1| + 1, n]$ by Corollary A.2.2; if $i > 2$ and z satisfies $x < z < y$ with $x, y \in \gamma(B^{\sigma}_{\underline{b}})$, $\underline{b} \in \{0,1\}^i$, then $x, y \in \gamma(B^{\sigma}_{\underline{b}\setminus\{b_i\}})$, which is an interval. Thus, $z \in \gamma(B^{\sigma}_{\underline{b}\setminus\{b_i\}})$. If $a'_{i,z} = 1 - b_i$, then $\gamma^{-1}(z) \in B^{\sigma}_{\underline{b}\setminus\{b_i\}|\{1-b_i\}}$ and $x, y \in B^{\sigma}_{\underline{b}\{b_i\}|\{b_i\}}$. Hence, if $b_i = 1$, then $y < z$, and, if $b_i = 0$, then $z < x$. In both cases we get a contradiction.

We notice that the first element of $\gamma(B^{\sigma}_{\langle i-1,j\rangle})$ is a break for i in M': if $j = 1$, the first element in $\gamma(B^{\sigma}_{i-1,1})$ is 1, which is always a break; if $j > 1$, let x be the first element in $\gamma(B^{\sigma}_{\langle i-1,j\rangle}) = \gamma(B^{\sigma}_{\underline{b}})$, $\underline{b} \in \{0,1\}^{i-1}$, then $x - 1 \in \gamma(B^{\sigma}_{\langle i-1,j-1\rangle}) = \gamma(B^{\sigma}_{\underline{b'}})$, $\underline{b'} \in \{0,1\}^{i-1}$. As $B^{\sigma}_{\langle i-1,j-1\rangle} < B^{\sigma}_{\langle i-1,j\rangle}$, the first non-zero element in $\underline{b'} - \underline{b}$ (let us denote it by p) is positive. So $a'_{p,x-1} = 1$ and $a'_{p,x} = 0$ with $p < i$ and x is a break for i in M'.

Moreover, $\gamma(B^{\sigma}_{\langle i-1,j\rangle})$ only contains one break for i in M': let $r'_1, r'_2 \in \gamma(B^{\sigma}_{\langle i-1,j\rangle}) \cap R'_i$ with $r'_1 < r'_2$, then $\exists i' < i \;/\; a'_{i',r'_2-1} = 1$ and $a'_{i',r'_2} = 0$, so $r'_2 - 1 \notin \gamma(B^{\sigma}_{\langle i-1,j\rangle})$, which is a contradiction with the fact that $\gamma(B^{\sigma}_{\langle i-1,j\rangle})$ is an interval.

Therefore, $l'_{i,j} = |X^{\sigma}_{i,j}|$ and we have $l_{i,j} = l'_{i,j}, \forall i < f$, $\forall j < k$ if $i = f$, and $l_{f,k} < l'_{f,k}$. Hence $M \leq_{\text{lex}} M'$.

$\Rightarrow$) Let us assume that there exist $\sigma \in \mathcal{S}_m$ and $\gamma \in \mathcal{S}_n$ such that $M <_{lex} P_\sigma M P_\gamma$, where $M' := P_\sigma M P_\gamma$ is a lexicographic representative. Then, there exist $f \in [m]$ and $k \in [t_f]$ such that $l_{i,j} = l'_{i,j}$, $\forall i < f$, $\forall j < k$ if $i = f$, and $l_{f,k} < l'_{f,k}$. Let us see $l'_{i,j} = |X^{\sigma}_{i,j}|$, $\forall i, j$.

Since M' is a lexicographic representative, it is, in particular, column-normalized. Then, by Lemma A.2.5, γ satisfies (A.2). Hence, as we saw in the first part of this proof, $[r'_{i,j}, r'_{i,j+1}) = \gamma(B^{\sigma}_{\langle i-1,j\rangle})$. Thus,

$$\begin{aligned} l'_{i,j} &= |\{c \in [r'_{i,j}, r'_{i,j+1}) : a'_{i,c} = 1\}| \\ &= |\{c \in \gamma(B^{\sigma}_{\langle i-1,j\rangle}) : a'_{i,c} = 1\}| \\ &= |\{x \in B^{\sigma}_{\langle i-1,j\rangle} : a_{\sigma^{-1}(i),x} = 1\}| \\ &= |B^{\sigma}_{i,1}(B^{\sigma}_{\langle i-1,j\rangle})| \\ &= |X^{\sigma}_{i,j}|. \end{aligned}$$

□

A.4 Connectivity Test

Lemma A.4.1. *Let $M = (a_{i,j}) \in \mathcal{M}_{m\times n}(\{0,1\})$ be a normalized matrix with lengths vector $L(M) = \{L_1, \dots, L_m\}$. Then, M is the non-connected biadjacency matrix of a bipartite graph G without isolated vertices if and only if $\exists i \in [m]$ / $a_{i',n} = 0$, $\forall i' < i$, and $L_i = \{0, \dots, 0, l_{i,t_i}\}$ with $l_{i,t_i} \neq 0$.*

Proof. $\Rightarrow$) Since G has no isolated vertices, M has no column with all its entries equal to 1. Then, there exists $i \in [n]$ such that $a_{i,n} = 1$. We set $N := min\{i \in [n] : a_{i,n} = 1\}$. We have $a_{N,j} = 1$, $\forall r_{N,t_N} \leq j \leq n$. If $N = 1$, then $a_{1,j} = 1$, $\forall j \in [n]$ (as M is column-normalized) and G is connected, what is a contradiction. Thus, $N \geq 2$.

We prove by reduction to the absurd that there exists $1 < I \leq N$ such that $l_{I,k} = 0$ $\forall k < t_I$. Let us suppose

$$\forall i,\ 1 < i \leq N,\ \exists k < t_i \ /\ l_{i,k} \neq 0\ . \tag{A.3}$$

We are going to reach the following contradiction: $G[W]$ is a connected graph where W is the set of vertices corresponding to the columns of M (recall G has no isolated vertex). Let us see that for every row $f > 1$, (Pf:) the vertices corresponding to the columns c with $r_{f,t_f} \leq c < r_{f,t_f} + l_{f,t_f}$ are connected to the vertices corresponding to $1 \leq c < r_{f,t_f}$. We prove it by induction on f. We have $a_{1,n} = 0$, so $L_1 = \{l_{1,1}\}$ and $L_2 = \{l_{2,1}, l_{2,2}\}$. By (A.3), $l_{2,1} \neq 0$, so (Pf) is true for $f = 2$. If (Pf) is true for f, then it is also true for $f + 1$ by A.3). For the case $f = N$, we obtain the desired result.

$\Leftarrow$) If $a_{i',n} = 0$, $\forall i' < i$ then $a_{i',j} = 0$, $\forall i' < i$, $\forall j \geq r_{i,t_i}$, otherwise there is a break for i greater than r_{i,t_i}. As $L_i = \{0, \dots, 0, l_{i,t_i}\}$, we have $a_{l,j} = 0$, $\forall l \geq i$, $\forall j < r_{i,t_i}$ since M is row-normalized. Thus, no vertex corresponding to a column c with $r_{i,t_i} \leq c \leq n$ or a row f with $i \leq f \leq m$ is connected to the rest of vertices. □

A.5 Main algorithm

Given two integers, m and n, we will work with two lists *lengths* and *breaks* representing the biadjacency matrix of a (m, n)-bipartite graph. Each list consists of m lists corresponding to the rows of the matrix.

1 Make sure the number of columns (n) is bigger than or iqual to the number of rows (m).

2 Initialize parameters in order to obtain a matrix with all estries iqual to 1 and add it to the list.

3 Find the last non-zero entry in *lengths*, let say *lengths_row_segment*.

4 Reduce its value one unit.

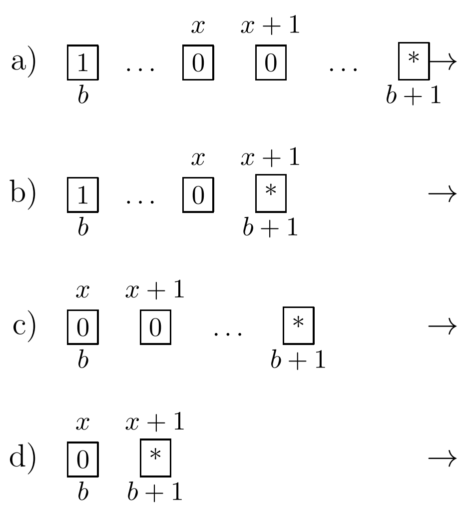
5 Modify the entries "on the right" of the vector *lengths_row* with the maximal possible values.

6 Check the first row of the associated matrix with a 1 in the last column.

7 Take a copy of *lengths_row* for the next vector in *lengths*.

8 Select the column x in the matrix where we have replaced a 1 with a 0 to compare it with the breaks for the corresponding row, *row* ($b \leq x < b + 1$ with $b \in$*breaks_row*).

9 Modify the vector *breaks_(row+1)* removing old breaks or introducing new ones. In the last case, also introduce a new entry in *lengths_(row+1)* iqual to 0 in the corresponding position. Tere are four posibilities:



a) → Replace old break with the new one. Keep 0 for the corresponding length.

b) → Insert new break. Insert 0 in the corresponding length.

c) → Remove old break. Remove the corresponding length.

d) → Do nothing.

10 Replace the entries of *lengths* and *breaks* after *row*+1 with copies of this one.

11 Check whether the associated graph has no isolated vertices.

12 Check whether the associated graph is connected.

13 Check whether the associated matrix M is a lexicographic leader.

14 If the matrix M is not square and 11,12,13 are verified then we add M to the list *list*.

15 If the matrix M is square we also have to check whether the matrix is greater than its transpose and all those equivalent to it before adding M to *list*.

Notation:

matrix(*breaks,lengths*) means the matrix coded by the two vectors,
insert(*object,vector,position*) means create a new position in *vector* after *position* and put there the value *object*,
remove(*position,vector*) means to delete the position *position* from *vector*,
last(*vector*) denote the element in the last position of *vector*.

A.6 Pseudocode

Representative Test's Pseudocode

Input: Two lists *breaks* and *lengths* and a logic parameter *transpose*;
Output: *TRUE* or *FALSE*;

```
M:=matrix(breaks,lengths);
finished:=FALSE;
usedRows:={};
realRow:=1;
candidate:=1;
B := {{1, . . . , n}} ;
while finished=FALSE do
   if candidate ∉ usedRows & candidate < m then
      continue:=TRUE;
      hip:=TRUE;
      newB:={};
      for j from 1 to —B_realRow— do
         if transpose=TRUE then
            B1:={c ∈ B_realRow_j : M_c_candidate=1};
            B0:={c ∈ B_realRow_j : M_c_candidate=0};
         else
            B1:={c ∈ B_realRow_j : M_candidate_c=1};
            B0:={c ∈ B_realRow_j : M_candidate_c=0};
         end if
         if B1 ≠ ∅ then newB:=newB∪{B1}; end if
         if B0 ≠ ∅ then newB:=newB∪{B0}; end if
         if |B1| <lengths_realRow_j then continue:=FALSE; break; end if
         if |B1| >lengths_realRow_j then
```

```
            continue:=FALSE;
            hip:=FALSE;
            break;
          end if
        end do
        if hip=TRUE then
          if continue=TRUE & realRow< m then
            usedRows:=usedRows∪{candidate};
            realRow:=realRow+1;
            candidate:=1;
            B := B∪{newB};
          end if
        else finished:=TRUE;
        end if
      else candidate:=candidate+1;
      end if
      if candidate ≥ m then
        if usedRows=∅ then
          candidate:=last(usedRows)+1;
          realRow:=realRow+1;
          remove(—usedRows—,usedRows);
          remove(|B|,B);
        else finished:=TRUE;
        end if
        if realRow≥ m then candidate:=last(usedRows)+1; end if
      end if
    end do
    return hip;
```

Main Algorithm's Pseudocode

Input: two positive integers m, n;

Output: The list of all non-isomorphic (m, n)-bipartite graphs;

```
1    if m > n then swap their values end if
2    { end1:=1;
     { lengths := {{n}, . . . , {n}}; (∗∗ |lengths| = m ∗∗)
     { breaks := {{1, n + 1}, . . . , {1, n + 1}}; (∗∗ |breaks| = m ∗∗)
     { list := {matrix(lengths, breaks)}
     while lengths≠ {{1}, {0} . . . , {0}} do
3    {   row:=max{i ∈ [m] : lengths_i ≠ 0̲};
     {   segment:=max{k : lengths_row_k ≠ 0};
4        lengths_row_segment:=lengths_row_segment-1;
5    {   for i from segment+1 to —lengths_row— do
     {       lengths_row_i:=breaks_row_(i+1)-breaks_row_i;
     {   end do
6    {   if segment<—lengths— & row<end1 then end1:=row; end if
     {   if segment=—lengths— & row=end1 then end1:=m+1; end if ;
7        lengths_(row+1):=lengths_row;
         if row< m then
8          x:=breaks_row_segment+lengths_row_segment;
9    {     if x ≠ breaks_row_segment then
     {       if x + 1 ≠ breaks_row_(segment+1) then
     {         breaks_(row+1)_(k+1):=x;
     {       else
     {         insert(x,breaks_(row+1),segment);
     {         insert (0,lengths_(row+1),segment);
     {       end if
     {     end if
     {     if x = breaks_row_segment & x + 1 ≠ breaks_row_(segment+1)
     {   then
     {       remove(segment,breaks_row);
     {       remove(segment,lengths_row);
     {     end if
10   {     for i from row+2 to m do
     {       lengths_i:=lengths_(row+1);
     {       breaks_i:=breaks_(row+1);
     {     end do
         end if
11       if (end1≤ m & ∃j s.t. lengths_m_j ≠ 0)
12         & ∀i = 1, . . . , end1, ∑_{j=1}^{|lengths_i|−1} lengths_i_j ≠ 0
13         & IsRlex(breaks,lengths,transpose=FALSE) then
14          if m ≠ n then
              list:=list ∪{M};
            else
15   {        if M^t ≤_lex M & IsRlex(breaks,lengths,transpose=TRUE) then
     {          list:=list∪{M};
     {        end if
            end if
         end if
     end do
     return list;
```

Bibliography

[1] P. Alexandroff. Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung. *Mathematische Annalen*, 98(1):617–635, 1928.

[2] L. L. Avramov, A. Conca, and S. B. Iyengar. Free resolutions over commutative koszul algebras. *Mathematical Research Letters*, 17:197–210, 2010.

[3] J. A. Barmak and E. G. Minian. Strong homotopy types, nerves and collapses. *arXiv:0907.2954*, July 2009.

[4] D. Bayer, I. Peeva, and B. Sturmfels. Monomial resolutions. *Mathematical Research Letters*, 5(1-2):31–46, 1998.

[5] I. Bermejo, P. Gimenez, and A. Simis. Polar syzygies in characteristic zero: the monomial case. *Journal of Pure and Applied Algebra*, 213(1):1–21, 2009.

[6] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[7] A. Björner. Nerves, fibers and homotopy groups. *Journal of Combinatorial Theory, Series A*, 102(1):88–93, 2003.

[8] A. Björner, B. Korte, and L. Lovász. Homotopy properties of greedoids. *Advances in Applied Mathematics*, 6(4):447–494, 1985.

[9] A. Björner, L. Lovász, S. T. Vrećica, and R. T. Živaljević. Chessboard complexes and matching complexes. *Journal of the London Mathematical Society. Second Series*, 49(1):25–39, 1994.

[10] B. Bollobás. *Graph theory. An introductory course*, volume 63 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.

[11] K. Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Polska Akademia Nauk. Fundamenta Mathematicae*, 35:217–234, 1948.

[12] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[13] A. Corso and U. Nagel. Monomial and toric ideals associated to ferrers graphs. *Transactions of the American Mathematical Society*, 361(3):1371–1395, 2009.

[14] D. A. Cox, J. Little, and D. O'Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2005.

[15] D. A. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007.

[16] P. Csorba. Subdivision yields Alexander duality on independence complexes. *The Electronic Journal of Combinatorics*, 16(2):R11, 2009.

[17] K. Dalili and M. Kummini. Dependence of Betti numbers on characteristic. *arXiv:1009.4243*, Sept. 2010.

[18] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[19] J. A. Eagon and V. Reiner. Resolutions of Stanley-Reisner rings and Alexander duality. *Journal of Pure and Applied Algebra*, 130(3):265–275, 1998.

[20] D. Eisenbud. *Commutative algebra. With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[21] D. Eisenbud. *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[22] D. Eisenbud, M. Green, K. Hulek, and S. Popescu. Restricting linear syzygies: algebra and geometry. *Compositio Mathematica*, 141(6):1460—1478, 2005.

[23] S. Eliahou and M. Kervaire. Minimal resolutions of some monomial ideals. *Journal of Algebra*, 129(1):1–25, 1990.

[24] A. Engström. Independence complexes of claw-free graphs. *European Journal of Combinatorics*, 29(1):234–241, 2008.

[25] A. Engström. Complexes of directed trees and independence complexes. *Discrete Mathematics*, 309(10):3299–3309, 2009.

[26] M. Estrada and R. H. Villarreal. Cohen-macaulay bipartite graphs. *Archiv der Mathematik*, 68(2):124–128, 1997.

[27] S. Faridi. The facet ideal of a simplicial complex. *Manuscripta Mathematica*, 109(2):159–174, 2002.

[28] S. Faridi. Monomial ideals via square-free monomial ideals. In *Commutative algebra*, volume 244 of *Lect. Notes Pure Appl. Math.*, page 85–114. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[29] O. Fernández-Ramos and P. Gimenez. First nonlinear syzygies of ideals associated to graphs. *Communications in Algebra*, 37(6):1921–1933, 2009.

[30] R. Fröberg. Determination of a class of poincaré series. *Mathematica Scandinavica*, 37(1):29–39, 1975.

[31] R. Fröberg. On Stanley-Reisner rings. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 57—70. PWN, Warsaw, 1990.

[32] V. Gasharov, T. Hibi, and I. Peeva. Resolutions of a-stable ideals. *Journal of Algebra*, 254(2):375–394, 2002.

[33] B. Grünbaum. Nerves of simplicial complexes. *Aequationes Mathematicae*, 4(1-2):63–73, 1970.

[34] H. T. Hà and A. Van Tuyl. Splittable ideals and the resolutions of monomial ideals. *Journal of Algebra*, 309(1):405–425, 2007.

[35] H. T. Hà and A. Van Tuyl. Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers. *Journal of Algebraic Combinatorics*, 27(2):215–245, 2007.

[36] H. T. Hà and A. Van Tuyl. Resolutions of square-free monomial ideals via facet ideals: a survey. In *Algebra, geometry and their interactions*, volume 448 of *Contemp. Math.*, pages 91–117. Amer. Math. Soc., 2007.

[37] F. Harary. *Graph theory*. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.

[38] S. Hell. On a topological fractional helly theorem. *arXiv:math/0506399*, June 2005.

[39] J. Herzog and T. Hibi. Distributive lattices, bipartite graphs and Alexander duality. *Journal of Algebraic Combinatorics*, 22(3):289–302, 2005.

[40] J. Herzog and T. Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London Ltd., London, 2011.

[41] T. Hibi. Buchsbaum complexes with linear resolutions. *Journal of Algebra*, 179(1):127–136, 1996.

[42] M. Hochster. Cohen-Macaulay rings, combinatorics, and simplicial complexes. In *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*, pages 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. Dekker, New York, 1977.

[43] S. Jacques and M. Katzman. The Betti numbers of forests. *arXiv:math/0501226*, Jan. 2005.

[44] J. Jonsson. *Simplicial Complexes of Graphs.* volume 1928 of Lecture notes in Mathematics. Springer, New York, 2007.

[45] G. Kalai and R. Meshulam. Intersections of leray complexes and regularity of monomial ideals. *Journal of Combinatorial Theory, Series A*, 113(7):1586–1592, 2006.

[46] M. Katzman. Characteristic-independence of Betti numbers of graph ideals. *Journal of Combinatorial Theory. Series A*, 113(3):435–454, 2006.

[47] M. Kreuzer and L. Robbiano. *Computational commutative algebra. 1.* Springer-Verlag, Berlin, 2000.

[48] M. Kummini. Regularity, depth and arithmetic rank of bipartite edge ideals. *Journal of Algebraic Combinatorics*, 30(4):429–445, 2009.

[49] J. Leray. Sur la forme des espaces topologiques et sur les points fixes des représentations. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 24:95–167, 1945.

[50] G. Lyubeznik. A new explicit finite free resolution of ideals generated by monomials in an r-sequence. *Journal of Pure and Applied Algebra*, 51(1-2):193–195, 1988.

[51] J. Matoušek. *Using the Borsuk-Ulam theorem.* Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

[52] R. Meshulam. Domination numbers and homology. *Journal of Combinatorial Theory, Series A*, 102(2):321–330, 2003.

[53] S. Morey and R. H. Villarreal. Edge ideals: algebraic and combinatorial properties. *arXiv:1012.5329*, 2010.

[54] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[55] I. Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications.* Springer-Verlag London Ltd., London, 2011.

[56] I. Peeva and M. Velasco. Frames and degenerations of monomial resolutions. *Transactions of the American Mathematical Society*, 363(4):2029–2046, 2011.

[57] G. A. Reisner. Cohen-Macaulay quotients of polynomial rings. *Advances in Mathematics*, 21(1):30–49, 1976.

[58] M. Roth and A. Van Tuyl. On the linear strand of an edge ideal. *Communications in Algebra*, 35(3):821–832, 2007.

[59] A. Simis, W. V. Vasconcelos, and R. H. Villarreal. On the ideal theory of graphs. *Journal of Algebra*, 167(2):389–416, 1994.

[60] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[61] M. Tancer. Strong d-collapsibility. *Contributions to Discrete Mathematics*, 6(2):32–35, 2011.

[62] D. K. Taylor. *Ideals generated by monomials in an R-sequence*. PhD thesis, University of Chicago, 1966.

[63] A. Van Tuyl. Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity. *Archiv der Mathematik*, 93(5):451–459, 2009.

[64] A. Van Tuyl. A beginner's guide to edge and cover ideals. Preprint. Available at: http://flash.lakeheadu.ca/~avantuyl/papers/MONICA_Lectures.pdf

[65] A. Van Tuyl and R. H. Villarreal. Shellable graphs and sequentially Cohen–Macaulay bipartite graphs. *Journal of Combinatorial Theory, Series A*, 115(5):799–814, 2008.

[66] M. Velasco. Minimal free resolutions that are not supported by a CW-complex. *Journal of Algebra*, 319(1):102–114, 2008.

[67] R. H. Villarreal. Cohen-macaulay graphs. *Manuscripta Mathematica*, 66(1):277–293, 1990.

[68] R. H. Villarreal. *Monomial algebras*, volume 238 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2001.

[69] G. Wegner. d-collapsing and nerves of families of convex sets. *Archiv der Mathematik*, 26:317–321, 1975.

[70] A. Weil. Sur les théorèmes de de Rham. *Commentarii Mathematici Helvetici*, 26:119–145, 1952.

[71] G. Whieldon. Jump sequences of edge ideals. *arXiv:1012.0108*, Dec. 2010.

[72] G. W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978.

[73] R. Woodroofe. Matchings, coverings, and Castelnuovo-Mumford regularity. *arXiv:1009.2756*, Sept. 2010.

[74] X. Zheng. Resolutions of facet ideals. *Communications in Algebra*, 32(6):2301–2324, 2004.