Chapter 1

MULTIDISTANCES AND DISPERSION MEASURES

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Abstract In this paper, we provide a formal notion of absolute dispersion measure that is satisfied by some classical dispersion measures used in Statistics, such as the range, the variance, the mean deviation and the standard deviation, among others, and also by the absolute Gini index, used in Welfare Economics for measuring inequality. The notion of absolute dispersion measure shares some properties with the notion of multidistance introduced and analyzed by Martín and Mayor in several recent papers. We compare absolute dispersion measures and multidistances and we establish that these two notions are compatible by showing some functions that are simultaneously absolute dispersion measures and multidistances. We also establish that remainders obtained through the dual decomposition of exponential means, introduced by García-Lapresta and Marques Pereira, are absolute dispersion measures up to sign.

Keywords: Dispersion measures, multidistances, aggregation functions, dual decomposition.

1. Introduction

In some simple situations, everybody seems to have an intuition about the notion of dispersion, being able to inform if some objects are or not more scattered than others. However, if we aim to measure the magnitude of their spread in order to provide a representation of such perception, even when dealing with mathematical objects or data, many troubles naturally arise. Of course, there exists a well-known approach from Descriptive Statistics, but recently some interesting attempts have been done by extending the usual binary concept of distance to more general settings. The tittle of the seminal paper by Martín and Mayor [15] introducing the so-called multidistances is significative in this sense: "How separated Palma, Inca and Manacor are?".

On the other hand, as suggested (without any formal definition) by García-Lapresta and Marques Pereira [9], the remainders of exponential means can be somehow considered as dispersion measures. We now establish this fact regarding the formal definition of an absolute dispersion measure introduced in this paper.

All in all, the mentioned concepts have common links in terms of closeness among different objects. This is the reason why in this paper we have considered their common background in order to establish their formal connections.

The rest of the paper is organized as follows. Section 2 introduces the notation and a comprehensive list of properties which will appear along the paper. Some of them are considered in our definition of absolute dispersion measure, in the third section, and the fulfillment of some other properties for this measures is checked or tested. Then, in the fourth section we relate the notion of multidistance with that of absolute dispersion measure, and this relationship is also analyzed, in the fifth section, in connection with the remainders of exponential means. The last section presents a synoptic diagram of the mentioned relationships, as well as some conjectures for further research and our concluding remarks.

2. Preliminaries

Let I be $[0,1]$ or $\mathbb{R}$, and $\mathbb{I} = \bigcup_{n\in\mathbb{N}} I^n$. Vectors in I^n are denoted as $\boldsymbol{x} = (x_1,\ldots,x_n)$, $\boldsymbol{0} = (0,\ldots,0)$, $\boldsymbol{1} = (1,\ldots,1)$. Accordingly, $x\cdot\boldsymbol{1} = (x,\ldots,x)$ for every $x \in I$.

Given $\boldsymbol{x},\boldsymbol{y} \in I^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1,\ldots,n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in I^n$, the increasing reordering of the coordinates of $\boldsymbol{x}$ is indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$. In particular, $x_{(1)} = \min\{x_1,\ldots,x_n\}$ and $x_{(n)} = \max\{x_1,\ldots,x_n\}$. The arithmetic mean of $\boldsymbol{x}$ is symbolized as usual by $\mu(\boldsymbol{x})$. Given a permutation π on $\{1,\ldots,n\}$, we denote $\boldsymbol{x}_\pi = (x_{\pi(1)},\ldots,x_{\pi(n)})$. Finally, the cardinality of the set $\{x_1,\ldots,x_n\}$ appears as $\#\{x_1,\ldots,x_n\}$.

We begin by defining standard properties of real functions on $\mathbb{R}^n$. For further details the interested reader is referred to Fodor and Roubens [7], Calvo *et al.* [6], Beliakov *et al.* [3], García-Lapresta and Marques Pereira [9], Grabisch *et al.* [11] and Beliakov *et al.* [2].

DEFINITION 1.1 *Let $A : I^n \longrightarrow \mathbb{R}$ be a function.*

1. *A is* idempotent *if for every $x \in I$ it holds $A(x\cdot\boldsymbol{1}) = x$.*
2. *A is* symmetric *if for every permutation π on $\{1,\ldots,n\}$ and every $\boldsymbol{x} \in I^n$ it holds $A(\boldsymbol{x}_\pi) = A(\boldsymbol{x})$.*

3 A *is* monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in I^n$ *it holds* $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$.

4 A *is* strictly monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in I^n$ *it holds* $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$.

5 A *is* compensative *if for every* $\boldsymbol{x} \in I^n$ *it holds* $x_{(1)} \leq A(\boldsymbol{x}) \leq x_{(n)}$.

6 A *is* anti-self-dual *if* $I = [0,1]$ *and for every* $\boldsymbol{x} \in [0,1]^n$ *it holds* $A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x})$.

7 A *is* even *if* $I = \mathbb{R}$ *and for every* $\boldsymbol{x} \in \mathbb{R}^n$ *it holds* $A(-\boldsymbol{x}) = A(\boldsymbol{x})$.

8 A *is* stable for translations *if for all* $\boldsymbol{x} \in I^n$ *and* $t \in \mathbb{R}$ *such that* $\boldsymbol{x} + t \cdot \mathbf{1} \in I^n$ *it holds* $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$.

9 A *is* invariant for translations *if for all* $\boldsymbol{x} \in I^n$ *and* $t \in \mathbb{R}$ *such that* $\boldsymbol{x} + t \cdot \mathbf{1} \in I^n$ *it holds* $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$.

10 A *is* invariant under positive scaling *(or* positively homogeneous of degree 0*) if for all* $\boldsymbol{x} \in I^n$ *and* $\lambda > 0$ *such that* $\lambda \cdot \boldsymbol{x} \in I^n$ *it holds* $A(\lambda \cdot \boldsymbol{x}) = A(\boldsymbol{x})$.

DEFINITION 1.2 *Let* $A : \mathbb{I} \longrightarrow \mathbb{R}$ *be a function.*

1 A *is* stable *if for all* $\boldsymbol{x} \in I^n$ *and* $i \in \{1, \ldots, n\}$ *it holds* $A(\boldsymbol{x}, x_i) = A(\boldsymbol{x})$.

2 A *is* contractive *if for all* $\boldsymbol{x} \in \mathbb{I}$ *there exists* $y \in I$ *such that* $A(\boldsymbol{x}, y) < A(\boldsymbol{x})$.

3 A *is* invariant for replications *if* $A(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A(\boldsymbol{x})$ *for every* $\boldsymbol{x} \in \mathbb{I}$ *and any number* $m \in \mathbb{N}$ *of replications of* $\boldsymbol{x}$.

3. Absolute dispersion measures

As far as we know, there is no an established notion of absolute dispersion measure in the literature. In this paper we gather some compelling properties that such concept should fulfill. Next, we show some classic statistic estimators which can be understood from this point of view (on this matter, Calot [4] is still useful for further details).

DEFINITION 1.3 *A function* $D : \mathbb{I} \longrightarrow \mathbb{R}$ *is an* absolute dispersion measure *if it satisfies the following conditions*

1 Positiveness*:* $D(\boldsymbol{x}) \geq 0$*, for every* $\boldsymbol{x} \in \mathbb{I}$.

2 Identity of indiscernibles*:* $D(\boldsymbol{x}) = 0 \Leftrightarrow x_1 = \cdots = x_n$*, for all* $n \in \mathbb{N}$ *and* $\boldsymbol{x} \in I^n$.

3 Symmetry: $D(\boldsymbol{x}_\pi) = D(\boldsymbol{x})$, *for all* $n \in \mathbb{N}$, $\boldsymbol{x} \in I^n$ *and permutation* π *on* $\{1,\ldots,n\}$.

4 Invariance for translations.

5 Invariance for replications.

6 Anti-self-duality *if* $I = [0,1]$ *(*evenness *if* $I = \mathbb{R}$*).*

REMARK 1.1 If invariance under positive scaling were imposed instead of invariance for translations, then we would move from the scenario of absolute dispersion measures to that of relative ones. However, in this paper we have focused our attention just on the first approach in order to establish connections with multidistances and remainders of exponential means, as will be shown along the paper.

Next we list some classical absolute dispersion measures commonly appearing in the literature. It is easy to check that they verify the above mentioned properties and hence they can also be examined under our approach. Some of them take into account the degree of clustering of the data considering the mean as reference. However, we do no pretend to be exhaustive. For example, some other less known possibilities taking into account the closeness to the median have been avoided.

DEFINITION 1.4 *Let* $\boldsymbol{x} \in I^n$.

1 *The* range *of* $\boldsymbol{x}$ *is defined as* $r(\boldsymbol{x}) = x_{(n)} - x_{(1)}$.

2 *The* variance *of* $\boldsymbol{x}$ *is defined as*

$$\sigma^2(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu(\boldsymbol{x}))^2.$$

3 *The* standard deviation *of* $\boldsymbol{x}$ *is defined as*

$$\sigma(\boldsymbol{x}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu(\boldsymbol{x}))^2}.$$

4 *The* mean deviation *of* $\boldsymbol{x}$ *is defined as*

$$md(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}|x_i - \mu(\boldsymbol{x})|.$$

5 *The Gini index ([10]), the most popular measure of inequality in welfare economics, was introduced by Corrado Gini in 1912. It is based*

on the average of the absolute differences between all possible pairs of observations (different formulations can be found in Yitzhaki [19] and Aristondo et al. *[1, Subsect. 3.1], among others).*

The relative Gini index *is defined as*

$$G(\boldsymbol{x}) = \frac{1}{2n^2\mu(\boldsymbol{x})} \sum_{i=1}^{n} \sum_{j=1}^{n} \left|x_i - x_j\right|, \text{ if } \mu(\boldsymbol{x}) \neq 0.$$

The absolute Gini index *is defined as*

$$G_a(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left|x_i - x_j\right|.$$

REMARK 1.2 For $n = 2$, it is interesting to note that both the standard deviation and the mean deviation of $\boldsymbol{x} = (x_1, \ldots, x_n)$ coincide with the *semi-range*, defined as $\frac{x_{(n)} - x_{(1)}}{2}$. In such case, also the mean coincides with the *mid-range* defined as $\frac{x_{(1)} + x_{(n)}}{2}$, and the range becomes four times the absolute Gini index.

REMARK 1.3 In the literature usually appears the *coefficient of variation* of $\boldsymbol{x} \in \mathbb{I}$ defined as $c_v(\boldsymbol{x}) = \dfrac{\sigma(\boldsymbol{x})}{|\mu(\boldsymbol{x})|}$, if $\mu(\boldsymbol{x}) \neq 0$. It is not properly an absolute dispersion measure according to the previous definition, because it vulnerates invariance for translations, and anti-self-duality when $I = [0,1]$. However, evenness is fulfilled when $I = \mathbb{R}$.

REMARK 1.4 It is easy to check that, in the list above, just the range is a stable absolute dispersion measure. On the other hand, according to computer simulations, the variance, standard deviation and mean deviation behave as contractive absolute dispersion measures. Obviously, an open problem consists on providing formal proofs of these facts. In the case of the absolute Gini index, its contractivity can be formally guaranteed in the next section under a multidistance approach.

4. Multidistances

The notion of multidistance was introduced by Martín and Mayor [15] from the classical definition of distance between two points (becoming elements in a metric space). In this way, from a more general point of view, these authors consider multidistances among any finite number of points by generalizying the usual triangle inequality (see also Martín and Mayor [16]).

DEFINITION 1.5 *A function* $M : \mathbb{I} \longrightarrow \mathbb{R}$ *is a* multidistance *if it satisfies the following conditions:*

1 Positiveness*:* $M(\boldsymbol{x}) \geq 0$*, for every* $\boldsymbol{x} \in \mathbb{I}$*.*

2 Identity of indiscernibles*:* $M(\boldsymbol{x}) = 0 \Leftrightarrow x_1 = \cdots = x_n$*, for all* $n \in \mathbb{N}$ *and* $\boldsymbol{x} \in I^n$*.*

3 Symmetry*:* $M(\boldsymbol{x}_\pi) = M(\boldsymbol{x})$*, for all* $n \in \mathbb{N}$*,* $\boldsymbol{x} \in I^n$ *and permutation* π *on* $\{1,\dots,n\}$*.*

4 Generalized triangle inequality*:* $M(\boldsymbol{x}) \leq M(x_1,y) + \cdots + M(x_n,y)$*, for all* $n \in \mathbb{N}$*,* $\boldsymbol{x} \in I^n$ *and* $y \in I$*.*

The first examples of multidistances proposed by Martín and Mayor [15] are the *drastic multidistances*. Among them, it is interesting the one defined as follows:

$$D(\boldsymbol{x}) = \#\{x_1,\dots,x_n\} - 1,$$

which also trivially fulfills all the conditions of absolute dispersion measures.

Another class of multidistances is that of *sum-based multidistances* (see Martín and Mayor [15, 16]), given by any function $D_\lambda : \mathbb{I} \longrightarrow \mathbb{R}$ such that

$$D_\lambda(\boldsymbol{x}) = \begin{cases} 0, & \text{if } n = 1, \\ \lambda(n) \sum\limits_{i<j} |x_i - x_j|, & \text{if } n \geq 2, \end{cases}$$

where $\lambda : \{2,3,\dots\} \longrightarrow \mathbb{R}$ is any discrete function such that $\lambda(2) = 1$ and $0 < \lambda(n) \leq \frac{1}{n-1}$ for $n > 2$ (this last condition stands for guaranteeing the generalized triangle inequality).

Notice that the absolute Gini index is just twice a multidistance of this family; even more, it is contractive (see Calvo *et al.* [5, Prop. 8]). However, not all sum-based multidistances are absolute dispersion measures. For example, considering $\lambda(n) = \frac{1}{n-1}$, invariance for replications fails:

$$D_\lambda(x_1,x_2) = \frac{1}{2-1}|x_2 - x_1| = |x_2 - x_1|,$$

whereas

$$D_\lambda(x_1,x_2,x_1,x_2) = \frac{1}{4-1}4|x_2 - x_1| = \frac{4}{3}|x_2 - x_1|.$$

Similarly, usual distances between all possible $\binom{n}{2}$ couples of n points can be used to achieve a multidistance by means of OWA operators. In order to define such *OWA-based multidistances* (see Martín *et al.* [17]), consider a weighting triangle where the entries are non-negative in each row and they add up to one, i.e.:

$$\begin{array}{ccccccccc}
 & & & & w_1^1 & & & & \\
 & & & w_1^2 & & w_2^2 & & & \\
 & & w_1^3 & & w_2^3 & & w_3^3 & & \\
 & w_1^4 & & w_2^4 & & w_3^4 & & w_4^4 & \\
w_1^5 & & w_2^5 & & w_3^5 & & w_4^5 & & w_5^5 \\
\dots\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\dots
\end{array}$$

where $w_j^i \geq 0$ and $\sum_{i=1}^{j} w_i^j = 1$.

Then we can define the function $D_W : \mathbb{I} \longrightarrow \mathbb{R}$ such that

$$D_W(\boldsymbol{x}) = \begin{cases} 0, & \text{if } n = 1, \\ W_n\overbrace{(|x_2 - x_1|, \dots, |x_n - x_{n-1}|)}^{\binom{n}{2}}, & \text{if } n \geq 2, \end{cases}$$

where W_n is the OWA operator whose weights are given by the $\binom{n}{2}$-th row and $w_1^{\binom{n}{2}} + \cdots + w_{n-1}^{\binom{n}{2}} > 0$ for all $n \geq 3$ (see Martín [14]).

REMARK 1.5 If all the left weights in the triangle are unitary, i.e., $w_1^{\binom{n}{2}} = 1$ for all $n \in \mathbb{N}$, we obtain the *maximum multidistance* (notice that this case coincides with the range, and hence it is an absolute dispersion measure). However, not all OWA-based multidistance are absolute dispersion measures. For example, if the weights are equal in each row, the corresponding OWA-based multidistance D_W becomes the sum-based multidistance with $\lambda(n) = \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$, which also vulnerates invariance for replications (in this case, it is easy to check that $D_W(x_1, x_2) = |x_2 - x_1| \neq D_W(x_1, x_2, x_1, x_2) = \frac{2}{3}|x_2 - x_1|$).

And also introduced by Martín and Mayor [15, 16], the *Fermat multidistance* $D_F : \mathbb{I} \longrightarrow \mathbb{R}$ is given by:

$$D_F(\boldsymbol{x}) = \min_{x \in I} \left\{ \sum_{i=1}^{n} |x_i - x| \right\} \qquad \text{where } \boldsymbol{x} \in I^n.$$

Once fixed $\boldsymbol{x} \in \mathbb{I}$, such minimum value is effectively reached by any of the points of the *Fermat set* associated with $\{x_1, \dots, x_n\}$. In our unidimensional context, a classic result (see, for instance, Jackson [12]) provides that such Fermat set is the singleton $\left\{ x_{\left(\frac{n+1}{2}\right)} \right\}$, i.e., the median, if n is odd; or the interval $\left[x_{\left(\frac{n}{2}\right)}, x_{\left(\frac{n+1}{2}\right)} \right]$, if n is even. In this case, it is usual to consider just any of the extremes: $x_{\left(\frac{n}{2}\right)}$, the lower median, or $x_{\left(\frac{n+1}{2}\right)}$, the higher median, or even the average of these two values. In what follows we will choose this last option, calling $\mathrm{med}(\boldsymbol{x})$ the median of $\{x_1,\ldots,x_n\}$. Notice that, as mentioned above, the argument of the minimum appearing in the expression of the Fermat multidistance can be rewritten as $\sum_{i=1}^{n} |x_i - \mathrm{med}(\boldsymbol{x})|$, which is exactly n times the mean deviation with respect to the median.

PROPOSITION 1.1 *The Fermat multidistance is an absolute dispersion measure. Moreover, it is an iterated sum of ranges of the data, where in each term the extreme values are sequentially withdrawn:*

$$D_F(\boldsymbol{x}) = \begin{cases} \left(x_{(n)} - x_{(1)}\right) + \left(x_{(n-1)} - x_{(2)}\right) + \cdots + \left(x_{\left(\frac{n}{2}+1\right)} - x_{\left(\frac{n}{2}\right)}\right), \\ \text{if } n \text{ is even}, \\ \left(x_{(n)} - x_{(1)}\right) + \left(x_{(n-1)} - x_{(2)}\right) + \cdots + \left(x_{\left(\frac{n+1}{2}+1\right)} - x_{\left(\frac{n+1}{2}-1\right)}\right), \\ \text{if } n \text{ is odd}. \end{cases}$$

Proof. The key idea is the very essence of the median, an intermediate value which divides de data in two sets on its left and right sides, each of them which exactly the same number of terms. Then we have

$$D_F(\boldsymbol{x}) = \sum_{i=1}^{n} |x_i - \mathrm{med}(\boldsymbol{x})| =$$
$$\left(x_{(n)} - \mathrm{med}(\boldsymbol{x})\right) + \left(\mathrm{med}(\boldsymbol{x}) - x_{(1)}\right) + \left(x_{(n-1)} - \mathrm{med}(\boldsymbol{x})\right) +$$
$$\left(\mathrm{med}(\boldsymbol{x}) - x_{(2)}\right) + \cdots = \left(x_{(n)} - x_{(1)}\right) + \left(x_{(n-1)} - x_{(2)}\right) + \cdots,$$

where the last terms in the corresponding two sums depend on the parity of n.
□

Thus, the Fermat multidistance inherits its condition of absolute dispersion measure from the range. And its is also true a sort of reciprocal: the above mentioned fact that the range is also a multidistance.

REMARK 1.6 The variance is not a multidistance because the triangle inequality fails:

$$\sigma^2(0,1) = 0.25 > \sigma^2(0,0.5) + \sigma^2(1,0.5) = 0.0625 + 0.0625 = 0.125.$$

On the other side, according to computer simulations, the standard deviation and the mean deviation behave as multidistances, but proofs of these conjectures are yet to be provided.

5. Remainders of exponential means

In a similar way to the previous scenarios, in what follows we can consider either aggregation functions with a fixed amount $n \in \mathbb{N}$ of input data in the unit interval, or extended aggregation functions defined for any $n \in \mathbb{N}$ (we will not distinguish the notation). Such number n is called the *arity* of the aggregation function.

DEFINITION 1.6

1. *A function* $A : [0,1]^n \longrightarrow [0,1]$ *is called an n-*ary aggregation function *if it is monotonic and satisfies* $A(\mathbf{1}) = 1$ *and* $A(\mathbf{0}) = 0$.
2. *A function* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \longrightarrow [0,1]$ *is called an* extended aggregation function *if* $A|_{[0,1]^n}$ *is an n-ary aggregation function for every* $n \in \mathbb{N}$.

For the sake of simplicity, the n-arity is omitted whenever it is clear from the context.

It is easy to see that for aggregation functions idempotentcy and compensativeness are equivalent concepts.

5.1 The remainder of an aggregation function

We now briefly recall the remainder of an aggregation function, due to García-Lapresta and Marques Pereira [8, 9].

DEFINITION 1.7 *Given an aggregation function* $A : [0,1]^n \longrightarrow [0,1]$*, the function*
$\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ *defined as*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A(\mathbf{1}-\boldsymbol{x}) - 1}{2}$$

is called the remainder *of A.*

Clearly, $\widetilde{A}$ is not an aggregation function: $\widetilde{A}(\mathbf{1}) = 0$.

The following result can be found in García-Lapresta and Marques Pereira [9] (excepting that invariance for replications is inherited by the remainder; the proof is immediate).

PROPOSITION 1.2 *The remainder $\widetilde{A}$ inherits from the aggregation function A the properties of continuity, symmetry, invariance for replications, whenever A has these properties.* □

The following result provide two more properties of the remainder (see García-Lapresta and Marques Pereira [9]).

PROPOSITION 1.3 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

1. *If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ for every $x \in [0,1]$.*
2. *If A is stable for translations, then $\widetilde{A}$ is invariant for translations.* □

The first statement establishes that remainders of idempotent aggregation functions are null on the main diagonal. The second statement applies to aggregation functions satisfying stability for translations. In such case, remainders are invariant for translations. These properties of the remainder $\widetilde{A}$ suggest that it may give some information about the dispersion of the coordinates of a vector in $[0,1]^n$.

5.2 Exponential means

Quasiarithmetic means are the only aggregation functions satisfying continuity, idempotency, symmetry, strict monotonicity and decomposability (see Kolmogoroff [13], Nagumo [18] and Fodor and Roubens [7, pp. 112-114]).

Exponential means are the only quasiarithmetic means satisfying stability for translations.

Given $\alpha \neq 0$, the *exponential mean* $A_\alpha : [0,1]^n \longrightarrow [0,1]$ is the aggregation function defined as

$$A_\alpha(\boldsymbol{x}) = \frac{1}{\alpha} \ln \frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{n}.$$

We now focus on the remainders of exponential means. Given $\alpha \neq 0$, the *remainder* of A_α is the mapping $\widetilde{A}_\alpha : [0,1]^n \longrightarrow \mathbb{R}$ defined as

$$\widetilde{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \frac{(e^{\alpha x_1} + \cdots + e^{\alpha x_n})(e^{-\alpha x_1} + \cdots + e^{-\alpha x_n})}{n^2}.$$

For every $\alpha \neq 0$, $\widetilde{A}$ satisfies identity of indiscernibles. Moreover, $\widetilde{A}_\alpha$ is continuous, symmetric, anti-self-dual, invariant for translations and invariant for replications (see García-Lapresta and Marques Pereira [9, Sect. 6] for details).

The following result presents the parameter limits of the remainders of exponential means (see García-Lapresta and Marques Pereira [9, Prop. 35]).

PROPOSITION 1.4 *For every $\boldsymbol{x} \in [0,1]^n$, the following statements hold:*

1. $\displaystyle\lim_{\alpha \to \infty} \widetilde{A}_\alpha(\boldsymbol{x}) = \frac{x_{(n)} - x_{(1)}}{2}.$
2. $\displaystyle\lim_{\alpha \to -\infty} \widetilde{A}_\alpha(\boldsymbol{x}) = -\frac{x_{(n)} - x_{(1)}}{2}.$

3 $\lim_{\alpha\to 0}\widetilde{A}_\alpha(\boldsymbol{x}) = 0.$ □

PROPOSITION 1.5 *$\widetilde{A}_\alpha$ is an absolute dispersion measure for every $\alpha > 0$.*

Proof. It will be shown the only remaining condition, i.e., that A_α is non-negative for every $\alpha > 0$. To this aim, it suffices to check that the argument of the logarithm appearing in the expression of the remainder is greater than or equal to 1. This happens because its numerator is greater than the denominator:

$$(e^{\alpha x_1} + \cdots + e^{\alpha x_n})(e^{-\alpha x_1} + \cdots + e^{-\alpha x_n}) =$$

$$\sum_{i=1}^{n} e^{\alpha(x_i - x_i)} + \sum_{i<j}(e^{\alpha(x_i-x_j)} + e^{\alpha(x_j-x_i)}) =$$

$$n + \sum_{i<j} 2\cosh(\alpha(x_i - x_j)) \geq n + 2\frac{(n-1)n}{2} = n + (n-1)n = n^2,$$

where the well known property $\cosh z = \frac{e^z+e^{-z}}{2} \geq 1$ of the hyperbolic cosine has been taken into account in each of the $1+2+\cdots+(n-1) = \frac{(n-1)n}{2}$ terms of the last summation. Hence, the logarithm is always non-negative and the exponential remainder stands as an absolute dispersion measure for $\alpha > 0$. □

Notice that, if $\alpha < 0$, then $\widetilde{A}_{-\alpha}(\boldsymbol{x}) = -\widetilde{A}_\alpha(\boldsymbol{x})$, and hence positiveness (required both for absolute dispersion measures and multidistances) is vulnerated. In fact, even for $\alpha > 0$, it is is easy to check that remainders of exponential means are not multidistances. For example, if $\alpha = 0.5$, we obtain:

$$\widetilde{A}_{0.5}(0,1) = 0.06186 > \widetilde{A}_{0.5}(0,0.5) + \widetilde{A}_{0.5}(1,0.5) =$$
$$0.01358 + 0.01358 = 0.02716$$

and, consequently, the triangle inequality does not hold.

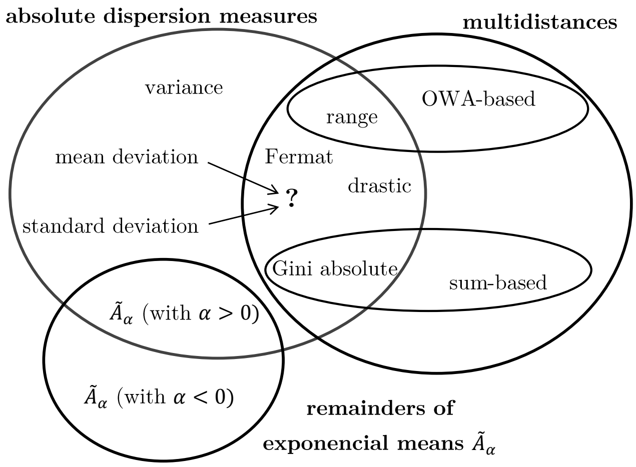
6. Concluding remarks

First, in Fig. 1.1 we show a panoramic diagram connecting the concepts appeared along the paper.

Next, we show a comprehensive table on the fulfillment of several properties considered above. Conjectures appear enclosed in parentheses.

	symm.	anti-self-dual.	evenness	trans. inv.	replic. inv.	contract.	stabil.
range	√	√	√	√	√	X	√
variance	√	√	√	√	√	(√)	X
standard deviation	√	√	√	√	√	(√)	X
mean deviation	√	√	√	√	√	(√)	X
absolute Gini ind.	√	√	√	√	√	√	X
coef. var.	√	X	X	X	√	(√)	X
exp. remainder	√	√	–	√	√	(√)	X

Figure 1.1. Relationship among concepts.



As commented above, although some open problems remain formally unsolved due to the complexity in their treatment, we consider that our approach opens up wider perspectives and sheds new light into the dispersion analysis.

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