GENERATION OF SHEET CURRENTS BY HIGH FREQUENCY FAST MHD WAVES

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Abstract

The evolution of fast magnetosonic waves of high frequency propagating into an axisymmetric equilibrium plasma is studied. By using the methods of weakly nonlinear geometrical optics, it is shown that the perturbation travels in the equatorial plane while satisfying a transport equation which enables us to predict the time and location of formation of shock waves. For plasmas of large magnetic Prandtl number, this would result into the creation of sheet currents which may give rise to magnetic reconnection and destruction of the original equilibrium.

1 Introduction

Linear stability of ideal MHD equilibria is one of the oldest and most satisfactorily solved topics in classical magnetohydrodynamics [1, 2]. No one expects, however, that even starting with a small perturbation of the equilibrium the linear predictions will hold for ever: nonlinear effects will appear and modify the plasma evolution, so that linear stability of a given equilibrium does not

guarantee that the plasma geometry will not break at some point. Nonlinear MHD, however, is a far more difficult subject and numerical methods play a large part in its predictions. We will consider one of the few instances where an analytic treatment exists and provides an accurate description of the plasma behavior: it concerns a high frequency perturbation propagating into the plasma as a fast magnetosonic wave. The success of this technique may be attributed to a satisfactory knowledge of the area of nonlinear geometrical optics. Although several special cases had been studied before [3,4], the first treatment for general hyperbolic systems is probably [5], later rigorously established and enormously extended. There exists a vast literature related to this method; we will make use of two excellent survey articles [6,7]. Among the books dealing in part with this subject we will cite the classical [8,9], plus two more recent ones [10,11]. In [10] the author advocates a modified approach to the asymptotic transport equation on which the rays themselves are variable. While his arguments are convincing enough, in our case the rays will be forced to be constant by the symmetry of the geometric configuration, so the old theory will be equally accurate. The reason why we choose fast magnetosonic waves is that we will deal with waves propagating across the magnetic field lines of the equilibrium, and neither slow nor Alfvén waves are able to do that. Formation of shock waves is a general feature of solutions of quasilinear hyperbolic systems, but in the case of ideal MHD it is equally important the creation of discontinuities of the magnetic field and therefore of sheet currents. The Rankine-Hugoniot relations for ideal MHD and the evolution of geometrically simple shocks are studied e.g in [12]. The importance of creation of current sheets and its outcome as rapid magnetic reconnection may be hardly overstated; solar flares are created by this process. In [13] it is shown how by varying the boundary condition of a simple solution of the Grad-Shafranov equation a current sheet and a magnetic eruption is eventually formed; although perhaps in a form not so spectacular, the probable outcome of the formation of a current sheet in an MHD equilibrium is its destruction, in a form not envisaged by linear stability analysis. Not only the magnetic field, but also the radial velocity and the density will be discontinuous at the shock, thus adding a further source of instability. To guess which of the two effects is stronger we must recall that we are dealing with ideal MHD, whereas all real

plasmas are diffusive (viscous and resistive) to some degree. Weakly nonlinear geometrical optics has been generalized to account for a small diffusivity [7], albeit for a single value of the diffusivity coefficient, but the analysis should hold with small modifications for the two coefficients of viscosity and resistivity. We must expect the shock to be replaced by a sharp, but continuous, gradient of the main quantities. The width of the region of rapid variation is proportional to the square root of the diffusivity, so that the magnetic jump is sharper than the kinetic one if the resistivity is smaller than the viscosity, and the opposite otherwise. This indicates that the magnetic disruption is more likely than the kinetic one if the magnetic Prandtl number Pr_m is large. It is known that Pr_m is large for hot thin diffuse plasmas such as the ones occurring in fusion devices, galaxies ($Pr_m \sim 10^{14}$) and galactic clusters ($Pr_m \sim 10^{19}$) [14], whereas Pr_m is small for denser environments such as liquid metal flows ($Pr_m \sim 10^{-5}$) and stellar convective zones ($Pr_m \sim 10^{-7} - 10^{-4}$ for the Sun) [15]. Thus for an axisymmetric equilibrium such as the ones posited for magnetic fusion devices, it is more likely that $Pr_m \gg 1$ and the effect of current sheets will be more relevant than the one of material shock waves.

2   Weakly nonlinear geometrical optics

The main methods of weakly nonlinear geometrical optics are well known, but both settings and construction vary somewhat in the literature. Most texts start from a state $\mathbf{u} = \mathbf{0}$, to which any quasilinear hyperbolic system may be reduced, but it is preferable to incorporate a nontrivial state $\mathbf{u}_0$ so that both magnitudes and constants correspond to the physical ones. Thus we reconstruct briefly the essential construction. We start from a quasilinear hyperbolic system

$$\frac{\partial \mathbf{u}}{\partial t} + A_j(t, \mathbf{x}, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x_j} + \mathbf{C}(t, \mathbf{x}, \mathbf{u}) = \mathbf{0}, \tag{1}$$

where we will follow the Einstein summation convention. In most cases the coefficients do not depend on t. The most important case occurs when the system proceeds from a conservation law,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}_j(\mathbf{u})}{\partial x_j} = \mathbf{0}, \tag{2}$$

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because in this case the asymptotic expansions hold everywhere, including the shock [7]. This is the case for ideal magnetohydrodynamics system [2], where in fact A_j and C are linear functions of the velocity and magnetic field, although not of the density. We look for small perturbations of a given static solution $\mathbf{u}_0$. The first step is to take the first term of the Taylor expansion of A_j near $\mathbf{u}_0$ (whence the term weakly nonlinear):

$$A_j(\mathbf{u}) \rightarrow A_j(\mathbf{u}_0) + \nabla_u A_j(\mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0), \tag{3}$$

where the last term means

$$\nabla_u A_j(\mathbf{u}_0) \cdot (\mathbf{u} - \mathbf{u}_0) = (\mathbf{u} - \mathbf{u}_0) \cdot \nabla_u A_j(\mathbf{u}_0) = (u_k - u_{0k}) \frac{\partial A_j}{\partial u_k}(\mathbf{u}_0). \tag{4}$$

The same notation is used for the vector $\mathbf{C}$. As asserted, for the MHD case the new expansion represents exactly the original term for all the variables except for the density. If we denote by $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$, system (1) may be written as

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla_u A_j(\mathbf{u}_0) \cdot \mathbf{v}) \frac{\partial \mathbf{v}}{\partial x_j} + A_j(\mathbf{u}_0) \frac{\partial \mathbf{v}}{\partial x_j} + D(\mathbf{v}) = \mathbf{0}, \tag{5}$$

where

$$D(\mathbf{v}) = (\nabla_u A_j(\mathbf{u}_0) \cdot \mathbf{v}) \frac{\partial \mathbf{u}_0}{\partial x_j} + \mathbf{v} \cdot \nabla_u \mathbf{C}(\mathbf{u}_0). \tag{6}$$

We look for oscillatory solutions of amplitude $\epsilon \ll 1$. An asymptotic formal expansion (really a multiple scale one) is posed:

$$\mathbf{v} = \epsilon \mathbf{v}_1(t, \mathbf{x}, \theta) + \epsilon^2 \mathbf{v}_2(t, \mathbf{x}, \theta) + \dots, \tag{7}$$

where $\theta = \tau/\epsilon$ and τ is the phase of the wave. The usual notation for τ is ϕ, but we reserve it for the azimuthal angle. Taking (7) to (5) and collecting terms of order zero, we find

$$\left(\frac{\partial \tau}{\partial t} + A_j(t, \mathbf{x}, \mathbf{u}_0) \frac{\partial \tau}{\partial x_j} \right) \frac{\partial \mathbf{v}_1}{\partial \theta} = \mathbf{0}. \tag{8}$$

Since we want a nontrivial dependence of $\mathbf{v}_1$ on θ, $\partial \mathbf{v}_1/\partial \theta$ must be a right eigenvector of the matrix on the left, which must satisfy the eikonal equation

$$\det \left(\frac{\partial \tau}{\partial t} I + A_j(t, \mathbf{x}, \mathbf{u}_0) \frac{\partial \tau}{\partial x_j} \right) = 0. \tag{9}$$

The system is assumed to be strictly hyperbolic, so the eigenvalues are different. Once one such eigenvalue $\lambda(t, \mathbf{x}, \mathbf{u}_0)$ and associated right eigenvector $\mathbf{R}(t, \mathbf{x}, \mathbf{u}_0)$ are chosen, depending smoothly on t and $\mathbf{x}$, one gets

$$\mathbf{v}_1(t, \mathbf{x}, \theta) = w(t, \mathbf{x}, \theta)\mathbf{R}(t, \mathbf{x}, \mathbf{u}_0). \tag{10}$$

The first order term in $D(\mathbf{v})$ is

$$\epsilon(\mathbf{v}_1 \cdot \nabla_u A_j(\mathbf{u}_0))\frac{\partial \mathbf{u}_0}{\partial x_j} + \epsilon \mathbf{v}_1 \cdot \nabla_u \mathbf{C}(\mathbf{u}_0). \tag{11}$$

The first order term in (5) is therefore

$$\left(\frac{\partial \tau}{\partial t} I + A_j(t, \mathbf{x}, \mathbf{u}_0)\frac{\partial \tau}{\partial x_j}\right)\frac{\partial \mathbf{v}_2}{\partial \theta} + \frac{\partial \mathbf{v}_1}{\partial t} + A_j(\mathbf{u}_0)\frac{\partial \mathbf{v}_1}{\partial x_j}$$

$$+\mathbf{v}_1 \cdot \nabla_u \mathbf{C}(\mathbf{u}_0) + (\mathbf{v}_1 \cdot \nabla_u A_j(\mathbf{u}_0))\frac{\partial \mathbf{u}_0}{\partial x_j} + \frac{\partial \tau}{\partial x_j}(\mathbf{v}_1 \cdot \nabla_u A_j(\mathbf{u}_0))\frac{\partial \mathbf{v}_1}{\partial \theta} = \mathbf{0}. \tag{12}$$

Substituting $\mathbf{v}_1$ by the formula in (10), and multiplying from the left by a left eigenvector $\mathbf{L}$ of the matrix in (9), one obtains

$$\mathbf{L} \cdot \mathbf{R}\frac{\partial w}{\partial t} + \mathbf{L} \cdot \left((A_j(\mathbf{u}_0) \cdot \mathbf{R})\frac{\partial w}{\partial x_j}\right) + p_0 w + q_0 w \frac{\partial w}{\partial \theta} = 0. \tag{13}$$

where

$$q_0 = \mathbf{L} \cdot (\mathbf{R} \cdot \nabla_u A_j(\mathbf{u}_0)(\mathbf{R}))\frac{\partial \tau}{\partial x_j}, \tag{14}$$

$$p_0 = \mathbf{L} \cdot \left[\frac{\partial \mathbf{R}}{\partial t} + A_j(\mathbf{u}_0)\frac{\partial \mathbf{R}}{\partial x_j} + \nabla_u \mathbf{C}(\mathbf{u}_0) \cdot \mathbf{R} + (\mathbf{R} \cdot \nabla_u A_j(\mathbf{u}_0))\frac{\partial \mathbf{u}_0}{\partial x_j}\right]. \tag{15}$$

The wavefronts are the surfaces $\tau = \text{const.}$, whereas the rays satisfy

$$\frac{dx_j}{dt} = \mathbf{L} \cdot A_j(\mathbf{u}_0)(\mathbf{R})[\mathbf{L} \cdot \mathbf{R}]^{-1}, \tag{16}$$

so that the first two terms in (13) represent a derivative along the ray. Since $\mathbf{L}$ and $\mathbf{R}$ are determined up to a multiplicative constant, this equation is not unique, but by choosing these eigenvectors smooth and bounded above and below, which is always possible, the location of the shocks is fixed. We will see that (13) may be set in the form of a multidimensional generalized Burgers equation (without dissipation), which seems to predict the presence of a shock; however, since v_1 is a zero order term in an asymptotic approximation, it could happen that this shock is an artifact of the approximation. That this is not true

is a consequence of the fact that the original equation may be set in conservative form [7].

The next problem lies in the character of the solution as a function of θ. Two main cases are usually studied: to take w almost periodic in θ or to assume the existence of limits

$$\lim_{\theta \to \pm\infty} w(t, \mathbf{x}, \theta). \tag{17}$$

Since we wish to study perturbations initially localized near the boundary of the plasma, or even in a part of the interior bounded by two wavefronts, the logical limit in (17) is 0. In fact the properties of Burgers' equation guarantee that if the initial condition has compact support as a function of θ, the same happens for all time. In this case [7] the asymptotic expansion is uniform, which means that the next term in the expansion of $\mathbf{v}$ is bounded; in particular, $\|\epsilon \mathbf{v}_2\| \ll \|\mathbf{v}_1\|$.

Other eigenvalues of the eikonal equation yield other phases; since the system is nonlinear these may interact with each other, giving rise to the phenomenon of resonance. This is a real physical possibility, but unfortunately the solutions become extremely complex [16]. Nevertheless, it has been shown that for w having compact support in θ the waves are non resonant [17]. Moreover, in our case the fast wave will travel into the equilibrium, whereas the slow and Alfvén ones cannot propagate across magnetic field lines and will be left behind.

It is well known that rays may collide and wavefronts fold into themselves, forming caustics which make the coefficients in (13) go to infinity. In our case, axisymmetry will force all wavefronts to be circular in the intersection with the equatorial plane, and rays radial. Depending on the initial condition for τ, caustics may occurs above and below this plane, but we will argue later that this is unlikely to occur for a judicious choice of equilibrium state and initial condition in the MHD case, which we now proceed to develop.

3   Propagation in axisymmetric equilibria

We first to state the equations of ideal MHD in cylindrical coordinates (z, r, ϕ). The main quantities are velocity, magnetic field and two of three thermodynamic quantities: density ρ, entropy S and pressure P, related by a state equation. We

will follow ([18], p. 16) and use the density and entropy as primary variables. The entropy in fact is merely transported by the flow and may be uncoupled from the rest of the equations, so we will ignore it and keep just seven variables; the three components of the velocity and the three components of the magnetic field, plus the density. We also assume axisymmetry ($\partial/\partial\phi = 0$). Let us denote the velocity and magnetic field by

$$\mathbf{v} = u\hat{z} + v\hat{r} + w\hat{\phi}, \tag{18}$$

$$\mathbf{B} = B_z\hat{z} + B_r\hat{r} + B_\phi\hat{\phi}. \tag{19}$$

The system occurring in [18] may be set in a more symmetric way by using the fact the divergence of the magnetic field is zero, that is

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\phi}{\partial\phi} + r\frac{\partial B_z}{\partial z}\right) = 0. \tag{20}$$

This implies

$$\frac{\partial B_r}{\partial r} = -\frac{B_r}{r} - \frac{\partial B_z}{\partial z}. \tag{21}$$

Let I_7 denote the 7×7 identity matrix, and we denote the derivative of the pressure P with respect to the density as P_ρ. Once we choose units so as to take the magnetic permeability of free space μ_0 as 1, the hyperbolic system of ideal MHD may be written as

$$\frac{\partial\mathbf{u}}{\partial t} + A_z\frac{\partial\mathbf{u}}{\partial z} + A_r\frac{\partial\mathbf{u}}{\partial r} + \mathbf{C} = \mathbf{0}, \tag{22}$$

where

$$\mathbf{u} = (u, v, w, B_z, B_r, B_\phi, \rho), \tag{23}$$

$$A_z = uI_7 + \begin{bmatrix} 0 & 0 & 0 & 0 & B_r/\rho & B_\phi/\rho & P_\rho/\rho \\ 0 & 0 & 0 & 0 & -B_z/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_z/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_r & -B_z & 0 & 0 & 0 & 0 & 0 \\ B_\phi & 0 & -B_z & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

$$A_r = vI_7 + \begin{bmatrix} 0 & 0 & 0 & -B_r/\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & B_z/\rho & 0 & B_\phi/\rho & P_\rho/\rho \\ 0 & 0 & 0 & 0 & 0 & -B_r/\rho & 0 \\ -B_r & B_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_\phi & -B_r & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (25)$$

$$\mathbf{C} = \begin{bmatrix} 0 \\ -w^2/r + B_\phi^2/\rho r \\ vw/r - B_r B_\phi/\rho r \\ B_z v/r \\ B_r v/r \\ \rho w/r \\ \rho v/r \end{bmatrix}. \qquad (26)$$

Without axisymmetry a further matrix A_ϕ would appear; its specific form will not be necessary. We will assume that the equilibrium quantities as well as the initial conditions for the perturbation are symmetric with respect to the middle plane $\Pi : z = 0$. With respect to the variable z, we could take B_z even, B_r and B_ϕ odd, and u odd, v and w even; or B_z and B_ϕ even, B_r odd, and u and w odd, v even. This last situation is the appropriate one for tokamaks; think of magnetic field lines coiling around axisymmetric tori, which means that B_ϕ is even. This implies

$$\frac{\partial B_z}{\partial z}(0, r) = 0, \qquad (27)$$

and the same for B_ϕ and ρ. As for B_r, since it is an odd function of z,

$$B_r(0, r) = 0. \qquad (28)$$

Let us consider now the eikonal equation in Π. Let us start with an axisymmetric initial condition $\tau(0, r, z)$ such that $\tau(0, r, z) = \tau(0, r, -z)$. The eikonal equation

for the fast mode is

$$\left(\frac{\partial \tau}{\partial t}\right)^2 = \frac{1}{2}\left(P_\rho + \frac{B^2}{\rho}\right)|\nabla\tau|^2$$

$$+\frac{1}{2}\left[\left(P_\rho + \frac{B^2}{\rho}\right)^2 |\nabla\tau|^4 - 4P_\rho \frac{(\mathbf{B}\cdot\nabla\tau)^2}{\rho}|\nabla\tau|^2\right]^{1/2}. \tag{29}$$

Since we have assumed that ρ and B_z are even functions of z, while B_r is odd, the solutions are even functions of z: thus $B_r \partial\tau/\partial z$ is even. In fact, Eq. (29) may be restricted to the equatorial plane because τ is even, so $\partial\tau/\partial z = 0$ at $z = 0$. Given the axisymmetry and the z-symmetry, wavefronts intersect the plane Π as circumferences and their normals are radial. The eigenvalue associated to the fast magnetosonic wave is given by

$$2\mu^2 = c^2 + \sqrt{c^4 - \frac{4B_r^2 P_\rho}{\rho}}, \tag{30}$$

where c is the total velocity, sum of the sound and Alfvén ones:

$$c^2 = P_\rho + \frac{B^2}{\rho}. \tag{31}$$

For $z = 0$, the eikonal equation reduces to

$$\frac{\partial\tau}{\partial t} \pm c(r)\frac{\partial\tau}{\partial r} = 0. \tag{32}$$

Thus the wavefronts move faster orthogonally to the magnetic field, where $\mathbf{B}\cdot\nabla\tau = 0$; and slower when parallel to it. If we start at $t = 0$ with a wedge-shaped wavefront symmetric with respect to the plane $z = 0$, it should move faster at the vertex, i.e. the equatorial plane, at slower at the sides, so that one should expect the wedge to sharpen as it advances into the plasma, a configuration which does not cross itself and therefore does not yield caustics. Obviously this is a rough argument and it depends on the characteristics of pressure and magnetic field outside the plane, but it indicates that the time where our analysis holds is likely to be long enough.

If we want the wavefront to move to the right as t grows we must choose the plus sign. With this convention,

$$\tau(t, r, 0) = F(r) - t, \quad F'(r) = \frac{1}{c(r)}. \tag{33}$$

Thus

$$\frac{\partial \tau}{\partial r}(t, r, 0) = \frac{1}{c(r)}, \tag{34}$$

whereas by the z-symmetry,

$$\frac{\partial \tau}{\partial z}(t, r, 0) = 0. \tag{35}$$

Provided the constant μ in (30) satisfies $\mu^2 \neq B_r^2/\rho$ (which could hold only in trivial cases, since it would imply $P_\rho = B_z = B_\phi = 0$), the right and left eigenvectors associated to the fast wave turn out to be

$$\mathbf{R} = \left(-\frac{B_z B_r}{\rho \mu}, \mu - \frac{B_r^2}{\rho \mu}, -\frac{B_r B_\phi}{\rho \mu}, B_z, 0, B_\phi, \frac{\rho}{\mu}\left(\mu - \frac{B_r^2}{\mu} \right) \right), \tag{36}$$

$$\mathbf{L} = \left(-\frac{B_z B_r}{\mu}, \rho\left(\mu - \frac{B_r^2}{\rho \mu} \right), -\frac{B_r B_\phi}{\mu}, B_z, 0, B_\phi, \frac{P_\rho}{\mu}\left(\mu - \frac{B_r^2}{\mu} \right) \right), \tag{37}$$

up to multiplication by a real constant. For convenience we write vectors as rows instead of columns. In the plane Π, where $B_r = 0$, those vectors simplify to

$$\mathbf{R} = (0, c, 0, B_z, 0, B_\phi, \rho), \tag{38}$$

$$\mathbf{L} = (0, \rho c, 0, B_z, 0, B_\phi, P_\rho), \tag{39}$$

so that

$$\mathbf{L} \cdot \mathbf{R} = \rho c^2 + B_z^2 + B_\phi^2 + \rho P_\rho = 2\rho c^2. \tag{40}$$

Let us start with a static equilibrium. Let us rewrite (13) for this axisymmetric case at the plane Π:

$$2\rho c^2 \left[\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial r} \right] + p_0 w + q_0 w \frac{\partial w}{\partial \theta} = 0, \tag{41}$$

with

$$q_0 = \frac{1}{c}\mathbf{L} \cdot [\mathbf{R} \cdot \nabla_u A_r(\mathbf{u}_0)(\mathbf{R})] \tag{42}$$

$$p_0 = \mathbf{L} \cdot \left[A_r(\mathbf{u}_0)\frac{\partial \mathbf{R}}{\partial r} + A_z(\mathbf{u}_0)\frac{\partial \mathbf{R}}{\partial z} + \nabla_u \mathbf{C}(\mathbf{u}_0) \cdot \mathbf{R} \right]$$

$$+\mathbf{L} \cdot \left[(\mathbf{R} \cdot \nabla_u A_r(\mathbf{u}_0)\frac{\partial \mathbf{u}_0}{\partial r} + \mathbf{R} \cdot \nabla_u A_z(\mathbf{u}_0)\frac{\partial \mathbf{u}_0}{\partial z} \right]. \tag{43}$$

The static state will have the form

$$\mathbf{u}_0 = (0, 0, 0, B_z, B_r, B_\phi, \rho).$$ (44)

We could study equilibria with flow, $\mathbf{v} \neq \mathbf{0}$, but besides adding complexity to the results, equilibria with flow are usually unstable and in fact the flows which allow for an ideal MHD equilibrium are very few [19]. The components of the magnetic field and the pressure satisfy the Grad-Shafranov equation. When calculating the terms in (42-43) we must take into account that to find derivatives with respect to z one must use the general form, not the one restricted to the equatorial plane Π; thus, for instance, in $\partial \mathbf{R}/\partial z$ we need (36), not (38). After some hard work one finds

$$\mathbf{L} \cdot A_r(\mathbf{u}_0) \frac{\partial \mathbf{R}}{\partial r} = c \frac{\partial}{\partial r} \left(P + \frac{B^2}{2} \right) + \rho c^2 \frac{\partial c}{\partial r}$$

$$\mathbf{L} \cdot A_z(\mathbf{u}_0) \frac{\partial \mathbf{R}}{\partial z} = \frac{B_z P_\rho}{c} \frac{\partial B_r}{\partial z}$$

$$\mathbf{L} \cdot \nabla_u \mathbf{C}(\mathbf{u}_0) \cdot \mathbf{R} = \frac{\rho c^3}{r}$$

$$\mathbf{L} \cdot \left(\mathbf{R} \cdot \nabla_u A_r(\mathbf{u}_0) \frac{\partial \mathbf{u}_0}{\partial r} \right) = c \left(\frac{1}{2} \frac{\partial B^2}{\partial r} + \rho P_{\rho\rho} \frac{\partial \rho}{\partial r} \right)$$

$$\mathbf{L} \cdot \left(\mathbf{R} \cdot \nabla_u A_z(\mathbf{u}_0) \frac{\partial \mathbf{u}_0}{\partial z} \right) = -B_z c \frac{\partial B_r}{\partial z}$$

$$\frac{1}{c} \mathbf{L} \cdot [\mathbf{R} \cdot \nabla_u A_r(\mathbf{u}_0)(\mathbf{R})] = 2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}.$$ (45)

Let us change now the variables (r, t) to (r', θ), where $r' = r$, $\theta = \tau(r, t)/\epsilon$. Then

$$\frac{D(r, t)}{D(r', \theta)} = \begin{bmatrix} 1 & 0 \\ 1/c & -\epsilon \end{bmatrix}$$

$$\frac{\partial}{\partial r'} = \frac{1}{c} \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial r} \right)$$

$$\frac{\partial}{\partial \theta} = -\epsilon \frac{\partial}{\partial t}.$$ (46)

Calling again r to r', and taking into account that P_ρ is the square of the sound

speed c_s^2, equation (41) becomes

$$\frac{\partial w}{\partial r} + \left(\frac{\partial}{\partial r} \ln \sqrt{\rho c^3 r} - \frac{B_z \rho (c_s^2 + c^2)}{2\rho^2 c^4} \frac{\partial B_r}{\partial z} \right) w$$
$$+ \frac{2\rho c^2 + B^2 + \rho^2 P_{\rho\rho}}{2\rho c^3} w \frac{\partial w}{\partial \theta} = 0. \tag{47}$$

We do not simplify ρ in the coefficient of w because ρP_ρ and ρc^2 are simpler than P_ρ and c^2. The derivative $\partial B_r / \partial z$ may be related to the curvature of the field line of the poloidal component of the magnetic field. If this is parametrized by $s \to (r(s), z(s))$, with $\dot{r} = B_r$, $\dot{z} = B_z$, the (signed) curvature may be written by the formula appropriate for any plane curve:

$$\kappa = \frac{B_r \dot{B}_z - B_z \dot{B}_r}{(B_z^2 + B_r^2)^{1/2}}, \tag{48}$$

which at the plane Π becomes

$$\kappa = -\frac{\dot{B}_r}{B_z^2}. \tag{49}$$

Thus

$$B_z \frac{\partial B_r}{\partial z} = \dot{B}_r = -B_z^2 \kappa, \tag{50}$$

and

$$-\frac{B_z \rho (c_s^2 + c^2)}{2\rho^2 c^4} \frac{\partial B_r}{\partial z} = \frac{B_z^2 \rho (c_s^2 + c^2)}{2\rho^2 c^4} \kappa. \tag{51}$$

We see that this is the only term where the values of the field outside Π are necessary. The curvature is negative when the poloidal field lines are concave with respect to Π, positive if convex. When the equilibrium configuration possesses as usual a magnetic axis, the first occurs in the left hand side, the second in the right hand one. We will see how this contributes to the formation of a shock wave in the next section.

4   Shock formation time

Let us abbreviate (47) to

$$\frac{\partial w}{\partial r} + pw + qw \frac{\partial w}{\partial \theta} = 0. \tag{52}$$

Let r_0 be any fixed radius; logically we should take r_0 as the point around which the original perturbation is concentrated. First we define

$$\sigma(r,\theta) = w(r,\theta) \exp\left(\int_{r_0}^r p(s)\,ds\right). \tag{53}$$

then σ satisfies

$$\frac{\partial\sigma}{\partial r} + q(r)\exp\left(\int_{r_0}^r p(s)\,ds\right)\sigma\frac{\partial\sigma}{\partial\theta} = 0. \tag{54}$$

Next we change the variable r to ℓ, given by

$$\ell(r) = \int_{r_0}^r q(s)\exp\left(\int_{r_0}^s p(\xi)\,d\xi\right)\,ds. \tag{55}$$

Notice that since $q > 0$, this is a valid variable. Equation (52) becomes

$$\frac{\partial\sigma}{\partial\ell} + \sigma\frac{\partial\sigma}{\partial\theta} = 0, \tag{56}$$

precisely the inviscid Burgers equation. Notice that

$$\sigma_0(\theta) = \sigma(\ell = 0, \theta) = \sigma(r = r_0, \theta) = w(r_0, \theta). \tag{57}$$

Although we could recover the full solution from the known implicit expression of Burgers' equation solution, this is not too useful because as asserted the choosing of $\mathbf{R}$ and $\mathbf{L}$ is arbitrary to a multiplicative factor and therefore w does not need to be a physically meaningful variable. What is definitely objective is the time and location of the formation of the shock wave. It is well known that the first value of the variable ℓ for which the solution ceases to be a function and becomes a shock is

$$\ell_{\text{break}} = -\left(\inf_{\theta\in\mathbb{R}}\frac{d\sigma_0}{d\theta}\right)^{-1} = -\left(\inf_{\theta\in\mathbb{R}}\frac{\partial w}{\partial\theta}(r_0,\theta)\right)^{-1}. \tag{58}$$

Given the definition of ℓ in (55), a shock occurs for $r = r_{\text{break}}$ with

$$\int_{r_0}^{r_{\text{break}}} q(s)\exp\left(-\int_{r_0}^s p(\xi)\,d\xi\right)\,ds = -\left(\inf_{\theta\in\mathbb{R}}\frac{\partial w}{\partial\theta}(r_0,\theta)\right)^{-1}. \tag{59}$$

We have found in (47) the coefficients p and q. Incorporating the form of p given by (51), which is more intuitive geometrically, we find that the location of the shock is given implicitly by

$$\left(\inf_{\theta\in\mathbb{R}}\frac{\partial w}{\partial\theta}(r_0,\theta)\right)^{-1} = -\int_{r_0}^{r_{\text{break}}}\left(\frac{\rho_0 c_0^3 r_0}{\rho c^3 r}\right)^{1/2}$$
$$\times\left(1 + \frac{B^2 + \rho^2 P_{\rho\rho}}{2\rho c^2}\right)\exp\left(-\int_{r_0}^s\frac{\alpha^2\rho(c_s^2 + c^2)}{2\rho^2 c^4}\kappa\,d\xi\right)\frac{1}{c}\,ds. \tag{60}$$

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As for the evolution of w in (t, r)-space, recall that the solution of the inviscid Burgers equation does not extend its original support set. If, say $\sigma(\ell = 0, \theta)$ vanishes outside an interval $[\theta_0, \theta_1]$, the same happens for all ℓ; however, the shape of the solution varies until its graph becomes vertical. In our case (see Eq. (33)),

$$\theta(t, \ell) = \frac{\tau(t, r(\ell))}{\epsilon} = \frac{1}{\epsilon}(F(r(\ell)) - t). \tag{61}$$

Hence, if for $t = 0$ the perturbation is limited by the wavefronts within the interval $[\tau_0, \tau_1]$, for the time t it is confined between the wavefronts $F^{-1}(\tau_0 + t)$ and $F^{-1}(\tau_1 + t)$. Since these advance into the plasma at velocity c, so does the perturbation. Thus it is not guaranteed that a shock wave will form within the plasma; for this to occur the value of r_{break} given by (60) must lie within the plasma, and the usual equilibrium configurations are limited by fixed radii, which may well be lower than r_{break}. A given initial condition fixes the left hand term in (60); the right hand one depends only on the equilibrium quantities. Those with subindex 0 refer to their values at $r = r_0$. The larger the integral in (60), the larger the probability that a shock will occur: in particular a negative curvature of poloidal magnetic field lines contributes to shock formation, while a positive curvature detracts from it. The first occurs when these field lines are curved in the sense of the wave propagation, which is very intuitive because this configuration tends to push the fluid towards the central plane, so increasing compression.

Once analyzed the behavior of w, we must consider how a perturbation of this precise type may occur. To give rise to a fast wave of this form, it must have the form $w\mathbf{R}$; thus the initial conditions must be proportional to the vector $\mathbf{R} = (0, c, 0, B_z, 0, B_\phi, \rho)$. For a static equilibrium of the type we consider, the velocity is zero and the magnetic field at the central plane has the form $(B_z, 0, B_\phi)$. Thus the initial state of the perturbation must correspond to a compressive push in the radial direction of velocity v, while magnetic field and density are kept as they are, only multiplied by v/c, where c is the total velocity of the plasma. It seems unlikely for a random perturbation to have this precise form; however, as long as this perturbation has a component in the direction of $\mathbf{R}$, it will give rise, among others MHD waves, to fast ones of the type studied

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here. Unlike the linear case, these waves will interact with one another, but the fast one is the only one able to travel across the equilibrium magnetic field lines, thus propagating into the plasma and therefore yielding the phenomena described before.

5   Conclusions

Among the several nonlinear effects which may endanger the stability of an ideal MHD equilibrium, the evolution of high frequency waves propagating into the plasma is one of the few amenable to theoretical analysis. The mathematical tool appropriate for this is the theory of weakly nonlinear geometrical optics, which poses an asymptotic expansion of any solution of small amplitude and high frequency. The main virtue of this approach is that for hyperbolic systems which may be set in conservation form, such as the one of Magnetohydrodynamics, the first order term of the expansion approaches the real solution up to and including any possible shock, thus making it adequate for the location of possible shock waves generated into the plasma. These shocks may be proved to be as usual the consequence of compressive waves, and besides creating a jump in the velocity of the fluid, they also generate a tangential discontinuity in the magnetic field and therefore create sheet currents. For weakly diffusing plasmas, these shocks smooth to narrow regions of rapid gradient; and which of the two effects is more relevant depends of the magnetic Prandtl number, which for plasmas of fusion importance favors the action of sheet currents. These sheets have already been both modeled and observed, showing that they usually lead to magnetic reconnection and rapid destruction of the equilibrium. We show that a perturbation whose initial form possesses a certain mathematical structure will travel in the equatorial plane of the equilibrium while satisfying a transport equation along the radii. This may be reduced to an inviscid Burgers equation, which enables us to predict the time and location where a shock will occur. This location may not lie within the physical confines of the plasma, so the presence of a shock is by no means certain. We present a formula detailing this phenomenon, and show that one of the factors that accelerate the formation of a shock is the presence of magnetic field lines curving in the direction of wave

propagation.

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