GENERATION OF SHOCKS BY THE BIERMANN BATTERY

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Abstract

The generation of magnetic field in shock surfaces separating regions of different electron density is a well known phenomenon. We study how this generation will affect the original structure of ionic flow. In a one-dimensional geometry, it turns out that the leading magnetosonic wavefront produced by the seed field may be compressional, ultimately evolving into a shock in a finite time. The time where this shock occurs depends on few parameters: sound velocity, Alfvén velocity and the variation of the magnetic field at the original surface at time zero. The alternative is that the magnetosonic wave may stabilize or damp out, which always happens if we start from a null magnetic field.

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1 Introduction

The fact that the magnetic induction equation is linear in the magnetic field shows that no magnetic field may be amplified or otherwise modified by any plasma flow unless it starts from a state different from zero. The need to provide a seed field for the numerous magnetic astrophysical magnetic fields induced L. Biermann [1] to propose a plausible mechanism, the so-called Biermann battery: since electrons are much lighter than ions, pressure forces impart them a greater acceleration, yielding a charge separation which creates an electric field which in its turn generates a magnetic field through Faraday's law. That this mechanism is both likely to occur and efficient in many astrophysical problems has been conclusively demonstrated [2–4]. Since the Biermann battery needs a gradient of electron pressure, an appropriate place to look for this effect is the vicinity of a shock of the plasma flow. That this is correct has been shown theoretically and in experiments [5, 6]; in fact a complex structure related to the Rayleigh-Taylor instability develops in the shock, something one wishes to ignore when trying to find analytic solutions and therefore aims for a geometry as simple as possible. The proper setting to study the shock behaviour would be the kinetic theory of plasmas, or failing that, a fully resistive and viscous MHD system. However, it has been shown [7] that the ideal system (inviscid fluid of infinite conductivity) yields answers concordant with the ones of more elaborate theories; in particular no unphysical infinities, neither theoretical nor numerical, occur. Thus we will use this simplified model in the confidence that it will provide a good guide for the large-scale behaviour of the flow.

It is known that at least for smooth flows, the Biermann battery corresponds to a forcing term in the magnetic induction equation. Its value, provided that the electron pressure follows the perfect gas law, is given by

$$\mathbf{F} = -\frac{ck_B}{e}\frac{\nabla n_e}{n_e} \times \nabla T_e, \tag{1}$$

where k_B is the Boltzmann constant, c the speed of light, e the electron charge, n_e the electron number density and T_e the electron temperature. For $\mathbf{F}$ not to vanish the gradients of density and temperature cannot be collinear, which excludes the usual symmetric configurations. Across a shock we must use the

MHD Rankine-Hugoniot relations (see e.g. [8, 9]). It is generally admitted that with good precision the electron temperature is continuous across the shock [10,11], so that the tangential derivative $\nabla T_e \times \mathbf{n}$, where $\mathbf{n}$ represents the normal vector at the shock surface, is continuous. In these conditions the jump condition for the induction equation becomes

$$[\mathbf{u}_n \times \mathbf{B}_T + \mathbf{u}_T \times \mathbf{B}_n + \mathbf{F}] = \mathbf{0}. \tag{2}$$

$\mathbf{u}$ represents the fluid velocity, $\mathbf{B}$ the magnetic field, the subindex T means tangential component and n the normal one; [ ] is the jump, i.e. the difference of the quantity between the right hand and the left hand side of the shock surface. This may be written as

$$[-u_n \mathbf{B}_T + B_n \mathbf{u}_T + \frac{ck_B}{e}(\ln n_e)\nabla T_e \times \mathbf{n}] = \mathbf{0}, \tag{3}$$

where as usual we have chosen a frame moving with the shock so that its velocity is zero. This result is proved in [7] in a rather cavalier manner (multiplying discontinuous functions by measures, i.e. distributions of order zero, which is not permitted), but the proof may easily be made rigorous given the continuity of $\nabla T_e \times \mathbf{n}$. The remaining jump relations are, taking $\mu_0 = 1$ for simplicity,

$$\begin{aligned} [\rho u_n] &= 0. \\ [\rho u_n \mathbf{u}_T - B_n \mathbf{B}_T] &= \mathbf{0} \\ \left[\rho u_n^2 + \Pi + \frac{B^2}{2}\right] &= 0 \\ \left[\left(\frac{1}{2}\rho u^2 + h + B^2\right) u_n - (\mathbf{u}\cdot\mathbf{B})B_n\right] &= 0 \\ [B_n] &= 0. \end{aligned} \tag{4}$$

Π is the kinetic pressure and h the enthalpy (equal to $\gamma\Pi/(\gamma-1)$ for polytropic fluids). If we assume that $\mathbf{u} = (u(t,x), 0, 0)$ and $\mathbf{B} = (0, 0, B(t,x))$, $\mathbf{F} = (0, 0, f(t))$, as it will occur in our model, the Rankine-Hugoniot relations simplify to

$$
\begin{aligned}
[\rho u] &= 0 \\
\left[\rho u^2 + \Pi + \frac{B^2}{2}\right] &= 0 \\
\left[\left(\frac{1}{2}\rho u^2 + h + B^2\right) u\right] &= 0 \\
[uB - f] &= 0.
\end{aligned} \tag{5}
$$

Thus we take the discontinuity at $x = 0$, and the Biermann forcing concentrated at this point. One way to achieve this is to take an electron temperature of the form

$$
T_e = f_0(t,x) + f_1(t,x)y. \tag{6}
$$

Thus

$$
\nabla T_e \times \mathbf{n} = (0, 0, -f_1), \tag{7}
$$

which means that the Biermann term in (3) becomes

$$
\frac{ck_B}{e} \ln \frac{n_{e2}}{n_{e1}} (0, 0, -f_1(t, 0)) = (0, 0, f(t)). \tag{8}
$$

In order to make the Biermann battery concentrated at the shock, one could assume the electron density constant for $x > 0$ and $x < 0$, $n_{ej}(t,x) = n_{ej}(t)$, so that the Biermann term vanishes elsewhere; or take $f_i = f_i(t)$ in (6) and $n_e = n_e(t,y)$. None of these hypotheses is highly likely from the physical viewpoint, but they are useful to simplify the model and Biermann forcings concentrated at shocks are very real, so that some distribution of electron density and temperature allowing for this must exist. Allowing for this hypothesis, the full model is as follows: In the whole domain, except for the shock at $x = 0$, the ideal MHD equations hold:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \mathbf{u} &= -\nabla \Pi + (\nabla \times \mathbf{B}) \times \mathbf{B} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) \\
\nabla \cdot \mathbf{B} &= 0,
\end{aligned} \tag{9}
$$

whereas in $x = 0$, the Rankine-Hugoniot relations with Biermann battery term (3-4) hold. A structure of fields consistent with our assumptions on $x = 0$ is as follows:

$$\begin{aligned} \rho &= \rho(t, x) \\ \mathbf{u} &= (u(t, x), 0, 0) \\ \mathbf{B} &= (0, 0, B(t, x)), \end{aligned} \tag{10}$$

plus the simplified relations (5). The Lorentz force may be written as

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \left(-\frac{1}{2} \frac{\partial B^2}{\partial x}, 0, 0 \right). \tag{11}$$

Let $\Pi = \Pi(\rho, S)$ be the kinetic pressure, where S represents the entropy. (9) becomes

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x} \left(\Pi + \frac{1}{2} B^2 \right) &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) &= 0 \\ \frac{\partial B}{\partial t} + \frac{\partial}{\partial x} (B u) &= 0 \\ \frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} &= 0. \end{aligned} \tag{12}$$

Initial conditions will be as follows: we start from a nonstationary equilibrium state $\rho = \rho_0$, $S = S_0$, $u = u_0 > 0$, $B = B_0$. From $t = 0$ on the Biermann battery starts acting at $x = 0$, inducing a variation on the magnetic field and therefore on the remaining variables. Since the system (12) is hyperbolic, this variation is not instantaneous: the leading perturbation is the fast magnetosonic wave starting at $t = 0$, $x = 0$. For points (t, x) not yet reached by it, the variables remain as in the equilibrium state.

For $B = 0$ and $S =$ const. (12) become the classical equations of isentropic one-dimensional flow (see e.g. [12, 13]). A change of variables simplifies (12). Let

$$R = \frac{B}{\rho}. \tag{13}$$

Then R satisfies the same transport equation as S:

$$\frac{\partial R}{\partial t} + u \frac{\partial R}{\partial x} = 0. \tag{14}$$

R is the one-dimensional version of the magnetic field per unit mass $\mathbf{B}/\rho$, which is known to be transported by the flow: the induction and continuity equations may be combined to yield

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right). \tag{15}$$

In the new variables,

$$P = \Pi(\rho, S) + \frac{1}{2}\rho^2 R^2. \tag{16}$$

Let the sound speed be defined as usual by

$$c^2 = \frac{\partial P}{\partial \rho} = \frac{\partial \Pi}{\partial \rho}(\rho, S) + \rho R^2 \geq 0. \tag{17}$$

This notation is classical; there should be no confusion with the speed of light, which will not be used henceforth. Then the system (12) may be written in the traditional form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \tag{18}$$

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + \rho c^2\frac{\partial u}{\partial x} = 0 \tag{19}$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0 \tag{20}$$

$$\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} = 0. \tag{21}$$

System (18-21) resembles the one for non isentropic flows, except by the fact that in addition to the entropy S we have a further quantity R transported by the flow as a passive scalar.

2 Evolution of the leading wavefront

System (18-21) possesses three families of characteristics: one double, given by the streamlines

$$C_0 : \frac{dx}{dt} = u(t, x), \tag{22}$$

and the slow and fast magnetosonic waves:

$$C_- : \frac{dx}{dt} = (u - c)(t, x) \tag{23}$$

$$C_+ : \frac{dx}{dt} = (u + c)(t, x). \tag{24}$$

Notice that if $u \geq c$ all the perturbations are transmitted in the positive sense of x, so that for $x < 0$ the state remains in equilibrium. Otherwise waves may travel backwards, in particular the slow magnetosonic ones; and recall that we have discounted the possible velocity of the initial shock by fixing it at $x = 0$. Thus $u_0 > 0$ means that the equilibrium flow velocity is larger than the one of the shock. The leading characteristic is the the fast magnetosonic one starting at $(0, 0)$: it is given by $C_+^0 : x = X(t)$,

$$\frac{dX}{dt} = (u + c)(t, X(t))$$
$$X(0) = 0. \tag{25}$$

As stated, for any (t, x) such that $x > X(t)$, all the magnitudes of the problem remain in equilibrium at this point. We will study the evolution of

$$\frac{\partial u}{\partial x}(t, X(t)). \tag{26}$$

The method consist in a kind of hodograph transformation using the characteristics C_+ and C_0 as coordinate curves. This procedure is not uncommon in this type of problems [14]. The new variables $\xi(t, x)$, $\tau(t, x)$ will therefore satisfy

$$\frac{\partial \xi}{\partial t} + (u + c)\frac{\partial \xi}{\partial x} = 0 \tag{27}$$
$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} = 0. \tag{28}$$

This does not completely characterize ξ and τ: they must be labeled at one point of each characteristic. Let us therefore take C_+^0 as $\xi = 0$, i.e. $\xi(0, 0) = 0$. For every point of C_+^0, a single C_0 characteristic intersects it. It will be labeled by t, i.e. we take $\tau(t, X(t)) = t$. This determines the value of τ for all the points lying between C_+^0 and the streamline C_0^0 starting at $(0, 0)$. The labeling of ξ for these points is not important, as we will concentrate in C_+^0. Anyway ξ is constant along the C_+ characteristics. The change of variables $(t, x) \to (\tau, \xi)$ will remain valid for as long as these characteristics remain transversal to each other. The jacobian matrix is

$$\frac{D(\xi, \tau)}{D(t, x)} = \begin{pmatrix} \frac{\partial \xi}{\partial t} & \frac{\partial \xi}{\partial x} \\ \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial x} \end{pmatrix} = \begin{pmatrix} -(u + c)\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial x} \\ -u\frac{\partial \tau}{\partial x} & \frac{\partial \tau}{\partial x} \end{pmatrix}, \tag{29}$$

whose inverse is

$$\frac{D(t,x)}{D(\xi,\tau)} = \begin{pmatrix} \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \end{pmatrix} = \left(\frac{\partial \xi}{\partial t}\frac{\partial \tau}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial \tau}{\partial t} \right)^{-1} \begin{pmatrix} \frac{\partial \tau}{\partial x} & -\frac{\partial \xi}{\partial x} \\ -\frac{\partial \tau}{\partial t} & \frac{\partial \xi}{\partial t} \end{pmatrix}$$
$$= \left(\frac{\partial \xi}{\partial t}\frac{\partial \tau}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial \tau}{\partial t} \right)^{-1} \begin{pmatrix} \frac{\partial \tau}{\partial x} & -\frac{\partial \xi}{\partial x} \\ u\frac{\partial \tau}{\partial x} & -(u+c)\frac{\partial \xi}{\partial x} \end{pmatrix}. \tag{30}$$

This implies

$$\frac{\partial x}{\partial \tau} = (u+c)\frac{\partial t}{\partial \tau} \tag{31}$$
$$\frac{\partial x}{\partial \xi} = u\frac{\partial t}{\partial \xi}. \tag{32}$$

Writing now

$$\frac{D(\xi,\tau)}{D(t,x)} = \left(\frac{D(t,x)}{D(\xi,\tau)}\right)^{-1} = \frac{1}{J}\begin{pmatrix} \frac{\partial x}{\partial \tau} & -\frac{\partial t}{\partial \tau} \\ -\frac{\partial x}{\partial \xi} & \frac{\partial t}{\partial \xi} \end{pmatrix}, \tag{33}$$

where J is the jacobian

$$J = -\frac{\partial t}{\partial \xi}\frac{\partial x}{\partial \tau} + \frac{\partial t}{\partial \tau}\frac{\partial x}{\partial \xi}, \tag{34}$$

we obtain

$$\frac{\partial \xi}{\partial t} = -\frac{1}{J}\frac{\partial x}{\partial \tau}, \qquad \frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial t}{\partial \tau}$$
$$\frac{\partial \tau}{\partial t} = \frac{1}{J}\frac{\partial x}{\partial \xi}, \qquad \frac{\partial \tau}{\partial x} = -\frac{1}{J}\frac{\partial t}{\partial \xi}. \tag{35}$$

Of course there is an abuse of notation here; the left hand side of these equations is taken at (t,x), whereas the right hand side is taken at $(\xi(t,x),\tau(t,x))$. Admitting this notation in the future, applying the chain rule to any magnitude Φ,

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{1}{J}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial t}{\partial \tau} - \frac{\partial \Phi}{\partial \tau}\frac{\partial t}{\partial \xi}\right) \tag{36}$$
$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \Phi}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{1}{J}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial x}{\partial \tau} - \frac{\partial \Phi}{\partial \tau}\frac{\partial x}{\partial \xi}\right). \tag{37}$$

This implies

$$\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} = -\frac{c}{J}\frac{\partial \Phi}{\partial \xi}\frac{\partial t}{\partial \tau}. \tag{38}$$

Using (38) on u and P, equations (18-21) may be written in the new coordinates. For (19), we obtain

$$-\frac{c}{J}\frac{\partial P}{\partial \xi}\frac{\partial t}{\partial \tau}+\frac{\rho c^2}{J}\left(\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \tau}-\frac{\partial u}{\partial \tau}\frac{\partial t}{\partial \xi}\right)=0, \tag{39}$$

i.e.

$$-\frac{\partial P}{\partial \xi}\frac{\partial t}{\partial \tau}+\rho c\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \tau}-\rho c\frac{\partial u}{\partial \tau}\frac{\partial t}{\partial \xi}=0. \tag{40}$$

Equation (18) becomes

$$-\frac{c}{J}\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \tau}+\frac{1}{\rho J}\left(\frac{\partial P}{\partial \xi}\frac{\partial t}{\partial \tau}-\frac{\partial P}{\partial \tau}\frac{\partial t}{\partial \xi}\right)=0, \tag{41}$$

i.e.

$$\frac{\partial P}{\partial \tau}=\left(\frac{\partial t}{\partial \tau}\right)\left(\frac{\partial f}{\partial \xi}\right)^{-1}\left(\frac{\partial P}{\partial \xi}-\rho c\frac{\partial u}{\partial \xi}\right). \tag{42}$$

Using (40), this becomes

$$\frac{\partial P}{\partial \tau}+\rho c\frac{\partial u}{\partial \tau}=0. \tag{43}$$

Finally, (20-21) become

$$\frac{\partial S}{\partial \xi}=\frac{\partial R}{\partial \xi}=0. \tag{44}$$

Since all the quantities are continuous in a characteristic, and C_+^0 is contiguous to the equilibrium zone, in this curve we have $\rho=\rho_0$, $u=u_0$, $S=S_0$, $R=R_0$, $c=c_0$. In particular (44) implies that both S and R are constant at all the C_0-characteristics starting at C_+^0, i.e. in the region limited by the leading magnetosonic wave and the leading streamline. This is logical as both magnitudes are transported by the flow and therefore do not change until the fluid starting at $x=0$ reaches the point. Since τ is a parameter along $C_+^0:\xi=0$, where $\tau=t$, we have

$$\frac{\partial u}{\partial \tau}=\frac{\partial \rho}{\partial \tau}=\frac{\partial P}{\partial \tau}=\frac{\partial S}{\partial \tau}=\frac{\partial R}{\partial \tau}=0, \qquad \frac{\partial t}{\partial \tau}=1, \tag{45}$$

at any point of the form $(0,\tau)$. Taking this to (40),

$$\frac{\partial P}{\partial \xi}(0,\tau)=\rho_0 c_0\frac{\partial u}{\partial \xi}(0,\tau), \tag{46}$$

and using (31, 32, 34),

$$\frac{\partial x}{\partial \xi} = u_0 \frac{\partial t}{\partial \xi}, \qquad \frac{\partial x}{\partial \tau} = u_0 + c_0, \qquad J = -c_0 \frac{\partial t}{\partial \xi}, \tag{47}$$

always at C_+^0. Hence, calling

$$F(\tau) = F(t) = \frac{\partial u}{\partial x}(0, \tau), \tag{48}$$

and using (36) with $\Phi = u$ plus (47), we obtain

$$F = \frac{1}{J} \frac{\partial u}{\partial \xi} = -\frac{1}{c_0} \frac{\partial u}{\partial \xi} \left(\frac{\partial t}{\partial \xi} \right)^{-1}. \tag{49}$$

Let us differentiate (31) and (43) with respect to ξ:

$$\frac{\partial^2 x}{\partial \xi \partial \tau} - \frac{\partial (u+c)}{\partial \xi} \frac{\partial t}{\partial \tau} - (u+c) \frac{\partial^2 t}{\partial \xi \partial \tau} = 0 \tag{50}$$

$$\frac{\partial^2 P}{\partial \xi \partial \tau} + \frac{\partial (\rho c)}{\partial \xi} \frac{\partial u}{\partial \tau} + \rho c \frac{\partial^2 u}{\partial \xi \partial \tau} = 0. \tag{51}$$

Evaluating these identities at C_+^0, we obtain

$$\left[\frac{\partial^2 x}{\partial \xi \partial \tau} \right]_{(0,\tau)} - \left[\frac{\partial (u+c)}{\partial \xi} \right]_{(0,\tau)} - (u_0 + c_0) \left[\frac{\partial^2 t}{\partial \xi \partial \tau} \right]_{(0,\tau)} = 0 \tag{52}$$

$$\left[\frac{\partial^2 P}{\partial \xi \partial \tau} \right]_{(0,\tau)} + \rho_0 c_0 \left[\frac{\partial^2 u}{\partial \xi \partial \tau} \right]_{(0,\tau)} = 0. \tag{53}$$

Let us differentiate now (32) and (40) with respect to τ:

$$\frac{\partial^2 x}{\partial \tau \partial \xi} - \frac{\partial u}{\partial \tau} \frac{\partial t}{\partial \xi} - u \frac{\partial^2 t}{\partial \tau \partial \xi} = 0 \tag{54}$$

$$\begin{aligned} -\frac{\partial^2 P}{\partial \tau \partial \xi} \frac{\partial t}{\partial \tau} - \frac{\partial P}{\partial \xi} \frac{\partial^2 t}{\partial \tau^2} + \frac{\partial (\rho c)}{\partial \tau} \frac{\partial u}{\partial \xi} \frac{\partial t}{\partial \tau} + \rho c \frac{\partial^2 u}{\partial \tau \partial \xi} \frac{\partial t}{\partial \tau} \\ + \rho c \frac{\partial u}{\partial \xi} \frac{\partial^2 t}{\partial \tau^2} - \frac{\partial (\rho c)}{\partial \tau} \frac{\partial u}{\partial \tau} \frac{\partial t}{\partial \xi} - \rho c \frac{\partial^2 u}{\partial \tau^2} \frac{\partial t}{\partial \xi} - \rho c \frac{\partial u}{\partial \tau} \frac{\partial^2 t}{\partial \tau \partial \xi} = 0. \end{aligned} \tag{55}$$

When evaluated at C_+^0, these relations simplify considerably:

$$\left[\frac{\partial^2 x}{\partial \tau \partial \xi} \right]_{(0,\tau)} - u_0 \left[\frac{\partial^2 t}{\partial \tau \partial \xi} \right]_{(0,\tau)} = 0 \tag{56}$$

$$-\left[\frac{\partial^2 P}{\partial \tau \partial \xi} \right]_{(0,\tau)} + \rho_0 c_0 \left[\frac{\partial^2 u}{\partial \tau \partial \xi} \right]_{(0,\tau)} = 0. \tag{57}$$

From (53) and (57) we deduce

$$\left[\frac{\partial^2 u}{\partial\tau\partial\xi}\right]_{(0,\tau)} = 0, \tag{58}$$

and from (52) and (56),

$$\left[\frac{\partial^2 t}{\partial\tau\partial\xi}\right]_{(0,\tau)} = -\frac{1}{c_0}\left[\frac{\partial(u+c)}{\partial\xi}\right]_{(0,\tau)}. \tag{59}$$

Since P is a function of the variables $(\rho,\, S,\, R)$, so is c. Thus

$$\frac{\partial c}{\partial\xi} = \frac{\partial c}{\partial\rho}\frac{\partial\rho}{\partial\xi} + \frac{\partial c}{\partial S}\frac{\partial S}{\partial\xi} + \frac{\partial c}{\partial R}\frac{\partial R}{\partial\xi}. \tag{60}$$

At C_+^0, the two last terms disappear. As for the first one, using (40) and the fact that in that curve

$$\frac{\partial P}{\partial\xi} = \frac{\partial P}{\partial\rho}\frac{\partial\rho}{\partial\xi} = c^2\frac{\partial\rho}{\partial\xi}, \tag{61}$$

we obtain

$$\left[\frac{\partial^2 t}{\partial\tau\partial\xi}\right]_{(0,\tau)} = -\frac{1}{c_0}\left(\frac{\rho_0}{c_0}\left[\frac{\partial c}{\partial\rho}\right]_{(0,\tau)} + 1\right)\left[\frac{\partial u}{\partial\xi}\right]_{(0,\tau)}. \tag{62}$$

Notice that since ρ, S and R are constant at C_+^0, so is the term within parentheses in (62). Let us call it M: obviously $M > 0$. Then, differentiating (49) with respect to τ (i.e. within the characteristic), we get

$$\frac{dF}{d\tau} = -c_0\left[\left(c_0\frac{\partial t}{\partial\xi}\right)^{-2}\left(\frac{\partial^2 u}{\partial\tau\partial\xi}\frac{\partial t}{\partial\xi} - \frac{\partial u}{\partial\xi}\frac{\partial^2 t}{\partial\tau\partial\xi}\right)\right]_{(0,\tau)}, \tag{63}$$

and using (58), (62) and the fact that $t = \tau$ at C_+^0, we finally obtain

$$\frac{dF}{dt} = -MF^2. \tag{64}$$

The solution to (64) is trivial: for $F(0) \neq 0$,

$$F(t) = \left(\frac{1}{F(0)} + Mt\right)^{-1}. \tag{65}$$

Therefore if $F(0) > 0$, F decreases to zero. Since F represents the slope of u at the leading magnetosonic wave, we interpret this as a damping of this wave. When $F(0) < 0$, F is always negative and tends to infinity at a time

$$t_\infty = -\frac{1}{MF(0)}. \tag{66}$$

Hence u reaches a step-like discontinuity, i.e. a shock wave. For $F(0) = 0$, F is always zero, which means that when the magnetosonic wave is so smooth that even the first derivatives of the velocity are continuous across it, this wave neither decays nor forms a shock in time. We will see later that this is a real possibility.

Although (65) and (66) are satisfactorily simple, we must identify both M and $F(0)$ in terms of the real physical parameters of the problem.

3 Initial conditions and shock formation time

Let us find first the value of M. From (62) we know that M is given by

$$M = 1 + \frac{\rho}{c}\frac{\partial c}{\partial \rho}, \tag{67}$$

and that this value is constant in C_+^0. By the formula (17),

$$\frac{\partial c}{\partial \rho} = \frac{1}{2}\left(\frac{\partial \Pi}{\partial \rho} + \rho R^2\right)^{-1/2}\left(\frac{\partial^2 \Pi}{\partial \rho^2} + R^2\right). \tag{68}$$

Thus

$$M = 1 + \frac{\rho}{2c^2}\left(\frac{\partial^2 \Pi}{\partial \rho^2} + R^2\right). \tag{69}$$

Let us consider the particular case of a polytropic gas,

$$\Pi(\rho, S) = A(S)\rho^\gamma. \tag{70}$$

For $A = 0$ the gas is pressureless and only the magnetic field exerts a pressure. For the general polytropic case, after some operations one finds

$$\begin{aligned} M &= 1 + \frac{(\gamma-1)A\gamma\rho^{\gamma-1} + \rho R^2}{2c^2} \\ &= 1 + \frac{\gamma-1}{2} + \left(1 - \frac{\gamma}{2}\right)\frac{\rho R^2}{c^2} = \frac{1+\gamma}{2} + \left(1 - \frac{\gamma}{2}\right)\frac{c_A^2}{c^2}, \end{aligned} \tag{71}$$

where $c_A = B/\sqrt{\rho}$ represents the Alfvén velocity of the flow. For $A = 0$, $c_A = c$ and $M = 3/2$, which is independent of any parameter. Since c and c_A remain constant at the wavefront C_+^0, their value is the same as the one at $(0, 0)$.

It remains to find $F(0) = (\partial u/\partial x)(0+, 0+)$. The sign + means that we must take the limit as $t \to 0$, $t > 0$, $x \to 0$, $x > 0$; from the left the limit is zero, as we start from an equilibrium state. Assuming a small initial variation of the magnetic field at the shock $x = 0$ given by the Biermann effect, for small time the solutions will be approximated by those of the linearized system, so that the value of $F(0)$ may be found by finding $(\partial u_1/\partial x)(0+, 0+)$, u_1 a perturbation of the equilibrium velocity u_0. Initial values are given, as stated, by $\rho = \rho_0$, $u = u_0 > 0$, $S = S_0$, $B = B_0$. Thus we consider small perturbations of these, $u = u_0 + u_1$, $\rho = \rho_0 + \rho_1$, $S = S_0 + S_1$, $B = B_0 + B_1$, which yield $\Pi = \Pi_0 + \Pi_1$. We will consider the polytropic case for concretion. The linearized versions of the Rankine-Hugoniot connection conditions in (5) are

$$[\rho_0 u_1 + \rho_1 u_0] = 0 \tag{72}$$

$$[\rho_1 u_0^2 + 2\rho_0 u_0 u_1 + \Pi_1 + B_0 B_1] = 0 \tag{73}$$

$$\left[\left(\frac{1}{2}\rho_1 u_0^2 + \rho_0 u_0 u_1 + \frac{\gamma \Pi_1}{\gamma - 1} + 2B_0 B_1\right) u_0 + \left(\frac{1}{2}\rho_0 u_0^2 + \frac{\gamma \Pi_0}{\gamma - 1} + B_0^2\right) u_1\right] = 0 \tag{74}$$

$$[u_0 B_1 + u_1 B_0 - f_1] = 0. \tag{75}$$

where f_1 is the perturbation of the Biermann term f; since we start from $f = 0$ for $t < 0$, we may as well take $f_1 = f$, at least while it remains small.

We will assume that $u_0 > c_0$, so that the fluid to the left of $x = 0$ remains unperturbed. In this case (72-75) yield the values of ρ_1, u_1, B_1 and Π_1 at $x = 0$. The linearized problem may therefore be set at the quadrant $x \geq 0$, $t \geq 0$, and the perturbed quantities are zero at the points $(x, 0)$ and have the values given by (72-75) at the points $(0, t)$. System (72-75) is linear, so with some work it may be solved. The perturbed velocity has the form

$$u_1 = \lambda B_0 f, \tag{76}$$

where

$$\lambda = \frac{(2 - \gamma)\rho_0}{c_0^2 - u_0^2}, \tag{77}$$

and

$$c_0^2 = \frac{\gamma \Pi_0}{\rho_0} + \frac{B_0^2}{\rho}. \tag{78}$$

Also

$$\rho_1 = -\frac{\rho_0 \lambda}{u_0} B_0 f. \tag{79}$$

Instead of writing B_1 it is better to use the perturbation of R, given by $B_1 = \rho_0 R_1 + \rho_1 R_0$. Using (72) and (75) we obtain $[u_0 \rho_0 R_1 - f] = 0$, i.e. for $t > 0$,

$$R_1 = \frac{f(t)}{u_0 \rho_0} = h(t). \tag{80}$$

In terms of h,

$$\begin{aligned} u_1 &= \lambda B_0 \rho_0 u_0 h \\ \rho_1 &= -\lambda B_0 \rho_0^2 h. \end{aligned} \tag{81}$$

It is clear that (76-80) occur only at the points of the form $(0, t)$. When $B_0 \neq 0$ (and therefore $R_0 \neq 0$) system (14-17), once linearized near the equilibrium, becomes

$$\frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + \frac{c_0^2}{\rho_0} \frac{\partial \rho_1}{\partial x} = -\rho_0 R_0 \frac{\partial R_1}{\partial x} \tag{82}$$

$$\frac{\partial \rho_1}{\partial t} + u_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial u_1}{\partial x} = 0 \tag{83}$$

$$\frac{\partial R_1}{\partial t} + u_0 \frac{\partial R_1}{\partial x} = 0, \tag{84}$$

Equation (84) means that R_1 is constant at the characteristics $x - u_0 t =$ const., i.e. it is a function of $x - u_0 t$. Since $R_1(t, 0) = h(t)$,

$$\begin{aligned} R_1(t, x) &= 0 \quad \text{for } x < 0 \\ R_1(t, x) &= h\left(t - \frac{x}{u_0}\right) \quad \text{for } x \geq 0. \end{aligned} \tag{85}$$

Notice that since $h(t) = 0$ for $t < 0$, $R_1(t, x)$ vanishes both for $x < 0$ and for $x > u_0 t$; that is, R_1 is limited to the domain lying between the original Biermann sheet and the characteristic C_0, which are the points reached by transport through the equilibrium flow u_0. We may see system (82-83) as a linear isentropic one with an independent term:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix} + \begin{pmatrix} u_0 & c_0^2/\rho_0 \\ \rho_0 & u_0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} -\rho_0 R_0(\partial R_1/\partial x) \\ 0 \end{pmatrix}. \tag{86}$$

Since we wish to find $\partial u/\partial x$ at the fast magnetosonic front C_+^0, and in the domain D lying between the characteristics C_0 and C_+^0 the independent term vanishes, we are left with the homogeneous equation. This may be written as

$$\begin{aligned} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix} &= -\begin{pmatrix} u_0 & c_0^2/\rho_0 \\ \rho_0 & u_0 \end{pmatrix}^{-1} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix} \\ &= \frac{1}{u_0^2 - c_0^2}\begin{pmatrix} -u_0 & c_0^2/\rho_0 \\ \rho_0 & -u_0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ \rho_1 \end{pmatrix}. \end{aligned} \tag{87}$$

Now, since $u_1(t,0)$, $\rho_1(t,0)$ are given by (81), the limit of the right hand side when $x \to 0$, $(t,x) \in D$, is

$$\frac{\lambda B_0 \rho_0}{u_0^2 - c_0^2}\begin{pmatrix} -u_0 & c_0^2/\rho_0 \\ \rho_0 & -u_0 \end{pmatrix}\begin{pmatrix} u_0 \\ -\rho_0 \end{pmatrix} h'(t) = \frac{(2-\gamma)\rho_0^2}{(u_0^2 - c_0^2)^2}\begin{pmatrix} u_0^2 + c_0^2 \\ -2\rho_0 u_0 \end{pmatrix} B_0 h'(t). \tag{88}$$

In particular

$$\frac{\partial u_1}{\partial x}(0+, 0+) = \frac{(2-\gamma)\rho_0^2(u_0^2 + c_0^2)}{(u_0^2 - c_0^2)^2} B_0 h'(0+). \tag{89}$$

The constant μ multiplying $B_0 h'(0+)$ in (89) is positive; hence the sign of $(\partial u_1/\partial x)(0+, 0+)$ is the one of $B_0 h'(0+)$. We see that the key parameter determining the fate of the leading wave is the variation of the Biermann forcing at the instant $t = 0$. Thus, if $B_0 h'(0+) < 0$, which means that the perturbation of the magnetic field at the sheet $x = 0$ decreases its size, we obtain that the time of formation of a shock for the leading fast magnetosonic wave is

$$t_\infty = -\frac{1}{M \mu B_0 h'(0+)}, \tag{90}$$

with M given by (69) and μ by (89). Notice that from (89) μ is very large when the equilibrium speed sound c_0 and the velocity u_0 are close; in that case the slow magnetosonic wave separates slowly from the sheet, which as we see accelerates the formation of a shock.

The reason for this shock formation is rather intuitive: since the size of the magnetic field tends to diminish, the Lorentz force $-\partial B^2/\partial x$ points towards the left, i.e. the flow tends to be retarded by the negative gradient of magnetic pressure; this means that the flow is compressive, which foresees the formation of a shock wave. On the contrary, if $B(0)h'(0) > 0$, the Lorentz force points outwards and tends to accelerate the flow, thus rarefying the fluid and damping the wave.

It remains to see what happens if $B_0 = R_0 = 0$, i.e. if we start from an unmagnetized state; this is the situation for which the Biermann battery was originally invoked. Then the independent term in the linearized momentum equation (82) is zero, so we should start from a second order term. An alternative is to take $B^2/2$ instead of B as a new variable. We leave this problem for future research, but we must expect a rarefaction wave and no fast magnetosonic shock.

4 Conclusions

One of the most successful mechanisms proposed to create a seed magnetic field in an originally field-free state is to assume a charge separation in the plasma due to the smaller inertia of electrons with respect to ions; thus kinetic pressure forces may give rise to a non conservative electric field and therefore a magnetic one. This so-called Biermann battery is more likely to occur where large gradients of electron pressure are present, i.e. in shocks. This hypothesis has been often invoked, with considerable success, to explain several features present in Astrophysics and in laser experimentation. Whereas the generation of magnetic field by this process has been extensively considered, the influence of this new field on the plasma flow itself has been so far less studied. This problem is analyzed in a simplified one-dimensional geometry, starting from a non static equilibrium state and generating a magnetic field in a fixed plasma sheet by the Biermann battery; once away from the sheet, the process is governed by the MHD equations. Thus the mathematical setting consists on the MHD equations in two domains separated by the original shock sheet, where the Rankine-Hugoniot formulas relate the value of the variables at both sides.

This system may be cast in the form of a purely hydrodynamic flow with two quantities transported by the flow: the entropy and the magnetic field per unit mass. We find that the evolution of the leading fast magnetosonic wave is very simple: either it damps out, remains at constant strength or gives rise to a new shock onwards from the original one. The last case occurs, as it could be expected, when this wave is compressional. The parameters governing both if this shock occurs as well as the time where it occurs, are few and may be related to the values of the original equilibrium state plus the behavior of the Biermann forcing at the initial instant. As a rule, when the Biermann current detracts from the size of the equilibrium magnetic field, a shock must be expected: a fact which may be intuitively explained by the braking effect of the Lorentz force on the flow. Conversely, if the magnetic field increases its size (which is the case why the Biermann mechanism was proposed), the Lorentz force accelerates the flow and the fluid becomes rarefied, so that the fast magnetosonic wave eventually damps out.

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