

Gamow states as solutions of a modified Lippmann-Schwinger equation.

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Abstract

In this paper we discuss the structure of Gamow states as solutions of Lippmann-Schwinger equation. The Friedrichs model is used to demonstrate it, both analytically and by applying perturbation theory to the extended spectrum of the Hamiltonian. The method presented here may be relevant to the inclusion of resonances in discrete basis, without the need of numerical constructions to define Gamow states, as entities depending on the choice of integrals contours, or as states resulting from ad-hoc discretizations of the continuum.

keywords: Lippmann-Schwinger equation, Gamow states, Friedrichs model.

1 Introduction

The study of the properties of Gamow states [1, 2], both from the mathematical and physical points of view, has attracted a renewed attention during the last decade, basically due to the need of concrete realizations of resonances in the context of treatments of the continuum. To the mathematically oriented descriptions of Gamow states [3], one may add the numerically finding of these states for a certain class of potentials [4], in the context of nuclear structure models [5] and reaction theories [6]. The use of Gamow states has been particularly useful to describe nuclear properties of nuclei far from the stability line, as well as for the calculation of very important physical processes like alpha decay [7], high-energy giant-resonances and particle emission rates from excited nuclear states [8]. The formulation of the scattering theory [9] in terms of incoming

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and outgoing distorted waves offers, as we shall show in this paper, a convenient framework where to insert the notion of resonant states rephrased in terms of Gamow states and their properties. Central to the calculation of scattering amplitudes is the formalism generally represented by the Lippmann-Schwinger equation [10], which establishes a very compact and mathematically elegant way to calculate the wave function of scattered states from the repeated application of the interaction upon states of the unperturbed portion of the Hamiltonian. The Lippmann-Schwinger equation then allows for a connection of states belonging to the spectrum of a chosen unperturbed Hamiltonian with out-going waves. It is the purpose of the present paper to demonstrate that Gamow states belong to the class of states described by the Lippmann-Schwinger equation. This result may indeed facilitate the use of Gamow states in an explicit manner, particularly in the description of physical systems where resonant states may play a role in the explanation of observed or expected features.

The paper is organized as follows. In Section 2 we introduce the essentials of the model due to Friedrichs [11], which is a model amenable to the identification of Gamow states resulting from the interaction between discrete and continuous components of the spectrum. In Section 3 we present the basic of Lippmann-Schwinger equation and make the connection between out-going waves and Gamow states. Section 4 is devoted to the development of an alternative demonstration based on the use of perturbation theory. Finally, our conclusions are presented in Section 5. In discussing the material presented in the paper we have tried to facilitate the access of readers more oriented towards aspects which are relevant to physics though without lost of mathematical rigour.

2 On the Friedrichs model

The Friedrichs model [11] has a Hamiltonian of the form $H = H_0 + \lambda V$, where H_0 is the free Hamiltonian and V is the interaction. We shall adopt, as degrees of freedom associated to H_0 , a discrete boson state $|1\rangle$ of energy ω_0 , and a free boson field $|\omega\rangle$, whose spectrum is defined by ω

$$H_0 = \omega_0|1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega| d\omega. \quad (1)$$

Let us consider the Hilbert space $\mathcal{H} = \mathbb{C} \oplus L^2(0, \infty)$. The vector $|1\rangle$ serves as a basis for \mathbb{C} and $L^2(0, \infty)$ has a continuous basis denoted as $|\omega\rangle$, $\omega \in \mathbb{R}^+$, so that any function $f \in L^2(0, \infty)$ can be written in the form $\int_0^\infty f(\omega)|\omega\rangle d\omega$.

In other words, the free Hamiltonian H_0 has $\mathbb{R}^+ \equiv [0, \infty)$ as continuous spectrum and an eigenvalue ω_0 embedded in the continuous spectrum, meaning that $H_0|1\rangle = \omega_0|1\rangle$ and $H_0|\omega\rangle = \omega|\omega\rangle$, with $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$, $\langle 1|1\rangle = 1$ and $\langle\omega|1\rangle = 0$ with $\omega \in \mathbb{R}^+$. The potential V produces an interaction between the discrete state and the field. The simplest form is by intertwining $|1\rangle$ and $|\omega\rangle$ by means of a square integrable function, the form-factor $f(\omega)$ which could be chosen real, so that

$$V = \int_0^\infty f(\omega) \{ |\omega\rangle\langle 1| + |1\rangle\langle \omega| \} d\omega. \quad (2)$$

The constant λ in $H = H_0 + \lambda V$ is a coupling constant and may have any real value. As a consequence of the interaction, the eigenvalue ω_0 of H_0 is dissolved in the continuum and yields to a resonance (under some mild conditions imposed on $f(\omega)$) [14, 12, 13]. This resonance is a pole at (along its complex conjugate) $z_R = E_R - i\Gamma/2$, of the analytic continuation of the reduced resolvent $\langle 1|(z-H)^{-1}|1\rangle$, or equivalently of the S matrix in the energy representation, and has the form:

$$z_R(\lambda) = [\omega_0 + \lambda^2 I(\omega_0, f) + o(\lambda^4)] - i[\pi\lambda^2 |f(\omega_0)|^2 + o(\lambda^4)], \quad (3)$$

where

$$I(\omega_0, f) = \mathcal{P} \int_0^\infty \frac{f^2(\omega) d\omega}{\omega_0 - \omega}, \quad (4)$$

where \mathcal{P} denotes Cauchy principal value.

The decaying Gamow vector has the form

$$|\psi^D\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_R - \omega + i0} |\omega\rangle d\omega. \quad (5)$$

Now, let us consider

$$\varphi(\lambda) := \frac{1}{z_R - \omega + i0} = \varphi(0) + \varphi'(0)\lambda + \frac{\varphi''(0)}{2}\lambda^2 + \dots, \quad (6)$$

where,

$$\varphi(0) = \frac{1}{\omega_0 - \omega + i0}, \quad (7)$$

$$\varphi'(0) = -\frac{z'_R(0)}{(\omega_0 - \omega + i0)^2} = 0, \quad (8)$$

$$\varphi''(0) = -\frac{z''_R(0)}{(\omega_0 - \omega + i0)^2} = -I(\omega_0, f) \frac{1}{(\omega_0 - \omega + i0)^2}. \quad (9)$$

Thus, up to third order in λ , the decaying Gamow-state $|\psi^D\rangle$ has the following form:

$$\begin{aligned} |\psi^D\rangle &= |1\rangle + \lambda \left[\int_0^\infty \frac{f(\omega)}{\omega_0 - \omega + i0} |\omega\rangle d\omega \right] \\ &- \frac{\lambda^3}{2} \left[I(\omega_0, f) \int_0^\infty \frac{f(\omega)}{(\omega_0 - \omega + i0)^2} |\omega\rangle d\omega \right] + \dots \end{aligned} \quad (10)$$

For notational convenience, we shall denote the sum of the first two terms in (10) as $|\psi_1^D\rangle$ and to the coefficient of λ^3 as $|\psi_3^D\rangle$, so that $|\psi^D\rangle = |\psi_1^D\rangle + \lambda^3|\psi_3^D\rangle + \dots$

3 Lippmann-Schwinger

The Lippmann-Schwinger equation gives the expression of the in- and out-perturbed state vectors in terms of the incoming free wave function in a scattering process given by the Hamiltonian pair $(H_0, H = H_0 + V)$. Although it may be written in terms of normalized wave functions, it is much simpler and useful to express it in terms of non-normalizable plane waves in the momentum representation as [2, 9, 10]

$$|\psi^\pm\rangle = |\phi\rangle + \frac{1}{\omega - H_0 \pm i0} V|\psi^\pm\rangle, \quad (11)$$

where $|\phi\rangle$ represents the free plane wave function in the momentum representation. In (11) the + and - signs represent in- and out-states, respectively. Taking (11) as our starting point, we shall demonstrate that $|\psi^D\rangle$ belongs to the class of states $|\psi^\pm\rangle$ which are solutions of the Lippmann-Schwinger equation.

Assume that $|\psi^D\rangle$ represents the decaying Gamow state, i.e., $(H_0 + \lambda V)|\psi^D\rangle = z_R|\psi^D\rangle$, where $z_R = E_R - i\Gamma/2$ is the resonance pole ($E_R, \Gamma > 0$). Following the same steps leading to equation (11), we have obtained a generalization of it which gives a similar expression concerning Gamow states for the Friedrichs model. In our case, $|\phi\rangle \equiv |1\rangle$ and $|\psi^+\rangle \equiv |\psi^D\rangle$, for the decaying Gamow state. The sequence is the following: we begin with by expressing the action of the interaction upon the state $|\psi^D\rangle$

$$(H_0 + \lambda V)|\psi^D\rangle = z_R|\psi^D\rangle \implies \lambda V|\psi^D\rangle = (z_R - H_0)|\psi^D\rangle, \quad (12)$$

since z_R has non-vanishing imaginary part, the operator $(z_R - H_0)$ is invertible, and it leads to

$$|\psi^D\rangle = (z_R - H_0)^{-1} \lambda V|\psi^D\rangle. \quad (13)$$

In absence of the interaction, that is for $\lambda = 0$, the system remains in the initial state $|1\rangle$, therefore, after adding it to (13), the complete solution of (12) is given by

$$|\psi^D\rangle = |1\rangle + (z_R - H_0)^{-1} \lambda V|\psi^D\rangle. \quad (14)$$

We should take into account that as $\lambda \rightarrow 0$, then, $z_R \rightarrow \omega_0$. To account for this limit (14) is written

$$|\psi^D\rangle = |1\rangle + \frac{\lambda}{z_R - (H_0 - i0)} V|\psi^D\rangle. \quad (15)$$

Inserting a complete set of kets $|\omega\rangle$ in (15), we have that

$$\begin{aligned} |\psi^D\rangle &= |1\rangle + \int_0^\infty \frac{\lambda d\omega}{z_R - (H_0 - i0)} |\omega\rangle \langle\omega| V |\psi^D\rangle \\ &= |1\rangle + \int_0^\infty \frac{\lambda d\omega}{z_R - (\omega - i0)} |\omega\rangle \langle\omega| V |\psi^D\rangle. \end{aligned} \quad (16)$$

The action of V upon the bra-state $\langle\omega|$ yields

$$\langle\omega|V = \int_0^\infty d\omega' \langle\omega| f(\omega') [|\omega'\rangle\langle 1| + |1\rangle\langle\omega'|] = f(\omega)\langle 1|. \quad (17)$$

Finally, inserting (17) in (16), we obtain

$$\begin{aligned} |\psi^D\rangle &= |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_R - (\omega - i0)} \langle 1|\psi^D\rangle |\omega\rangle d\omega \\ &= |1\rangle + \int_0^\infty \frac{\lambda f(\omega)|\omega\rangle}{z_R - \omega + i0} d\omega. \end{aligned} \quad (18)$$

Since $\langle 1|1\rangle = 1$, e.g. the bound state of H_0 is assumed to be normalized, and $\langle 1|\omega\rangle = 0$, we conclude that $\langle 1|\psi^D\rangle = 1$.

Thus, we have arrived to (6) and proved that it is the solution of a Lipmann-Schwinger type-equation. The subsequent developments are identical as in the previous section. To explain for the appearance of the term $i0$ in the above equations, let us go back at the first line of (18). Usually, it is assumed that the form-factor $f(\omega)$ admits an analytical continuation to the lower half-plane. Then, the function under the integral in (18) represents the limit from below, hence the minus sign in $\omega \rightarrow \omega - i0$ of the denominator in the first row of (18), of this analytic continuation.

This can also be seen by considering the S matrix description in the energy representation. In the case of the Friedrichs model, poles of the analytic continuation of the S matrix and of the reduced resolvent coincide [12]. Both analytic continuations are supported by a two sheeted Riemann surface [2]. The value z_R appears as a pole of this analytic continuation located in the lower half plane of the second sheet. This half plane is connected with the upper half plane in the first sheet through the *upper rim* of the cut. Then, if we take $\text{Im } z_R \mapsto 0$, z_R , i.e., $z_R \mapsto \omega_0$ will go to the upper rim. The boundary values of the analytic continuation of the S matrix, $S(z)$, on the upper rim, $S(\omega + i0)$, $\omega > 0$, are the limits from *above to below* of $S(z)$, where z lies on the upper half plane in the first sheet. The same situation occurs for the reduced resolvent.

We have noted that the decaying Gamow state $|\psi^D\rangle$ corresponds to the “in” Lipmann Schwinger equation characterized with the plus sign in the term $+i0$ in the denominator. In fact, one may define the origin of times, $t = 0$, as the moment at which the creation of the decaying state $|\psi^D\rangle$ is completed and starts

to decay [2]. Thereafter, we know that $|\psi^D\rangle$ undergoes a purely time exponential decay at times $t > 0$ [18]. The “in” character of the decaying Gamow vector has been made explicit in another models, like one involving unstable interactions in field theory [19].

Correspondingly, there is an “out” state, the growing Gamow state $|\psi^G\rangle$, which is characterized by the property $(H_0 + V)|\psi^G\rangle = z_R^*|\psi^G\rangle$, where the star denotes complex conjugation, i.e., $z_R^* = E_r + i\Gamma/2$. Then, a similar analysis as the one performed to arrive to equation (18) leads to

$$|\psi^G\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_R^* - \omega - i0} |\omega\rangle d\omega. \quad (19)$$

In fact, the growing Gamow vector $|\psi^G\rangle$ is nothing else than the time reversal of $|\psi^D\rangle$ [20].

4 Perturbative analysis

In the sequel, we consider the perturbative analysis developed in Section 2 from another point of view. To begin with we have an initial state $\psi_0(0)$ that evolves under the action of the free Hamiltonian H_0 to become $\psi_0(\beta) = e^{-\beta H_0} \psi_0(0)$. Here, the variable β either denotes $\beta = it$, where t is time or $1/(kT)$, where k is the Boltzmann constant and T the temperature. In the first case, $\psi_0(\beta)$ represents the state at time t , provided that $\psi_0(0)$ is the state at time $t = 0$. Let us assume that the total Hamiltonian is given by $H = H_0 + \lambda V$ as in the Friedrichs model. When $\beta = 0$, the initial state is $\psi_0(0)$. For finite values of β the state is $\psi(\beta) := e^{-\beta H} \psi_0(0)$. The theory of perturbations gives us the following expression for $\psi(\beta)$ in terms of $\psi_0(\beta)$:

$$\psi(\beta) = \psi_0(\beta) - \lambda \int_0^\beta d\beta_1 e^{-(\beta-\beta_1)H_0} V \psi(\beta_1). \quad (20)$$

This is a typical integral equation which can be solved perturbatively, by folding each term with the previous one, namely:

$$\begin{aligned} \psi(\beta) = & \psi_0(\beta) - \lambda \int_0^\beta d\beta_1 e^{-(\beta-\beta_1)H_0} V \psi_0(\beta_1) \\ & - (-1)\lambda^2 \int_0^\beta d\beta_1 e^{-(\beta-\beta_1)H_0} V \int_0^{\beta_1} d\beta_2 e^{-(\beta_1-\beta_2)H_0} V \psi_0(\beta_2) \\ & \dots - (-1)^{n-1} \lambda^n \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \dots \int_0^{\beta_{n-1}} d\beta_n \times \\ & \times e^{-(\beta-\beta_1)H_0} V e^{-(\beta_1-\beta_2)H_0} V e^{-(\beta_2-\beta_3)H_0} V \dots V e^{-(\beta_{n-1}-\beta_n)H_0} \psi_0(\beta_n) + \dots \end{aligned} \quad (21)$$

In the Friedrichs model, the most general form of $\psi_0(0)$ ($\beta = 0$) is

$$\psi_0(0) = \alpha|1\rangle + \int_0^\infty \psi(\omega) |\omega\rangle d\omega. \quad (22)$$

Note that $|1\rangle$ is indeed $|1\rangle \otimes |0\rangle$ and $|\omega\rangle = |0\rangle \otimes |\omega\rangle$, where the ket $|0\rangle$ denotes the vacuum of the field in the first case and the absence of bound state in the second. This is important for the following developments. Although (22) is the most general initial state, subsequent calculations starting with it are too complicated. Furthermore, it is reasonable to begin with the bound state $|1\rangle$ as initial state, so that we shall assume from now on that $\psi_0(0) \equiv |1\rangle$. Then, let us apply $e^{-\beta_1 H_0}$ to $|1\rangle$ as in (22). Taking into account the form of H_0 (1), we obtain:

$$\psi(\beta_1) = e^{-\beta_1 H_0} \psi_0(0) = e^{-\beta_1 \omega_0 a^\dagger a} e^{-\beta_1 \int_0^\infty \omega b_\omega^\dagger b_\omega d\omega} |1\rangle = e^{-\beta_1 \omega_0} |1\rangle. \quad (23)$$

In order to evaluate the first integral in (21), let us write:

$$\begin{aligned} V\psi_0(\beta_1) &= \int_0^\infty f(\omega) [a^\dagger b_\omega + ab_\omega^\dagger] d\omega [e^{-\beta_1 \omega_0} |1\rangle] \\ &= \alpha e^{-\beta_1 \omega_0} \int_0^\infty f(\omega) |\omega\rangle d\omega. \end{aligned} \quad (24)$$

and make use of the definition of the variable $\tau = -(\beta - \beta_1)$, so that

$$\begin{aligned} &e^{\tau H_0} V\psi(\beta_1) \\ &= e^{\tau \omega_0 a^\dagger a} e^{\tau \int_0^\infty \omega b_\omega^\dagger b_\omega d\omega} \left[e^{-\beta_1 \omega_0} \int_0^\infty f(\omega) |\omega\rangle d\omega \right] \\ &= e^{-\beta_1 \omega_0} \int_0^\infty e^{\tau \omega} f(\omega) |\omega\rangle d\omega = e^{-\beta_1 \omega_0} \int_0^\infty f(\omega) e^{-\beta \omega} e^{\beta_1 \omega} |\omega\rangle d\omega \\ &= \int_0^\infty f(\omega) e^{-\beta \omega} e^{\beta_1(\omega - \omega_0)} |\omega\rangle d\omega. \end{aligned} \quad (25)$$

Let us integrate with respect to β_1 , it yields the result

$$\int_0^\infty e^{-\beta \omega} \frac{e^{\beta(\omega - \omega_0)} - 1}{\omega - \omega_0 + i0} f(\omega) |\omega\rangle d\omega = \int_0^\infty \frac{e^{-\beta \omega_0} - e^{-\beta \omega}}{\omega - \omega_0 + i0} f(\omega) |\omega\rangle d\omega. \quad (26)$$

Then, up to first order in λ , equation (21) gives:

$$\begin{aligned}
\psi(\beta) &= e^{-\beta\omega_0}|1\rangle - \lambda \int_0^\infty \frac{e^{-\beta\omega_0} - e^{-\beta\omega}}{\omega - \omega_0 + i0} f(\omega) |\omega\rangle d\omega \\
&= e^{-\beta\omega_0} \left[|1\rangle - \int_0^\infty \frac{\lambda f(\omega)}{\omega - \omega_0 + i0} |\omega\rangle d\omega + \lambda \int_0^\infty \frac{e^{-\beta(\omega-\omega_0)} f(\omega)}{\omega - \omega_0 + i0} |\omega\rangle d\omega \right], \quad (27)
\end{aligned}$$

which is equivalent to (18). Next, let us calculate the second order in λ . In order to do it, we recall that the second integral in the second line in (21) has been already evaluated and the result is given in (26), where we have to replace β by β_1 . Then, the second line in (21) becomes:

$$\lambda^2 \int_0^\beta d\beta_1 e^{-(\beta-\beta_1)H_0} V \int_0^\infty \frac{e^{-\beta_1\omega_0} - e^{-\beta_1\omega}}{\omega - \omega_0 + i0} f(\omega) |\omega\rangle d\omega \quad (28)$$

To evaluate this term, note that

$$V|\omega\rangle = \int_0^\infty f(\omega') [a^\dagger b_{\omega'} + ab_{\omega'}^\dagger] |\omega\rangle d\omega' = \left[\int_0^\infty f(\omega) d\omega \right] |1\rangle, \quad (29)$$

leading to

$$\begin{aligned}
V \int_0^\infty \frac{e^{-\beta_1\omega_0} - e^{-\beta_1\omega}}{\omega - \omega_0 + i0} f(\omega) |\omega\rangle d\omega &= \int_0^\infty \frac{e^{-\beta_1\omega_0} - e^{-\beta_1\omega}}{\omega - \omega_0 + i0} f(\omega) V |\omega\rangle d\omega \\
&= \left[\int_0^\infty d\omega f^2(\omega) \frac{e^{-\beta_1\omega_0} - e^{-\beta_1\omega}}{\omega - \omega_0 + i0} \right] |1\rangle = I_2(\beta_1)|1\rangle, \quad (30)
\end{aligned}$$

so that the second line in (21) has the form ($\tau = \beta - \beta_1$):

$$\lambda^2 \int_0^\infty d\omega \int_0^\beta d\beta_1 f^2(\omega) \frac{e^{-\beta_1\omega_0} - e^{-\beta_1\omega}}{\omega - \omega_0 + i0} e^{\tau H_0} |1\rangle, \quad (31)$$

with

$$e^{\tau H_0} |1\rangle = e^{-(\beta-\beta_1)\omega_0} |1\rangle. \quad (32)$$

Carrying (32) into (31), we obtain the following relation:

$$\begin{aligned}
&\lambda^2 \int_0^\infty d\omega \frac{f^2(\omega)}{\omega - \omega_0 + i0} \int_0^\beta d\beta_1 e^{\beta_1\omega_0} (e^{-\beta_1\omega_0} - e^{-\beta_1\omega}) |1\rangle \\
&= \lambda^2 \int_0^\infty d\omega \frac{f^2(\omega)}{\omega - \omega_0 + i0} \left[\beta + \frac{e^{\beta(\omega-\omega_0)} - 1}{\omega - \omega_0 + i0} \right] |1\rangle. \quad (33)
\end{aligned}$$

Note that this expression vanishes when $\beta = 0$.

Next, let us go to the third order. The corresponding term in (21) reads:

$$\begin{aligned}
& -\lambda^3 \int_0^\beta d\beta_1 e^{-(\beta-\beta_1)H_0} V \int_0^{\beta_1} d\beta_2 e^{-(\beta_1-\beta_2)H_0} V \int_0^{\beta_3} d\beta_3 e^{-(\beta_2-\beta_3)H_0} V \psi_0(\beta_3) \\
& = -\lambda^3 \int_0^\beta d\beta_1 G(\omega, \omega_0, \beta_1) e^{-(\beta-\beta_1)H_0} V |1\rangle,
\end{aligned} \tag{34}$$

with

$$G(\omega, \omega_0, \beta_1) = e^{-\beta_1 \omega_0} \int_0^\infty d\omega \frac{f^2(\omega)}{\omega - \omega_0 + i0} \left[\beta_1 + \frac{e^{\beta_1(\omega - \omega_0)} - 1}{\omega - \omega_0 + i0} \right]. \tag{35}$$

Repeating the same arguments as in (24) and (25), we obtain that (35) is equal to

$$-\lambda^3 \int_0^\beta d\beta_1 G(\omega, \omega_0, \beta_1) \int_0^\infty e^{-(\beta-\beta_1)\omega'} f(\omega') |\omega'\rangle d\omega'. \tag{36}$$

Due to the form of $G(\omega, \omega_0, \beta_1)$ given in (35), (36) is a sum of three terms. The first one is given by

$$\begin{aligned}
& -A\lambda^3 \int_0^\infty e^{-\beta\omega'} f(\omega') |\omega'\rangle d\omega' \int_0^\beta \beta_1 e^{-\beta_1(\omega_0 - \omega')} d\beta_1 \\
& = -A\lambda^3 \int_0^\infty e^{-\beta\omega'} f(\omega') \left\{ \frac{1}{(\omega_0 - \omega' + i0)^2} \left[1 - e^{-\beta(\omega_0 - \omega')} \right] - \frac{\beta e^{-\beta(\omega_0 - \omega')}}{\omega_0 - \omega' + i0} \right\} |\omega'\rangle d\omega', \tag{37}
\end{aligned}$$

with

$$A = \int_0^\infty d\omega \frac{f^2(\omega)}{\omega - \omega_0 + i0}. \tag{38}$$

The second one is:

$$\begin{aligned}
& \lambda^3 \int_0^\beta d\beta_1 e^{-\beta_1 \omega_0} \int_0^\infty d\omega \frac{f^2(\omega)}{(\omega - \omega_0 + i0)^2} \int_0^\infty d\omega' e^{-(\beta-\beta_1)\omega} f(\omega') |\omega'\rangle d\omega' \\
& = \lambda^3 B \int_0^\infty d\omega' f(\omega') |\omega'\rangle \int_0^\beta d\beta_1 e^{-\beta\omega'} e^{-\beta_1(\omega_0 - \omega')} \\
& = \lambda^3 B \int_0^\infty \frac{1}{\omega' - \omega_0 + i0} (e^{-\beta_1(\omega_0 - \omega')} - 1) f(\omega') |\omega'\rangle, \tag{39}
\end{aligned}$$

with

$$B := \int_0^\infty d\omega \frac{f^2(\omega)}{(\omega - \omega_0 + i0)^2}. \quad (40)$$

Finally, the third term reads:

$$\begin{aligned} & -\lambda^3 \int_0^\beta d\beta_1 e^{-\beta_1 \omega_0} \int_0^\infty d\omega \frac{f^2(\omega)}{(\omega - \omega_0 + i0)^2} e^{-\beta_1(\omega - \omega_0)} \int_0^\infty d\omega' e^{-(\beta - \beta_1)\omega'} f(\omega') |\omega'\rangle \\ & = \lambda^3 \int_0^\infty d\omega \frac{f^2(\omega)}{(\omega - \omega_0 + i0)^2} \int_0^\infty d\omega' f(\omega') \frac{1}{\omega - \omega' + i0} e^{\beta\omega'} (e^{-\beta(\omega - \omega')} - 1) |\omega'\rangle. \end{aligned} \quad (41)$$

Before to proceed with our analysis, let us discuss on the integrals on which expressions of the form $(\omega - \omega_1 \pm i0)^{-1}$ or $(\omega - \omega_1 \pm i0)^{-2}$ appear. These are well known distributions over spaces of test functions. If $f^2(\omega)$ belongs to one of these spaces, which is the case if for instance $f(\omega)$ belong to a Schwartz space [21], formula (38) has a clear meaning (see [21]). Note that $-(\omega - \omega_1 \pm i0)^{-2}$ is the distributional derivative of $(\omega - \omega_1 \pm i0)^{-1}$, so that (40) is meaningful in this distributional sense.

In other formulas like in the second integral in the second row of (41), we could have omitted the term $i0$ in the denominator, since we have a term in the numerator which goes to zero as $\omega \mapsto \omega'$. In (41), this is $e^{-\beta(\omega - \omega')} - 1$. This type of integrals have a meaning as functionals [15].

4.1 Comments

Equation (27) can be written as:

$$\psi(\beta) = e^{-\beta\omega_0} \left[|\psi_1^D\rangle + \lambda \int_0^\infty \frac{e^{-\beta(\omega - \omega_0)} f(\omega)}{\omega - \omega_0 + i0} |\omega\rangle d\omega \right]. \quad (42)$$

The value of the decaying Gamow at third order, $|\psi_3^D\rangle$ is hidden in the above calculations. First, note that A in (38) can be written as

$$A = -\mathcal{P} \int_0^\infty \frac{f^2(\omega)}{\omega_0 - \omega} d\omega - i\pi f(\omega_0) = -I(\omega_0, f) - i\pi f(\omega_0). \quad (43)$$

Then, observe that in (37), there exists a term of the form

$$e^{-\beta\omega_0} \int_0^\infty \frac{f(\omega')}{(\omega_0 - \omega' + i0)^2} |\omega'\rangle d\omega'. \quad (44)$$

As before, the contribution proportional to $|\psi_3^D\rangle$ is hidden in (37), as well as higher order terms. Therefore, the structure of (42) is just of the form (18).

5 Conclusions

In this paper we have presented the mathematical steps needed to demonstrate the correspondence between the functional representation of Gamow states and a given class of solutions of the Lippmann-Schwinger equation. As a convenient formalism to work with we have chosen the Friedrichs model, which consists of a discrete state interacting with a continuum. In this model, Gamow states are easily constructed. We have demonstrated, both exactly and perturbatively, that their structure is identical to the solutions of a Lippmann-Schwinger equation. This finding may greatly simplified the use of Gamow states in systems where the description of resonances is relevant for the calculation of observables, as was the case for the evaluation of the complex entropy attached to a Gamow state [22], and it is also expected to be important in future formulation of many-body quantum unstable states.

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