A new consensus ranking approach for correlated ordinal information based on Mahalanobis distance

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Abstract

We investigate from a global point of view the existence of cohesiveness among experts' opinions. We address this general issue from three basic essentials: the management of experts' opinions when they are expressed by ordinal information; the measurement of the degree of dissensus among such opinions; and the achievement of a group solution that conveys the minimum dissensus to the experts' group.

Accordingly, we propose and characterize a new procedure to codify ordinal information. We also define a new measurement of the degree of dissensus among individual preferences based on the Mahalanobis distance. It is especially designed for the case of possibly correlated alternatives. Finally, we investigate a procedure to obtain a social consensus solution that also includes the possibility of alternatives that are correlated. In addition, we examine the main traits of the dissensus measurement as well as the social solution proposed. The operational character and intuitive interpretation of our approaches are illustrated by an explanatory example.

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1. Introduction

A considerable amount of literature has contributed to the research issue of obtaining consensus in group decision making problems. This issue is an active subject in several areas such as Social Choice Theory and Decision Making Theory. From the Social Choice perspective several contributions can be emphasized, e.g., [9], [49], [25], [2], [4] and [26], among others. From the Decision Making Theory, it has been successfully tackled by a great amount of contributions, e.g., [28], [32], [36], [35] and [58], among others. Besides these main areas, there are some other methodologies that proposed different definitions of the consensus concept. It is worth mentioning the work of González Jaime et al. [38] and López Molina, De Baets and Bustince [47].

Any group decision making problem focused on obtaining consensus involves at least three key pillars. The first one is the way in which experts give their opinions on a set of alternatives and how such an information is managed. Once the opinions of the agents have been gathered it seems natural to measure how much cohesiveness these opinions generate. Thus, the second pillar is to establish a mechanism able to provide such measurements. Apart from determining the degree of consensus among experts the main aim of a group decision making problem is to determine a solution. The better solution the greater agreement this solution generates among experts. Consequently, supplying a method to achieve a group consensus solution is the third pillar.

We now briefly review the previous literature related to each basic essentials.

Information formats. Generally speaking, experts can express their opinions by means of ordinal or cardinal information, the former being more extensively used in the research issue addressed in this work. Nonetheless, contributions dealing with cardinal information include the approaches proposed by Herrera-Viedma, Herrera and Chiclana [36], González-Pachón and Romero [30] and González-Arteaga, Alcantud and de Andrés Calle [26]. The representation of ordinal information has been a subject of study for over two centuries for linear orders (see e.g., [9] and [3]), weak orders(see e.g., [16],

[24] and [4]) and fuzzy preferences (see e.g., [12],[22], [51] and [56], among others).

Regardless of the experts' information format, it is necessary to manipulate it in order to make suitable computations. In the literature several procedures to codify linear and complete preorders into numerical values can be found (see [8], [7], [14] and [25], among others), Borda [8] being the first author to manage ordinal preferences in such way.

Consensus measurement. This topic was initiated by Bosch [9] from the Social Choice perspective. In this vein McMorris and Powers [49] characterized consensus rules defined on hierarchies, while García-Lapresta and Pérez-Román [25] introduced a class of consensus measures based on distances. Subsequently, Alcalde-Unzu and Vorsatz [2, 3] proposed and characterized a family of linear and additive consensus measures based on measuring similarity among preferences. From another point of view, Alcantud, de Andrés Calle and Cascón [5, 6] introduced the analysis when opinions are dichotomous.

The use of distance and similarity functions has provided interesting insights about cohesiveness measurement. We highlight the role of the Kemeny, Mannhattan, Jacard, Dice and Cosine distance functions (see e.g., [15],[25],[5] and [13]). Moreover, it is also possible to apply some association measures to that purpose (see e.g., [55], [33], [14], [42], [21] and [27]).

Group consensus solution. Finding the best option or solution from alternatives is the main aim in group decision making problems. Recently, various approaches have been developed to solve this problem from a variety of science areas: *Operational Research* (see e.g., [17] and [20]), *Statistical Analysis* (see e.g., [45], [23] and [1]), *Fuzzy Theory* (see e.g., [18], [59] and [46]), and *Computational Analysis* (see e.g., [37] and [60]).

Traditionally, the achievement of a global solution has been considered as an aggregation problem of experts' opinions in order to obtain a social solution. Different methods have been proposed and analyzed for aggregating agents' opinions (preferences in the case of ordinal information) into a social solution. Borda [8] first examined this problem in a voting context and Kendall [41] subsequently revised Borda's method in a statistical framework.

Other authors also proposed alternative distance-based aggregation rules e.g., Eckert and Klamler [19], Klamler [44, 43], Meskanen and Nurmi [50], Ratliff [52, 53], and Saari and Merlin [54], even though Kemeny's rule [39] could be considered as a landmark in aggregation procedures based on distances. Following Kemeny's rule, Cook and Seiford [14] established an equivalence between the Borda-Kendall method [40] and their approach. González-Pachón and Romero [28] developed a general framework for distance-based consensus models under the assumption of a generic l_p metric. These authors have recently designed socially optimal decisions in a consensus scenario [31].

Once we have reviewed the related literature we now summarize the main contributions of this paper.

- We focus on group decision making problems where agents or experts provide their opinions on a set of alternatives by complete preorders. In this regard, we propose a new codification procedure to transform the original opinions/preferences of agents into numerical vectors in order to manage them. For the purpose of better understanding this process we investigate exactly which vectors are realizations by a *canonical codification procedure* of generic complete preorders. The characterization of the new codification procedure is a key point because it ensures consistency of our approach and its use in any methodology.

- In order to measure the degree of cohesiveness among agents' preferences, we design an indicator of dissensus for a finite collection of complete preorders on a finite set of alternatives based on the Mahalonobis distance, which is dependent on a positive definite matrix (the parameter) that captures the importance and possible cross-relations of each alternative, namely, the *Mahalanobis dissensus measures*. Any such indicator ranks the profiles of complete preorders (in the form of codified matrices) according to their inherent cohesiveness. The strength of our measurement unlike other aforementioned approaches based on distances is the inclusion of the relationships among alternatives. Then, the new measure incorporates relevant information that in other way is ignored. Moreover, we investigate the main characteristics of the novel measure and prove that a partial order can be naturally induced on the parametric class of all *Mahalanobis dissensus measures*.

- Then we exploit these measures in order to propose a consensus solution especially designed for profiles of preferences on possibly correlated alternatives and to overcome the drawbacks of the aforementioned distance-based methodologies. That solution aggregates individual opinions into a social preference on the alternatives by minimizing dissensus with respect to the original profile of preferences. In order to facilitate the computation of such compromise solution we prove that the problem is equivalent to minimizing the Mahalanobis distance to a single average vector. Whatever the statement of the minimization problem, the objective function is restricted to feasible codified vectors, which emphasizes the importance of our characterization for the canonical codification procedure. Some properties of our *Mahalanobis consensus solution* are proven and discussed.

In addition, an explanatory example illustrates the operational characteristics and intuitive interpretation of our approaches to find rankings that best agree with the original opinions.

This paper is organized as follows. Section 2 is devoted to the problem of transforming ordinal information about individual preferences into numerical vectors as well as essential notation. Section 3 introduces the basic definition of dissensus measure and the Mahalanobis class of dissensus measures. Here we also explore their main traits too. In Section 4 we set forth the definition of our proposal of Mahalanobis social consensus solutions, prove some of its properties, and solve a visually appealing example. Finally, some concluding remarks are pointed out in Section 5.

2. Ordinal information

Most group decision making problems can usually manage different types of information. In this contribution we focus on the representation of agents' opinions by means of rankings allowing ties since most real situations involve such a kind of information. Dealing with this type of information necessarily entails determining how it is represented. In the specialized literature it is possible to find several approaches or procedures to codify ordinal information into numerical values (see [8], [7], [14] and [25], among others).

Due to the importance of the choice of the codification procedure to accomplish any methodology over ordinal information, it should be relevant to dispose of a consistent codification procedure. Accordingly, in this section we provide and characterize a new method to handle ordinal information as well as the basic notation of our proposal.

2.1. Notation

Consider a society of agents or experts $\mathbf{N} = \{1, 2, ..., N\}$, $N > 1$. Let $X = \{x_1, ..., x_k\}$ be a finite set of k issues, options or alternatives $|X| \geqslant 2$. Abusing notation, on occasions we refer to issue x_s as issue s for convenience.

Assume experts grade alternatives by means of complete preorders (also known as weak orders). Technically speaking, a complete preorder R on X means a complete and transitive binary relation on X. We write $W(X)$ to denote the set of all complete preorders on X. [1]

Let $R \in W(X)$ be a complete preorder on X, then $x_s \succ_R x_k$ means x_s is strictly preferred to x_k, $x_s \sim_R x_k$ means x_s and x_k are equally preferred and $x_s \succcurlyeq_R x_k$ means alternative x_s is at last as good as x_k. For a complete preorder $R \in W(X)$, let R^{-1} be the inverse of R such that $x_s \succ_{R^{-1}} x_k \Leftrightarrow x_k \succ_R x_s$ for all $x_s, x_k \in X$.

A *profile* $\mathcal{P} = (R_1, ..., R_N) \in W(X) \times \ldots \times W(X) = W(X)^N$ of the society $\mathbf{N}$ on the set of alternatives X is a collection of N complete preorders, where R_i represents the preferences of the individual i on the k alternatives for each $i = 1, ..., N$. Given a profile $\mathcal{P} = (R_1, \ldots, R_N)$, its inverse is denoted by $\mathcal{P}^{-1} = (R_1^{-1}, \ldots, R_N^{-1})$.

Any permutation σ of the agents/experts $\{1, 2, ..., N\}$ determines a permutation of $\mathcal{P}$ by $\mathcal{P}^\sigma = (R_{\sigma(1)}, ..., R_{\sigma(N)})$. Analogously, any permutation π of the alternatives $\{1, 2, ..., k\}$ determines a permutation of every complete preorder $R \in W(X)$ such that the permuted profile is denoted by ${}^\pi\mathcal{P} = ({}^\pi R_1, ..., {}^\pi R_N)$. We write $P(X) = \cup_{N \geqslant 1} W(X)^N$ to denote the set of all profiles for arbitrary societies.

The codification of preferences by numerical vectors has been used extensively in both theoretical and practical situations. Borda [8] was first to manage ordinal preferences in such way. His method, known as the "method of marks" or "Borda-Kendall method", has been widely disseminated in several areas.

Following the Social Choice tradition, the components of a numerical vector represent the rank or priority assigned to each alternative, or their average in case of ties. This convention has been exemplified by Black [7], Cook and Seiford [14] and García-Lapresta and Pérez-Román [25].

[1] It is assumed that a linear order on X is an antisymmetric weak order on X.

We now introduce notation related to the codification of linear and complete preorders by means of numerical vectors.

Let $R \in W(X)$ be a complete preorder on X, a *codified complete preorder* is a real-valued vector $M_R = (m_1, \ldots, m_k)$ where m_j represents the codification value corresponding to alternative x_j. It relates to R in the sense that $x_i \succcurlyeq_R x_j \Leftrightarrow m_i \geq m_j$.

A *codified profile* of $\mathcal{P}$ is a $N \times k$ real-valued matrix

$$M_{\mathcal{P}} = \begin{pmatrix} m_{11} & \ldots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{N1} & \ldots & m_{Nk} \end{pmatrix}_{N \times k}$$

where m_{ij} is the codification value of expert i over the alternative x_j. We write $\mathbb{M}_{N\times k}$ for the set of all $N \times k$ real-valued matrices. Thus $M_{\mathcal{P}} = (M_{R_1}, \ldots, M_{R_N}) \in \mathbb{M}_{N\times k}$ produces a unique profile $\mathcal{P}$ of complete preorders, although every profile of complete preorders can be associated with infinitely many matrices from $\mathbb{M}_{N\times k}$. For simplicity, on occasions we refer to $M_{\mathcal{P}}$ as M.

Row i of the profile $M_{\mathcal{P}}$ is identified by M_i. It describes the codification preferences of expert i over all alternatives, $M_i = M_{R_i} \in \mathbb{M}_{1\times k}$. Similarly, column j of the codification profile $M_{\mathcal{P}}$ captures the codification of agents' preferences on the alternative j, and it is denoted by $M^j \in \mathbb{M}_{N\times 1}$.

Any permutation σ of the experts $\{1, 2, ..., N\}$ determines a codified profile $M^\sigma = (M_{\sigma(1)}, \ldots, M_{\sigma(N)}) \in \mathbb{M}_{N\times k}$ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile $M \in \mathbb{M}_{N\times k}$. Similarly, any permutation π of the alternatives $\{1, 2, ..., k\}$ determines a codified profile ${}^\pi M \in \mathbb{M}_{N\times k}$ by permutation of the columns of $M \in \mathbb{M}_{N\times k}$: column j of the profile ${}^\pi M$ is column $\pi(j)$ of the codification profile M. Notice that $M_{\mathcal{P}^\sigma} = (M_{\mathcal{P}})^\sigma$ and $M_{{}^\pi\mathcal{P}} = {}^\pi(M_{\mathcal{P}})$.

2.2. The canonical codification. Definition and characterization

In this subsection we define a new way to represent ordinal preferences by numerical vectors, namely, the *canonical codification.* Moreover, we characterize the new codification procedure to associate every profile of complete preorders with a unique matrix. Therefore, the use of this particular codification procedure is consistence and it could be used in any approach or methodology. Along this section, some illustrative examples are included to put it in practice.

Definition 1. *The* canonical codified complete preorder *associated with* $R \in W(X)$ *is defined by the numerical vector* $K_R = (c_1, ..., c_k) \in (\{1, \dots, k\})^k$ *where* $c_j = |\{q : x_j \succcurlyeq_R x_q\}|$ *and therefore* c_j *accounts for the number of alternatives that are graded at most as good as* x_j*.*

A *canonical codified profile* associated with $\mathcal{P} = (R_1, ..., R_N) \in W(X)^N$ is an $N \times k$ real-valued matrix denoted as $K_{\mathcal{P}} = (K_{R_1}, ..., K_{R_N}) \in \mathbb{M}_{N\times k}$. Each K_{R_i} is row i in $K_{\mathcal{P}}$ and it corresponds to the canonical codified complete preorder associated with R_i.

Let us now provide an example in order to improve the understanding of our codification proposal.

Example 1. Let R_1, R_2, R_3 be the complete preorders on $\{x_1, x_2, x_3\}$ such that:

$$\begin{aligned} R_1 :&\quad x_1 \succ_{R_1} x_2 \sim_{R_1} x_3, \\ R_2 :&\quad x_2 \succ_{R_2} x_1 \succ_{R_2} x_3, \\ R_3 :&\quad x_3 \succ_{R_3} x_1 \sim_{R_3} x_2. \end{aligned}$$

Following Definition 1 their respective canonical codifications are $K_{R_1} = (3, 2, 2)$, $K_{R_2} = (2, 3, 1)$, and $K_{R_3} = (2, 2, 3)$. We consider only for illustration that these complete preorders define a profile, $\mathcal{P} = (R_1, R_2, R_3)$. Then its respective canonical codified profile is

$$K_{\mathcal{P}} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

In order to motivate the main result of this section, let us observe that not all vectors of natural values are feasible canonical codified complete preorders. For example, by means of the canonical codification it is not possible to get $K_R = (1, 1, 1)$ with $k = 3$ because if there is a tie among the three alternatives Definition 1 produces $(3, 3, 3)$.

Considering these limitations, we now proceed to identify exactly which vectors correspond to a canonical codified complete preorder.

Proposition 1. *Given a vector* $c = (c_1, \dots, c_k) \in (\{1, \dots, k\})^k$*, this vector is the canonical codified complete preorder* K_R *associated with* $R \in W(X)$ *if and only if the increasingly ordered vector* $(c_{(1)}, \dots, c_{(k)})$ *verifies* *(i)* $c_{(1)} = t_1$,

(ii) $c_{(j+1)} = c_{(j)} + t_{j+1} \cdot D_{j+1}, \quad j \in \{1, \dots, k-1\}$,

where t_j *is the number of values equal to* $c_{(j)}$ *among the components of* c *and*

$$D_{j+1} = \begin{cases} 0 & \text{if} \quad c_{(j+1)} = c_{(j)}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof 1. *Let* $R \in W(X)$ *be a complete preorder whose canonical codification is* $K_R = (c_1, \dots, c_k)$. *Given a permutation on the alternatives* τ, R^τ *denotes the permutation* τ *on the complete preorder* $R \in W(X)$ *such that* $K_{R^\tau} = (c_{(1)}, \dots, c_{(k)})$ *and* $c_{(1)} \leqslant \dots \leqslant c_{(k)}$.

Throughout the proof, $t \in \mathbb{N}^k$ *stands for the vector containing the number of coincidences for the elements of* K_R, $t = (t_1, t_2, \dots, t_k) = (|T_1|, |T_2|, \dots, |T_k|)$ *where* $T_j = \{\, t \in \{1, \dots, k\} \mid c_t = c_{(j)} \,\}$ *for* $j \in \{1, \dots, k\}$. *Thus* $|T_j|$ *is the number of ties equal to* $c_{(j)}$. *The* t *vector is also called the ties vector of* K_R.

Let us first examine necessity. *Given a canonical codified complete preorder* $K_R = (c_1, \dots, c_k) \in (\{1, \dots, k\})^k$ *of* $R \in W(X)$, *let us check conditions* (i) *and* (ii).

(i) *To deduce* $c_{(1)} = t_1$, *we consider Definition 1*

$$c_{(1)} = |\{q : x_{(1)} \succcurlyeq x_q\}|,$$

where $x_{(1)}$ *is the alternative associated with* $c_{(1)}$. *Then,* $c_{(1)} = t_1$ *due to the fact that* $c_{(1)}$ *is the number of alternatives equally preferred to* $x_{(1)}$.

(ii) *To deduce* $c_{(j+1)} = c_{(j)} + t_{j+1} \cdot D_{j+1}, \quad j \in \{1, \dots, k-1\}$, *we claim that if alternative* $x_{(j+1)}$ *is equally preferred to alternative* $x_{(j)}$, *then* $c_{(j+1)} = c_{(j)}$. *In other case, by Definition 1*

$$c_{(j+1)} = |\{q : x_{(j+1)} \succcurlyeq x_q\}|.$$

Hence, $c_{(j+1)}$ *is the sum of the number of the strictly less preferred alternatives to* $x_{(j+1)}$ *plus the number of equally preferred alternatives to* $x_{(j+1)}$. *Formally,*

$$c_{(j+1)} = |\{q : x_{(j+1)} \succ x_q\}| + |\{q : x_{(j+1)} \sim x_q\}|.$$

Then, $c_{(j+1)} = c_{(j)} + t_{j+1}$.

We now proceed to prove sufficiency. *Suppose a numerical vector that verifies conditions* (i) *and* (ii), $c = (c_1, \ldots, c_k) \in (\{1, \ldots, k\})^k$. *We are in a position to build a complete preorder* $R \in W(X)$ *such that* $K_R = c$ *as follows.*

By ordering in an increasing order the vector c, we obtain an ordered vector, $c_{()} = (c_{(1)}, \ldots, c_{(k)})$ *and it is easy to compute its associated ties vector* $t = (t_1, \ldots, t_k) = (|T_1|, \ldots, |T_k|)$. *Then*

$$c_{()} = (c_{(1)}, \ldots, c_{(k)}) = (\overbrace{t_1, \ldots, t_1}^{t_1 \text{ times}}, \overbrace{t_1 + t_2, \ldots, t_1 + t_2}^{t_2 \text{ times}}, \ldots, \overbrace{k, \ldots, k}^{t_k \text{ times}})$$

and consequently, we can deduce the complete preorder R^{τ}*:*

$$x_{k-t_k+1} \sim \ldots \sim x_k \succ \ldots \succ x_{t_1+1} \sim \ldots \sim x_{t_1+t_2} \succ x_1 \sim \ldots \sim x_{t_1}$$

whose associated canonical codification is $c_{()}$. *The proof is completed due to* $K_R = c$. □

Now we proceed to exemplify the relevance of this result.

Example 2. In order to verify the necessity of establishing a characterization of the codification procedure, let us check if some numerical vectors can actually represent codified complete preorders for the case of four alternatives.

- Consider the numerical vector $c = (3, 4, 1, 1)$. First, its increasingly ordered vector and its corresponding ties vector are determined, $c_{()} = (1, 1, 3, 4)$ and $(|T_1|, |T_2|, |T_3|, |T_4|) = (2, 2, 1, 1)$, respectively. Second, by Proposition 1, we check that if c represents a canonical codification K_R for some complete preorder $R \in W(X)$ then the first element of K_R should be 2. Therefore, c is not a canonical codified complete preorder.

- We repeat the previous exercise for the numerical vector $c = (2, 3, 3, 1)$. Then, $c_{()} = (1, 2, 3, 3)$ is its increasingly ordered vector and $(|T_1|, |T_2|, |T_3|, |T_4|) = (1, 1, 2, 2)$ is its ties vector. Using Proposition 1, the first, second and third element of K_R should be 1, 2 and $2 + 2 = 4$, respectively. However, the latter is not true since $c_{(3)} = 3$. Thus, the vector c does not represent any complete preorder by the canonical codification.

- Finally, given $c = (4, 2, 2, 1)$ a numerical vector, being its corresponding increasingly ordered vector $c_{()} = (1, 2, 2, 4)$ and its ties vector $(|T_1|, |T_2|, |T_3|, |T_4|) = (1, 2, 2, 1)$. By means of Proposition 1, if c represents a canonical codification K_R for some complete preorder $R \in W(X)$, the first and second element of K_R should be 1 and $1+2 = 3$ respectively, but it is not true because $c_{(2)} = 2$. Therefore, c does not represent any complete preorder by the canonical codification.

3. A new dissensus measure for ordinal information: The class of Mahalanobis dissensus measures

A considerable amount of the most cited contributions on consensus measurement have addressed this topic considering functions that assign to every ranking profile a real number *from the unit interval*. Therefore, the higher the assignment, the more coherence among agents' preferences.

In this contribution we focus on the notion of dissensus measurement, concretely, our approach resembles the notion of a "measure of statistical dispersion", in the sense that 0 captures the natural notion of unanimity as total lack of variability, and then increasingly higher numbers mean more disagreement among rankings in the profile. Then, we introduce a new broad class of dissensus measures associated with a reference matrix, namely the *Mahalanobis dissensus measures* that includes the possibility of cross-related alternatives. Moreover, some important properties of the new measurement are included.

Definition 2. *A* dissensus measure *is a mapping* $\delta : W(X)^N \to [0, \infty)$ *given by*

$$\delta(\mathcal{P}) = \delta^*(M_{\mathcal{P}})$$

for each profile $\mathcal{P} \in W(X)^N$ *and its codified profile* $M_{\mathcal{P}} \in \mathbb{M}_{N\times k}$*, where* δ^* *is a mapping* $\delta^* : \mathbb{M}_{N\times k} \to [0, \infty)$ *with the property:*

(I) $\delta(\mathcal{P}) = 0$ *if and only if* $\mathcal{P}$ *is unanimous. In other words,* $\delta^*(M_{\mathcal{P}}) = 0$ *if and only if* $M_{\mathcal{P}}$ *is unanimous.*

Henceforth we also deal with dissensus measures that are normal, in the following sense:

Definition 3. *A dissensus measure is* normal *if it further verifies:*

(II) Anonymity*:* $\delta(\mathcal{P}^\sigma) = \delta^*\left((M_\mathcal{P})^\sigma\right) = \delta^*(M_\mathcal{P}) = \delta(\mathcal{P})$ *for each permutation* σ *of the agents and* $M_\mathcal{P} \in \mathbb{M}_{N\times k}$*.*

(III) Neutrality*:* $\delta({}^\pi\mathcal{P}) = \delta^*\left({}^\pi(M_\mathcal{P})\right) = \delta^*(M_\mathcal{P}) = \delta(\mathcal{P})$ *for each permutation* π *of the alternatives and* $M_\mathcal{P} \in \mathbb{M}_{N\times k}$*.*

Before providing our main definition, we recall the *Mahalanobis distance* [48] on which our measure is based. This distance is a common tool in multivariate statistical analysis, e.g., in regression models. We select it in our proposal because it allows to take into account cross relations among alternatives which is frequent in real situations.

Definition 4. *Let* $\Sigma \in \mathbb{M}_{k\times k}$ *be a positive definite matrix and* $x, y \in \mathbb{R}^k$ *be two row vectors. The Mahalanobis (squared) distance on* $\mathbb{R}^k$ *associated with* Σ *is defined by* [2]

$$d_\Sigma(x, y) = (x - y)\Sigma^{-1}(x - y)^t$$

The Mahalanobis distance includes some particular distances such as the (squared) Euclidean distance when Σ is the identity matrix.

Definition 5. *Let* $\Sigma \in \mathbb{M}_{k\times k}$ *be a positive definite matrix and let us fix a codification procedure for profiles of complete preorders,* $\mathcal{P} \in W(X)^N$*. The* Mahalanobis dissensus measure *associated with* Σ *is the mapping* $\delta_\Sigma : W(X)^N \to [0, \infty)$ *given by*

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}),$$

for each profile $\mathcal{P} \in W(X)^N$ *and its codified profile* $M_\mathcal{P} \in \mathbb{M}_{N\times k}$*, where* δ^*_Σ *is the mapping* $\delta^*_\Sigma : \mathbb{M}_{N\times k} \to [0, \infty)$ *given by*

$$\delta^*_\Sigma(M_\mathcal{P}) = \frac{1}{C^2_N} \cdot \sum_{i<j} d_\Sigma(M_i, M_j),$$

and $C^2_N = \frac{N(N-1)}{2}$ *is the number of unordered pairs of the* N *agents.*

[2] Our choice of $d_\Sigma(x, y)$ coincides with Mahalanobis' original definition [48].

Notice that δ^*_Σ is the arithmetic mean of the Mahalanobis distances between each pair of codified complete preorders for each agent following Hays's approach [34].

Remark 1. *The Mahalanobis dissensus measure satisfies the assumption of Definition 2 because*

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = 0$$

if and only if $\mathcal{P}$ is unanimous. This fact is easy to prove since d_Σ is a distance.

Along this contribution we use the codification procedure given in Section 2 even though the Mahalanobis dissensus measure is compatible with different codification procedures.

To emphasize the advantages of our proposal, it could be interesting not only to obtain values but to compare them in order to rank the original profiles attending to their degree of dissensus. In this sense, next definition is provided.

Definition 6. *Each dissensus measure δ_Σ, for a positive definite matrix $\Sigma \in \mathbb{M}_{k\times k}$, produces* a ranking of profiles *of complete preorders $\succcurlyeq_{\delta_\Sigma}$ by establishing that*

$$\mathcal{P} \succcurlyeq_{\delta_\Sigma} \mathcal{P}' \quad \textit{iff} \quad \delta^*_\Sigma(M_{\mathcal{P}'}) \geqslant \delta^*_\Sigma(M_\mathcal{P}),$$

for $\mathcal{P}, \mathcal{P}' \in W(X)^N$ two profiles with codified profiles $M_\mathcal{P}, M_{\mathcal{P}'} \in \mathbb{M}_{N\times k}$.

This is to say, a profile $\mathcal{P}$ conveys at least as much consensus as the profile $\mathcal{P}'$ when the dissensus measure of codified profile of $\mathcal{P}'$ is at least as large as the dissensus measure of the codified profile of $\mathcal{P}$.[3]

By way of illustration, we present the following example.

Example 3. Let Σ be the identity matrix and $\mathcal{P}_1, \mathcal{P}_2 \in W(X)^2$ be two profiles whose numerical codifications are

$$M_{\mathcal{P}_1} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \text{ and } M_{\mathcal{P}_2} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Their Mahalanobis dissensus measures are computed as:

[3] As is standard practice, the asymmetric part of the complete preorder $\succcurlyeq_{\delta_\Sigma}$ is denoted by $\succ_{\delta_\Sigma}$.

- $\delta_\Sigma(\mathcal{P}_1) = \delta^*_\Sigma(M_{\mathcal{P}_1}) = (1,0,-1)\Sigma^{-1}(1,0,-1)^t = 2.$
- $\delta_\Sigma(\mathcal{P}_2) = \delta^*_\Sigma(M_{\mathcal{P}_2}) = (1,-1,1)\Sigma^{-1}(1,-1,1)^t = 3$.

Assuming Definition 6 we can conclude $\mathcal{P}_1 \succ_{\delta_\Sigma} \mathcal{P}_2$.

The major source of uncertainty in the Mahalanobis dissensus measure is the choice of the Σ matrix. In this regard, we propose next definition for establishing a partial order on the set of all Mahalanobis dissensus measures and then to overcome this possible drawback.

Definition 7. *Let Δ be the set of all Mahalanobis dissensus measures. For any $\delta_{\Sigma_1}, \delta_{\Sigma_2} \in \Delta$ associated with $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$, a binary relation R_Δ is defined by*

$$\delta_{\Sigma_1} \succcurlyeq_{R_\Delta} \delta_{\Sigma_2} \Leftrightarrow \delta^*_{\Sigma_1}(M) \geq \delta^*_{\Sigma_2}(M),$$

for each N and for all codified profile $M \in \mathbb{M}_{N\times k}$.

This relation verifies the property of reflexivity, antisymmetry and transitivity. Therefore, R_Δ is a partial order in Δ.

In order to analize the properties of the Mahalanobis dissensus measures, it seems reasonable that we initially explore if these measures satisfy anonymity and neutrality, that is, if the Mahalanobis dissensus measures are normal dissensus measures and then the rest of their properties.

Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and let us fix a codification procedure for profiles of complete preorders, $\mathcal{P} \in W(X)^N$ such that for each profile $\mathcal{P}$ produces its codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$. The Mahalanobis dissensus measures verify:

Anonymity. Given permutation σ of the agents in the profile $\mathcal{P}$, a Mahalanobis dissensus measure δ_Σ verifies *anonymity* since

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_\Sigma\left((M_\mathcal{P})^\sigma\right) = \delta_\Sigma(\mathcal{P}^\sigma)$$

for any codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$.

Neutrality. A Mahalanobis dissensus measure δ_Σ verifies *neutrality* if and only if the associated Σ matrix is a diagonal matrix whose diagonal elements have to be equal among them. Formally:

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_\Sigma({}^\pi M_\mathcal{P}) = \delta_\Sigma({}^\pi \mathcal{P}),$$

for any codified profile $M_\mathcal{P} \in \mathbb{M}_{N\times k}$ and for any permutation π of $\{1,\dots,k\}$ if and only if $\Sigma = diag\{\lambda,\dots,\lambda\}$ for a value $\lambda > 0$. [4]

Noting the previous result and being critical of our measure, it could be considered as a drawback the fact that *neutrality* is only verified when Σ matrix is so specific. Thinking about it, we can point out that the main contribution of our approach is to allow different roles for alternatives. This fact produces that traditional *neutrality* property is only verified when alternatives are not related and are exchangeable.

In order to overcome this drawback and emphasize the advantages of the Mahalanobis dissensus measures (cross relations among alternatives allowed), we propose to recall the neutrality property. If the alternatives are relabeled, there exists a way to recover the same value of the Mahalanobis dissensus measure, δ_Σ for each profile, as Proposition 2 shows.

Proposition 2. (Weak neutrality). *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $\mathcal{P} \in W(X)^N$, its codified profile $M \in \mathbb{M}_{N\times k}$ and for each permutation π of the alternatives, it is verified*

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_\mathcal{P}) = \delta^*_{\Sigma^\pi}({}^\pi M_\mathcal{P}) = \delta_{\Sigma^\pi}({}^\pi \mathcal{P}),$$

where $\Sigma^\pi = \Pi^t_\pi\, \Sigma\, \Pi_\pi$ and $\Pi_\pi \in \mathbb{M}_{k\times k}$ the permutation matrix corresponding to π.[5]

[4] A diagonal matrix Σ with diagonal elements $\{\lambda,\dots,\lambda\}$ is represented as $\Sigma = diag(\lambda,\dots,\lambda)$.

[5] Let π be a permutation of $\{1,2,...,k\}$ and e_i be the i-th vector of the canonical base of $\mathbb{R}^n$, that is, $e_{ij} = 1$ if $i = j$, $e_{ij} = 0$ otherwise. The matrix $\Pi_\pi \in \mathbb{M}_{k\times k}$ whose rows are $e_{\pi(i)}$ is called the permutation matrix associated to π. The rearrangement of the corresponding rows (resp. columns) of a matrix A using π is obtained by left (resp., right) multiplication of Π_π, $\Pi_\pi A$ (resp., $A\,\Pi_\pi$).

Proof 2. *Proposition 2 proof is similar to analogous result in González-Arteaga, Alcantud and de Andrés Calle [26].*

Compatibility. Let $\mathcal{P}, \mathcal{P}' \in W(X)^N$ be two profiles and $M_{\mathcal{P}}, M_{\mathcal{P}'} \in \mathbb{M}_{N\times k}$ be their respective codified profiles. A Mahalanobis dissensus measure δ_Σ is *compatible* with linear transformations of codified profiles if

$$\delta^*_\Sigma(M_{\mathcal{P}'}) \geqslant \delta^*_\Sigma(M_{\mathcal{P}}) \;\Leftrightarrow\; \delta^*_\Sigma(f(M_{\mathcal{P}'})) \geqslant \delta^*_\Sigma(f(M_{\mathcal{P}}))$$

where $f(M_{\mathcal{P}})$, $f(M_{\mathcal{P}'})$ are respective cell-by-cell transformations of the codified profiles $M_{\mathcal{P}}$ and $M_{\mathcal{P}'}$ by any linear transformation f.

Note compatibility refers to the behavior of the ranking of the profiles previously provided in Definition 6.

Proof 3. *Let f be a linear transformation $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = a+bx$. Using this transformation cell-by-cell on $M_{\mathcal{P}}$ and $M_{\mathcal{P}'}$, it is obtained $f(M_{\mathcal{P}}) = a\cdot\ \boldsymbol{1}_N+b\cdot M_{\mathcal{P}}$ and $f(M_{\mathcal{P}'}) = a\cdot\ \boldsymbol{1}_N+b\cdot M_{\mathcal{P}'}$, where $\boldsymbol{1}_N = (1, 1, \dots, 1)$. Then, $(f(M_{\mathcal{P}}))_i = a\cdot\ \boldsymbol{1}_N + b\cdot(M_{\mathcal{P}})_i = f((M_{\mathcal{P}})_i)$ and analogously for $M_{\mathcal{P}'}$. This implies*

$$f((M_{\mathcal{P}})_i) - f((M_{\mathcal{P}})_j) = a\cdot\boldsymbol{1}_N + b\cdot(M_{\mathcal{P}})_i - a\cdot\boldsymbol{1}_N + b\cdot(M_{\mathcal{P}})_j = b\cdot((M_{\mathcal{P}})_i - (M_{\mathcal{P}})_j).$$

$$\begin{aligned}
\delta^*_\Sigma(f(M_{\mathcal{P}})) &= \frac{1}{C^2_N}\sum_{i<j} d_\Sigma\,[\,(f(M_{\mathcal{P}}))_i,\ (f(M_{\mathcal{P}}))_j\,] = \\
&= \frac{1}{C^2_N}\sum_{i<j} d_\Sigma\,[\,f((M_{\mathcal{P}})_i),\ f((M_{\mathcal{P}})_j)\,] = \\
&= \frac{1}{C^2_N}\sum_{i<j} d_\Sigma\,(\,a\cdot\boldsymbol{1}_N + b\cdot(M_{\mathcal{P}})_i\,,\ a\cdot\boldsymbol{1}_N + b\cdot(M_{\mathcal{P}})_j\,) = \\
&= \frac{1}{C^2_N}\sum_{i<j}\Big[(\ f((M_{\mathcal{P}})_i) - f((M_{\mathcal{P}})_j)\ )\,\Sigma^{-1}\,(\ f((M_{\mathcal{P}})_i) - f((M_{\mathcal{P}})_j)\ )^t\Big] = \\
&= \frac{1}{C^2_N}\sum_{i<j}\Big[(\ b\cdot(M_{\mathcal{P}})_i - b\cdot(M_{\mathcal{P}})_j\ )\,\Sigma^{-1}\,(\ b\cdot(M_{\mathcal{P}})_i - b\cdot(M_{\mathcal{P}})_j\ )^t\Big] = \\
&= \frac{1}{C^2_N}\sum_{i<j} b^2\Big[(\ (M_{\mathcal{P}})_i - (M_{\mathcal{P}})_j\ )\,\Sigma^{-1}\,(\ (M_{\mathcal{P}})_i - (M_{\mathcal{P}})_j\ )^t\Big] = \\
&= b^2\,\frac{1}{C^2_N}\sum_{i<j} d_\Sigma\,[\,(M_{\mathcal{P}})_i,\ (M_{\mathcal{P}})_j\,] = \\
&= b^2\,\delta^*_\Sigma(M_{\mathcal{P}}).
\end{aligned}$$

Therefore, we have $\delta^*_\Sigma(f(M_{\mathcal{P}})) = b^2\,\delta^*_\Sigma(M_{\mathcal{P}})$ *and* $\delta^*_\Sigma(f(M_{\mathcal{P}'})) = b^2\,\delta^*_\Sigma(M_{\mathcal{P}'})$. *Now, it is easy to complete the proof.* □

Reciprocity. Reciprocity means that if all individual complete preorders are reversed, then the degree of dissensus does not change. A Mahalanobis dissensus measure δ_Σ is *reciprocal* if

$$\delta_\Sigma(\mathcal{P}) = \delta^*_\Sigma(M_{\mathcal{P}}) = \delta^*_\Sigma(M_{\mathcal{P}^{-1}}) = \delta_\Sigma(\mathcal{P}^{-1})$$

for all $\mathcal{P} = (R_1, ..., R_N) \in W(X)^N$ and a codification procedure such that $M_{\mathcal{P}^{-1}} = (k+1)\cdot \mathbf{1}_N - M_{\mathcal{P}}$ where $\mathbf{1}_N = (1, 1, \ldots, 1)$.

Proof 4. *Let* $\mathcal{P} = (R_1, ..., R_N) \in W(X)^N$ *be a profile whose codified profile is* $M_{\mathcal{P}} \in \mathbb{M}_{N\times k}$. *The reverse of the complete preorders produces a new profile* $\mathcal{P}^{-1} = (R_1^{-1}, ..., R_N^{-1}) \in W(X)^N$ *whose codified profile is* $M_{\mathcal{P}^{-1}} \in \mathbb{M}_{N\times k}$. *The proof is easy from*

$$\begin{aligned} d_\Sigma(M_{R_i^{-1}}, M_{R_j^{-1}}) &= d_\Sigma((k+1)\cdot \boldsymbol{1}_N - M_{R_i},\, (k+1)\cdot \boldsymbol{1}_N - M_{R_j}) = \\ &= d_\Sigma(M_{R_i}, M_{R_j}). \end{aligned}$$

□

4. Reaching a social consensus solution based on Mahalanobis distance

The problem of reaching a social consensus solution intends to determine the ranking of alternatives that best agrees with individual preferences, or in other words, the ranking that minimizes the disagreement among individuals.

In this section we present a new proposal to obtain a social consensus solution based on the Mahalanobis distance as well as its properties. The Mahalanobis social consensus solution preserves the advantages of the Mahalanobis distance since it takes into account the correlation among alternatives. In addition, an illustrative example is included to show the graphical interpretation of our proposal.

4.1. Our proposal: The Mahalanobis social consensus solution

Our aim is to determine a complete preorder $\hat{R}$ that provides the best agreement for N rankings taking into account the Mahaloanobis distance. This relation $\hat{R}$ is called the *Mahalanobis social consensus solution*.

Following the traditional approaches and in order to obtain a consensus solution, first of all it is necessary to establish the objective function to optimize. In this contribution, this function is called *Mahalanobis consensus distance function* (MCDF).

Definition 8. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a definite positive matrix and $\mathcal{P} = (R_1, \ldots, R_N) \in W(X)^N$ be a profile of complete preorders. Given a codification procedure, $M_{\mathcal{P}} = (M_{R_1}, \ldots, M_{R_N}) \in \mathbb{M}_{N\times k}$ is the codified profile of $\mathcal{P}$. The* Mahalanobis consensus distance function (MCDF) *is a mapping $\mathcal{C}_{\Sigma,\mathcal{P}} : \mathbb{M}_{N\times k} \longrightarrow [0, \infty)$ defined by*

$$\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R)\Sigma^{-1}(M_{R_i} - M_R)^t$$

and it regards the sum of the Mahalanobis distances from each of the N agent's preferences to a complete preorder R whose codification is M_R.

Once the Mahalanobis consensus distance function has been defined, we proceed to establish our optimization problem:

$$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

where the feasible set F is the set with elements M_R that are codified complete preorders, so that $M_R = (m_1, \ldots, m_k)$.

Solving the above optimization problem we obtain the following solution, $M_{\hat{R}}$.

Definition 9. *A* Mahalanobis consensus solution *is an ordinal ranking of the alternatives obtained by solving*

$$\begin{array}{llcll} \underset{M_R}{min} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \underset{M_R}{min} & \displaystyle\sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \ \mathcal{C}_{\Sigma,\mathcal{P}}(M_{\hat{R}}) \\ s.t. & M_R \in F & & s.t. & M_R \in F \end{array}$$

where $M_{\hat{R}} = (\hat{m}_1, \ldots, \hat{m}_k)$ is a vector which minimizes the Mahalanobis consensus distance function.

The proposed optimization problem can generate complete preorders or linear orders like ranking solutions. If no ties are required, the set of constraints in F has to provide for.

In order to simplify and facilitate the computation of Mahalanobis consensus solutions we present Theorem 1. This new result allows to establish an equivalence between rankings obtained by the method of minimized Mahalanobis consensus distance function (MCDF) and rankings closest to the mean vector $\overline{M}$ defined by the component-wise averages. This theorem makes the method analytically rigorous and provides an intuitively appealing approach. [6]

Theorem 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and $M_{\mathcal{P}} \in \mathbb{M}_{N\times k}$ be a codified profile. The following statements are equivalent:*

1. *$M_{\hat{R}}$ minimizes $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \displaystyle\sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R)$.*

2. *$M_{\hat{R}}$ minimizes $d_\Sigma(\overline{M_{\mathcal{P}}}, M_R)$ being*

$$\overline{M_{\mathcal{P}}} = (\overline{M^1}, \ldots, \overline{M^k}) = (\frac{1}{N}\sum_{i=1}^{N} m_{i1}, \ldots, \frac{1}{N}\sum_{i=1}^{N} m_{ik}).$$

[6] The degree of computational complexity of our approach is not higher than other related well-known approaches [11]. Nowadays there are several powerful computational tools able to solve this kind of problems for a reasonable size (see e.g., [10] and [57], among others).

Proof 5.

$$\begin{aligned}
\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) &= \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R)\Sigma^{-1}(M_{R_i} - M_R)^t = \\
&= \sum_{i=1}^{N} \left(M_{R_i}\,\Sigma^{-1} M_{R_i}^t - 2M_{R_i}\Sigma^{-1}M_R^t + M_R\Sigma^{-1}M_R^t\right) = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\,\Sigma^{-1}M_{R_i}^t\right) - 2\left(\sum_{i=1}^{N} M_{R_i}\Sigma^{-1}M_R^t\right) + \left(\sum_{i=1}^{N} M_R\Sigma^{-1}M_R^t\right) = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\,\Sigma^{-1}M_{R_i}^t\right) - 2\left(\sum_{i=1}^{N} M_{R_i}\right)\Sigma^{-1}M_R^t + NM_R\Sigma^{-1}M_R^t = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\,\Sigma^{-1}M_{R_i}^t\right) - 2N\overline{M}\,\Sigma^{-1}M_R^t + NM_R\,\Sigma^{-1}M_R^t = \\
&= \left(\sum_{i=1}^{N} M_{R_i}\,\Sigma^{-1}M_{R_i}^t\right) + N\left(-2\overline{M}\,\Sigma^{-1}M_R^t + M_R\,\Sigma^{-1}M_R^t\right)
\end{aligned}$$

$$\begin{aligned}
d_\Sigma(\overline{M_{\mathcal{P}}}, M_R) &= (\overline{M_{\mathcal{P}}} - M_R)\Sigma^{-1}(\overline{M_{\mathcal{P}}} - M_R)^t = \\
&= \overline{M_{\mathcal{P}}}\;\Sigma^{-1}\;\overline{M_{\mathcal{P}}}^t + \left(-2\overline{M_{\mathcal{P}}}\;\Sigma^{-1}M_R^t + M_R\;\Sigma^{-1}M_R^t\right)
\end{aligned}$$

As we can observe the minimization of $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R)$ *and* $d_\Sigma(\overline{M_{\mathcal{P}}}, M_R)$ *only depends, in both cases, on* $-2\overline{M_{\mathcal{P}}}\;\Sigma^{-1}\,M_R^t + M_R\,\Sigma^{-1}\,M_R^t$. *Then, both problems are equivalent.* □

A strong evidence of the strength of our proposal to obtain a social consensus solution is given through the consistency between the methodology proposed by Cook and Seiford [14] based on Euclidean distance and ours based on Mahalanobis distance. Concretely, Cook and Seiford [14] formalized the so-called *monotone non-decreasing* property for the case of the *Minimum Variance* method in order to realize the potential of the alignment between the average point and the ranking that minimizes the Euclidean distance. If we apply Cook and Seiford's idea but using a Mahalanobis distance associated with $\Sigma \in \mathbb{M}_{k\times k}$, then their relationship is satisfied when the vectors are expressed in the space of the eigenvectors of the matriz Σ. More precisely, the Mahalanobis consensus solution and the mean vector are linked like the next proposition shows.

Proposition 3. *A Mahalanobis consensus solution* $M_{\hat{R}} = (\hat{m}_1, \dots, \hat{m}_k)$ *does not reverse preferences given by the average point* $\overline{M} = (\overline{M^1}, \dots, \overline{M^k})$ *when* *both are expressed in the basis of the eigenvectors of the matrix* Σ, $\overline{M}^e$ *and* $M^e_{\hat{R}}$, *respectively. More precisely:*

$$\overline{M^i}^{\,e} < \overline{M^j}^{\,e} \Longrightarrow \hat{m}^e_i < \hat{m}^e_j \qquad \textit{for } i,j \in \{1,\ldots k\},\ \ i \neq j$$

Proof 6. *Consider the spectral decomposition of the matrix* $\Sigma = \Gamma^t D_\lambda \Gamma$ *where* Γ *and* D_λ *contain eigenvectors (by columns) and the corresponding eigenvalues of* Σ *as diagonal elements, respectively.*

Let $E = \Gamma\, D_\lambda^{-\frac{1}{2}}$ *be the matrix that defines the linear transformation in order to change* N*-dimensional vectors to coordinates of the eigenspace of* Σ. *Applying the aforementioned transformation on the vectors* $\overline{M}$, *and* $M_{\hat{R}}$, *it yields* $\overline{M}^e = \overline{M}\, E = (\overline{M^1}^{\,e}, \ldots, \overline{M^k}^{\,e})$ *and* $M^e_{\hat{R}} = M_{\hat{R}}\, E = (\hat{m}^e_1, \ldots, \hat{m}^e_k)$, *respectively.*

We must prove that if $\overline{M^i}^{\,e} < \overline{M^j}^{\,e}$ *then* $\hat{m}^e_i \leq \hat{m}^e_j$ *for* $i,j \in \{1,\ldots k\}$, $i \neq j$. *Suppose* $\overline{M^i}^{\,e} < \overline{M^j}^{\,e}$ *and* $\hat{m}^e_i > \hat{m}^e_j$. *Let* M' *be a vector such that* $M'^e = M'\, E = (m'^e_1, \ldots, m'^e_k)$ *and its elements are obtained from* $M^e_{\hat{R}}$ *by interchanging* $\hat{m}^e_i$ *and* $\hat{m}^e_j$, *i.e.,*

$$m'^e_r = \begin{cases} \hat{m}^e_j & \textit{if} \quad r = i, \\ \hat{m}^e_i & \textit{if} \quad r = j, \\ \hat{m}^e_r & \textit{otherwise.} \end{cases}$$

First, we obtain $d_\Sigma(\overline{M}, M_{\hat{R}})$:

$$\begin{aligned} d_\Sigma(\overline{M}, M_{\hat{R}}) &= (\overline{M} - M_{\hat{R}})\Sigma^{-1}(\overline{M} - M_{\hat{R}})^t = \\ &= (\overline{M} - M_{\hat{R}})\Gamma D_\lambda^{-1}\Gamma^t(\overline{M} - M_{\hat{R}})^t = \\ &= (\overline{M} - M_{\hat{R}})\Gamma D_\lambda^{-\frac{1}{2}} D_\lambda^{-\frac{1}{2}}\Gamma^t(\overline{M} - M_{\hat{R}})^t = \\ &= (\overline{M} - M_{\hat{R}})E\, E^t(\overline{M} - M_{\hat{R}})^t = \\ &= (\overline{M}\, E - M_{\hat{R}}\, E)(\overline{M}\, E - M_{\hat{R}}\, E)^t = \\ &= (\overline{M}^e - M^e_{\hat{R}})(\overline{M}^e - M^e_{\hat{R}})^t \end{aligned}$$

Analogously, we compute $d_\Sigma(\overline{M}, M')$:

$$d_\Sigma(\overline{M}, M') = (\overline{M} - M')\Sigma^{-1}(\overline{M} - M')^t = (\overline{M}^e - M'^e)(\overline{M}^e - M'^e)^t.$$

Next, we must get $d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M')$:

$$
\begin{aligned}
&d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M') = \\
&= (\overline{M}^e - M^e_{\hat{R}})(\overline{M}^e - M^e_{\hat{R}})^t - (\overline{M}^e - M'^e)(\overline{M}^e - M'^e)^t = \\
&= \left(\overline{M}^e\,\overline{M}^{e\,t} - 2\,\overline{M}^e\,M^{e\,t}_{\hat{R}} + M^e_{\hat{R}}\,M^{e\,t}_{\hat{R}}\right) - \left(\overline{M}^e\,\overline{M}^{e\,t} - 2\,\overline{M}^e\,M'^{\,e\,t} + M'^{\,e}\,M'^{\,e\,t}\right) = \\
&= (\overline{M}^e_1, \ldots, \overline{M}^e_i, \ldots \overline{M}^e_j, \ldots \overline{M}^e_k)\left(m_1'^{\,e}, \ldots, m_i'^{\,e}, \ldots, m_j'^{\,e}, \ldots, m_k'^{\,e}\right) - \\
&-(\overline{M}^e_1, \ldots, \overline{M}^e_i, \ldots \overline{M}^e_j, \ldots \overline{M}^e_k)\left(\hat{m}^e_1, \ldots, \hat{m}^e_i, \ldots, \hat{m}^e_j, \ldots, \hat{m}^e_k\right) = \\
&= (\overline{M}^e_1, \ldots, \overline{M}^e_i, \ldots \overline{M}^e_j, \ldots \overline{M}^e_k)\left(\hat{m}^e_1, \ldots, \hat{m}^e_j, \ldots, \hat{m}^e_i, \ldots, \hat{m}^e_k\right) - \\
&-(\overline{M}^e_1, \ldots, \overline{M}^e_i, \ldots \overline{M}^e_j, \ldots \overline{M}^e_k)\left(\hat{m}^e_1, \ldots, \hat{m}^e_i, \ldots, \hat{m}^e_j, \ldots, \hat{m}^e_k\right) = \\
&= 2\left(\overline{M}^e\,M'^{\,e\,t} - \overline{M}^e\,M^{e\,t}_{\hat{R}}\right) = 2\left(m^e_i \hat{m}^e_j + m^e_j \hat{m}^e_i - m^e_i \hat{m}^e_i - m^e_j \hat{m}^e_j\right) = \\
&= 2(\hat{m}^e_i - \hat{m}^e_j)(m^e_j - m^e_i).
\end{aligned}
$$

Then, $d_\Sigma(\overline{M}, M_{\hat{R}}) - d_\Sigma(\overline{M}, M') = 2\,(\hat{m}^e_i - \hat{m}^e_j)(m^e_j - m^e_i) > 0$, *so* $d_\Sigma(\overline{M}, M_{\hat{R}}) > d_\Sigma(\overline{M}, M')$ *and, thus* $d_\Sigma(\overline{M}, M_{\hat{R}})$ *is not minimal and* $M_{\hat{R}}$ *is not the Mahalanobis consensus solution. In that way, a contradiction is reached. Consequently, the hypothesis* $\overline{M^{i\,e}} < \overline{M^{j\,e}} \Longrightarrow \hat{m}^e_i < \hat{m}^e_j$ *is verified.* □

Additionally to the previous results it should be interesting to study if Mahalanobis social consensus solutions satisfy other properties usually claimed in Social Choice Theory. In the next subsection we explore some of them.

4.2. Properties of the Mahalanobis social consensus solution

We now proceed to define and prove the main properties of the Mahalanobis social consensus solution. These properties ensure the suitability and avoid weird behaviors of the new approach. Moreover, these good theoretical properties make it easier to accept the social solution obtained for the group.

- *Anonymity*. Any member's ranking is considered equal in importance to the ranking preferred by any other member. More precisely, given a profile $\mathcal{P} \in W(X)^N$, a Mahalanobis social consensus solution does not change for each permutation σ of the agents. The problem to solve then is

$$
\begin{aligned}
\min_{M_R} \quad & \sum_{i=1}^{N} d_\Sigma(M_{\sigma(i)}, M_R) \\
\text{s.t.} \quad & M_R \in F
\end{aligned}
$$

Proof 7. *It is straightforward that a ranking $\hat{R}$ whose codified complete preorder is $M_{\hat{R}}$ given by Definition 9 is also a solution to the above problem since $\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \mathcal{C}_{\Sigma,\mathcal{P}^\sigma}(M_R^\sigma)$ for all M_R and σ.* $\square$

- *Unanimity.* If all agents show the same preferences on all alternatives, then a Mahalanobis social consensus solution coincides with such common complete preorder.

  Proof 8. *This is easily seen since the column means of the codified profile is equal to that common codified complete preorder. That means, it belongs to the feasible set F and Theorem 1 produces the result.* $\square$

- *Neutrality.* Generally speaking, this property means all alternatives are treated strictly equal. More precisely, any relabelling of the alternatives or issues induces the corresponding permutation of a Mahalanobis social consensus solution. Due to the fact that our proposal presents a collection of functions MCDFs, relying on Σ matrix, it should be reasonable that the verification of this property depends on Σ.

  Consider a Mahalanobis social consensus solution $M_{\hat{R}}$ obtaining by Definition 9. Given π a permutation of the set of alternatives. The MCDF after permuting the alternatives can be written as

$$\mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) = \sum_{i=1}^{N} d_\Sigma({}^\pi M_{R_i}, M_R)$$

  Due to the previous reasoning, the property to prove is:

$$\begin{array}{lllll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) & = & \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_{{}^\pi\hat{R}}) \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F & & \end{array}$$

  where $M_{{}^\pi\hat{R}} ={}^\pi M_{\hat{R}} = (\hat{m}_{\pi(1)}, \ldots, \hat{m}_{\pi(k)})$ is a consensus solution for this problem if and only if $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for some $\lambda > 0$.

  Proof 9. *We consider the following two problems to solve:*

- $$\begin{array}{lcl} \min\limits_{M_R} \ \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \min\limits_{M_R} \ \sum\limits_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) \\ s.t. \quad M_R \in F & & s.t. \quad M_R \in F \end{array}$$

- $$\begin{array}{lcl} \min\limits_{M_R} \ \mathcal{C}_{\Sigma,{}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} \ \sum\limits_{i=1}^{N} d_\Sigma(M_{{}^\pi R_i}, M_R) \\ s.t. \quad M_R \in F & & s.t. \quad M_R \in F \end{array}$$

By Theorem 1 the resolution of these problems can be reduced to minimize $d_\Sigma(\overline{M}\,,\, M_R)$ and $d_\Sigma({}^\pi\overline{M}, M_R)$, respectively.

$$d_\Sigma(\overline{M},\, M_R) \ = \ (\overline{M} - M_R)\Sigma^{-1}(\overline{M} - M_R)^t.$$

In order to simplify the notation and due to the equivalence among the set of complete preorders and the set of their permutations, we can write ${}^\pi M_R$ for some M_R. Thus,

$$\begin{aligned} d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R) &= ({}^\pi\overline{M} -{}^\pi M_R)\ \Sigma^{-1}\ ({}^\pi M_i -{}^\pi M_R)^t = \\ &= (\overline{M}\Pi_\pi - M_R\Pi_\pi)\,\Sigma^{-1}\ (\overline{M}\Pi_\pi - M_R\Pi_\pi)^t = \\ &= (\overline{M} - M_R)\ \Pi_\pi\,\Sigma^{-1}\,\Pi_\pi^t\ (\overline{M} - M_R)^t. \end{aligned}$$

Let us first prove sufficiency*. If $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for a value $\lambda > 0$, then $\Pi_\pi\,\Sigma^{-1}\,\Pi_\pi^t = \Sigma^{-1}$ and consequently,*

$$d_\Sigma(\overline{M},\, M_R) = d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R)$$

that is, the distance to minimize coincides for both problems and the result is straightforward.[7]

Let us now prove necessity*. Assuming that given a codified profile $M \in \mathbb{M}_{N\times k}$ and for each π, $d_\Sigma(\overline{M},\, M_R) = d_\Sigma({}^\pi\overline{M},\ {}^\pi M_R)$, therefore $\Pi_\pi\,\Sigma^{-1}\,\Pi_\pi^t = \Sigma^{-1}$, we must prove that $\Sigma = diag\{\lambda, \ldots, \lambda\}$.*

[7] Recall Π_π is the permutation matrix corresponding to π.

The proof of this point is similar to the demonstration included in González-Arteaga, Alcantud and de Andrés Calle [26, Appendix A, Proof of Property 1]. □

In the same way that happens to the Mahalanobis dissensus measure, the Mahalanobis social consensus solution preserves the advantages of the Mahalanobis distance, concretely, it takes into account the correlation among alternatives. Therefore, on the question of *neutrality* for the consensus solution, the reasons and the clarifications aforementioned in Section 3 (Property 2) are maintained. We now present a weak version of the neutrality property.

- *Weak neutrality.* Any relabelling of the alternatives or issues induces the corresponding permutation of the Mahalanobis social consensus solution associated to the appropriate permutation on Σ. Formally:

Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $\mathcal{P} \in W(X)^N$ whose codified profile is $M \in \mathbb{M}_{N\times k}$ and for each permutation π of the alternatives, the problem to solve is

$$\begin{array}{llcll} \min\limits_{M_R} & \mathcal{C}_{\Sigma^\pi,\,{}^\pi\mathcal{P}}(M_R) & = & \min\limits_{M_R} & \sum\limits_{i=1}^{N} d_{\Sigma^\pi}({}^\pi M_{R_i}, M_R), \\ \text{s.t.} & M_R \in F & & \text{s.t.} & M_R \in F \end{array}$$

where $\Sigma^\pi = \Pi_\pi^t \,\Sigma\, \Pi_\pi$.

Therefore, the minimization of the MCDF, $\mathcal{C}_{\Sigma^\pi,\,{}^\pi\mathcal{P}}(M_R)$ produces a Mahalanobis social consensus solution ${}^\pi M_{\hat{R}} = M_{{}^\pi\hat{R}}$ obtained from $M_{\hat{R}}$.

Proof 10. *Let us consider the set of codified complete preorders in the form ${}^\pi M_R = M_{{}^\pi R}$ like possible solutions. Since Definition 9, it is sufficient to prove*

$$d_{\Sigma^\pi}({}^\pi M_{R_i}, {}^\pi M_R) = d_\Sigma(M_{R_i}, M_R) \quad \textit{for} \quad i = 1, \ldots N.$$

Using the fact that the permutation matrix Π_π is orthogonal

$$
\begin{aligned}
d_{\Sigma^\pi}({}^\pi M_{R_i}, {}^\pi M_R) &= ({}^\pi M_{R_i} - {}^\pi M_R)\ (\Sigma^\pi)^{-1}\ ({}^\pi M_{R_i} - {}^\pi M_R)^t = \\
&= (M_{R_i}\Pi_\pi - M_R\Pi_\pi)\ (\Pi_\pi^t \Sigma \Pi_\pi)^{-1}\ (M_{R_i}\Pi_\pi - M_R\Pi_\pi)^t = \\
&= (M_{R_i} - M_R)\ \Pi_\pi\, \Pi_\pi^t \Sigma^{-1} \Pi_\pi\, \Pi_\pi^t\ (M_{R_i} - M_R)^t = \\
&= (M_{R_i} - M_R)\Sigma^{-1}(M_i - M_R)^t = \\
&= d_\Sigma(M_{R_i}, M_R).
\end{aligned}
$$

Then, the proof is straightforward. □

- *Consistency.* Given a set of agents divided in two disjoint subcommittees. Suppose that Mahalanobis social consensus solutions obtained for each subcommittee coincide. Then, Mahalanobis social consensus solutions derived from the original set of agents are the same that the obtained for the subcommittees.

  Proof 11. *Let $\mathbf{N} = \mathbf{N}^{(1)} \cup \mathbf{N}^{(2)}$ be a partition of the set $\mathbf{N}$ of agents in two disjoint subcommittees. The Mahalanobis social consensus solutions for each subcommittee are $M_{\hat{R}^{(1)}}$ and $M_{\hat{R}^{(2)}}$, respectively. According to the hypothesis: $M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$. The MCDF for the set of agents $\mathbf{N}$ can be written as*

$$
\begin{aligned}
\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) &= \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \\
&= \sum_{i\in\mathbf{N}^{(1)}} d_\Sigma(M_{R_i}, M_R) + \sum_{i\in\mathbf{N}^{(2)}} d_\Sigma(M_{R_i}, M_R) = \\
&= \mathcal{C}_{\Sigma,\mathcal{P}^{(1)}}(M_{R^{(1)}}) + \mathcal{C}_{\Sigma,\mathcal{P}^{(2)}}(M_{R^{(2)}}).
\end{aligned}
$$

  $M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$ minimizes the first and the second summand. Therefore, the minimum of the first term in the above equality is reached in $M_{\hat{R}^{(1)}} = M_{\hat{R}^{(2)}}$ because both summands are positive. □

- *Compatibility.* Let $M^* = a \cdot \mathbf{1}_N + b \cdot M$ the matrix arising from an affine transformation of $M \in \mathbb{M}_{N\times k}$ which represents the codified profile associated with $\mathcal{P} \in W(X)^N$. The Mahalanobis social consensus solution obtained for M^* is $M^*_{\hat{R}} = a \cdot \mathbf{1}_N + b \cdot M_{\hat{R}}$ being $M_{\hat{R}}$ the corresponding Mahalanobis social consensus solution for M.

Proof 12. *The problem to solve is*

$$\begin{array}{ll} \underset{M_R^*}{min} & \sum_{i=1}^{N} d_\Sigma(M_{R_i}^*, M_R^*) \\ s.t. & M_R^* \in F^* \end{array}$$

where F^* is the set of all possible vectors that represent codified complete preorders using the affine transformation.

Replacing $M_{R_i}^* = a \cdot \boldsymbol{1}_N + b \cdot M_{R_i}$ and $M_R^* = a \cdot \boldsymbol{1}_N + b \cdot M_R$, we obtain:

$$\begin{aligned} \sum_{i=1}^{N} d_\Sigma(M_i^*, M_R^*) &= \sum_{i=1}^{N} (M_i^* - M_R^*)\Sigma^{-1}(M_i^* - M_R^*)^t = \\ &= b^2 \sum_{i=1}^{N} (M_i - M_R)\Sigma^{-1}(M_i - M_R)^t = \\ &= b^2 \sum_{i=1}^{N} d_\Sigma(M_i, M_R). \end{aligned}$$

Therefore, $M_{\hat{R}}^* \in F^*$ if and only if there exists $M_{\hat{R}} \in F$ such that $M_{\hat{R}}^* = a \cdot \boldsymbol{1}_N + b \cdot M_{\hat{R}}$. □

- *Reciprocity*. Reciprocity means that if all individual rankings in a profile are reversed, then the consensus solution is obtained by reversing the original solution. This is true for Mahalanobis social consensus solution under a basic condition on the codification procedure. Formally:

  Let $\mathcal{P} \in W(X)^N$ be a profile and $\mathcal{P}^{-1} \in W(X)^N$ be its reverse, whose associated codified profiles are $M_\mathcal{P}$, $M_{\mathcal{P}^{-1}} \in \mathbb{M}_{N\times k}$, respectively. Fixed a positive definite matrix $\Sigma \in \mathbb{M}_{k\times k}$, the problems to solve are the following:

$$\begin{array}{llll} (P1) & \underset{M_R}{\min} \ \mathcal{C}_{\Sigma,\mathcal{P}^{-1}}(M_R) & = & \underset{M_R}{\min} \ \sum_{i=1}^{N} d_\Sigma((M_{\mathcal{P}^{-1}})_i, M_R) \\ & \text{s.t.} \quad M_R \in F & & \text{s.t.} \quad M_R \in F \end{array}$$

$$\begin{array}{llll} (P2) & \underset{M_R}{\min} \ \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) & = & \underset{M_R}{\min} \ \sum_{i=1}^{N} d_\Sigma((M_{\mathcal{P}})_i, M_R) \\ & \text{s.t.} \quad M_R \in F & & \text{s.t.} \quad M_R \in F \end{array}$$

Then, the solution of the problem $(P1)$ has to be the reverse of the solution of the problem $(P2)$.

Reciprocity is fulfilled if the codification procedure used on $R \in W(X)$ verifies $M_{R^{-1}} = a \cdot \mathbf{1}_N + b \cdot M_R = (a + bm_1, \ldots, a + bm_k)$ for $a, b \in \mathbb{R}$. Therefore, $(M_{\mathcal{P}^{-1}})_i = a \cdot \mathbf{1}_N + b \cdot (M_{\mathcal{P}})_i$.

Notice that our codification proposal, the canonical codification, satisfies the aforementioned condition since

$$\begin{aligned} M_{R^{-1}} = a \cdot \mathbf{1}_N + b \cdot M_R &= (a + bm_1, \ldots, a + bm_k) = \\ &= (n + 1 - m_1, \ldots, n + 1 - m_k) = \\ &= (n + 1) \cdot \mathbf{1}_N - M_R. \end{aligned}$$

Proof 13. *In order to solve problems $(P1)$ and $(P2)$, Theorem 1 is used. Thus, it is enough to minimize $d_\Sigma(\overline{M}_{\mathcal{P}^{-1}}, M_R)$ and $d_\Sigma(\overline{M}_{\mathcal{P}}, M_R)$ subject to $M_R \in F$, respectively.*

Considering that $\overline{M}_{\mathcal{P}^{-1}} = a \cdot \boldsymbol{1}_N + b \cdot \overline{M}_{\mathcal{P}}$ and being $M_{\hat{R}}$ a solution of problem $(P1)$, it is easy to check that $a \cdot \boldsymbol{1}_N + b \cdot M_{\hat{R}}$ is a solution of problem $(P2)$ since

$$\begin{aligned} d_\Sigma(\overline{M}_{\mathcal{P}^{-1}},\ a \cdot \boldsymbol{1}_N + b \cdot M_{\hat{R}}) &= d_\Sigma(a \cdot \boldsymbol{1}_N + b \cdot \overline{M}_{\mathcal{P}},\ a \cdot \boldsymbol{1}_N + b \cdot M_{\hat{R}}) = \\ &= b^2\, d_\Sigma(\overline{M}_{\mathcal{P}},\ M_{\hat{R}}). \end{aligned}$$

□

- *Non-dictatorship.* The Mahalanobis social consensus solution is never dictatorial. Recall that in a dictatorship, social choices are based on the preferences of only one expert or agent. Formally, an agent $j \in \mathbf{N}$ exists such that for all alternatives $x_r, x_s \in X$ and for all profiles $\mathcal{P} \in W(X)^N$

$$x_r \succcurlyeq_{R_j} x_s \Longrightarrow x_r \succcurlyeq_{\hat{R}} x_s.$$

Proof 14. *Immediate from Theorem 1.* □

4.3. Graphical interpretation and discussion: An illustrative example

To clarify and discuss the new approach presented in Subsection 4.1, we develop an explanatory example.

By way of illustration, we suppose the following group decision making problem: a set of students have to choose the destination of their graduation trip. Students should order destinations offered by a travel agency.

We consider a set of three students of the Faculty of Sciences (experts) $\mathbf{N} = \{e_1, e_2, e_3\}$ and a set of three destinations (alternatives) $X = \{x_1 =$ Paris, $x_2 =$ Berlin, $x_3 =$ Istanbul$\}$. Each student participates in a survey about her/his preferences on the trip destinations where she/he is asked to order them.

Their responses are summarized as follows:

$$\begin{array}{rl} \text{Student}\, e_1: & x_3 \succ_{R_{e_1}} x_1 \sim_{R_{e_1}} x_2 \\ \text{Student}\, e_2: & x_3 \succ_{R_{e_2}} x_1 \sim_{R_{e_2}} x_2 \\ \text{Student}\, e_3: & x_1 \sim_{R_{e_3}} x_2 \succ_{R_{e_3}} x_3 \end{array}$$

The previous complete preorders generate a particular profile $\mathcal{P}$. Applying Definition 1 to each complete preorder the codified profile for $\mathcal{P}$ is

$$M_{\mathcal{P}} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix},$$

or also $M_{\mathcal{P}} = (M_{R_{e_1}}, M_{R_{e_2}}, M_{R_{e_3}})$.

In order to obtain a group solution that captures the minimum possible dissensus among students' preferences (i.e., the maximum possible consensus), we must solve the following general optimization problem:

$$\begin{array}{ll} \min\limits_{M_R} & \mathcal{C}_{\Sigma,\mathcal{P}}(M_R) \\ \text{s.t.} & M_R \in F \end{array}$$

where F is the feasible set computed by Proposition 1 and

$$\mathcal{C}_{\Sigma,\mathcal{P}}(M_R) = \sum_{i=1}^{N} d_\Sigma(M_{R_i}, M_R) = \sum_{i=1}^{N} (M_{R_i} - M_R)\Sigma^{-1}(M_{R_i} - M_R)^t.$$

This problem adapted to our specific case takes the form gathered in Table 1. Moreover, the corresponding feasible set is displayed in Figure 1.

Min $d_\Sigma(M_{R_{e_1}}, M_R)+$	$d_\Sigma(M_{R_{e_2}}, M_R)+$	$d_\Sigma(M_{R_{e_3}}, M_R)$	
Subject to M_R belongs to:			
	$(3,3,3)$	$(3,3,1)$	$(1,3,2)$
	$(2,2,3)$	$(3,1,3)$	$(2,3,1)$
	$(3,2,2)$	$(1,3,3)$	$(2,1,3)$
	$(2,3,2)$	$(1,2,3)$	$(3,1,2)$
	$(3,2,1)$		

Table 1: Formulation of the optimization problem.

This optimization problem can be simplified by means of Theorem 1 hence it boils down to:

Min $d_\Sigma(\overline{M}, M_R) = $ Min $(\overline{M} - M_R)\Sigma^{-1}(\overline{M} - M_R)^t$
Subject to M_R belongs to:

$(3,3,3)$	$(3,3,1)$	$(1,3,2)$
$(2,2,3)$	$(3,1,3)$	$(2,3,1)$
$(3,2,2)$	$(1,3,3)$	$(2,1,3)$
$(2,3,2)$	$(1,2,3)$	$(3,1,2)$
$(3,2,1)$		

where $\overline{M} = (2.34, 1.67, 2.34)$ is the vector of column means of $M_{\mathcal{P}}$.

The solution of this problem hinges on the Σ matrix. We now provide Mahalanobis social consensus solutions under the assumption of three different Σ matrices to enrich the case of study and promote the discussion:

1. Case 1. In the simplest case, the Σ matrix is the identity matrix, $\Sigma = I = diag(1,1,1)$. In our example this means that all destinations are equally treated. By solving the corresponding optimization problem the following solutions are obtained (see Table 2):

   - $M_{R_2} = (2,2,3)$, that is, $x_3 \succ x_1 \sim x_2$.
   - $M_{R_3} = (3,2,2)$, that is, $x_1 \succ x_2 \sim x_3$.

   About graphical interpretation, on the left of Figure 2 the elements of the feasible set F are displayed like dots using a color scale. Dots have different colors depending on their Mahalanobis distance, d_I, to the mean point $\overline{M}$ (black triangle). Associated distance values are shown in Table 2.

Additionally, Figure 3 shows the minimum equidistant surface to $\overline{M}$, that in this case is a blue sphere centered at $\overline{M}$. Moreover, Figure 3 includes two different perspectives in order to improve the view.

Notice that considering the Σ matrix as the identity matrix is equivalent to using the Euclidean distance ($l_p = l_2$) to compute a solution. The Euclidean distance has been extensively used in other approaches like [14], [29] and [30]. Then, Case 1 could be used to compare our approach with other methods and to show its efficiency. Next cases include the importance and the cross-relations of alternatives by means of several Σ matrices.

2. Case 2. Now we account for a case where alternatives are considered differently by means of a diagonal Σ matrix. In our example this means that all destinations are not equally treated. Suppose for instance $\Sigma = D = diag(0.3, 0.8, 1.2)$ where the third alternative has the biggest significance. In Table 2 we can find the social consensus solution for this particular case:

   - $M_{R_2} = (2, 2, 3)$, that is, $x_3 \succ x_1 \sim x_2$.

   Analogously to the previous case, on the right of Figure 2 the elements of the feasible set are shown. The color scale is built for the Mahalanobis distance, d_D, between dots and the mean point $\overline{M}$ (black triangle). Such distance values are also shown in Table 2.

   In addition, the aforementioned social solution can be found in Figure 4 from two perspectives. It shows the minimum equidistant surface to $\overline{M}$, that is a blue ellipsoid centered at $\overline{M}$. On the right, after rotating the ellipsoid, our figure makes clear that dots with labels $v1$ and $v3$ are outside of the ellipsoid, farther away than the dot $v2$.

3. Case 3. Finally, we examine the case of a non-diagonal matrix, which allows to incorporate the interdependence of the alternatives because the role of Σ in the Mahalanobis distance. Let us assume the following particular matrix

$$\Sigma = \Sigma_1 = \begin{pmatrix} 0.30 & 0.37 & -0.36 \\ 0.37 & 0.80 & -0.29 \\ -0.36 & -0.29 & 1.20 \end{pmatrix}.$$

Since Σ matrix can be considered as a variance-covarince matrix in the Mahalanobis distance, it is easy to compute the corresponding correlation matrix $Corr$, that is, the correlation among the alternatives.[8]

$$Corr = Corr_1 = \begin{pmatrix} 1.00 & 0.75 & -0.60 \\ 0.75 & 1.00 & -0.30 \\ -0.60 & -0.30 & 1.00 \end{pmatrix}.$$

In our example this matrix implies not only that all destinations are not equally treated but they are also correlated. Alternatives x_1 and x_2 are highly positively correlated whereas alternatives x_1 and x_3, and x_2 and x_3, are negatively correlated. Therefore, it is assumed that Paris and Berlin are "positively" correlated destinations. However, the preferences relative to Paris versus Istanbul are more intensively opposite than Berlin versus Istanbul.

In order to solve the optimization problem for this case we observe the corresponding distance values, d_{Σ_1}, contained in Table 2. In this case, we conclude that the solution is:

- $M_{R_{11}} = (2, 1, 3)$, that is, $x_3 \succ x_1 \succ x_2$

Regarding graphical interpretation, Figure 5 shows the minimum equidistant surface to $\overline{M}$. In this case, it is a blue oriented ellipsoid centered at $\overline{M}$. After a rotation, the graph on the right reveals that dots $v2$ and $v4$ are outside of the ellipsoid, farther than the dot $v11$.

5. Concluding remarks

This study is aimed at proposing a new approach to obtain a group consensus solution under the assumption of ordinal information. A new procedure based on an optimization model has been developed, obtaining a *social*

[8]The element ij of the corelation matrix $Corr$ is $\frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{jj}}}$, where Σ_{ij} is the element ij of variance-covariance matrix Σ.

consensus solution based on the Mahalanobis distance. To accomplish such target two new contributions have been developed in addition of the main result: the characterization of a codification procedure for ordinal information, namely, the *canonical codification* and the definition and analysis of a new dissensus measure, namely, the *Mahalanobis dissensus measure*. The use of the Mahalanobis distance as a base of our approaches brings advantage by considering possible cross relations among alternatives. Moreover, the operational character and intuitive interpretation of our approaches have been illustrated by an explanatory example.

The findings of this study have a number of important implications for future practice. Many problems from a variety of fields can be managed by our methods such as the performance of consumers' preferences, Clinical Decision Making problems, allocation of projects, Human Resources Department problems, etc.

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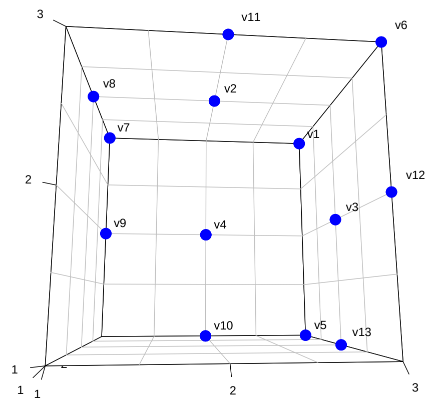

Figure 1: Graphical display of complete preorders included in Table 1 and Table 2. M_{R_i} is labeled by v_i.

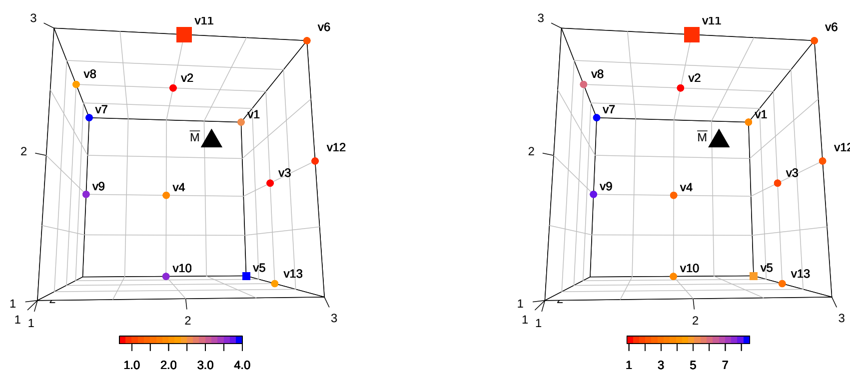

Figure 2: A display of elements in F with colored dots depending on the distance (on the left d_I and on the right d_D) to the mean point $\overline{M}$ (black triangle). In addition, squares denotes the codified complete preorders (M_{R_5} and $M_{R_{11}}$) included in M.

Complete preorders	Codified complete preorders	Graphic labels	d_I	d_D	d_{Σ_1}
$R_1: \ x_1 \sim x_2 \sim x_3$	$M_{R_1} = (3,3,3)$	v1	2.67	4.07	4.43
$R_2: \ x_3 \succ x_1 \sim x_2$	$M_{R_2} = (2,2,3)$	v2	**0.67**	**0.88**	1.99
$R_3: \ x_1 \succ x_2 \sim x_3$	$M_{R_3} = (3,2,2)$	v3	**0.67**	1.71	2.85
$R_4: \ x_2 \succ x_1 \sim x_3$	$M_{R_4} = (2,3,2)$	v4	2.00	2.69	12.51
$R_5: \ x_1 \sim x_2 \succ x_1$	$M_{R_5} = (3,3,1)$	v5	4.00	5.19	3.07
$R_6: \ x_1 \sim x_3 \succ x_2$	$M_{R_6} = (3,1,3)$	v6	1.33	2.41	14.45
$R_7: \ x_2 \sim x_3 \succ x_1$	$M_{R_7} = (1,3,3)$	v7	4.00	8.52	39.08
$R_8: \ x_3 \succ x_2 \succ x_1$	$M_{R_8} = (1,2,3)$	v8	2.33	6.44	22.04
$R_9: \ x_2 \succ x_3 \succ x_1$	$M_{R_9} = (1,3,2)$	v9	3.67	8.24	47.00
$R_{10}: \ x_2 \succ x_1 \succ x_3$	$M_{R_{10}} = (2,3,1)$	v10	3.67	4.07	18.27
$R_{11}: \ x_3 \succ x_1 \succ x_2$	$M_{R_{11}} = (2,1,3)$	v11	1.00	1.30	**0.77**
$R_{12}: \ x_1 \succ x_3 \succ x_2$	$M_{R_{12}} = (3,1,2)$	v12	1.00	2.13	9.77
$R_{13}: \ x_1 \succ x_2 \succ x_3$	$M_{R_{13}} = (3,2,1)$	v13	2.33	3.10	2.32

Table 2: Mahalanobis distances $d_I(M_{R_i}, \overline{M})$, $d_D(M_{R_i}, \overline{M})$ and $d_{\Sigma_1}(M_{R_i}, \overline{M})$ from codified complete preorders M_{R_i} (elements in F) to the mean point $\overline{M}$.

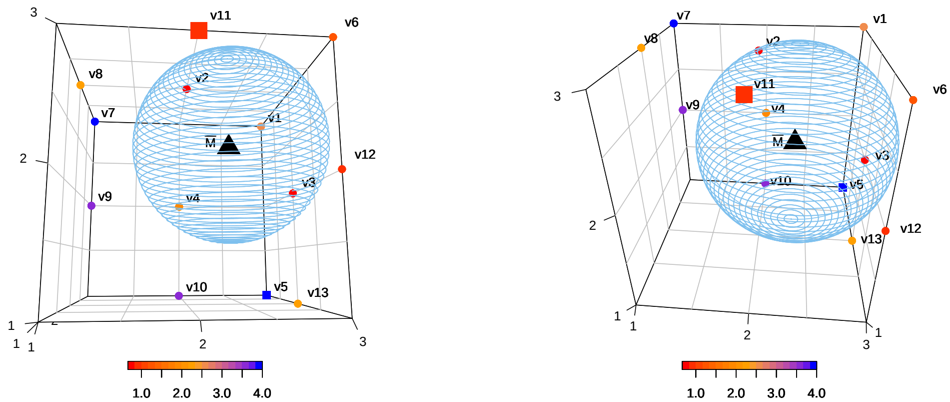

Figure 3: Graphical interpretation of case 1 in Subsection 4.3 using d_I.

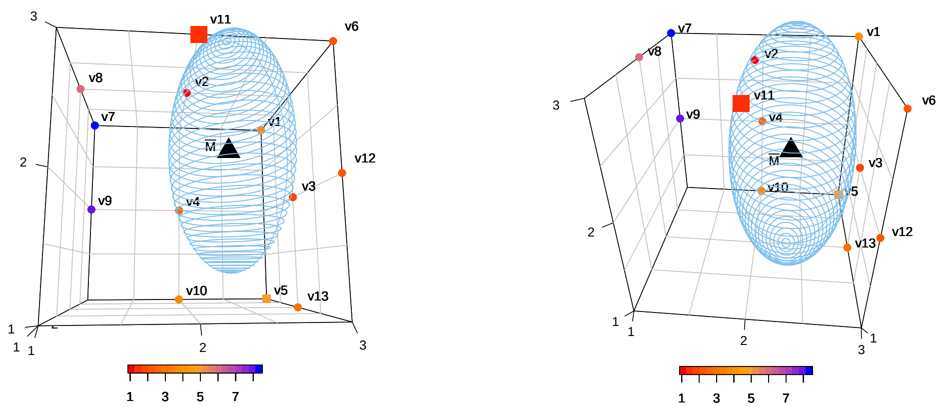

Figure 4: Graphical interpretation of case 2 in Subsection 4.3 using d_D.

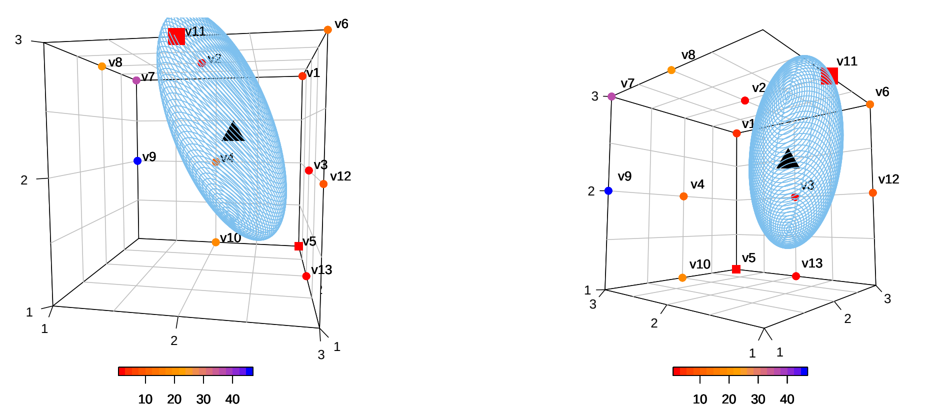

Figure 5: Graphical interpretation of case 3 in Subsection 4.3 using d_{Σ_1}.