Time exponential splitting technique for the Klein-Gordon equation with Hagstrom-Warburton high-order absorbing boundary conditions

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Abstract

Klein-Gordon equations on an unbounded domain are considered in one dimensional and two dimensional cases. Numerical computation is reduced to a finite domain by using the Hagstrom-Warburton (H-W) high-order absorbing boundary conditions (ABCs). Time integration is made by means of exponential splitting schemes that are efficient and easy to implement. In this way, it is possible to achieve a negligible error due to the time integration and to study the behavior of the absorption error. Numerical experiments displaying the accuracy of the numerical solution for the two dimensional case are provided. The influence of the dispersion coefficient on the error is also studied.

Keywords:
Splitting methods, Absorbing boundary conditions, Dispersive waves, Auxiliary variables, Artificial boundary, Finite differences
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1. Introduction

Many applications such as weather prediction, underwater acoustic or earth geophysics concern with wave problems in unbounded domains.

We consider dispersive waves propagating in $(-\infty, \infty) \times [a, b]$, a two dimensional strip. The south and north boundaries of the strip are denoted by Γ_S and Γ_N. Inside the strip, we consider the Klein-Gordon equation,

$$\partial_t^2 u - c^2 \nabla^2 u + s^2 u = f. \tag{1}$$

Here $c = c(x, y)$ is the given wave speed, $s = s(x, y)$ the medium dispersion coefficient and $f(x, y, t)$ is a given source.

Our results hold when, outside a compact region Ω_0, the speed c and the dispersion coefficient are constant and the source f vanishes. However, for the sake of simplicity, we only consider examples which satisfy this assumption in the whole strip $(-\infty, \infty) \times [a, b]$.

On the south and north boundaries we consider Neumann boundary conditions,

$$\partial_y u = 0, \quad \text{on } \Gamma_S \text{ and } \Gamma_N. \tag{2}$$

Finally, we consider the initial conditions,

$$u(x, y, 0) = u_0(x, y), \quad \partial_t u(x, y, 0) = v_0(x, y), \tag{3}$$

which satisfy the boundary conditions on Γ_S and Γ_N, and vanish outside Ω_0.

The numerical approximations of these problems need to reduce the computation to a finite domain. Therefore, we truncate the infinite domain by introducing the west artificial boundary Γ_W, located at $x = x_W$, $a \leq y \leq b$, and the east artificial boundary Γ_E at $x = x_E$, $a \leq y \leq b$. We denote by Ω the computational domain bounded by $\Gamma_N \cup \Gamma_W \cup \Gamma_S \cup \Gamma_E$, such that $\Omega_0 \subset \Omega$.

The function u satisfies the Klein-Gordon equation (1) inside Ω, the Neumann boundary condition (2) on Γ_S and Γ_N, and the initial conditions (3) in Ω. It is necessary to define suitable artificial boundary conditions on the artificial boundaries Γ_W and Γ_E. For this, there are several possibilities, but we have focused on the so called Absorbing Boundary Conditions, which are designed to produce small reflections inside the computational domain and to have local character. The ABCs are built in order to achieve, after the discretization, a stable, accurate, efficient and easy to implement scheme. There exists a wide literature on this subject, see the works [5, 6, 10, 13, 14, 15, 16, 17] and the review papers [9, 11, 12, 26].

As an alternative, it can be considered first a space discretization of the problem and then obtain ABCs for the discrete problem. We have worked in this sense for example in [2, 3, 4].

In this paper, we have considered the Hagstrom-Warburton high-order ABCs, a modified version of the Higdon ABCs [18, 19]. These ABCs use auxiliary variables to avoid high derivatives in their formulation [16, 17]. Arbitrary order of absorption P can be achieved by introducing P auxiliary variables ϕ_j, $j = 1, \ldots, P$, satisfying the recursive relations

$$\begin{array}{rcl} (a_0\partial_t + c\partial_x)u & = & a_0\partial_t\phi_1, \\ (a_j\partial_t + c\partial_x)\phi_j & = & (a_j\partial_t - c\partial_x)\phi_{j+1}, \quad j = 1, \ldots, P, \\ \phi_{P+1} = 0, & & \end{array} \tag{4}$$

in the vicinity of Γ_E. The parameters a_j have to be chosen. We consider $a_j = 1$ for all j, following the recommendation done in [8] and the remark in [13] in which the authors find this choice satisfactory in general. Other choices of the parameters could be considered, see for example [17] and Section 6 in this paper.

From the assumption that the initial conditions have compact support away from Γ_W and Γ_E, we have,

$$\phi_j(y, 0) = 0, \quad \partial_t\phi_j(y, 0) = 0 \quad \text{on} \quad \Gamma_W, \Gamma_E.$$

From (2) and the recursive relations (4), the conditions

$$\partial_y\phi_j(a, t) = \partial_y\phi_j(b, t) = 0,$$

can be deduced.

In [15], it is established that the H-W ABCs of order P given by (4) may be rewritten as

$$(\partial_t + c\partial_x)u = \partial_t\phi_1, \tag{5}$$

$$\partial_t^2\phi_1 = c^2(\frac{1}{2}\partial_y^2\phi_0 + \frac{1}{4}\partial_y^2\phi_1 + \frac{1}{4}\partial_y^2\phi_2) - s^2(\frac{1}{2}\phi_0 + \frac{1}{4}\phi_1 + \frac{1}{4}\phi_2), \tag{6}$$

$$\partial_t^2\phi_j = c^2(\frac{1}{4}\partial_y^2\phi_{j-1} + \frac{1}{2}\partial_y^2\phi_j + \frac{1}{4}\partial_y^2\phi_{j+1}) - s^2(\frac{1}{4}\phi_{j-1} + \frac{1}{2}\phi_j + \frac{1}{4}\phi_{j+1}), j = 2, \ldots, P, \tag{7}$$

$$u = \phi_0, \quad \phi_{P+1} = 0. \tag{8}$$

On the boundary Γ_W, the equation (5) has to be replaced by $(\partial_t - c\partial_x)u = \partial_t\phi_1$, but the other conditions (6)-(8) are the same.

The H-W high-order ABCs permit to approach the exact solution with an arbitrary order of absorption P. Increasing the order P causes only a linear rising of the computational cost. However, sometimes the accuracy of the numerical solution is limited by the space and time discretization error, above all for the two dimensional case.

The space discretization is reached by means of finite differences on a uniform grid, with fourth order inside the computational domain. For the time discretization, we propose a fourth order exponential splitting method which improves the computational efficiency of the time integration. The combination of the exponential splitting scheme with H-W ABCs is not a trivial matter and, as far as we know, it has not made before. The splitting choice is specific for the problem here considered and it involves to make several momentous decisions. We split the auxiliary variables and the function u on the spatial nodes and, we make a splitting in order to obtain an explicit and simple to implement method. We show the improvement by comparing our splitting scheme with the standard fourth-order four-stage Runge-Kutta method used in [17]. Useful overviews of splitting methods can be found in the review papers [7, 22]. Other interesting references are [23, 24].

We notice that it is possible to use higher order space discretizations, (cf. [15]), along with higher order exponential splitting methods designed similarly to the one used in this work.

The paper is organized as follows. Section 2 is devoted to introduce the spatial discretization and the time exponential splitting method which we propose, along with some properties of stability in the discrete energy norm which show the feasibility of this full discretization. For this, a simple one dimensional problem with periodic boundary conditions is used. In Section 3, we build the exponential splitting time method for the one dimensional case problem when the ABCs are added. A similar splitting method, for the two dimensional case, also with ABCs, is proposed in Section 4. Numerical experiments for one and two dimensional Klein-Gordon equations are presented in Section 5. Finally, a brief Section 6 is devoted to conclusions and future works.

2. Preliminaries

In this section, we present the space discretization and the splitting technique used for the time integration. For the sake of simplicity, we consider the

one dimensional Klein-Gordon equation with periodic boundary conditions. The H-W ABCs are added in the following sections.

Therefore, the problem studied here is

$$\begin{aligned} \partial_t^2 u(x,t) &= c^2 \partial_x^2 u(x,t) - s^2 u(x,t), \quad x_W \le x \le x_E, \\ u(x_W,t) &= u(x_E,t), \\ \partial_x u(x_W,t) &= \partial_x u(x_E,t), \end{aligned}$$

with initial conditions

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x).$$

2.1. Spatial discretization

For the spatial discretization, second order derivatives are approximated by fourth order finite differences. For $N > 0$, let $h = \dfrac{x_E - x_W}{N}$ be the size step and $x_j = x_W + (j-1)h$, $j = 1, \ldots, N+1$, the nodes of the spatial discretization. We denote $u_j(t) = u(x_j, t)$, then

$$\begin{aligned} \frac{d^2}{dt^2}u_1 &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_N + \frac{4}{3}u_{N+1} - \frac{5}{2}u_1 + \frac{4}{3}u_2 - \frac{1}{12}u_3\right) - s^2 u_1, \\ \frac{d^2}{dt^2}u_2 &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{N+1} + \frac{4}{3}u_1 - \frac{5}{2}u_2 + \frac{4}{3}u_3 - \frac{1}{12}u_4\right) - s^2 u_2, \\ \frac{d^2}{dt^2}u_j &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{j-2} + \frac{4}{3}u_{j-1} - \frac{5}{2}u_j + \frac{4}{3}u_{j+1} - \frac{1}{12}u_{j+2}\right) - s^2 u_j, j = 3, \ldots, N-1, \\ \frac{d^2}{dt^2}u_N &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{N-2} + \frac{4}{3}u_{N-1} - \frac{5}{2}u_N + \frac{4}{3}u_{N+1} - \frac{1}{12}u_1\right) - s^2 u_N, \\ \frac{d^2}{dt^2}u_{N+1} &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{N-1} + \frac{4}{3}u_N - \frac{5}{2}u_{N+1} + \frac{4}{3}u_1 - \frac{1}{12}u_2\right) - s^2 u_{N+1}. \end{aligned}$$

We rewrite this problem as a first order ordinary differential system,

$$\frac{d}{dt}\begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}\begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \end{bmatrix}, \tag{9}$$

where $\mathbf{u}_h = [u_1, \ldots, u_{N+1}]^T$, $\mathbf{v}_h = \frac{d}{dt}\mathbf{u}_h$, I is the identity matrix of dimension

$N+1$, $A = \dfrac{c^2}{h^2}B - s^2 I$, and

$$B = \begin{bmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & & & & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & & & & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \\ -\frac{1}{12} & & & & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & & & & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} \end{bmatrix}. \quad (10)$$

Proposition 1. *The matrix*

$$A = \frac{c^2}{h^2}B - s^2 I$$

is symmetric negative definite.

Proof. Since the matrix B in (10) is symmetric, this is also true for the matrix A. Moreover, B is a circulant matrix, and the exact value of its eigenvalues is given by (see [20]),

$$\begin{aligned} \lambda_j &= -\frac{5}{2} + \frac{4}{3}\exp(\frac{2\pi ij}{N+1}) - \frac{1}{12}\exp(\frac{4\pi ij}{N+1}) \\ &\quad -\frac{1}{12}\exp(\frac{(N-1)2\pi ij}{N+1}) + \frac{4}{3}\exp(\frac{N2\pi ij}{N+1}),\ j = 0, \ldots, N. \end{aligned}$$

By using a straightforward calculation, we deduce that

$$\begin{aligned} \lambda_j &= -\frac{5}{2} + \frac{8}{3}\cos(\frac{2\pi j}{N+1}) - \frac{1}{6}\cos(\frac{4\pi j}{N+1}) \\ &= -\frac{5}{2} + \frac{8}{3}\cos(\frac{2\pi j}{N+1}) - \frac{1}{6}\left(2\cos^2(\frac{2\pi j}{N+1}) - 1\right) \\ &= -\frac{7}{3} + \frac{8}{3}\cos(\frac{2\pi j}{N+1}) - \frac{1}{3}\cos^2(\frac{2\pi j}{N+1}). \end{aligned}$$

Then, if $p(x) = -\frac{7}{3} + \frac{8}{3}x - \frac{1}{3}x^2$,

$$\sigma(B) \subset \{p(x) : x \in [-1,1]\} = [-5 - \frac{1}{3}, 0].$$

Finally, the eigenvalues of the matrix A are

$$\sigma(A) = \{ \frac{c^2}{h^2}\,\mu - s^2, \quad \mu \in \sigma(B)\},$$

and we deduce the result. □

From Proposition 1, the discrete energy

$$E_h(t)(\mathbf{u},\mathbf{v}) \;=\; \frac{h}{2}(\mathbf{v}^T\mathbf{v} - \mathbf{u}^T A\mathbf{u}), \tag{11}$$

is a norm and, since, for the solution of (9),

$$\begin{aligned}\frac{dE_h}{dt}(t) &= h\,\frac{d\mathbf{u}_h}{dt}^T \frac{d^2\mathbf{u}_h}{dt^2} - h\,\frac{d\mathbf{u}_h}{dt}^T A\mathbf{u}_h \\ &= h\,\frac{d\mathbf{u}_h}{dt}^T \left(\frac{d^2\mathbf{u}_h}{dt^2} - A\mathbf{u}_h\right) = 0,\end{aligned}$$

the discrete energy norm is conserved and the problem (9) is well posed. Its exact solution is given by

$$\left[\begin{array}{c}\mathbf{u}(t)\\ \mathbf{v}(t)\end{array}\right] = \exp\left(t\left[\begin{array}{cc}0 & I\\ A & 0\end{array}\right]\right)\left[\begin{array}{c}\mathbf{u}(0)\\ \mathbf{v}(0)\end{array}\right], \quad t \geq 0.$$

2.2. Time discretization: exponential splitting method

By using the exact solution of (9), we deduce that

$$\left[\begin{array}{c}\mathbf{u}(t+k)\\ \mathbf{v}(t+k)\end{array}\right] = \exp\left(k\left[\begin{array}{cc}0 & I\\ A & 0\end{array}\right]\right)\left[\begin{array}{c}\mathbf{u}(t)\\ \mathbf{v}(t)\end{array}\right],\; t \geq 0,\; k > 0.$$

The main idea of splitting methods for the time integration of ordinary differential equations involves to separate the system into several parts, being each of them easily integrable. More precisely, given the initial value problem

$$\frac{d}{dt}u(t) = g(u), \quad u(0) = u_0 \in \mathbb{R}^D, \tag{12}$$

with $g : \mathbb{R}^D \longrightarrow \mathbb{R}^D$, splitting methods require three steps [22]:

1. choosing the functions g_i such that $g = \sum_{i=1}^{n} g_i$, and the equations

$$\frac{d}{dt}u(t) = g_i(u), \quad u(0) = u_0, \quad i = 1, \ldots, n$$

   can be easily integrated.
2. solving exactly each equation $\frac{d}{dt}u(t) = g_i(u)$,
3. combining the solutions of the intermediate problems to achieve a good approximation of (12).

We propose to split

$$\begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} = M_1 + M_2.$$

Now, in order to solve exactly the problems

$$\frac{d}{dt}\begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \end{bmatrix} = M_i \begin{bmatrix} \mathbf{u}_h \\ \mathbf{v}_h \end{bmatrix}, \quad i = 1, 2,$$

we use that $M_i^2 = 0$ for $i = 1, 2$ and then,

$$\exp(kM_1) = \begin{bmatrix} I & kI \\ 0 & I \end{bmatrix}, \quad \exp(kM_2) = \begin{bmatrix} I & 0 \\ kA & I \end{bmatrix}.$$

To advance a step of size k in time, we use the second order Strang splitting

$$\psi_k : \exp(k(M_1 + M_2)) \approx \exp(\frac{k}{2}M_1)\exp(kM_2)\exp(\frac{k}{2}M_1),$$

which, applied to $[\mathbf{u}, \mathbf{v}]^T$, leaves to

$$\begin{aligned}
\psi_{k/2}^{[1]}: \quad & \exp(\tfrac{k}{2}M_1)\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} &&= \begin{bmatrix} \mathbf{u} + \frac{k}{2}\mathbf{v} \\ \mathbf{v} \end{bmatrix}, \\
\psi_{k}^{[2]}: \quad & \exp(kM_2)\begin{bmatrix} \mathbf{u} + \frac{k}{2}\mathbf{v} \\ \mathbf{v} \end{bmatrix} &&= \begin{bmatrix} \mathbf{u} + \frac{k}{2}\mathbf{v} \\ \mathbf{v} + kA\mathbf{u} + \frac{k^2}{2}A\mathbf{v} \end{bmatrix}, \\
\psi_{k/2}^{[1]}: \quad & \exp(\tfrac{k}{2}M_1)\begin{bmatrix} \mathbf{u} + \frac{k}{2}\mathbf{v} \\ \mathbf{v} + kA\mathbf{u} + \frac{k^2}{2}A\mathbf{v} \end{bmatrix} &&= \begin{bmatrix} \mathbf{u} + k\mathbf{v} + \frac{k^2}{2}A\mathbf{u} + \frac{k^3}{4}A\mathbf{v} \\ \mathbf{v} + kA\mathbf{u} + \frac{k^2}{2}A\mathbf{v} \end{bmatrix}.
\end{aligned}$$

That is, the Strang splitting time integrator is given by

$$\exp(k\begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}) \approx \begin{bmatrix} I + \frac{k^2}{2}A & kI + \frac{k^3}{4}A \\ kA & I + \frac{k^2}{2}A \end{bmatrix} := M(k, -A).$$

In this way, we have obtained an explicit method which is easily implemented. However, it is not unconditionally stable and the stability has to be studied. That is, the size of the powers of $M(k, -A)$ has to be bounded in the matrix norm associated to the discrete energy norm. For this, we write the discrete energy (11) as

$$\begin{aligned} E_h(t)(\mathbf{u}, \mathbf{v}) &= \frac{h}{2}\left(\mathbf{v}^T \cdot \mathbf{v} + ((-A)^{1/2}\mathbf{u})^T((-A)^{1/2}\mathbf{u})\right) \\ &= |||[(-A)^{1/2}\mathbf{u}, \mathbf{v}]|||_2 = ||Q[\mathbf{u}, \mathbf{v}]^T||_2 := |||[\mathbf{u}, \mathbf{v}]|||_Q, \end{aligned}$$

where,

$$Q = \begin{bmatrix} (-A)^{1/2} & 0 \\ 0 & I \end{bmatrix}.$$

Therefore,

$$\begin{aligned} ||M^n(k, -A)||_Q &= ||QM^n(k, -A)Q^{-1}||_2 \\ &= ||\left(QM(k, -A)Q^{-1}\right)^n||_2 \\ &= ||R^n(k(-A)^{1/2})||_2, \end{aligned} \tag{13}$$

where,

$$R(\omega) = \begin{bmatrix} 1 - \frac{\omega^2}{2} & \omega - \frac{\omega^3}{4} \\ -\omega & 1 - \frac{\omega^2}{2} \end{bmatrix},$$

is called the stability matrix.

The study of the boundedness of the powers (13) is not easy in general (cf. [1]) but, in order to accomplish this, a necessary condition is that the eigenvalues of $k(-A)^{1/2} = \frac{ck}{h}(-B + \frac{s^2h^2}{c^2}I)^{1/2}$ must be in the *stability interval*

$$\{\omega > 0 : \rho(R(\omega)) \leq 1\},$$

where $\rho(R(\omega))$ is the spectral radius of $R(\omega)$. In our case, $\rho(R(\omega))$ can be easily calculated and we can deduce the interval of stability.

Proposition 2. *The spectral radius of $R(\omega)$ satisfies*

$$\rho(R(\omega)) \leq 1,$$

if, and only if, $\omega \in [0, 2]$.

Proof. Since

$$R(\omega) = (1 - \frac{\omega^2}{2})I + \begin{bmatrix} 0 & \omega - \frac{\omega^3}{4} \\ -\omega & 0 \end{bmatrix},$$

we deduce that, when $\omega \in [0, 2]$, the eigenvalues of $R(\omega)$ are the complex numbers given by

$$\begin{aligned} \lambda_1(\omega) &= (1 - \frac{\omega^2}{2}) + i\omega\sqrt{1 - \frac{\omega^2}{4}} \\ \lambda_2(\omega) &= (1 - \frac{\omega^2}{2}) - i\omega\sqrt{1 - \frac{\omega^2}{4}}. \end{aligned}$$

It is clear that these eigenvalues satisfy $|\lambda_j(\omega)| = 1$ for $j = 1, 2$ and any $\omega \in [0, 2]$.

On the other hand, when $\omega > 2$, the eigenvalues of $R(\omega)$ are the real numbers given by

$$\begin{aligned} \lambda_1(\omega) &= (1 - \frac{\omega^2}{2}) + \omega\sqrt{\frac{\omega^2}{4} - 1} \\ \lambda_2(\omega) &= (1 - \frac{\omega^2}{2}) - \omega\sqrt{\frac{\omega^2}{4} - 1}. \end{aligned}$$

Since $\lambda_1(\omega)\lambda_2(\omega) = 1$, we deduce that $\rho(R(\omega)) > 1$. □

From Propositions 1 and 2, we obtain that the stability condition

$$\frac{ck}{h}\sqrt{5 + \frac{1}{3} + \frac{s^2h^2}{c^2}} < 2$$

has to be satisfied. This will be reached, for sh small enough, when

$$\frac{ck}{h} < \frac{2}{\sqrt{5 + 1/3}} \approx 0.866. \tag{14}$$

Our numerical experiments corroborate that stability is achieved when the time step k is chosen to meet (14).

In following sections, where ABCs will be added, a more complicated splitting has to be used, and a similar condition for stability will be established numerically.

3. One dimensional Klein-Gordon equation with H-W ABCs

The one dimensional Klein-Gordon equation has its own interest. Among its scope of applications are the propagation of transverse waves in strings, cables on linearly elastic springs and torsional and axial waves in beams on elastic substrate [21]. On the other hand, it is convenient to start considering the exponential splitting method in this simpler equation and translate the method to the two-dimensional case.

The problem in one dimension reduces to

$$\begin{aligned}
\partial_t u(x_W,t) &= c\partial_x u(x_W,t) + \partial_t \phi_1^W, && (15)\\
\partial_t^2 u(x,t) &= c^2\partial_x^2 u(x,t) - s^2 u(x,t), \quad x_W < x < x_E, && (16)\\
\partial_t u(x_E,t) &= -c\partial_x u(x_E,t) + \partial_t \phi_1^E, && (17)\\
\frac{d^2}{dt^2}\phi_1^W &= -s^2(\frac{1}{2}\phi_0^W + \frac{1}{4}\phi_1^W + \frac{1}{4}\phi_2^W), && (18)\\
\frac{d^2}{dt^2}\phi_j^W &= -s^2(\frac{1}{4}\phi_{j-1}^W + \frac{1}{2}\phi_j^W + \frac{1}{4}\phi_{j+1}^W), j = 2,\ldots,P, && (19)\\
\frac{d^2}{dt^2}\phi_1^E &= -s^2(\frac{1}{2}\phi_0^E + \frac{1}{4}\phi_1^E + \frac{1}{4}\phi_2^E), && (20)\\
\frac{d^2}{dt^2}\phi_j^E &= -s^2(\frac{1}{4}\phi_{j-1}^E + \frac{1}{2}\phi_j^E + \frac{1}{4}\phi_{j+1}^E), j = 2,\ldots,P, && (21)\\
u(x_W,t) &= \phi_0^W, \quad \phi_{P+1}^W = 0, \quad u(x_E,t) = \phi_0^E, \quad \phi_{P+1}^E = 0, && (22)
\end{aligned}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x),$$

$$\phi_j^W(0) = 0, \quad \phi_j^E(0) = 0, \quad \partial_t\phi_j^W(0) = 0, \quad \partial_t\phi_j^W(0) = 0.$$

3.1. Spatial discretization

For $N > 0$, let $h = \dfrac{x_E - x_W}{N}$ be the size step and $x_j = x_W + (j-1)h$, $j = 1,\ldots,N+1$, the nodes of the spatial discretization. We denote $u_j(t) = u(x_j,t)$. Spatial derivatives from u_3 to u_{N-1} in (16) are approximated by fourth order central finite differences as in Section 2, and for u_2 and u_N by fourth order one-sided finite differences. For u_1 and u_{N+1} spatial derivatives

in (15) and (17) are approximated by fourth order one-sided finite differences. In this way, we obtain a semidiscrete ordinary differential problem given by,

$$\begin{aligned}
\frac{d}{dt}u_1 &= \frac{c}{h}\left(-\frac{25}{12}u_1 + 4u_2 - 3u_3 + \frac{4}{3}u_4 - \frac{1}{4}u_5\right) + \frac{d}{dt}\phi_1^W,\\
\frac{d^2}{dt^2}u_2 &= \frac{c^2}{h^2}\left(\frac{11}{12}u_1 - \frac{5}{3}u_2 + \frac{1}{2}u_3 + \frac{1}{3}u_4 - \frac{1}{12}u_5\right) - s^2u_2,\\
\frac{d^2}{dt^2}u_j &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{j-2} + \frac{4}{3}u_{j-1} - \frac{5}{2}u_j + \frac{4}{3}u_{j+1} - \frac{1}{12}u_{j+2}\right) - s^2u_j, j = 3,\ldots,N-1,\\
\frac{d^2}{dt^2}u_N &= \frac{c^2}{h^2}\left(-\frac{1}{12}u_{N-3} + \frac{1}{3}u_{N-2} + \frac{1}{2}u_{N-1} - \frac{5}{3}u_N + \frac{11}{12}u_{N+1}\right) - s^2u_N,\\
\frac{d}{dt}u_{N+1} &= \frac{c}{h}\left(-\frac{1}{4}u_{N-3} + \frac{4}{3}u_{N-2} - 3u_{N-1} + 4u_N - \frac{25}{12}u_{N+1}\right) + \frac{d}{dt}\phi_1^E,
\end{aligned}$$

and the ordinary differential equations (18)-(21) for the auxiliary variables.

3.2. Time integration

We start rewriting the problem of Subsection 3.1 as a first order ordinary differential system

$$\frac{d}{dt}\left[\begin{array}{c}\mathbf{u}\\ \boldsymbol{\phi}^W\\ \boldsymbol{\phi}^E\\ \hline \mathbf{u}'\\ (\boldsymbol{\phi}^W)'\\ (\boldsymbol{\phi}^E)'\end{array}\right] = M\left[\begin{array}{c}\mathbf{u}\\ \boldsymbol{\phi}^W\\ \boldsymbol{\phi}^E\\ \hline \mathbf{u}'\\ (\boldsymbol{\phi}^W)'\\ (\boldsymbol{\phi}^E)'\end{array}\right] = \left[\begin{array}{c|c} M_{11} & M_{12}\\ \hline M_{21} & 0\end{array}\right]\left[\begin{array}{c}\mathbf{u}\\ \boldsymbol{\phi}^W\\ \boldsymbol{\phi}^E\\ \hline \mathbf{u}'\\ (\boldsymbol{\phi}^W)'\\ (\boldsymbol{\phi}^E)'\end{array}\right], \qquad (23)$$

where $\mathbf{u} = [u_1,\ldots,u_{N+1}]^T$, $\mathbf{u}' = [\frac{d}{dt}u_2,\ldots,\frac{d}{dt}u_N]^T$, $\boldsymbol{\phi}^W = [\phi_1^W,\ldots,\phi_P^W]^T$, $\boldsymbol{\phi}^E = [\phi_1^E,\ldots,\phi_P^E]^T$, $(\boldsymbol{\phi}^W)' = [\frac{d}{dt}\phi_1^W,\ldots,\frac{d}{dt}\phi_P^W]$, $(\boldsymbol{\phi}^E)' = [\frac{d}{dt}\phi_1^E,\ldots,\frac{d}{dt}\phi_P^E]^T$ and M a square matrix of dimension $2N+4P$, with submatrices

$$M_{11} = \left[\begin{array}{c|c|c}\frac{c}{h}A_1 & 0 & 0\\ \hline 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right], M_{12} = \left[\begin{array}{c|c|c}A_2 & A_3 & A_4\\ \hline 0 & I_P & 0\\ \hline 0 & 0 & I_P\end{array}\right], M_{21} = \left[\begin{array}{c|c|c}A_5 & 0 & 0\\ \hline -s^2A_6 & -s^2A_7 & 0\\ \hline -s^2A_8 & 0 & -s^2A_7\end{array}\right],$$

in which A_1 is a $(N+1)\times(N+1)$ matrix with all elements zero except in the first and last rows,

$$A_1 = \begin{bmatrix} -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} & & & & \\ & & & & 0 & & & & \\ & & & & -\frac{1}{4} & \frac{4}{3} & -3 & 4 & -\frac{25}{12} \end{bmatrix};$$

A_2 is the $(N+1)\times(N-1)$ matrix,

$$A_2 = \begin{bmatrix} 0 \\ I_{N-1} \\ 0 \end{bmatrix},$$

where 0 denotes a row vector of $N-1$ zeros, I_{N-1} is the identity matrix of dimension $N-1$; A_3 is the $(N+1)\times P$ matrix with all elements zero except the position $(1,1)$ which is equal to 1; A_4 is the $(N+1)\times P$ matrix with all elements zero except the position $(N+1,1)$ which is equal to 1; A_5 is the matrix given by

$$A_5 = \frac{c^2}{h^2}M_x - s^2 A_2^T,$$

where M_x is the $(N-1)\times(N+1)$ matrix

$$M_x = \begin{bmatrix} \frac{11}{12} & -\frac{5}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{1}{12} & & & \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & \\ & & -\frac{1}{12} & \frac{1}{3} & \frac{1}{2} & -\frac{5}{3} & \frac{11}{12} & \end{bmatrix};$$

A_6 is the $P \times (N+1)$ matrix with all elements zero except the position $(1,1)$ which is equal to $1/2$; A_7 is a $P \times P$ given by

$$A_7 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ \ddots & \ddots & \ddots & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & & \frac{1}{4} & \frac{1}{2} & \end{bmatrix},$$

and finally A_8 is the $P \times (N+1)$ matrix with all elements zero except the position $(1, N+1)$ which is equal to $1/2$.

We propose one splitting so that the matrix of each intermediate problem has a simpler exponential, in a similar way to the one explained in Section 2. Concretely, we consider two steps $g = g_1 + g_2$. The step 1 is

$$\frac{d}{dt}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{bmatrix} = M_1 \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{bmatrix} = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{bmatrix}.$$

Now we try to reach $\exp(kM_1) = I_{2N+4P} + kM_1 + \sum_{n=2}^{\infty} \frac{k^n}{n!} M_1^n$.

Taking into account that $A_1^n = \left(-\frac{25}{12}\right)^{n-1} A_1$, then for $n \geq 2$,

$$M_1^n = \left[\begin{array}{ccc|cc} (\frac{c}{h})^n A_1^n & 0 & 0 & (\frac{c}{h})^{n-1} A_1^{n-1} A_2 & (\frac{c}{h})^{n-1} A_1^{n-1} A_3 & (\frac{c}{h})^{n-1} A_1^{n-1} A_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{3}{c|}{0} & \multicolumn{3}{c}{0} \end{array}\right] =$$

$$\left[\begin{array}{ccc|ccc} (\frac{c}{h})^n (-\frac{25}{12})^{n-1} A_1 & 0 & 0 & (\frac{c}{h})^{n-1} (-\frac{25}{12})^{n-2} A_1 A_2 & (\frac{c}{h})^{n-1} (-\frac{25}{12})^{n-2} A_1 A_3 & (\frac{c}{h})^{n-1} (-\frac{25}{12})^{n-2} A_1 A_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{3}{c|}{0} & \multicolumn{3}{c}{0} \end{array}\right].$$

Therefore,

$$\begin{aligned}\exp(c\frac{k}{h}A_1) &= I_{N+1} + \sum_{n=1}^{\infty} \frac{k^n}{n!} \left(\frac{c}{h}\right)^n \left(-\frac{25}{12}\right)^{n-1} A_1 \\ &= I_{N+1} - \frac{12}{25}\left(-1 + \exp\left(-\frac{25}{12}c\frac{k}{h}\right)\right) A_1 \\ &= I_{N+1} + \eta_k A_1,\end{aligned}$$

being $\eta_k = -\frac{12}{25}(-1 + \exp(-\frac{25}{12}c\frac{k}{h}))$.

Denoting $B = A_1A_2$ the $(N+1)\times(N-1)$ matrix

$$B = \begin{bmatrix} 4 & -3 & \frac{4}{3} & -\frac{1}{4} & & & \\ & & & 0 & & & \\ & & & -\frac{1}{4} & \frac{4}{3} & -3 & 4 \end{bmatrix},$$

$$\begin{aligned} & kA_2 + \sum_{n=2}^{\infty} \frac{k^n}{n!} \left(\frac{c}{h}\right)^{n-1} \left(-\frac{25}{12}\right)^{n-2} B \\ = \; & kA_2 + \frac{h}{c}\left(-\frac{12}{25}\right)^2 \sum_{n=2}^{\infty} \frac{k^n}{n!} \left(\frac{c}{h}\right)^n \left(-\frac{25}{12}\right)^n B \\ = \; & kA_2 + \frac{h}{c}\left(-\frac{12}{25}\right)^2 \left(-1 + k\frac{c}{h}\frac{25}{12} + \exp\left(-\frac{25}{12}c\frac{k}{h}\right)\right) B \\ = \; & kA_2 + \frac{h}{c}\left(-\frac{12}{25}\right)\left(\eta_k - k\frac{c}{h}\right) B := B_k. \end{aligned}$$

Noting that $A_1A_i = -\frac{25}{12}A_i$, for $i = 3, 4$,

$$\begin{aligned} & kA_i + \sum_{n=2}^{\infty} \frac{k^n}{n!} \left(\frac{c}{h}\right)^{n-1} \left(-\frac{25}{12}\right)^{n-1} A_i \\ = \; & \frac{h}{c}\eta_k A_i. \end{aligned}$$

Summarizing,

$$\exp(kM_1) = \left[\begin{array}{c|c|c|c|c|c} I_{N+1} + \eta_k A_1 & 0 & 0 & B_k & \frac{h}{c}\eta_k A_3 & \frac{h}{c}\eta_k A_4 \\ \hline 0 & I_P & 0 & 0 & kI_P & 0 \\ \hline 0 & 0 & I_P & 0 & 0 & kI_P \\ \hline \multicolumn{3}{c|}{0} & \multicolumn{3}{c}{I_{N-1+2P}} \end{array}\right].$$

Therefore, flow $\psi_k^{[1]}$ (its non constant part) is

$$\psi_k^{[1]}: \quad \begin{array}{rcl} \mathbf{u}(t+k) & = & \mathbf{u}(t) + \eta_k A_1 \mathbf{u}(t) + B_k \mathbf{u}'(t) + \frac{h}{c}\eta_k A_3 (\boldsymbol{\phi}^W)'(t) + \frac{h}{c}\eta_k A_4 (\boldsymbol{\phi}^E)'(t), \\ \boldsymbol{\phi}^W(t+k) & = & \boldsymbol{\phi}^W(t) + k(\boldsymbol{\phi}^W)'(t), \\ \boldsymbol{\phi}^E(t+k) & = & \boldsymbol{\phi}^E(t) + k(\boldsymbol{\phi}^E)'(t). \end{array}$$

The step 2 corresponds to

$$\frac{d}{dt}\left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right] = M_2 \left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline M_{21} & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right].$$

In this case,

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline M_{21} & 0 \end{array}\right]^2 = 0 \text{ and, } \exp(k \left[\begin{array}{c|c} 0 & 0 \\ \hline M_{21} & 0 \end{array}\right]) = \left[\begin{array}{c|c} I_{N+1+2P} & 0 \\ \hline kM_{21} & I_{N-1+2P} \end{array}\right].$$

Hence the flow $\psi_k^{[C]}$ means

$$\psi_k^{[2]}: \quad \begin{array}{rcl} \mathbf{u}'(t+k) & = & \mathbf{u}'(t) + kA_5 \mathbf{u}(t), \\ (\boldsymbol{\phi}^W)'(t+k) & = & (\boldsymbol{\phi}^W)'(t) + k(-s^2)A_6 \mathbf{u}(t) + k(-s^2)A_7 \boldsymbol{\phi}^W(t), \\ (\boldsymbol{\phi}^E)'(t+k) & = & (\boldsymbol{\phi}^E)'(t) + k(-s^2)A_8 \mathbf{u}(t) + k(-s^2)A_7 \boldsymbol{\phi}^E(t). \end{array}$$

3.2.1. Combining the two steps and increasing the order by composition

Once we have chosen the steps and we have solved exactly each step, there is still missing combining these solutions to obtain an approximation of the solution of (23). We consider the second order Strang splitting

$$\mathcal{S}_k^{[2]} = \psi_{k/2}^{[1]} \circ \psi_k^{[2]} \circ \psi_{k/2}^{[1]}, \tag{24}$$

used in Section 2.

Numerical integrators of arbitrarily high order can be obtained by composition of $\mathcal{S}^{[2]}$ [25, 27]. For example, in the numerical experiments of Section 5 we have considered the fourth order integrator $\mathcal{S}^{[4]}$

$$\mathcal{S}_k^{[4]} = \mathcal{S}_{\alpha k}^{[2]} \circ \mathcal{S}_{\beta k}^{[2]} \circ \mathcal{S}_{\alpha k}^{[2]}, \quad \text{with} \quad \alpha = \frac{1}{2 - 2^{1/3}}, \quad \beta = 1 - 2\alpha. \tag{25}$$

We remark that it is possible to save some computational cost in (25) by join together the last step in the composition of $\mathcal{S}^{[2]}_{\alpha k}$ and the first one in $\mathcal{S}^{[2]}_{\beta k}$ and similarly, the last one in the composition of $\mathcal{S}^{[2]}_{\beta k}$ and the first one in $\mathcal{S}^{[2]}_{\alpha k}$. That is,

$$\begin{aligned}\mathcal{S}^{[4]}_{k} &= \psi^{[1]}_{\alpha k/2}\circ\psi^{[2]}_{\alpha k}\circ\psi^{[1]}_{\alpha k/2}\circ\psi^{[1]}_{\beta k/2}\circ\psi^{[2]}_{\beta k}\circ\psi^{[1]}_{\beta k/2}\circ\psi^{[1]}_{\alpha k/2}\circ\psi^{[2]}_{\alpha k}\circ\psi^{[1]}_{\alpha k/2},\\ &= \psi^{[1]}_{\alpha k/2}\circ\psi^{[2]}_{\alpha k}\circ\psi^{[1]}_{(\alpha+\beta) k/2}\circ\psi^{[2]}_{\beta k}\circ\psi^{[1]}_{(\alpha+\beta) k/2}\circ\psi^{[2]}_{\alpha k}\circ\psi^{[1]}_{\alpha k/2}. \qquad (26)\end{aligned}$$

4. Two dimensional Klein-Gordon equation with H-W ABCs

We approach here the two dimensional case, more computationally expensive than the previous one dimensional case.

4.1. Spatial discretization

For the sake of simplicity, we consider the same size step in both directions x and y, that is, for a value of N, $h = \dfrac{x_E - x_W}{N}$ and $M = \dfrac{b-a}{h}$. Let $x_j = x_W + (j-1)h$, $j = 1, \ldots, N+1$, and $y_l = a + (l-1)h$, $l = 1, \ldots, M+1$, be the nodes of the spatial discretization. This produces a uniform grid in the computational domain with $M+1$ rows and $N+1$ columns. We denote $u_{jl}(t) = u(x_j, y_l, t)$. In this way, there is a matrix of unknowns. On the other hand, we consider $\phi_{rl} = \phi_r(y_l)$, $r = 1, \ldots, P$, and $l = 1, \ldots, M+1$, on west and east boundaries.

As in the one dimensional case, second order spatial derivatives in the direction x, $\partial_x^2 u_{jl}$, from $j = 3$ to $N-1$, are approximated by fourth order central finite differences and, for $j = 2$ and N, by fourth order one-sided finite differences. First order spatial derivatives $\partial_x u_{1l}$ and $\partial_x u_{N+1,l}$ are approximated by fourth order one-sided finite differences.

We assume that the unknowns associated with nodes on south and north boundaries have been removed using Neumann boundary conditions. Spatial derivative in the direction y, $\partial_y^2 u_{jl}$, from $l = 3$ to $M-1$, are approximated by fourth order central finite differences. For $l = 2$ and M, fourth order one-sided finite differences and Neumann boundary condition are used to obtain the approximation to $\partial_y^2 u_{j2}$ and $\partial_y^2 u_{jM}$. We explain this process in the case of $\partial_y^2 u_{j2}$. First, fourth order one-sided finite differences for $\partial_y^2 u_{j2}$ are considered,

$$\partial_y^2 u_{j2} \approx \frac{1}{h^2}\left(\frac{11}{12}u_{j1} - \frac{5}{3}u_{j2} + \frac{1}{2}u_{j3} + \frac{1}{3}u_{j4} - \frac{1}{12}u_{j5}\right), \qquad (27)$$

and third order one-sided finite differences for $\partial_y u_{j1}$,

$$\partial_y u_{j1} \approx \frac{1}{h}\left(-\frac{11}{6}u_{j1} + 3u_{j2} - \frac{3}{2}u_{j3} + \frac{1}{3}u_{j4}\right).$$

Since $\partial_y u_{j1} = 0$, u_{j1} is substituted by $\frac{6}{11}(3u_{j2} - \frac{3}{2}u_{j3} + \frac{1}{3}u_{j4})$ in (27), and we achieve

$$\partial_y^2 u_{j2} \approx \frac{1}{h^2}\left(-\frac{1}{6}u_{j2} - \frac{1}{4}u_{j3} + \frac{1}{2}u_{j4} - \frac{1}{12}u_{j5}\right).$$

For the auxiliary variables corresponding to the west and east boundaries, spatial derivatives $\partial_y^2 \phi_r(y_l)$ are approximated following the same criterion.

4.2. Time integration

Let it be $\mathbf{u}_j$ the column j without the first and last components, $\boldsymbol{\phi}_j^W$ and $\boldsymbol{\phi}_j^E$ the column vectors corresponding to the auxiliary variable j on west and east boundary respectively, without the first and last components. We consider $\mathbf{u} = [\mathbf{u}_1^T, \ldots, \mathbf{u}_{N+1}^T]^T$, $\mathbf{u}' = [\frac{d}{dt}\mathbf{u}_2^T, \ldots, \frac{d}{dt}\mathbf{u}_N^T]^T$, $\boldsymbol{\phi}^W = [(\boldsymbol{\phi}_1^W)^T, \ldots, (\boldsymbol{\phi}_P^W)^T]^T$, $\boldsymbol{\phi}^E = [(\boldsymbol{\phi}_1^E)^T, \ldots, (\boldsymbol{\phi}_P^E)^T]^T$, $(\boldsymbol{\phi}^W)' = [\frac{d}{dt}(\boldsymbol{\phi}_1^W)^T, \ldots, \frac{d}{dt}(\boldsymbol{\phi}_P^W)^T]^T$ and finally $(\boldsymbol{\phi}^E)' = [\frac{d}{dt}(\boldsymbol{\phi}_1^E)^T, \ldots, \frac{d}{dt}(\boldsymbol{\phi}_P^E)^T]^T$.

Rewriting the problem of Subsection 4.1 as a first order ordinary differential system

$$\frac{d}{dt}\left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right] = \mathcal{M}\left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right] = \left[\begin{array}{c|c} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \hline \mathcal{M}_{21} & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \boldsymbol{\phi}^W \\ \boldsymbol{\phi}^E \\ \hline \mathbf{u}' \\ (\boldsymbol{\phi}^W)' \\ (\boldsymbol{\phi}^E)' \end{array}\right], \qquad (28)$$

and $\mathcal{M}$ a square matrix of dimension $(2N + 4P)(M - 1)$, with submatrices

$$\mathcal{M}_{11} = \left[\begin{array}{c|c|c} \frac{c}{h}\mathcal{A}_1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \mathcal{M}_{12} = \left[\begin{array}{c|c|c} \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 \\ \hline 0 & I_{P(M-1)} & 0 \\ \hline 0 & 0 & I_{P(M-1)} \end{array}\right], \mathcal{M}_{21} = \left[\begin{array}{c|c|c} \mathcal{A}_5 & 0 & 0 \\ \hline \mathcal{A}_6 & \mathcal{A}_7 & 0 \\ \hline \mathcal{A}_8 & 0 & \mathcal{A}_7 \end{array}\right],$$

where, by denoting $\otimes$ to the Kronecker product of matrices, $\mathcal{A}_1 = A_1 \otimes I_{M-1}$, $\mathcal{A}_2 = A_2 \otimes I_{M-1}$, $\mathcal{A}_3 = A_3 \otimes I_{M-1}$, $\mathcal{A}_4 = A_4 \otimes I_{M-1}$ and $\mathcal{A}_5 = \dfrac{c^2}{h^2} M_x \otimes I_{M-1} + A_2^T \otimes A_9$, $\mathcal{A}_6 = A_6 \otimes A_9$, $\mathcal{A}_7 = A_7 \otimes A_9$, $\mathcal{A}_8 = A_8 \otimes A_9$,

$$A_9 = \frac{c^2}{h^2} M_y - s^2 I_{M-1},$$

$$M_y = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{12} & & & & \\ \frac{11}{12} & -\frac{5}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{1}{12} & & & \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & \\ & & -\frac{1}{12} & \frac{1}{3} & \frac{1}{2} & -\frac{5}{3} & \frac{11}{12} & \\ & & & -\frac{1}{12} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{6} & \end{bmatrix}.$$

In order to obtain an explicit method with intermediate problems with simple exponentials, we consider an exponential splitting method based in two steps similar to the ones for the one dimensional case.

$$\begin{array}{l} \psi_k^{[1]}: \\ \mathbf{u}(t+k) = \mathbf{u}(t) + \eta_k \mathcal{A}_1 \mathbf{u}(t) + \mathcal{B}_k \mathbf{u}'(t) + \frac{h}{c}\eta_k \mathcal{A}_3 (\boldsymbol{\phi}^W)'(t) + \frac{h}{c}\eta_k \mathcal{A}_4 (\boldsymbol{\phi}^E)'(t), \\ \boldsymbol{\phi}^W(t+k) = \boldsymbol{\phi}^W(t) + k(\boldsymbol{\phi}^W)'(t), \\ \boldsymbol{\phi}^E(t+k) = \boldsymbol{\phi}^E(t) + k(\boldsymbol{\phi}^E)'(t), \\ \\ \psi_k^{[2]}: \\ (\boldsymbol{\phi}^W)'(t+k) = (\boldsymbol{\phi}^W)'(t) + k\mathcal{A}_6 \mathbf{u}(t) + k\mathcal{A}_7 \boldsymbol{\phi}^W(t), \\ (\boldsymbol{\phi}^E)'(t+k) = (\boldsymbol{\phi}^E)'(t) + k\mathcal{A}_8 \mathbf{u}(t) + k\mathcal{A}_7 \boldsymbol{\phi}^E(t), \end{array} \tag{29}$$

where $\mathcal{B}_k = B_k \otimes I_{M-1}$. The formulas (29) are very similar to the ones displayed in the one dimensional case.

Finally, we combine the previous steps to obtain a fourth order in time splitting method by using the same formulas (24), (25) and (26) used for the one dimensional case in Section 3.

4.3. Analysis of the efficiency of the algorithm

Here, we only take into account the matrix vector products in the algorithm described in Subsection 4.2. Step 1 requires $(N + 2P + 19)(M - 1)$ products, that is $NM + 2PM + O(N) + O(M) + O(P)$ products. Regarding step 2, $\mathcal{A}_5\mathbf{u}$, $(N-1)(16+9(M-3))$ products are needed. As for $\mathcal{A}_6\mathbf{u}$, $\mathcal{A}_7\boldsymbol{\phi}^W$, $\mathcal{A}_8\mathbf{u}$ and $\mathcal{A}_7\boldsymbol{\phi}^E$, $6P(8 + 5(M - 3))$ products are necessary. Altogether, step 2 requires $9NM + 30PM + O(N) + O(M) + O(P)$ products.

When we combine steps 1 and 2 using the second order Strang splitting (24) and (25) to obtain fourth order in time, using the formula (26), we repeat four times step 1 and three times the step 2. Therefore, we need $31NM + 98PM + O(N) + O(M) + O(P)$ products for every step in time. Moreover, if the last step in the composition of $\mathcal{S}^{[2]}_{\alpha k}$ for one step and the first one in $\mathcal{S}^{[2]}_{\alpha k}$ for the next step are joined together ($\psi^{[1]}_{\alpha k/2} \circ \psi^{[1]}_{\alpha k/2} = \psi^{[1]}_{\alpha k}$), only three times of step 1 are needed, and then the products for one step are $30NM + 96PM + O(N) + O(M) + O(P)$.

5. Numerical experiments

In this section, we want to study the stability, the efficiency and the performance of H-W ABCs when the dispersion coefficient varies. We use the full discretization explained in the previous sections taking advantage of the efficiency of the fourth order in time exponential splitting integrator, above all in the two dimensional case.

5.1. One dimensional case

We consider the discretization of the problem (15)-(22), described in Section 3, with initial conditions

$$u_0(x) = \begin{cases} \dfrac{(x+0.2)^3(0.2-x)^3}{(0.2)^6}, & -0.2 < x < 0.2, \\ 0, & \text{otherwise}, \end{cases} \qquad v_0(x) = 0,$$

with compact support contained in the computational interval $[-1/4, 1/4]$. The sixth order polynomial in u_0 is chosen so that $u_0 \in C^1([-1/4, 1/4])$. We have set $c = 1$ and final time $T = 4$.

We have computed numerically the stability limit time step k_l. In the considered domain we take $h = 1/2N$. Table 1 shows that for $N = 50$ the influence of s^2 in the stability limit time step can be seen for big values

	N=50	N=100	N=200
$s^2 = 0$	0.006808	0.003404	0.001702
$s^2 = 1$	0.006808	0.003404	0.001702
$s^2 = 20$	0.006808	0.003404	0.001702
$s^2 = 50$	0.006808	0.003404	0.001702
$s^2 = 100$	0.006808	0.003404	0.001702
$s^2 = 200$	0.006804	0.003404	0.001702
$s^2 = 400$	0.006792	0.003403	0.001702

Table 1: Stability limit time step k_l.

	N=50	N=100	N=200
$s^2 = 0$	0.6808	0.6808	0.6808
$s^2 = 1$	0.6808	0.6808	0.6808
$s^2 = 20$	0.6808	0.6808	0.6808
$s^2 = 50$	0.6808	0.6808	0.6808
$s^2 = 100$	0.6808	0.6808	0.6808
$s^2 = 200$	0.6804	0.6808	0.6808
$s^2 = 400$	0.6792	0.6806	0.6808

Table 2: Ratio stability limit k_l/h.

as $s^2 = 200$ and $s^2 = 400$. For $N = 100$, the influence of s^2 is noticed for $s^2 = 400$ and, for $N = 200$ the influence is not significant for the values of s^2 considered. In short, when sh is small enough, the influence of s^2 in the stability is negligible, which is coherent with the stability condition established for the one dimensional periodic case. Table 2 displays the ratio k_l/h as function of s^2 and N.

The error of the full discrete problem with H-W ABCs is made up of three types of errors. A first error depends on the accuracy of the spatial discretization, a second error is due to the time numerical integration, and a third error is caused by the incorporation of the ABCs. In the following experiments, we consider the time step $k = 10^{-4}$. In this way, the error coming from the time numerical integration is negligible compared to the other two errors.

In order to measure the error of absorption associated with the ABCs, for each numerical solution u^h of the test problem, a reference solution $u^{h,ref}$, in

a longer domain $[-1/4-5/2, 1/4+5/2]$ is generated. For the final time considered, no reflections of the reference solution reenter in the computational domain $[-1/4, 1/4]$. We use $Emax = ||u^h - u^{h,ref}||_\infty$ to measure the error in each time where the numerical solution is calculated.

We want to study the influence of the dispersion coefficient. In Figure 1

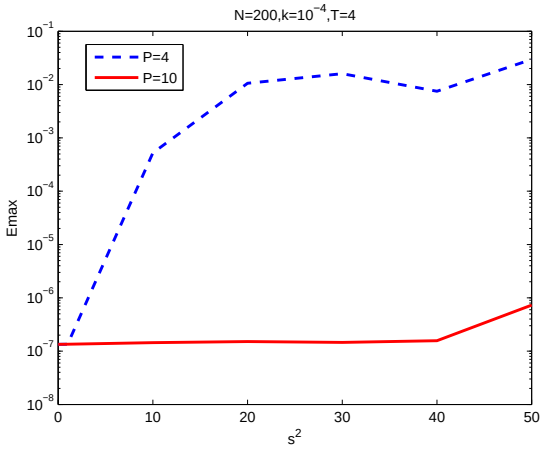

Figure 1: $Emax$ versus s^2 for $N = 200$, $k = 10^{-4}$, $T = 4$ for $P = 4$ and $P = 10$.

we have set $N = 200$ and we have displayed the maximum error in $T = 4$ versus s^2, for P equal to 4 and 10. Taking $P = 4$, it can be seen that the error increases when the dispersion does. This behavior is similar to the one observed in [14]. However, for $P = 10$, a better behavior of the error can be noted.

It seems that the ABCs are able to absorb the solution for large values of the dispersion coefficient if we use a big enough order of absorption. To corroborate this, we carry out numerical experiments with $s^2 = 20$. In Figure 2, we have shown, for $s^2 = 20$, $Emax$ versus t for $N = 50$, $N = 100$ and $N = 200$ for the values of $P = 4, 6, 8$. For fixed N, the error decreases when P increases until the level of accuracy of the discretization is achieved. If one wants reduce the error, one must refine the space discretization. Until about $t = 1.5$, the errors for $P = 4$, $P = 6$ and $P = 8$ are similar, but afterward, the error for $P = 4$ is larger. Similarly, until about $t = 2.5$ the errors for $P = 6$ and $P = 8$ are similar, but later the error for $P = 6$ is larger. For small times, a low order P is sufficient to achieve the level of accuracy of the spatial discretization. Whenever longer times are considered higher order P are needed to saturate the accuracy of the spatial discretization.

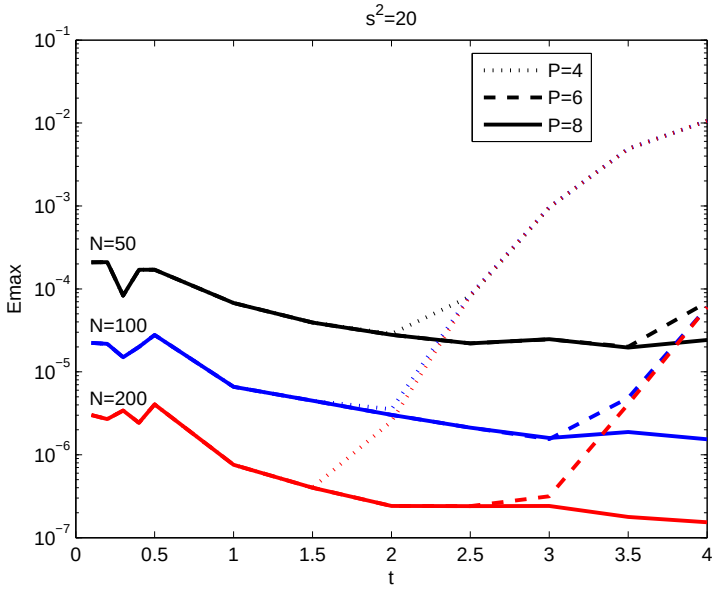

Figure 2: *Emax* versus t for $N = 50$, $N = 100$ and $N = 200$.

5.2. Two dimensional case

We consider the problem described in Section 4 with initial conditions

$$u_0(x,y) = \begin{cases} \dfrac{(x+0.2)^3(0.2-x)^3(y+0.2)^3(0.2-y)^3}{(0.2)^{12}}, & -0.2 < x, y < 0.2, \\ 0, & \text{otherwise}, \end{cases}$$

and $v_0(x,y) = 0$, with compact support contained in the computational domain $[-1/4, 1/4] \times [-1/4, 1/4]$. The polynomial in u_0 is chosen so that $u_0 \in C^1([-1/4, 1/4] \times [-1/4, 1/4])$.

We set $c = 1$ and final time $T = 4$. The level curves of the numerical solution u^h, for the step size h associated to $N = 200$, have been displayed in Figure 3, for $P = 40$, and the time values 0, 0.25, 0.3, 0.6, 1 and 4.

We have computed numerically the stability limit time step k_l in a similar way to the one made for the one dimensional case. In the considered domain we take $h = 1/2N$. From Table 3, it can be observed that, the influence of s^2 in the stability limit time step is not significant when sh is small enough, which is coherent with the stability condition established for the one dimensional periodic case. Table 4 displays the ratio k_l/h as function of s^2 and N.

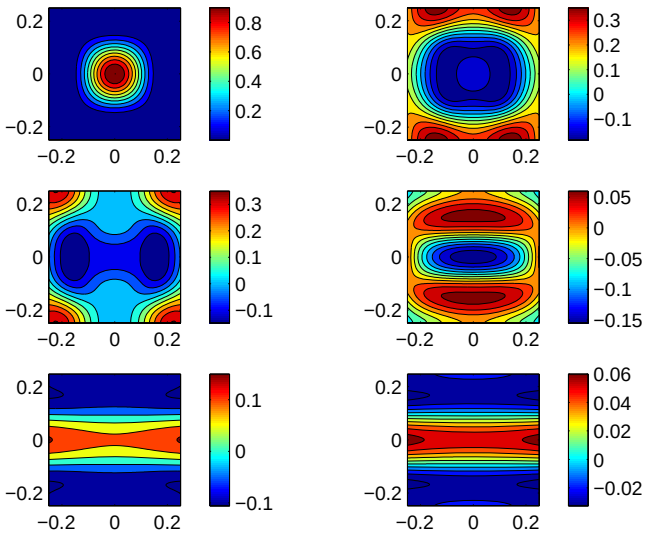

Figure 3: u^h for times 0, 0.25, 0.3, 0.6, 1 and 4.

Now, we study the efficiency of the splitting scheme by comparing with the fourth-order four-stage Runge-Kutta method. Since we take $N = M$, we deduce from Subsection 4.3 that the computational cost of one step in time for the algorithm described in Subsection 4.2 is $3(10N^2 + 32PN) + O(N) + O(P) = 30N^2 + 96PN + O(N) + O(P)$ products. For the Runge-Kutta the computational cost of the product $\mathcal{M}\mathbf{w}$ requires $10N^2+32PN+O(N)+O(P)$ products too. But for the splitting method this has to be done three times for each step, while for the four-stage Runge-Kutta method it is necessary do it four times for each step, $40N^2 + 128PN + O(N) + O(P)$.

Figure 4 displays the maximum error in $T = 4$ versus CPU time for the exponential splitting integrator and the fourth-order four-stage Runge-Kutta method, for $s^2 = 1$. A reference solution has been computed by using the fourth-order four-stage Runge-Kutta method with $N = 1600$, $k = 4/27200$ and $P = 50$. For both methods we have used the following values of N and k: $N = 50$ and $k = 4/850$, $N = 100$ and $k = 4/1700$ and $N = 200$, $k = 4/3400$. It can be seen that, for the same level of accuracy, the splitting method is less costly than the Runge-Kutta method.

We have ran both algorithms for several values of N and we have measured the computational cost in terms of CPU time. Table 5 shows the CPU ratio

	N=50	N=100	N=200
$s^2 = 0$	0.004821	0.002410	0.001204
$s^2 = 1$	0.004821	0.002410	0.001204
$s^2 = 20$	0.004820	0.002410	0.001204
$s^2 = 50$	0.004820	0.002408	0.001204
$s^2 = 100$	0.004818	0.002408	0.001204
$s^2 = 200$	0.004816	0.002408	0.001204
$s^2 = 400$	0.004812	0.002408	0.001204

Table 3: Stability limit time step k_l.

	N=50	N=100	N=200
$s^2 = 0$	0.4821	0.4820	0.4816
$s^2 = 1$	0.4821	0.4820	0.4816
$s^2 = 20$	0.4820	0.4820	0.4816
$s^2 = 50$	0.4820	0.4816	0.4816
$s^2 = 100$	0.4818	0.4816	0.4816
$s^2 = 200$	0.4816	0.4816	0.4816
$s^2 = 400$	0.4812	0.4816	0.4816

Table 4: Ratio stability limit k_l/h.

r for the Runge-Kutta method and the splitting method. It can be seen that the behavior for the splitting method is better than the expected from the analysis of the products required. Maybe, this can be due to the fact that the splitting method uses smaller sparse matrices, which permits faster location of non zero elements.

Finally, we are again interested on checking the error of absorption committed by the ABCs in a similar way to the one dimensional case. We take $k = 10^{-4}$ to achieve a time integration error negligible. The reflections are measured comparing the numerical solution u^h on the computational domain $[-1/4, 1/4] \times [-1/4, 1/4]$ with the numerical solution on a longer domain $[-1/4 - 5/2, 1/4 + 5/2] \times [-1/4, 1/4]$ for the same step size h. In the interval of times $[0, 4]$, there are not reflections of $u^{h,ref}$ on the computational domain. We denote by $Emax$ the maximum of the absolute value of the difference between u^h and $u^{h,ref}$ on the computational domain.

In the two dimensional case we are also interested in the influence of the

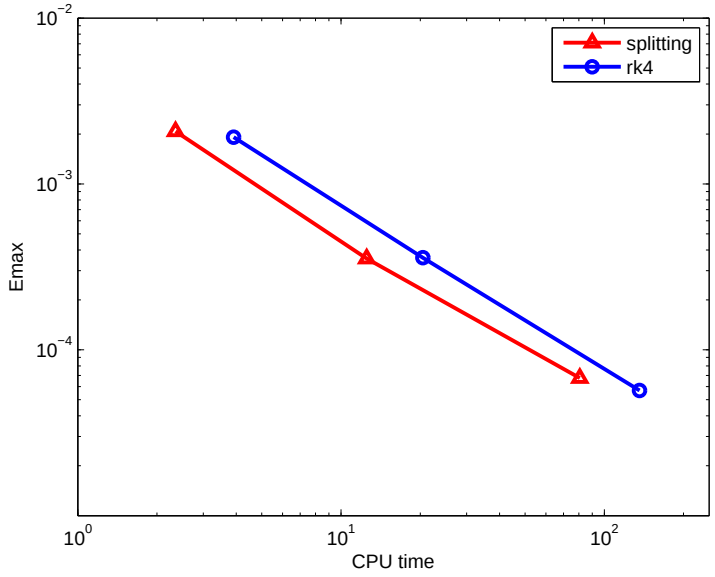

Figure 4: Emax versus CPU time for the exponential splitting integrator and the fourth-order four-stage Runge-Kutta method.

N	50	100	200	400	800	1600
r	1.6638	1.6334	1.6864	1.5192	1.6034	1.5239

Table 5: CPU ratio for the Runge-Kutta method and the splitting method.

dispersion coefficient. We have displayed in Figure 5 the maximum error in $T = 4$ versus s^2, for $N = 200$, and P equal to 10 and 40. It can be observed that there are almost no difference among distinct values of s^2. For $P = 10$, errors are large, but taking $P = 40$, errors decrease in a suitable way. If enough big order of absorption is used, the ABCs are able to absorb the solution also in the two dimensional case.

To study in more detail the evolution of the error versus the time, we have set $s^2 = 1$ and we have shown in Figure 6, the time evolution of $Emax$, for $N = 50$, $N = 100$, and $N = 200$, for the values of $P = 10$, $P = 20$ and $P = 40$. In this two dimensional case, the use of the exponential splitting method of Section 4 has done possible to reach very small errors due to the time integration (cf. [14, 15]). Then, the situation is similar to the one dimensional case. For fixed N, the error decreases when P increases until

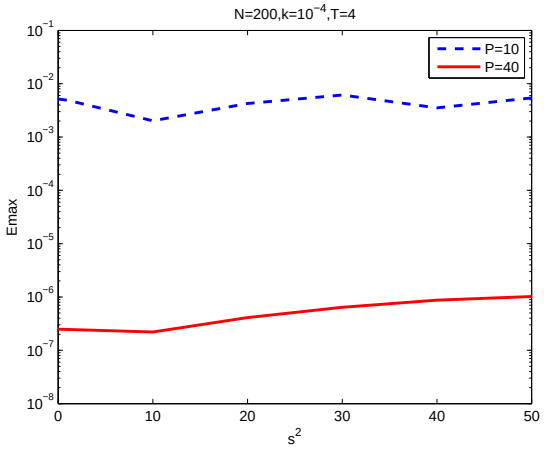

Figure 5: *Emax* versus s^2 for $N = 200$, $k = 10^{-4}$, $T = 4$ for $P = 10$ and $P = 40$.

the level of accuracy of the spatial discretization is achieved. However, higher orders of P are needed. Until about $t = 0.5$, the errors for $P = 10$, $P = 20$ and $P = 40$, almost coincide, but for bigger times, errors for $P = 10$ grow. During a time later, the errors for $P = 20$ and $P = 40$, are similar, whereas between $t = 1.5$ and $t = 2.5$, the errors for $P = 20$ increase. It seems that, in this interval of time, the level of accuracy of the spatial discretization is reached with $P = 40$.

6. Conclusions and future work

We have obtained a time integrator for the Klein-Gordon equation with Hagstrom-Warburton high-order absorbing boundary conditions based on exponential splitting techniques. This time scheme has good stability properties and is more efficient than other standard time integrators. Using this method we are able to study the influence of the dispersion coefficient.

As future work, we consider the study of the properties of this method as geometric integrator. For this, the incorporation of the optimal choice of the parameters of the ABCs for long time computation obtained in [17] is essential. Moreover, the use of exponential splitting time integrators for other equations could be considered.

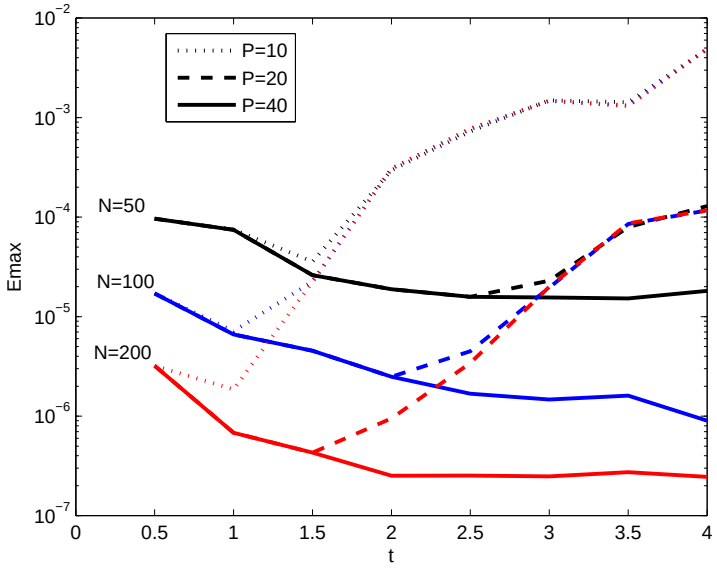

Figure 6: *Emax* versus time for $N = 50$, $N = 100$ and $N = 200$.

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