On the topological type of a set of plane valuations with symmetries *

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Abstract

Let $\{C_i : i = 1, \ldots, r\}$ be a set of irreducible plane curve singularities. For an action of a finite group G, let $\Delta^L(\{t_{ai}\})$ be the Alexander polynomial in $r|G|$ variables of the algebraic link $(\bigcup_{i=1}^{r} \bigcup_{a \in G} aC_i) \cap S^3_\varepsilon$ and let $\zeta(t_1, \ldots, t_r) = \Delta^L(t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_r, \ldots, t_r)$ with $|G|$ identical variables in each group. (If $r = 1$, $\zeta(t)$ is the monodromy zeta function of the function germ $\prod_{a \in G} a^* f$, where $f = 0$ is an equation defining the curve C_1.) We prove that $\zeta(t_1, \ldots, t_r)$ determines the topological type of the link L. We prove an analogous statement for plane divisorial valuations formulated in terms of the Poincaré series of a set of valuations.

1 Introduction

An equivariant (with respect to an action of a finite group G) version of the Poincaré series of a multi-index filtration (defined by a collection of valuations $\{\nu_i\}$, $i = 1, \ldots, r$) was defined in [4] as an element $P^G_{\{\nu_i\}}(t_1, \ldots, t_r)$ of the ring $\widetilde{A}(G)[[t_1, \ldots, t_r]]$ of power series with coefficients in a certain modification $\widetilde{A}(G)$ of the Burnside ring $A(G)$ of the group G. In [5] it was shown that, for a filtration on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of functions in two variables defined either by a collection of divisorial valuations or by a collection of curve valuations (in

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the latter case with certain exceptions), the series $P^G_{\{\nu_i\}}(t_1, \ldots, t_r)$ determines the (weak) equivariant topological type of the set of valuations. (In the case of curve valuations this means the equivariant embedded topological type of the curve or of the corresponding algebraic link.) (In [3] it was shown that even in the non-equivariant case (i. e., for $G = \{e\}$) the Poincaré series of a collection including both divisorial and curve valuations does not determine, in general, the topological type of the set of valuations.)

One has a natural homomorphism φ from the ring $\widetilde{A}(G)$ to the ring $\mathbb{Z}$ of integers which sends a finite G-set (with additional structure) to its number of elements. This homomorphism can be extended to a homomorphism from the ring $\widetilde{A}(G)[[t_1, \ldots, t_r]]$ to the ring $\mathbb{Z}[[t_1, \ldots, t_r]]$: φ is applied to the coefficients of the corresponding series.

Definition: The series $\zeta(t_1, \ldots, t_r) = \varphi(P^G_{\{\nu_i\}}(t_1, \ldots, t_r))$ is called the (equivariant) *zeta function* of the set of valuations $\{\nu_i\}$.

One has the following obvious statement, which shows that the zeta function $\zeta(t_1, \ldots, t_r)$ is determined in terms of the "usual" (non-equivariant) Poincaré series.

Proposition 1 *One has*

$$\zeta(t_1, \ldots, t_r) = P(t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_r, \ldots, t_r),$$

where the number of identical variables in each group is equal to $|G|$, $P(\{t_{ai}\})$ *is the usual Poincaré series (in* $r|G|$ *variables) of the collection of valuations* $\{a^*\nu_i\}$, $a \in G$, $i = 1, \ldots, r$.

Remark. We permit that the collection $\{a^*\nu_i\}$ contains equal valuations, i. e., $a_1^*\nu_i = a_2^*\nu_i$ for some $a_1 \neq a_2$. The usual definition of the Poincaré series of a collection of valuation holds in this situation as well. (In fact in this case the Poincaré series $P(\{t_{ai}\})$ is determined by the one for the collection where the repeated valuations are deleted.)

If ν_i, $i = 1, \ldots, r$, are the curve valuations corresponding to irreducible plane curve singularities $(C_i, 0) \subset (\mathbb{C}^2, 0)$, then $P(\{t_{ai}\})$ coincides with the Alexander polynomial $\Delta^L(\{t_{ai}\})$ of the algebraic link $L = \left(\bigcup_{i=1}^{r} \bigcup_{a \in G} aC_i\right) \cap S^3_\varepsilon$, where S^3_ε is the sphere of small radius ε centred at the origin in $\mathbb{C}^2$: [2]. Strictly speaking this holds if all the curves aC_i are different. If, in a collection of plane curve singularities $(Y_i, 0) \subset (\mathbb{C}^2, 0)$, $i = 1, \ldots, s$, two curves coincide (say, Y_{s-1} and Y_s and only they), the Alexander polynomial of the corresponding link $(\bigcup_{i=1}^{s} Y_i) \cap S^3_\varepsilon$ should be defined as $\Delta^{L'}(t_1, \ldots, t_{s-2}, t_{s-1} \cdot t_s)$, where $\Delta^{L'}(t_1, \ldots, t_{s-2}, t_{s-1})$ is the usual Alexander polynomial of the link $L' = \left(\bigcup_{i=1}^{s-1} Y_i\right) \cap S^3_\varepsilon$ with $(s-1)$ components. If $r = 1$ and the curve C_1 is defined by an equation $f_1 = 0$ ($f_1 \in \mathcal{O}_{\mathbb{C}^2,0}$), the series $P(t, \ldots, t)$ coincides with the monodromy zeta function of the germ $\prod\limits_{a\in G} a^* f_1$ (see, e. g., [1]).

For a collection $\{\nu_i\}$, $i = 1, \ldots, r$, consisting of divisorial and curve valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ one has the following A'Campo type formula for the Poincaré series: [2, 6]. Let $\pi : (X, \mathcal{D}) \to (\mathbb{C}^2, 0)$, $\mathcal{D} = \pi^{-1}(0)$, be a resolution of the collection $\{\nu_i\}$ of valuations. This means that π is a modification of the plane (by a finite number of blow-ups of points) such that all the divisors defining the divisorial valuations from the collection are present in the exceptional divisor $\mathcal{D}$ and the strict transforms of all the curves defining the curve valuations are smooth, do not intersect each other in X and are transversal to $\mathcal{D}$ (at its smooth points). All the components E_σ of the exceptional divisor $\mathcal{D}$ are isomorphic to the complex projective line. Let $\overset{\circ}{E}_\sigma$ be "the smooth part of E_σ in the resolution", that is E_σ itself minus the intersection points with other components of the exceptional divisor $\mathcal{D}$ and with the strict transforms of the curves defining the curve valuation. A *curvette* at the component E_σ is the image in $(\mathbb{C}^2, 0)$ of a smooth curve germ transversal to $\overset{\circ}{E}_\sigma$ at a point of it. Let $\varphi_\sigma = 0$ be an equation of a curvette at E_σ and let $m_{\sigma i} := \nu_i(\varphi_\sigma)$, $\underline{m}_\sigma := (m_{\sigma 1}, \ldots, m_{\sigma r}) \in \mathbb{Z}^r_{\geq 0}$.

Proposition 2 *One has*

$$P_{\{\nu_i\}}(\underline{t}) = \prod_\sigma (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}, \tag{1}$$

where $\underline{t} = (t_1, \ldots, t_r)$, $\underline{t}^{\underline{m}_\sigma} = t_1^{m_{\sigma 1}} \cdots t_r^{m_{\sigma r}}$ *and* $\chi(\cdot)$ *is the Euler characteristic.*

A formula for the Alexander polynomial $\Delta^L(t_1, \ldots, t_r)$ in several variables of an algebraic link L in terms of a resolution of the corresponding curve can be found in [7]. If all the valuations in the collection $\{\nu_i\}$ are curve ones, the equation (1) and the formula from [7] for the corresponding algebraic link L give the same results, i. e.,

$$P_{\{\nu_i\}}(t_1, \ldots, t_r) = \Delta^L(t_1, \ldots, t_r).$$

Here we show, in particular, that the "usual" (not equivariant) topological type of the curve $\bigcup\limits_{i=1}^{r} \bigcup\limits_{a\in G} aC_i$ is determined by the zeta function $\zeta(t_1, \ldots, t_r)$ (that is by the described reduction of the Alexander polynomial in $r|G|$ variables). We also prove an analogous to the statement above for a collection of plane divisorial valuations (with certain precisely described exceptions).

One can get the impression that the results here are somewhat weaker than those in [5] because from formal point of view they describe the usual (not equivariant) topology of the set of valuations (of the curve if all the valuations are curve ones). However, the difference here is not too big. In the setting of [5], i.e., if the action of the group is induced by its action (a representation) on $(\mathbb{C}^2, 0)$, the zeta function $\zeta(\ldots)$ permits to restore the (minimal) equivariant resolution graph of the set of valuations. The only object which is missed is the representation of the group on $\mathbb{C}^2$. This cannot be read from the zeta function $\zeta(\ldots)$ and for that in [5] one used the equivariant Poincaré series. (Even in that case this was possible not always, but with certain exceptions.) Thus, if the representation on $\mathbb{C}^2$ is given in advance, the outputs of the results of [5] and those here are essentially the same. On the other hand here the information is extracted from a considerably "smaller" invariant: a series with coefficients in $\mathbb{Z}$, not in the Burnside ring of the group. We believe that this makes the results considerably stronger. (This also makes the proofs somewhat more complicated.)

It is well known that the Alexander polynomial determines the topological type of an algebraic knot. Moreover, the topological type of an algebraic link $L = C \cap S^3_\varepsilon$ with r components ($(C, 0) \subset (\mathbb{C}^2, 0)$ is a plane curve singularity) is determined by its Alexander polynomial $\Delta^L(t_1, \ldots, t_r)$ in several variables (the number of variables being the number r of components of the link): [8]. On the other hand it is known that the Alexander polynomial in one variable (that is $\Delta^L(t, \ldots, t)$) does not determine the topological type of an algebraic link with at least two components (see, e. g., an example in [8]).

The statement above says, in particular, that, if the curve $(C, 0) \subset (\mathbb{C}^2, 0)$ defining the link L consists of the a-shifts of an irreducible plane curve singularity for all $a \in G$, its Alexander polynomial in one variable determines the topological type of the curve (or of the link). In this case all the components of the curve C are equisingular, that is have the same topological type. The attempt to understand whether it is really necessary to have a symmetry (defined by a group) between the branches of the curve, or it is sufficient that all the components of the curve are equisingular, led to the example of two algebraic links with unknotted components and with equal Alexander polynomials in one variable (see Section 5). In terms of the Singularity Theory this example can be interpreted in the following way. The link corresponding to a curve $\{f = 0\}$ consists of unknotted components if and only if f is the product of function germs without critical points (that is of germs right equivalent to a coordinate function). In this way the example gives two functions of this sort with equal monodromy zeta functions.

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2 Topology of divisorial valuations

In [5] we considered collections of curve (or divisorial) plane valuations consisting of the orbits of some of them under an action of a finite group G on the plane $(\mathbb{C}^2, 0)$. Here we consider a slightly more general setting which can be applied to some other situations.

Let $\pi : (X, \mathcal{D}) \to (\mathbb{C}^2, 0)$, $\mathcal{D} = \pi^{-1}(0)$, be a modification of the plane by a finite number of blow-ups of points. All the components E_σ of the exceptional divisor $\mathcal{D}$ are isomorphic to the complex projective line. The (dual) graph Γ of the modification is defined in the following way. Its vertices are in one-to-one correspondence with the components E_σ of the exceptional divisor $\mathcal{D}$. Two vertices are connected by an edge if and only if the corresponding components intersect. The graph Γ is a tree. Each vertex σ of Γ (that is a component E_σ of the exceptional divisor $\mathcal{D}$) has its age: the minimal number of blow-ups needed to create the component. The dual graph of a modification with the ages of the vertices determines the combinatorics of the modification.

A divisorial valuation on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of functions in two variables is defined by a component of the exceptional divisor of a modification. This modification is called a *resolution* of the (divisorial) valuation. A modification which is a resolution of each divisorial valuation from a (finite) collection $\{\nu_i\}$ is called a *resolution of the collection*.

Definition: Two collections $\{\nu_i\}$ and $\{\nu'_i\}$ of divisorial valuations are called *topologically equivalent* if they have isomorphic minimal resolution graphs Γ and Γ', i. e., if there exists an isomorphism between the abstract graphs Γ and Γ' preserving the ages and sending the vertices corresponding to the valuations ν_i to the vertices corresponding to the valuations ν'_i.

Assume that the graph Γ of a modification carries an action of a finite group G preserving the ages of the vertices. Let ν_i, $i = 1, \ldots, r$, be divisorial valuations corresponding to some vertices of Γ, and let $\nu_{ai} := a^*\nu_i$, $a \in G$, be the divisorial valuation defined by the a-shift of the corresponding vertex. An important example of this situation (treated in [5]) is the case when the group G acts (analytically) on $(\mathbb{C}^2, 0)$ and the modification π is G-equivariant. In fact, in the constructions below, the structure of the group G is not really important. We use only the order h_0 of the group G (this order is assumed to be known) and the orders of its subgroups.

Let $\check{\Gamma}$ be the quotient Γ/G of the modification graph Γ by the group action. It is a graph of a modification. (One can say that the modification π above is "an equivariant extension" of this one.) To avoid some difficulties (and/or ambiguities) in the descriptions and in the notations below, we shall usually choose an embedding of the graph $\check{\Gamma} = \Gamma/G$ into the graph Γ (as a "section" of the quotient map). This can be made in many ways, but we shall fix one embedding. This permits us to assume that all the vertices corresponding to the valuations ν_i lie in $\check{\Gamma}$. As above, for a vertex $\delta \in \Gamma$, let $\varphi_\delta = 0$, $\varphi_\delta \in \mathcal{O}_{\mathbb{C}^2,0}$, be an equation of a curvette at the component E_δ, $m_{\delta i} := \nu_i(\varphi_\delta)$. Let $M_{\delta i} := \sum_{a \in G} m_{(a\delta)i} = \sum_{a\in G}(a^*\nu_i)(\varphi_\delta)$ and $\underline{M}_\delta = (M_{\delta 1}, \ldots, M_{\delta r}) \in \mathbb{Z}^r_{\geq 0}$. The "multiplicities" $\underline{M}_\delta$ are the same for the vertices from one G-orbit. Therefore they depend only on the corresponding vertex in the quotient graph $\check{\Gamma}$. All the multiplicities $\underline{M}_\sigma$, $\sigma \in \check{\Gamma}$, are different and for $\sigma, \tau \in \Gamma$ one has $\underline{M}_\sigma = \underline{M}_\tau$ if and only if $\tau = a\sigma$ for some $a \in G$.

Let $P(\{t_{ai}\})$ be the Poincaré series of the collection $\{\nu_{ai}\}$ of $r|G|$ valuations and let

$$\zeta(t_1, \ldots, t_r) := P(t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_r, \ldots, t_r)$$

with $|G|$ identical variables in each group (recall that we permit that the collection $\{\nu_{ai}\}$ contains repeated valuations). We shall assume that either the number of edges at the initial vertex of the (minimal) modification graph (that is of the only vertex with the age 1) is different from 2, or it is equal to 2, but these two edges are not interchanged by the group action.

Let us show that the last condition is necessary, i.e. if it is not satisfied, the zeta function $\zeta(\cdot)$ does not determine, in general, the topological type of a set of divisorial valuations.

Example. Let us consider two modification graphs shown on Figure 1 with the obvious (non-trivial) actions of the group $\mathbb{Z}_2$ with 2 elements. The numbers at the vertices are the ages, the divisors defining the valuations under consideration are marked by the circles. In the both cases one has $\zeta(t) = (1-t^5)^{-2}$ (see

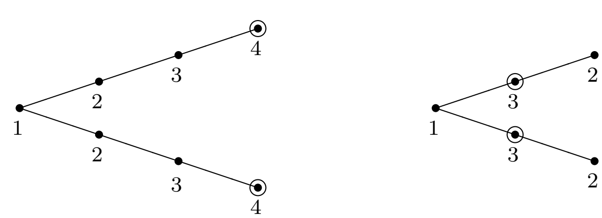


Figure 1: The modification graphs defining the divisorial valuations.

equation (2) below). Thus in these cases one cannot determine the topological type of the set of valuations from the series $\zeta(t)$.

The main feature of this example is the fact that, for the initial vertex σ_0 of the modification graph (the only vertex with the age equal to 1), the Euler characteristic $\chi(\overset{\circ}{E}_{\sigma_0})$ is equal to zero. Therefore, in the A'Campo type formula, the binomial $(1 - t^{M_{\sigma_0}})$ is absent (it is with the zero exponent) and one cannot determine the multiplicity M_{σ_0} from the zeta function $\zeta(t)$. This problem does not appear in the considerations in [5] since the equivariant Euler characteristic of $\overset{\circ}{E}_{\sigma_0}$ (with values in the Burnside ring of the group G) is equal to $2[G/G] - [G/H]$ where H is a subgroup of G of index 2.

Theorem 1 *Assume that, either the number of edges at the initial vertex of the (minimal) modification graph is different from 2, or it is equal to 2, but these two edges are not interchanged by the group action. Then, the zeta function* $\zeta(t_1, \ldots, t_r)$ *determines the topological type of the collection* $\{\nu_{ai}\}$ *of divisorial valuations.*

Proof. The statement follows from Propositions 3 and 4 below.

Let $\nu = \nu_i$ be one of the valuations under consideration. Without loss of generality we can assume that $i = 1$. The zeta function $\zeta_\nu(t)$ corresponding to this valuation is determined from $\zeta(t_1, \ldots, t_r)$ by the following "projection formula":

$$\zeta_\nu(t) = \zeta(t, 1, \ldots, 1).$$

Proposition 3 *Under the described conditions, the minimal resolution graph of the collection of valuations* $\{a^*\nu\}$, $a \in G$, *is determined by the zeta function* $\zeta(t_1, \ldots, t_r)$.

Proof. We shall show that the minimal resolution graph is essentially determined by the zeta function $\zeta_\nu(t)$. However, in a certain situation we shall look back at $\zeta(t_1, \ldots, t_r)$. The dual graph Γ of the minimal resolution of the divisorial valuations $\{a^*\nu\}$ is shown in Figure 2. The quotient Γ/G of this graph by the action of the group G is the minimal resolution graph $\check{\Gamma}$ of the valuation ν shown in Figure 3. Here σ_q, $q = 0, 1, \ldots, g$, are called the *dead ends* of the graph, τ_q, $q = 1, \ldots, g$, are called the *rupture points* and g is the number of the Puiseux pairs of a curvette corresponding to the divisor defining ν. The graph Γ can be obtained from the graph $\check{\Gamma}$ by the following construction. There are several vertices ρ_1, ..., ρ_ℓ in the graph $\check{\Gamma}$ lying on the geodesic from σ_0 (the only vertex with the age 1) to ν, $\rho_1 < \rho_2 < \ldots < \rho_\ell$, and some numbers $h_0 = |G| > h_1 > \ldots > h_\ell$ such that $h_{i+1}|h_i$. The vertices ρ_j (we shall call them the *splitting points*) are the images under the quotient map of the points in Γ in whose neighbourhoods the quotient map is not an isomorphism. The number h_j is the order of the isotropy subgroup for the vertices inbetween ρ_{j-1}

and ρ_j (ρ_{j-1} excluded and ρ_j included). (Not all sequences $\rho_1 < \rho_2 < \ldots < \rho_\ell$ are permitted.) To get the graph Γ from the graph $\check{\Gamma}$ one takes $|G|$ copies of the latter one. The parts of all of them preceding ρ_1 (ρ_1 included) are identified. The remaining parts preceding ρ_2 (ρ_2 included) are identified in groups containing h_0/h_1 copies each. The remaining parts preceding ρ_3 (ρ_3 included) are identified in groups containing h_0/h_2 copies each, etc. Notice that h_0/h_ℓ is the cardinality of the orbit of ν.

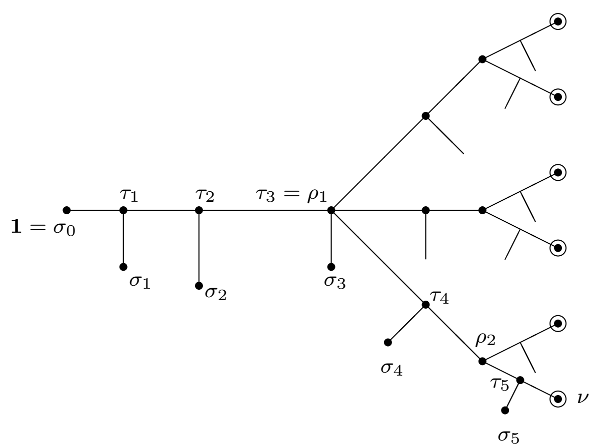


Figure 2: The resolution graph Γ of the valuations $\{a^*\nu\}$.

As above, we shall choose an embedding of the graph $\check{\Gamma} = \Gamma/G$ into the graph Γ (as a "section" of the quotient map). (The number of embeddings is equal to h_0/h_ℓ, we shall assume that one embedding is fixed.) This embedding is shown in Figure 2 by indicating the same names for some vertices in Γ and in $\check{\Gamma}$.

Also as above, for a vertex $\delta \in \check{\Gamma}$, let $\varphi_\delta = 0$, $\varphi_\delta \in \mathcal{O}_{\mathbb{C}^2,0}$, be an equation of a curvette at the component E_δ, $m_\delta = \nu(\varphi_\delta)$. (We use the same notations $m_\bullet$ as for the corresponding multiplicities in the graph Γ, since for the vertices in the image of the graph $\check{\Gamma}$ they coincide.) It is known that the numbers m_{σ_0}, $m_{\sigma_1}, \ldots, m_{\sigma_g}$ form the minimal set of generators of the semigroup of values

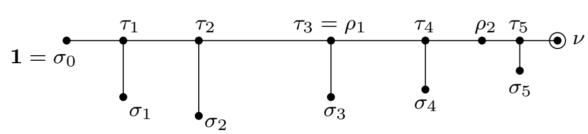


Figure 3: The resolution graph $\check{\Gamma} = \Gamma/G$ of the valuation ν.

of the valuation ν, m_{τ_i} is a multiple of m_{σ_i}, $i = 1, \ldots, g$: $m_{\tau_i} = N_i m_{\sigma_i}$, and $m_{\sigma_0} = N_1 \cdots N_g$.

For a vertex $\delta \in \Gamma$, let $M_\delta := \sum_{a \in G} m_{(a\delta)} = \sum_{a \in G} (a^*\nu)(\varphi_\delta)$. The multiplicities M_δ are the same for the vertices from one G-orbit. Therefore they depend only on the corresponding vertex in the quotient graph $\check{\Gamma}$. All the multiplicities M_σ, $\sigma \in \check{\Gamma}$, are different and for $\sigma, \tau \in \Gamma$ one has $M_\sigma = M_\tau$ if and only if $\tau = a\sigma$ for some $a \in G$.

The A'Campo type formula for the Poincaré series of a collection of plane divisorial valuations (see (1)) implies that the zeta function $\zeta_\nu(t)$ is given by the equation

$$\begin{aligned} \zeta_\nu(t) &= \prod_{q=0}^{g} \left(1 - t^{M_{\sigma_q}}\right)^{-h_0/h_{j(q)}} \cdot \prod_{q=1}^{g} \left(1 - t^{M_{\tau_q}}\right)^{h_0/h_{j(q)}} \cdot \\ &\cdot \prod_{j=1}^{\ell} \left(1 - t^{M_{\rho_j}}\right)^{(h_0/h_j)-(h_0/h_{j-1})} \cdot \left(1 - t^{M_\nu}\right)^{-h_0/h_\ell}, \end{aligned} \tag{2}$$

where $h_{j(q)}$ is the order of the isotropy subgroup of the vertex σ_q, $M_{\tau_q} = N_q M_{\sigma_q}$.

Any series in t with integer coefficients and with the initial term 1 can be in a unique way written as the product $\prod\limits_{m \geq 1} (1 - t^m)^{s(m)}$ with $s(m) \in \mathbb{Z}$ (in general an infinite one). The same statement holds for series in several variables. The equation (2) implies that the product in the right hand side of the equation

$$\zeta_\nu(t) = \prod_{m \geq 1} (1 - t^m)^{s(m)} \tag{3}$$

is finite. In (2) some exponents of t in the binomials may coincide. This happens if and only if the corresponding vertices coincide. Namely, ρ_1 may coincide with σ_0, each ρ_i, $i \geq 1$, may coincide with a certain τ_q, and, finally, ν may coincide with τ_g. In the first and in the latter cases this can lead to the situation when the corresponding binomial "is not seen" in the decomposition (3) because the corresponding exponents cancel. Moreover, if $\rho_1 = \sigma_0$, the corresponding factors cancel if and only if $h_0/h_1 = 2$.

Let us take the minimal m such that the binomial $(1 - t^m)$ appears in the decomposition (3) (i. e., $s(m) \neq 0$). If $s(m) = -1$, one has $m = M_{\sigma_0}$ (and $\rho_1 \neq \sigma_0$). If $s(m) < -1$, then $s(m) = -2$, $\rho_1 = \sigma_0$, $h_0/h_1 = 2$, and either $m = M_{\sigma_1}$, or $g = 0$, $\ell = 1$ and $m = M_\nu$. (The latter option is a degenerate one and is analogous to the one shown on the left hand side of Figure 1.) If $s(m) > 0$, one has $\rho_1 = \sigma_0$. There are two options. Either $h_0/h_1 = 2$ and the binomial $\left(1 - t^{M_{\sigma_0}}\right)$ does not appear in the decomposition (3), or $h_0/h_1 > 2$, $m = M_{\sigma_0}$, $s(m) = (h_0/h_1) - 2$.

If $\rho_1 = \sigma_0$ and the binomial $\left(1 - t^{M_{\sigma_0}}\right)$ does not appear in the decomposition (3), the binomial of the form $\left(1 - t_1^{M_{\sigma_0}} t_2^{k_2} \cdots t_r^{k_r}\right)$ is present in the corresponding decomposition of $\zeta(t_1, t_2, \ldots, t_r)$ (due to the conditions imposed on the resolution graph before Theorem 1: in this case there are more than two edges of the graph Γ at the vertex σ_0) and the corresponding multi-exponent $(M_{\sigma_0}, k_2, \cdots, k_r)$ is the minimal one in the decomposition. Therefore M_{σ_0} can be determined in this case as well.

Now let us find all the exponents M_{σ_i}, M_{τ_i} for $i \geq 1$ (maybe except M_{τ_g}), M_{ρ_j}, $j = 1, \ldots, \ell$, and the ratios h_{j-1}/h_j. Let us take all the binomials $(1-t^m)$ with $m > M_{\sigma_0}$ in the decomposition (3) with negative exponents $s(m)$. All of them except possibly the biggest one are the multiplicities M_{σ_i}. (The biggest one may coincide with M_ν.) Let $M_{\sigma_1} < M_{\sigma_2} < \ldots < M_{\sigma_p}$ be these exponents in the increasing order. One has either $p = g$ (in this case $\nu = \tau_g$) or $p = g+1$ (and $\sigma_p = \nu$). Moreover, one has $s(M_{\sigma_p}) = h_0/h_\ell$ (the number of different valuations in the orbit $\{a^*\nu\}$ of the valuation ν).

Let us take the exponents m in (3) such that $M_{\sigma_0} < m < M_{\sigma_1}$ with non-zero (and thus positive) $s(m)$. All these exponents correspond to the splitting points (up to $\rho_{j(1)}$). This gives the values M_{ρ_j} for these j. Moreover, one has $s(M_{\rho_j}) = (h_0/h_j) - (h_0/h_{j-1})$. This gives all the ratios h_0/h_j for all $j \leq j(1)$.

Assume that we have detected all M_{ρ_j} ($j \leq j(q)$) and M_{τ_i} ($i < q$) smaller than M_{σ_q} and all the ratios h_0/h_j for $j \leq j(q)$. Let us take all the exponents m in (3) such that $M_{\sigma_q} < m < M_{\sigma_{q+1}}$ with non-zero (and thus positive) $s(m)$. The smallest among them is M_{τ_q}. The vertex τ_q can either be the splitting point $\rho_{j(q)+1}$ or not. The vertex τ_q is the splitting point $\rho_{j(q)+1}$ if and only if $s(M_{\tau_q}) > -s(M_{\sigma_q})$. In this case $s(M_{\tau_q}) = 2(h_0/h_{j(q)}) - (h_0/h_{j(q)+1})$. This equation gives $h_0/h_{j(q)+1}$. All the remaining exponents m inbetween M_{τ_q} and $M_{\sigma_{q+1}}$ (with $s(m)$ positive) correspond to the splitting points ρ_j (with j up to $j(q+1)$). As above the ratio h_0/h_j is determined by $s(M_{\rho_j}) = (h_0/h_j) - (h_0/h_{j-1})$.

For $1 \leq q < p$ one has M_{τ_q} is a multiple of M_{σ_q} and moreover $M_{\tau_q}/M_{\sigma_q} = N_q$. Now we compute the multiplicities m_{σ_q} for $0 \leq q \leq p$ and m_{ρ_j} for $1 \leq j \leq \ell$ (and finally determine m_ν).

For $\sigma_q \leq \rho_1$ one has $M_{\sigma_q} = h_0 m_{\sigma_q}$ and $M_{\rho_1} = h_0 m_{\rho_1}$. These equations give all the generators m_{σ_q} of the semigroup of values of the valuation with $\sigma_q \leq \rho_1$ and also m_{ρ_1}.

For $j \geq 1$, let $\sigma_{q(j)}$ be the minimal dead end greater than ρ_j (i. e. there are the dead ends $\sigma_{q(j)}$, …, $\sigma_{q(j+1)-1}$ inbetween ρ_j and ρ_{j+1}). Let us consider the dead ends σ_q such that $\rho_1 < \sigma_q < \rho_2$. One has

$$M_{\sigma_{q(1)}} = h_1 m_{\sigma_{q(1)}} + (h_0 - h_1) m_{\rho_1} = h_1 m_{\sigma_{q(1)}} + (M_{\rho_1} - h_1 m_{\rho_1})\,.$$

For $\rho_1 < \sigma_{q(1)} < \sigma_{q(1)+1} < \sigma_{q(1)+2} < \cdots < \sigma_{q(2)-1} < \rho_2$, one has

$$\begin{array}{rcl} M_{\sigma_{q(1)+1}} & = & h_1 m_{\sigma_{q(1)+1}} + (M_{\rho_1} - h_1 m_{\rho_1}) N_{q(1)} \,, \\ M_{\sigma_{q(1)+2}} & = & h_1 m_{\sigma_{q(1)+2}} + (M_{\rho_1} - h_1 m_{\rho_1}) N_{q(1)} N_{q(1)+1} \,, \\ & & \cdots \\ M_{\sigma_{q(2)-1}} & = & h_1 m_{\sigma_{q(2)-1}} + (M_{\rho_1} - h_1 m_{\rho_1}) N_{q(1)} N_{q(1)+1} \cdot \ldots \cdot N_{q(2)-2} \\ M_{\rho_2} & = & h_1 m_{\rho_2} + (M_{\rho_1} - h_1 m_{\rho_1}) N_{q(1)} N_{q(1)+1} \cdot \ldots \cdot N_{q(2)-1} \,. \end{array}$$

These equations give us the numbers m_{σ_q} with $\sigma_q < \rho_2$ and also m_{ρ_2}.

Assume that we have determined all the numbers m_{σ_q} for $q < q(j)$ and also the number m_{ρ_j}. Let us consider the dead ends σ_q such that $\rho_j < \sigma_q < \rho_{j+1}$. One has

$$\begin{array}{rcl} M_{\sigma_{q(j)}} & = & h_j m_{\sigma_{q(j)}} + (M_{\rho_j} - h_j m_{\rho_j}) \,, \\ M_{\sigma_{q(j)+1}} & = & h_j m_{\sigma_{q(j)+1}} + (M_{\rho_j} - h_j m_{\rho_j}) N_{q(j)} \,, \\ M_{\sigma_{q(j)+2}} & = & h_j m_{\sigma_{q(j)+2}} + (M_{\rho_j} - h_j m_{\rho_j}) N_{q(j)} N_{q(j)+1} \,, \\ & & \cdots \\ M_{\sigma_{q(j+1)-1}} & = & h_j m_{\sigma_{q(j+1)-1}} + (M_{\rho_j} - h_j m_{\rho_j}) N_{q(j)} N_{q(j)+1} \cdots N_{q(j+1)-2} \,, \\ M_{\rho_{j+1}} & = & h_j m_{\rho_j} + (M_{\rho_j} - h_j m_{\rho_j}) N_{q(j)} N_{q(j)+1} \cdot \ldots \cdot N_{q(j+1)-1} \,. \end{array}$$

These equations give all the numbers m_{σ_q} with $q < q(j+1)$ and also $m_{\rho_{j+1}}$.

This procedure gives us the numbers m_{σ_q} for all $q \leq p$. If $\gcd(m_{\sigma_0}, m_{\sigma_1}, \ldots, m_{\sigma_{p-1}}) = 1$, then $p = g+1$, $\sigma_p = \nu$. Otherwise $p = g$, $\nu = \tau_g$. In this way one determines all the numbers m_{σ_q} for $0 \leq q \leq g$ (i.e. the generators of the semigroup of values of the valuation ν); m_{ρ_j} and h_j for $1 \leq j \leq \ell$ and m_ν. This gives us the minimal resolution graph of the set of valuations $\{a^*\nu | a \in G\}$. $\square$

To complete the proof of Theorem 1 one has to show that the zeta function $\zeta(t_1, \ldots, t_r)$ determines the (minimal) resolution graph of the collection of valuations $\{a^*\nu_i | i = 1 \ldots, r; a \in G\}$. For that one has to show that this zeta function determines the separation points between the valuations.

Proposition 4 *For any pair (i,j), $1 \leq i, j \leq r$, the zeta function $\zeta(t_1, \ldots, t_r)$ determines the separation point δ_{ij} between the valuations ν_i and ν_j in the graph $\check{\Gamma} = \Gamma/G$.*

Proof. For convenience let us assume that $i = 1$ and $j = 2$. Let

$$\zeta(t_1, t_2, 1 \ldots, 1) = \prod (1 - t_1^{M_1} t_2^{M_2})^{s(M_1, M_2)}, \tag{4}$$

$s(M_1, M_2) \in \mathbb{Z}$, be the decomposition of the zeta function $\zeta(t_1, t_2, 1, \ldots, 1)$ into the product of binomials. The separation point δ_{12} corresponds to the maximal exponent present in the decomposition (4) (i. e., such that $s(M_1, M_2) \neq 0$) with

$$\frac{M_2}{M_1} = \frac{M_{\sigma_0 2}}{M_{\sigma_0 1}}.$$

(If there is no such (M_1, M_2), one has $\delta_{12} = \sigma_0$.) To reduce the situation to the standard statements about topology of curves one has to determine the multiplicities m_1 and m_2 of the separation point in the graph $\check{\Gamma} = \Gamma/G$. Let δ' (respectively δ'') be the vertex of the minimal resolution graph Γ_1 (respectively Γ_2) of the valuation ν_1 (respectively ν_2) such that $M_{\delta'} = M_1$ in Γ_1 ($M_{\delta''} = M_2$ in Γ_2). Such vertex δ' (respectively δ'') either does not exist or is a unique one. Moreover, either δ', or δ'' (or both) are present in the graphs Γ_1 and Γ_2 respectively. If exactly one of δ' and δ'' exists, without loss of generality we can assume that this is δ'. If both of them exist, we can assume (again without loss of generality) that the age of δ' is smaller or equal to the age of δ''. One can see that δ' is the separation point in the resolution graph $\Gamma_{\{12\}}$ of the pair $\{\nu_1, \nu_2\}$. The corresponding multiplicity $m_{\delta'}$ in Γ_1/G can be found in the same way as m_{σ_i}, m_{τ_i} and m_{ρ_j} above. The multiplicity $m_{\delta''}$ of the separation point in the graph $\Gamma_{\{12\}}$ corresponding to the valuation ν_2 is determined from the equality $m_{\delta''}/m_{\delta'} = M_2/M_1$. $\square$

This completes the proof of Theorem 1. $\square$

3 Topology of curve valuations

Here we shall discuss collections of curve valuations on $\mathcal{O}_{\mathbb{C}^2,0}$. Let $(C_i, 0) \subset (\mathbb{C}^2, 0)$, $i = 1, \ldots, s$, be irreducible plane curve singularities and let ν_i be the (curve) valuation on $\mathcal{O}_{\mathbb{C}^2,0}$ defined by the branch C_i. Let $\pi : (X, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be an embedded resolution of the curve $C = \bigcup_{i=1}^{s} C_i$. The exceptional divisor $\mathcal{D}$ is the union of its irreducible components E_σ. For $i = 1, \ldots, s$, let E_{α_i} be the component of $\mathcal{D}$ intersecting the strict transform of the branch C_i. The dual graph Γ of the resolution π is the graph whose vertices correspond to the components E_σ of the exceptional divisor $\mathcal{D}$ and to the components C_i of the curve C; the latter ones are depicted by arrows. Two vertices are connected by an edge if and only if the corresponding components intersect. Pay attention that several arrows may be connected with one and the same vertex. Each vertex σ (corresponding to the divisor E_σ) has its age.

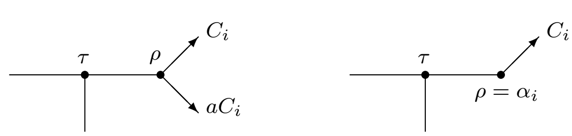
As in Section 2, let us assume that the graph Γ carries an action of a finite group G preserving the ages of the vertices. In particular, this means that the group acts on the set of arrows, that is on the components of C_i of the curve for $i = 1, \ldots, s$. Assume that the components $C_1, \ldots, C_r$ ($r < s$) are representatives of all the orbits of the G-action on the curve components. The component obtained from C_i by the a-shift ($a \in G$) will be denoted by aC_i or by C_{ai} and the corresponding curve valuation will be denoted by $a^*\nu_i$ or by ν_{ai}.

Let $P(\{t_{ai}\})$ be the Poincaré series of the collection $\{\nu_{ai}\}$ of the $r|G|$ curve valuations (defined by the components of the curve C) and let $\zeta(t_1, \ldots, t_r) := P(t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_r, \ldots, t_r)$ (with $|G|$ identical variables in each group) be the corresponding (equivariant) zeta function. (As in Section 2 we permit that the collection $\{\nu_{ai}\}$ contains equal valuations, i. e., that $C_{a_1 i} = C_{a_2 i}$ for some $a_1 \neq a_2$.)

Theorem 2 *The zeta function $\zeta(t_1, \ldots, t_r)$ determines the topological type of the curve $C = \bigcup_{a \in G} \bigcup_{i=1}^{r} C_{ai}$.*

Pay attention that we do not impose additional conditions like in the divisorial case in Section 2.

Proof. As in the proof of Theorem 1, we have to show that the zeta function $\zeta(t_1, \ldots, t_r)$ determines the minimal resolution graph Γ of the curve C. The graph Γ looks essentially like the resolution graph of the divisorial valuations corresponding to the vertices $a\alpha_i$ ($i = 1, \ldots, r$, $a \in G$) (see Figure 2 for $r = 1$) with several (possibly one) arrows attached to each vertex $a\alpha_i$. Let $\check{\Gamma} = \Gamma/G$ be the quotient of the graph Γ by the G-action (like in Figure 3 for $r = 1$); some arrows have to be added. In general (if among the arrows attached to one vertex $a\alpha_i$ one has different representatives of one G-orbit), the graph $\check{\Gamma}$ is not the minimal resolution graph of a curve, but may be somewhat enlarged (see Figure 4; in the minimal resolution graph the arrow C_i is attached to the vertex τ and the "tail" between τ and ρ does not exist).



(a) Γ (b) $\check{\Gamma}$

Figure 4: Parts of the graphs Γ and $\check{\Gamma} = \Gamma/G$.

As above we fix an embedding of the graph $\check{\Gamma}$ into the graph Γ. (This embedding determines, in particular, the choice of representatives C_i, $i = 1, \ldots, r$, from the G-orbits by $C_j \in \check{\Gamma}$.)

Let m_{δ_i}, M_{δ_i} and $\underline{M}_\delta$ be defined as in Section 2. Let k_i be the order of the isotropy group of the branch C_i, $i = 1, \ldots, r$. One can see that, for $i, j \in \{1, \ldots, r\}$, $m_{\alpha_i j}$ is just the intersection multiplicity between the curves C_i and C_j and

$$M_{\alpha_i j} = \sum_{a\in G} m_{(a\alpha_i)j} = \sum_{a\in G} (a^*\nu_j)(\varphi_{\alpha_i}) = (C_i, \sum_{a\in G} aC_j) = (C_j, \sum_{a\in G} aC_i) = M_{\alpha_j i}\,.$$

In the case of curve valuations the "projection formula" is different from that for divisorial valuations. Namely, for $i_0 \in \{1, \ldots, r\}$ one has:

$$\zeta_{\{\nu_i\}}(\underline{t})_{|t_{i_0}=1} = (1 - \underline{t}^{\underline{M}_{\alpha_{i_0}}})^{|G|/k_{i_0}}_{|t_{i_0}=1} \cdot \zeta_{\{\nu_i\}_{i\neq i_0}}(t_1, \ldots, t_{i_0-1}, t_{i_0+1}, \ldots, t_r)\,. \qquad (5)$$

(This can be easily deduced from the A'Campo type formula for the Poincaré series of a collection of curve valuations.) Using (5) several times for the indices $i \neq i_0$, one gets:

$$\zeta_{\{\nu_i\}}(\underline{t})_{|t_i=1, i\neq i_0} = \zeta_{\nu_{i_0}}(t_{i_0}) \cdot \prod_{i\neq i_0} (1 - t_{i_0}^{M_{\alpha_i i_0}})^{|G|/k_i}\,. \qquad (6)$$

Using the fact that $M_{\alpha_i i_0} = M_{\alpha_{i_0} i}$, one sees that all the multiplicities $M_{\alpha_i i_0}$, $i \neq i_0$, are contained in $\underline{M}_{\alpha_{i_0}}$ as the components of it.

Using the induction on the number r of the curve orbits, equations (5) and (6) mean that in order to describe the (minimal) resolution graph Γ of the curve C, we have:

1) To describe the minimal resolution graph of the curve $\bigcup\limits_{a\in G} aC_{i_0}$ (i. e., for $r = 1$) through the zeta function $\zeta(t) = \zeta_{\nu_{i_0}}(t)$ (i.e to adapt the corresponding description for one divisorial valuation in the proof of Theorem 1 to the curves case). This will be proved in Proposition 5

2) To detect the binomial $(1-\underline{t}^{\underline{M}_{\alpha_{i_0}}})$ and the number $|G|/k_{i_0}$ corresponding to (at least) one index $i_0 \in \{1, \ldots, r\}$ (see Proposition 6 below). Using (5) this permits to get the series zeta for the remaining $r - 1$ valuations $\{\nu_i\}_{i\neq i_0}$. The induction assumes that the minimal resolution graph of the collection $\{\nu_i\}_{i\neq i_0}$ is determined by the zeta function $\zeta(\ldots)$. This gives, in particular, all the numbers $|G|/k_i$ for $i \neq i_0$. Using (6) one gets the zeta function and thus the minimal resolution graph for the valuation ν_{i_0}.

3) To determine the separation point of the curves C_{i_0} and C_i ($i \neq i_0$) in the minimal resolution graph of these two curves.

Proposition 5 *The zeta function $\zeta(t) =$ of the curve valuation ν corresponding to an irreducible curve C determines the minimal resolution graph of the curve $\bigcup\limits_{a\in G} aC$.*

Proof. Let us recall that $\zeta(t) = P_{\{a\nu_1\}}(t, \ldots, t)$ with $|G|$ identical variables in the right hand side, where $P_{\{a\nu\}}(t_1, \ldots, t_{|G|})$ is the Poincaré series of the set $\{a\nu\}$ of curve valuations coinciding with the Alexander polynomial of the curve $\bigcup\limits_{a \in G} aC$. It is possible that $a\nu = a'\nu$ for $a \neq a'$.

Mostly the way to determine the minimal resolution graph from the zeta function $\zeta(t)$ repeats the one for the divisorial case. There is only one essential difference. In the divisorial case we had to assume that the multiplicity M_{σ_0} can be determined at the first step of the consideration. This led to the exceptions which had to be excluded. The knowledge of M_{σ_0} permitted us to determine m_{σ_i}, m_{τ_i} for $i \geq 1$, m_{ρ_j}, ... In particular this permitted us to determine m_{τ_g} (even if the corresponding binomial was absent in the decomposition of $\zeta(t)$). However, if, by a chance, one knows M_{τ_g} (in the divisorial case this means that $\tau_g \neq \nu$), one can recover M_{σ_0} from the values M_{σ_i} and M_{τ_i} for $i = 1, \ldots, g$. This happens in the curve case since the binomial $(1 - t^{M_{\tau_g}})$ in the decomposition of $\zeta(t)$ is always present with a positive exponent.

The procedure described in the proof of Theorem 1 permits us to determine all M_{σ_i}, M_{τ_i} for $i = 1, \ldots, g$ (including M_{τ_g}!) and also M_{ρ_j} and h_0/h_j for all j except possibly the first one if $\rho_1 = \sigma_0$ and $h_0/h_1 = 2$. We have $M_{\tau_i} = N_i M_{\sigma_i}$, $i = 1, \ldots, g$. It is known that $m_{\sigma_0} = N_1 \cdots N_g$ and therefore $M_{\sigma_0} = h_0 m_{\sigma_0} = h_0 N_1 \cdots N_g$. When M_{σ_0} is known, all the multiplicities m_{σ_i}, m_{τ_i} and m_{ρ_j} can be determined in the same way as in Section 2. This gives the minimal resolution graph for one curve valuation. $\square$

Proposition 6 *The zeta function $\zeta(t_1, \ldots, t_r)$ permits to detect the binomial $(1 - \underline{t}^{\underline{M}_{\alpha_{i_0}}})$ corresponding to an index $i_0 \in \{1, \ldots, r\}$ and the corresponding number $|G|/k_{i_0}$.*

Proof. Let us assume $r > 1$ and let us fix $j, k \in \{1, \ldots, r\}$. The separation point $s(\alpha_j, \alpha_k) \in \Gamma$ of α_j and α_k is defined by the condition $[\sigma_0, \alpha_j] \cap [\sigma_0, \alpha_k] = [\sigma_0, s(\alpha_j, \alpha_k)]$. Here $[\sigma_0, \sigma]$ is the geodesic in the graph Γ joining the first vertex σ_0 with the vertex σ. Let us recall that the graph $\check{\Gamma}$ is embedded into the graph Γ and $\alpha_i \in \check{\Gamma}$ for all i. This implies that $s(j,k) := s(\alpha_j, \alpha_k) \in \check{\Gamma}$ (and $s(\alpha_j, \alpha_k) \geq s(\alpha_j, a\alpha_k)$ for $a \in G$).

The whole graph $\check{\Gamma}$ is constituted by its "skeleton": $SK = \bigcup\limits_{i=1}^{r} [\sigma_0, \alpha_i]$ and the segments connecting the dead ends (i. e., the vertices σ such that $\chi(\overset{\circ}{E}_\sigma) = 1$) with SK. The ratio $M_{\sigma j}/M_{\sigma k}$ is constant for σ in $[\sigma_0, s(j,k)]$ and is a strictly increasing function for $\sigma \in [s(i,j), \alpha_j] \subset \check{\Gamma}$. Moreover, this ratio is also constant along the segments connecting the dead ends with SK. The described properties of the ratios $M_{\sigma i}/M_{\tau i}$ imply that, for each index $i \in \{1, \ldots, r\}$, one has:

$$\begin{aligned} \frac{1}{M_{\alpha_i i}} \underline{M}_{\alpha_i} &\leq \frac{1}{M_{\delta i}} \underline{M}_{\delta} \quad \forall \delta \in \check{\Gamma} \; , \\ \frac{1}{M_{\alpha_i i}} \underline{M}_{\alpha_i} &< \frac{1}{M_{\tau i}} \underline{M}_{\tau} \quad \forall \tau \in SK; \; \tau \neq \alpha_i \; . \end{aligned} \tag{7}$$

(Here we use the partial order $\underline{M} = (M_1, \ldots, M_r) \leq \underline{M}' = (M'_1, \ldots, M'_r)$ if $M_i \leq M'_i$ for all $i = 1, \ldots, r$; $\underline{M} < \underline{M}'$ if $\underline{M} \leq \underline{M}'$ and $M_i < M'_i$ for at least one i.)

Let $\sigma \in \check{\Gamma}$ be such that the multiplicity $\underline{M}_{\sigma} = (M_{\sigma 1}, \ldots, M_{\sigma r})$ is a maximal one among the set of exponents $\underline{M}_{\delta}$ appearing in the factorization

$$\zeta(\underline{t}) = \prod_{\delta \in \check{\Gamma} \;,\; s(\underline{M}_{\delta}) \neq 0} (1 - \underline{t}^{\underline{M}_{\delta}})^{s(\underline{M}_{\delta})} \; , \tag{8}$$

i. e., such that $s(\underline{M}_{\delta}) \neq 0$. (An element $\underline{M}$ from a subset of $\mathbb{Z}^r_{\geq 0}$ is called *maximal* if there are no elements greater than $\underline{M}$.) The maximality of $\underline{M}_{\sigma}$ implies that, in the (minimal) resolution process, we do not blow-up a point of the corresponding divisor E_{σ}. Therefore there exists an index $i \in \{1, \ldots, r\}$ such that $\sigma = \alpha_j$. Let $B(\sigma)$ be the set of such indices. Note that the exponent $s(\underline{M}_{\sigma})$ of the binomial $(1 - \underline{t}^{\underline{M}_{\sigma}})$ is $s(\underline{M}_{\sigma}) = -n\chi(\overset{\circ}{E}_{\sigma})$ for some positive integer n and therefore, if $\sigma = \alpha_j$ for some j, one has $s(\underline{M}_{\sigma}) > 0$.

Let $A(\sigma) \subset \{1, \ldots, r\}$ be the set of indices i, $1 \leq i \leq r$, such that

$$\frac{1}{M_{\sigma i}} \underline{M}_{\sigma} \leq \frac{1}{M_{\delta i}} \underline{M}_{\delta}$$

for all $\delta \in \check{\Gamma}$ with $s(\underline{M}_{\delta}) \neq 0$. The equations (7) imply that $B(\sigma) \subset A(\sigma)$, however $A(\sigma)$ could contain some indices ℓ such that $\alpha_{\ell} \neq \sigma$.

Let us assume that there exists $\ell \in A(\sigma)$ such that $\alpha_{\ell} \neq \sigma$. By the equations (7) this implies that $\sigma \in [\sigma_0, \alpha_{\ell}]$ and for all $\delta \in [\sigma, \alpha_{\ell}]$, $\delta \neq \sigma$, one has that $s(\underline{M}_{\delta}) = 0$ and so $\chi(\overset{\circ}{E}_{\delta}) = 0$. As a consequence the age of α_{ℓ} is smaller than the one of σ and α_{ℓ} is an end point of the resolution graph of a branch C_j with $j \in B(\sigma)$. In this case one has that $M_{\sigma \ell} < M_{\sigma j}$. On the other hand it is clear that $M_{\sigma j} = M_{\sigma i}$ if $i, j \in B(\sigma)$ and so one can detect the indices of $B(\sigma)$ as $j \in A(\sigma)$ such that $M_{\sigma j} \geq M_{\sigma \ell}$ for all $\ell \in A(\sigma)$.

For $i_0 \in B(\sigma)$, the binomial $(1 - \underline{t}^{\underline{M}_{\sigma}})$ appears in (8) with the exponent $s(\underline{M}_{\sigma}) = -\chi(\overset{\circ}{E}_{\sigma})(|G|/k_{i_0})$. Thus in order to finish the proof one has to find an index $i_0 \in B(\sigma)$ and to compute $\chi(\overset{\circ}{E}_{\sigma})$.

First of all, if (σ, δ) is an edge at σ, the maximality of $\underline{M}_\sigma$ implies that the age of δ is smaller than the one of σ. So, the number $\epsilon(\sigma)$ of edges at σ is ≤ 2. The case $\epsilon(\sigma) = 0$ is only possible in the trivial case of two smooth and transversal branches, in this case the Poincaré series (and so $\zeta(\underline{t})$) is equal to 1 and we can omit this situation.

Let us assume that $A(\sigma) \neq B(\sigma)$, then $\epsilon(\sigma) \geq \#(A(\sigma) \setminus B(\sigma))$ and so if $\#(A(\sigma) \setminus B(\sigma)) = 2$ (see Figure 5(a)) then $-\chi(\overset{\circ}{E}_\sigma) = \#B(\sigma)$. The case $\#(A(\sigma) \setminus B(\sigma)) = \epsilon(\sigma) = 1$ is possible only if $A(\sigma) = \{1, \ldots, r\}$, all the branches are smooth, all the branches of $B(\sigma)$ split at the same vertex σ and the one in $A(\sigma) \setminus B(\sigma)$ is transversal to the others (see Figure 5(b)). This case is characterized by the fact that $\zeta(\underline{t}) = (1 - \underline{t}^{\underline{M}_\sigma})^{(r-2)|G|}$, $M_{\sigma j} = M_{\sigma j'}$ if $j, j' \in B(\sigma)$; $M_{\sigma_\ell} = 1$ for $\ell \notin B(\sigma)$. In the remaining case one has $\#(A(\sigma) \setminus B(\sigma)) = 1$, $\epsilon(\sigma) = 2$ and so $-\chi(\overset{\circ}{E}_\sigma) = \#B(\sigma)$. (See Figure 5(c)).

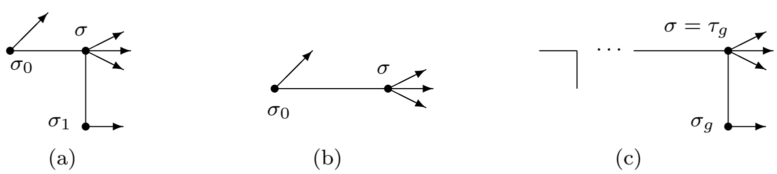


Figure 5: The cases with $A(\sigma) \neq B(\sigma)$.

As a consequence of the above discussion we can assume that $A(\sigma) = B(\sigma)$. First, assume that there exists $\delta \in \check{\Gamma}$ such that $s(\underline{M}_\delta) < 0$ and $\underline{M}_\sigma = n\underline{M}_\delta$ for some positive integer n. The case in which the first vertex $\sigma_0 = \delta$ is the only one with this condition is only reached if $A(\sigma) = \{1, \ldots, r\}$ and all the branches are smooth and split at σ (see Figure 6(b)). This case is characterized by the fact that $\zeta(\underline{t}) = (1 - \underline{t}^{\underline{M}_\sigma})^{(r-1)|G|}(1 - \underline{t}^{\underline{M}\delta})^{-|G|}$, $\underline{M}_\delta = (1, \ldots, 1)$ and $\underline{M}_\delta = (k, \ldots, k)$. Otherwise one has $-\chi(\overset{\circ}{E}_\sigma) = \#A(\sigma)$ (see Figures 6(a) and (c)).

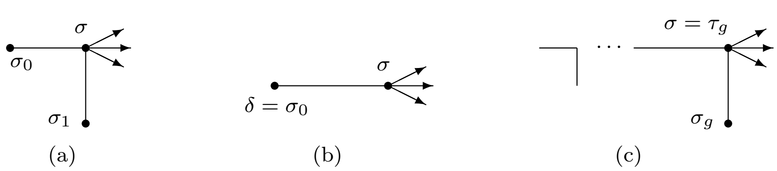


Figure 6: The cases with $\underline{M}_\sigma = n\underline{M}_\delta$.

Let us assume that there exists $\tau \in \check{\Gamma}$ with $s(\underline{M}_\tau) > 0$ such that the difference $\underline{M}_{\sigma i} - \underline{M}_{\tau i}$ is equal to 0 for all the coordinates $i \notin A(\sigma)$ and is equal to one and the same constant for those $i \in A(\sigma)$. In this case the number of edges at σ, $\epsilon(\sigma)$, is equal to 1 and so we finish because $-\chi(\overset{\circ}{E}_\sigma) = \#A(\sigma) - 1$.

The only situation when $\epsilon(\sigma) = 1$ and the above mentioned element τ does not exists is the following one: all the branches from $A(\sigma)$ are smooth and split at the vertex σ; moreover there is no vertices $\delta \neq \sigma$ on the geodesic $[\sigma_0, \sigma]$ with $s(\underline{M}_\delta) \neq 0$. Thus, in particular, $\chi(\overset{\circ}{E}_{\sigma_0}) = 0$ and so there are two edges on Γ starting at σ_0. The case when this edges are conjugate by the action of the group G is characterized by the fact that $\zeta(\underline{t}) = (1 - \underline{t}^{\underline{M}_\sigma})^{(r-1)|G|/2}$. Otherwise both edges are invariant by the action of the group, one of them corresponds to the indices from $A(\sigma)$ and the other one to the remaining ones. In this case we proceed as follows: let us consider an element $\underline{M}_{\sigma'}$, maximal among $\underline{M}_\delta$ with $s(\underline{M}_\delta) \neq 0$ different from $\underline{M}_\sigma$. For this new element we reproduce the steps we made before for σ. Note that all the indices involved in this new process are in the complement $A'(\sigma)$ of $A(\sigma)$ so, there is no conflict with the previous ones. If, by the previous methods, we are able to determine an index $i_0 \in A'(\sigma)$ and the Euler characteristic $\chi(\overset{\circ}{E}_{\alpha_{i_0}})$ then we finish. Otherwise we have just a similar situation for the vertex σ', that is we have $\epsilon(\sigma') = 1$, the branches of $A(\sigma')$ are all the branches of $A'(\sigma)$; all of them are smooth and split at σ' and, moreover, there is no vertex δ on the geodesic $[1, \sigma']$ such that $s(\underline{M}_\delta) \neq 0$ (see Figure 7). This case is characterized by the fact that the function zeta is

$$\zeta(\underline{t}) = (1 - \underline{t}^{\underline{M}_\sigma})^{(\#A(\sigma)-1)|G|}(1 - \underline{t}^{\underline{M}_{\sigma'}})^{(\#A(\sigma')-1)|G|}$$

with $\underline{M}_{\sigma i} = 1$ (respectively $\underline{M}_{\sigma' i} = 1$) if $i \notin A(\sigma)$ (respectively $i \notin A(\sigma')$) and with $\underline{M}_{\sigma i} = k$ (respectively $\underline{M}_{\sigma' i} = k'$) if $i \in A(\sigma)$ (respectively $i \in A(\sigma')$).

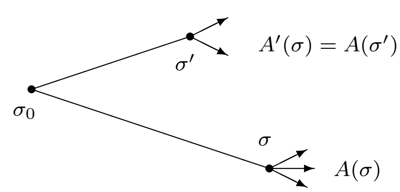


Figure 7: The degenerate case with $A(\sigma') = A'(\sigma)$ and $\epsilon(\sigma) = \epsilon(\sigma') = 1$.

If the above procedure did not produce an index $i_0 \in \{1, \dots, r\}$ and its Euler characteristic, then for the index σ one has that $\epsilon(\sigma) = 2$ and so $-\chi(\overset{\circ}{E}_\sigma) = \#A(\sigma)$. This finishes the determination of an index i_0 from $\{1, \dots, r\}$, of the Euler characteristic of $E_{\alpha_{i_0}}$, and thus of the exponent $|G|/k_{i_0}$. $\square$

The method to determine the separation points between the curve C_{i_0} and the curves C_i with $i \neq i_0$ is literally the same as in the divisorial case. This completes the proof of Theorem 2. $\square$

4 The order of the group and the function zeta

In Sections 2 and 3 we assumed that the group G was known. (In fact we used only the order of the group G). This requirement is really needed in some cases. Let us discuss situations when the knowledge of the order of the group is not necessary and the required information (first of all the order of G) is determined by the zeta function itself.

In Theorem 1 (under the described conditions) the knowledge of the order of the group is not necessary. One does not use the order of the group to find the multiplicities M_{σ_i}, M_{τ_i} and M_{ρ_j} (for $r = 1$). The same procedure does not give the orders h_j of the isotropy subgroups, but gives all the ratios h_0/h_j. One should modify the equations for the multiplicities m_{σ_i} and m_{ρ_j} used in the proof of Theorem 1 so that they become equations with respect to $h_\ell \cdot m_{\sigma_i}$ for $i = 0, \ldots, p$ and $h_\ell \cdot m_{\rho_j}$ for $j = 1, \ldots, \ell$. The integer h_ℓ (the order of the isotropy subgroup of the valuation ν) is nothing else but $\gcd(h_\ell m_{\sigma_0}, h_\ell m_{\sigma_1}, \ldots, h_\ell m_{\sigma_p})$. The knowledge of h_ℓ and of the ratio h_0/h_ℓ gives the order h_0 of the group G.

In the setting of Theorem 2, in some situations the multiplicity M_{σ_0} can be determined just at the very beginning. Namely, this is possible if, in the resolution graph of the curve $C = \bigcup_{a \in G} \bigcup_{i=1}^{\infty} aC_i$, either the number of edges at the (initial) vertex σ_0 is different from 2, or it is equal to 2, but these two edges are not interchanged by the group action. (This means that either the number of lines in the tangent cone of the curve C is different from 2, or the tangent cone consists of two lines not in the same G-orbit.) If the multiplicity M_{σ_0} is known, the way to determine the multiplicities m_{σ_i}, m_{τ_i} and m_{ρ_j} and therefore the resolution graph is the same as in the divisorial case above. The following example shows that, if the order of the group G is not assumed to be known and the multiplicity M_{σ_0} cannot be determined in the described way, the topological type of the curve singularity (in fact already with $r = 1$) is not determined by the zeta function $\zeta(t)$.

Example. Let C' be the (non-reduced) curve defined by the equation $(y^2 - x^3)^7(x^2 - y^3)^7 = 0$ with the natural (non-trivial) action of the group of order 14 on its components and let C'' be the curve defined by the equation $(y^2 - x^5)^5(x^2 - y^5)^5 = 0$ with the natural (non-trivial) action of the group of order

10 on its components. The (minimal) resolution graphs of the curves C' and C'' are shown in Figure 8.

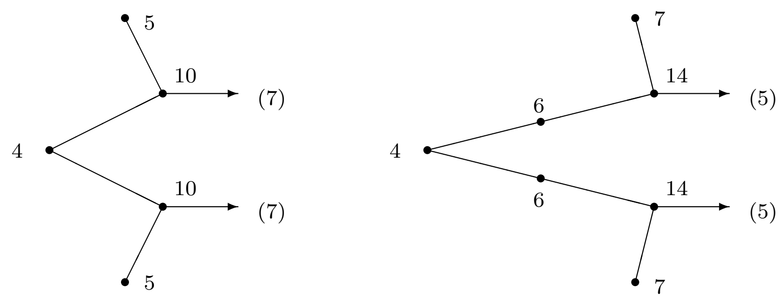


Figure 8: The resolution graphs of the curves C' and C''

The numbers at the vertices (including the arrows) are the multiplicities of the corresponding components in the zero divisors of the liftings of the corresponding functions (the left hand sides of the equations) to the surfaces of resolution. (These multiplicities define the ages of the vertices and thus the combinatorics of the resolutions in an obvious way.) The A'Campo formula gives $\zeta(t) = (1-t^{35})^{-2}(1-t^{70})^2$ in the both cases.

In the setting of Theorem 1 (i. e., for divisorial valuations) the possibility to find the multiplicity M_{τ_g} in some cases (namely when the binomial $(1-t^{M_{\tau_g}})$ is present in the decomposition (3)) does not permit, in general, to restore the resolution graph and/or the order of the group. This is shown by the following example.

Example. Let us consider two modification graphs shown on Figure 9 with the divisorial valuations corresponding to the vertices marked by the circles with the groups of orders 14 and 10 in the left and in the right hand sides respectively (exchanging the two valuations in each case).

The A'Campo type formula gives

$$\zeta(t) = (1-t^{35})^{-2}(1-t^{70})^2(1-t^{105})^{-2}$$

in the both cases.

5 Unknotted links with the same Alexander polynomials

In the setting of Theorem 2 with $r=1$, all the components aC_1 of the curve $C = \bigcup\limits_{a\in G} aC_1$ are equisingular, that is have the same topological type. The

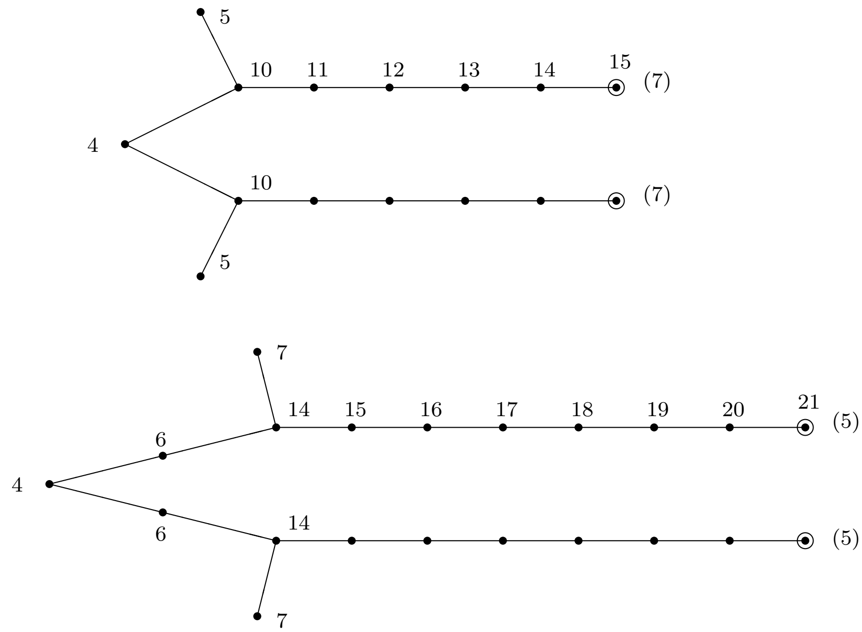


Figure 9: The modification graphs definning the divisorial valuations.

attempt to understand whether for the statement to hold it is really necessary that the components of the curve C are obtained from one of them by a group action on a modification graph or it is sufficient that all the components are equisingular led to the following example. Let $f'(x,y) := (x^3+y^{12})(y+x^2)(y+x^2)(y^3+x^{12})$, $f''(x,y) := (x^3+y^{15})(y+x^2)(y^4+x^{12})$. The function germs f' and f'' are products of function germs right equivalent to the function x. Therefore all the components of the curves $C' = \{f'=0\}$ and $C'' = \{f''=0\}$ are equisingular, moreover the components of the curve C' and the components of the curve C'' have the same topological type (all of them are smooth). The algebraic links $L' = C' \cap S^3_\varepsilon$ and $L'' = C'' \cap S^3_\varepsilon$ consist of unknotted components (of 8 of them each).

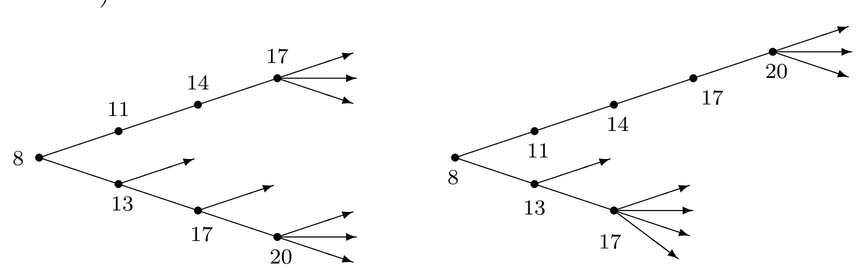


Figure 10: The resolution graphs of the functions f' and f''.

Proposition 7 *The curve singularities C' and C'' (and thus the links L' and L'') are topologically different. The monodromy zeta functions of the germs f' and f'' (and thus the Alexander polynomials in one variable of the links L' and L'') coincide.*

Proof. The resolution graphs of the function germs f' and f'' are shown in Fig 10. The numbers at the vertices are the multiplicities of the corresponding components in the zero divisor of the liftings of the functions f' and f'' to the corresponding surfaces of resolution. The graphs are topologically different and therefore the curve germs C' and C'' (and the links L' and L'') are topologically different as well. (In particular, these links have different Alexander polynomials in several variables.) The A'Campo formula gives that in the both cases

$$\zeta(t) = (1 - t^{13})(1 - t^{17})^3(1 - t^{20})^2.$$

□

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