FAVARD THEORY AND FREDHOLM ALTERNATIVE FOR DISCONJUGATED RECURRENT SECOND ORDER EQUATIONS

JUAN CAMPOS, RAFAEL OBAYA AND MASSIMO TARALLO

ABSTRACT. We discuss the existence of a Fredholm–type Alternative for a recurrent second order linear equation, which is disconjugate in a strong sense. The basic result is about bounded solutions of equations with bounded coefficients: it depends on kinematic similarities that allow to reduce the problem to a pair of very simple normal forms. Then the result is specialized to recurrent equations, by means of Favard theory.

1. INTRODUCTION

We are concerned with a boundary value problem for a class of the scalar equations of the type:

$$(1.1) \qquad x'' + \lambda(t)x' + a(t)x = g(t)$$

in the *recurrent* framework. We assume that $a(t)$ and $\lambda(t)$ are jointly recurrent, in the sense that their joint hull:

$$H(a,\lambda) \;=\; \mathrm{cls}\big\{(a_\tau, \lambda_\tau) : \tau \in \mathbb{R}\big\}$$

is compact minimal, where subscripts stand for translation factors and closure is taken with repect to the compact–open topology. The question we would like to answer is the following: among the inhomogeneous terms $g(t)$ *representable* on $H(a,\lambda)$, for which ones the equation (1.1) admits a solution $x(t)$ which is again *representable* on $H(a,\lambda)$?

Saying that $x(t)$ is representable on $H(a,\lambda)$ is much more than saying that it is recurrent: roughly speaking, it has also to inherit the joint recurrence properties of $a(t)$ and $\lambda(t)$. When for instance both these coefficients are T–periodic, the solutions must be T–periodic too. The answer to this classical periodic problem is universally known: solvability is decided by the so–called *periodic Fredholm Alternative*, see for instance Hale textbook [8]. That is, given an arbitrary inhomogeneous term $g(t)$ which is also T–periodic, the equation (1.1) admits a T–periodic solution if and only if the following orthogonality condition:

$$(1.2) \qquad \int_0^T g(t)y(t)\,dt \;=\; 0$$

---

is satisfied for every T–periodic solution of the *adjoint equation*:

$$(1.3) \qquad \{y' - \lambda(t)y\}' + a(t)y = 0 .$$

The nonstandard form of the adjoint equation is due to the possible lack of smoothness of the coefficients of the equation: see Section 2.

The existence of a Fredholm–type Alternative in the recurrent but possibly aperiodic case is a much more delicate question. A result in this direction has been obtained in [14] for a constant damping:

$$(1.4) \qquad \lambda(t) = \lambda_0 \neq 0$$

and under the key assumption that the homogeneous part of (1.1):

$$(1.5) \qquad x'' + \lambda(t)x' + a(t)x = 0$$

is *disconjugate* in a strong sense, namely it admits a bounded solution $\varphi(t)$ which is separated from zero:

$$(1.6) \qquad 0 < \inf_t \varphi(t) \leq \sup_t \varphi(t) < +\infty .$$

Because of (1.4) this is actually shown to be the unique bounded solution of (1.5) up to scalar multiples. Moreover, the adjoint equation (1.3) is shown to be behave exactly as the direct one: it is disconjugate and its only bounded solutions are the scalar multiples of a single solution $\psi(t)$, which is bounded and separated from zero. Finally, the authors of [14] prove that, given an arbitrary $g(t)$ which is representable on $H(a, \lambda) = H(a)$, the inhomogeneous equation (1.1) has a solution $x(t)$ representable on the same hull if and only if :

$$(1.7) \qquad \psi\, g \in BP(\mathbb{R})$$

where BP stands for having bounded primitive. We call this fact a *representable Fredholm Alternative* for the involved equation.

A couple of comments about the above result are probably worth. The first one is to legitimate the name of Fredholm Alternative: in the periodic case, condition (1.7) is equivalent to (1.2), so that we get exactly the classical periodic one. See Section 4 for a more detailed explanation. The second comment is that, because of disconjugacy, standard arguments provide an integral formula for the general solution of the inhomogeneous equation (1.1). However the point is that, as usual in the aperiodic case, such formula is not readily usable to detect bounded solutions and their possible recurrence.

For sake of precision, in [14] the focus is on bounded solutions more than on representable solutions: the latter are obtained from the former, via the so–called Favard theory. This is actually the common approach of all the literature in the field, and the present paper is not an exception. The price to pay for using such theory is the so–called *Favard separation condition*, which we recall in Appendix B. Here we refer the Favard separation condition to scalar second order equations like (1.5) or (1.3), but it actually should be referred to the associated planar first order systems. In the case of equation (1.5) this is the system:

$$(1.8) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -\lambda(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

while the association is less obvious for the equation (1.3). The point is that, because of the combined action of the assumptions (1.4) and (1.6), both the direct

and the adjoint equations are shown to satisfy the Favard separation condition. As a consequence, both the distingushed solutions $\varphi(t)$ and $\psi(t)$ result to be representable on $H(a, \lambda)$: this fact is not mentioned in [14] but is relevant in order to think of condition (1.7) as to a reasonable Fredholm–type Alternative in the given context.

The aim of the present paper is to discuss what happens when the disconjugacy assumption (1.6) is maintained but (1.4) is removed, wondering whether:

(1) $\varphi(t)$ is representable on $H(a, \lambda)$;

(2) the adjoint equation is disconjugate and $\psi(t)$ is representable too;

(3) the direct and adjoint equations satisfy the Favard separation condition;

(4) at least when all previous questions have positive answers, the representable Fredholm Alternative holds for (1.5)

As we will see, the all questions are tightly interconnected and the answers are driven by the spectral and the Favard properties of the damping coefficient $\lambda(t)$. The spectrum $\sigma(\lambda)$ is the *dichotomy spectrum* introduced by Sacker and Sell in [17]: we recall this notion (in the general higher dimensional context) and its consequences in Appendix A. Next result answer to questions (1) and (2).

Theorem 1.1. *The adjoint equation (1.3) is disconjugate if and only if $\lambda(t)$ satisfies the following condition:*

$$(1.9) \qquad\qquad 0 \notin \sigma(\lambda) \qquad or \qquad \lambda \in BP(\mathbb{R}) \,.$$

In this case moreover, up to scalar multiples $\varphi(t)$ and $\psi(t)$ are the unique bounded solutions and they are both representable on $H(a, \lambda)$.

What the above theorem does not say, is whether or not $\varphi(t)$ is representable on $H(a, \lambda)$ when condition (1.9) fails, that is:

$$(1.10) \qquad\qquad \begin{cases} 0 \in \sigma(\lambda) \\ \lambda \notin BP(\mathbb{R}) \end{cases}$$

Of course there are cases where everything works fine, like for instance the trivial equation:

$$(1.11) \qquad\qquad x'' + \lambda(t)x' \;=\; 0$$

but we suspect that non representable $\varphi(t)$'s may also appear for some suitable choice of the coefficients. The situation is similar for question (3).

Theorem 1.2. *The adjoint equation (1.3) satisfies the Favard separation condition if and only if $\lambda(t)$ satisfies condition (1.9). In this case moreover, also the direct equation (1.5) satisfies the Favard separation condition.*

This time, however, when $\lambda(t)$ satisfies (1.10) we know that both the possibilities are open: we may arrange $\lambda(t)$ in such a way that the Favard separation condition for the direct equation (1.5) holds or fails.

It remains to discuss question (4). The two previous theorems give the appropriate setup for the discussion: all the ingredients for cooking the representable Fredholm Alternative are ready, as soon as condition (1.9) is satisfied. However, for the answer we have to distinguish the two parts this condition is made of.

Theorem 1.3. *Assume that $0 \notin \sigma(\lambda)$ and $g(t)$ is representable on $H(a, \lambda)$. Then equation (1.1) admits a representable solution on $H(a, \lambda)$ if and only if condition (1.7) is satisfied.*

By taking into account the periodic Fredholm Alternative, we have the following result for the opposite case.

Theorem 1.4. *Assume that $\lambda \in BP(\mathbb{R})$ and is not jointly periodic with $a(t)$. Then there exists a $g(t)$ representable on $H(a, \lambda)$ for which, at the same time:*

  - *condition (1.7) is satisfied*
  - *equation (1.1) has no bounded solutions.*

The proofs of the theorems are spread over the next three sections: each statement has indeed an equivalent in the bounded framework, which is proved in Section 2 or in Section 3, while representability is added in Section 4. Our approach rests on the use of kinematic similarities that transform the planar system (1.8) in a much more handy system. They are special changes of variables, which does not affect the spectral and the Favard properties of the system: we recall them in Appendix D. The ultimate reason for Theorem 1.3 and Theorem 1.4 is the following results, which combines the main results of Section 3 with Section 4.

Theorem 1.5. *If $0 \notin \sigma(\lambda)$ then system (1.8) is kinematically similar to:*

$$(1.12) \qquad \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda(t) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

while for $\lambda \in BP(\mathbb{R})$ we get:

$$(1.13) \qquad \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)\psi(t)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

In both cases, the kinematic similarities are representable on $H(a, \lambda)$.

Let us point out that the involved kinematic similarities are very explicit: see Proposition 3.3 and Proposition 3.6 in Section 3. The resulting normal forms (1.12) and (1.13) permit an accurate description of the whole dynamics of (1.8) whose extent, we believe, goes beyond the questions we treated in the present paper.

We conclude by commenting the exiting literature on the subject, besides [14]. As far as we know, the existence of a Fredholm Alternative in recurrent systems has been investigated in few other papers, namely [1], [2], [4] and [23]. All of them are concerned with first order systems:

$$x' = A(t)x + f(t)$$

in any dimension but with some special properties. These properties are transversal to those on which the present paper rest, as far as [2], [4] and [23] are concerned. Actually [4] refers to almost periodic systems for which the quadratic form associated to $A(t)$ has a sign, while [2] gives a nontrivial extension of these results to the recurrent case. The paper [23] also refers to almost periodic systems, but all the arguments extend trivially to the recurrent case. The main assumption there is that all the solutions of:

$$x' = A(t)x$$

are bounded and separated from zero. This is always false for system (1.8), whatever $\lambda(t)$ we take. When (1.9) the space of the bounded solutions has dimension one, instead of two. In the opposite case (1.10), it can happen that all the solution

are bounded, but some them are not separated from zero: this follows for Proposition 4.1 combined with the arguments of Appendix B.

It remains to comment [1]. We recall the main result in Appendix B: roughly speaking, Theorem C.4 gives a general prescription for the validity of the Fredholm Alternative in the recurrent setting, which turns out to be a kind of common root of all the quoted papers. One of the goals of the present paper is to show that Theorem 1.3 is also covered by the result of [1]. However, while [1] rests on a block–diagonalization procedure which is not very explicit and destroys the representing hull $H(a, \lambda)$, the change of variables on which Theorem 1.5 is very explicit and preserves the hull, so allowing to prove Theorem 1.4.

Notations. By $|\xi|$ we mean the Euclidean norm in $\mathbb{R}^N$ while $\|A\|$ stands for the operatorial norm of an $N \times N$ matrix A, considered as a linear operator on $\mathbb{R}^N$. The symbols $C(\mathbb{R})$, $BC(\mathbb{R})$, $BUC(\mathbb{R})$ and $AP(\mathbb{R})$ denote the spaces of the continuous, bounded continuous, bounded uniformly continuous and almost periodic functions on $\mathbb{R}$, with value in some finite dimensional normed space. By adding a superscript k like in $C^k(\mathbb{R}), BC^k(\mathbb{R})$ etc. we mean that the defining property of the class also applies to the first k derivatives.

When $\in BC(\mathbb{R})$ we denote by $\|f\|_\infty$ the standard least upper bound norm. Given any $f \in C(\mathbb{R})$ we introduce a selected primitive by:

$$\widehat{f}(t) \ = \ \int_0^t f(s)\, ds \ .$$

By disconjugacy we always mean the strong form of disconjugacy considered in the Introduction. The symbols $\varphi(t)$ and $\psi(t)$ will always denote separating solution of the direct equation (1.5) and the adjoint equation (1.3) respectively, when they are disconjugate.

2. Disconjiugacy in the bounded framework

In this section we consider the same equations of the Introduction, and in particular the homogeneous one:

$$(2.1) \qquad\qquad\qquad x'' + \lambda(t)x' + a(t)x = 0$$

but we relax the recurrence of the involved coefficients, by asking only that:

$$\lambda, a \in BC(\mathbb{R}) \ .$$

By solution we mean a classical solution $x \in C^2$. We are mainly interested in solutions which are bounded. Concerning them, let us recall that standard a priori estimates show that $x \in BC(\mathbb{R})$ actually implies $x \in BC^2(\mathbb{R})$.

Notice that the joint hull $H(a, \lambda)$ may be noncompact and all the recurrence properties are loosen, but some results still persist and are indeed useful to understand the recurrent framework. Here and in the later sections, all the results relay on the discongiugacy condition we already made in the Introduction: hereafter we stipulate it once for all.

<u>Standing Assumption</u>　*Equation (2.1) has a separating solution $\varphi(t)$*

where by *separating* we mean that:

$$0 \;<\; \inf_t \varphi(t) \;\leq\; \sup_t \varphi(t) \;<\; +\infty \,.$$

Though discongiugate equations are somewhat rare, they are by no means unfrequent. Indeed, by reversing the perspective of the definition, one finds that discongiugate equations are exactly the equation where:

$$(2.2) \qquad a(t) \;=\; -\frac{1}{\varphi(t)} \left\{ \varphi''(t) + \lambda(t)\varphi'(t) \right\}$$

for an arbitrary $\varphi \in BC^2(\mathbb{R})$ satisfying (1.6) and an arbitrary $\lambda \in BC(\mathbb{R})$.
Discongiugate equations are rather simple, inasmuch the knowledge of the separating solution $\varphi(t)$ allows to reduce the order of the equation (2.1) and yields to the following integral formula for the general solution:

$$(2.3) \qquad x(t) \;=\; \alpha\varphi(t) + \beta\varphi(t) \int_0^t \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2} \, ds$$

where $\alpha, \beta \in \mathbb{R}$. As a trivial consequence, all the solutions are bounded if and only if it happens that:

$$(2.4) \qquad e^{-\widehat{\lambda}} \in BP(\mathbb{R}) \,.$$

When on the contrary this condition fails, the solution $\varphi(t)$ is, up to scalar multiples, the only bounded solution of the equation (2.1). Next lemma points out a consequence of (2.4) which will be relevant in Section 4.

Lemma 2.1. *If condition (2.4) is satisfied then $\widehat{\lambda}(\pm\infty) = +\infty$.*

Proof. We only prove that $\lambda(+\infty) = +\infty$, the other case being similar. If by contradiction the claim is false, then there exists a value M and a time sequence $0 < \tau_n \to +\infty$ such that $f(\tau_n) \leq M$. But standard Lipschitz estimates show that:

$$\widehat{\lambda}(t) \leq M + \|\lambda\|_\infty \qquad \forall t \in [\tau_n, \tau_n + 1]$$

implying that $e^{-\widehat{\lambda}(t)}$ is non integrable at $+\infty$. □

Let us now introduce the adjoint equation of (2.1). A bit of care is due, in view of the fact that $\lambda(t)$ is possibly nondifferentiable. The most appropriate notion here happens to be the following:

$$(2.5) \qquad \left\{ y' - \lambda(t)y \right\}' + a(t)y = 0$$

where solutions are not required to have the usual C^2 regularity but instead:

$$(2.6) \qquad y \in C^1(\mathbb{R}) \qquad y' - \lambda y \in C^1(\mathbb{R}) \,.$$

We denote this space with the symbol $C^\lambda(\mathbb{R})$ to remark its dependence on the coefficient $\lambda(t)$. This space may appear rather exotic but actually is not. For instance, it is clear that $C^\lambda(\mathbb{R}) = C^2(\mathbb{R})$ as soon $\lambda \in C^1(\mathbb{R})$: see Lemma 2.4 for the case of a nonsmooth $\lambda(t)$.
To justify the unusual notion of adjoint, let us look at the second order differential operators associated to (2.1) and (2.5) respectively:

$$\begin{aligned} Lx &= x'' + \lambda(t)x' + a(t)x \\ L^*y &= \left\{ y' - \lambda(t)y \right\}' + a(t)y \end{aligned}$$

as defined on $C^2(\mathbb{R})$ and $C^\lambda(\mathbb{R})$. Consider now the quadratic differential operator:

$$(2.7) \qquad\qquad R(x,y) \;=\; x'y - x(y' - \lambda y)$$

which is again defined on the same for $x \in C^2(\mathbb{R})$ and $y \in C^\lambda(\mathbb{R})$ but only depends on first derivatives. The following property holds and is the desired justification.

Lemma 2.2. *Assume that $x \in C^2(\mathbb{R})$ and $y \in C^\lambda(\mathbb{R})$. Then:*

$$\{Lx\}y - x\{L^*y\} \;=\; \frac{d}{dt}\{R(x,y)\}$$

In particular, if $x(t)$ and $y(t)$ are solutions of the direct and the adjoint equation respectively, then:

$$\frac{d}{dt}\{R(x,y)\} \;\equiv\; 0$$

that is, the quantity $R(x,y)$ becomes an invariant of motion. As we will see in the proof of Proposition 2.5, the value of this invariant has some relevant consequences on dynamics.

Proof. The proof is trivial, and we sketch it just point out the role of (2.6). Because of $y \in C^1(\mathbb{R})$ we have:

$$x''y \;=\; \frac{d}{dt}(x'y) - x'y'$$

and hence $y' - \lambda y \in C^1(\mathbb{R})$ implies:

$$\begin{aligned}
x''y + \lambda x'y \;&=\; \frac{d}{dt}\{x'y\} - x'y' + \lambda x'y \;=\; \frac{d}{dt}\{x'y\} + x'\{\lambda y - y'\} \\
&=\; \frac{d}{dt}\{x'y\} + \frac{d}{dt}\{x\{\lambda y - y'\}\} - x\{\lambda y - y'\}'
\end{aligned}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As for the direct equation (2.1), also for the adjoint equation (2.5) we are mainly interested in bounded solutions: next lemma provides explicitly for the latter the a priori estimates we mentioned for the former.

Lemma 2.3. *If $y \in BC(\mathbb{R})$ is a solutions of the adjoint equation (2.5) then:*

$$(2.8) \qquad\qquad y \in BC^1(\mathbb{R}) \qquad\qquad y' - \lambda y \in BC^1(\mathbb{R})$$

Proof. For writing convenience, let us set $Y = y' - \lambda y$. We know $Y \in C^1(\mathbb{R})$ and moreover $Y' \in BC(\mathbb{R})$ from the equation, since:

$$\|Y'\|_\infty \;\leq\; \|a\|_\infty \|y\|_\infty \;.$$

We claim that actually $Y \in BC(\mathbb{R})$, which in turn trivially implies (2.8). To prove the claim, let us start fixing $\tau > 0$. Given an arbitrary $t \in \mathbb{R}$ there exist $\xi = \xi(t,\tau)$ such that:

$$t - \tau < \xi < t \qquad y(t) - y(t - \tau) = \tau\, y'(\xi) \;.$$

Thus we have:

$$\begin{aligned}
|Y(t)| \;&=\; |Y(\xi)| + \int_\xi^t |Y'(s)|\, ds \;\leq\; \frac{2}{\tau}\|y\|_\infty + \|\lambda\|_\infty\|y\|_\infty + (t-\xi)\|Y'\|_\infty \\
&\leq\; \left\{\frac{2}{\tau} + \|\lambda\|_\infty + \tau\|a\|_\infty\right\}\|y\|_\infty \;.
\end{aligned}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since the adjoint equation is not a priori disconjugate, deciding how many bounded solutions it has seems more difficult than it was for the direct equation: in spite of that, next lemma shows that the solutions of the former can be obtained from the latter. The lemma is a modified version of a result in [14].

Lemma 2.4. *The time–dependent change of variables:*

$$x = y\,e^{-\widehat{\lambda}(t)}$$

defines a one–to–one correspondence between $C^2(\mathbb{R})$ *and* $C^\lambda(\mathbb{R})$ *and also between the solutions of (2.1) and (2.5).*

In fact, it is not difficult to check that $C^2(\mathbb{R}) \cong C^\lambda(\mathbb{R})$ at a linear and topological level, when we endow the two spaces with the topology of the uniform convergence on compact sets of functions and the involved derivatives. Notice moreover that, depending on the properties of $\widehat{\lambda}(t)$, boundedness may be not respected along the change of variables.

Proof. Since $\widehat{\lambda} \in C^1(\mathbb{R})$, it is clear that $x \in C^1(\mathbb{R})$ if and only if $y \in C^1(\mathbb{R})$. Suppose this is the case and differentiate obtaining:

$$(2.9) \qquad\qquad x' = y'e^{-\widehat{\lambda}} - \lambda y e^{-\widehat{\lambda}} = (y' - \lambda y)e^{-\widehat{\lambda}}$$

Thus $x' \in C^1(\mathbb{R})$ if and only if $y' - \lambda y \in C^1(\mathbb{R})$. Summing up, $x \in C^2(\mathbb{R})$ if and only if the smoothness conditions (2.6) are satisfied by $y(t)$. This completes the proof of the first claim: the second claim follows now by a straightforward computation. □

Because of the previous lemma and formula (2.3), the general solution of the adjoint equation writes as:

$$(2.10) \qquad\qquad y(t) = \alpha\,\varphi(t)\,e^{\widehat{\lambda}(t)} + \beta\,\varphi(t)\,e^{\widehat{\lambda}(t)}\int_0^t \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2}\,ds$$

where $\alpha, \beta \in \mathbb{R}$. Notice that this formula does not steam from an order reduction into the adjoint equation, as formula (2.3) does for the direct equation by exploiting disconjugacy. On the contrary, we now use formula (2.10) to discuss the possible disconjugacy of the adjoint equation. The symbol $\sigma(\lambda)$ in the statement stand for the dichotomy spectrum of $\lambda(t)$: see Appendix A for the definition and the main properties.

Proposition 2.5. *The adjoint equation (2.5) admits a separating solution $\psi(t)$ if and only if $\lambda(t)$ satisfies the following condition:*

$$(2.11) \qquad\qquad 0 \notin \sigma(\lambda) \qquad or \qquad \lambda \in BP(\mathbb{R})\,.$$

In the former case $R(\varphi, \psi) = 0$ while in the latter $R(\varphi, \psi) \neq 0$. In both cases there are no bounded solutions of (2.5) but the scalar multiples of $\psi(t)$.

Along the whole paper, we freely quote $\psi(t)$ when the adjoint equation is disconjugate, with the meaning of any given separating solution. Its analytic expression is driven by the two parts of condition (2.11), which are mutually exclusive.

Proof. From Lemma 2.2 we know that $R(\varphi, \psi)$ is an invariant, namely there exists $c \in \mathbb{R}$ such that:

$$(2.12) \qquad \varphi'(t)\psi(t) - \varphi(t)\psi'(t) - \lambda(t)\varphi(t)\psi(t) = c \qquad\qquad \forall t\,.$$

When $\psi(t)$ is separating, the above can be read as a prescription on $\lambda(t)$, which is then written as the sum of two terms:

$$\lambda(t) \;=\; \frac{c}{\varphi(t)\psi(t)} + \left\{ \frac{\psi'(t)}{\psi(t)} - \frac{\varphi'(t)}{\varphi(t)} \right\} \;.$$

Observe that the term multiplied by c is separating, while the term inside the curly brackets has bounded primitive. Thus:

$$\begin{array}{ccc} c = 0 & \Leftrightarrow & \lambda \in BP(\mathbb{R}) \\ c \neq 0 & \Leftrightarrow & 0 \notin \sigma(\lambda) \end{array}$$

as a consequence condition (A.7) in Appendix A. In particular, if the adjoint equation is disconjugate, then condition (2.11) must be satisfied.

To prove the converse, start assuming that $\lambda \in BP(\mathbb{R})$. The first term in the right hand side of (2.10) is bounded, while the second is unbounded. Thus the only bounded solutions of (2.5) are the scalar multiples of:

$$(2.13) \qquad\qquad \psi(t) \;=\; \varphi(t)\, e^{\widehat{\lambda}(t)}$$

which is clearly separating.

Let us now consider the case $0 \notin \sigma(\lambda)$. For definiteness, we assume that the interval $\sigma(\lambda)$ lies to the right of zero: the other case can be handled in a similar way. The choice we made implies that $\widehat{\lambda}(+\infty) = +\infty$. In turn, inserting this information into (2.10), we find that the only candidates to be bounded solutions of (2.5) are the scalar multiples of:

$$(2.14) \qquad\qquad \psi(t) \;=\; \varphi(t)\, e^{\widehat{\lambda}(t)} \int_{t}^{+\infty} \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2}\, ds \;.$$

The solution (2.14) is positive and we claim that it is separating. These conclusions can be obtained directly by some length computations, but we prefer to give here a shorter though undirect proof. First of all, it is clear that the desired features can be equivalently proved for the auxiliary function:

$$(2.15) \qquad\qquad p(t) \;=\; e^{\widehat{\lambda}(t)} \int_{t}^{+\infty} e^{-\widehat{\lambda}(s)}\, ds \;.$$

Notice that $p(t) > 0$ for every t and that is solves the first order equation:

$$(2.16) \qquad\qquad \dot{p} - \lambda(t)p \;=\; -1 \;.$$

Actually, it is the only candidate to be a bounded solution, for the very same arguments already used before. But $0 \notin \sigma(\lambda)$ implies that this equation has an exponential dichotomy, and then we know that it really admits a bounded solution. Assume now by contradiction that $p(t_n) \to 0$ along some sequence t_n. Then there exists another sequence τ_n such that:

$$p(\tau_n) \to 0 \qquad\qquad \dot{p}(\tau_n) \to 0 \;.$$

This is a standard variational principle, proved by Ekeland in [6] in a much more general form. Inserting the above information into (2.16) give the desired contradictions, proving our claim. $\qquad\square$

We now start with the main stream of the paper, that is characterizing the existence of bounded solutions for the complete equation:

$$(2.17) \qquad\qquad x'' + \lambda(t)x' + a(t)x = g(t)$$

and understanding if and how this fact is related to the disconjugacy of the adjoint equation. For the moment, the inhomogeneous term is just bounded and continuous and, contrarily to the Introduction, we are not dealing with a boundary value problem: the effect of recurrence will be considered in Section 4 only. The following result gives a necessary condition, which does not depend on the possible disconjugacy of of the adjoint equation (2.1).

Proposition 2.6. *If the inhomogeneous equation (2.17) admits a bounded solution for a given $g \in BC(\mathbb{R})$, then*

$$(2.18) \qquad\qquad\qquad\qquad yg \in BP(\mathbb{R})$$

for every bounded $y(t)$ solving the adjoint equation(2.5).

Proof. Let $x(t)$ be a bounded solutions to (2.17). Since $Lx \equiv g$ and $L^*y \equiv 0$, Lemma 2.2 implies:

$$\int_0^t g(s)y(s)\,ds \;=\; \left[x'(s)y(s) - x(s)y'(s) + \lambda(s)x(s)y(s)\right]_0^t$$

which is a bounded function of t. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The point is now to decide whether or not (2.18) is also sufficient in order (2.17) to admit bounded solutions. A positive answer has been given in [14] under the assumption that:

$$\lambda(t) = \lambda_0 \neq 0 \;.$$

In this case, Proposition 2.5 guarantees that the adjoint equation (2.5) is disconjugate and that the only admissible $y(t)$ to test condition (2.18) are the scalar multiples of the separating solution $\psi(t)$. Hence condition (2.18) takes the more comfortable form:

$$(2.19) \qquad\qquad\qquad\qquad \psi g \in BP(\mathbb{R})$$

which we already used in the Introduction.
As we will see, the general answer is actually negative in both case, when the adjoint equation (2.5) is disconjugate and when it is not. However, the results we get are rather different in the two cases: they are more complete in the former case, which is also the most relevant for the recurrent framework. The negative and positive results for the bounded framework will be presented in Section 3: there scalar second order equations will be replaced by first order systems, to exploit of a more flexible notion of change of variables. The recurrent framework will be analyzed in Section 4.

3. Fredholm Alternative for bounded planar systems

Here, as in the previous section, we will work in the $BC(\mathbb{R})$ framework only. The standard way to get a planar system from the equation (2.1), is to set:

$$x_1 = x \qquad\qquad x_2 = x'$$

obtaining:

$$(3.1) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -\lambda(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

Bounded solutions are preserved along the process in the sense that, due to standard a priori estimates, the map $x \mapsto (x, x')$ defines an isomorphism between the corresponding classes.

Take now any $f, g \in BC(\mathbb{R})$ and consider the inhomogeneous system:

$$(3.2) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -\lambda(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

which, when $f = 0$, is equivalent to the inhomogeneous equation (2.17). The existence of bounded solutions to (3.2) have been already considered in literature and the known results are summarized in the Appendix A. The natural *necessary condition* is that:

$$(3.3) \qquad y_1 \, f + y_2 \, g \ \in \ BP(\mathbb{R})$$

for every bounded solution (y_1, y_2) to:

$$(3.4) \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & a(t) \\ -1 & \lambda(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

which is the adjoint system to (3.1). Next lemma makes clear how such system is related to the adjoint equation (2.5) and shows that (3.3) becomes (2.18) when both apply, namely when $f = 0$. The proof depends on Lemma 2.3 and is straightforward.

Lemma 3.1. *Setting:*

$$y_1 = \lambda(t)y - y' \qquad\qquad y_2 = y$$

transforms equation (2.5) into system (3.4) and preserves bounded solutions.

Consider now what happen in the case:

$$0 \notin \sigma(\lambda) \qquad \text{or} \qquad \lambda \in BP(\mathbb{R}) \ .$$

We know from Proposition 2.5 that the adjoint equation (2.5) is disconjugate and we denote by $\psi(t)$ any given separating solution. The same proposition also guarantees that the scalar multiples of $\psi(t)$ are the only bounded solutions to (2.5) and hence, due to Lemma 3.1, the necessary condition (3.3) takes the very convenient form:

$$(3.5) \qquad (\lambda\psi - \psi')f + \psi g \ \in \ BP(\mathbb{R})$$

which, as expected, reduces to (2.19) as soon as $f = 0$.

Our aim is to understand when the above necessary condition becomes also sufficient for the inhomogeneous system (3.2) to admit bounded solutions. To investigate this problem we will use kinematic similarities. They are special changes of variables which preserve the spectral properties of a system and the class of bounded solutions: see Appendix D for an introduction to this classical subject. The first kinetic similarity we introduce takes advantage of the disconjugacy of the scalar equation (2.1): roughly speaking, it does the analogous of an order reduction for the planar system (3.1).

Proposition 3.2. *The change of variables:*

$$(3.6) \qquad x_1 = \varphi(t)u_1 \qquad\qquad x_2 = \varphi'(t)u_1 + \frac{1}{\varphi(t)}u_2$$

is a kinematic similarity transforming the system (3.1) into:

$$(3.7) \qquad \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)^2} \\ 0 & -\lambda(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The proof is obtained by inspection and we omit it. Let us only point out that the proposed change of variable is actually a kinematical similarity bue to the boundedness of $\varphi(t)$.

From now one, we denote by σ, d_S, d_B and σ^*, d_S^*, d_B^* the dichotomy spectrum, the Sacker–Sell dimension and the bounded dimension of the original system (3.1) and of the adjoint system (3.4), respectively. Because of Appendix D, they are the same when computed for the new system (3.7) and its adjoint. By exploiting in a standard way the triangular form of (3.7) we then obtain:

$$(3.8) \qquad \sigma = \{0\} \cup \sigma(-\lambda) \qquad d_S = \begin{cases} 1 & \text{if } 0 \notin \sigma(\lambda) \\ 2 & \text{if } 0 \in \sigma(\lambda) \end{cases}.$$

while the bounded dimension follows directly from Section 2:

$$(3.9) \qquad d_B = \begin{cases} 1 & \text{if } e^{\widehat{\lambda}} \notin BP(\mathbb{R}) \\ 2 & \text{if } e^{\widehat{\lambda}} \in BP(\mathbb{R}) \end{cases}.$$

Passing to the adjoint system, the spectral features changes in a predictable way:

$$\sigma^* = -\sigma \qquad\qquad d_S^* = d_S$$

while the bounded dimensions d_B^* seems to be unrelated to d_B.

The normal form (3.7) and its spectral consequences are not to fully answer the questions we are interested in: we need further changes of variables, which however depends on the disconjugacy of the adjoint equation (2.5). Because of that, we distinguish three different cases which will be treated separately.

THE ADJOINT NONDISCONJUGATE CASE $\qquad \boxed{\begin{cases} 0 \in \sigma(\lambda) \\ \lambda \notin BP(\mathbb{R}) \end{cases}}$

No further change of variables are avalaible. Our aim is to provide an example of system (3.1) for which condition (3.3) does not act as a Fredholm–type Alternative, without attempting to show that such failure is indeed a general property, as we indeed suspect.

As it can be easily guessed, the most convenient strategy is looking first at the normal form (3.7), managing to have:

$$(3.10) \qquad\qquad\qquad d_B^* = 0$$

which turns (3.3) into an empty condition. To this aim, we take $\lambda \in BC(\mathbb{R})$ such that:

$$-\lambda(-t) = \lambda(t) > 0 \qquad \forall t > 0$$

and that moreover satisfies:

$$\lambda(+\infty) = 0 \qquad\qquad \widehat{\lambda}(+\infty) = +\infty .$$

That $0 \in \sigma(\lambda)$ follows easily from (A.7). The adjoint equation of (3.7) is:

$$(3.11) \qquad \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{\varphi(t)^2} & \lambda(t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and its general solution is given by:

$$v_1(t) = \alpha \qquad v_2(t) = e^{\widehat{\lambda}(t)} \left\{ \beta + \alpha \int_0^t \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2} \, ds \right\}$$

where α and β are constants of integration. Suppose now that $v_2(t)$ is bounded. Since $\widehat{\lambda}(\pm\infty)$ we must have:

$$\beta + \alpha \int_0^{+\infty} \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2} \, ds \; = \; 0 \; = \; \beta + \alpha \int_0^{-\infty} \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2} \, ds \; .$$

But this implies

$$\alpha \int_{-\infty}^{+\infty} \frac{e^{-\widehat{\lambda}(s)}}{\varphi(s)^2} \, ds \; = \; 0$$

showing that $\alpha = 0$. That $\beta = 0$ follows now trivially, proving (3.10).

It remains to prove that there exist $p, q \in BC(\mathbb{R})$ such that the inhomogeneous system:

$$(3.12) \qquad \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)^2} \\ 0 & -\lambda(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \; .$$

does not admit bounded solutions. On the one hand, this follows from $0 \in \sigma$: see the functional characterization of exponential dichotomies in Appendix A. On the other hand, it would be preferable to have a counter–example that originates from a second order equation: this requires some more work. Start noticing that, undoing the change of variables (3.6), the inhomogeneous system (3.12) becomes (3.2) with:

$$f(t) = \varphi(t)p(t) \qquad g(t) = \varphi'(t)p(t) + \frac{1}{\varphi(t)} \, q(t) \; .$$

Thus $f = 0$ is equivalent to $p = 0$ and, in this case, the system (3.2) is equivalent to the equation:

$$x'' + \lambda(t)x' + a(t)x \; = \; q(t)/\varphi(t) \; .$$

The simplest concrete case is:

$$x'' + \lambda(t)x' \; = \; q(t)$$

which corresponds to $a = 0$ and is disconjugate with $\varphi = 1$. We will use it the next section to produce a counter–example in the recurrent framework.

Summing up, to conclude the construction of the counter–example, we claim that (3.12) has no bounded solutions as soon as we choose $q(t)$ such that:

$$\lim_{t \to +\infty} \frac{q(t)}{\lambda(t)} \; = \; +\infty \; .$$

Too see why observe that, if system (3.12) has a bounded solution, then its second component:

$$u_2(t) \; = \; e^{-\widehat{\lambda}(t)} \left\{ \alpha + \int_0^t e^{\widehat{\lambda}(s)} q(s) \, ds \right\}$$

must be bounded for some $\alpha \in \mathbb{R}$. But de l'Hôpital's rule applies to show that:

$$\lim_{t \to +\infty} u_2(t) \;=\; \lim_{t \to +\infty} \frac{\alpha + \displaystyle\int_0^t e^{\widehat{\lambda}(s)} q(s)\, ds}{e^{\widehat{\lambda}(t)}} \;=\; \lim_{t \to +\infty} \frac{q(t)}{\lambda(t)} = +\infty\,.$$

THE ADJOINT DISCONJUGATE CASE $\boxed{\lambda \in BP(\mathbb{R})}$

Due to Proposition 2.5 in Section 2, the adjoint equation (2.5) is disconjugate and moreover we know that:

$$d_B \;=\; 1 \;=\; d_B^*\,.$$

In the next proposition, we use the separating solution $\psi(t)$ to perform a further and very convenient change of variables.

Proposition 3.3. *Assume $\lambda \in BP(\mathbb{R})$. Then the change of variable:*

$$(3.13) \qquad\qquad u_1 = w_1 \qquad\qquad u_2 = \frac{\varphi(t)}{\psi(t)}\, w_2$$

is a kinematic similarity transforming the system (3.7) into:

$$(3.14) \qquad\qquad \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)\psi(t)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\,.$$

Proof. That (3.13) is a kinematic similarity follows form the boundedness of $\varphi(t)$ and $\psi(t)$ with their first derivatives. Concerning the resulting system (3.14), invert (3.13) and then differentiate to get:

$$w_2' \;=\; \frac{\psi'\varphi - \varphi'\psi}{\varphi^2}\, u_2 + \frac{\psi}{\varphi}\, (-\lambda u_2) \;=\; -\frac{1}{\varphi\psi}\, R(\varphi, \psi)\, w_2$$

where $R(\varphi, \psi) = 0$ from Proposition 2.5. $\square$

Notice that the planar system (3.14) is not so far to be selfadjoint. Indeed the adjoint system is:

$$(3.15) \qquad\qquad \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{\varphi(t)\psi(t)} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and becomes (3.14) by the kinematic similarity defined by the time–independent change of variables:

$$z_1 = w_2 \qquad\qquad x_2 = -w_1\,.$$

Of course, the same conclusions must hold also for the pair (3.1)–(3.4): this is just a different way to express part of the content of Lemma 2.4 in Section 2.
The general solution of the adjoint equation (3.15) is:

$$z_1(t) = \alpha \qquad\qquad z_2(t) = \beta + \alpha \int_0^t \frac{ds}{\varphi(s)\psi(s)}$$

and is bounded if and only if $\alpha = 0$. Thus, the standard necessary condition for the inhomogeneous system:

$$(3.16) \qquad \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)\psi(t)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

to admit bounded solutions writes:

$$(3.17) \qquad\qquad q \in BP(\mathbb{R}) \,.$$

On the other hand, the inhomogeneous system (3.16) is now so simple, that we can completely solve it.

Proposition 3.4. *System (3.16) has a bounded solution if and only if condition (3.17) is satisfied and moreover there exists $c \in \mathbb{R}$ such that:*

$$(3.18) \qquad\qquad p + \frac{1}{\varphi\psi}\left\{c + \widehat{q}\right\} \ \in \ BP(\mathbb{R}) \,.$$

In this case, all the solutions to (3.16) are bounded.

It is not difficult to check that condition (3.18) can be satisfied for at most a single value of c.

Proof. The equation for the second component solves as $w_2(t) = c + \widehat{q}(t)$ where $c \in \mathbb{R}$. Thus condition (3.17) decides the boundedness of $w_2(t)$. Moreover, the equation for the first component writes:

$$w_1' \ = \ \frac{1}{\varphi\psi}\left\{c + \widehat{q}\right\} + p$$

and the conclusion follows trivially. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

As it can be easily guessed, condition (3.18) does not follow from the necessary condition (3.17) for free, so showing that the latter does not act as a Fredholm–type Alternative. A main point is that, contrarily to the previous paragraph, this is a general failure: it happens *for every $\lambda \in BP(\mathbb{R})$ we take.*

The easiest to construct a counter–example originates from the presence of $p(t)$ in (3.18), which on the contrary does not appear in (3.17). Assume for instance that $q = 0$, so that the necessary condition (3.17) is trivially true. On the other hand, because of Lemma A.1, to make (3.18) failing it is enough to take any $p(t)$ such that:

$$\begin{cases} 0 \in \sigma(p) \\ p \notin BP(\mathbb{R}) \end{cases} \,.$$

To construct a counter–example with $p = 0$ is a bit more involved, but also more interesting for the same reason of the previous paragraph. Start assuming that $q \in BC(\mathbb{R})$ satisfies:

$$(3.19) \qquad\qquad -q(-t) = q(t) < 0 \qquad \forall t > 0 \,.$$

Then $\widehat{q}(t)$ is odd, strictly negative for $t \neq 0$ and the common limit $c_\infty = \widehat{q}(\pm\infty)$ exists by monotonicity and is strictly negative. Our further and last assumptions on $q(t)$ are that c_∞ is finite and:

$$(3.20) \qquad\qquad \widehat{q} - c_\infty \ \notin BP(\mathbb{R}) \,.$$

The necessary condition $q \in BP(\mathbb{R})$ is satisfied by definition. Observe now that, whatever c we take, the sign of the function:

$$\widehat{q}(t) + c = \{\widehat{q}(t) - c_\infty\} + \{c + c_\infty\}$$

is eventually constant as $t \to \pm\infty$. As a consequence, condition (3.18) becomes equivalent to:

$$\widehat{q} + c \in BP(\mathbb{R}) .$$

Now, this is false by definition when $c = -c_\infty$. When on the contrary $c \neq -c_\infty$, using de l'Hôpital rule it is easy to check that:

$$\int_0^t \left[\widehat{q}(s) + c\right] ds = \left\{\frac{1}{t} \int_0^t \left[\widehat{q}(s) - c_\infty\right] ds + (c + c_\infty)\right\} t \sim (c + c_\infty)t$$

as $t \pm \infty$. The primitive is again unbounded, concluding the construction of the desired counter–example.

Coming the original variables, we obtain the following result.

Theorem 3.5. *Assume $\lambda \in BP(\mathbb{R})$. Then system (3.2) has bounded solutions if the necessary condition (3.5) is satisfied and moreover there exists $c \in \mathbb{R}$ such that:*

$$(3.21) \qquad \frac{1}{\varphi} f + \frac{1}{\varphi\psi}\left\{c + \overline{(\lambda\psi - \psi')f + \psi g}\right\} \in BP(\mathbb{R}) .$$

Proof. Composing the change of variables (3.6) with (3.13), we get:

$$x_1 = \frac{1}{\varphi(t)} w_1 \qquad x_2 = \varphi'(t)w_1 + \frac{1}{\psi(t)} w_2 .$$

From Proposition 2.11 we know that $R(\varphi, \psi) = 0$. Using this information to express the inverse of the above change of variables, we find that the inhomogeneous system (3.2) is transformed into (3.16) with:

$$(3.22) \qquad p(t) = \frac{1}{\varphi(t)} f(t) \qquad q(t) = \left\{\lambda(t)\psi(t) - \psi'(t)\right\}f(t) + \psi(t)g(t) .$$

Conclusion then follows from Proposition 3.4. □

Notice that $f = 0$ is equivalent to $p = 0$ once again, In this case, condition (3.5) becomes:

$$q = \psi g \in BP(\mathbb{R}) .$$

This is the usual necessary condition for the scalar second order equation:

$$(3.23) \qquad x'' + \lambda(t)x' + a(t)x = \frac{q(t)}{\psi(t)}$$

to admit bounded solutions, which fails to be sufficient as soon as we take $q(t)$ as in (3.19)–(3.20). The simplest concrete case is given by:

$$x'' = q(t)$$

which is obtained for $a = \lambda = 0$ and the corresponding choice $\varphi = \psi = 1$.

THE ADJOINT DISCONJUGATE CASE $\boxed{0 \notin \sigma(\lambda)}$

As in the previous paragraph, we have:

$$d_B \;=\; 1 \;=\; d_B^*$$

and we exploit the disconjugacy of the adjoint equation (2.5) to change variables again.

Proposition 3.6. *Assume $0 \notin \sigma(\lambda)$. Then the change of variable:*

(3.24) $$u_1 = w_1 - \frac{\psi(t)}{\varphi(t)}\, w_2 \qquad\qquad u_2 = R(\varphi, \psi) w_2$$

is a kinematic similarity transforming the system (3.7) into:

(3.25) $$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda(t) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} .$$

Proof. Remember that $R(\varphi, \psi) = c \neq 0$ from Proposition 2.5, so that (3.24) is a kinematic similarity due to the boundedness of $\varphi(t)$ and $\psi(t)$ with their first derivatives. Concerning the resulting system (3.25), start inverting (3.25) to get:

$$w_1 \;=\; u_1 + \frac{\psi}{\varphi}\, w_2 \;=\; u_1 + \frac{\psi}{c\varphi}\, u_2 \;.$$

Then differentiate and use system (3.7) to show that:

$$w_1' \;=\; \frac{1}{\varphi^2}\, u_2 + \frac{1}{c} \left\{ \frac{\psi'\varphi - \varphi'\psi}{\varphi^2}\, u_2 + \frac{\psi}{\varphi}\, (-\lambda u_2) \right\} \;=\; \frac{1}{\varphi^2} \left\{ c - R(\varphi, \psi) \right\} w_2$$

which vanishes again due to $R(\varphi, \psi) = c$. $\qquad\qquad\qquad\square$

The adjoint system is:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and the general solution:

$$z_1(t) = \alpha \qquad\qquad z_2(t) = \beta e^{\widehat{\lambda}(t)}$$

is bounded if and only if $\beta = 0$. The necessary condition for:

(3.26) $$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda(t) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

to admit bounded solutions then writes:

(3.27) $$p \in BP(\mathbb{R}) \;.$$

This is also sufficient to the same aim since, whatever $q \in BC(\mathbb{R})$ we take, the equation:

$$w_2' \;=\; -\lambda(t) w_2 + q(t)$$

admits a unique bounded solution. Coming back to the original variables, we proved the following result.

Theorem 3.7. *Assume that $0 \notin \sigma(\lambda)$. Whatever $f, g \in BC(\mathbb{R})$ we take, the necessary condition (3.5) is also sufficient for system (3.2) to admit bounded solutions.*

Proof. Using the expression of $R(\varphi, \psi)$, check that the composition of the two changes of variables (3.6) and (3.24) transforms (3.2) into (3.26) with:

$$p(t) = \frac{1}{R(\varphi, \psi)} \left\{ \{\lambda(t)\psi(t) - \psi'(t)\} + \psi(t)g(t) \right\}$$

$$q(t) = \frac{1}{R(\varphi, \psi)} \left\{ -\varphi'(t)f(t) + \varphi(t)g(t) \right\}$$

$\square$

4. Introducing recurrence

In this section we restrict the scope of our investigation, replacing bounded and continuous functions with a special type of recurrence functions. We stipulate from the beginning that the coefficient $a(t)$ and $\lambda(t)$ of the homogeneous planar system:

$$(4.1) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -\lambda(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

are jointly recurrent, namely their joint hull satisfies:

$$(4.2) \qquad\qquad H(a, \lambda) \text{ is compact minimal}$$

and we focus on functions which are *representable* on $H(a, \lambda)$. See Appendix B for the definition and the main properties; since the hull does not vary in the section, we often omit mentioning it.

The idea is to get results for the recurrent framework, by specializing those we obtained in the previous sections for the bounded framework. On the one hand, this suggests that distinguishing:

$$(4.3) \qquad\qquad 0 \notin \sigma(\lambda) \qquad \text{or} \qquad \lambda \in BP(\mathbb{R})$$

from the opposite case:

$$(4.4) \qquad\qquad \begin{cases} 0 \in \sigma(\lambda) \\ \lambda \notin BP(\mathbb{R}) \end{cases}$$

will play a key role. On the other hand, however, specializing is not always for free for a couple of reasons. The first one is that we have to convert existence results from bounded solutions to representable solutions: this is a matter of Favard theory, which is recalled in Appendix B. The second problem concerns the use we made of kinematic similarities: as explained in Appendix D, they are useful in the new context only when they preserve representability. These two questions are indeed the main topics of the present section.

We start exploring the validity of the Favard separation condition for the direct system (4.1). This will be done by means of the dimensional approach introduced in Appendix B, which requires the knowledge of the dimension of the subspace of the bounded solutions to the homogeneous system:

$$(4.5) \qquad\qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b(t) & -\mu(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

when we let varying $(b, \mu) \in H(a, \lambda)$. As in the Appendix A, we refer to this dimension as to the *bounded dimension* of a given equation.

It is clear that the bounded dimension of (4.5) can be equivalently computed for the second order scalar equation:

$$(4.6) \qquad x'' + \mu(t)x' + b(t)x = 0 .$$

Standard compactness arguments show that, for every $(b, \mu) \in H(a, \lambda)$, this equation admits a solutions $x(t)$ such that:

$$(4.7) \qquad 0 < \inf_t \varphi(t) =: m \le x(t) \le M := \sup_t \varphi(t) < +\infty .$$

Because of disconjugacy, all the results of Section 2 apply: this fact seems quite promising in view of Favard theory, but the real picture a bit more involved than expected.

Proposition 4.1. *Assume that (4.2) is verified. Then the Favard dimension of the direct system (4.1) is always $d_F = 1$. Moreover, the Favard separation condition:*

- *a) holds when (4.3) is satisfied*
- *b) may hold or fail when (4.4) is satisfied*

Proof. Due to disconjugacy, the bounded dimension (4.6) is at least for every $(b, \mu) \in H(a, \lambda)$ and hence $d_F \ge 1$. Moreover we know from Section 2 that the bounded dimension is two exactly at those (b, μ) for which:

$$(4.8) \qquad e^{-\widehat{\mu}} \in BP(\mathbb{R}) .$$

Assume now that that (4.3) is satisfied. Then clearly $e^{-\widehat{\lambda}} \notin BP(\mathbb{R})$. Since moreover (4.3) is preserved throughout the hull, condition (4.8) must fail for every $\mu \in H(\lambda)$: the claim is trivial to prove, but it can be also seen as a consequence of Lemma 2.1. The bounded dimension of the equation (4.6) is then one for every $(b, \mu) \in H(a, \lambda)$, which in turn implies that the Favard condition is satisfied with $d_F = 1$.

Let us now consider the case (4.4). It is well known that there exists $\mu \in H(\lambda)$ such that $\widehat{\mu}(t)$ is bounded from below: see Appendix B. For such $\mu(t)$ condition (4.8) fails and the corresponding bounded dimension in one: this implies that $d_F = 1$ once more.

The Favard separation condition is then satisfied if and only if (4.8) fails for every $\mu \in H(\lambda)$. This is for instance true if $\lambda(t)$ is a Kozlov function in the sense of Appendix B. Using indeed the full force of Lemma 2.1, we know that (4.8) implies $\widehat{\mu}(\pm\infty) = -\infty$: but this contradicts condition B.3.

To give an example where the Favard separation condition fails, it is sufficient to consider an almost periodic $\lambda(t)$ whose primitive verifies condition (B.2). $\qquad \square$

Next we worry about the representability of $\varphi(t)$. The parametrization (2.2) shows that we can find disconjugate equations with representable $\varphi(t)$ in both the cases (4.3) and (4.4). The best we can say is the following.

Proposition 4.2. *Assume that (4.2) is verified and that the direct system (4.1) satisfies the Favard separation condition. Then $\varphi(t)$ is representable on $H(a, \lambda)$ together with $\varphi'(t)$ and $\varphi''(t)$.*

We suspect that the validity of the Favard condition result is optimal for the representability of $\varphi(t)$, but we have no explicit counter–examples.

Proof. If $\varphi(t)$ is representable, then so is $\varphi'(t)$ due to the boundedness of $\varphi''(t)$ and Lemma B.2. The representability of $\varphi''(t)$ follows now from the equation. To prove that $\varphi(t)$ is representable, define:

$$\delta(b,\mu) \;=\; \inf\big\{\|x\|_\infty : \; x(t) \text{ solves } (4.6) \text{ and satisfies } (4.7)\big\}$$

and observe that the least upper bound is attained, due to standard compactness arguments. We claim that, because of the discongiugacy of (4.6) it is uniquely attained, say at $\varphi_{(b,\mu)}$.

To prove the claim, we recall form Proposition 4.1 that $d_F = 1$. Since the Favard condition holds by hypothesis, the bounded dimension of (4.6) must be one. As a consequence, if $\delta(b,\mu)$ is attained also at another $\widetilde{\varphi}_{(b,\mu)}$ then we must have

$$\widetilde{\varphi}_{(b,\mu)}(t) \;=\; c\,\varphi_{(b,\mu)}(t)$$

for some scalar $c > 0$. Since the two involved functions must have the same norm, this is possible only when $c = 1$ so proving the claim.

The rest of the proof is standard in Favard theory: we sketch it for the reader convenience. Because of $H(a,\lambda)$ is minimal, one finds that actually $\delta(b,\mu) = \delta$ independently of $(b,\mu) \in H(A,\lambda)$. Together with the uniqueness proved just above, this shows that:

$$(4.9) \qquad \varphi_{(b_\tau,\mu_\tau)}(t) = \varphi_{(b,\mu)}(t+\tau) \qquad \forall t, \tau$$

and that $\varphi_{(b,\mu)}$ depends continuously on $(b,\mu) \in H(a,\lambda)$. In particular, the rule:

$$\phi(b,\mu) \;=\; \varphi_{(b,\mu)}(0)$$

defines a continuous map on $H(a,\lambda)$. Because of (4.9), we have the representation $\varphi(t) = \phi(a_t,\lambda_t)$ and hence conclusion follows from the point (3) of Lemma B.1. $\square$

The picture becomes much sharper when we look at the adjoint system:

$$(4.10) \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & a(t) \\ -1 & \lambda(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or, which is equivalent due to Proposition 3.1, at the adjoint equation:

$$(4.11) \qquad \big\{y' - \lambda(t)y\big\}' + a(t)y = 0 \,.$$

Proposition 4.3. *Assume that (4.2) is verified. Then the Favard dimension of the adjoint system (4.10) verifies $d_F^* \leq 1$ and the following statements are equivalent:*

    a) *$d_F^* = 1$*
    b) *condition (4.3) is satisfied*
    c) *the adjoint system (4.10) satisfies the Favard separation condition*

In this case moreover, the adjoint equation (4.11) is disconjugate and the corresponding $\psi(t)$ is representable on $H(a,\lambda)$ together with $\psi'(t)$ and the derivative of $\psi'(t) - \lambda(t)\psi(t)$.

Proof. We start exploring what happens when (4.4) holds. We take first $\mu \in H(\lambda)$ such that there exist two sequences $s_n \to -\infty$ and $t_n \to +\infty$ satisfying:

$$(4.12) \qquad \widehat{\mu}(s_n) \to +\infty \qquad \widehat{\mu}(t_n) \to +\infty \,.$$

We recall that this is generically true in $H(\lambda)$: see Appendix B. Then we choose any $b \in H(a)$ such that $(b,\mu) \in H(a,\lambda)$. We claim that the corresponding equation:

$$(4.13) \qquad \big\{y' - \mu(t)y\big\}' + b(t)y = 0$$

has no nontrivial bounded solutions. To prove it, we recall from Section 2 that the general solution of (4.13) is:

$$(4.14) \qquad y(t) \; = \; \varphi_{(b,\mu)}(t)\, e^{\widehat{\mu}(t)} \left\{ \alpha \,+\, \beta \int_0^t \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds \right\}$$

where $\varphi_{(b,\mu)}(t)$ now denotes a bounded positive solution separated from zero of the equation (4.6). Assume now that $y(t)$ bounded. From (4.12) we deduce:

$$\alpha \,+\, \beta \int_0^{s_n} \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds \,\to\, 0 \,\leftarrow\, \alpha \,+\, \beta \int_0^{t_n} \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds$$

and hence by monotonicity arguments:

$$\alpha \,+\, \beta \int_0^{-\infty} \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds \,=\, 0 \,=\, \alpha \,+\, \beta \int_0^{+\infty} \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds$$

Eliminating α we deduce:

$$\beta \int_{-\infty}^{+\infty} \frac{e^{-\widehat{\mu}(s)}}{\varphi_{(b,\mu)}(s)^2} \, ds \;=\; 0 \,.$$

which finally implies $\beta = 0$ and hence also $\alpha = 0$. Summing up, when (4.4) holds we have $d_F^* = 0$. But we know from Section 3 that $0 \in \sigma = \sigma^*$ and hence, as explained in Appendix B, the adjoint system (4.10) cannot satisfy the Favard separation condition.

We now consider the opposite case (4.3). Proposition 2.5 applies to equation (4.13) showing that, whatever $(b,\mu) \in H(\lambda)$ we take, the equation is disconjugate and its bounded dimension is exactly one. Thus the adjoint system (4.10) satisfies the Favard separation condition with $d_F^* = 1$. The conclusion about $\psi(t)$ follows along the same steps we already used for $\varphi(t)$ in the proof of Proposition 4.2. $\qquad \Box$

Remark 4.4. It may be of some interest to notice that the adjoint equation (4.11) maintains a weak form of disconjugacy, even in the adverse case (4.4). More precisely, let us consider a $\mu \in H(\lambda)$ such that $\widehat{\mu}(t)$ is bounded from above. Then:

$$y(t) \;=\; \varphi(t)\, e^{\widehat{\mu}(t)}$$

is a bounded positive solution of the equation (4.13) as soon as $(b,\mu) \in H(a,\lambda)$. Since $\widehat{\mu}(t)$ must be unbounded from below, we have:

$$\inf_t y(t) \;=\; 0$$

as predicted by Proposition 2.5. With the help of Ekeland principle, it can be easily checked that such $y(t)$ is actually responsible for the failure of the Favard separation condition for the adjoint system (4.10).

With previous proposition, we completed the proof of the results stated in the Introduction which do not refer to Fredholm Alternative. More precisely:

$$\text{Theorem 1.1} \;\subset\; \text{Propositions } 2.5 + 4.1 + 4.2 + 4.3$$
$$\text{Theorem 1.2} \;\subset\; \text{Propositions } 4.1 + 4.3$$
$$\text{Theorem 1.5} \;\subset\; \text{Propositions } 3.3 + 3.6 + 4.2 + 4.3$$

Moreover, we have now available all the necessary ingredients to discuss the existence of representable solutions to the inhomogeneous system:

$$(4.15) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -\lambda(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

when the inhomogeneous term is also representable. It is clear that the same condition we used in the bounded framework, that it:

$$(4.16) \qquad\qquad y_1 f + y_2\, g \ \in \ BP(\mathbb{R})$$

for every *bounded* solution (y_1, y_2) of the adjoint system (4.10), is again necessary in order (4.15) to admit representable solutions. It might seem more natural testing (4.16) on the representable solutions of the adjoint system, instead of bounded ones, but the former may fail where the latter work fine: an example is given in [2] and commented in Appendix B.
We next show that condition (4.16) may be not sufficient for representable solutions of (4.15) to exist, when we are in the case described by (4.4). The example we provided for the analogous situation in the bounded framework does not work here: there the involved functions are not even recurrent, while we need they are representable. Concerning the Favard separation condition, we know from Proposition 4.3 that it must fail for the adjoint system, and we manage to make it holding for the direct one: this way we show that the Favard separation condition for the adjoint system is optimal for the sufficiency of condition (4.16).

Example 4.5. To keep the construction as simple as possible, we look for an example in the class of second order equations:

$$(4.17) \qquad\qquad x'' + \lambda(t)x' = g(t) \ .$$

We choose the as $\lambda(t)$ a quasi–periodic Kozlov function, as defined in Appendix B. Condition (4.4) is then satisfied by construction. Moreover, as in the proof of part *b)* of Proposition 4.1, the Favard separation codnition is satisfied by the planar system associated to the scalar equation:

$$x'' + \lambda(t)x' = 0 \ .$$

Finally, in addition we assume that the adjoint equation:

$$\left\{ y' - \lambda(t)y \right\}' = 0$$

has no nontrivial bounded solutions. This is possible since $d_F^* = 0$ from Proposition 4.3: then either the claim is true from the beginning, or it becomes true after re–defining $\lambda(t)$ as a suitable element of $H(\lambda)$. Of course, the Kozlov character of $\lambda(t)$ is not affected by the possible re–definition. As a consequence of the assumption we made, condition (4.16) is empty and then trivially satisfied for every $g(t)$ representable on $H(\lambda)$.
It remains to show that there exists a $g(t)$ which is representable on $H(\lambda)$ and such that (4.17) has no representable solutions. To this aim notice that, if $x(t)$ is a representable solution (4.17), then $x'(t)$ is also representable: see Lemma B.2. But $x'(t)$ is a solution of:

$$z' + \lambda(t)z = g(t)$$

and Theorem C.2 in Appendix C guarantees that, due to $0 \notin \sigma(\lambda)$, there is a representable $g(t)$ such that the above equation has no representable solutions.

We consider now the most favorable case (4.3). The Favard condition is satisfied by the direct system (4.1) and the adjoint system (4.10), in both case with Favard dimension one. Moreover, from Section 3 we know that the necessary condition (4.16) takes the simpler form:

$$(4.18) \qquad\qquad (\lambda\psi - \psi')f + \psi g \ \in \ BP(\mathbb{R})$$

where $\psi(t)$ and $\psi'(t)$ are both representable on $H(a, \lambda)$.

Notice that (4.3) includes the case where $a(t)$ and $\lambda(t)$ are both T–periodic. In this case representability means having period T and then (4.18) is clearly equivalent to the classical orthogonality condition:

$$\int_0^T \left\{ \{\lambda(s)\psi(s) - \psi'(s)\} f(s) + \psi(s)g(s) \right\} ds \;=\; 0 \,.$$

The periodic Fredholm Alternative says this is (necessary and) sufficient in order (4.15) to admit T–periodic solutions. As Section 3 suggests, the situation is much more delicate in the recurrent case. The next is our positive result.

Theorem 4.6. *Assume that $H(a, \lambda, f, g)$ is compact minimal and that $0 \notin \sigma(\lambda)$. Then condition (4.18) is necessary and sufficient for (4.15) to admit bounded solutions, all of which being representable on $H(a, \lambda, f, g)$.*

Theorem 1.3 in the Introduction follows when $f(t)$ and $g(t)$ are representable on $H(a, \lambda)$, under the only assumption that $H(a, \lambda)$ is compact minimal. This assumption is weaker than $H(a, \lambda, f, g)$ compact minimal: see Appendix B.

Proof. The characterization of the existence of bounded solutions is provided by Theorem 3.7. Since the direct system (4.1) satisfies the Favard separation condition, Favard Theorem C.1 applies to guarantee that (4.15) has representable solutions if and only it has bounded ones. To conclude that all the bounded solutions are representable on $H(a, \lambda)$, observe that the difference between two of them must be a scalar multiple of $\varphi(t)$, which is representable due to Propositin 4.2. □

It is worth to point out that the same conclusions of Theorem 4.6 could have been obtained by using a more general result of [1]: see Theorem C.4 in Appendix C. The proof of that theorem is also based on kinematic similarities, which however do not preserve the hull in general, and rests on the equality between the Sacker–Sell and the Favard dimensions. When $0 \notin \sigma(\lambda)$, such equality follows from (3.9) and Proposition 4.1 that say respectively:

$$d_F \;=\; 1 \qquad\qquad d_S = 1 \,.$$

The situation is different when $\lambda \in BP(\mathbb{R})$, since we now have:

$$d_F \;=\; 1 \qquad\qquad d_S = 2 \,.$$

Another result of [1] concerns this situation (see Theorem C.5) but, always because of the non preservation of the hull, it does not cover the following result.

Theorem 4.7. *Assume that $H(a, \lambda)$ is not periodic and satisfies (4.2). If moreover $\lambda \in BP(\mathbb{R})$ then there exist functions $f(t)$ and $g(t)$ representable on $H(a, \lambda)$ such that, at the same time:*

  a) condition (4.18) is satisfied
  b) system (4.15) has no bounded solutions

hold at the same time. Moreover, the same conclusion remains valid when we restrict to the case $f = 0$.

Theorem 1.4 in the Introduction corresponds to the last claim.

Proof. As explained in Appendix D, the representable kinematic similarities does not affect the representable Fredholm Alternative. Because of Theorem 1.5, when looking for a counter–example we can always assume to deal with the normal form:

$$
\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{\varphi(t)\psi(t)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} .
$$

We recall from Section 3 that the necessary condition (4.18) becomes $q \in BP(\mathbb{R})$ and that it becomes sufficient for the existence of bounded solutions when, in addition, there exists $c \in \mathbb{R}$ such that:

$$
(4.19) \qquad\qquad p + \frac{1}{\varphi\psi} \left\{ c + \widehat{q} \right\} \ \in \ BP(\mathbb{R}) .
$$

When $a(t)$ and $\lambda(t)$ are both T–periodic, we already know indirectly that the additional condition (4.19) must be superfluous. To check this fact directly, notice that (4.19) is now equivalent to say that the involved function has mean value zero. So it's enough to choose c such that:

$$
\int_0^T \frac{c + \widehat{q}(s)}{\varphi(s)\psi(s)} \, ds \ = \ -\int_0^T p(s) \, ds .
$$

From now on, we assume that $a(t)$ and $\lambda(t)$ are not jointly periodic. As in Section 3 we construct two types of counter–examples. When we suppose $q = 0$, is enough taking a representable $p(t)$ satisfying:

$$
\begin{cases} 0 \in \sigma(p) \\ p \notin BP(\mathbb{R}) \end{cases} .
$$

This is never a problem due to the aperiodicity of $H(a, \lambda)$ (see Appendix B) and Lemma A.1 applies to show that (4.19) fails for every $c \in \mathbb{R}$.

Suppose now $p = 0$, which in Section 3 we proved to correspond to $f = 0$ in the original variables. Again due to the aperiodicity of $H(a, \lambda)$ we known there exists a continuously differentiable function $h(t)$ which is representable on $H(a, \lambda)$ together with $h'(t)$ and such that:

$$
\begin{cases} 0 \in \sigma(h) \\ h \notin BP(\mathbb{R}) \end{cases} .
$$

When $a(t)$ and $\lambda(t)$ are almost periodic, the function $h(t)$ can be constructed by hand, as generalized Fourier series. This is no longer possible in the general recurrent case, but the existence of functions of this type is nevertheless common knowledge. For instance, in [1] the existence of $h \in C(\mathbb{R})$ is shown, but minor modifications in the proof allow to construct $h \in C^1(\mathbb{R})$.

Define now:

$$
q(t) \ = \ (\varphi\psi h)'(t)
$$

and observe that $q(t)$ is representable on $H(a, \lambda)$ and satisfy $q \in BP(\mathbb{R})$. The conclude the proof, we have to show that condition (4.19) is satisfied, with $p = 0$. To this aim notice that $\widehat{q}(t) = \varphi(t)\psi(t)h(t) - \alpha$ where $\alpha = \varphi(0)\psi(0)h(0)$ and hence:

$$
\frac{1}{\varphi(t)\psi(t)} \left\{ c + \widehat{q}(t) \right\} \ = \ h(t) + \frac{c - \alpha}{\varphi(t)\psi(t)} .
$$

The primitive is unbounded for every $c \in \mathbb{R}$, once more due to Lemma A.1. $\square$

Appendix A. Summary of spectral theory

Consider the linear homogeneous system:

$$(A.1) \qquad\qquad x' = A(t)x$$

where $A \in BC(\mathbb{R})$ is and $N \times N$ matrix and denote by $X_A(t)$ the principal matrix solution, namely that satisfying $X_A(0) = I$. The system has an *exponential dichotomy* (over the whole $\mathbb{R}$) when there is a (time independent) projector P in $\mathbb{R}^N$ and constants $K \geq 1$ and $\delta > 0$ such that:

$$(A.2) \qquad \begin{aligned} \left\| X_A(t) P\, X_A(s)^{-1} \right\| &\leq K e^{-\delta(t-s)} && \forall t \geq s \\ \left\| X_A(t)(I-P) X_A(s)^{-1} \right\| &\leq K e^{-\delta(s-t)} && \forall s \geq t \end{aligned}$$

See Coppel's book [5] for a good introduction to the subject: we refer to it for the proofs of the properties we need. The projector P is uniquely defined, inasmuch $\ker(P)$ and $\operatorname{Im}(P)$ are made by the initial data $\xi \in \mathbb{R}^N$ such that $X_A(-\infty) = 0$ and $X_A(+\infty) = 0$ respectively. In particular, system (A.1) no bounded solution nut the trivial one.

The *dichotomy spectrum* $\sigma(A)$ of the system (A.1) is defined as the set of $\gamma \in \mathbb{R}$ such that:

$$x' = \big[A(t) - \gamma I\big]x$$

has not an exponential dichotomy. The notion of spectrum has been first introduced and studied by Sacker and Sell in [17] when $H(A)$ is compact, while the noncompact case has been considered by Siegmund in [22]. Hereafter we recall only the few facts of spectral theory that we need.

The spectrum is made by $1 \leq n \leq N$ closed, bounded and pairwise disjoint (possibly degenerate) intervals:

$$\sigma(A) \;=\; [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

usually called the *spectral intervals* of A. Each spectral interval $[a_k, b_k]$ takes with it a spectral linear subspace $\mathcal{W}_k(A) \subset \mathbb{R}^N$. It consists of the initial data of solutions that, roughly speaking, have Lyapunov exponents in the interval. More precisely $\xi \in \mathcal{W}_k(A)$ if and only if:

$$(A.3) \qquad \lim_{t \to -\infty} e^{-\alpha t} X_A(t)\xi \;=\; 0 \;=\; \lim_{t \to +\infty} e^{-\beta t} X_A(t)\xi$$

where $\alpha < a_k \leq b_k < \beta$ is any open neighborhood of the spectral interval $[a_k, b_k]$ which avoids all the other spectral intervals. The Spectral Theorem states that the spectral subspaces span the whole $\mathbb{R}^N$, namely:

$$\mathcal{W}_1(A) \oplus \;\cdots\; \oplus \mathcal{W}_n(A) \;=\; \mathbb{R}^N \;.$$

For comparison use only, we are interested in the dimension of one of these subspaces, namely:

$$(A.4) \qquad\qquad d_S(A) \;=\; \dim \mathcal{W}_k(A)$$

where k is the unique index such that $0 \in [a_k, b_k]$. Of course, we agree that $d_S(A) = 0$ when $0 \notin \sigma(A)$. As in [1], we give $d_S(A)$ the name of *Sacker–Sell dimension* of $A(t)$.

Let us now introduce a second dimension, which we call *bounded dimension* of $A(t)$, in the following way:

$$(A.5) \qquad\qquad d_B(A) = \dim \mathcal{V}(A)$$

where $\mathcal{V}(A)$ is the subspace of $\mathbb{R}^N$ of the initial data giving rise to bounded solutions of (A.1), that is:

$$\mathcal{V}(A) = \left\{\xi \in \mathbb{R}^N : \sup_t |X_A(t)\xi| < +\infty\right\}.$$

If $0 \in \sigma(\lambda)$ the clearly $\mathcal{V}(A) \subset \mathcal{W}_k(A)$, the superset being the same we used in (A.4). This yields the following inequality:

$$(A.6) \qquad\qquad d_B(A) \leq d_S(A).$$

Whether this inequality is strict or not is very relevant for our aim, for reason which will be presented in the next appendix.

When $A(t)$ gives rise to an exponential dichotomy then the same is true also for its time translations $A_\tau(t) = A(t + \tau)$ and all their uniform limits on compacts sets, namely the elements of:

$$H(A) = \mathrm{cls}\big\{A_\tau : \tau \in \mathbb{R}\big\}.$$

Notice that $H(A) \subset BC(\mathbb{R})$ and that, in general, it is not compact: see Appendix B for more details on the argument. Like the exponential dichotomy, also the dichotomy spectrum and the dimensions of the spectral subspaces are hull invariants. That is, in particular:

$$\sigma(B) = \sigma(A) \qquad\qquad d_S(B) = d_S(A)$$

for every $B \in H(A)$. For the bounded dimension, it is cleat that $d_B(A_\tau) = d_B(A)$ holds again for every τ, but it may happen that $d_B(A)$ differs form $d_B(A)$ for some $B \in H(A)$: this fact is deeply related to Favard theory, which will be recalled in the next appendix too.

We now specialize spectral theory to the *scalar case* $N = 1$. The spectrum $\sigma(A)$ consists of a single interval and the dimension $d_S(A)$ is either 0 or 1, depending on the fact that either $0 \notin \sigma(A)$ or $0 \in \sigma(A)$ respectively. Concerning the consistency of the spectrum, notice that $P = 0$ and $P = I$ are the only two possible projectors on $\mathbb{R}$ and:

$$X_{A-\lambda}(t) = e^{\int_0^t A(s)\, ds - \lambda t}.$$

Thus one has $\lambda \notin \sigma(A)$ if and only one of the following alternatives:

$$(A.7) \qquad \liminf_{T \to +\infty} \frac{1}{T}\int_s^{s+T} A(t)\, dt > \lambda \qquad \text{or} \qquad \limsup_{T \to +\infty} \frac{1}{T}\int_s^{s+T} A(t)\, dt < \lambda$$

holds uniformly for $s \in \mathbb{R}$. If for instance $A \in BP(\mathbb{R})$ then $\sigma(A) = \{0\}$ and hence $d_S(A) = 1$; moreover it is clear that also $d_B(A) = 1$. In the paper we often bump into the case:

$$(A.8) \qquad\qquad \begin{cases} 0 \in \sigma(A) \\ A \notin BP(\mathbb{R}) \end{cases}$$

that corresponds to $d_S(A) = 1$ and $d_B(A) = 0$. Next lemma says what happens when we add scalar multiples of an $\omega \in C(\mathbb{R})$ satisfying:

$$(A.9) \qquad\qquad 0 < \inf_t \omega(t) \leq \sup_t \omega(t) < +\infty.$$

Lemma A.1. *Assume (A.8) and (A.9) hold. Then for every $c \in \mathbb{R}$ one has:*

$$A + c\,\omega \notin BP(\mathbb{R}) \ .$$

Proof. Because of (A.8), the claim is true for $c = 0$. Assume now by contradiction that the claim is false for some $c \neq 0$, namely:

$$A + c\,\omega = h \in BP(\mathbb{R}) \ .$$

As a consequence of (A.9) and the characterization (A.7), we then have:

$$0 \notin \sigma\,(-c\,\omega) = \sigma(A - h) \ .$$

But (A.7) applies again to show that $\sigma(A - h) = \sigma(A)$. Thus (A.8) gives the contradiction $0 \in \sigma(A - h)$. $\qquad\square$

Let us coming back to the general higher dimensional case, by looking at spectral properties of the adjoint system:

$$(A.10) \qquad\qquad\qquad y' \ = \ -A(t)^T y \ .$$

In the paper we will mark with a superscript * any mathematical object, when refers to the adjoint system (A.10) instead of the direct system (A.1). Thus we have:

$$X_A^*(t) \ = \ \left(X_A(t)^T \right)^{-1}$$

for the principal matrix solution of (A.10). Because of that, the adjunction is shown to preserve exponential dichotomies together with all the main spectral features. In particular one has:

$$(A.11) \qquad\qquad \sigma^*(A) = -\sigma(A) \qquad\qquad d_S^*(A) = d_S(A) \ .$$

On the contrary, the bounded dimensions $d_B(A)$ and $d_B^*(A)$ seems to be largely unrelated.

We conclude the appendix by discussing the existence of bounded solutions for the inhomogeneous system:

$$(A.12) \qquad\qquad\qquad x' = A(t)x + f(t) \ .$$

It is well known [12] that $0 \notin \sigma(A)$ if and only if, whichever $f \in BC(\mathbb{R})$ we take, system (A.12) admits a bounded solution. In fact such solution is unique and writes:

$$x(t) \ = \ \int_{-\infty}^{t} X(t) P X(s)^{-1} f(s)\, ds \ - \ \int_{t}^{+\infty} X(t)(I - P)X(s)^{-1} f(s)\, ds$$

where P is the projector that appears in (A.2). This formula can be used to show that $x(t)$ inherits some important features from $A(t)$ and $f(t)$, like for instance a recurrence: see Appendix B.

More in general, a necessary condition for (A.10) to admit bounded solutions is easily obtained by invoking the bounded solutions of the adjoint system (A.10), that is:

$$(A.13) \qquad\qquad \left\langle X_A^*(\cdot)\zeta, f \right\rangle \in BP(\mathbb{R}) \qquad\qquad \forall \zeta \in \mathcal{V}^*(A) \ .$$

This condition becomes sufficient, though somewhat trivially, when for instance $0 \notin \sigma(A)$. Indeed $0 \notin \sigma^*(A) = -\sigma(A)$ and hence:

$$d_B(A) = d_B^*(A) = 0 \ .$$

Thus (A.13) is empty, but we know that (A.12) has nevertheless a bounded solution for every $f \in BC(\mathbb{R})$. The idea that (A.13) may be actually determine a Fredholm–type Alternative has been suggested in [23], after proving that it is sufficient also when $A(t)$ is such that:

$$0 \; < \; \inf_t |X_A(t)\xi| \; \leq \; \inf_t |X_A(t)\xi| \; < \; +\infty \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\} \; .$$

Notice that this case is diametrically opposite to the previous one, inasmuch:

$$\sigma(A) = \sigma^*(A) = \{0\} \qquad\qquad d_B(A) = d_B^*(A) = N$$

The question is to see what happen for bounded dimensions which are intermediate between 0 and N. Section 3 makes clear that the concern of the present paper is one of this cases. The paper [14] consider is a special case of the present one, and there everything works fine. The same is true also for [4] and [2]. The paper [1] makes clear that sufficiency may fail in general, but at the same time suggests which is the class of $A(t)$'s where it works: we recall this result in Section 4, since it concerns the recurrent framework.

Appendix B. Representability on minimal hulls

The Bebutov flow in the space $C(\mathbb{R})$ with a fixed codomain is defined by:

$$u_\tau(t) = u(t + \tau) \qquad \forall t$$

and is continuous $\mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R})$ as soon as $C(\mathbb{R})$ is endowed with the compact–open topology. Given an $u \in C(\mathbb{R})$, the closure of its flow line is the hull $H(u)$ as defined in the previous appendix. It is well know that $H(u)$ is compact if and only if $u \in BUC(\mathbb{R})$: see [21] for a proof. Moreover, it is not difficult to see that $BC(\mathbb{R})$ and $BUC(\mathbb{R})$ are closed with respect to the operation of taking hulls.

The function $u(t)$ is said *recurrent* when $H(u)$ is compact and minimal. This is for instance true when $u(t)$ is continuous and periodic or almost periodic, in the sense of Bohr.

We next consider the case where $u(t)$ is a *scalar recurrent* function. It is easy to check that properties like $0 \notin \sigma(u)$ or $u \in BP(\mathbb{R})$ are *hull invariants*: that is, either they are possessed by every element $v \in H(u)$ or by no one. Moreover the primitive $\widehat{u}(t)$ is bounded in the whole $\mathbb{R}$ if and only if it bounded in $\mathbb{R}^+$ or in $\mathbb{R}^-$. The most interesting case is however:

(B.1) $$\begin{cases} 0 \in \sigma(u) \\ u \notin BP(\mathbb{R}) \end{cases} .$$

This cannot happen when $u(t)$ is periodic, but it is well known to be the most frequent case in the recurrent aperiodic framework: see after for a more precise statement. For the moment, we fix a function $u(t)$ as in (B.1) and summarize some set properties of $\widehat{v}(t)$ when $v \in H(u)$. Since $0 \in \sigma(u)$ we know from [18] that there exists $v \in \sigma(u)$ such that:

$$\inf_t \widehat{v}(t) \; > \; -\infty$$

and then automatically:

$$\sup_{t \leq 0} \widehat{v}(t) \ = \ +\infty \ = \ \sup_{t \geq 0} \widehat{v}(t) \ .$$

Since $0 \in \sigma^*(u) = -\sigma(u)$, another $v \in H(u)$ exists for which the same conclusions hold with reversed signs. Moreover, Johnson proved in [9] that these occurrences are rare facts, inasmuch:

$$\liminf_{t \to \pm\infty} \widehat{v}(t) = -\infty \qquad\qquad \limsup_{t \to \pm\infty} \widehat{v}(t) = +\infty$$

for a generic $v \in H(u)$.

Besides these general properties, suitable scalar recurrent $u(t)$ can be constructed, whose primitive has some special additional properties. Given for instance any $0 < \alpha < 1$, it is well know (see [16] and [25]) that there exist quasi periodic $u(t)$ such that:

(B.2) $$\liminf_{|t| \to +\infty} \ |t|^{-\alpha} \widehat{u}(t) \ > \ 0 \ .$$

Finally, it is much less known but as much relevant, that such asymptotic behavior is impossible when the quasi periodic $u(t)$ is represented by a sufficiently smooth function on its own hull $H(u)$, which is a torus. More precisely, in this case $\widehat{u}(t)$ is Poisson stable in the future and in the past. This was first proved by Kozlov in[10] for the two–dimensional torus and then generalized to higher dimensional tori by Moshchevitin in [13]. See also [1] for a more complete description of these results and for the trivial consequence were are interested in, that is:

(B.3) $$\forall v \in H(u) \qquad \lim_{|t| \to +\infty} \ \widehat{v}(t) \ \text{ does not exist.}$$

We call *Kozlov functions* the scalar recurrent functions $u(t)$ satisfying conditions (B.1) and (B.3).

Coming back to the general possibly nonscalar case, assume that $u(t)$ is recurrent and consider another function $v \in C(\mathbb{R})$. We say that $v(t)$ is *representable* on $H(u)$ when a continuous flow homomorphism exists:

$$\chi : H(u) \to H(v) \qquad \text{with} \qquad \chi(u) = v \ .$$

Since $\chi(u_\tau) = v_\tau$ must be true for every τ, it is clear that χ is unique and is actually an epimorphism. In particular $H(v)$ is minimal and hence $v(t)$ is recurrent. Next lemma is a variations of some results by Ščerbakov in [20] and says that $v(t)$ actually inherits the recurrence type of $u(t)$. The proof is standard and then omitted, but it can be found in [1].

Lemma B.1. *Let $u(t)$ be recurrent and $v \in C(\mathbb{R})$, Then the following statements are equivalent:*

    (1) *$v(t)$ is representable on $H(u)$*
    (2) *for every choice of the involved time sequences, if $u_{\tau_n} - u_{s_n} \to 0$ then also $v_{\tau_n} - v_{s_n} \to 0$*
    (3) *there exists (a unique) $V \in C\big(H(u)\big)$ such that $v(t) = V(u(t))$ for every t*

As a trivial consequence, if for instance $u(t)$ is T–periodic then $v(t)$ is T–periodic too. In other words, representability is the key to introduce a notion of boundary data in the recurrent aperiodic context. Using the equivalences of Lemma B.1 it is easy to check that representability is preserved under a number of algebraic operation, like sums or product or composition with continuous functions, when

they are well defined. Next two lemmas worry about analytic operations and are classical.

Lemma B.2. *Let $u(t)$ be recurrent and differentiable everywhere. If $u'(t)$ is uniformly continuous, then $u'(t)$ is representable on $H(u)$.*

Lemma B.3. *Let $u(t)$ be recurrent. If $u \in BP(\mathbb{R})$ the its primitive $\widehat{u}(t)$ is representable on $H(u)$.*

A function $v(t)$ is said *jointly recurrent* with $u(t)$ when the pair $(u(t), v(t))$ is recurrent, namely their joint hull $H(u, v) \subset H(u) \times H(v)$ is a compact minimal set. In this case, the restricted projections:

$$H(u, v) \to H(u) \qquad\qquad H(u, v) \to H(v)$$

which are continuous flow homomorphisms that preserve the base points of the hulls, are indeed epimorphisms. As a consequence, the minimality of $H(u, v)$ implies that of $H(u)$ and $H(v)$, that is the recurrence of $u(t)$ and $v(t)$. Moreover, by composing the involved flows, it is clear that a function $w(t)$ which is representable on $H(u)$ or $H(v)$ is also representable on $H(u, v)$.

Next lemma goes in the opposite direction, showing that representability is a particular case of joint recurrence.

Lemma B.4. *Let $u(t)$ be recurrent and $v(t)$ representable on $H(u)$. Then $v(t)$ is jointly recurrent with $u(t)$ and the projection $H(u, v) \to H(u)$ is actually a flow isomorphism.*

Proof. Due to representability, the map $u_\tau \mapsto (u_\tau, v_\tau)$ is well defined and uniformly continuous. Thus it extends uniquely and continuously to $H(u)$ and, since the initial map is a flow homomorphism, the same is true for the extension. The extension is actually the inverse of the projection $H(u, v) \to H(u)$. $\square$

APPENDIX C. SUMMARY OF FAVARD THEORY

The notion of representability is at the very core of Favard theory, whose natural scope is the recurrent framework. To introduce this theory, instead of the single homogeneous system (A.1) we have to consider the class of homogeneous systems:

$$\text{(C.1)} \qquad\qquad\qquad\qquad x' = B(t)x$$

where $B \in H(A)$. The so-called *Favard separation condition* requires that, for every $B \in H(A)$, the following condition on the bounded solutions to (C.1) is satisfied:

$$\text{(C.2)} \qquad\qquad \inf_t |X_B(t)\xi| > 0 \qquad\qquad \forall \xi \in \mathcal{V}(B) \setminus \{0\} \,.$$

Later on, we express this fact by saying that (F_A) is satisfied.

Theorem C.1. *Let $A(t)$ and $f(t)$ be jointly recurrent and assume that (F_A) is satisfied. If the inhomogeneous system:*

$$\text{(C.3)} \qquad\qquad\qquad x' = A(t)x + f(t)$$

has a bounded solution, then it also has a representable solution on $H(A, f)$.

Let us stress that representability is the core of the result: recurrent solutions, even jointly with $A(t)$ and $f(t)$, always exist due to standard compactness arguments, independently of the Favard separation condition. The theorem was proved by Favard in [7] in the almost periodic case, but the proof extends without changes to the recurrent case: see also Palmer in [15]. On the contrary, turning down the minimality of $H(A, f)$ seems to break down the main results of the theory: see [3] and [1] for more details.

Next we assume that $H(A)$ *is minimal* and discuss the validity of condition (F_A). It is clear that (F_A) is trivially satisfied when $0 \notin \sigma(A)$, since in this case the bounded space of $B(t)$ satisfies:

$$\mathcal{V}(B) = \{0\} \qquad \forall B \in H(A)$$

and there is nothing to test. Sacker and Sell proved in [18] that, due to the minimality of $H(A)$, the above condition is actually equivalent to $0 \notin \sigma(A)$. There is another characterization that we need for the construction of some counter–example, which is due to Massera and Scäffer, see Theorem 103.A in [12].

Theorem C.2. *Let $A \in AP(\mathbb{R})$ but aperiodic. Then $0 \notin \sigma(A)$ if and only if and only if, for every $f(t)$ representable on $H(A)$, system (C.3) has a solution representable on $H(A)$.*

In general, it may happen that condition (C.2) is satisfied for some $B \in H(A)$ but not for all of them: this is a main difference with spectral features, which are typically hull invariant. A description of the way the Favard separation condition breaks down has been provided in [3]. There the authors introduce the *Favard dimension* of A as the minimal bounded dimension along the hull, that is:

$$d_F(A) = \min_{B \in H(A)} d_B(B)$$

and prove that such minimal dimension is attained exactly at those B's for which condition (C.2) is satisfied. Moreover, they show that such B's describe a topologically large subset of $H(A)$. Thus condition (F_A) is satisfied if and only if:

(C.4) $\qquad\qquad d_B(B) = d_F(A) \qquad \forall B \in H(A)$.

This often represents a convenient way to test the Favard separation condition: we will use this approach in Section 4.

The Favard Theorem C.1 allows to re–use condition (A.13) for building up a Fredholm–type Alternative in the recurrent framework. To this aim, we take a recurrent matrix $A(t)$ satisfying (F_A) and we keep it fixed. We say that this matrix, or the corresponding homogeneous system:

(C.5) $\qquad\qquad\qquad\qquad x' = A(t)x$

satisfies some kind of Fredholm–type alternative in the recurrent framework, when the necessary condition (A.13) is also sufficient in order (A.12) to admit bounded solutions, and this happens for all the inhomogeneous terms $f(t)$ which belongs to some suitable class of recurrent functions. Depending of the class, we distinguish two different Fredholm–type Alternatives:

    (a) the *representable Fredholm Alternative*, where $f(t)$ is representable on $H(A)$ and from a bounded solution of (C.3) one gets, via Favard theory, a solution representable on $H(A)$;

(b) the *recurrent Fredholm Alternative*, where $f(t)$ is jointly recurrent with A(t) and from a bounded solution of (C.3) one gets, via Favard theory, a solution representable on $H(A, f)$.

Remark C.3. It is natural wondering what happens when condition (A.13) is weakened, by asking that:

$$(C.6) \qquad\qquad \langle y, f \rangle \in BP(\mathbb{R})$$

only for those solutions of the adjoint system (A.10) that are representable on $H(A)$. The best answer is that there are matrices $A \in AP(\mathbb{R})$ such that: the recurrent Fredholm Alternative works fine when formulated with (A.10), while the representable Fredholm Alternative fails when formulated with (C.6). A concrete example is provided by the planar system:

$$(C.7) \qquad\qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha(t) \\ \alpha(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

where $\alpha \in AP(\mathbb{R})$ si such that $\overline{\alpha} = 0$ and $\alpha \notin BP(\mathbb{R})$. The associated homogeneous system is self–adjoint and all its solutions have constant Euclidean norm in $\mathbb{R}^2$, but Lillo proved in [11] that none of them is almost periodic. Thus the weak condition (C.6) is empty, but Theorem C.2 guarantees that there is at least one pair $(f_1, f_2) \in AP(\mathbb{R})$ and representable on $H(\alpha)$, such that (C.7) has no bounded solutions. On the other hand, the results of [23] or [1] apply to prove that, whatever pair of jointly recurrent functions $f_1(t), f_2(t)$ we take, condition (A.13) is sufficient in order (C.7) to admit solutions representable on $H(\alpha, f_1, f_2)$.

The notion of representable Fredholm Alternative is probably the most natural one. Consider indeed the case where $A(t)$ is purely periodic. In spite of the fact that condition (A.13) seems to require much more than the classical periodic orthogonality condition, it is not difficult to see that the representable Fredholm Alternative is completely equivalent to the classical periodic one: see [2] for more on this subject. On the contrary, the recurrent Fredholm Alternative is much stronger than the representable one in any context: if for instance $A(t)$ is periodic, with the recurrent Fredholm Alternative we pretend to have the better of, at least, all the $f(t)$'s which are almost periodic.

Roughly speaking, the aim of paper [1] is to characterize the recurrent Fredholm Alternative. Before introducing the concrete results, it is worth pointing out that, besides (F_A), also the Favard separation condition (F_A^*) for the adjoint system (A.10) plays a relevant role in the program of [1]. The point is that, contrarily to the spectral feature, in general the Favard–type features are not preserved under adjunction: for instance, [1] provides an example where (F_A) and (F_A^*) have not the same truth values. We will not discuss further this question here, passing instead to state the two results of [1] we are interested in. The key condition involves the Sacker–Sell dimension $d_S(A)$ introduced in Appendix A, which of course always satisfies:

$$d_F(A) \ \leq \ d_B(A) \ \leq \ d_S(A) \, .$$

In general, all the above inequalities can be strict. The first result of [1] says that, when they are not strict, everything work fine with the Fredholm Alternative.

Theorem C.4. *Let $A(t)$ be recurrent and assume that:*

$$(C.8) \qquad\qquad d_F(A) \ = \ d_S(A) \, .$$

Then (F_A) and (F_A^*) are both satisfied with Favard dimension $d_S(A)$ and the recurrent Fredholm Alternative holds.

The second result is a kind of low dimensional converse of the previous one: it proves the optimality of (C.8) for the Fredholm Alternative and suggests it may be necessary.

Theorem C.5. *Let $A(t)$ be recurrent and (F_A) and (F_A^*) be satisfied. If $d_S(A) \leq 2$ and the recurrent Fredholm Alternative holds, then $d_F(A) = d_S(A)$.*

The proofs of both the results rest on a kinematic similarities that, in general, do not preserve the hull $H(A)$: because of that, they are not suitable to deal with the representable Fredholm Alternative, as explained in the next appendix. In particular, Theorem 1.4 in the Introduction is not covered by Theorem C.5.

Appendix D. Kinematic similarities

Let $Q(t)$ be a time dependent matrix which is invertible and continuously differentiable. By setting:

$$\text{(D.1)} \qquad\qquad x \;=\; Q(t)u$$

we transform the inhomogeneous system (A.12) into:

$$\text{(D.2)} \qquad\qquad u' \;=\; C(t)u + g(t)$$

where of course:

$$\text{(D.3)} \qquad C(t) \;=\; Q(t)^{-1}\bigl\{A(t)Q(t) - Q'(t)\bigr\} \qquad g(t) = Q(t)^{-1}f(t) \ .$$

As usual in the literature, we say that (D.1) is a *kinematic similarity* when in addition:

$$\text{(D.4)} \qquad\qquad Q, Q^{-1}, Q' \;\in\; BC(\mathbb{R}) \ .$$

The name Lyapunov–Perron transformations is also used in the literature to denote the same change of variables. When (D.4) holds, it is clear that $A \in BC(\mathbb{R})$ is equivalent to $C \in BC(\mathbb{R})$ and that the same is true for the pair $f(t)$ and $g(t)$, the corresponding spaces actually being isomorphic. From now on, *we will implicitly assume that $A, f \in BC(\mathbb{R})$ and that (D.4) is verified.*

First of all, it is clear that the change of variables (D.1) maps isomorphically the bounded solution to (D.2) onto the bounded solutions to (A.12). When $f = g = 0$ this implies that:

$$d_B(C) = d_B(A) \ .$$

Actually, the principal matrix solutions corresponding to the homogeneous systems associated to (A.12) and (D.2) are related by the formula:

$$\text{(D.5)} \qquad\qquad X_A(t) \;=\; Q(t)X_C(t)Q(0)^{-1} \ .$$

It is not difficult to see that exponential dichotomies are preserved and how the involved projections are related. The final effect is that the spectral features are unaffected by kinematic similarities, which we summarize by saying:

$$\sigma(C) \;=\; \sigma(A) \qquad\qquad d_S(C) = d_S(A) \ .$$

Another standard fact is that the adjoint systems are also kinematically similar. Precisely, the new adjoint system:

$$(D.6) \qquad\qquad v' \ = \ -C(t)^T v$$

is obtained from the old one (A.10) via the change of variables:

$$(D.7) \qquad\qquad y \ = \ \left\{Q(t)^{-1}\right\}^T v$$

which again satisfies (D.4). Thus all the aforementioned arguments and equalities have their own starred counterpart. Moreover, observe that:

$$\big\langle v(t), g(t) \big\rangle = \big\langle v(t), Q(t)^{-1} f(t) \big\rangle \big\langle \{Q(t)^{-1}\}^T v(t), f(t) \big\rangle = \big\langle y(t), f(t) \big\rangle$$

for every solution $v(t)$ of (D.6), being $y(t)$ the corresponding solution of (A.10) obtained via (D.7). As a consequence, the necessary condition (A.13) for the existence of bounded solutions to (A.12) translates word–by–word into the analogous condition for (D.2).

We now introduce recurrence into play, by assuming that the matrix $A(t)$ is recurrent. In order the matrix $C(t)$ to be recurrent, condition (D.4) is no longer sufficient.

Lemma D.1. *Assume the kinematical similarity $Q(t)$ is jointly recurrent with $A(t)$ and that moreover $Q' \in BUC(\mathbb{R})$. Then $C(t)$ is representable on $H(A, Q)$ and:*

(1) *$d_F(A) = d_F(C)$;*
(2) *(F_A) is satisfied if and only if (F_C) is.*

Proof. Because of Lemma B.2 we know that $Q'(t)$ is representable on $H(Q)$ and hence also in $H(A, Q)$, which is minimal by assumption. Thus (D.3) shows that $C(t)$ is representable on $H(A, Q)$. Observe now that:

$$(D.8) \qquad (B, R) \in H(A, Q) \ \mapsto \ \big(R^{-1}\{BR - R'\}, R\big) \in H(C, Q)$$

is a flow isomorphism preserving the base points. Consider now $B \in H(A)$. Since the projection $H(A, Q) \to H(A)$ is a flow epimorphism, there exists $R \in H(Q)$ such that $(B, R) \in H(A, Q)$. Let be $D(t) = R(t)^{-1}\{B(t)R(t) - R'(t)\}$ and observe that $R(t)$ defines a kinematical similarity between (C.1) and:

$$v' = -D(t)^T v \ .$$

As a consequence $d_B(B) = d_B(D)$ and, when B is allowed to vary in the whole $H(A)$, this implies $d_F(A) \geq d_F(C)$. By reversing the arrow (D.8) we get, along similar steps, the inverse inequality $d_F(C) \geq d_F(A)$. The dimensional characterization (C.4) of the Favard separation condition shows now that (F_A) and (F_C) are equivalent. $\qquad\square$

We say that a kinematic similarity is *representable on $H(A)$* when the corresponding $Q(t)$ is and $Q' \in BUC(\mathbb{R})$. Because of Lemma B.4, by specializing the previous lemma we get that $C(t)$ is also representable on $H(A)$. Moreover, it is clear that the map:

$$f \mapsto Q^{-1} f$$

is a linear isomorphism of the space of the representable functions on $H(A)$. Together with what we said above for the bounded framework, this fact allows to conclude that the *representable Fredholm Alternative* holds for the system (A.1) if and only if it does for:

$$u' = C(t)u \ .$$

This conclusion may fail when $C(t)$ happens to be representable but $Q(t)$ is not. In this case, the recurrent Fredholm Alternative is a more appropriate notion: see [1] for more information on the subject.

References

[1] J. Campos, R. Obaya and M. Tarallo, *Favard theory for the adjoint equation and Fredholm Alternative*, preprint 2015.

[2] J. Campos, R. Obaya and M. Tarallo, *Recurrent equations with wign wnd Fredholm Alternative*, preprint 2015.

[3] J. Campos and M. Tarallo, *Almost automorphic linear dynamics by Favard theory*, J. Differential Equations, **256** (2014), 1350–1367.

[4] P. Cieutat and A. Haraux *Exponential decay and existence of almost periodic solutions for some linear forced differential equations*, Port. Math., **59** (2002), 141–159.

[5] W.A. Coppel, "Dichotomies in stability theory", Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York, 1978.

[6] I. Ekeland, *On the variational principle.*, J. Math. Anal. Appl., **47** (1974), 324–353.

[7] J. Favard, *Sur les equations différentielles linéairesà coefficients presque–périodiques*, (French) [On the linear differential equations with almost periodic coefficients], Acta Math., **51** (1927), 31–81.

[8] J.K. Hale, "Ordinary differential equations", Pure and Applied Mathematics, Vol. XXI, Wiley-Interscience, New York, 1969.

[9] R.A. Johnson, *Minimal functions with unbounded integral*, Israel J. Math., **31** (1978), 133-141.

[10] V.V. Kozlov, *On a problem by Poincaré*, J. Appl. Math. Mech., **40** (1976), 326-329.

[11] J.C. Lillo, *Approximate similarity and almost periodic matrices*, Proc. Amer. Math. Soc., **12** (1961), 400-407.

[12] J.K. Massera and J.J. Schäffer, "Linear differential equations and function spaces", Pure and Applied Mathematics, Vol. XXI, Academic Press, New York, 1966.

[13] N.G. Moshchevitin, *recurrence of an integral of a smooth conditionally periodic function*, Math. Notes, **63** (1998), 648-657.

[14] R. Ortega and M. Tarallo, *Almost periodic equations and conditions of Ambrosetti–Prodi type*, Math. Proc. Camb. Phil. Soc., **135** (2003), 239-254.

[15] K.J. Palmer, *On bounded solutions of almost periodic linear differential systems*, J. Math. Anal. Appl., **103** (1984), 16–25.

[16] H.Poincaré, *Sur le séries trigonométriques*, C.R. Acad. Sci., **101** (1885), 1131–1134.

[17] R.J. Sacker and G.R. Sell, *A spectral theory for linear differential systems*, J. Differential Equations, **27** (1978), 320–358.

[18] R.J. Sacker and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. I.*, J. Differential Equations, **15** (1974), 429–458.

[19] R.J. Sacker and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems. III.*, J. Differential Equations, **22** (1976), 497–522.

[20] B.A. Ščerbakov, *The comparability of the motions of dynamical systems with regard to the nature of their recurrence*, Diff. Uravnenija, **11** (1975), 1246–1255.

[21] G.R. Sell, "Topological dynamics and ordinary differential equations", Lecture Notes No. 33, Van Nostrand Reinhold, 1971.

[22] S. Siegmund, *Dichotomy spectrum for nonautonomous differential equations*, J. Dynam. Differential Equations, **14** (2002), 243–258.

[23] M. Tarallo, *Fredholms alternative for a class of almost periodic linear systems*, Discrete Contin. Dyn. Syst., **32** (2012), 2301–2313.

[24] M. Tarallo, *The Favard separation condition as a purely dimensional fact*, J. Dyn. Diff. Equations, **25** (2013), 291–304.

[25] V.V. Zhikov and B.M. Levitan, *Favard theory*, Russian Math. Surveys, **32** (1977), 129-180.

Universidad de Granada, 18071 Granada, Spain
E-mail address: `campos@ugr.es`

Universidad de Valladolid, Paseo del Cauce s/n, 47011 Valladolid, Spain
E-mail address: `rafoba@wmatem.eis.uva.es`

Università di Milano, via Saldini 50, 20133 Milano, Italy
E-mail address: `massimo.tarallo@unimi.it`