DYNAMICAL PROPERTIES OF NONAUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract. A type of nonautonomous n-dimensional state-dependent delay differential equation (SDDE) is studied. The evolution law is supposed to satisfy standard conditions ensuring that it can be imbedded, via the Bebutov hull construction, in a new map which determines a family of SDDEs when it is evaluated along the orbits of a flow on a compact metric space. Additional conditions on the initial equation, inherited by those of the family, ensure the existence and uniqueness of the maximal solution of each initial value problem. The solutions give rise to a skew-product semiflow which may be noncontinuous, but which satisfies strong continuity properties. In addition, the solutions of the variational equation associated to the SDDE determine the Fréchet differential with respect to the initial state of the orbits of the semiflow at the compatibility points. These results are key points to start using topological tools in the analysis of the long-term behavior of the solution of this type of nonautonomous SDDEs.

1. Introduction

Functional differential equations of state-dependent delay type (SDDEs for short) have been object of active analysis during the last years, due in part to the high theoretical interest of this study, but mainly to the increasing number of models of applied sciences which respond to this pattern: see e.g. Hartung [6], Wu [19], Hartung *et al.* [7], Mallet-Paret and Nussbaum [13], Barbarossa and Walther [1], He and de la Llave [8], and Krisztin and Rezounenko [12], as well as the many references therein.

In this setting, the regularity properties required on the vector field to guarantee existence, uniqueness, and continuous variation of solutions of initial value problems are much more exigent than in the case of fixed delay or even time-dependent delay differential equations. Especially complex is the nonautonomous case: due to the time-dependence, the solutions do not generate a semiflow on the state space, and more sophisticated tools must be designed in order to use the methods of the topological dynamics in the analysis of the dynamical properties of the solutions. A detailed description of some of these methods can be found in Hale [3] and Sell and You [16]. To establish the bases for the use of these tools in the analysis of nonautonomous SDDEs is the global purpose of this paper.

Let C and $W^{1,\infty}$ respectively represent the spaces of continuous and Lipschitz-continuous n-dimensional real functions on $[-r, 0]$. Hartung analyzes in [5, 6] the

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nonautonomous n-dimensional SDDE

$$\dot y(t) = f(t, y(t), y(t-\widetilde\tau(t, y_t)))\,, \qquad t \geq 0 \tag{1.1}$$

(where $y_t(s) := y(t+s)$ for $s \in [-r, 0]$), and the associated initial value problems, given by $y_0 = x$ for $x \in W^{1,\infty}$. He establishes regularity conditions on the vector field $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and on the delay $\widetilde\tau\colon \mathbb{R} \times C \to [0, r]$ guaranteeing the local existence and uniqueness of the solutions $y(t, x)$ of the initial value problem, as well as the fact that the map $[-r, 0] \to \mathbb{R}^n,\ s \mapsto y(t+s, x)$ belongs to $W^{1,\infty}$ for those values of (t, x) for which it is defined.

We have already mentioned that our global purpose is to describe a scenario on which the methods coming from the topological dynamics can be applied in the analysis of the long-term behavior of the solutions of (1.1). Let us describe our approach. Standard conditions on the temporal variation of the map

$$(f, \widetilde\tau)\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times C \to \mathbb{R}^n \times [0, r]\,, \quad (t, y_1, y_2, v) \mapsto (f(t, y_1, y_2), \widetilde\tau(t, v))$$

(which are satisfied in the uniformly almost-periodic case, but also in much more general situations), ensure that its *hull* Ω (which is defined as the closure in the compact open topology of the set of time-translated functions $(f, \widetilde\tau)_t(s, y_1, y_2, v) := (f, \widetilde\tau)(t+s, y_1, y_2, v)$ for t varying in $\mathbb{R}$) is a compact metric space. Its elements are functions

$$\omega = (\omega_1, \omega_2)\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times C \to \mathbb{R}^n \times [0, r],\ (t, y_1, y_2, v) \mapsto (\omega_1(t, y_1, y_2), \omega_2(t, v))\,.$$

In addition, the map $\mathbb{R} \times \Omega \to \Omega\,,\ (t, (\omega_1, \omega_2)) \mapsto (\omega_1, \omega_2){\cdot}t$ given by time-translation (i.e., $((\omega_1, \omega_2){\cdot}t)(s, y_1, y_2, v) = (\omega_1(t+s, y_1, y_2), \omega_2(t+s, v))$) defines a continuous flow on Ω. These conditions also ensure that the maps $F(\omega, y_1, y_2) = \omega_1(0, y_1, y_2)$ and $\tau(\omega, x) = \omega_2(0, x)$ for $\omega = (\omega_1, \omega_2)$ are continuous operators: see Hino *et al.* [9]. This procedure (designed by Bebutov around 1940) takes us to consider the family of nonautonomous SDDEs

$$\dot y(t) = F(\omega{\cdot}t, y(t), y(t-\tau(\omega{\cdot}t, y_t)))\,, \qquad t \geq 0\,, \tag{1.2}$$

for $\omega \in \Omega$. Note that the initial equation is included in this one: just take $\omega = (f, \widetilde\tau)$ (which in particular has a dense orbit in Ω). In addition, it turns out that any of the equations of the family satisfies the hypotheses assumed on the initial one. The great advantage of having this family of equations is that its solutions will allow us to define a semiflow of skew-product type on a suitable product space with base Ω.

As a matter of fact, we will take a family of the type (1.2) as starting point, without assuming that it comes from the single equation (1.1): Ω will simply be a compact metric space supporting a continuous flow, without further recurrence property (as the existence of a dense orbit on it). In this way, our framework is more general. The conditions that we will impose on F and τ are intended to ensure that each one of the equations of the family satisfied those of [5].

Our first purpose, carried out in Section 3, is to establish a global version of the fundamental Hartung's result: we will show the existence and uniqueness of a maximal solution $y(t, \omega, x)$ of the equation (1.2) corresponding to ω with $y_0 = x \in W^{1,\infty}$, which is defined on a right-open interval $[-r, \beta_{\omega,x})$ with $0 < \beta_{\omega,x} \leq \infty$. We will also prove that $\beta_{\omega,x} = \infty$ if $y(t, \omega, x)$ is norm-bounded. As before, it turns out that the map $u(t, \omega, x)$ defined by $u(t, \omega, x)(s) := y(t+s, \omega, x)$ for $s \in [-r, 0]$ belongs to $W^{1,\infty}$ whenever it is defined. We will show that $\Pi\colon \mathbb{R}^+ \times \Omega \times W^{1,\infty} \mapsto \Omega \times W^{1,\infty},\ (t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))$, which is locally defined, determines what we call a *pseudo-continuous* semiflow: a possibly noncontinuous semiflow but with strong continuity properties, as the continuity of $\Pi\colon \mathbb{R}^+ \times \Omega \times W^{1,\infty} \to \Omega \times C$; of the restriction $\widetilde{\Pi}\colon [r,\infty) \times \Omega \times W^{1,\infty} \mapsto \Omega \times W^{1,\infty}$; of the section $\Pi_t\colon \Omega \times W^{1,\infty} \to \Omega \times W^{1,\infty}$ for any fixed time t; and of the section $\Pi_{\omega,x}\colon \mathbb{R}^+ \to \Omega \times W^{1,\infty}$ for any (ω,x) satisfying the *compatibility condition* "$x \in C^1([-r,0],\mathbb{R}^n)$ and $\dot{x}(0^-) = F(\omega,x(0),x(-\tau(\omega,x)))$". For further purposes we represent by $\mathcal{C}_0$ the set of points (ω,x) satisfying this compatibility property. These results can be easily extended to the continuous dependence with respect to parameters. A consequence of the previous properties is that the restriction of Π to any compact Π-invariant set $\mathcal{K} \subset \Omega \times W^{1,\infty}$ defines a global continuous semiflow. Section 3 also describes the Lipschitz variation of the solutions of a particular equation with respect to the initial data.

Our second purpose, carried out in Section 4, concerns the existence and regularity properties of the Fréchet differential of the solutions with respect to the state variable x. We begin by analyzing the properties of the family of (linear) *variational equations*

$$\dot{z}(t) = L(\Pi(t,\omega,x))z_t\,, \quad t \in [0,\beta_{\omega,x}) \tag{1.3}$$

for $(\omega,x) \in \mathcal{C}_0$, where

$$\begin{aligned}L(\omega,x)v := {} & D_2F(\omega,x(0),x(-\tau(\omega,x)))v(0) + D_3F(\omega,x(0),x(-\tau(\omega,x)))v(-\tau(\omega,x)) \\ & - D_3F(\omega,x(0),x(-\tau(\omega,x)))\dot{x}(-\tau(\omega,x))\cdot D_2\tau(\omega,x)v\end{aligned}$$

for $(\omega,x) \in \mathcal{C}_0$ and $v \in C$. Note that the equation is nonautonomous, linear, and just time-dependent. We begin by analyzing the continuity properties of the maps $\mathcal{C}_0 \to \mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$, $(\omega,x) \mapsto L(\omega,x)$ and $\mathcal{C}_0 \times C \to \mathbb{R}^n$, $(\omega,x,v) \mapsto L(\omega,x)v$. These properties are one of the key points required to prove that, if $z(t,\omega,x,v)$ represents the solution of (1.3) agreeing with $v \in W^{1,\infty}$ in $[-r,0]$, and $w(t,\omega,x,v)(s) := z(t+s,\omega,x,v)$ for $s \in [-r,0]$, then the map $(t,\omega,x,v) \mapsto (\Pi(t,\omega,x), w(t,\omega,x,v))$ defines a new pseudo-continuous semiflow on $\mathcal{K} \times W^{1,\infty}$ (linear in this case), where $\mathcal{K}$ is any compact Π-invariant subset of $\Omega \times W^{1,\infty}$. The importance of this result relies on the fact that $w(t,\omega,x,v) = u_x(t,\omega,x)v$; that is, that $w(t,\omega,x,\cdot)$ represents the differential (in the Fréchet sense, as a matter of fact) with respect to the state variable of the Π-semiorbit corresponding to a compatibility point. This last equality concerning the map $u_x(t,\omega,x)\colon W^{1,\infty} \to W^{1,\infty}$ is proved for the local solution by Hartung in [5]. For the sake of completeness we include in Section 4 some steps of the proof adapted to our setting, since they are relevant to understand the regularity properties of the pseudo-continuous semiflow generated by u_x. These properties mean that $u_x(t,\omega,x)$ has full dynamical sense, as we will explain in the next paragraph. An in-deep analysis of some additional regularity properties of $u_x(t,\omega,x)$ completes the section, and the paper.

In order to see that these results give indeed form to a scenario in which the topological dynamical methods can be applied in the analysis of nonautonomous SDDEs, we mention some of their consequences. The restriction of the pseudo-continuous semiflow Π to any positively invariant compact set $\mathcal{K} \subset \Omega \times W^{1,\infty}$ determines a continuous semiflow on $\mathcal{K}$. If, in addition, the points of $\mathcal{K}$ satisfy the compatibility condition previously mentioned, the solutions of the family of linearized equations determines the usually so-called *linearized* semiflow of Π along the semiorbits of $\mathcal{K}$, namely $\widetilde{\Pi}^L\colon \mathbb{R}^+ \times \mathcal{K} \times W^{1,\infty} \to \mathcal{K} \times W^{1,\infty}$, $(t,\omega,x,v) \mapsto (\Pi(t,\omega,x), u_x(t,\omega,x)v)$. These results, also included on Sections 3 and 4, will be the starting point for the analysis of long-term dynamics of the orbits of $\mathcal{K}$, for which we can make use of: the properties of the linear pseudo-continuous semiflow $\widetilde{\Pi}^L$; and the properties of the continuous discrete semiflows given by the iteration of the continuous map $\widetilde{\Pi}^L_t$ for any $t > 0$. These and other questions are developed in [14]. In turn, all these results, combined with techniques of monotone systems (also new in the case of SDDEs) can be applied in the description of applied models, as that of a biological neural network: see [15].

We close this introduction by remarking that some authors consider different formulations providing different properties of regularity. Let us mention some of them. Walther studies in [18, 17] autonomous SDDEs defined by a continuously differentiable vector field $F\colon \mathcal{U} \subset C^1([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ satisfying mild smoothness conditions; his phase space is $\{\, x \in C^1([-r,0],\mathbb{R}^n) \,|\, \dot x(0^-) = F(x_t)\,\}$ endowed with the structure of C^1-manifold. Hartung proves in [6] the existence of the linearized map $u_x(t,x)\colon W^{1,\infty} \to C$ for every $x \in \Omega \times W^{1,\infty}$ and every $v \in W^{1,\infty}$ when $(d/dt)(t - \tau(t,u(t,x))) > \rho > 0$ for every t. If this inequality is globally satisfied, then the map $\mathcal{U} \to C$, $(t,x) \mapsto u(t,x)$ is differentiable with respect to the initial data in the complete domain $\mathcal{U}$ of F. A similar approach is used by Chen *et al.* in [2], where the state-dependent delay is supposed to satisfy an ordinary differential equation given by a vector field which is bounded above by a constant $\rho^* < 1$. Properties of regularity of the semiflow are used by Hu and Wu in [10] and by Hu *et al.* in [11] in order to investigate the Hopf-bifurcation of one-parametric families of SDDEs as well as the global continuation of the periodic solutions. And He and de la Llave use in [8] the parameterization method in order to construct quasi-periodic solutions of quasi-periodic SDDEs, which are defined as the ε-perturbation of an hyperbolic family of ordinary differential equations.

2. Basic notions on topological dynamics

Let Ω be a complete metric space. A continuous map $\sigma\colon \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \sigma(t,\omega) =: \sigma_t(\omega)$ satisfying

(f1) $\sigma_0 = \mathrm{Id}$

(f2) $\sigma_{t+l} = \sigma_t \circ \sigma_l$,

for $t,l \in \mathbb{R}$ in the case of (f2), defines a *real continuous flow* $(\Omega,\sigma,\mathbb{R})$. The *orbit* of the point $\omega \in \Omega$ is the set $\{\sigma_t(\omega) \,|\, t \in \mathbb{R}\}$. A subset $\mathcal{M} \subseteq \Omega$ is *σ-invariant* if $\sigma_t(\mathcal{M}) = \mathcal{M}$ for every $t \in \mathbb{R}$. The flow is *local* if the map σ is defined, continuous, and satisfies (f1) and (f2) (whenever it makes sense) on an open set $\mathcal{O} \supset \{0\}\times\Omega$.

Let us represent $\mathbb{R}^+ := \{t \in \mathbb{R} \,|\, t \ge 0\}$. If $\sigma\colon \mathbb{R}^+\times\Omega \to \Omega$, $(t,\omega) \mapsto \sigma(t,\omega)$ is a continuous map which satisfies the properties (f1) and (f2) for all $t,l \in \mathbb{R}^+$, then $(\Omega,\sigma,\mathbb{R}^+)$ is a *real continuous semiflow*. The set $\{\sigma_t(\omega) \,|\, t \ge 0\}$ is the (*positive*) *semiorbit* of the point $\omega \in \Omega$. A subset $\mathcal{M} \subseteq \Omega$ is *positively σ-invariant* if $\sigma_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \ge 0$. The semiflow is *local* if the map σ is defined, continuous, and satisfies (f1) and (f2) (whenever it makes sense) on an open subset $\mathcal{O} \subseteq \mathbb{R}^+\times\Omega$ containing $\{0\}\times\Omega$.

Let $(\Omega,\sigma,\mathbb{R}^+)$ be a global semiflow on a *compact* metric space Ω, and let X be a Banach space. We denote $\omega{\cdot}t := \sigma_t(\omega) = \sigma(t,\omega)$. A local semiflow $(\Omega\times X,\Pi,\mathbb{R}^+)$ is a *skew-product semiflow with base* $(\Omega,\sigma,\mathbb{R}^+)$ *and fiber* X if it takes the form

$$\Pi\colon \mathcal{U} \subseteq \mathbb{R}^+\times\Omega\times X \to \Omega\times X\,, \qquad (t,\omega,x) \mapsto (\omega{\cdot}t, u(t,\omega,x))\,.$$

Property (f2) above (with $\Omega\times X$ instead of Ω) means that u satisfies the *cocycle property* $u(t+l,\omega,x) = u(t,\omega{\cdot}l,u(l,\omega,x))$ whenever the right-hand function is defined. A global skew-product semiflow Π is *linear* if it takes the form $\Pi\colon \mathbb{R}^+\times\Omega\times X\to\Omega\times X$, $(t,\omega,x)\mapsto(\omega{\cdot}t,\phi(t,\omega)\,x)$, where $\phi(t,\omega)$ is a bounded linear operator on X; in particular, $u(t,\omega,x)$ is linear in x for each $(t,\omega)\in\mathbb{R}^+\times\Omega$.

We end this short section by fixing some notation. Given two Banach spaces $(X,\|{\cdot}\|_X)$ and $(Y,\|{\cdot}\|_Y)$, we represent by $\mathrm{Lin}(X,Y)$ the set of bounded linear maps $\phi\colon X\to Y$ equipped with the operator norm $\|\phi\|_{\mathrm{Lin}(X,Y)}=\sup_{\|x\|_X=1}\|\phi(x)\|_Y$. Let us fix $r>0$. The set C represents the Banach space of continuous functions $C([-r,0],\mathbb{R}^n)$ equipped with $\|\psi\|_C:=\sup_{s\in[-r,0]}|\psi(s)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The subset $C^1\subset C$ is given by the functions which have continuous derivative on $[-r,0]$ (one-sided derivatives at the edges). The set L^∞ is the space of Lebesgue-measurable functions $\psi\colon[-r,0]\to\mathbb{R}^n$ which are *essentially bounded*; i.e., for which there exists $k\ge 0$ such that the set $\{\,x\in[-r,0]\mid|\psi(x)|>k\,\}$ has zero measure. The norm on L^∞, which is denoted by $\|{\cdot}\|_{L^\infty}$, is defined as the inferior of the set of real numbers $k\ge0$ with the previous property. The set $W^{1,\infty}$ is the Banach space of Lipschitz-continuous functions $\psi\colon[-r,0]\to\mathbb{R}^n$ equipped with the Lipschitz norm $\|\psi\|_{W^{1,\infty}}:=\max\{\,\|\psi\|_C,\|\dot\psi\|_{L^\infty}\,\}$. Finally, given a continuous function $x\colon[-r,\gamma]\to\mathbb{R}^n$ for $\gamma>0$ and a time $t\in[0,\gamma]$, we denote by $x_t\in C$ the function defined by $x_t(s)=x(t+s)$ for $s\in[-r,0]$.

3. State-dependent delay differential equations

Let $(\Omega,\sigma,\mathbb{R})$ be a continuous flow on a compact metric space, and let us represent $\omega{\cdot}t=\sigma(t,\omega)$. Given two maps $F\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $\tau\colon\Omega\times C\to[0,r]$, we consider the family of nonautonomous SDDEs

$$\dot y(t)=F(\omega{\cdot}t,y(t),y(t-\tau(\omega{\cdot}t,y_t)))\,,\qquad t\ge0\,,\tag{3.1}$$

for $\omega\in\Omega$. The derivative at $t=0$ is the right-hand derivative. It has been explained in the Introduction the way in which one of this families may arise from one of its equations, via the hull procedure. We have also mentioned that if this is the case, at least one of the elements $\omega\in\Omega$ has a dense orbit. But recall that we are not assuming this fact here: we work in the most general case.

The conditions on F and τ which we will assume are

H1 $F\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, and its partial derivatives w.r.t. its second and third arguments exist and are continuous on $\Omega\times\mathbb{R}^n\times\mathbb{R}^n$; and

H2 $\tau\colon\Omega\times C\to[0,r]$ is continuous and differentiable w.r.t. its second argument, and the map $D_2\tau\colon\Omega\times C\to\mathrm{Lin}(C,\mathbb{R})$ is continuous.

Remarks 3.1. 1. Note that H2 ensures the next property: τ is locally Lipschitz-continuous in the sense that, for every compact subset $\mathcal{K}\subset\Omega\times C$, there exists a constant $L_1=L_1(\mathcal{K})>0$ such that

$$|\tau(\omega,x_1)-\tau(\omega,x_2)|\le L_1\|x_1-x_2\|_C\quad\text{for all }\;(\omega,x_1)\;\text{ and }\;(\omega,x_2)\;\text{ in }\;\mathcal{K}\,.$$

In order to check this assertion, we take a compact subset $\mathcal{K}\subset\Omega\times C$ and note that the set $\bar{\mathcal{K}}:=\{(\omega,s\,x_1+(1-s)\,x_2)\mid(\omega,x_1),\,(\omega,x_2)\in\mathcal{K}\text{ and }s\in[0,1]\}$ is also compact in $\Omega\times C$. We define $L_1=L_1(\mathcal{K}):=\sup\{\|D_2\tau(\omega,\bar x)\|_{\mathrm{Lin}(C,\mathbb{R})}\mid(\omega,\bar x)\in\bar{\mathcal{K}}\,\}$: condition H2 ensures that L_1 is finite. Then,

$$|\tau(\omega,x_1)-\tau(\omega,x_2)|\le\left|\int_0^1D_2\tau(\omega,s\,x_1+(1-s)\,x_2)(x_1-x_2)\,ds\right|\le L_1\|x_1-x_2\|_C$$

whenever (ω,x_1) and (ω,x_2) belong to $\mathcal{K}$, as asserted.

2. Having in mind the previous remark, it is easy to check that each one of the equations of the family satisfies the conditions A1 and A2 (adapted to our setting) assumed by Hartung in [5]. Therefore, all his local results may be applied.

Theorem 3.3 summarizes the dynamical properties of the solutions of the family (3.1). A key role is played by the set of pairs "(equation, initial datum)" which satisfy the *compatibility condition*,

$$\mathcal{C}_0 = \{(\omega, x) \in \Omega \times C^1 \mid \dot{x}(0^-) = F(\omega, x(0), x(-\tau(\omega, x)))\}\,. \tag{3.2}$$

Remark 3.2. As a matter of fact, the continuous differentiability properties of F and τ required in conditions H1 and H2 can be weakened for Theorem 3.3, in the line of conditions A1(i)&(ii) and A2(i)&(ii) of [5].

Theorem 3.3. *Suppose that conditions* H1 *and* H2 *hold. Then,*

(i) *for $\omega \in \Omega$ and $x \in W^{1,\infty}$, there exists a unique maximal solution $y(t, \omega, x)$ of the equation* (3.1) *corresponding to ω satisfying $y(s, \omega, x) = x(s)$ for $s \in [-r, 0]$, which is defined for $t \in [-r, \beta_{\omega,x})$ with $0 < \beta_{\omega,x} \le \infty$. In particular, $y(t, \omega, x)$ is continuous on $[-r, \beta_{\omega,x})$ and satisfies* (3.1) *on $(0, \beta_{\omega,x})$, and there exists the lateral derivative $\dot{y}(0^+, \omega, x) = F(\omega, x(0), x(-\tau(\omega, x)))$.*

Let us define $u(t, \omega, x)(s) := y(t + s, \omega, x)$ for $(\omega, x) \in \Omega \times W^{1,\infty}$, $t \in [0, \beta_{\omega,x})$, and $s \in [-r, 0]$. Then,

(ii) *$u(t, \omega, x) \in W^{1,\infty}$ for all $t \in [0, \beta_{\omega,x})$.*
(iii) *If $\sup_{t\in[0,\beta_{\omega,x})} \|u(t, \omega, x)\|_C < \infty$ then $\beta_{\omega,x} = \infty$.*

Let us further define $\mathcal{C}_0 \subset \Omega \times W^{1,\infty}$ by (3.2) *and*

$$\begin{aligned}
&\mathcal{U} := \{(t, \omega, x) \mid (\omega, x) \in \Omega \times W^{1,\infty},\ t \in [0, \beta_{\omega,x})\} \subset \mathbb{R}^+ \times \Omega \times W^{1,\infty},\\
&\Pi\colon \mathcal{U} \to \Omega \times W^{1,\infty}\,, \quad (t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))\,,\\
&\widetilde{\mathcal{U}} := \{(t, \omega, x) \in \mathcal{U} \mid t \ge r\} \subset \mathbb{R}^+ \times \Omega \times W^{1,\infty},\\
&\mathcal{U}^0 := \{(t, \omega, x) \mid (\omega, x) \in \mathcal{C}_0,\ t \in [0, \beta_{\omega,x})\} \subset \mathbb{R}^+ \times \Omega \times W^{1,\infty},
\end{aligned} \tag{3.3}$$

and provide $\mathcal{U}$, $\widetilde{\mathcal{U}}$, $\mathcal{C}_0$ and $\mathcal{U}^0$ with the respective subspace topologies. Then,

(iv) *the set $\mathcal{U}$ is open in $\mathbb{R}^+ \times \Omega \times W^{1,\infty}$ and Π satisfies conditions* (f1) *and* (f2) *of Section* 2 *(wherever it makes sense, and with Ω replaced by $\Omega \times W^{1,\infty}$).*
(v) *The map $\mathcal{U} \to \Omega \times C$, $(t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))$ is continuous.*
(vi) *The map $\widetilde{\mathcal{U}} \to \Omega \times W^{1,\infty}$, $(t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))$ is continuous.*
(vii) *Let us fix $\widetilde{t} \ge 0$ with $\mathcal{U}_{\widetilde{t}} := \{(\omega, x) \mid (\widetilde{t}, \omega, x) \in \mathcal{U}\}$ nonempty. Then the map $\mathcal{U}_{\widetilde{t}} \to \Omega \times W^{1,\infty}$, $(\omega, x) \mapsto (\omega{\cdot}\widetilde{t}, u(\widetilde{t}, \omega, x))$ is continuous.*
(viii) *The map $\mathcal{U}^0 \to \mathcal{C}_0 \subset \Omega \times W^{1,\infty}$, $(t, \omega, x) \mapsto (\omega{\cdot}t, u(t, \omega, x))$ is continuous.*
(ix) *If $\sup_{t\in[0,\infty)} \|u(t, \omega, x)\|_C < \infty$, then the set $\{(\omega{\cdot}t, u(t, \omega, x)) \mid t \in [r, \infty)\} \subset \Omega \times W^{1,\infty}$ is relatively compact.*

Proof. (i)&(ii) Let us fix $(\omega, x) \in \Omega \times W^{1,\infty}$. Theorem 1 of [5] proves the existence of a constant $\delta > 0$ and a map $[-r, \delta] \to \mathbb{R}^n$, $t \mapsto y(t, \omega, x)$ satisfying: $y(s, \omega, x) = x(s)$ for $s \in [-r, 0]$; and equation (3.1) on $[0, \delta]$ with $\dot{y}(0, \omega, x)$ and $\dot{y}(\delta, \omega, x)$ replaced by $\dot{y}(0^+, \omega, x)$ and $\dot{y}(\delta^-, \omega, x)$. It also proves that $u(t, \omega, x) \in W^{1,\infty}$. The classical method of prolongability of solutions outside any right-closed interval shows the existence of maximal solution defined on a right-open interval. The details are left to the reader.

(iii) Assume that $\sup_{t\in[0,\beta)}\|u(t,\omega,x)\|_C =: c_0<\infty$ for a point $(\omega,x)\in\Omega\times W^{1,\infty}$ and, for contradiction, that $\beta:=\beta_{\omega,x}<\infty$. We will prove that $y(t,\omega,x)$ exists on $[-r,\beta]$ and satisfies (3.1) on $[0,\beta]$: as indicated in (i), this contradicts the definition of β. Recall that the derivatives at the edge points are one-sided.

It follows from $x\in W^{1,\infty}$ and from H1 that $|\dot y(t,\omega,x)|\le c_1$ for Lebesgue-a.a. $t\in[-r,0]$ and all $t\in[0,\beta)$, which means that $\sup_{t\in[0,\beta)}\|u(t,\omega,x)\|_{W^{1,\infty}} =: c_2<\infty$. We take $(t_m)\uparrow\beta$ and note that the previous properties allow us to use Arzelá–Ascoli theorem in order to find a subsequence (t_k) and an element $v\in C$ with $v=\lim_{k\to\infty}u(t_k,\omega,x)$ in the $\|{\cdot}\|_C$-topology. It is easy to check that $v(s)=y(\beta+s,\omega,x)$ for $s\in[-r,0)$. In particular, $|\dot v(s)|\le c_2$ for Lebesgue-a.a. $s\in[-r,0]$, so that $v\in W^{1,\infty}$. In addition, $v(0)=\lim_{s\to 0^-}y(\beta+s,\omega,x)$. We define $y(\beta,\omega,x):=v(0)$, so that we get a function $y(t,\omega,x)$ defined and continuous on $[-r,\beta]$ and satisfying (3.1) on $[0,\beta)$. We also define $u(\beta,\omega,x):=v\in W^{1,\infty}$. If $s\in(-r,0)\cap(-\beta,0)$, then there exists $\dot v(s)=\dot y(s+\beta,\omega,x)$. By H1 and H2, $\lim_{s\to 0^-}\dot y(s+\beta,\omega,x)=F(\omega{\cdot}\beta,y(\beta,\omega,x),y(\beta-\tau(\omega{\cdot}\beta,u(\beta,\omega,x)),\omega,x))$. Finally, the existence of the limit yields the existence of the left-side derivative $\dot y(\beta^-)$, with $\dot y(\beta^-)=\lim_{s\to 0^-}\dot y(s+\beta,\omega,x)$. The last two equalities complete the proof of the first assertion, and hence that of (iii).

(iv)&(v) We will first prove these properties under the assumption that F is bounded, which we will remove later.

An easy contradiction argument using property (iii) shows that $\beta_{\omega,x}=\infty$ for all $(\omega,x)\in\Omega\times W^{1,\infty}$. Therefore, $\mathcal U=\mathbb R^+\times\Omega\times W^{1,\infty}$, and hence is open. It is obvious that Π satisfies (f1). The uniqueness of solutions ensured by (i) implies $y(t+l,\omega,x)=y(t,\omega{\cdot}l,u(l,\omega,x))$, so that also (f2) holds for $t\ge 0$ and $l\ge 0$: (iv) is proved.

Now let us check that, if $(\widetilde\omega,\widetilde x)=\lim_{m\to\infty}(\omega_m,x_m)$ in $\Omega\times W^{1,\infty}$ and $T>0$, then $y(t,\widetilde\omega,\widetilde x)=\lim_{m\to\infty}y(t,\omega_m,x_m)$ uniformly in $t\in[-r,T]$. Since F is bounded, so is $\{\dot y(t,\omega_m,x_m)\mid t\in[0,T]\text{ and }m\in\mathbb N\}$. An application of Arzelá–Ascoli theorem shows that any subsequence of (y_m), where $y_m(t):=y(t,\omega_m,x_m)$, has in turn a subsequence (y_k) which converges to a limit $\widetilde y$ uniformly on $[-r,T]$. Our goal is to prove that $\widetilde y(t)=y(t,\widetilde\omega,\widetilde x)$ for $t\in[-r,T]$: if $t\in[-r,0]$, $\widetilde y(t)=\lim x_k(t)=\widetilde x(t)=y(t,\widetilde\omega,\widetilde x)$; and if $t\in(0,T]$, then

$$\begin{aligned}\widetilde y(t)=\lim_{k\to\infty}y_k(t)&=\lim_{k\to\infty}\left(x_k(0)+\int_0^t F(\omega_k{\cdot}l,y_k(l),y_k(l-\tau(\omega_k{\cdot}l,(y_k)_l)))\,dl\right)\\&=\widetilde x(0)+\int_0^t F(\widetilde\omega{\cdot}l,\widetilde y(l),\widetilde y(l-\tau(\widetilde\omega{\cdot}l,\widetilde y_l)))\,dl\,.\end{aligned}$$

(Here we have used H1 and H2.) So, $\widetilde y(\cdot)$ and $y(\cdot,\widetilde\omega,\widetilde x)$ agree on $[-r,T]$, as asserted.

Now we assume that $(\widetilde t,\widetilde\omega,\widetilde x)=\lim_{m\to\infty}(t_m,\omega_m,x_m)$ in $\mathcal U$, take $T\ge\sup_{m\in\mathbb N}t_m$, fix $\varepsilon>0$, and write

$$\begin{aligned}&\|u(\widetilde t,\widetilde\omega,\widetilde x)-u(t_m,\omega_m,x_m)\|_C\\&\qquad\le\|u(\widetilde t,\widetilde\omega,\widetilde x)-u(t_m,\widetilde\omega,\widetilde x)\|_C+\|u(t_m,\widetilde\omega,\widetilde x)-u(t_m,\omega_m,x_m)\|_C\,.\end{aligned}$$

Since $y(\cdot,\widetilde\omega,\widetilde x)$ is uniformly continuous on $[-r,T]$, $\|u(t,\widetilde\omega,\widetilde x)-u(t_m,\widetilde\omega,\widetilde x)\|_C\le\varepsilon/2$ if m is larger than an m_0. And the property shown in the preceding paragraph ensures that $\|u(t,\widetilde\omega,\widetilde x)-u(t,\omega_m,x_m)\|_C\le\varepsilon/2$ for all $t\in[0,T]$ if m is larger than an $m_1\ge m_0$. Altogether, $\|u(\widetilde t,\widetilde\omega,\widetilde x)-u(t_m,\omega_m,x_m)\|_C\le\varepsilon$ if $m\ge m_1$, so that (v) is proved.

The boundedness of F is not assumed from now on. Let us fix $(\widetilde{t},\widetilde{\omega},\widetilde{x})\in\mathcal{U}$; take $T\in(\widetilde{t},\beta_{\widetilde{\omega},\widetilde{x}})$; define $c:=\max_{t\in[-r,T]}|y(t,\widetilde{\omega},\widetilde{x})|$; take a C^1 function $h\colon\mathbb{R}^n\to[0,1]$ such that $h(x)=1$ if $|x|\le 2c$ and $h(x)=0$ if $|x|\ge 3c$; and define $G(\omega,y_1,y_2)=F(\omega,y_1,y_2)h(y_1)h(y_2)$, which is bounded and satisfies H1. We denote by $y_G(t,\omega,x)$ the solution of the problem given by the equation (3.1) corresponding to G and the initial data x, which is defined on $[-r,\infty)$; and we denote $u_G(t,\omega,x)(s)=y_G(t+s,\omega,x)$ for $t\in[0,\infty)$ and $s\in[-r,0]$. Applying (v) to G we find $\delta>0$ such that $\|u_G(t,\omega,x)-u_G(t,\widetilde{\omega},\widetilde{x})\|_C\le c$ for all $t\in[0,T]$ whenever $d_\Omega(\omega,\widetilde{\omega})\le\delta$ (where d_Ω is the distance in Ω) and $\|x-\widetilde{x}\|_{W^{1,\infty}}\le\delta$. In particular, for these values of (t,ω,x), $\|u_G(t,\omega,x)\|_C\le 2c$, which ensures that $y_G(t,\omega,x)$ satisfies (3.1) (i.e., $y_G(t,\omega,x)=y(t,\omega,x)$). Using this property, it is easy to complete the proofs of (iv) and (v).

(vi) We assume that the sequence $((t_m,\omega_m,x_m))$ of elements of $\widetilde{\mathcal{U}}$ converges to $(\widetilde{t},\widetilde{\omega},\widetilde{x})\in\widetilde{\mathcal{U}}$, and fix $\varepsilon>0$. Since $\mathcal{U}$ is open, there is no restriction in assuming the existence of $t_0\in(\widetilde{t},\beta_{\widetilde{\omega},\widetilde{x}})$ such that $t_m\le t_0<\beta_{\omega_m,x_m}$ for all $m\in\mathbb{N}$. We define $y_m(t):=y(t,\omega_m,x_n)$ and $\widetilde{y}(t):=y(t,\widetilde{\omega},\widetilde{x})$ for $t\in[-r,t_0]$, and $u_m(t):=u(t,\omega_m,x_n)$ and $\widetilde{u}(t):=u(t,\widetilde{\omega},\widetilde{x})$ for $t\in[0,t_0]$. Note also that the set

$$\mathcal{S}:=\{(\omega_m,x_m)\mid m\in\mathbb{N}\}\cup\{(\widetilde{\omega},\widetilde{x})\} \tag{3.4}$$

is compact in $\Omega\times W^{1,\infty}$.

Let us fix $\varepsilon>0$. According to (v), the map $u\colon[0,t_0]\times\mathcal{S}\to\Omega\times C$ is uniformly continuous. This uniform continuity guarantees that the families $\mathcal{F}_1:=\{u_m\mid m\in\mathbb{N}\}\cup\{\widetilde{u}\}\subset C([0,t_0],C)$ and $\mathcal{F}_2:\{y_m\mid m\in\mathbb{N}\}\cup\{\widetilde{y}\}\subset C([-r,t_0],\mathbb{R}^n)$ are equicontinuous. Let us take $s_1,s_2\in[0,t_0]$. Then

$$\begin{aligned}\dot{\widetilde{y}}(s_1)-\dot y_m(s_2)&=F(\widetilde{\omega}{\cdot}s_1,\widetilde{y}(s_1),\widetilde{y}(s_1-\tau(\widetilde{\omega}{\cdot}s_1,\widetilde{u}(s_1))))\\&\quad-F(\omega_m{\cdot}s_2,y_m(s_2),y_m(s_2-\tau(\omega_m{\cdot}s_2,u_m(s_2))))\,.\end{aligned}$$

The set $\mathcal{K}:=\{u(t,\omega,x)\mid\ t\in[0,t_0]\text{ and }(\omega,x)\in\mathcal{S}\}$ is compact in C, so that $k:=\sup\{\|u(t,\omega,x)\|_C\mid t\in[0,t_0]\text{ and }(\omega,x)\in\mathcal{S}\}$ is finite. We define $\overline{\mathcal{B}}:=\{y\in\mathbb{R}^n\mid\|y\|\le k\}$. Then the map

$$[0,t_0]\times\Omega\times\overline{\mathcal{B}}\times\overline{\mathcal{B}}\to\mathbb{R}^n\,,\quad(s,\omega,y_1,y_2)\mapsto F(\omega{\cdot}s,y_1,y_2)$$

is uniformly continuous: that is, there exists $\delta_1>0$ such that, if $|s_1-s_2|\le\delta_1$, $|y_1^1-y_1^2|\le\delta_1$, and $|y_2^1-y_2^2|\le\delta_1$, then $|F(\omega{\cdot}s_1,y_1^1,y_2^1)-F(\omega{\cdot}s_2,y_1^2,y_2^2)|<\varepsilon$ for all $\omega\in\Omega$. Since the family $\mathcal{F}_1$ is equicontinuous on $[-r,t_0]$, there exists δ_2 such that if $|s_1-s_2|\le\delta_2$, then $|\widetilde{y}(s_1)-y_m(s_2)|\le\delta_1$ for all $m\in\mathbb{N}$. And, since the family $\mathcal{F}_2$ is equicontinuous on $[0,t_0]$ and $[0,t_0]\times\Omega\times\mathcal{K}\to\mathbb{R}$, $(t,\omega,x)\mapsto\tau(\omega{\cdot}t,x)$ is uniformly continuous, there is δ_3 and $m_0\in\mathbb{N}$ such that if $|s_1-s_2|\le\delta_3$ and $m\ge m_0$ then $|\tau(\widetilde{\omega}{\cdot}s_1,\widetilde{u}(s_1))-\tau(\omega_m{\cdot}s_2,u_m(s_2))|\le\delta_2$ for all $\omega\in\Omega$. Altogether, we take $\delta=\min(\delta_1,\delta_2,\delta_3)$ and conclude that

$$\text{if }|s_1-s_2|\le\delta\text{ and }m\ge m_0,\text{ then }|\dot{\widetilde{y}}(s_1)-\dot y_m(s_2)|\le\varepsilon\text{ for }m\ge m_0\,. \tag{3.5}$$

Now we take $m_1\ge m_0$ such that $|\widetilde{t}-t_m|\le\delta$ for all $m\ge m_1$, and recall that $t_m\ge r$ and $\widetilde{t}\ge r$ to deduce from (3.5) that

$$\|\dot{\widetilde{u}}(\widetilde{t})-\dot u_m(t_m)\|_{L^\infty}=\sup_{s\in[-r,0]}|\dot{\widetilde{y}}(\widetilde{t}+s)-\dot y_m(t_m+s)|\le\varepsilon$$

whenever $m\ge m_1$. Since, by (v), $\|\widetilde{u}(\widetilde{t})-u_m(t_m)\|_C\le\varepsilon$ for large enough m, we conclude that the same happens with $\|\widetilde{u}(\widetilde{t})-u_m(t_m)\|_{W^{1,\infty}}$, which proves (vi).

(vii) In the case that $\widetilde{t} \ge r$, property (vii) follows from (vi), and if $\widetilde{t}=0$ the assertion is trivial. So that assume that $\widetilde{t} \in (0,r)$. Let us take a sequence $((\omega_m,x_m))$ in $\mathcal{U}_{\widetilde{t}}$ with limit $(\widetilde{\omega},\widetilde{x}) \in \mathcal{U}_{\widetilde{t}}$. Let us fix $\varepsilon>0$ and $\delta \in (0,\varepsilon]$. We call $y_m(t) := y(t,\omega_m,x_m)$ and $\widetilde{y}(t) := y(t,\widetilde{\omega},\widetilde{x})$ for $t \in [-r,\widetilde{t}]$, and , $u_m(t) := u(t,\omega_m,x_m)$ and $\widetilde{u}(t) := u(t,\widetilde{\omega},\widetilde{x})$ for $t \in [0,\widetilde{t}]$. According to (v), $\|\widetilde{u}(\widetilde{t})-u_m(\widetilde{t})\|_C$ is as small as desired if m is large enough. Therefore, there exists m_0 such that, if $m \ge m_0$, then

$$\|\widetilde{u}(t)-u_m(t)\|_C \le \|\widetilde{u}(\widetilde{t})-u_m(\widetilde{t})\|_C + \|\widetilde{x}-x_m\|_C \le \delta \le \varepsilon \quad \text{for } t \in [0,\widetilde{t}]\,. \tag{3.6}$$

On the other hand,

$$\|\dot{\widetilde{u}}(\widetilde{t})-\dot{u}_m(\widetilde{t})\|_{L^\infty} \le \|\dot{\widetilde{x}}-\dot{\widetilde{x}}_m\|_{L^\infty} + \sup_{t\in[0,\widetilde{t}]} |\dot{y}(t,\widetilde{\omega},\widetilde{x})-\dot{y}(t,\omega_m,x_m)|\,,$$

and

$$\begin{aligned}|\dot{\widetilde{y}}(t)-\dot{y}_m(t)| = \big|&F(\widetilde{\omega}{\cdot}t,\widetilde{y}(t),\widetilde{y}(t-\tau(\widetilde{\omega}{\cdot}t,\widetilde{u}(t))))\\ &- F(\omega_m{\cdot}t,y_m(t),y_m(t-\tau(\omega_m{\cdot}t,u_m(t))))\big|\end{aligned}$$

for $t \in [0,\widetilde{t}]$. It follows easily from the continuity of F and τ guaranteed by H1 and H2 and from (3.6) (which is valid for a δ which can be prefixed from the properties of F and τ) that $|\dot{\widetilde{y}}(t)-\dot{y}_m(t)| \le \varepsilon/2$ for all $t \in [0,\widetilde{t}]$ if m is large enough, and clearly the same happens with $\|\dot{\widetilde{x}}-\dot{\widetilde{x}}_m\|_{L^\infty}$. This ensures that $\|\dot{\widetilde{u}}(\widetilde{t})-\dot{u}_m(\widetilde{t})\|_{L^\infty} \le \varepsilon$ for large enough m, which together with (3.6) proves the result.

(viii) Let us take a sequence $((t_m,\omega_m,x_m))$ in $\mathcal{U}^0$ with limit $(\widetilde{t},\widetilde{\omega},\widetilde{x}) \in \mathcal{U}^0$ and define t_0, $\mathcal{S}$, y_m, $\widetilde{y}$, u_m and $\widetilde{u}$ as at the beginning of the proof of (vi). Note that

$$\|\widetilde{u}(\widetilde{t})-u_m(t_m)\|_{W^{1,\infty}} \le \|\widetilde{u}(\widetilde{t})-u_m(\widetilde{t})\|_{W^{1,\infty}} + \|u_m(\widetilde{t})-u_m(t_m)\|_{W^{1,\infty}}$$

and that we already know, by (vii) and (v), that $\lim_{m\to\infty}\|\widetilde{u}(\widetilde{t})-u_m(\widetilde{t})\|_{W^{1,\infty}}=0$ and $\lim_{m\to\infty}\|u_m(\widetilde{t})-u_m(t_m)\|_C=0$. Hence, our goal is to prove that

$$\lim_{m\to\infty}\|u_m(\widetilde{t})-u_m(t_m)\|_{L^\infty}=0\,.$$

It is very easy to check that this property follows from the equicontinuity of the family $\{\dot{y}_m \mid m \in \mathbb{N}\}$ on $[-r,t_0]$. It is also easy to deduce from the fact that $\lim_{m\to\infty}\dot{x}_m = \dot{\widetilde{x}}$ in C that the family $\{\dot{y}_m \mid m \in \mathbb{N}\}$ is equicontinuous on $[-r,0]$. On the other hand, given $\varepsilon>0$, we conclude by repeating step by step the argument used in the proof of (vi) that there exists $\delta>0$ such that, if $s_1,s_2 \in [0,t_0]$ and $|s_1-s_2| \le \delta$, then $|\dot{y}_m(s_1)-\dot{y}_m(s_2)| \le \varepsilon$ all $m \in \mathbb{N}$. This means that the family $\{\dot{y}_m \mid m \in \mathbb{N}\}$ is equicontinuous on $[0,t_0]$, and hence on $[-r,t_0]$, which completes the proof of (viii).

(ix) We fix (ω,x) with $c_0 := \sup_{t\in[0,\beta_{\omega,x})}\|u(t,\omega,x)\|_C < \infty$, which according to (iii) ensures that $\beta_{\omega,x}=\infty$. Property (vi) ensures that $\{(\omega{\cdot}t,u(t,\omega,x)) \mid t \in [r,2r]\}$ is compact, so that it suffices to prove that $\{(\omega{\cdot}t,u(t,\omega,x)) \mid t \in [2r,\infty)\}$ is relatively compact. In order to check it, given a sequence (t_m) in $[0,\infty)$, we look for a convergent subsequence of $((\omega{\cdot}(t_m+2r),u(t_m+2r,\omega,x)))$ in $\Omega\times W^{1,\infty}$. Since Ω is compact, there is no restriction in assuming the existence of $\omega^* := \lim_{m\to\infty}\omega{\cdot}t_m$, and hence of $\omega^*{\cdot}(2r) = \lim_{m\to\infty}\omega{\cdot}(t_m+2r)$. We represent

$$y_m\colon [0,2r]\to\mathbb{R}^n,\ t\mapsto y(t_m+t,\omega,x) \qquad \text{for } m \in \mathbb{N}\,,$$

so that we obtain a sequence (y_m) in $C([0,2r],\mathbb{R}^n)$ which is uniformly bounded by c_0. As in the proof of (iv)&(v), Arzelá–Ascoli theorem provides a subsequence (y_k) which converges uniformly on $[0,2r]$ to $y^* \in C([0,2r],\mathbb{R}^n)$. By condition H2,

$\lim_{k\to\infty}\tau(\omega{\cdot}(t_k{+}t),(y_k)_t)=\tau(\omega^*{\cdot}t,(y^*)_t)$ uniformly on $[r,2r]$, which together with H1 yields

$$\begin{aligned}\lim_{k\to\infty} F(\omega{\cdot}(t_k+t),y_k(t),y(t_k+t-\tau(\omega{\cdot}(t_k+t),(y_k)_t),\omega,x))\\ = F(\omega^*{\cdot}t,y^*(t),y^*(t-\tau(\omega^*{\cdot}t,(y^*)_t)))\end{aligned} \tag{3.7}$$

uniformly on $[r,2r]$. Now, the sequence (y_k) satisfies

$$y_k(t)=y_k(r)+\int_r^t F(\omega{\cdot}(t_k{+}s),y_k(s),y(t_k{+}s-\tau(\omega{\cdot}(t_k{+}s),(y_k)_s),\omega,x))\,ds\,. \tag{3.8}$$

So, on the one hand, $y^*(t)=\lim_{k\to\infty}y_k(t)$ for $t\in[r,2r]$; and on the other hand, by (3.7), the right hand term of (3.8) converges to

$$y^*(r)+\int_r^t F(\omega^*{\cdot}s,y^*(s),y^*(s-\tau(\omega^*{\cdot}s,(y^*)_s)))\,ds\,.$$

Therefore y^* solves the equation (3.1) on $[r,2r]$. Consequently, $(\dot y_k)$ converges to $\dot y^*$ uniformly on $[r,2r]$. Altogether, we have checked the sequence $(u(t_k+2r,\omega,x))$ converges to y^*_{2r} in $W^{1,\infty}$, which completes the proof of (ix) and of the theorem. □

Corollary 3.4. *Suppose that conditions H1 and H2 hold, and let Π be defined by (3.3). Let $\mathcal{K}\subset\Omega\times W^{1,\infty}$ be a positively Π-invariant compact set. Then the restriction of Π to $\mathcal{K}$ defines a global continuous semiflow on $\mathcal{K}$.*

Proof. Note that $\mathbb{R}^+\times\mathcal{K}\subset\mathcal{U}$ and $\Pi(t,\omega,x)\in\mathcal{K}$ for all $(t,\omega,x)\in\mathbb{R}^+\times\mathcal{K}$, so that the restriction $\Pi\colon\mathbb{R}^+\times\mathcal{K}\to\mathcal{K}$ is well defined and globally defined. And it is easy to check that the topologies induced by $\|\cdot\|_C$ and $\|\cdot\|_{W^{1,\infty}}$ on $\mathcal{K}$ are the same, so that the continuity follows from Theorem 3.3(v). □

Remark 3.5. We can repeat the arguments of the proofs of points (v), (vi) and (vii) of Theorem 3.3 in order to prove analogous results on the joint continuity with respect to (t,ω,x,λ) for the solutions of the family of equations $\dot y(t)=F(\omega{\cdot}t,y(t),y(t-\tau(\omega{\cdot}t,y_t,\lambda),\lambda)$ when λ belongs to a Banach space and F and τ satisfy the corresponding jointly continuity properties included in H1 and H2. The details are left to the reader, whom is referred to [5] for a more exhaustive analysis of the regularity properties with respect to parameters of the solution of SDDEs.

We complete this section by analyzing the Lipschitz behaviour of the map u defined in the statement of Theorem 3.3 with respect to the initial condition x. This result is a global version, adapted to our setting, of the local property given by Theorem 1(iv) of [5]. Recall that d_Ω represents the distance in Ω.

Theorem 3.6. *Suppose that conditions H1 and H2 hold, and define $\mathcal{U}$ and $u\colon\mathcal{U}\to W^{1,\infty}$ as in the statement of Theorem 3.3. Let us fix $\widetilde t>0$ such that the set $\mathcal{U}_{\widetilde t}:=\{(\omega,x)\mid(\widetilde t,\omega,x)\in\mathcal{U}\}$ is nonempty. Let us also fix $(\widetilde\omega,\widetilde x)\in\mathcal{U}_{\widetilde t}$. Then,*

- (i) *there exists $\rho>0$ small enough to guarantee that*
  1. *$u(t,\omega,x)$ is defined (i.e., $(t,\omega,x)\in\mathcal{U}$) whenever $t\in[0,\widetilde t\,]$ and $(\omega,x)\in\mathcal{B}^\rho_{\widetilde\omega,\widetilde x}:=\{(\omega,x)\in\Omega\times W^{1,\infty}\mid d_\Omega(\omega,\widetilde\omega)<\rho$ and $\|x-\widetilde x\|_{W^{1,\infty}}<\rho\}$;*
  2. *$\sup\{\|u(t,\omega,x)\|_C\mid t\in[0,\widetilde t\,]$ and $(\omega,x)\in\mathcal{B}^\rho_{\widetilde\omega,\widetilde x}\}=:\widetilde c<\infty\,.$*
- (ii) *Let us fix a value of ρ for which 1 and 2 hold. Then there exists $L=L(\widetilde t,\widetilde\omega,\widetilde x,\rho)$ such that, if (ω,x_1) and (ω,x_2) belong to $\mathcal{B}^\rho_{\widetilde\omega,\widetilde x}$, then*

$$\|u(t,\omega,x_1)-u(t,\omega,x_2)\|_{W^{1,\infty}}\le L\,\|x_1-x_2\|_{W^{1,\infty}}\quad \textit{for all }\ t\in[0,\widetilde t\,]\,.$$

Proof. (i) Let us fix $\widetilde{t}$, $\widetilde{\omega}$, and $\widetilde{x}$ as in the statement. The existence of $\widetilde{\rho}>0$ such that 1 holds follows immediately from the open character of $\mathcal{U}$ ensured by Theorem 3.3(iv). In order to check the existence of $\rho\in(0,\widetilde{\rho}]$ such that also 2 holds, we assume for contradiction that, for any $m\in\mathbb{N}$, there exist $t_m\in[0,\widetilde{t}\,]$ and $(\omega_m,x_m)\in\Omega\times W^{1,\infty}$ such that $d_\Omega(\omega_m,\widetilde{\omega})<1/m$, $\|x_m-\widetilde{x}\|_{W^{1,\infty}}<1/m$, and $|y(t_m,\omega_m,x_m)|\geq m$; take a subsequence (t_k) with limit t^*; observe that $((t_k,\omega_k,x_k))$ converges to $(t^*,\widetilde{\omega},\widetilde{x})$ in $[0,\widetilde{t}\,]\times\Omega\times W^{1,\infty}$; and conclude from Theorem 3.3(v) that $|y(t^*,\widetilde{\omega},\widetilde{x})|=\infty$, which is impossible.

(ii) The points $\widetilde{t}$, $\widetilde{\omega}$, and $\widetilde{x}$ will be fixed in the whole proof, as well as a constant ρ for which 1 and 2 hold. We begin by proving properties (3.9), (3.10) and (3.14), which will be used below. Let $\widetilde{c}$ be the constant appearing in condition 2. Then

$$|\,s\,y(t_1,\omega,x_1)+(1-s)\,y(t_2,\omega,x_2)\,|\leq\widetilde{c} \tag{3.9}$$

for all $t_1,t_2\in[-r,\widetilde{t}]$, (ω,x_1) and (ω,x_2) in $\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}$, and $s\in[0,1]$. On the other hand, the continuity of $D_iF\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^n\to\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)$ for $i=2,3$ ensured by H1 guarantees that these maps take compact set in compact sets, so that there exists $L_1<\infty$ such that

$$\|D_iF(\omega,y_1,y_2)\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\leq L_1\quad\text{for all }\ \omega\in\Omega\ \text{ if }\ |y_1|\leq\widetilde{c}\ \text{ and }\ |y_2|\leq\widetilde{c} \tag{3.10}$$

for $i=2,3$. Note also that the continuity of $F\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ ensured by H1 combined with condition 2 and equation (3.1) ensures that the set

$$\{\,\dot y(t,\omega,x)\,|\,t\in[0,\widetilde{t}\,]\text{ and }(\omega,x)\in\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}\,\}\subset\mathbb{R}^n$$

is bounded. In addition, $|\dot y(s,\omega,x)|\leq\|x\|_{W^{1,\infty}}<\rho+\|\widetilde{x}\|_{W^{1,\infty}}$ for Lebesgue-a.a. $s\in[-r,0]$ whenever $(\omega,x)\in\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}$. These two properties and 2 yield

$$\sup\{\|u(t,\omega,x)\|_{W^{1,\infty}}\,|\,t\in[0,\widetilde{t}\,]\ \text{ and }\ (\omega,x)\in\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}\}=:c^*<\infty\,, \tag{3.11}$$

which together with Arzelá-Ascoli theorem ensures that the set

$$\mathcal{K}=\Omega\times\mathrm{closure}_C\{\,u(t,\omega,x)\,|\,t\in[0,\widetilde{t}\,]\ \text{ and }\ (\omega,x)\in\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}\,\}\subset\Omega\times C \tag{3.12}$$

is compact. As seen in Remark 3.1.1, there exists L_2 such that

$$|\tau(\omega,x_1)-\tau(\omega,x_2)|\leq L_2\|x_1-x_2\|_C\quad\text{for all }\ (\omega,x_1)\ \text{ and }\ (\omega,x_2)\ \text{ in }\ \mathcal{K}\,. \tag{3.13}$$

Let us take (ω,x_1) and (ω,x_2) in $\mathcal{B}^{\rho}_{\widetilde{\omega},\widetilde{x}}$, and denote $y_1(t):=y(t,\omega,x_1)$ and $y_2(t):=y(t,\omega,x_2)$ for $t\in[-r,\widetilde{t}]$, and $u_1(t):=u(t,\omega,x_1)$, and $u_2(t):=u(t,\omega,x_2)$ for $t\in[0,\widetilde{t}\,]$. Note that $u_1(t)$ and $u_2(t)$ belong to the $\mathcal{K}$ given by (3.12) for all $t\in[0,\widetilde{t}\,]$. Let c^* and L_2 be the constants appearing in (3.11) and (3.13). Then,

$$\begin{aligned}
&|y_1(t-\tau(\omega{\cdot}t,u_1(t)))-y_2(t-\tau(\omega{\cdot}t,u_2(t)))|\\
&\qquad\leq|y_1(t-\tau(\omega{\cdot}t,u_1(t)))-y_2(t-\tau(\omega{\cdot}t,u_1(t)))|\\
&\qquad\quad+|y_2(t-\tau(\omega{\cdot}t,u_1(t)))-y_2(t-\tau(\omega{\cdot}t,u_2(t)))|\\
&\qquad\leq\|u_1(t)-u_2(t)\|_C+\|u_2(t)\|_{W^{1,\infty}}|\tau(\omega{\cdot}t,u_1(t))-\tau(\omega{\cdot}t,u_2(t))|\\
&\qquad\leq\|u_1(t)-u_2(t)\|_C+c^*\,L_2\,\|u_1(t)-u_2(t)\|_C=L_3\,\|u_1(t)-u_2(t)\|_C
\end{aligned} \tag{3.14}$$

for all $t\in[0,\widetilde{t}\,]$, where $L_3:=1+c^*L_2$.

Now we can proceed with the proof. With the previous notation, if $t\in[0,\widetilde{t}\,]$,

$$\begin{aligned}
&F(\omega{\cdot}t,y_1(t),y_1(t-\tau(\omega{\cdot}t,u_1(t))))-F(\omega{\cdot}t,y_2(t),y_2(t-\tau(\omega{\cdot}t,u_2(t))))\\
&\quad=\int_0^1 D_2F\Big(\omega{\cdot}t,s\,y_1(t)+(1-s)\,y_2(t),s\,y_1(t-\tau(\omega{\cdot}t,u_1(t)))\\
&\qquad\qquad+(1-s)\,y_2(t-\tau(\omega{\cdot}t,u_2(t)))\Big)\big(y_1(t)-y_2(t)\big)\,ds\\
&\qquad+\int_0^1 D_3F\Big(\omega{\cdot}t,s\,y_1(t)+(1-s)y_2(t),s\,y_1(t-\tau(\omega{\cdot}t,u_1(t)))\\
&\qquad\qquad+(1-s)\,y_2(t-\tau(\omega{\cdot}t,u_2(t)))\Big)\\
&\qquad\qquad\big(y_1(t-\tau(\omega{\cdot}t,u_1(t)))-y_2(t-\tau(\omega{\cdot}t,u_2(t)))\big)\,ds\,.
\end{aligned}$$

This equality together with (3.9), (3.10) and (3.14) ensures that

$$\begin{aligned}
&\big|F(\omega{\cdot}t,y_1(t),y_1(t-\tau(\omega{\cdot}t,u_1(t))))\\
&\quad-F(\omega{\cdot}t,y_2(t),y_2(t-\tau(\omega{\cdot}t,u_2(t))))\big|\le L_4\,\|u_1(t)-u_2(t)\|_C
\end{aligned}\tag{3.15}$$

for all $t\in[0,\widetilde{t}\,]$, where $L_4:=L_1(1+L_3)$. Now, it follows from the integral form of equation (3.1) that $y_1(t)-y_2(t)$ satisfies

$$\begin{aligned}
y_1(t)-y_2(t)=x_1(0)-x_2(0)+\int_0^t\Big(&F(\omega{\cdot}l,y_1(l),y_1(l-\tau(\omega{\cdot}l,u_1(l))))\\
&-F(\omega{\cdot}l,y_2(l),y_2(l-\tau(\omega{\cdot}l,u_2(l))))\Big)\,dl
\end{aligned}$$

for $t\in[0,\widetilde{t}\,]$, which together with (3.15) yields

$$|y_1(t)-y_2(t)|\le\|x_1-x_2\|_C+\int_0^t L_4\,\|u_1(l)-u_2(l)\|_C\,dl$$

for all $t\in[0,\widetilde{t}\,]$. And $|y_1(t)-y_2(t)|\le\|x_1-x_2\|_C$ for $t\in[-r,0]$, so that

$$\|u_1(t)-u_2(t)\|_C\le\|x_1-x_2\|_C+\int_0^t L_4\,\|u_1(l)-u_2(l)\|_C\,dl$$

for $t\in[0,\widetilde{t}\,]$. Applying the Gronwall lemma we obtain

$$\|u_1(t)-u_2(t)\|_C\le L_5\,\|x_1-x_2\|_C\tag{3.16}$$

for $t\in[0,\widetilde{t}\,]$, where $L_5=\exp(L_4\,\widetilde{t}\,)\ge1$. Combining now (3.1), (3.15) and (3.16), we obtain

$$\begin{aligned}
&|\dot y_1(t)-\dot y_2(t)|\\
&\quad=\big|F(\omega{\cdot}t,y_1(t),y_1(t-\tau(\omega{\cdot}t,u_1(t))))-F(\omega{\cdot}t,y_2(t),y_2(t-\tau(\omega{\cdot}t,u_2(t))))\big|\\
&\quad\le\;L_4\,\|u_1(t)-u_2(t)\|_C\;\le\;L_4\,L_5\,\|x_1-x_2\|_C\;\le\;L_4\,L_5\,\|x_1-x_2\|_{W^{1,\infty}}
\end{aligned}$$

for $t\in[0,\widetilde{t}\,]$. Since $|\dot y_1(t)-\dot y_2(t)|\le\|x_1-x_2\|_{W^{1,\infty}}\le L_5\|x_1-x_2\|_{W^{1,\infty}}$ for Lebesgue a.a. $t\in[-r,0]$, we obtain

$$\|u(t,\omega,x_1)-u(t,\omega,x_2)\|_{W^{1,\infty}}\le L\,\|x_1-x_2\|_{W^{1,\infty}}\quad\text{ for all }\;t\in[0,\widetilde{t}\,]\,,$$

where $L:=\max(L_5,L_4L_5)$. This completes the proof of (ii). □

4. Differentiability with respect to the initial state

Throughout this section, we assume that H1 and H2 hold, and use the notation established in the previous one. Recall that the compatibility set $\mathcal{C}_0$ and the closely related set $\mathcal{U}^0$ are defined by

$$\mathcal{C}_0 := \{(\omega, x) \in \Omega \times C^1 \mid \dot{x}(0^-) = F(\omega, x(0), x(-\tau(\omega, x)))\}\,, \tag{4.1}$$

$$\mathcal{U}^0 := \{(t, \omega, x) \mid (\omega, x) \in \mathcal{C}_0,\ t \in [0, \beta_{\omega,x})\}\,, \tag{4.2}$$

and that we provide them with the topologies induced by those of $\Omega \times W^{1,\infty}$ and $\mathbb{R}^+ \times \Omega \times W^{1,\infty}$, respectively. It is very easy to deduce from the definition (3.3) of the semiflow Π that $\Pi(\mathcal{U}^0) = \mathcal{C}_0$, which is a fundamental property for what follows.

Let us consider the family of (linear) *variational equations*

$$\dot{z}(t) = L(\Pi(t, \omega, x))z_t\,, \quad t \in [0, \beta_{\omega,x}) \tag{4.3}$$

for $(\omega, x) \in \mathcal{C}_0$, where

$$\begin{aligned} L(\omega, x)v := {} & D_2F(\omega, x(0), x(-\tau(\omega, x)))v(0) + D_3F(\omega, x(0), x(-\tau(\omega, x)))v(-\tau(\omega, x)) \\ & - D_3F(\omega, x(0), x(-\tau(\omega, x)))\dot{x}(-\tau(\omega, x)){\cdot}D_2\tau(\omega, x)v \end{aligned}$$

for $(\omega, x) \in \mathcal{C}_0$ and $v \in C$. Note that each equation of the family (4.3) is evaluated along one of the positive Π-semiorbits lying on $\mathcal{C}_0$, and that it is not state-dependent, but just time-dependent. This section presents an analysis of the solutions of this family of delay equations, in the line of that made in Section 3 for the family (3.1). Its importance will be clarified by the properties stated in Corollary 4.3 and Theorem 4.4.

All the results of this section depend on the continuity properties of the maps

$$\mathcal{C}_0 \to \mathrm{Lin}(W^{1,\infty}, \mathbb{R}^n)\,, \quad (\omega, x) \mapsto L(\omega, x)\,,$$

and

$$\mathcal{C}_0 \times C \to \mathbb{R}^n, \quad (\omega, x, v) \mapsto L(\omega, x)v\,,$$

which we analyze in the next proposition.

Proposition 4.1. *Suppose that conditions H1 and H2 hold. Then,*

(i) *the map* $\mathcal{C}_0 \to \mathrm{Lin}(W^{1,\infty}, \mathbb{R}^n)$, $(\omega, x) \mapsto L(\omega, x)$ *is continuous.*
(ii) *The map* $\mathcal{U}^0 \to \mathrm{Lin}(W^{1,\infty}, \mathbb{R}^n)$, $(t, \omega, x) \mapsto L(\Pi(t, \omega, x))$ *is continuous.*
(iii) *Let us fix* $(\omega, x) \in \mathcal{C}_0$. *The map* $C \to \mathbb{R}^n\,,\quad v \mapsto L(\omega, x)v$ *is a bounded linear operator. In addition, for each* $k > 0$,

$$\sup\left\{\|L(\omega, x)\|_{\mathrm{Lin}(C,\mathbb{R}^n)} \;\middle|\; (\omega, x) \in \mathcal{C}_0 \text{ and } \|x\|_{W^{1,\infty}} \le k\right\} < \infty\,.$$

(iv) *The map* $\mathcal{C}_0 \times C \to \mathbb{R}^n\,,\ (\omega, x, v) \mapsto L(\omega, x)v$ *is continuous.*
(v) *The map* $\mathcal{U}^0 \times C \to \mathbb{R}^n\,,\ (t, \omega, x, v) \mapsto L(\Pi(t, \omega, x))v$ *is continuous.*

Proof. Recall that H1 and H2 ensure the continuity of $\tau\colon \Omega \times C \to \mathbb{R}$ and the existence and continuity of $D_iF\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ for $i = 2, 3$ and of $D_2\tau\colon \Omega \times C \to \mathrm{Lin}(C, \mathbb{R})$.

(i) Let us take a sequence $((\omega_m, x_m))$ in $\mathcal{C}_0$ with limit $(\widetilde{\omega}, \widetilde{x}) \in \mathcal{C}_0$. We will check that $L(\widetilde{\omega}, \widetilde{x})v = \lim_{m\to\infty} L(\omega_m, x_m)v$ by proving this property for each one of the terms appearing in the expression of L. So, we write $L(\omega, x) = L_1(\omega, x) + L_2(\omega, x) +$

$L_3(\omega,x)$. We take $\varepsilon>0$ and $v\in W^{1,\infty}$ with $\|v\|_{W^{1,\infty}}=1$, and call $\widetilde\tau:=\tau(\widetilde\omega,\widetilde x)$ and $\tau_m:=\tau(\omega_m,x_m)$. For $|L_1(\widetilde\omega,\widetilde x)-L_1(\omega_m,x_m)|$, since $|v(0)|\le 1$, we have

$$\begin{aligned}&\big|\big(D_2F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_2F(\omega_m,x_m(0),x_m(-\tau_m))\big)v(0)\big|\\&\qquad\le\|D_2F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_2F(\omega_m,x_m(0),x_m(-\tau_m))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\end{aligned}$$

so that the continuity of τ and D_2F shows that it is smaller than ε if m is large enough. Therefore, $L_1(\widetilde\omega,\widetilde x)=\lim_{m\to\infty}L_1(\omega_m,x_m)$ in $\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$.

The norm $|L_2(\widetilde\omega,\widetilde x)v-L_2(\omega_m,x_m)v|$ is bounded by the sum of two:

$$\begin{aligned}&\big|\big(D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_3F(\omega_m,x_m(0),x_m(-\tau_m))\big)v(-\tau_m)\big|\\&\qquad\le\|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_3F(\omega_m,x_m(0),x_m(-\tau_m))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\,,\end{aligned}$$

which is in the same situation as the previous term; and

$$\begin{aligned}&\big|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))\big(v(-\widetilde\tau)-v(-\tau_m)\big)\big|\\&\qquad\le\|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\|\dot v\|_{L^\infty}|\widetilde\tau-\tau_m|\\&\qquad\le\|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}|\widetilde\tau-\tau_m|\,,\end{aligned}\tag{4.4}$$

which is smaller than ε for large enough m due to the continuity of τ on $\mathcal{C}_0\subset\Omega\times C$. Here we have used that $v(-\widetilde\tau)-v(-\tau_m)=\int_0^1\dot v(-s\widetilde\tau+(s-1)\tau_m)(\tau_m-\widetilde\tau)\,ds$ and that $\|\dot v\|_{L^\infty}\le\|v\|_{W^{1,\infty}}=1$. (Incidentally: note that the proof would fail at this point if $\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$ were replaced by $\mathrm{Lin}(C,\mathbb{R}^n)$ as codomain of L.) Altogether, $L_2(\widetilde\omega,\widetilde x)=\lim_{m\to\infty}L_2(\omega_m,x_m)$ in $\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$.

The term $L_3(\omega,x)$ has in turn two factors. For the second one, we have

$$\big|\big(D_2\tau(\widetilde\omega,\widetilde x))-D_2\tau(\omega_m,x_m)\big)v|\le\|D_2\tau(\widetilde\omega,\widetilde x)-D_2\tau(\omega_m,x_m)\|_{\mathrm{Lin}(C,\mathbb{R})}\,,$$

and we can use the continuity of $D_2\tau$ to bound it by ε for large enough m. Finally,

$$\begin{aligned}&\big|\big(D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_3F(\omega_m,x_m(0),x_m(-\tau_m))\big)\dot x_m(-\tau_m)\big|\\&\qquad\le\|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))-D_3F(\omega_m,x_m(0),x_m(-\tau_m))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\|x_m\|_{W^{1,\infty}}\end{aligned}$$

and

$$\begin{aligned}&\big|D_3F(\widetilde\omega,\dot{\widetilde x}(0),\widetilde x(-\widetilde\tau))\big(\dot{\widetilde x}(-\widetilde\tau)-\dot x_m(-\tau_m)\big)\big|\\&\qquad\le\|D_3F(\widetilde\omega,\widetilde x(0),\widetilde x(-\widetilde\tau))\|_{\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^n)}\big(|\dot{\widetilde x}(-\widetilde\tau)-\dot{\widetilde x}(-\tau_m)|+\|\widetilde x-x_m\|_{W^{1,\infty}}\big)\,,\end{aligned}$$

and both terms can be easily bounded by ε if m is large enough. It follows easily that $L_3(\widetilde\omega,\widetilde x)=\lim_{m\to\infty}L_3(\omega_m,x_m)$ in $\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$. Altogether, we have checked that

$$\lim_{m\to\infty}\|\widetilde L(\widetilde\omega,\widetilde x)-\widetilde L(\omega_m,x_m)\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)}=0\,,$$

so that (i) is proved.

(ii) We have already pointed out that $\Pi(t,\omega,x)\in\mathcal{C}_0$ whenever $(t,\omega,x)\in\mathcal{U}^0$, so that $L(\Pi(t,\omega,x))$ is a well-defined map. Property (ii) follows from (i) and Theorem 3.3(viii).

(iii) The first assertion of (iii) is an easy consequence of the continuity of D_2F, D_3F and $D_2\tau$. The second also follows easily from hypotheses H1 and H2.

(iv) We take a sequence $((\omega_m,x_m,v_m))$ of points of $\mathcal{C}_0\times C$ with limit $(\widetilde\omega,\widetilde x,\widetilde v)$ in $\mathcal{U}^0\times C$, and repeat step by step the proof of (i) (no matter the fact that v_m and $\widetilde v$ belongs to C instead of $W^{1,\infty}$): note that the sequence $(\|v_m\|_C)$ is bounded, and that no uniformity in v_m is required. The only slightly different point is the analogous of (4.4), which is simpler in the current situation.

(v) As said in (ii), $L(\Pi(t,\omega,x))$ is a well-defined map. Property (iv) follows from (iv) and Theorem 3.3(viii). □

The next results (Theorem 4.2 and Corollary 4.3) constitute the analogues of Theorems 3.3 and 3.6 for the family of variational equations (4.3). In particular, we show that this family induces a pseudo-continuous semiflow on $\mathcal{K}\times W^{1,\infty}$, where $\mathcal{K}$ is any positively Π-invariant compact subset on $\mathcal{C}_0$. We will also prove that this semiflow is the one usually so-called *linearized* semiflow of Π along the semiorbits of $\mathcal{K}$: see Theorem 4.4.

Recall that the Π-semiorbit $\Pi(t,\omega,x)$ starting at $(\omega,x)\in\Omega\times W^{1,\infty}$ is defined on a maximal interval represented by $[-r,\beta_{\omega,x})$.

Theorem 4.2. *Suppose that conditions H1 and H2 hold. Then,*

(i) *for $(\omega,x)\in\mathcal{C}_0$ and $v\in C$, there exists a unique maximal solution $z(t,\omega,x,v)$ of the equation (4.3) corresponding to (ω,x) satisfying $z(s,\omega,x,v)=v(s)$ for $s\in[-r,0]$, which is defined for $t\in[-r,\beta_{\omega,x})$. In addition, the map $C\to\mathbb{R}^n,\ v\mapsto z(t,\omega,x,v)$ is linear and continuous for all $t\in[-r,\beta_{\omega,x})$.*

Let us define

$$w(t,\omega,x,v)(s):=z(t+s,\omega,x,v)\quad \text{for }\ (\omega,x)\in\mathcal{C}_0\,,\ t\in[0,\beta_{\omega,x}),\ v\in C\,,\ \text{and } s\in[-r,0]\,. \tag{4.5}$$

Then,

(ii) *$w(t+l,\omega,x,v)=w(t,\Pi(l,\omega,x),w(l,\omega,x,v))$ whenever the right-hand term exists.*

(iii) *The map $\mathcal{U}^0\times C\to C\,,\ (t,\omega,x,v)\mapsto w(t,\omega,x,v)$ is continuous.*

(iv) *If $v\in W^{1,\infty}$, then $w(t,\omega,x,v)\in W^{1,\infty}$ for all $t\in[0,\beta_{\omega,x})$.*

(v) *Let us define*

$$\widetilde{\mathcal{U}}^0:=\{(t,\omega,x)\in\mathcal{U}^0\mid t\ge r\}\,. \tag{4.6}$$

The map $\widetilde{\mathcal{U}}^0\times W^{1,\infty}\to W^{1,\infty},\ (t,\omega,x,v)\mapsto w(t,\omega,x,v)$ is continuous.

(vi) *Let us fix $\widetilde{t}\ge 0$ with*

$$\mathcal{U}^0_{\widetilde{t}}:=\{(\omega,x)\mid(\widetilde{t},\omega,x)\in\mathcal{U}^0\}=\{(\omega,x)\in\mathcal{C}_0\mid\widetilde{t}<\beta_{\omega,x}\} \tag{4.7}$$

nonempty. Then the map $\mathcal{U}^0_{\widetilde{t}}\times W^{1,\infty}\to W^{1,\infty},\ (\omega,x,v)\mapsto w(\widetilde{t},\omega,x,v)$ is continuous.

(vii) *Let us define $\mathcal{V}^0:=\{(t,\omega,x,v)\mid(t,\omega,x)\in\mathcal{U}^0$ and $v\in C^1$ with $\dot v(0^-)=L(\omega,x)v\}$. Then the map $\mathcal{V}^0\to W^{1,\infty},\ (t,\omega,x,v)\mapsto w(t,\omega,x,v)$ is continuous.*

(viii) *Let $\mathcal{U}^0_{\widetilde{t}}$ be defined by (4.7). Let us fix $\widetilde{t}>0$ with $\mathcal{U}^0_{\widetilde{t}}$ nonempty, and $(\widetilde{\omega},\widetilde{x})\in\mathcal{U}^0_{\widetilde{t}}$. Let us also fix $\rho>0$ be small enough to guarantee that*
  1. *$u(t,\omega,x)$ is defined (i.e., (t,ω,x) belongs to $\mathcal{U}^0$) whenever $t\in[0,\widetilde{t}]$ and $(\omega,x)\in\mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}:=\{(\omega,x)\in\mathcal{C}_0\mid d_\Omega(\omega,\widetilde{\omega})<\rho$ and $\|x-\widetilde{x}\|_{W^{1,\infty}}<\rho\}$;*
  2. *$\sup\{\|u(t,\omega,x)\|_C\mid t\in[0,\widetilde{t}]$ and $(\omega,x)\in\mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}\}=:\widetilde{c}<\infty$.*

  Then there exists $M=M(\widetilde{t},\widetilde{\omega},\widetilde{x},\rho)$ such that, if $(\omega,x)\in\mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}$, then

$$\|w(t,\omega,x,v)\|_{W^{1,\infty}}\le M\,\|v\|_{W^{1,\infty}}\qquad\text{for all }\ t\in[0,\widetilde{t}]\,.$$

Proof. (i) The properties proved in Proposition 4.1(iii)-(v) allow us to apply the general theory for finite-delay differential equations of Hale and Verdyun Lunel [4], Section 2.2, in order to ensure the existence and uniqueness of $z(t,\omega,x,v)$ and its continuity with respect to $v \in C$. The classical method of continuation of solutions for linear differential equations allows us to prove that $z(t,\omega,x,v)$ is defined for all $t \in [-r, \beta_{\omega,x})$; and since the equation is linear, so is the map $v \mapsto z(t,\omega,x,v)$.

(ii) This cocycle property follows from the uniqueness established in (i).

(iii) Let us take a sequence $((t_m,\omega_m,x_m,v_m))$ in $\mathcal{U}^0 \times C$ with limit $(\widetilde{t},\widetilde{\omega},\widetilde{x},\widetilde{v}) \in \mathcal{U}^0 \times C$. The open character of $\mathcal{U}$ allows us to assume without restriction the existence of $t_0 \in (\widetilde{t}, \beta_{\widetilde{\omega},\widetilde{x}})$ such that $t_m \le t_0 < \beta_{\omega_m,x_m}$ for all $m \in \mathbb{N}$. Let us represent $z_m(t) := z(t,\omega_m,x_m,v_m)$ for $t \in [-r,t_0]$ and $w_m(t) := w(t,\omega_m,x_m,v_m)$ for $t \in [0,t_0]$ and $m \in \mathbb{N}$. The integral form of (4.3) shows that

$$z_m(t) = v_m(0) + \int_0^t L(\Pi(l,\omega_m,x_m))w_m(l)\,dl$$

for $t \in [0,t_0]$. On the other hand, the set $\mathcal{S}$ defined by (3.4) is contained in $\mathcal{C}_0$ and compact in $\Omega\times W^{1,\infty}$; therefore, Proposition 4.1(ii) ensures that $(L\circ\Pi)([0,t_0]\times\mathcal{S}) \subset \mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$ is compact; and hence $k_0 := \sup\{\|L(\Pi(t,\omega_m,x_m))\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)} \mid t \in [0,t_0]$ and $m \in \mathbb{N}\}$ is finite. Let us call $\alpha := \sup_{m\in\mathbb{N}} \|v_m\|_C$. Then,

$$\|w_m(t)\|_C \le \alpha + \int_0^{t_0} k_0\,\|w_m(l)\|_C\,dl$$

for all $t \in [0,t_0]$, and the Gronwall lemma shows that $k := \sup\{\|w_m(t)\|_C \mid t \in [0,t_0]$ and $m \in \mathbb{N}\}$ is finite. In turn, this fact and (4.3) ensure that the set $\{\dot z_m(t) \mid t \in [0,t_0]$ and $m \in \mathbb{N}\}$ is uniformly bounded.

Now we follow the scheme of the proof of Theorem 3.3(v). First, we deduce from Arzelá-Ascoli theorem that $\lim_{m\to\infty} z(t,\omega_m,x_m,v_m) = z(t,\widetilde{\omega},\widetilde{x},\widetilde{v})$ uniformly in $[-r,t_0]$. And second, we write $\|w(t,\widetilde{\omega},\widetilde{x},\widetilde{v}) - w(t_m,\omega_m,x_m,v_m)\|_C \le \|w(t,\widetilde{\omega},\widetilde{x},\widetilde{v}) - w(t_m,\widetilde{\omega},\widetilde{x},\widetilde{v})\|_C + \|w(t_m,\widetilde{\omega},\widetilde{x},\widetilde{v}) - w(t_m,\omega_m,x_m,v_m)\|_C$ and note that: the term $\|w(t,\widetilde{\omega},\widetilde{x},\widetilde{v}) - w(t_m,\widetilde{\omega},\widetilde{x},\widetilde{v})\|_C$ is as small as desired for large enough m due to the uniform continuity of $t \mapsto z(t,\widetilde{\omega},\widetilde{x},\widetilde{v})$ in $[-r,t_0]$; and the term $\|w(t,\widetilde{\omega},\widetilde{x},\widetilde{v}) - w(t,\omega_m,x_m,v_m)\|_C$ is as small as desired for large enough m for all $t \in [0,t_0]$ due to the previously proved uniform convergence. Thus, (iii) is proved.

(iv) Let us take $v \in W^{1,\infty}$. We know that $\dot z(t,\omega,x,v) = L(\Pi(t,\omega,x))w(t,\omega,x,v)$ for $t \ge 0$. This ensures that, if $t \ge r$, $\|w(t,\omega,x,v)\|_{L^\infty}$ is finite, so that $w(t,\omega,x,v) \in W^{1,\infty}$. If $t \in [0,r]$, then $\|w(t,\omega,x,v)\|_{W^{1,\infty}} \le \|v\|_{W^{1,\infty}} + \|w(r,\omega,x,v)\|_{W^{1,\infty}}$.

(v)&(vi) These properties can be checked with the arguments used to prove Theorem 3.3(vi)&(vii).

(vii) The proof of this point can be done following the ideas of Theorem 3.3(viii), and is simpler: if $\lim_{m\to\infty}(t_m,\omega_m,x_m,v_m) = (\widetilde{t},\widetilde{\omega},\widetilde{x},\widetilde{v})$ in $\mathcal{V}^0$, if $t_m \le t_0 < \beta_{\omega_m,x_m}$ for all $m \in \mathbb{N}$, and if $z_m(t) := z(t,\omega_m,x_m,v_m)$ for $t \in [-r,t_0]$, then the equicontinuity of the family $\{\dot z_m(t) \mid m \in \mathbb{N}\}$ on $[0,t_0]$ follows easily from (4.3), Proposition (4.1)(i), and the uniform continuity of Π on $[0,t_0]\times\mathcal{S}$, where $\mathcal{S}$ is given by (3.4).

(viii) The point $(\widetilde{t},\widetilde{\omega},\widetilde{x})$ will be fixed in this proof. The existence of $\rho > 0$ for which conditions 1 and 2 hold is proved by Theorem 3.6(i), where we also checked that (see (3.11))

$$\sup\{\|u(t,\omega,x)\|_{W^{1,\infty}} \mid t \in [0,\widetilde{t}] \;\text{ and }\; (\omega,x) \in \mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}\} =: c^* < \infty\,.$$

Therefore, according to Proposition 4.1(iii),

$$\sup\{\|L(\Pi(t,\omega,x))\|_{\mathrm{Lin}(C,\mathbb{R}^n)} \mid t \in [0,\widetilde{t}] \text{ and } x \in \mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}\} =: \widehat{c} < \infty\,.$$

In particular, conditions 1 and 2 hold. We take $t \in [0,\widetilde{t}]$, $(\omega,x) \in \mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}$, and $v \in W^{1,\infty}$, and make use of (4.3) in order to write

$$
\begin{aligned}
|z(t,\omega,x,v)| &\le |v(0| + \int_0^t \|L(\Pi(l,\omega,x))\|_{\mathrm{Lin}(C,\mathbb{R}^n)} \|w(l,\omega,x,v)\|_C \, dl \\
&\le \|v\|_{W^{1,\infty}} + \int_0^t \widehat{c}\, \|w(l,\omega,x,v)\|_C \,.
\end{aligned}
$$

It follows easily that $\|w(t,\omega,x,v)\|_C \le \|v\|_{W^{1,\infty}} + \int_0^t \widehat{c}\, \|w(l,\omega,x,v)\|_C$, so that the Gronwall lemma ensures that $\|w(t,\omega,x,v)\|_C \le e^{\widehat{c}t}\,\|v\|_{W^{1,\infty}}$. To finish the proof of (i) is now easy: see for instance the end of the proof of Theorem 3.6(ii). ☐

Corollary 4.3. *Suppose that conditions H1 and H2 hold. Let $\mathcal{K} \subset \mathcal{C}_0$ be a positively Π-invariant compact set. We define by (4.5) the function $w(t,\omega,x,v)$ for $t \in \mathbb{R}^+$, $(\omega,x) \in \mathcal{K}$, and $v \in C$. Then*

(i) *the map*

$$
\Pi^L \colon \mathbb{R}^+ \times \mathcal{K} \times C \to \mathcal{K} \times C\,, \quad (t,\omega,x,v) \mapsto (\Pi(t,\omega,x), w(t,\omega,x,v))
$$

is a continuous linear skew-product semiflow with base $(\mathcal{K},\Pi,\mathbb{R}^+)$.

(ii) *The map*

$$
\widetilde{\Pi}^L \colon \mathbb{R}^+ \times \mathcal{K} \times W^{1,\infty} \to \mathcal{K} \times W^{1,\infty},\ (t,\omega,x,v) \mapsto (\Pi(t,\omega,x), w(t,\omega,x,v))
$$

satisfies properties (f1) and (f2) with Ω replaced by $\mathcal{K} \times W^{1,\infty}$ (for all $t \ge 0$ and all $l \ge 0$ in the case of (f2)). In addition,

- *$[r,\infty) \times \mathcal{K} \times W^{1,\infty} \to \mathcal{K} \times W^{1,\infty}$, $(t,\omega,x,v) \mapsto (\Pi(t,\omega,x), w(t,\omega,x,v))$ is a continuous map.*
- *For each $\widetilde{t} \ge 0$, the map $\Pi^L_{\widetilde{t}} \colon \mathcal{K} \times W^{1,\infty} \to \mathcal{K} \times W^{1,\infty}$, $(\omega,x,v) \mapsto (\Pi(\widetilde{t},\omega,x), w(\widetilde{t},\omega,x,v))$ is continuous.*
- *Let us define*

$$
\begin{aligned}
\mathcal{V}^0_{\mathcal{K}} &:= \{(t,\omega,x,v) \in \mathcal{V}^0 \,|(\omega,x) \in \mathcal{K}\} \\
&= \{(t,\omega,x,v) \in \mathbb{R}^+ \times \mathcal{K} \times C^1 \mid v(0^-) = L(\omega,x)v\}\,.
\end{aligned}
$$

The map $\mathcal{V}^0_{\mathcal{K}} \to \mathcal{K} \times W^{1,\infty}$, $t \mapsto (\Pi(t,\omega,x), w(t,\omega,x,v))$ is continuous.

Proof. Corollary 3.4 shows that $(\mathcal{K},\Pi,\mathbb{R}^+)$ is a global continuous semiflow. Having this in mind, all the assertions are trivial consequences of Theorem 4.2. ☐

As we anticipated, our next result, Theorem 4.4, will show that, as a matter of fact, $\widetilde{\Pi}^L \colon \mathbb{R}^+ \times \mathcal{K} \times W^{1,\infty} \to \mathcal{K} \times W^{1,\infty}$ is the *linearized semiflow of* Π, in the sense that each one of its semiorbits determine the differential with respect to the state variable of the semiorbits of Π. The first assertion in the theorem is proved (in a slightly different setting) in [5], Theorem 4. For the sake of completeness we give here part of the details of the proof, since they help the reader to understand the dynamical meaning of the function $u_x(t,\omega,x)$.

Note that the uniformity of the limit (4.8) with respect to the elements of the unit ball means that $u_x(t,\omega,x)$ is the classical Fréchet differential with respect to the initial state of the function $u(t,\omega,x)$, which provides it with full dynamical meaning.

The sets $\mathcal{C}_0$ and $\mathcal{U}^0$ appearing in the next statement are given by (4.1) and (4.2).

Theorem 4.4. *Suppose that H1 and H2 hold. Let us fix $(\omega,x)\in\mathcal{C}_0$. If $t\in[0,\beta_{\omega,x})$, then there exists*

$$u_x(t,\omega,x)v=\lim_{\varepsilon\to 0}\frac{u(t,\omega,x+\varepsilon v)-u(t,\omega,x)}{\varepsilon}\quad \text{in } W^{1,\infty} \\ \text{uniformly in } v\in\overline{\mathcal{B}}_1\,, \tag{4.8}$$

where $\overline{\mathcal{B}}_1:=\{v\in W^{1,\infty}\mid \|v\|_{W^{1,\infty}}=1\}$. In addition, the map

$$[-r,\beta_{\omega,x})\to W^{1,\infty},\quad t\mapsto (u_x(t,\omega,x)v)(t)$$

is the unique solution of (4.3) agreeing with v on $[-r,0]$. That is, $u_x(t,\omega,x)v=w(t,\omega,x,v)$, this last map being defined by (4.5).

Consequently, the map $(t,\omega,x,v)\mapsto u_x(t,\omega,x)v$ satisfies all the continuity properties described in Theorem 4.2.

Proof. In the whole proof, the point $(\omega,x)\in\mathcal{C}_0$ will be fixed, and any v will belong to $\overline{\mathcal{B}}_1$. We also fix an arbitrary time $T\in(0,\beta_{\omega,x})$. Theorem 3.3(v) provides $\delta>0$ such that, for $\varepsilon\in[-\delta,\delta]$, there exists $y_v^\varepsilon(t):=y(t,\omega,x+\varepsilon v)$ for $t\in[-r,T]$, and $v\in\overline{\mathcal{B}}_1$. We also represent $u_v^\varepsilon(t):=u(t,\omega,x+\varepsilon v)$, $\tau_v^\varepsilon(t):=\tau(\omega{\cdot}t,u_v^\varepsilon(t))$, $z_v(t):=z(t,\omega,x,v)$, and $w_v(t):=w(t,\omega,x,v)$ for $|\varepsilon|\le\delta$, $t\in[0,T]$, and $v\in\overline{\mathcal{B}}_1$. (Recall that $z(t,\omega,x,v)$ is the solution of (4.3) agreeing with v on $[-r,0]$, and that $w(t,\omega,x,v)$ is defined by (4.5).) Note that $y_v^0(t)$, $u_v^0(t)$, and $\tau_v^0(t)$ are independent of $v\in\overline{\mathcal{B}}_1$. For this reason, we omit the subscript when $\varepsilon=0$.

Theorem 2 and Corollary 1 of Hartung [5] prove that, if $(\omega,x)\in\mathcal{C}_0$ and $t\in[0,\beta_{\omega,x})$, then

$$u_x(t,\omega,x)v=\lim_{\varepsilon\to 0}\frac{u(t,\omega,x+\varepsilon v)-u(t,\omega,x)}{\varepsilon}\quad \text{in } C \text{ unif. in } v\in\overline{\mathcal{B}}_1\,, \tag{4.9}$$

where $\overline{\mathcal{B}}_1:=\{v\in W^{1,\infty}\mid \|v\|_{W^{1,\infty}}=1\}$. They also prove that

$$(u_x(t,\omega,x)v)=w(t,\omega,x,v)\,,$$

where $w(t,\omega,x,v)$ is defined by (4.5). In particular, the map $u_x(t,\omega,x)\colon W^{1,\infty}\to C$ is linear. And we have seen in Theorem 4.2(iv) that $u_x(t,\omega,x)(v)=w(t,\omega,x,v)\in W^{1,\infty}$ if $v\in W^{1,\infty}$, so that $u_x(t,\omega,x)\colon W^{1,\infty}\to W^{1,\infty}$ is well defined. So that the goal is to prove that (4.9) is still true in the topology of $W^{1,\infty}$ instead of that of C. (See also Theorem 4 of [5].)

Since T is arbitrarily chosen, a standard compactness argument shows that (4.9) can be rewritten as

$$\lim_{\varepsilon\to 0}\left\|\frac{u_v^\varepsilon(t)-u^0(t)}{\varepsilon}-w_v(t)\right\|_C=0\qquad \text{for all } t\in[0,T] \text{ unif. in } v\in\overline{\mathcal{B}}_1\,. \tag{4.10}$$

For the same reason, in order to prove (4.8), we must prove that

$$\lim_{\varepsilon\to 0}\left\|\frac{u_v^\varepsilon(t)-u^0(t)}{\varepsilon}-w_v(t)\right\|_{W^{1,\infty}}=0\qquad \text{for all } t\in[0,T] \text{ unif. in } v\in\overline{\mathcal{B}}_1\,.$$

Note that $\big(\dot y_v^\varepsilon(s)-\dot y^0(s)\big)/\varepsilon=\dot v(s)=\dot z_v(s)$ for Lebesgue-a.a. $s\in[-r,0]$. This property, (4.10), and the definition of $\|{\cdot}\|_{W^{1,\infty}}$, show that our goal will be achieved once we have proved that

$$\lim_{\varepsilon\to 0}\frac{\dot y_v^\varepsilon(t)-\dot y^0(t)}{\varepsilon}=\dot z_v(t)\qquad \text{uniformly in } t\in[0,T] \text{ and } v\in\overline{\mathcal{B}}_1\,. \tag{4.11}$$

Equation (3.1) satisfied by $y_v^\varepsilon(t)$ combined with H1 and H2 yield

$$\begin{aligned}\frac{\dot y_v^\varepsilon(t)-\dot y^0(t)}{\varepsilon}&=\frac{F(\omega{\cdot}t,y_v^\varepsilon(t),y_v^\varepsilon(t-\tau_v^\varepsilon(t)))-F(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t)))}{\varepsilon}\\&=\int_0^1\bigg(D_2F\big(\omega{\cdot}t,s\,y_v^\varepsilon(t)+(1-s)\,y^0(t),sy_v^\varepsilon(t-\tau_v^\varepsilon(t))\\&\qquad+(1-s)\,y^0(t-\tau^0(t))\big)\left(\frac{y_v^\varepsilon(t)-y^0(t)}{\varepsilon}\right)\bigg)ds\\&\quad+\int_0^1\bigg(D_3F\big(\omega{\cdot}t,s\,y_v^\varepsilon(t)+(1-s)\,y^0(t),sy_v^\varepsilon(t-\tau_v^\varepsilon(t))\\&\qquad+(1-s)\,y^0(t-\tau^0(t))\big)\left(\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau^0(t))}{\varepsilon}\right)\bigg)\,ds\,.\end{aligned}$$

The proof of (i) will be a consequence of the following property: the limits of the first and second integrands as $\varepsilon\to0$ are, respectively,

$$l_1=D_2F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)z_v(t)\,, \tag{4.12}$$

$$\begin{aligned}l_2&=D_3F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)z_v(t-\tau^0(t))\\&\quad-D_3F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)\;\dot y^0(t-\tau^0(t)){\cdot}D_2\tau^0(t)w_v(t)\end{aligned} \tag{4.13}$$

uniformly in $t\in[0,T]$, $s\in[0,1]$, and $v\in\overline{\mathcal B}_1$. In order to check that this uniform limiting behavior yields indeed the result, we assume for the moment being that (4.12) and (4.13) hold. Then we can combine the continuity of D_2F and D_3F ensured by H1 in order to deduce that

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{\dot y_v^\varepsilon(t)-\dot y^0(t)}{\varepsilon}&=D_2F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)z_v(t)\\&\quad+D_3F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)z_v(t-\tau^0(t))\\&\quad-D_3F\big(\omega{\cdot}t,y^0(t),y^0(t-\tau^0(t))\big)\dot y^0(t-\tau^0(t)){\cdot}D_2\tau^0(t)w_v(t)\\&=L(\Pi(t,\omega,x))w_v(t)\end{aligned}$$

uniformly in $t\in[0,T]$ and $v\in\overline{\mathcal B}_1$. Since, according to (4.3), the last expression agrees with $\dot z_v(t)$, the equality (4.11) (and hence assertion (i)) will be proved, as asserted.

It is easy to deduce from the continuity $\Pi\colon\mathbb R^+\times\Omega\times W^{1,\infty}\to\Omega\times C$ ensured by Theorem 3.3(v) that, given $\rho>0$, there exists $\delta=\delta(\rho)>0$ such that, if $|\varepsilon|\le\delta$, then $\|u_v^\varepsilon(t)-u_v^0(t)\|_C\le\rho$ for all $t\in[0,T]$ and all $v\in\overline{\mathcal B}_1$. In other words, $\lim_{\varepsilon\to0}u_v^\varepsilon(t)=u^0(t)$ uniformly in $t\in[0,T]$ and $v\in\overline{\mathcal B}_1$. This property guarantees that the following limits exist and are uniform in $t\in[0,T]$ and $v\in\overline{\mathcal B}_1$:

$$\begin{aligned}y^0(t)&=\lim_{\varepsilon\to0}y_v^\varepsilon(t)\,,\\\tau^0(t)&=\lim_{\varepsilon\to0}\tau_v^\varepsilon(t)\,,\\y^0(t-\tau^0(t))&=\lim_{\varepsilon\to0}y_v^\varepsilon(t-\tau_v^\varepsilon(t))\,.\end{aligned} \tag{4.14}$$

The last limit follows from the previous ones and $\|y^0(t-\tau^0(t))-y_v^\varepsilon(t-\tau_v^\varepsilon(t))\|\le\|y^0(t-\tau^0(t))-y^0(t-\tau_v^\varepsilon(t))\|+\|y^0(t-\tau_v^\varepsilon(t))-y_v^\varepsilon(t-\tau_v^\varepsilon(t))\|$.

On the other hand, (4.10) yields

$$\lim_{\varepsilon\to 0}\frac{y_v^\varepsilon(t)-y^0(t)}{\varepsilon}=z_v(t)\qquad\text{uniformly in }\; t\in[-r,T]\;\text{ and }\; v\in\overline{\mathcal{B}}_1\,. \tag{4.15}$$

These facts and the continuity of D_2F guaranteed by H1 allow us to check (4.12). To deal with the second integrand, we will prove that

$$\lim_{\varepsilon\to 0}\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau^0(t))}{\varepsilon}=z_v(t-\tau^0(t))-\dot y^0(t-\tau^0(t))\cdot D_2\tau^0(t)w_v(t) \tag{4.16}$$

uniformly in $t\in[0,T]$ and $v\in\overline{\mathcal{B}}_1$. Let us write

$$\begin{aligned}&\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau^0(t))}{\varepsilon}\\&\qquad=\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau_v^\varepsilon(t))}{\varepsilon}+\frac{y^0(t-\tau_v^\varepsilon(t))-y^0(t-\tau^0(t))}{\varepsilon}\,,\end{aligned}$$

and deal with each term to obtain (4.16).

We first check that

$$\lim_{\varepsilon\to 0}\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau_v^\varepsilon(t))}{\varepsilon}=z_v(t-\tau^0(t))\quad\text{unif. in }\; t\in[0,T]\;\text{ and }\; v\in\overline{\mathcal{B}}_1\,,$$

which in turn requires

$$\lim_{\varepsilon\to 0}\big|z_v(t-\tau_v^\varepsilon(t))-z_v(t-\tau^0(t))\big|=0\quad\text{unif. in }\; t\in[0,T]\;\text{ and }\; v\in\overline{\mathcal{B}}_1\,. \tag{4.17}$$

To prove this last property, we use the convergence of $\tau_v^\varepsilon(t)$ to $\tau^0(t)$ as $\varepsilon\to 0$ (see (4.14)), which is uniform in $t\in[0,T]$ and $v\in\overline{\mathcal{B}}_1$, together with the equicontinuity of $\{z_v\mid v\in\mathcal{B}_1\}$ on $[-r,T]$: note first that the family is equicontinuous on $[-r,0]$ (where $|z(t_1)-z(t_2)|\le\|\dot v\|_{L^\infty}|t_1-t_2|\le|t_1-t_2|$); and second, that

$$|\dot z_v(t)|\le\|L(\Pi(t,\omega,x))\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)}\|w_v(t)\|_{W^{1,\infty}}$$

for $t>0$, so that the uniform continuity of the map $[0,T]\to\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)$, $t\mapsto L(\Pi(t,\omega,x))$ (ensured by Proposition 4.1(ii)) and the Lipschitz behavior of $w_v(t)$ (ensured by Theorem 4.2(viii)) provide a constant M such that $|\dot z_v(t)|\le M$ for all $t\in[0,T]$ and $v\in\mathcal{B}_1$. So, (4.17) is proved, and we can write

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left|\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau_v^\varepsilon(t))}{\varepsilon}-z_v(t-\tau^0(t))\right|\\&\qquad\le\lim_{\varepsilon\to 0}\left|\frac{y_v^\varepsilon(t-\tau_v^\varepsilon(t))-y^0(t-\tau_v^\varepsilon(t))}{\varepsilon}-z_v(t-\tau_v^\varepsilon(t))\right|\\&\qquad\quad+\lim_{\varepsilon\to 0}\big|z_v(t-\tau_v^\varepsilon(t))-z_v(t-\tau^0(t))\big|=0+0\,,\end{aligned}$$

the last limits being uniform in $t\in[0,T]$ and $v\in\overline{\mathcal{B}}_1$. (The assertion concerning the first limit is 0 is an easy consequence of (4.15).)

The remaining limit to compute is

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{y^0(t-\tau_v^\varepsilon(t))-y^0(t-\tau^0(t))}{\varepsilon}\\&\qquad=\lim_{\varepsilon\to 0}\int_0^1\dot y^0(t-s\tau_v^\varepsilon(t)-(1-s)\tau^0(t))\,ds\cdot\lim_{\varepsilon\to 0}\frac{\tau^0(t)-\tau_v^\varepsilon(t)}{\varepsilon}\,.\end{aligned}$$

Since $(\omega,x)\in\mathcal{C}_0$, it follows from Theorem 3.3(vii) that $\dot y^0$ is uniformly continuous on $[-r,T]$, from where it follows easily that $\lim_{\varepsilon\to 0}\dot y^0(t-s\tau_v^\varepsilon(t)-(1-s)\tau^0(t))=\dot y^0(t-\tau^0(t))$ uniformly in $t\in[0,T]$, $s\in[0,1]$ and $v\in\overline{\mathcal{B}}_1$. These properties yield

$$\lim_{\varepsilon\to 0}\int_0^1 \dot y^0(t-s\tau_v^\varepsilon(t)-(1-s)\tau^0(t))\,ds=\dot y^0(t-\tau^0(t))$$

uniformly in $t\in[0,T]$, $s\in[0,1]$ and $v\in\overline{\mathcal{B}}_1$. On the other hand, we can write

$$\lim_{\varepsilon\to 0}\frac{\tau^0(t)-\tau_v^\varepsilon(t)}{\varepsilon}=\lim_{\varepsilon\to 0}\int_0^1 D_2\tau\big(\omega{\cdot}t,s\,u^0(t)+(1-s)\,u_v^\varepsilon(t)\big)\frac{u^0(t)-u_v^\varepsilon(t)}{\varepsilon}\,ds\,.$$

We deduce from H2 and from the property $\lim_{\varepsilon\to 0}u_v^\varepsilon(t)=u^0(t)$ uniformly in $t\in[0,T]$ and $v\in\overline{\mathcal{B}}_1$ (see above) that $\lim_{\varepsilon\to 0}D_2\tau\big(\omega{\cdot}t,s\,u_v^\varepsilon(t)+(1-s)\,u^0(t)\big)=D_2\tau(\omega{\cdot}t,u^0(t))$ in $\mathrm{Lin}(C,\mathbb{R}^n)$ uniformly in $t\in[0,T]$, $s\in[0,1]$ and $v\in\overline{\mathcal{B}}_1$. Finally, according to (4.10), $\lim_{\varepsilon\to 0}\big(u_v^\varepsilon(t)-u^0(t)\big)/\varepsilon=w_v(t)$ in C uniformly in $t\in[0,T]$ and $v\in\overline{\mathcal{B}}_1$. These facts ensure that

$$\lim_{\varepsilon\to 0}\frac{\tau^0(t)-\tau_\varepsilon(t)}{\varepsilon}=-D_2\tau(\omega{\cdot}t,u^0(t))w_v(t)\quad\text{uniformly in } t\in[0,T]\ \text{ and } v\in\overline{\mathcal{B}}_1\,.$$

Altogether, we see that (4.16) is proved. Now we can check the assertion concerning the second integrand by combining equalities (4.14), the continuity of D_3F ensured by H1, and (4.16). The proof is complete. □

We complete the paper with a deeper analysis of the regularity properties of the map $u_x(t,\omega,x)v$.

Theorem 4.5. *Suppose that conditions H1 and H2 hold, and define the sets* $\mathcal{U}^0_t$ *and* $\widetilde{\mathcal{U}}^0$ *by (4.7) and (4.6).*

(i) *Let us fix* $\widetilde t\ge 0$ *with* $\mathcal{U}^0_{\widetilde t}$ *nonempty. Then the map* $\mathcal{U}^0_{\widetilde t}\to\mathrm{Lin}(W^{1,\infty},W^{1,\infty})$, $(\omega,x)\mapsto u_x(\widetilde t,\omega,x)$ *is continuous.*

(ii) *The map* $\widetilde{\mathcal{U}}^0\to\mathrm{Lin}(W^{1,\infty},W^{1,\infty})$, $(t,\omega,x)\mapsto u_x(t,\omega,x)$ *is continuous.*

Proof. In the whole proof, we will use the notation $u_x(t,\omega,x)v=w(t,\omega,x,v)$, since we will permanently use the fact that the function $t\mapsto w(t,\omega,x,v)(0)=z(t,\omega,x,v)$ solves (4.3).

(i) Let us take a sequence $((\omega_m,x_m))$ in $\mathcal{U}^0_{\widetilde t}$ with limit $(\widetilde\omega,\widetilde x)\in\mathcal{U}^0_{\widetilde t}$. We also take a constant ρ satisfying conditions 1 and 2 of Theorem 4.2(viii), and assume without restriction that $(\omega_m,x_m)\in\mathcal{B}^{\rho,0}_{\widetilde\omega,\widetilde x}$ for all $m\in\mathbb{N}$. For any $v\in W^{1,\infty}$, we represent $z_m(t,v):=z(t,\omega_m,x_m,v)$ and $\widetilde z(t,v):=z(t,\widetilde\omega,\widetilde x,v)$ for $t\in[-r,\widetilde t]$, and $w_m(t,v):=w(t,\omega_m,x_m,v)$ and $\widetilde w(t,v):=w(t,\widetilde\omega,\widetilde x,v)$ for $t\in[0,\widetilde t]$. Equation (4.3) yields

$$\begin{aligned}|\widetilde z(t,v)-z_m(t,v)|\le&\left|\int_0^t\big(L(\Pi(l,\omega_m,\widetilde x_m))-L(\Pi(l,\widetilde\omega,\widetilde x))\big)w_m(l,v)\,dl\right|\\&+\left|\int_0^t L(\Pi(l,\widetilde\omega,\widetilde x))(\widetilde w(l,v)-w_m(l,v))\,dl\right|\end{aligned}\tag{4.18}$$

for all $t\in[0,\widetilde t]$. Let us fix $\varepsilon>0$, define $\mathcal{S}$ by (3.4), note that $\mathcal{S}\subset\mathcal{C}_0$, and use Proposition 4.1(ii) and the compactness of $[0,\widetilde t]\times\mathcal{S}$ to find m_0 such that $\|L(\Pi(t,\omega_m,\widetilde x_m))-L(\Pi(t,\widetilde\omega,\widetilde x))\|_{\mathrm{Lin}(\Omega,R_n)}\le\varepsilon$ for $t\in[0,\widetilde t]$ and $m\ge m_0$. According to Theorem 4.2(viii), there exists $M>0$ such that $\|w_m(t,v)\|_{W^{1,\infty}}\le M\|v\|_{W^{1,\infty}}$

for $t \in [0, \widetilde{t}\,]$. And Proposition 4.1(iii) yields $k := \sup_{t\in[0,\widetilde{t}\,]} \|L(\Pi(l, \widetilde{\omega}, \widetilde{x}))\|_{\mathrm{Lin}(C,\mathbb{R}^n)}$ is finite. Therefore, by (4.18),

$$|\widetilde{z}(t, v) - z_m(t, v)| \le \varepsilon\, \widetilde{t}\, M\, \|v\|_{W^{1,\infty}} + \int_0^t k\, \|\widetilde{w}(l, v) - w_m(l, v)\|_C \, dl$$

for $t \in [0, \widetilde{t}\,]$ and $m \ge m_0$. Since $|\widetilde{z}(t, v) - \widetilde{z}_m(t, v)| = 0$ for $t \in [-r, 0]$, we conclude that

$$\|\widetilde{w}(t, v) - w_m(t, v)\|_C \le \varepsilon\, \widetilde{t}\, M\, \|v\|_C + \int_0^t k\, \|\widetilde{w}(l, v) - w_m(l, v)\|_C \, dl$$

for $t \in [0, \widetilde{t}\,]$ and $m \ge m_0$. The Gronwall lemma yields

$$\|\widetilde{w}(t, v) - w_m(t, v)\|_C \le \varepsilon \widetilde{t}\, M\, \|v\|_{W^{1,\infty}} e^{k\widetilde{t}} = \varepsilon\, \widetilde{M}\, \|v\|_{W^{1,\infty}}$$

for $t \in [0, \widetilde{t}\,]$ and $m \ge m_0$, where $\widetilde{M} = \widetilde{t}\, M\, e^{k\widetilde{t}}$. Now,

$$\begin{aligned}
|\dot{\widetilde{z}}(t, v) - \dot{z}_m(t, v)| &\le \|L(\Pi(t, \widetilde{\omega}, \widetilde{x})) - L(\Pi(t, \omega_m, \widetilde{x}_m))\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)} \|w_m(t, v)\|_{W^{1,\infty}} \\
&\quad + \|L(\Pi(t, \widetilde{\omega}, \widetilde{x}))\|_{\mathrm{Lin}(C,\mathbb{R}^n)} \|\widetilde{w}(t, v) - w_m(t, v)\|_C \\
&\le \varepsilon\, M\, \|v\|_{W^{1,\infty}} + \varepsilon\, k\, \widetilde{M}\, \|v\|_{W^{1,\infty}} = \varepsilon\, M^*\, \|v\|_{W^{1,\infty}}
\end{aligned}$$

for $t \in [0, \widetilde{t}\,]$ and $m \ge m_0$, where $M^* = M + k\widetilde{M}$. And $|\dot{\widetilde{z}}(t, v) - \dot{z}_m(t, v)| = 0$ for $t \in [-r, 0]$. Therefore, if $m \ge m_0$, we have

$$\|\widetilde{w}(\widetilde{t}, v) - w_m(\widetilde{t}, v)\|_{W^{1,\infty}} \le \varepsilon\, (\widetilde{M} + M^*)\, \|v\|_{W^{1,\infty}}\,.$$

Since the constants $\widetilde{M}$ and M^* can be defined from the beginning, (ii) is proved.

(ii) Let us take a sequence $((t_m, \omega_m, x_m))$ in $\widetilde{\mathcal{U}}^0$ with limit $(t_0, \widetilde{\omega}, \widetilde{x}) \in \widetilde{\mathcal{U}}^0$ and assume without restriction the existence of of $\widetilde{t} \in (t_0, \beta_{\widetilde{\omega},\widetilde{x}})$ with $2r \le t_m \le \widetilde{t} < \beta_{\omega_m,x_m}$ for all $m \in \mathbb{N}$. We also take a constant ρ satisfying conditions 1 and 2 of Theorem 4.2(viii) for $(\widetilde{t}, \widetilde{\omega}, \widetilde{x})$, and assume without restriction that $(\omega_m, x_m) \in \mathcal{B}^{\rho,0}_{\widetilde{\omega},\widetilde{x}}$ for all $m \in \mathbb{N}$. For any $v \in W^{1,\infty}$, we represent $z_m(t, v) := z(t, \omega_m, x_m, v)$ and $\widetilde{z}(t, v) := z(t, \widetilde{\omega}, \widetilde{x}, v)$ for $t \in [-r, \widetilde{t}\,]$, and $w_m(t, v) := w(t, \omega_m, x_m, v)$ and $\widetilde{w}(t, v) := w(t, \widetilde{\omega}, \widetilde{x}, v)$ for $t \in [0, \widetilde{t}\,]$. And we fix $\varepsilon > 0$. Note that

$$\begin{aligned}
&\|\widetilde{w}(t_0, v) - w_m(t_m, v)\|_{W^{1,\infty}} \\
&\qquad \le \|\widetilde{w}(t_0, v) - w_m(t_0, v)\|_{W^{1,\infty}} + \|w_m(t_0, v) - w_m(t_m, v)\|_{W^{1,\infty}}\,,
\end{aligned}$$

so that (i) allows us to focus just on the second term. Equation (4.3) yields

$$|z_m(t, v) - z_m(t^*, v)| = \left| \int_{t^*}^t L(\Pi(l, \omega_m, x_m)) w(l, v)\, dl \right|$$

for all t and t^* in $[r, \widetilde{t}\,]$ and $m \in \mathbb{N}$. Let us define $\mathcal{S}$ by (3.4), note that $\mathcal{S} \subset \mathcal{C}_0$, and use Proposition 4.1(ii) and the compactness of $[0, \widetilde{t}\,] \times \mathcal{S}$ to ensure: first, that

$$k^* := \sup\{\|L(\Pi(t, \omega_m, x_m))\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)} \mid t \in [r, \widetilde{t}\,] \ \text{ and } \ m \in \mathbb{N}\}$$

is finite; and second, that there exists $m_0 \in \mathbb{N}$ such that

$$\|L(\Pi(t_0 + s, \omega_m, x_m)) - L(\Pi(t_m + s, \omega_m, x_m))\|_{\mathrm{Lin}(W^{1,\infty},\mathbb{R}^n)} \le \varepsilon$$

for all $m \geq m_0$ and $s \in [-r, 0]$. According to Theorem 4.2(viii), there exists $M > 0$ such that $\|w_m(t,v)\|_{W^{1,\infty}} \leq M \|v\|_{W^{1,\infty}}$ for $t \in [0, \widetilde{t}\,]$. For next purposes, we assume without restriction that $k^* M > 1$. Therefore,

$$|z_m(t,v) - z_m(t^*,v)| \leq k^* M \, |t - t^*| \, \|v\|_{W^{1,\infty}}$$

for all t and t^* in $[0, \widetilde{t}\,]$. On the other hand, if t and t^* in $[-r, 0]$, then

$$|z_m(t,v) - z_m(t^*,v)| = |v(t) - v(t^*)| \leq |t - t^*| \, \|\dot{v}\|_{L^\infty} \leq k^* M \, |t - t^*| \, \|v\|_{W^{1,\infty}} \, ,$$

and, if $-r \leq t \leq 0 \leq t^* \leq \widetilde{t}$,

$$\begin{aligned} |z_m(t,v) - z_m(t^*,v)| &\leq |z_m(t,v) - z_m(0,v)| + |z_m(0,v) - z_m(t^*,v)| \\ &\leq -t\, k^* M \, \|v\|_{W^{1,\infty}} + t^* k^* M \, \|v\|_{W^{1,\infty}} = |t^* - t| \, k^* M \, \|v\|_{W^{1,\infty}} \, . \end{aligned}$$

Consequently, if t and t^* belong to $[0, \widetilde{t}\,]$ and $m \in \mathbb{N}$,

$$\|w_m(t,v) - w_m(t^*,v)\|_C \leq k^* M \, |t - t^*| \, \|v\|_{W^{1,\infty}} \, .$$

Let us take $m_1 \geq m_0$ such that $|t_m - t_0| \leq \varepsilon$ if $m \geq m_1$, and recall that $t_0 \geq r$ and $t_m \geq r$. Then, if $m \geq m_1$ and $s \in [-r, 0]$,

$$\begin{aligned} &|\dot{z}_m(t_0 + s, v) - \dot{z}_m(t_m + s, v)| \\ &\leq \|L(\Pi(t_0 + s, \omega_m, x_m)) - L(\Pi(t_m + s, \omega_m, x_m))\|_{\mathrm{Lin}(W^{1,\infty}, \mathbb{R}^n)} \|w_m(t_0 + s, v)\|_{W^{1,\infty}} \\ &\quad + \|L(\Pi(t_m + s, \omega_m, x_m))\|_{\mathrm{Lin}(C, \mathbb{R}^n)} \|w_m(t_0 + s, v) - w_m(t_m + s, v)\|_C \\ &\leq \varepsilon \, M \, \|v\|_{W^{1,\infty}} + \varepsilon \, (k^*)^2 \, M \, \|v\|_{W^{1,\infty}} = \varepsilon \, M^* \, \|v\|_{W^{1,\infty}} , \end{aligned}$$

where $M^* = (1 + (k^*)^2) \, M$. Therefore, if $m \geq m_1$, we have

$$\|w_m(t_0, v) - w_m(t_m, v)\|_{W^{1,\infty}} \leq \varepsilon \, (k^* M + M^*) \, \|v\|_{W^{1,\infty}} \, .$$

Since k^*, M and M^* can be defined from the beginning, (ii) is proved. □

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