Unbounded growth in the Neoclassical growth model with non-constant discounting

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Abstract

For a Neoclassical growth model, the literature highlights that exponential discounting is observationally equivalent to quasi-hyperbolic discounting, if the instantaneous discount rate decreases asymptotically towards a positive value. Conversely, in this paper a zero long-run value allows a solution without stagnation. We prove that a less than exponential but unbounded growth can be attained, even without technological progress. The growth rate of consumption decreases asymptotically towards zero, although so slowly that consumption grows unboundedly. The asymptotic convergence towards a non-hyperbolic steady-state which saving rate matches the intertemporal elasticity of substitution and the speed of convergence towards this equilibrium are analyzed.

JEL Classification: O40, C61, C62, D90.

Keywords Non-constant discounting; less than exponential unbounded growth; non-hyperbolic equilibrium; center manifold.

1. Introduction

The way individuals discount the future plays an important role in intertemporal decision making and, in consequence, it may have an impact on economic growth. This paper aims to prove that non-exponential discounting facilitates the appearance of patterns of growth distinct from the standards in a Neoclassical growth model without technological change.

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More specifically, this paper shows that the hypothesis of consumers discounting the future at non-constant rates might help to avoid stagnation even if technological progress is nil. In particular, this hypothesis makes a new pattern of growth feasible somewhere in between stagnation (the standard outcome in Neoclassical growth models) and exponential growth (obtained in endogenous growth models).

This less than exponential pattern of growth in a Neoclassical growth model has been analyzed in the literature through the concept of quasi-arithmetic growth. This type of growth is characterized by a rate of decline of the growth rate proportional to itself. Under this regularity condition economic growth is unbounded although the growth rate of the economy decreases asymptotically towards zero. Quasi-arithmetic growth was first introduced by Mitra (1983) and later developed by Asheim et al. (2007), Pezzey (2004) and Groth et al. (2010). The existence of this type of growth relies on the assumption of a quasi-arithmetic technical progress in the long-run.

In our paper the center of gravity moves from production or technology to time preferences. We discard any source of long-run growth such as technology or population growth and focus on human impatience. We prove that, if consumers discount the future at a non-constant rate, less than exponential growth in a Neoclassical growth model is feasible even if technical progress is nil. Moving away from the standard assumption of a constant rate of time preference we adopt the empirically supported hypothesis of non-constant discounting with decreasing impatience. Specifically, we assume an instantaneous time discount rate which decreases towards zero with the time-distance from the present.

There exists a wide consensus on the appropriateness of time-varying discount rates.[4] According to Laibson (1997), individuals are highly impatient about consumption in the near future, but much more patient when confronted with choices in the distant future. However, time-varying discount rates have not been widely used in economics because they imply an important drawback: the time-inconsistency problem. With non-constant discounting, preferences change with time and committed choices typically differ from those chosen sequentially (this had already been pointed out by Ramsey 1928 and later by Strotz 1956, Pollak 1968 and Goldman 1980).[5] One way to solve this problem is to consider a so-

[4]A very interesting discussion on time discounting can be found in Frederick et al. (2002). The opinion that a non-constant discounting better fits reality is not unanimous in the literature, see, for example, Andersen et al. (2014) for a recent criticism.

[5]The empirical analysis of this time inconsistency and its consequences on welfare are analyzed in Fang

phisticated agent that, being aware that he cannot precommit his future behavior, adopts a strategy of optimal planning against its future self, a game among a succession of future planners. Using this approach, Karp (2007) proposed a method for analyzing the solutions in a non-constant discounting infinite time horizon setting, and Marín-Solano and Navas (2009) extended the results to a non-constant discounting finite horizon problem.

In his seminal paper, Barro (1999) introduces a varying rate of time preference in a Neoclassical growth model with sophisticated agents to avoid time-inconsistency.[6] The instantaneous time discount rate decreases with the time-distance from the present, and it is assumed that it converges towards a positive constant. The consequence of this assumption of quasi-hyperbolic discounting is that, the model is observationally equivalent[7] to the standard Neoclassical growth model with a positive constant rate of time preference. In consequence, stagnation is the forced outcome in the long-term if exogenous technological improvements are absent. In this paper, we also consider a sophisticated agent but, unlike Barro (1999), we assume that the instantaneous time discount rate asymptotically declines towards zero. Our assumption means that a present individual does not discount a delay in the very distant future. This hypothesis is already present in the hyperbolic discount functions proposed in the classical psychological literature (Ainslie, 1992). Moreover this property is characteristic of many discount functions proposed in the economic literature on non-constant discounting, like, for example, hyperbolic discounting, proportional discounting, power discounting and constant sensitivity (see, Abdellaoui et al. 2010). In a continuous setting, a time-varying discount rate asymptotically converging to zero also appears in the logarithmic discounting proposed by Heal (2007) (and considered by Heal 1998 and Li and Lofgren 2000 to analyze sustainable resource management), and in the regular discounting used by Pezzey (2004) and Farzin and Wendner (2014). Moreover, we assume that the discount function never vanishes, meaning that individuals are not shortsighted in the sense that they do not disregard long-term consequences. Under these assumptions, our model predicts similar results to those of the Neoclassical growth model with a zero time-discount rate.

and Silverman (2009), Fang and Wang (2015).

[6] For endogenous growth models, the effects of a time-varying discount rate on long-run growth is analyzed in Strulik (2014) and Cabo et al. (2015).

[7] The observational equivalence is proved for a log utility. For more general utility functions, when the discount rate depends on calendar time, the result does not generally hold. See, for example, Farzin and Wendner (2014).

Our first aim is, therefore, to analyze the benchmark Neoclassical growth model without technological change and with a zero time-discount rate. The Neoclassical hypothesis of decreasing returns to capital implies an ever decreasing rate of return on capital and, therefore, a continuously decreasing growth rate of consumption. However, if population is constant and the capital depreciation rate is null (or a net-of-capital-depreciation production function is assumed), a regular or quasi-arithmetic growth is obtained.[8] Under this regularity condition it can be analytically proved that the growth rate of the economy decays towards zero, but this decay is so slow that consumption grows asymptotically unbounded.

The general model we propose in this paper, with a non-constant discount rate converging asymptotically towards zero, avoids stagnation under conditions equivalent to those in the benchmark model. The economy grows at a decreasing rate converging towards zero, but again consumption grows asymptotically unbounded. We find that the neoclassical economy, when consumers show non-constant time discounting, can be described by a four autonomous differential-equation system. This system presents non-hyperbolic asymptotic equilibria for which the standard techniques to study non-linear dynamical systems are not applicable. The trajectories converging towards the equilibria lie on a center manifold. By analyzing the dynamics within this manifold we manage to prove that, even though the pattern of growth displayed by our general model is not quasi-arithmetic (the regularity condition is not satisfied), it shares the two main properties of this type of growth. First, from a given time on, the growth rate remains positive while decreasing asymptotically towards zero (less than exponential growth); and second, the decay in the growth rate is so slow as to ensure an unlimited amount of consumption as time goes to infinity (unbounded growth). Thus, we call this new pattern of growth less than exponential unbounded growth (LEUG).

The benchmark Neoclassical growth model without discounting is studied in Section 2. In Section 3 we analyze the general model with non-constant discounting. Both problems are solved taking into account the catching-up optimality criterium. Finally, Section 4 concludes. All proofs are given in the Appendix.

[8] A similar result has been recently obtained by Bazhanov (2013) considering the constant-utility criterion with social progress.

2. Neoclassical growth model with a zero time-discount rate

As commented on the introduction, in order to have a better insight of the pattern of growth under non-constant discounting with an instantaneous time discount rate converging towards zero, we start by analyzing the Neoclassical growth model without technological change in the extreme case of a zero time-discount rate.

As is well known (see, for example, Barro and Sala-i-Martin 2004 or Acemoglu 2009), the Neoclassical growth model with a Cobb-Douglas production function and a isoelastic utility function is driven by two equations:[9]

$$\dot{k} = f(k) - c - (n+\delta)k = Ak^{\alpha} - c - (n+\delta)k, \quad k(0) = k_0, \tag{1}$$

$$\gamma_c \equiv \frac{\dot{c}}{c} = \frac{1}{\sigma}\left[f'(k) - (\delta+\rho)\right] = \frac{1}{\sigma}\left[\alpha A k^{\alpha-1} - (\delta+\rho)\right], \tag{2}$$

with $k(t)$ capital per capita, $c(t)$ consumption per capita, and constant $A > 0$ the total factor productivity. Constants $n \geq 0$ the growth rate of population, $\delta \geq 0$ the depreciation rate of capital, $\alpha \in (0,1)$ the output elasticity of capital, $k_0 \geq 0$ the initial capital per capita, $\sigma > 0$ the inverse of the intertemporal elasticity of substitution, and $\rho \geq 0$ the instantaneous time-discount rate.

In this section we center our analysis on the extreme case with a zero time-discount rate, $\rho = 0$. Since the objective function could be unbounded, we use the catching-up optimality criterium (see, for example, Sydster and Seierstad 1987). First-order conditions for optimality are given by equations (1) and (2), together with the transversality condition:

$$\lim_{t\to\infty} \inf \mu(t)[\tilde{k}(t) - k(t)] \geq 0,$$

for all admissible $\tilde{k}(t)$, where $\mu(t) = c(t)^{-\sigma}$ is the shadow price of the capital per capita for the representative consumer.

With a zero discount rate, the well-known phase diagram of the Neoclassical growth model (left graph in Figure 1) is still valid. The $\dot{k} = 0$ locus is described by the curve $c = Ak^{\alpha} - (n+\delta)k$, and the $\dot{c} = 0$ locus by the curves $k = k^* \equiv (A\alpha/\delta)^{1/(1-\alpha)}$ and $c = 0$. This diagram depicts a unique steady-state saddle point with positive consumption. If the capital per capita is initially low, the transitional dynamics is characterized by growing

[9] Time argument is omitted henceforth when no confusion can arise. γ_c denotes the growth rate of consumption per capita.

consumption and capital per capita, converging towards their steady-state values. In consequence, in the absence of technological progress, the assumption of a zero discount rate in the standard model with diminishing returns in capital per capita does not modify the standard result: the growth in capital per worker must necessarily come to a halt; it cannot grow unboundedly.

Nevertheless, stagnation is no longer the obliged outcome under the assumptions of constant population, $n = 0$, and a net-of-capital production function, $\delta = 0$. In the phase diagram in Figure 1 (right), the $\dot{k} = 0$ locus is now concave, although ever-increasing, function $c = Ak^{\alpha}$. Meanwhile, the $\dot{c} = 0$ locus is exclusively defined by the horizontal-axis $c = 0$, which cannot be reached if $c_0 > 0$. Thus, given the initial capital stock, if consumption does not exhaust all output, capital starts growing and it might continue growing as long as the economy remains below the $\dot{k} = 0$ locus. Correspondingly, consumption would grow at a strictly positive rate, proportional to the marginal productivity of capital per capita. Since technology exhibits diminishing returns to scale, increments in capital would induce a continuous reduction in its marginal productivity and hence on the growth rate of consumption.

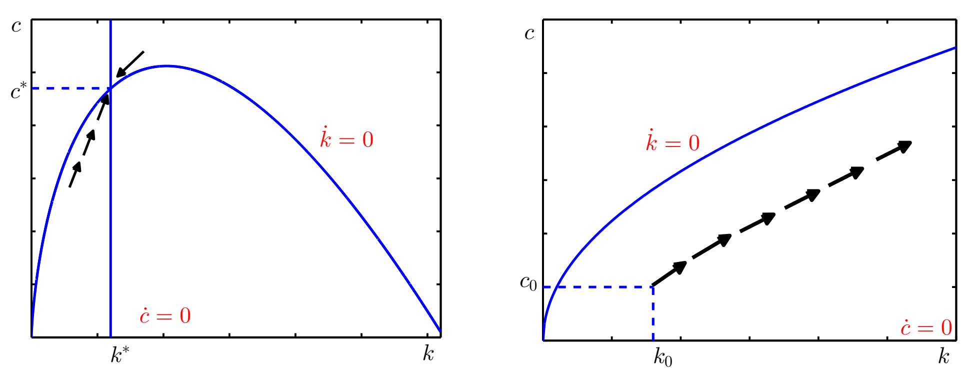

Figure 1: Phase diagrams $k - c$; $\rho = 0,\ n, \delta > 0$ (left); $\rho = n = \delta = 0$ (right)

To analyze the feasibility of this type of solution, we first write the system dynamics (1)-(2) in the particular case $\rho = \delta = n = 0$:

$$\dot{k} = f(k) - c = Ak^{\alpha} - c, \quad k(0) = k_0, \tag{3}$$

$$\gamma_c = \frac{f'(k)}{\sigma} = \frac{\alpha A}{\sigma} k^{\alpha - 1}. \tag{4}$$

Defining s as the share of savings, $s \in [0,1]$, the consumption can be written as

$$c = (1-s)f(k) = (1-s)Ak^{\alpha}, \tag{5}$$

and defining

$$r = f'(k) = \alpha A k^{\alpha-1} \tag{6}$$

as the rate of return of capital, then system (3)-(4) can be rewritten as:

$$\dot{r} = -\frac{1-\alpha}{\alpha} s r^2, \qquad r(0) = r_0 = \alpha A k_0^{\alpha-1}, \tag{7}$$

$$\dot{s} = -(1-s)\left(\frac{1}{\sigma} - s\right) r. \tag{8}$$

From equation (8) the $\dot{s} = 0$ locus is defined by the vertical-axis $r = 0$, and the horizontal straight lines $s = 1$ and $s = 1/\sigma$. The $\dot{r} = 0$ locus is defined by the horizontal ($s = 0$) and the vertical ($r = 0$) axes.

Under the assumption $\sigma > 1$ we study the phase diagram in variables $r - s$ in Figure 2 which is equivalent to the phase diagram in variables $k - c$ in system (3)-(4). There exist infinitely many possible equilibria located in the unit interval between the origin and point $\mathbb{E} = (0,1)$. By computing the integral curves it can be shown that point $\mathbb{E}$ is a stable equilibrium for any initial situation with $s_0 \in (1/\sigma, 1)$ and $r_0 > 0$. The equilibria $(0, s^*)$ with $s^* \in [0,1), s^* \neq 1/\sigma$ are unstable. The equilibrium $\mathbb{E}^{\sigma} = (0, 1/\sigma)$ is saddle-path stable and it will only be reached if the saving rate lies initially at $s_0 = 1/\sigma$. Alternatively the same analysis for $\sigma < 1$ shows that there is no stable equilibrium satisfying $s \in [0,1]$. In consequence, here and henceforth, we make the frequent assumption in the literature, of

$$\sigma > 1.$$

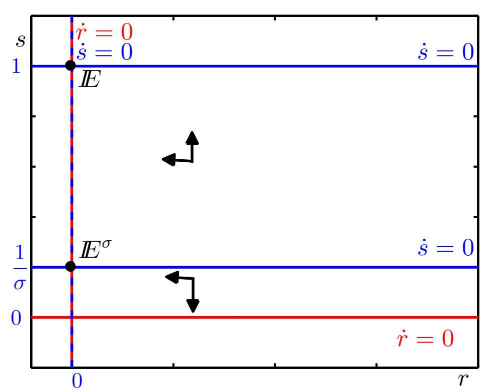

Figure 2: Phase diagram $r - s$ ($\rho = \delta = n = 0$)

If the saving rate initially matches the intertemporal elasticity of substitution, $s(0) = s_0 = 1/\sigma$, it remains constant and equation (3) can be written as the fundamental dynamic equation in the Solow-Swan model, $\dot{k} = Ak^{1-(1-\alpha)}/\sigma$. This is an autonomous Bernoulli equation whose solution reads

$$k(t) = k_0 \left(1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right)^{\frac{1}{1-\alpha}}, \qquad \text{with } \gamma_0 = \frac{\alpha A}{\sigma} k_0^{\alpha-1}. \tag{9}$$

Note that γ_0 is the growth rate of consumption at $t = 0$, then from equations (4) and (5) immediately follows:

$$\gamma_c(t) = \frac{\gamma_0}{1 + \gamma_0 \frac{1-\alpha}{\alpha} t}, \tag{10}$$

$$c(t) = c_0 \left(1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right)^{\frac{\alpha}{1-\alpha}}, \qquad \text{with } c_0 = \frac{\sigma - 1}{\sigma} A k_0^{\alpha}. \tag{11}$$

Note that c_0 is the consumption at $t = 0$ when $s_0 = 1/\sigma$.

The growth rate of consumption in equation (10) fulfils the regularity condition for a quasi-arithmetic growth path $\dot{\gamma}_c = -(1-\alpha)/\alpha\gamma_c^2$ (for example, in Groth et al. (2010)). This pattern of growth satisfies that the decay in the growth rate is so slow that both capital and consumption grow unboundedly as time goes to infinity. Trajectories for consumption and capital would collapse in the exponential case as α tends to 1.

From (9) and (11) it is easy to verify that the condition $\lim_{t\to\infty} \mu(t)k(t) = 0$, sufficient to guarantee the transversality condition, is satisfied under condition $1/\sigma < \alpha$, which states that the output elasticity of capital surpasses the intertemporal elasticity of substitution. If this condition is not fulfilled, the output grows too slowly with the capital stock, or the marginal utility decays too slowly with consumption. In either case the capital stock as time goes to infinity would be positively valued. This could mean that the economy should have accumulated more capital.

These results are summarized in the following proposition.

Proposition 1. *If $s_0 = 1/\sigma$, then $s(t)$ remains constant, $\lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty} \gamma_c(t) = 0$ and k and c grow following a regular pattern with $\lim\limits_{t\to\infty} k(t) = \lim\limits_{t\to\infty} c(t) = \infty$.*

If the saving rate is initially greater than the intertemporal elasticity of substitution, it will grow towards 1, while consumption and capital per capita still grow unboundedly. However, this solution would be suboptimal, as it is proved in the following proposition:

Proposition 2. *If $s_0 > 1/\sigma$, then $\lim_{t\to\infty} s(t) = 1$, and $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \gamma_c(t) = 0$. Moreover, $r(t)$ and $\gamma_c(t)$ converge towards zero as function t^{-1}, hence $\lim_{t\to\infty} k(t) = \lim_{t\to\infty} c(t) = \infty$. However these trajectories are suboptimal.*

From this proposition follows that the solution given in (11) with $s(t) = 1/\sigma$ for all $t \geq 0$ is the unique solution to the Neoclassical growth without technological change and model with a zero time-discount rate; and this solution displays a regular growth path. The next section analyzes the more general model in which, while maintaining the assumptions of constant population and no capital depreciation, we give entrance to impatience in the consumers' preferences. We consider individuals whose discount rate decreases as the time distance from the present widens. The question would be whether a similar pattern of unlimited growth could emerge. In particular, we study the conditions under which the growth rate of consumption, although tending towards zero, allows consumption to grow unboundedly as time goes to infinity. The growth rate of consumption remains positive, at least from a given time on:

$$\gamma_c(t) > 0,\ \forall t \geq \bar{t} \geq 0, \quad \lim_{t\to\infty} \gamma_c(t) = 0, \quad \lim_{t\to\infty} c(t) = +\infty. \tag{12}$$

We denote this pattern of growth as less than exponential unbounded growth (LEUG). Regular growth paths satisfy this definition, which is more general in the sense that no regularity condition is imposed.

3. The Neoclassical growth model with non-constant discounting

Economic growth theory has relied on the assumption that households have a constant rate of time preference, ρ. However, the rationale for this assumption is unclear and experimental studies of human behavior suggest that this assumption is unrealistic (see, for example, the excellent review by Frederick et al. (2002)). To asses this issue, we define the consumer's objective function taking into account non-constant time preferences:

$$U(t) = \int_t^\infty u\,[c(h)]\ \theta(h-t)\,dh,$$

where $u(c)$ represents the isoelastic utility function with an intertemporal elasticity of substitution, $1/\sigma$, t is the current date, h measures the time distance from the present and $\theta(j) \geq 0$ is the discount function which measures the time preference. By definition, the discount function, $\theta(j)$ is a decreasing function, initially equal to one. Defining the instantaneous discount rate as $\rho(j) = -\dot{\theta}(j)/\theta(j) > 0$, the literature on hyperbolic discounting considers that $\rho(j)$ decreases through time, that is

$$\theta(j) \geq 0, \quad \dot{\theta}(j) < 0, \quad \theta(0) = 1, \quad \dot{\rho}(j) < 0, \quad \forall j \geq 0. \tag{13}$$

Moreover, we assume that the instantaneous discount rate $\rho(j)$ also satisfies

$$\lim_{j\to\infty} \rho(j) = 0. \tag{14}$$

The seminal paper on economic growth and non-constant discounting by Barro (1990) considered that the instantaneous discount rate converges towards a positive value, $\lim_{j\to\infty} \rho(j) = \bar{\rho} > 0$. This is the assumption supported by the literature on quasi-hyperbolic discounting (it is made, for example, in Karp and Tsur 2011). In contrast, we follow the other strand of the literature, already commented on in the introduction, which considers a vanishing instantaneous discount rate in the limit. Thus, we assume that delays are not discounted in the very long run.

Assuming a sophisticated agent to avoid the time-inconsistency problem, at each date t, the t-agent solves the following problem:

$$\max_{c_t(h)} \int_t^\infty u\left[c_t(h)\right] \theta(h-t)\, dh \tag{15}$$

$$\text{s.t.:} \qquad \dot{k}_t(h) = f(k_t(h)) - c_t(h), \quad k_t(t) = \bar{k}_t. \tag{16}$$

As proved in Barro (1999), in the absence of any commitment, the usual Ramsey rule for the growth rate of consumption is modified to

$$\gamma_c(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(r(t) - \lambda(t)\right), \tag{17}$$

where $r(t) = f'(k(t)) = \alpha A k(t)^{\alpha-1}$ in the case of a Cobb-Douglas production function, and $\lambda(t)$, the effective rate of time preference, is given by

$$\begin{gathered} \lambda(t) = B(t) \int_0^\infty \rho(j) w(t,j) dj, \\ B(t) = \left[\int_0^\infty w(t,j)dj\right]^{-1} > 0, \qquad w(t,j) = \theta(j) e^{(1-\sigma)\int_t^{t+j} \gamma_c(\tau)d\tau} > 0. \end{gathered} \tag{18}$$

The function $\lambda(t) > 0$ can be interpreted as a weighted mean of the instantaneous discount rates, $\rho(j)$, with weights given by $w(t,j)$.

Due to diminishing returns to capital, the rate of return r will decline as the capital stock k grows, with $\lim_{k\to\infty} r(k) = 0$. However, when λ is given by (18), it might decrease as well, so the growth rates for consumption and capital may be kept in positive figures.

Furthermore, we will show that these growth rates, although decreasing, are sufficient to guarantee unbounded consumption as time goes to infinity.

With non-constant discounting, the system dynamics is given by

$$\begin{aligned} \dot{k} &= Ak^{\alpha} - c, \quad k(0) = k_0, \\ \gamma_c &= \frac{1}{\sigma}(\alpha A k^{\alpha-1} - \lambda), \end{aligned}$$

Proceeding as in Section 2, this system dynamics can be rewritten in terms of s and r:

$$\dot{r} = -\frac{1-\alpha}{\alpha} s r^2, \quad r(0) = r_0, \tag{19}$$

$$\dot{s} = -(1-s)\left[\left(\frac{1}{\sigma} - s\right) r - \frac{\lambda}{\sigma}\right]. \tag{20}$$

This system depends on λ as defined in (18). As it is proved in Lemma 2, in the Appendix, the temporal evolution of λ and B are given by the system:

$$\dot{\lambda} = -\rho_0 B - \lambda(\lambda - B) - B\int_0^{\infty}\left[\dot{\rho}(j) - \rho^2(j)\right] w(t,j)dj, \tag{21}$$

$$\dot{B} = \frac{1-\sigma}{\sigma}(r-\lambda)B + B(B-\lambda), \tag{22}$$

where $\rho(0) = \rho_0$. Note that $\rho(j)$ is usually given as an ad-hoc function and different functional forms would give different patterns of growth for λ, and hence, for consumption. To simplify the analysis of the system dynamics we consider a discount function satisfying the following condition:

$$\frac{\dot{\rho}(j)}{\rho(j)} = \rho(j) - \phi, \quad \phi > \rho(0) = \rho_0 > 0. \tag{23}$$

Under this assumption, since $B\int_0^{\infty}\left[\dot{\rho}(j) - \rho^2(j)\right] w(t,j)dj = -\phi\lambda$, the dynamics of the effective rate of time preference, λ, is analytically tractable. At the same time our desirable conditions in (13) and (14) are met. According to (23), the instantaneous discount rate decreases at an increasing rate (in absolute terms). This rate of decay, $-\dot{\rho}(j)/\rho(j)$, converges towards ϕ as the instantaneous discount rate, ρ, tends to zero. Thus, the decay in the discount rate is initially mild, stepping up as the time distance from the present widens. While we make this assumption for tractability, whether other specifications with constant or decreasing [10] decay rate might lead to similar results is a challenging research question.

[10] Farzin and Wendner (2014), consider that the rate of decay of the instantaneous discount rate is proportional to itself: $\dot{\rho}(j)/\rho(j) = \beta\rho(j)$, with $\beta < 0$. Whether the rate of decay speeds up or slows down is, to the best of our knowledge, still a matter for research.

On integrating equation (23), the discount function and the instantaneous discount rate read

$$\theta(j) = 1 - \frac{\rho_0}{\phi}\left(1 - e^{-\phi j}\right), \quad \rho(j) = \frac{\rho_0 \phi}{\rho_0 + (\phi - \rho_0)e^{\phi j}}. \tag{24}$$

Under the assumption $\rho_0 < \phi$, conditions in (13) and (14) immediately follow. Furthermore, $\lim_{j\to\infty} \theta(j) = 1 - \rho_0/\phi > 0$, meaning that consumers, although being impatient with the near future (in the sense of Laibson 1997), are not shortsighted in the sense of disregarding the consequences in the very long run.

For the family of discount functions in (24), the dynamics of λ given in the differential equation (21) has a simple expression. Thus, the temporal evolution of the Neoclassical growth model with non-constant discounting can be described by a system of four autonomous differential equations:

$$\begin{aligned}
\dot{r} &= -\frac{1-\alpha}{\alpha} s r^2, \quad r(0) = r_0, & (25)\\
\dot{s} &= -(1-s)\left[\left(\frac{1}{\sigma} - s\right) r - \frac{\lambda}{\sigma}\right], & (26)\\
\dot{B} &= \frac{1-\sigma}{\sigma}\left(r - \lambda\right) B + B\left(B - \lambda\right), & (27)\\
\dot{\lambda} &= -\rho_0 B - \lambda\left(\lambda - B\right) + \phi\lambda. & (28)
\end{aligned}$$

Moreover, for the family of discount functions given in (24), Lemma 3 in the Appendix proves that a LEUG path must fulfill the following final condition:

$$\lim_{t\to\infty} \frac{\lambda(t)}{B(t)} = \frac{\rho_0}{\phi}. \tag{29}$$

Since we are interested in patterns of growth avoiding stagnation, and particularly in LEUG paths, from now on we impose the final condition (29) to system (25)-(28).[11] Note that any point of the form $(r, s, B, \lambda) = (0, s^*, 0, 0)$ with $s^* \in [0, 1]$ is a steady state of the system (25)-(28).[12] Moreover, these steady states are not hyperbolic, contrary to the usual equilibria in economic growth models with constant temporal discount rates. The following proposition proves this result.

[11] Furthermore, trajectories $\lambda(t)$ and $B(t)$ arising from the system (25)-(28) would coincide with those of the system (19)-(20), when this condition is satisfied.

[12] Additionally, system (25)-(28) has other steady states:

$$(0, 1, 0, \phi), \quad \left(0, 1, \frac{\sigma\phi - \rho_0}{\sigma(\sigma - 1)}, \frac{\sigma\phi - \rho_0}{\sigma - 1}\right), \quad (\phi, 0, 0, \phi).$$

However, none of them satisfies the final condition (29). We are not interested in trajectories converging to them, because they will not be LEUG paths.

Proposition 3. *The steady states of the form $(0, s^*, 0, 0)$ are non-hyperbolic equilibria, and are characterized by a one-dimensional unstable manifold and a three-dimensional center manifold.*

Economic growth models are often characterized by saddle-path stability, which implies that economies will diverge from the equilibrium unless the initial conditions lie on the stable manifold. Similarly, for the non-hyperbolic equilibria of the form $(0, s^*, 0, 0)$, if the initial conditions do not lie on the center manifold, the optimal trajectories will diverge from the steady state. In contrast, those trajectories starting on the center manifold will never leave it. To understand the behaviour within this center manifold we compute a second order approximation, (see Lemma 4 in the Appendix):

$$\lambda = \frac{\rho_0}{\phi} B \left[1 - \frac{\sigma - 1}{\phi \sigma} \left(r - \frac{\rho_0}{\phi} B \right) \right]. \tag{30}$$

From equation (17), the growth rate of consumption reads

$$\gamma_c = \frac{1}{\sigma} \left(r - \frac{\rho_0}{\phi} B \right) \left(1 + \frac{\sigma - 1}{\phi \sigma} \frac{\rho_0}{\phi} B \right). \tag{31}$$

In consequence, trajectories starting on the center manifold must initially satisfy

$$\gamma_c(0) \equiv \frac{1}{\sigma} (r_0 - \lambda_0) = \frac{1}{\sigma} \left(r_0 - \frac{\rho_0}{\phi} B_0 \right) \left(1 + \frac{\sigma - 1}{\phi \sigma} \frac{\rho_0}{\phi} B_0 \right). \tag{32}$$

From (30) it easily follows that those trajectories on the center manifold converging to $(0, s^*, 0, 0)$ already satisfy the final condition (29).

Once the economy is located within the center manifold we still have to prove the convergence towards the equilibrium. The following subsection studies the dynamics in the center manifold.

3.1. Convergence on the center manifold

Once the steady-states have been identified, two questions arise: Are there initial conditions in the center manifold from which the trajectories asymptotically converge to the steady states? And if so, are these convergent trajectories LEUG paths? Since the steady-state equilibria are not hyperbolic, we analyze the dynamics in the center manifold.

On substituting λ for (30) in system (25)-(28), the flow on the center manifold can be described by

$$\dot{r} = -\frac{(1-\alpha) r^2}{\alpha} s, \quad r(0) = r_0, \tag{33}$$

$$\dot{s} = -(1-s) \left[\frac{1}{\sigma} \left(r - \frac{\rho_0}{\phi} B \right) \left(1 + \frac{\sigma - 1}{\sigma \phi} \frac{\rho_0}{\phi} B \right) - sr \right], \tag{34}$$

$$\dot{B} = B \left[\frac{1}{\sigma} \left(r - \frac{\rho_0}{\phi} B \right) \left(1 + \frac{\sigma - 1}{\sigma \phi} \frac{\rho_0}{\phi} B \right) - (r - B) \right], \quad B(0) = B_0. \tag{35}$$

The initial rate of return, r_0, is given and the initial condition B_0 is the unique positive value satisfying (32), which guarantees that variables lie on the center manifold. Equilibria points $(r, s, B) = (0, s^*, 0)$ are still steady states of system (33)-(35). Since the saving rate, s, is a control variable, convergence towards any of these points requires the existence of a stable manifold of, at least, dimension two. In consequence, we consider as unstable any equilibrium characterized by a one-dimensional stable manifold. The reason for this is that for a three-dimensional dynamical system the solution cannot be placed on the one-dimensional stable manifold when only the saving rate at the initial time can be chosen. Lying in this stable manifold is only a knife-edge possibility.

Proposition 4. *For $1/\sigma < \alpha$, the solution $(r(t), s(t), B(t))$ of the dynamical system (33)-(35), can asymptotically converge either towards $E^\sigma = (0, 1/\sigma, 0)$ or towards $E = (0, 1, 0)$.*

As stated in expression (12) a necessary characteristic of a LEUG path is a positive growth rate from a finite time on, which converges towards zero asymptotically. From equation (31) the asymptotic convergence is guaranteed for any equilibria in which the rate of return and the auxiliary variable, B, are null. Moreover, the next proposition proves that along any trajectory converging towards the steady-state equilibria in Proposition 4, the growth rate of consumption remains strictly positive from a specific moment onwards.

Proposition 5. *Any solution converging towards the steady-state equilibria $E^\sigma = (0, 1/\sigma, 0)$ or $E = (0, 1, 0)$ satisfies that from a finite time on, both the growth rate of consumption, γ_c, and the saving rate, s, are strictly positive.*

Likewise, as in the case with a zero time-discount rate, the first equilibrium with a steady-state saving rate $s^* = 1/\sigma$, can be approached asymptotically. However, conversely to the case with a zero time-discount rate, the initial value for the saving rate which guarantees convergence is not the constant, $1/\sigma$, but a lower value which depends on the initial value of the capital stock. Because this initial saving rate is unique, there is no indeterminacy. Again, replicating the case with a zero time-discount rate the equilibrium with full savings $s^* = 1$ will be proved suboptimal. All these results are shown in the proof of the following proposition.

Proposition 6. *Assuming $1/\sigma < \alpha$, given an initial value of r_0, there exists a unique value s_0, lower than $1/\sigma$, from which the solution asymptotically converges to $E^\sigma = (0, 1/\sigma, 0)$.*

In the following section we shall prove that the trajectory converging to this steady-state equilibrium satisfies the last requirement for a LEUG path in (12): the consumption grows unboundedly as time goes to infinity.

3.2. Speed of convergence

Having a positive consumption growth rate (from a given time on) does not guarantee an unbounded consumption when the growth rate tends to zero. Additionally, it is necessary that this growth rate decreases no faster than function t^{-1}. This subsection proves that the rate of return of the economy, as well as the growth rate of consumption, lies between two functions which converge towards zero as function t^{-1}.

Proposition 7. *For the trajectory converging towards $E^\sigma = (0, 1/\sigma, 0)$, the rate of return converges towards zero as time goes to infinity as function t^{-1} .*

It is now possible to prove that the growth rate of consumption is upper and lower bounded by two functions which converge to zero as time goes to infinity as function t^{-1}.

Proposition 8. *For the trajectory converging towards $E^\sigma = (0, 1/\sigma, 0)$, there exist a finite $\bar{t}$ and a lower bound for the saving rate, $s_{\min} \in (0, 1/\sigma)$, such that the growth rate of consumption along this trajectory satisfies*

$$\frac{1}{\sigma}\left(1 - \frac{\rho_0}{\phi}\right)\frac{\bar{r}}{1 + \bar{r}\frac{1}{\sigma}\frac{1-\alpha}{\alpha}(t - \bar{t})} < \gamma_c(t) < \frac{1}{\sigma}\frac{\bar{r}}{1 + \bar{r}s_{\min}\frac{1-\alpha}{\alpha}(t - \bar{t})}, \quad \forall t \geq \bar{t}, \tag{36}$$

where $\bar{r} = r(\bar{t})$.

The previous result, together with Proposition 6, allow us to answer the main research question of this paper positively.

Theorem 1. *If $1/\sigma < \alpha$, there exists a unique initial saving rate s_0 starting from which the economy asymptotically converges towards the steady state $E^\sigma = (0, 1/\sigma, 0)$, and the solution is a LEUG path.*

Note that the condition of an output elasticity of capital greater than the intertemporal elasticity of substitution $\alpha > 1/\sigma$, in Theorem 1 for a LEUG path, also guarantees the transversality condition. Recall that this condition is also valid to prove the fulfillment of the transversality condition in the Neoclassical economy with no temporal discount.

3.3. Other discount functions

3.3.1. Biparametric family of discount functions

The discount function satisfying condition (23) has been chosen in order to make the dynamical system (19)-(22) analytically tractable. This subsection checks whether a LEUG path is still feasible for more general discount functions. With this aim, we define a biparametric family of discount functions of the form:

$$\frac{\dot{\rho}(j)}{\rho(j)} = \beta\rho(j) - \phi, \quad \text{with} \quad (\phi > \beta\rho_0 > 0) \quad \text{or} \quad (\phi = 0 \text{ and } \beta < 0). \tag{37}$$

This biparametric specification encompasses our previous uniparametric function when $\beta = 1$. It also allows regular discounting[13] when $\phi = 0$ and $\beta < 0$. In any case it satisfies the main property of an instantaneous discount rate converging asymptotically towards zero.

There is no hope of having analytical results for these discount functions. Therefore, we resort to a numerical analysis. Necessarily, the time horizon in the numerical analysis must be finite, which makes the generalization to the infinite case not straightforward. The analysis has been carried out in Matlab.[14] The system of differential equations (19)-(22) has been discretized. The time discretization of equation (21) requires the discretization of the improper integral, which in turn requires the discretization of the integral in the definition of weights $w(t, j)$ in (18).

For the discount function in (23), the interesting non-hyperbolic equilibrium towards which the system evolves is $(0, 1/\sigma, 0, 0)$. Therefore, the discretized system is solved backwards (on reverse time) starting from a initial condition near this equilibrium. This initial condition satisfies also the second-order approximation of the center manifold given in (30).

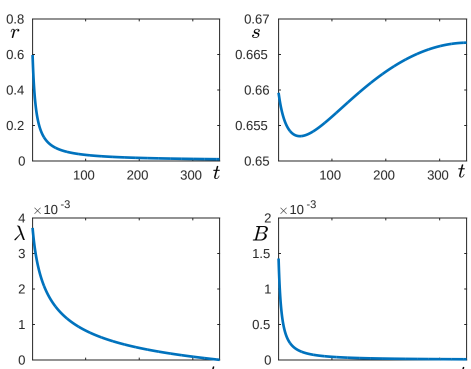

Figure 3: Optimal time paths.

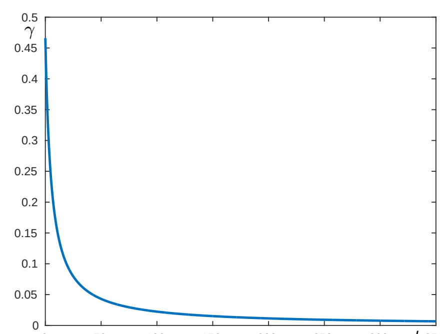

Figure 4: Growth rate of c.

For the particular case $\beta = 1$ (the discount function in (23)) the qualitative behavior of the time paths of the four variables (r, s, λ, B), and the growth rate of consumption is consistent with the behavior described by Propositions 4 to 8, as shown in Figures 3 and 4.

[13]Recently used in the literature as for example in Pezzey (2004) and Farzin and Wedner (2014).

[14]The code is available upon request.

The plots in Figures 3 and 4 are generated for $\beta = 1$ and $\phi = 0.2$, henceforth considered as a benchmark. Similarly we have carried out the numerical analysis for different values for β and ϕ. Firstly, we compute two scenarios in which the instantaneous discount decreases faster towards zero, either due to a higher ϕ or a lower β. Secondly, we analyze the regular discounting with a much softer decay in the instantaneous discount rate ($\beta < 0$ and $\phi = 0$). In all cases, we have maintained the same initial condition (values near the steady-state equilibrium) both for comparability and conjecturing that the steady state equilibrium is the same for all discount functions in the biparametric family.

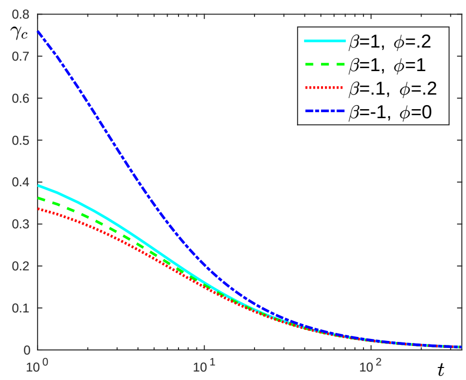

Figure 5: Growth rate time paths of consumption.

Trajectories of the main variables are qualitatively similar to those presented in Figure 3. Specifically, the growth rate time paths for all four cases are compared in Figure 5. The cyan solid curve represents the benchmark case, which is a LEUG as proved in Proposition 8. The green-dashed ($\beta = 1$ and $\phi = 0.1$) and the red-dotted ($\beta = 0.1$ and $\phi = 0.2$) curves run smoother and below the benchmark. Therefore, if the three paths had started from the same initial condition (considering the time forward), the softer the decay in the growth rate, the greater the economic growth. Thus, knowing that the benchmark is a LEUG, the two curves running below are also LEUG. This supports the intuition that the faster the decay in the instantaneous discount rate, the stronger the economic growth. By contrast, regular discounting ($\beta < 0$ and $\phi = 0$) represents a case with a much softer decay of instantaneous discount rate:

$$\rho(j) = \frac{\rho_0}{1 - \rho_0 \beta j}, \quad \beta < 0.$$

In particular we assume $\beta = -1$. The time path for the growth rate (blue dash-dotted curve) is much stepper than the benchmark and hence, assuming again identical initial growth rate in forward time, the economic growth would be weaker. Numerically, it cannot be proved that the growth rate decreases no faster than a function t^{-1}, and therefore we cannot ascertain stagnation in finite time, but certainly LEUG is less plausible for regular discounting.

3.3.2. Separable discount function

The assumption used as starting point for all discount functions satisfying (37) is that the discount depends exclusively on the time distance from the present. This assumption leads to problems of time-inconsistency, unless the decision maker behaves sophisticatedly, as considered up to now. An alternative approach, suggested by Drouhin (2009), is to assume a multiplicatively separable discount function of the form:

$$\theta(h,t) = \varphi(h)\eta(t).$$

An agent with this type of preferences is time consistent. Decisions taken by a current individual will remain optimal for individuals from successive cohorts. Therefore, there is no need to assume that he behaves sophisticatedly playing a game again its future self. Thus the optimal growth rate of consumption is independent of the date t when the t-agent optimally decides the consumption path. Thus the growth rate of the economy along time is given by the modified Ramsey rule:

$$\frac{\dot{c}(h)}{c(h)} = \frac{1}{\sigma}\left(r(h) - \lambda(h)\right) \text{ with } \lambda = -\frac{\dot{\varphi}(h)}{\varphi(h)}.$$

For the specific case $\varphi(h) = b/(1+bh)$, with $b \in (0,1)$, we have that $\lambda(h) = b/(1+bh)$ and $\dot{\lambda} = -\lambda^2$. Therefore, the system that describes the temporal evolution in the Neoclassical growth model would be given by (19), (20) and $\dot{\lambda} = -\lambda^2$. This system presents non-hyperbolic equilibria of the form $(0, s^*, 0)$ with a three-dimensional center manifold. Following a similar analysis as described in Section 3, it can be easily proved that the main variables behaves similarly and a less than exponential unbounded growth path exists.

4. Conclusions

Barro (1999) was the first author who introduced a time-varying discount rate in a Neoclassical growth model, taking into account sophisticated agents in order to avoid time-inconsistency problems. He considered quasi-hyperbolic preferences with an instantaneous time-discount rate decreasing towards a positive constant. The conclusion is that the model's predictions are similar to those of the standard model with exponential discounting and stagnation is the unavoidable result in the absence of technological progress. In this paper we have proved that a time-discount rate which decays in the long run towards zero enables a growth rate of consumption which although decreasing towards zero, is compatible with unbounded growth. This less than exponential pattern of growth has been already found in the literature if quasi-arithmetic technological progress is assumed, at least in the long run. Here, despite the fact that we discard technological progress and any other source of long-run growth, we attain a similar result. Our result is exclusively based on the assumption of non-constant discounting with an instantaneous discount rate tending towards zero with the time distance from the present, and farsighted consumers who are concerned on the consequences in the very long run.

To gain a better insight of the more general case, we started by analyzing a first scenario considering consumers with no time preference. We have shown that the hypotheses of unvarying population and a production function net-of-capital-depreciation do not necessarily lead to stagnation in a Neoclassical growth model. This result requires an intertemporal elasticity of substitution lower than the output elasticity of capital, implying individuals are not very willing to substitute consumption intertemporally. Under this assumption, a solution with a constant saving rate equal to the intertemporal elasticity is optimal and shows a regular pattern of growth. That is, the growth rate of the economy, although positive, decays at a rate proportional to itself, which guarantees that consumption grows unboundedly.

In a more realistic second scenario, we have assumed that consumers do discount the future, although they have a declining rate of time preference. In contrast to the assumption in Barro (1999) of a positive asymptotic discount rate, we have assumed an instantaneous discount rate approaching zero in the very long run. That is, consumers are impatient about immediate consumption, but their degree of impatience decreases with the time distance from the present. This assumption is already present in the economic literature on non-constant discounting and the classical psychological literature. Additionally, we also have considered that the agents are not shortsighted in the sense of ignoring long-term consequences. Under these assumptions, and the same conditions considered in the case with a zero time-discount rate in the first scenario, there exists a unique initial value lower than the intertemporal elasticity of substitution from which the saving rate converges asymptotically towards this value from below. Thus, our model with non-constant discounting displays a pattern of growth similar to the model with zero time-discount rate. Although the rate of return approaches zero at infinity (due to diminishing returns) the effective rate of time preference (a weighted mean of instantaneous discount rates) also decays to zero. In consequence, the growth rate of consumption as defined by the modified Ramsey rule can remain positive. We have proved that from a given time on, consumption grows at a positive growth rate which converges towards zero. The decay in the growth rate of consumption does not satisfy a regularity condition (like the regular pattern of growth). Nonetheless, we can prove that this decay is again slow enough to imply that the consumption does not stagnate, but it grows unboundedly as time goes to infinity. This pattern of growth is defined as less than exponential unbounded growth (LEUG).

We have proved that the asymptotic equilibrium characterized by a saving rate equal to the intertemporal elasticity of substitution is a non-hyperbolic equilibrium with a one-dimensional unstable manifold and a three-dimensional center manifold. For an economy initially in the center manifold we have shown that there exists a unique value of the saving rate starting from which the economy will eventually converge to the asymptotic equilibrium. From a given time on the growth rate of consumption is upper- and lower-bounded by two functions which converge towards zero as time goes to infinity as function t^{-1}, which means that consumption, although growing at a decreasing rate, is asymptotically unbounded.

Our results are obtained for a specific family of temporal discounting functions which allows a substantial simplification of the analysis. Whether different specifications might lead to similar results deserves future research efforts.

The results are crucially dependent on two assumptions: a zero population growth and no capital depreciation. Nevertheless, we believe that positive but decreasing rates of population growth and/or capital depreciation would, presumably, give rise to the same kind of results. A probably feasible specification could be to assume that these rates decrease no slower than the marginal productivity of the capital stock. This is a question for further research. An additional limitation of our analysis stems from the assumption of an instantaneous discount rate which converges towards zero but only asymptotically. This may give rise to robustness problems when replacing an infinite-horizon economy with individuals with a finite life span, as in overlapping generation models.

Appendix A.

Proof of Proposition 2

If $s_0 > 1/\sigma$, according to (7)-(8), $s(t)$ will increase towards 1, while $r(t)$ will diminish towards zero at a rate satisfying:

$$-\frac{1-\alpha}{\alpha}r < \frac{\dot{r}}{r} < -\frac{1-\alpha}{\alpha}\frac{1}{\sigma}r.$$

From the above inequalities, we have

$$\frac{r_0}{1+r_0\frac{1-\alpha}{\alpha}t} < r(t) < \frac{r_0}{1+r_0\frac{1-\alpha}{\alpha}\frac{1}{\sigma}t}.$$

Then, taking into account (6),

$$k_0\left(1+\gamma_0\frac{1-\alpha}{\alpha}t\right)^{\frac{1}{1-\alpha}} < k(t) < k_0\left(1+\sigma\gamma_0\frac{1-\alpha}{\alpha}t\right)^{\frac{1}{1-\alpha}}, \quad \text{with } \gamma_0 = \frac{\alpha A}{\sigma}k_0^{\alpha-1}.$$

As before, taking into account (4), the growth rate of consumption is upper and lower bounded by two functions which converge to zero as function t^{-1}:

$$\frac{\gamma_0}{1+\sigma\gamma_0\frac{1-\alpha}{\alpha}t} < \frac{\dot{c}(t)}{c(t)} < \frac{\gamma_0}{1+\gamma_0\frac{1-\alpha}{\alpha}t} \tag{A.1}$$

By integrating these expressions:

$$c(0)\left(1+\sigma\gamma_0\frac{1-\alpha}{\alpha}t\right)^{\frac{\alpha}{\sigma(1-\alpha)}} < c(t) < c(0)\left(1+\gamma_0\frac{1-\alpha}{\alpha}t\right)^{\frac{\alpha}{1-\alpha}}, \tag{A.2}$$

it follows that consumption per capita grows unboundedly at a positive growth rate which tends towards zero. Moreover, by comparing (11) with (A.2) it immediately becomes clear that consumption at any time is lower under the scenario with $s_0 > 1/\sigma$, and therefore, the solution satisfying this initial condition will be suboptimal according to the catching-up criterium.

Lemmas 2, 3 and 4 and Proof of Proposition 3

Lemma 2. *The time derivatives of* λ *and* B *are given by equations*

$$\begin{aligned}\dot{\lambda} &= -\rho_0 B - \lambda(\lambda - B) - B\int_0^\infty \left[\dot{\rho}(j) - \rho^2(j)\right] w(t,j)dj,\\ \dot{B} &= \frac{1-\sigma}{\sigma}(r-\lambda)B + B(B-\lambda),\end{aligned}$$

where $\rho(0) = \rho_0$.

Proof. Note that we can write

$$B(t) = \frac{\epsilon(t)}{\int_t^\infty \theta(h-t)\epsilon(h)dh} \qquad \text{with} \qquad \epsilon(t) = e^{(1-\sigma)\int_0^t \gamma_c(\tau)d\tau},$$

then

$$\dot{B} = \frac{(1-\sigma)\gamma_c(t)\epsilon(t)}{\int_t^\infty \theta(h-t)\epsilon(h)dh} - \frac{\epsilon^2(t)\left(-1+N(t)\right)}{\left(\int_t^\infty \theta(h-t)\epsilon(h)dh\right)^2} = (1-\sigma)\gamma_c B + B^2 - \lambda B, \tag{A.3}$$

where $N(t) = \int_0^\infty \rho(j)w(t,j)dj$. Moreover,

$$N(t) = \frac{\int_t^\infty \rho(h-t)\theta(h-t)\epsilon(h)dh}{\epsilon(t)},$$

and, then

$$\dot{N} = -\rho_0 - (1-\sigma)\gamma_c N - \frac{\int_t^\infty \left[\dot{\rho}(h-t) - \rho^2(h-t)\right]\theta(h-t)\epsilon(h)dh}{\epsilon(t)}.$$

Since $\lambda = NB$, the result follows. ■

Lemma 3. *If the discount function is of the form in (24), a LEUG path of system (19)-(20) satisfies*

$$\lim_{t\to\infty} \frac{\lambda(t)}{B(t)} = \frac{\rho_0}{\phi}.$$

Proof. Note that, by definition

$$\frac{\lambda(t)}{B(t)} = N(t) = \int_0^\infty -\dot{\theta}(j)e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau}dj.$$

Along a LEUG path $\int_t^{t+j}\gamma_c(\tau)d\tau > 0$ for all $t > \bar{t}$, then

$$\mid -\dot{\theta}(j)e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau} \mid \leq \mid -\dot{\theta}(j) \mid \quad \text{for all } t > \bar{t}.$$

Because function $\dot{\theta}(j)$ is Lebesgue integrable by the Lebesgue's dominated convergence theorem (see, for example, Apostol (1991)), it follows that

$$\lim_{t\to\infty} \int_0^\infty -\dot{\theta}(j)e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau}dj = \int_0^\infty \lim_{t\to\infty}\left[-\dot{\theta}(j)e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau}\right]dj.$$

From the mean value theorem there exists an intermediate $\omega \in [t, t+j]$ such that

$$\int_t^{t+j}\gamma_c(\tau)d\tau = \gamma_c(\omega)j.$$

Then, along a LEUG path

$$\lim_{t\to\infty}\int_t^{t+j}\gamma_c(\tau)d\tau = \lim_{\omega\to\infty}\gamma_c(\omega)j = 0.$$

As a consequence

$$\lim_{t\to\infty}\frac{\lambda(t)}{B(t)} = \int_0^\infty -\dot{\theta}(j)\,dj = \lim_{j\to\infty} -\theta(j) + \theta(0) = -1 + \frac{\rho_0}{\phi} + 1 = \frac{\rho_0}{\phi}.$$

■

Proof of Proposition 3

The Jacobian matrix of system (25)-(28) evaluated at the steady state $(0, s^*, 0, 0)$ presents a positive eigenvalue and a zero eigenvalue with algebraic multiplicity equal to 3.

Lemma 4. *A second-order approximation of the center manifold is given by*

$$\lambda = \frac{\rho_0}{\phi} B \left[1 - \frac{\sigma - 1}{\phi\sigma} \left(r - \frac{\rho_0}{\phi} B \right) \right].$$

Proof. For mathematical tractability, variable s is changed to $x = 1 - s$, so that the new system would have a fixed point at the origin $(0, 0, 0, 0)$. Note that $s \in [0, 1]$ implies $x \in [0, 1]$. The first two equations (25) and (26) in the dynamical system would now read

$$\dot{r} = -\frac{1-\alpha}{\alpha}(1-x)r^2, \tag{A.4}$$

$$\dot{x} = x \left[\left(\frac{\sigma - 1}{\sigma} - x \right) r + \frac{\lambda}{\sigma} \right]. \tag{A.5}$$

The Jacobian matrix of system (27), (28), (A.4) and (A.5) evaluated at the steady-state of type $(0, 0, 0, 0)$ presents a positive eigenvalue and a zero eigenvalue with algebraic multiplicity equal to 3.

By choosing the adequate matrix of eigenvectors we can reduce the Jacobian to its diagonal form, in new variables (u, r, x, z) given by

$$(u, r, x, z) = \left(\frac{\rho_0}{\phi} B, \quad r, \quad x, \quad \lambda - \frac{\rho_0}{\phi} B \right). \tag{A.6}$$

The Jacobian matrix for the new variables is

$$\left(\begin{array}{ccc|c} & & & 0 \\ & \mathrm{O}_{3x3} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \phi \end{array} \right).$$

Therefore, the system of equations can be written as

$$\begin{pmatrix} \dot{u} \\ \dot{r} \\ \dot{x} \end{pmatrix} = \mathrm{O}_{3x3} \begin{pmatrix} u \\ r \\ x \end{pmatrix} + \bar{g}(u, r, x, z), \tag{A.7}$$

$$\dot{z} = \phi z + g^z(u, r, x, z), \tag{A.8}$$

where the vectorial function $\bar{g} = (g^u, g^r, g^x)$ and the scalar function g^z collect the nonlinear terms of the dynamical system for variables u, r, x, z. The center manifold can be defined as the set:

$$W^c = \left\{(u, r, x, z) |\ z = f(u, r, x), \quad f(\bar{0}) = 0,\ \nabla f(\bar{0}) = \bar{0}\right\}.$$

As usual, this center manifold cannot be analytically characterized. However, it can be approximated by substituting a Taylor expansion of this function, $\hat{f}(u, r, x)$, in the partial differential equation:

$$\nabla \hat{f}(u, r, x) \left[\mathrm{O}_{3x3} \begin{pmatrix} u \\ r \\ x \end{pmatrix} + \bar{g}(u, r, x, \hat{f}(u, r, x)) \right] = \phi \hat{f}(u, r, x) + g^z(u, r, x, \hat{f}(u, r, x)),$$

and by identifying coefficients. The center manifold for a second-order approximation[15] is given by

$$z = \hat{f}(u, r, x) = \frac{\sigma - 1}{\sigma \phi} u(u - r). \tag{A.9}$$

This characterization can be rewritten in terms of the original variables as

$$\lambda - \frac{\rho_0}{\phi} B = \frac{\sigma - 1}{\sigma} \frac{\rho_0}{\phi} B \left(\frac{\rho_0}{\phi} B - r \right).$$

■

Proof of Proposition 4, 5 and 6

Proof of Proposition 4

The equilibria in variables (r, s, B) are non hyperbolic. Hence, to transform system (33)-(35) into a new system with hyperbolic equilibria, we first make the change of variable $y = B/r$, obtaining the following dynamical system in variables y, r, B:

$$\begin{aligned}
\dot{y} &= B \left[\frac{1}{\sigma} \left(1 - \frac{\rho_0}{\phi} y \right) \left(1 + \frac{\sigma - 1}{\sigma \phi} \frac{\rho_0}{\phi} B \right) - (1 - y) \right] + \frac{1 - \alpha}{\alpha} s B, \\
\dot{s} &= -(1 - s) \frac{B}{y} \left[\frac{1}{\sigma} \left(1 - \frac{\rho_0}{\phi} y \right) \left(1 + \frac{\sigma - 1}{\sigma \phi} \frac{\rho_0}{\phi} B \right) - s \right], \\
\dot{B} &= \frac{B^2}{y} \left[\frac{1}{\sigma} \left(1 - \frac{\rho_0}{\phi} y \right) \left(1 + \frac{\sigma - 1}{\sigma \phi} \frac{\rho_0}{\phi} B \right) - (1 - y) \right].
\end{aligned}$$

[15] An approximation of a higher order can be easily computed. We do not present it here for the clarity of exposition.

Secondly, we change the time scale by defining a new variable $\tau = -\ln(r/r_0)$. From (33), $r(t)$ is a decreasing function[16] and hence, $\tau = 0$ for $r = r_0$, and $\tau \to \infty$ as $r \to 0$. Moreover;

$$\frac{d\tau}{dt} = \frac{1-\alpha}{\alpha}\frac{B}{y}s.$$

Using the time-elimination method,

$$\frac{dy}{d\tau} = \frac{dy/dt}{d\tau/dt} = \frac{\alpha}{1-\alpha}\frac{y}{s}\left[\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}y\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-(1-y)\right]+y, \tag{A.10}$$

$$\frac{ds}{d\tau} = \frac{ds/dt}{d\tau/dt} = -\frac{\alpha}{1-\alpha}\frac{1-s}{s}\left[\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}y\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-s\right], \tag{A.11}$$

$$\frac{dB}{d\tau} = \frac{dB/dt}{d\tau/dt} = \frac{\alpha}{1-\alpha}\frac{B}{s}\left[\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}y\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-(1-y)\right]. \tag{A.12}$$

The hyperbolic steady states for this system are $(0, 1/\sigma, 0)$, $(0, 1, 0)$, $(y_3^*, 1, 0)$, $(y_4^*, s_4^*, 0)$, where

$$y_3^* = \frac{\phi\,(\alpha(\sigma-1)-(1-\alpha)\sigma)}{(\phi\sigma-\rho_0)\alpha}, \quad y_4^* = \frac{\phi\,(\sigma\alpha-1)}{\phi\sigma\alpha-\rho_0}, \quad s_4^* = \alpha\frac{\phi-\rho_0}{\phi\sigma\alpha-\rho_0}.$$

The steady state $(y_3^*, 1, 0)$ is feasible if and only if $\alpha > 1/2$ and $\sigma \geq \alpha/(2\alpha-1)$, whereas the steady state $(y_4^*, s_4^*, 0)$ is a feasible solution if and only if $\alpha \geq 1/\sigma$.

The Jacobian matrix of system (A.10)-(A.12) evaluated at the steady state $(0, 1/\sigma, 0)$ is

$$J^*_{(0,1/\sigma,0)} = \begin{pmatrix} \frac{\alpha(1-\sigma)}{1-\alpha}+1 & 0 & 0 \\ \frac{\alpha}{1-\alpha}\frac{\sigma-1}{\sigma}\frac{\rho_0}{\phi} & \frac{\alpha(\sigma-1)}{1-\alpha} & -\frac{\alpha}{1-\alpha}\left(\frac{\sigma-1}{\sigma}\right)^2\frac{\rho_0}{\phi^2} \\ 0 & 0 & \frac{\alpha(1-\sigma)}{1-\alpha} \end{pmatrix}.$$

For $1/\sigma < \alpha$, this matrix has one positive and two negative eigenvalues. Hence this equilibrium is a saddle path with a two-dimensional stable manifold.

Following the same procedure, it can be easily proved that under condition $1/\sigma < \alpha$ the steady state $(0, 1, 0)$ is either a saddle path with a two-dimensional stable manifold, or locally asymptotically stable. Likewise, the steady state $(y_3^*, 1, 0)$, when feasible, is a saddle path with a two-dimensional stable manifold.

Finally, the Jacobian matrix at the steady state $(y_4^*, s_4^*, 0)$ is

$$J^*_{(y_4^*,s_4^*,0)} = \begin{pmatrix} \frac{\alpha}{1-\alpha}\frac{y_4^*}{s_4^*}(1-\frac{\rho_0}{\sigma\phi}) & \frac{y_4^*}{s_4^*} & \frac{\alpha}{1-\alpha}\frac{y_4^*}{s_4^*}\frac{\rho_0(\sigma-1)}{\sigma^2\phi^2}(1-\frac{\rho_0}{\phi}y_4^*) \\ \frac{\alpha}{1-\alpha}\frac{1-s_4^*}{s_4^*}\frac{\rho_0}{\sigma\phi} & \frac{\alpha}{1-\alpha}\frac{1-s_4^*}{s_4^*} & -\frac{\alpha}{1-\alpha}\frac{1-s_4^*}{s_4^*}\frac{\rho_0(\sigma-1)}{\sigma^2\phi^2}(1-\frac{\rho_0}{\phi}y_4^*) \\ 0 & 0 & -1 \end{pmatrix},$$

[16] For any solution converging towards $s^* \in (0, 1]$, there exists a finite time above which $r(t)$ decreases strictly and the change in the time scale is well defined.

which for $1/\sigma < \alpha$ always has one negative eigenvalue and two positive eigenvalues. This equilibrium is unstable in the sense that lying in a one-dimensional stable manifold is only a knife-edge possibility.

Thus, going back to system (33)-(35) in variables (r, s, B), it has been proved that among the equilibria $(r, s, B) = (0, s^*, 0)$, those with $s^* = 1/\sigma$ or $s^* = 1$ can be asymptotically approached.

Proof of Proposition 5

From the Ramsey rule (17) and equation (30), the consumption growth rate on the center manifold can be written in terms of variables r, y and B:

$$\gamma_c = \frac{r}{\sigma}\left(1 - \frac{\rho_0}{\phi}y\right)\left(1 + \frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right). \tag{A.13}$$

Whenever y converges either to 0 or $y_3^* < \phi/\rho$, there exists a finite time $\bar{t}$ such that $y(t) < \phi/\rho$ for any $t \geq \bar{t}$. In consequence $\gamma_c(t) > 0$ for any $t \geq \bar{t}$.

From (34) and (31) if $s(t) = 0$ for $t \in (\bar{t}, +\infty)$, then $\dot{s}(t) = -\gamma_c(t) < 0$, which would imply an unfeasible negative value of the saving rate. This proves the statement that $s(t) > 0$ for $t \geq \bar{t}$.

Proof of Proposition 6

Let us denote $(\hat{s}^1, \hat{r}^1, \hat{k}^1, \hat{c}^1)$ and $(\hat{s}, \hat{r}, \hat{k}, \hat{c})$ the time paths of s, r, k and c which correspond to the solution for which the saving rate converges to 1 and $1/\sigma$, respectively. From Proposition 5 there exist two finite times $\bar{t}^1$, $\bar{t}$ and two lower bounds $s^1_{\min}, s_{\min} > 0$, such that $s^1_{\min} \leq \hat{s}^1(t) < 1$ for any $t \geq \bar{t}^1$ and $s_{\min} \leq \hat{s}(t) < 1/\sigma$ for any $t \geq \bar{t}$. In consequence, in each case the speed of convergence of the rate of return towards zero would satisfy

$$\begin{aligned} -\frac{(1-\alpha)}{\alpha}r < \frac{\dot{r}}{r} &\leq -\frac{(1-\alpha)}{\alpha}s^1_{\min}r \quad \text{for all} \quad t \geq \bar{t}^1, \\ -\frac{(1-\alpha)}{\alpha}\frac{1}{\sigma}r < \frac{\dot{r}}{r} &\leq -\frac{(1-\alpha)}{\alpha}s_{\min}r \quad \text{for all} \quad t \geq \bar{t}. \end{aligned} \tag{A.14}$$

By integrating these expressions from $\bar{t}^1$ and $\bar{t}$ on we obtain

$$\begin{aligned} \frac{\bar{r}^1}{1 + \bar{r}^1\frac{1-\alpha}{\alpha}(t - \bar{t}^1)} < \hat{r}^1(t) &\leq \frac{\bar{r}^1}{1 + \bar{r}^1 s^1_{\min}\frac{1-\alpha}{\alpha}(t - \bar{t}^1)}, \quad \text{for all} \quad t \geq \bar{t}^1, \\ \frac{\bar{r}}{1 + \bar{r}\frac{1}{\sigma}\frac{1-\alpha}{\alpha}(t - \bar{t})} < \hat{r}(t) &\leq \frac{\bar{r}}{1 + \bar{r}s_{\min}\frac{1-\alpha}{\alpha}(t - \bar{t})}, \quad \text{for all} \quad t \geq \bar{t}, \end{aligned} \tag{A.15}$$

with $r(\bar{t}^1) = \bar{r}^1$ and $r(\bar{t}) = \bar{r}$. From the definition of the rate of return, it is immediately obvious that

$$\bar{k}^1 \left(1 + \bar{r}^1 s^1_{\min} \frac{1-\alpha}{\alpha}(t - \bar{t}^1)\right)^{\frac{1}{1-\alpha}} < \hat{k}^1(t) < \bar{k}^1 \left(1 + \bar{r}^1 \frac{1-\alpha}{\alpha}(t - \bar{t}^1)\right)^{\frac{1}{1-\alpha}}, \quad \text{for all} \quad t \geq \bar{t}^1,$$
$$\bar{k} \left(1 + \bar{r} s_{\min} \frac{1-\alpha}{\alpha}(t - \bar{t})\right)^{\frac{1}{1-\alpha}} < \hat{k}(t) < \bar{k} \left(1 + \bar{r} \frac{1}{\sigma} \frac{1-\alpha}{\alpha}(t - \bar{t})\right)^{\frac{1}{1-\alpha}}, \quad \text{for all} \quad t \geq \bar{t},$$

with $\bar{k}^1 = \left(\alpha A/\bar{r}^1\right)^{1/(1-\alpha)}$ and $\bar{k} = (\alpha A/\bar{r})^{1/(1-\alpha)}$. From (5), $c = (1-s)Ak^\alpha$, then we can compute the limit:

$$\lim_{t\to\infty} \frac{\hat{c}^1(t)}{\hat{c}(t)} = \lim_{t\to\infty} \frac{(1-\hat{s}^1(t))(\hat{k}^1(t))^\alpha}{(1-\hat{s}(t))(\hat{k}(t))^\alpha} < \lim_{t\to\infty} \frac{(1-\hat{s}^1(t))(\bar{k}^1)^\alpha \left(1 + \bar{r}^1 \frac{1-\alpha}{\alpha}(t-\bar{t}^1)\right)^{\frac{\alpha}{1-\alpha}}}{\left(1-\frac{1}{\sigma}\right)(\bar{k})^\alpha \left(1 + \bar{r} s_{\min} \frac{1-\alpha}{\alpha}(t-\bar{t})\right)^{\frac{\alpha}{1-\alpha}}} =$$
$$\frac{\sigma}{\sigma-1} \left(\frac{\bar{k}^1}{\bar{k}}\right)^\alpha \lim_{t\to\infty} (1-\hat{s}^1(t)) \left(\frac{1 + \bar{r}^1 \frac{1-\alpha}{\alpha}(t-\bar{t}^1)}{1 + \bar{r} s_{\min} \frac{1-\alpha}{\alpha}(t-\bar{t})}\right)^{\frac{\alpha}{1-\alpha}} = 0.$$

In consequence, there exists a finite time T such that $\hat{c}^1(t) < \hat{c}(t)$ for any $t \geq T$, and the solutions converging towards $s^* = 1$ are suboptimal under the catching-up optimality criterium.

Proofs of Proposition 7 and 8

Proof of Proposition 7

See equations (A.14) and (A.15).

Proof of Proposition 8

A trajectory converging towards $(0, 1/\sigma, 0)$ satisfies that for any $\bar{y} > 0$ there exists a finite $\bar{t}$ such that $y(t) \leq \bar{y}$ for any $t \geq \bar{t}$. In particular, considering $\bar{y} = 1 < \phi/\rho_0$, there exists a finite $\bar{t}$ such that $y(t) < 1 < \phi/\rho_0$ for any $t \geq \bar{t}$. Therefore, taking into account (A.13),

$$\gamma_c(t) > \frac{1}{\sigma}\left(1 - \frac{\rho_0}{\phi}\right) r(t) \left(1 + \frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi} B(t)\right) > \frac{1}{\sigma}\left(1 - \frac{\rho_0}{\phi}\right) r(t), \quad \forall t \geq \bar{t},$$

which proves the first inequality in (36), taking into account (A.15).

Moreover, since $\lambda(t) > 0$ by definition, from (17) and (A.15) the second inequality in (36) follows.

Acknowledgements

The authors are grateful to Javier de Frutos, Victor Jiménez, Jesús Marín-Solano and two anonymous referees for their useful discussions and comments. This work was partially supported by MEC [project ECO2014-52343-P], co-financed by FEDER funds; and by the EU Framework Programme Horizon 2020 [COST Action IS1104 "The EU in the new economic complex geography: models, tools and policy evaluation"].

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