Brunella's Local Alternative

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0 Introduction

It is a question of M. Brunella to decide if the following alternative is true:

> *Let $\mathcal{F}$ be a singular holomorphic foliation of codimension one in the projective space $\mathbb{P}^3$. If there is no projective algebraic surface invariant by $\mathcal{F}$, each leaf is a union of algebraic curves.*

The answer to this question is known [10] to be positive in the case of a generic pencil of foliations.

This work concerns a local version of the above alternative. Consider a germ $\mathcal{F}$ of singular holomorphic foliation of codimension one in $(\mathbb{C}^3, \underline{0})$ and assume that it has no invariant germ of analytic surface. We prove, under some conditions on the foliation, that there exists a neighborhood of the origin which is a union of *semi-transcendental leaves.*

A key remark for understanding germs of foliations without invariant germs of surface is that they must be *dicritical.* In a general we say that $\mathcal{F}$ is dicritical if there exists a holomorphic germ of map

$$\begin{array}{rcl} \phi : (\mathbb{C}^2, \underline{0}) & \to & (\mathbb{C}^3, \underline{0}) \\ (x, y) & \mapsto & (\phi_1(x, y), \phi_2(x, y), \phi_3(x, y)) \end{array}$$

such that $\phi((y = 0))$ is invariant by $\mathcal{F}$ and the pullback $\phi^*\mathcal{F}$ of the foliation $\mathcal{F}$ coincides with the foliation $dx = 0$ in $(\mathbb{C}^2, \underline{0})$. In [5] it is proved that any nondicritical foliation in $(\mathbb{C}^3, \underline{0})$ has an invariant germ of analytic hypersurface; this is also true in any ambient dimension [8].

In this paper we consider only *Relatively Isolated Complex Hyperbolic* germs of foliations in $(\mathbb{C}^3, \underline{0})$, that we shall refer to as "RICH foliations", for short. A germ $\mathcal{F}$ of singular holomorphic foliation of codimension one in $(\mathbb{C}^3, \underline{0})$ is a RICH foliation if there exists a reduction of singularities for $\mathcal{F}$

$$\mathcal{S} : \quad (\mathbb{C}^3, \underline{0}) = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

such that for any $1 \leq k \leq N$ we have

1. The center $Y_{k-1} \subset M_{k-1}$ of the blow-up π_k is nonsingular, has normal crossings with the total exceptional divisor $E^{k-1} \subset M_{k-1}$ and is invariant by the transform $\mathcal{F}_{k-1}$ of $\mathcal{F}$.

2. The intersection $Y_{k-1} \cap (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{k-1})^{-1}(\underline{0})$ is a single point.

Moreover, we ask (Complex Hyperbolic) that all the points of M_N are simple and without saddle-nodes in the sense of the general reduction of singularities in dimension three [4].

The condition "Complex Hyperbolic" has been frequently considered since the publication of the paper [2], where the authors consider germs of foliations in dimension two, called "generalized curves", without saddle-nodes in the reduction of singularities.

The condition "Relatively Isolated" is less restrictive than "Absolutely Isolated". It contains as examples the case of equireduction along a curve and the foliations of the type $df = 0$, where $f = 0$ defines a germ of surface with absolutely isolated singularity. The absolutely isolated singularities of vector fields have been studied in [1], whereas for the case of codimension one foliations on $(\mathbb{C}^3, \underline{0})$ the singular locus has codimension two unless we have a holomorphic first integral as proved in [14]. Anyway, in the paper [7], the authors consider foliations desingularized essentially by punctual blow-ups, which gives also a condition more restrictive than being Relatively Isolated.

Let us recall, see for instance [16], that a germ of foliation $\mathcal{G}$ on $(\mathbb{C}^2, \underline{0})$ contains a *nodal separator* if in the reduction of singularities there is a singularity analytically equivalent to $dy - \lambda dx = 0$ where λ is a non rational positive real number.

Consider a germ of curve Γ contained in the singular locus of $\mathcal{F}$. We say that $\mathcal{F}$ is *generically dicritical along* Γ if it is dicritical at a generic point of Γ. This is equivalent to saying that in the reduction of singularities $\mathcal{S}$ there exists a dicritical (generically transversal) component D of the exceptional divisor $E^N \subset M_N$ such that $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_N(D) = \Gamma$. Moreover, we can verify this fact at the equireduction points of Γ by doing an essentially two-dimensional reduction of singularities [4]. If $\mathcal{F}$ is not generically dicritical along Γ, it is known [4] that the equiredution along Γ is given by the (nondicritical) reduction of singularities of the restriction $\mathcal{G}$ of $\mathcal{F}$ to a plane section transversal to Γ at a generic point. In this case, we say that $\mathcal{F}$ *is generically nodal along* Γ if this is such a plane transversal section $\mathcal{G}$ has a nodal separator.

The main result in this work my be stated as follows:

Theorem *Let $\mathcal{F}$ be a RICH foliation in $(\mathbb{C}^3, \underline{0})$. Assume that there is no germ of invariant analytic surface for $\mathcal{F}$. Then one of the two properties holds:*

1. *There exists a neighborhood W of the origin $\underline{0} \in \mathbb{C}^3$ such that for each leaf $L \subset W$ of $\mathcal{F}$ in W there is an analytic curve $\gamma \subset L$ with $\underline{0} \in \gamma$.*

2. *There is an analytic curve Γ contained in the singular locus Sing $\mathcal{F}$ such that $\mathcal{F}$ is generically dicritical or generically nodal along Γ.*

Note that in order to verify possibility 2 it is enough to perform finitely many blow-ups with center in the irreducible components of Sing $\mathcal{F}$.

1 Preliminaries

1.1 Codimension one holomorphic foliations

Let M be a complex manifold of dimension n and let Ω_M be its cotangent sheaf - that is to say, it is the sheaf of germs of differential holomorphic 1-forms over M. A *holomorphic singular foliation of codimension one* $\mathcal{F}$, over M, is an integrable and invertible $\mathcal{O}_M$-submodule of Ω_M such that the quotient $\Omega_M/\mathcal{F}$ is torsion-free. This means that for each point $p \in M$ we can find local coordinates $x_1, x_2, \ldots, x_n$ such that the stalk $\mathcal{F}_p$ is generated by a differential 1-form

$$\Omega = \sum_{i-1}^{n} b_i dx_i,\ b_i \in \mathcal{O}_{M,p}$$

where $\Omega \wedge d\Omega = 0$ and the coefficients $b_1, b_2, \ldots, b_n$ have no common factor. The *singular locus* Sing $\mathcal{F}$ is locally given by

$$\text{Sing } \mathcal{F} = \{b_1 = b_2 = \cdots = b_n = 0\}\ .$$

It is a closed analytic subset of M of codimension ≥ 2. An irreducible element $f \in \mathcal{O}_{M,p}$ (resp. $\hat{\mathcal{O}}_{M,p}$) is a *separatrix* (resp. *formal separatrix*) if, and only if, f divides $\Omega \wedge df$. This means that, outside Sing $\mathcal{F}$, the closed analytic hypersurface $(f = 0)$ is contained in a leaf of $\mathcal{F}$.

Though the description of $\mathcal{F}$ near a singular point can be quite complicated, the theorem below asserts that, on the other hand, in a neighborhood of a regular point this task is much simpler:

Theorem 1 (Frobenius) *Let Ω be an integrable 1-form over M and p a point such that $\Omega(p) \neq 0$. There exist two germs of functions $u, f \in \mathcal{O}_{M,p}$ such that $u(p) \neq 0,\ df(p) \neq 0$ and*

$$\Omega_p = udf\ .$$

It is sometimes useful to regard a foliation $\mathcal{F}$ as adapted to a normal crossings divisor $E \subset M$.

A subset $E \subset M$ is a *normal crossings divisor* on M is a union of finitely many nonsingular hypersurfaces such that at each point $p \in M$ we can find local coordinates $x_1, x_2, \ldots, x_n$ such that

$$E = \left(\prod_{i=1}^{e} x_i = 0\right),\ e \in \{1, 2, \ldots, n\}\ .$$

Let $\Omega_M[-E]$ be the sheaf of germs of differential meromorphic 1-forms over M which have at most simple poles along E. A *holomorphic codimension one foliation adapted to E over M* is a pair $(\mathcal{F}, E)$ where $\mathcal{F}$ is an $\mathcal{O}_M$-submodule of $\Omega_M[-E]$ such that

(a) $\mathcal{F}$ is locally free of rank one.

(b) $\mathcal{F} \wedge d\mathcal{F} = 0$.

(c) $\Omega_M[-E]/\mathcal{F}$ is torsion-free.

Let's take a moment to explain the consequences of this definition at each point of M. Let J_E be the sheaf of ideals that define the divisor $E \subset M$ and fix a point $p \in M$; we may choose local coordinates $x_1, x_2, \ldots, x_n$ (which are simply a regular system of parameters of the local ring $\mathcal{O}_{M,p}$) such that

$$J_{E,p} = \left(\prod_{i \in A} x_i\right) \cdot \mathcal{O}_{M,p}, \ \ A \subset \{1, 2, \ldots, n\} \ .$$

Then the stalk $\Omega_{M,p}[-E]$ is generated by

$$\left\{\frac{dx_i}{x_i}\right\}_{i \in A} \cup \ \{dx_i\}_{i \notin A} \ .$$

Therefore, $\mathcal{F}_p$ is generated by a differential meromorphic 1-form

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i dx_i, \ \ a_i \in \mathcal{O}_{M,p}$$

such that $\omega \wedge d\omega = 0$ and $a_1, a_2, \ldots, a_n$ have no common factor.

Let $\mathcal{F}(M, E)$ be the space of holomorphic codimension one foliations adapted to E. Given $(\mathcal{F}, E) \in \mathcal{F}(M, E)$ and a point $p \in M$, the *adapted order* $\nu_p(\mathcal{F}, E)$ is (using the notation above)

$$\nu_p(\mathcal{F}, E) = \min\{\nu_p(a_i); i = 1, 2, \ldots, n\} \ .$$

The singular locus of $(\mathcal{F}, E)$ is given by

$$\text{Sing } (\mathcal{F}, E) = \{p \in M; \ \nu_p(\mathcal{F}, E) \geq 1\} \ .$$

It is a closed analytic subset of X and since $\Omega_M[-E]/\mathcal{F}$ has no torsion, it has codimension ≥ 2.

If $E = \emptyset$, we recover the usual notion of holomorphic codimension one foliation. Furthermore, there is a **bijection**

$$\mathrm{hol} : \mathcal{F}(M, E) \to \mathcal{F}(M, \emptyset)$$

defined by the following property:

$$\text{If } (\mathcal{G}, \emptyset) = \mathrm{hol}(\mathcal{F}, E), \text{ then } \mathcal{G}\Big|_{M-E} = \mathcal{F}\Big|_{M-E} .$$

This implies that if $\mathcal{F}_p$ is generated by the 1-form ω above, then $\mathcal{G}_p$ is generated by

$$\Omega = \left(\prod_{i \in A^*} x_i \right) \omega \ ,$$

where $A^* = \{i \in A; x_i \text{ does not divide } a_i\}$. Note that $x_i = 0$ where $i \in A^*$ are precisely the components of E that are separatrices.

Now fix $(\mathcal{G}, \emptyset) \in \mathcal{F}(M, \emptyset)$ and a point $p \in M$. Assume $\mathcal{G}_p$ is generated by the 1-form Ω above. We have already defined what is a separatrix (resp. formal separatrix) of $(\mathcal{G}, \emptyset)$. An *invariant analytic space* of $(\mathcal{G}, \emptyset)$ is an irreducible closed analytic space $K \subset M$ such that $\Omega\Big|_K = 0$ at nonsingular points of K. In this case, we'll say that K is *invariant by* $\mathcal{G}$. If $H \subset M$ is an analytic hypersurface which is invariant for $\mathcal{G}$, then it defines, at each point $p \in H$, a separatrix of $\mathcal{G}$. Conversely, an irreducible hypersurface $H \subset M$ is invariant for $\mathcal{G}$ if and only if it defines a separatrix at each point $p \in H$.

Let $(\mathcal{F}, E) \subset \mathcal{F}(M, E)$ and fix and irreducible component D of E. We say F is a *nondicritical component* of E for $(\mathcal{F}, E)$ if and only if D is invariant for $\mathrm{hol}(\mathcal{F}, E)$. Otherwise we say that F is a *dicritical component* of E for $(\mathcal{F}, E)$. Therefore, using the notation above, we have that

$$A^* = \Big\{ i \in A; (x_i = 0) \text{ is a nondicritical component for } (\mathcal{F}, E) \Big\} .$$

Let $Y \subset M$ be a nonsingular analytic subspace of M. We say that Y *has normal crossings* with E if the following holds: at each point $p \in Y$ there are local coordinates $x_1, x_2, \ldots, x_n$ and sets $A, B \subset \{1, 2, \ldots, n\}$ such that

$$E = \left(\prod_{i \in A} x_i = 0 \right) \text{ and } Y = \bigcap \Big\{ x_i = 0; \ i \in B \Big\}$$

locally at p. Assume Y and E have normal crossings and let $\pi : M' \to M$ be a blow-up centered at Y. Call $E' = \pi^{-1}(E \cup Y)$ (with reduced structure); then $E' \subset M'$ is a normal crossings divisor on M'. Now, if $(\mathcal{F}, E) \in \mathcal{F}(M, E)$ and $\mathrm{hol}(\mathcal{F}, E) = (\mathcal{G}, \emptyset)$, there exist unique $(\mathcal{F}', E') \in \mathcal{F}(M', E')$ and $(\mathcal{G}', \emptyset) \in \mathcal{F}(M', \emptyset)$ such that

$$\mathcal{F}'\Big|_{M'-\pi^{-1}(Y)} = \mathcal{F}\Big|_{M-Y} \text{ and } \mathcal{G}'\Big|_{M'-\pi^{-1}(Y)} = \mathcal{G}\Big|_{M-Y}$$

under the isomorphism $\pi : M' - \pi^{-1}(Y) \to M - Y$. Furthermore $\mathrm{hol}(\mathcal{F}', E') = (\mathcal{G}', \emptyset)$. In this situation, we say $(\mathcal{F}', E')$ is the *adapted strict transform* of $(\mathcal{F}, E)$ by π and that $(\mathcal{G}', \emptyset)$ is the *strict transform* of $(\mathcal{G}, \emptyset)$ by π. We denote $\pi^*\mathcal{F} = \mathcal{F}'$, $\pi^*\mathcal{G} = \mathcal{G}'$.

We will go into more detail about the properties of blow-up morphisms in the next section.

In this work, we will consider holomorphic codimension one foliations of $(\mathbb{C}^3, \underline{0}) = M$. At some points, however, we will regard the restriction of these foliations to a non-invariant transversal two-dimensional section, which results in a codimension one foliation of $\mathbb{C}^2$. Thus in this chapter we also recall some concepts, definitions and results concerning foliations in dimension two.

1.2 Blow-up morphisms

Let M be a complex manifold, dim $M = n$. In this section, we recall the definition of the blow-up of a point $p \in M$, and the definition of the blow-up of a smooth analytic subset $S \subset M$ that has normal crossings with M and such that codim $S \geq 2$. We focus our attention in the local equations. We refer to the vast literature for the universal property of the blow-up, the properness and other intrinsic properties of these morphisms.

Consider the set

$$\Sigma = \Big\{ (x, X) \in \mathbb{C}^n \times \mathbb{P}^{n-1};\ x \in X \Big\} .$$

Let's write $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, $X = [X_1 : X_2 : \cdots : X_n] \in \mathbb{P}^{n-1}$. So $x \in X$ means that $[x] = [(x_1, x_2, \ldots, x_n)] \in \mathbb{P}^{n-1}$ is precisely

$$X = [X_1 : X_2 : \cdots : X_n] = [(X_1, X_2, \ldots, X_n)] .$$

Since $(x_1, x_2, \ldots, x_n) \sim (X_1, X_2, \ldots, X_n)$ if and only if there exists a $\lambda \in \mathbb{C}^*$ such that $(x_1, x_2, \ldots, x_n) = \lambda(X_1, X_2, \ldots, X_n)$, we get that, whenever $x_j, X_j \neq 0$,

$$\frac{x_1}{X_1} = \frac{x_2}{X_2} = \cdots = \frac{x_n}{X_n} = \lambda \ .$$

So whenever $x_j, X_j \neq 0$, the equations

$$\frac{x_i}{x_j} = \frac{X_i}{X_j} \ , i \neq j$$

define the set $\Sigma \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. We regard Σ with the induced topology of $\mathbb{C}^n \times \mathbb{P}^{n-1}$.

Consider the first projection

$$\begin{array}{rccc} \pi : & \Sigma & \rightarrow & \mathbb{C}^n \\ & (x, X) & \mapsto & x \end{array} .$$

Suppose $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n, x \neq \underline{0}$: there exists a $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq 0$. Therefore $[x] \in \mathbb{P}^{n-1}$ is well defined, and we may put $\pi^{-1}(x) = (x, [x]) \in \sum$. So apart from the choice of representant of the class $[x]$, π is injective. Naturally, π is surjective. Therefore

$$\pi : \Sigma - \pi^{-1}(\underline{0}) \rightarrow \mathbb{C}^n - \{\underline{0}\}$$

is a isomorphism. We have that $\pi^{-1}(\underline{0}) = (\underline{0}, [a_1 : a_2 : \cdots : a_n])$ such that $\underline{0} \in \lambda(a_1, a_2, \ldots, a_n)$; thus

$$\pi^{-1}(\underline{0}) = \{\underline{0}\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1} \ .$$

The map π is called the *blow-up of the origin of* $\mathbb{C}^n$, the set $\pi^{-1}(\underline{0})$ is called the *exceptional divisor* and the set $\Sigma \cup \ \pi^{-1}(\underline{0})$ is the new ambient space, also of dimension n.

Now we would like to write the map π in *local charts*. Let

$$H_j = \Big\{[a_1 : a_2 : \cdots : a_n] \in \mathbb{P}^{n-1}; \ a_j \neq 0\Big\} \ .$$

Note that $H_j \simeq \mathbb{C}^{n-1}$. We put

$$\Sigma_j = \Sigma \cap (\mathbb{C}^n \times H_j) = \Big\{(x, X); \ X_j \neq 0\Big\} \ .$$

Finally, we define

$$\begin{array}{rccc} \Phi_j : & \Sigma_j & \rightarrow & \mathbb{C}^n \\ & (x, X) & \mapsto & \left(\frac{X_1}{X_j}, \frac{X_2}{X_j}, \ldots, \frac{X_{j-1}}{X_j}, x_j, \frac{X_{j+1}}{X_j}, \ldots, \frac{X_n}{X_j}\right) \end{array} .$$

The map $\pi \circ \Phi_j^{-1}$ will give the expression of π in the local chart Σ_j. For instance, in the case $n = 2$, we have

$$\begin{array}{cccc} \Phi_1 : & \Sigma_1 & \longrightarrow & \mathbb{C}^2 \\ & ((x_1, x_2), [X_1 : X_2]) & \mapsto & \left(x_1, \frac{X_2}{X_1}\right) \end{array}$$

and

$$\begin{array}{cccc} \Phi_2 : & \Sigma_1 & \longrightarrow & \mathbb{C}^2 \\ & ((x_1, x_2), [X_1 : X_2]) & \mapsto & \left(\frac{X_1}{X_2}, x_2\right) \end{array} .$$

So

$$\begin{array}{cccc} \Phi_1^{-1} : & \mathbb{C}^2 - \{(0, y)\} & \longrightarrow & \Sigma_1 \\ & (a, b) & \mapsto & \Big((a, ab), [1 : b]\Big) \end{array}$$

and

$$\begin{array}{cccc} \Phi_2^{-1} : & \mathbb{C}^2 - \{(x, 0)\} & \longrightarrow & \Sigma_2 \\ & (a, b) & \mapsto & \Big((ab, b), [a : 1]\Big) \end{array} .$$

Hence

$$\pi \circ \Phi_1^{-1}((a, b)) = (a, ab) \text{ is the } \textit{first local chart} \ ,$$

$$\pi \circ \Phi_2^{-1}((a, b)) = (ab, b) \text{ is the } \textit{second local chart} \ .$$

If $n = 3$, we have

$$\begin{array}{cccc} \Phi_1 : & \Sigma_1 & \longrightarrow & \mathbb{C}^3 \\ & ((x_1, x_2, x_3), [X_1 : X_2 : X_3]) & \mapsto & \left(x_1, \frac{X_2}{X_1}, \frac{X_3}{X_1}\right) \end{array} ,$$

$$\begin{array}{cccc} \Phi_2 : & \Sigma_2 & \longrightarrow & \mathbb{C}^3 \\ & ((x_1, x_2, x_3), [X_1 : X_2 : X_3]) & \mapsto & \left(\frac{X_1}{X_2}, x_2, \frac{X_3}{X_2}\right) \end{array} ,$$

$$\begin{array}{cccc} \Phi_3 : & \Sigma_3 & \longrightarrow & \mathbb{C}^3 \\ & ((x_1, x_2, x_3), [X_1 : X_2 : X_3]) & \mapsto & \left(\frac{X_1}{X_3}, \frac{X_2}{X_3}, x_3\right) \end{array} .$$

So

$$\begin{array}{cccc} \Phi_1^{-1} : & \mathbb{C}^3 - \{(0, y, z)\} & \longrightarrow & \Sigma_1 \\ & (a, b, c) & \mapsto & \Big((a, ab, ac), [1 : b : c]\Big) \end{array} ,$$

$$\begin{array}{cccc} \Phi_2^{-1} : & \mathbb{C}^3 - \{(x, 0, z)\} & \longrightarrow & \Sigma_2 \\ & (a, b, c) & \mapsto & \Big((ab, b, bc), [a : 1 : c]\Big) \end{array} ,$$

$$\begin{array}{rccc} \Phi_3^{-1}: & \mathbb{C}^3 - \{(x,y,0)\} & \longrightarrow & \Sigma_3 \\ & (a,b,c) & \mapsto & \Big((ac,bc,c),[a:b:1]\Big) \end{array} .$$

Therefore

$$\pi \circ \Phi_1^{-1}((a,b,c)) = (a, ab, ac) \text{ is the } \textit{first local chart} \ ,$$

$$\pi \circ \Phi_2^{-1}((a,b,c)) = (ab, b, bc) \text{ is the } \textit{second local chart} \ ,$$

$$\pi \circ \Phi_3^{-1}((a,b,c)) = (ac, bc, c) \text{ is the } \textit{third local chart} \ .$$

So in dimension n, we will have

$$\begin{array}{rccl} \Phi_j^{-1}: & \mathbb{C}^n - \{x_j = 0\} & \to & \Sigma_j \\ & (x_1,\ldots,x_n) & \mapsto & \Big((x_1x_j, x_2x_j, \ldots, x_{j-1}x_j, x_j, x_{j+1}x_j, \ldots, x_nx_j), \\ & & & [x_1 : \cdots : x_{j-1} : 1 : x_{j+1} : \cdots : x_n]\Big) \end{array}$$

and therefore

$$\pi \circ \Phi_j^{-1}((x_1,\ldots,x_n)) = (x_1x_j, x_2x_j, \ldots, x_{j-1}x_j, x_j, x_{j+1}x_j, \ldots, x_nx_j)$$

is the *j-th local chart* of the blow-up of the origin $\underline{0} \in \mathbb{C}^n$.

Now we wish to perform the blow-up of an analytic subset $S \subset M$ that has normal crossings with M and such that codim $S \geq 2$. For each point $p \in S$, we can find local coordinates at p such that

$$S = \left(\prod_{i \in A_p} x_i = 0 \right) \text{ where } A_p \subset \{1, 2, \ldots, n\} \ .$$

To make the notation easier, we will write

$$S = \Big(x_1 = x_1 = \cdots = x_k = 0\Big), \ k \leq n-2.$$

The only coordinates we will modify will be $x_1, x_2, \cdots, x_k$; the others will be kept as they are. Naturally, when we perform the blow-up of the origin, we modify all coordinates given that

$$\{\underline{0}\} = \{x_1 = x_2 = \cdots = x_n = 0\} \ .$$

For $j = 1, 2, \ldots, k$, we will once again consider the sets

$$\Sigma_j = \Sigma \cap (\mathbb{C}^n \times H_j) = \Big\{((x_1,\ldots,x_n),[X_1 : \cdots : X_n]); \ X_j \neq 0\Big\} \ .$$

Now we will define the maps

$$\begin{array}{rccl} \Psi_j: & \Sigma_j & \to & \mathbb{C}^n \\ & (x,X) & \mapsto & \left(\frac{X_1}{X_j}, \frac{X_2}{X_j}, \ldots, \frac{X_{j-1}}{X_j}, x_j, \frac{X_{j+1}}{X_j}, \ldots, \frac{X_k}{X_j}, x_{k+1}, \ldots, x_n\right) \end{array} .$$

So

$$\begin{array}{rccl} \Psi_j^{-1}: & \mathbb{C}^n - \{x_j = 0\} & \to & \Sigma_j \\ & (x_1, \ldots, x_n) & \mapsto & \Big((x_1x_j, \ldots, x_{j-1}x_j, x_j, x_{j+1}x_j, \ldots, x_kx_j, x_{k+1}, \ldots, x_n), \\ & & & [x_1 : \cdots : x_{j-1} : 1 : x_{j+1} : \cdots : x_k : \frac{x_{k+1}}{x_j} : \ldots : \frac{x_n}{x_j}]\Big) \end{array}$$

and

$$\pi \circ \Psi_j^{-1}((x_1, \ldots, x_n)) = (x_1x_j, \ldots, x_{j-1}x_j, x_j, x_{j+1}x_j, \ldots, x_kx_j, x_{k+1}, \ldots, x_n)$$

is the *j-th local chart* of the blow-up of $S \in \mathbb{C}^n$. For each point $p \in S$, we have that

$$\pi \circ \Psi_j^{-1}(p) \simeq \mathbb{P}^{k-1} .$$

For example, suppose we want to blow-up the z-axis of $\mathbb{C}^3$, $Z = \{x = y = 0\}$. We will consider the maps

$$\begin{array}{rccl} \Psi_1: & \Sigma_1 & \to & \mathbb{C}^3 \\ & ((x_1, x_2, x_3), [X_1 : X_2 : X_3]) & \mapsto & \left(x_1, \frac{X_2}{X_1}, x_3\right) \end{array}$$

and

$$\begin{array}{rccl} \Psi_2: & \Sigma_2 & \to & \mathbb{C}^3 \\ & ((x_1, x_2, x_3), [X_1 : X_2 : X_3]) & \mapsto & \left(\frac{X_1}{X_2}, x_2, x_3\right) \end{array} .$$

Therefore

$$\begin{array}{rccl} \Psi_1^{-1}: & \mathbb{C}^3 - \{(0, y, z)\} & \to & \Sigma_1 \\ & (a, b, c) & \mapsto & \left((a, ab, c), [1 : b : \frac{c}{a}]\right) \end{array}$$

and

$$\begin{array}{rccl} \Psi_2^{-1}: & \mathbb{C}^3 - \{(x, 0, z)\} & \to & \Sigma_2 \\ & (a, b, c) & \mapsto & \left((ab, b, c), [a : 1 : \frac{c}{b}]\right) \end{array} ,$$

thus we have

$$\pi \circ \Psi_1^{-1}((a, b, c)) = (a, ab, c) \text{ is the } \textit{first local chart} ,$$

$$\pi \circ \Phi_2^{-1}((a, b, c)) = (ab, b, c) \text{ is the } \textit{second local chart} .$$

Note that for each point $p \in Z$, $\pi^{-1}(p) \simeq \mathbb{P}^1$. So $\pi^{-1}(Z) \simeq Z \times \mathbb{P}^1$.

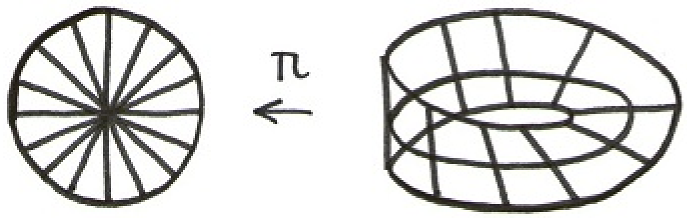


Figure 1: Explosion of the origin of $\mathbb{R}^2$

For example, if $\pi : M' \to M = (\mathbb{R}^2, \underline{0})$ is the blow up of the origin in $\mathbb{R}^2$ we have that

$$\pi \circ \phi_1^{-1}(x,y) = (x', x'y') \ ,$$
$$\pi \circ \phi_2^{-1}(x,y) = (x''y'', y'') \ .$$

Hence the change of coordinate is given by $y' \mapsto \dfrac{1}{x''}$ and $\pi^{-1}(\underline{0}) \simeq \mathbb{P}^1$.

1.3 Simple singularities in dimension two

Throughout this section, M will denote a complex manifold of dimension two. Let $\mathcal{F}$ be a holomorphic codimension one foliation of M, and let $p \in M$ be a singular point of $\mathcal{F}$. Given local coordinates x, y at p, let $\omega = a(x,y)dx + b(x,y)dy$ be a generator of $\mathcal{F}$.

Definition 1 *We say that p is a* simple singularity *of $\mathcal{F}$ if the jacobian matrix*

$$J_p(\omega; x, y) = \begin{bmatrix} -\dfrac{\partial b}{\partial x}(p) & -\dfrac{\partial b}{\partial y}(p) \\ \\ \dfrac{\partial a}{\partial x}(p) & \dfrac{\partial a}{\partial y}(p) \end{bmatrix}$$

has two eigenvalues, $(\lambda, \mu) \neq (0,0)$, such that if $\lambda\mu \neq 0$ then $\lambda/\mu \notin \mathbb{Q}_{>0}$. We say that the origin is a saddle-node *singularity if $\lambda\mu = 0$; in the case $\lambda\mu \neq 0$, we say the origin is a* complex hyperbolic *singularity.*

This definition depends neither on the choice of generator ω nor on the choice of local coordinates. Thus we can rewrite the local generator of $\mathcal{F}$ as

$$\omega = (\lambda x dy - \mu y dx) + \omega_1,$$

where the coefficients of ω_1 have order ≥ 2.

Remark 1 There are exactly two formal invariant curves at the origin, Γ_x and Γ_y, tangent to $L_x = T_{\underline{0}}(x = 0)$ and $L_y = T_{\underline{0}}(y = 0)$ respectively. The directions L_x and L_y are called the *proper directions* of the singular point $\underline{0} \in \mathbb{C}^2$. If $\mu \neq 0$ then L_x is a *strong* proper direction, and it is *weak* otherwise; the same for L_y. Briot-Bouquet's Theorem asserts that if L_x is strong, then Γ_x is convergent.

This discussion is resumed in the following lemma, whose proof we omit (it may be found in [6]):

Proposition 1 *Through a simple singularity pass exactly two formal curves, at least one of them convergent.*

One very important characteristic of the simple singularities is that they are stable under blow-up.

Proposition 2 *Let $\pi : M_1 \to M_0 = M$ be the blow-up morphism centered at p. Suppose that p is a simple singularity of $\mathcal{F}$ such that the quotients of the eigenvalues are $\{\alpha, 1/\alpha\}$. Let $\mathcal{F}_1$ be the strict transform of $\mathcal{F}$. Then*

1. *the exceptional divisor $E = \pi^{-1}(p)$ is invariant by $\mathcal{F}$;*
2. *the foliation $\mathcal{F}_1$ has exactly two singular points, p_1, p_2 in E, and the quotients of the eigenvalues are, respectively,*

$$\left\{\alpha - 1, \frac{1}{\alpha - 1}\right\}, \ \left\{\frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha}\right\} \ .$$

In particular, the points p_1, p_2 are simple singularities of $\mathcal{F}_1$.

In [16], J-F. Mattei and D. Marín give the following definition:

Definition 2 *Let $\mathcal{F}$ be a codimension one foliation of M, $\dim M = 2$. A point $p \in$ Sing $\mathcal{F}$ is a* nodal singularity *if the 1-form that generates $\mathcal{F}$ locally at p is given by*

$$\omega_p = (\lambda_1 y + \cdots)dx + (\lambda_2 x + \cdots)dy$$

with $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{R}_{<0} \setminus \mathbb{Q}_{<0}$.

Remark 2 The topological characterization of a nodal singularity is the existence of a saturated closed set whose complement is disconnected and such that each connected component is a neighborhood of one of the two separatrices (without the origin). That is to say, this saturated closed set acts like a separator of leaves of the foliation near the separatrices at the point. Suppose $\Gamma_x = (x = 0)$, $\Gamma_y = (y = 0)$ are the two separatrices at p. So if $\Delta \subset M$ is a one-dimensional section transversal to Γ_y at a regular point q and not invariant by the foliation (say, for instance, that $\Delta = \{1\} \times \mathbb{D}$), we have that $\mathrm{Sat}_{\mathcal{F}}(\Delta)$ is not a neighborhood of the nodal point p. This phenomenon is not seen in the complex hyperbolic singularities which are not nodal: at those points, if Δ is like before, then $\mathrm{Sat}_{\mathcal{F}}(\Delta)$ is in fact a neighborhood of the singular point. We will study this situation in detail in Chapter 4.

1.4 Reduction of singularities in dimension two

This section is devoted to recalling, without much detail, the proof of a very known and important result due to Seidenberg [24], which can be stated as follows:

Theorem 2 *Let $\mathcal{F}$ be a codimension one singular foliation of M, dim $M = 2$. There exists a morphism $\pi : M \to M_0 = M$, composition of finitely many blow-ups centered at points, such that every singularity of $\pi^*\mathcal{F}$ is simple.*

The proof of Theorem 2 is split in two parts: first we perform finitely many blow-ups centered at points in order to obtain *pre-simple singularities*; second, we make the passage from pre-simple singularities to simple ones, also by performing finitely many blow-ups.

Definition 3 *Let $\mathcal{F}$ be a codimension one foliation of M and let $p \in M$ be a singular point of $\mathcal{F}$. We will say p is a* pre-simple singularity *if, given local coordinates x, y at p such that $\mathcal{F}$ is generated by the 1-form $\omega = a(x,y)dx + b(x,y)dy$, the matrix*

$$J_p(\omega; x, y) = \begin{bmatrix} -\dfrac{\partial b}{\partial x}(p) & -\dfrac{\partial b}{\partial y}(p) \\ \\ \dfrac{\partial a}{\partial x}(p) & \dfrac{\partial a}{\partial y}(p) \end{bmatrix}$$

is non-nilpotent.

Though we have fixed the local coordinates at the singular point this definition does not depend on them, nor on the choice of the local generator ω. As a consequence of the definition, we have that p is a pre-simple singularity if the linear part of the vector field

$$X(x,y) = b(x,y)\frac{\partial}{\partial x} + a(x,y)\frac{\partial}{\partial y}$$

has one nonzero eigenvalue. Like the simple singularities, the pre-simple singularities are also stable under blow-up. Let $p \in M$ be a pre-simple point of $\mathcal{F}$ which is not a simple singularity. Then the vector field X locally at p has one of the following types:

1. $X = mx\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y} + \cdots\ ,\ m, n \in \mathbb{Z}_{>0}$;

2. $X = x\frac{\partial}{\partial x} + (y+x)\frac{\partial}{\partial y} + \cdots$.

We will now exhibit a series of arguments which will lead to the proof of Theorem 2, but beforehand, let's fix some notation.

Remark 3 - Notation Let $\mathcal{F}$ be a codimension one foliation of M. We will repeatedly work with a sequence

$$M = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_n \leftarrow \cdots \tag{1}$$

composition of blow-ups morphisms such that:

- the center of π_1 is a singular point of $\mathcal{F}$, $p \in M = M_0$;
- the center of π_s is a point $p_{s-1} \in M_{s-1}, s \geq 2$;
- $D_s^s = \pi_s^{-1}(p_{s-1}) \simeq \mathbb{P}^1$;
- D_i^s is the strict transform by π_s of $D_i^{s-1},\ i < s$;
- $E^s = D_1^s \cup D_2^s \cup \cdots \cup D_{s-1}^s \cup D_s^s$ is the exceptional divisor;
- $\mathcal{F}_1 = \pi_1^*\mathcal{F},\ \ldots\ , \mathcal{F}_s = \pi_i^*\mathcal{F}_{s-1}, i \geq 2$, where $\pi_i^*\mathcal{F}_{i-1}$ denotes the transform of $\mathcal{F}_{i-1}$ by π_i.

Note that each D_i^s is isomorphic to $\mathbb{P}^1$ and that at each stage, the exceptional divisor E^s has normal crossings with M_s. We will fix $E^0 \subset M = M_0$, the first divisor, to be the empty set. If $\Gamma_{s-1} \subset M_{s-1}$ is a curve, then Γ_s will denote its strict transform by π_s.

The first result concerning we would like to exhibit is the following:

Lemma 3 *Let $\mathcal{F}$ be a codimension one foliation of M, p a singular point of $\mathcal{F}$, and let Γ be a formal nonsingular curve passing through p. Consider a sequence like (1) such that $p_i = \Gamma_i \cap \pi_i^{-1}(p_{i-1})$. If p_i is a singular point of $\mathcal{F}_i$ for every $i \in \mathbb{N}$, then Γ is an invariant curve of $\mathcal{F}$.*

Proof: Assume that $\Gamma = (y = 0)$ locally at p. We wish to show that Γ_1 is invariant by $\mathcal{F}_1$ and therefore Γ will be invariant by $\mathcal{F}$.

Take local coordinates x', y' at p_1 given by $x' = x, y' = y/x$. Note that $\Gamma_1 = (y' = 0)$. The exceptional divisor E^1 is given at p_1 by $x' = 0$. Even if $x' = 0$ is dicritical, we write a generator of $(\mathcal{F}_1, E^1)$ as

$$\omega_1 = a_1(x', y')\frac{dx'}{x'} + b_1(x', y')dy' \ ,$$

where a_1, b_1 have no common factor. The fact that p_1 is singular implies that

$$\nu_{p_1}(a_1) \geq 1 \ .$$

We perform the second blow-up and we obtain

$$\omega_2 = a_2(x'', y'')\frac{dx''}{x''} + b_2(x''y'')dy''$$

where $x'' = x' = x, y'' = y'/x'$ and, putting $\nu_2 = \min \{\nu_{p_1}(a_1), \nu_{p_1}(b_1) + 1\}$,

$$a_2 = \left[\frac{1}{x''}\right]^{\nu_2} a_1 + y''b_1$$

$$b_2 = \left[\frac{1}{x''}\right]^{\nu_2 - 1} b_1 \ .$$

Note that $\nu_2 \geq 1$. We repeat the argument. Note that $y' = 0$ is invariant by $\mathcal{F}_1$ if and only if we have $a_1 = y'\tilde{a}_1$. Suppose, by absurd, that we have the contrary; thus we can write

$$a_1 = x^{m_1}u_1(x) + y'\tilde{a}_1, \ u_1(0) \neq 0 \ .$$

$$a_2 = x^{m_2}u_2(x) + y''\tilde{a}_2, \ u_2(0) \neq 0 \ .$$

$$\cdots \qquad , \qquad \cdots$$

The computations above show that $m_2 = m_1 - \nu_2 < m_1$, but this cannot occur indefinitely and we arrive to a contradiction.

$\square$

Remark 4 The result is also valid if Γ is a singular curve.

Given a point $p \in M$ and a divisor with normal crossings $E \subset M$, we will denote $e_p(E)$ as the number of components of E passing through p. Given a foliation $\mathcal{F}$ and a normal crossings divisor E, we denote E_{inv}, respectively E_{dic}, the normal crossings union of the invariant components of E by $\mathcal{F}$, respectively dicritical components of E by $\mathcal{F}$.

Now we recall the definition of the Milnor number of a foliation at a point $p \in M$. Let $\mathcal{F}$ be a codimension one foliation in M such that, given local coordinates x, y at p, $\mathcal{F}$ is generated by the 1-form

$$\omega = a(x,y)dx + b(x,y)dy \ .$$

The *Milnor number of* $\mathcal{F}$ *at* p, $\mu_p(\mathcal{F})$, is the intersection multiplicity of the curves $\{a = 0\}$ and $\{b = 0\}$ at p,

$$\mu_p(\mathcal{F}) = i_p(a,b) \ .$$

Suppose $\pi_1 : M_1 \to M$ is a blow-up centered at p, and put $E^1 = \pi_1^{-1}(p)$. Noether's formula combines the multiplicity of intersection before and after π_1:

$$i_p(a,b) = \nu_p(a) \cdot \nu_p(b) + \sum_{p' \in E_1} i_{p'}(a', b') \ ,$$

where $\nu_p(a)$, $\nu_p(b)$ are the orders of a, b at p and (we are considering the first local chart, $x = x', y = x'y'$)

$$a'(x', y') = \frac{1}{x'^{\nu_p(a)}} a(x', x'y')$$

$$b'(x', y') = \frac{1}{x'^{\nu_p(b)}} b(x', x'y') \ .$$

We use Noether's formula to achieve the next result, whose proof we will omit but may be found in several places, such as in [6].

Lemma 4 *Suppose* $\pi_1 : M_1 \to M$ *is a blow-up centered at* p, $E_1 = \pi_1^{-1}(p)$. *If* m *be the minimum of the multiplicities of* a, b, *then*

1. $\mu_p(\mathcal{F}) = m^2 - (m+1) + \sum_{p' \in E_1} \mu_{p'}(\mathcal{F}_1)$ *if* π_1 *is nondicritical ;*

2. $\mu_p(\mathcal{F}) = (m+1)^2 - (m+2) + \sum_{p' \in E_1} \mu_{p'}(\mathcal{F}_1)$ *if* π_1 *is dicritical.*

Let us start the proof of Theorem 2. In order to do that, let's assume it is false; thus we can find an infinite sequence of blow-ups

$$\mathcal{S}: \quad M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots$$

as in Remark 3 with the following conditions:

1. The center p_{i-1} of π_i is not a simple point of $\mathcal{F}_{i-1}$;
2. Each $p_i \in \pi_i^{-1}(p_{i-1})$.

Let us show that $\mathcal{S}$ cannot exist.

Let $I_q = \mu_q(\mathcal{F}_i) - e_q(E^i_{inv})$; we wish to see the behavior of I_q under blow-ups. Due to Lemma 4, if $m \geq 2$ and π_{i+1} is the blow-up centered at p_i, dicritical or not, we have that

$$\sum_{p' \in D^{i+1}_{i+1}} \mu_{p'}(\mathcal{F}_{i+1}) < \mu_{p_i}(\mathcal{F}_i) \ .$$

Thus for every point $p' \in D^{i+1}_{i+1}$, $\mu_{p'}(\mathcal{F}_{i+1}) < \mu_{p_i}(\mathcal{F}_i)$ and $I_{p'} < I_{p_i}$; that is to say, if $m \geq 2$, I_{p_i} decreases with each blow-up. So let's see what happens when $m = 1$.

Lemma 5 *In the situation above,*

- *if* $m = 1$ *and* π_{i+1} *is dicritical, then* p_i *is pre-simple;*
- *if* $m = 1$, $e_{p_i}(E^i) = 2$, *then* p_i *is pre-simple;*
- *if* p_i *is not pre-simple, then* $I_{p_i} \geq I_{p'}$ $\forall$ $p' \in D^{i+1}_{i+1}$. *Furthermore, we have a strict inequality if* π_{i+1} *is dicritical.*

Proof: For the first assertion, consider the dicritical divisor D^{i+1}_{i+1}: there exist two distinct points in D^{i+1}_{i+1} and two smooth curves, Γ_1 and Γ_2, invariant by $\mathcal{F}_{i+1}$ and transversal to D^{i+1}_{i+1} at these points. Take local coordinates x, y at p_i such that $\pi_{i+1}(\Gamma_1 \cup \Gamma_2) = (xy = 0)$. Then $\mathcal{F}_i$ is generated by the 1-form

$$\omega = ya(x,y)dx + xb(x,y)dy \ .$$

Since $m = 1$, either $a(p_i) \neq 0$ or $b(p_i) \neq 0$; suppose $a(p_i) \neq 0$. Then we can divide ω by a and obtain another generator for $\mathcal{F}_i$, $\omega^* = ydx + xb^*(x,y)dy$. Thus the matrix

$$J_{p_i}(\omega^*_1; x, y) = \begin{pmatrix} \star & \star \\ 0 & 1 \end{pmatrix}$$

has at least one nonzero eigenvalue, and p_i is pre-simple.

Now for the second assertion: there are two invariant components of E^i passing through p_i, thus $\omega = ya(x,y)dx + xb(x,y)dy$ is a local generator of $\mathcal{F}_i$ at p_i; we repeat the argument above.

For the last assertion, we need only consider the case where $m = 1$, $e_{p_i}(E^i) = 1$ and π_{i+1} is nondicritical. The points in D_{i+1}^{i+1} either have $e_{p'}(E^{i+1}) = 1$ or 2. Consider a point $q \in D_{i+1}^{i+1}$ such that $e_q(E^{i+1}) = 1$. There exists another point $q' \in D_{i+1}^{i+1}$ such that $e_{q'}(E^{i+1}) = 2$ (for example, $q' = D_i^{i+1} \cap D_{i+1}^{i+1}$). Then

$$\mu_{p_i}(\mathcal{F}_i) = -1 + \sum_{p' \in D_{i+1}^{i+1}} \mu_{p'}(\mathcal{F}_{i+1}) \geq -1 + \mu_q(\mathcal{F}_{i+1}) + \mu_{q'}(\mathcal{F}_{i+1}) \geq \mu_q(\mathcal{F}_{i+1}) \ ,$$

which implies $I_{p_i} \geq I_q$ since $e_{p_i}(E^i) = e_q(E^{i+1}) = 1$. Now, for the points that, like q', have $e_Q(E^{i+1}) = 2$, note that $\mu_{p_i}(\mathcal{F}_i) \geq -1 + \mu_{q'}(\mathcal{F}_{i+1})$, therefore $I_{p_i} = \mu_{p_i}(\mathcal{F}_i) - 1 \geq \mu_{q'}(\mathcal{F}_{i+1}) - 2 = I_{q'}$.

□

Corollary 3 *Given any sequence S as in Remark 3, there exists an index k such that p_k is pre-simple.*

Proof: Suppose, by absurd, that there exists a sequence S as in Remark 3 such that p_i is not pre-simple for every $i \in \mathbb{N}$. If there exists an index $s \in \mathbb{N}$ such that π_i is dicritical for $i \geq s$, then I_{p_i} decreases infinitely, which is not possible. Therefore we may assume that, apart from finitely many indices, π_i is nondicritical. Then I_{p_i} must stabilize; thus except for finitely many indices, we may assume $I_{p_i} = I_{p_{i+1}}$ for all i. Due to Lemma 5, this implies that for all i, $e_{p_i}(E^i) = 1$ and $m = 1$. Then the points p_i are the points of intersection of the strict transform (at each stage M_i) of a nonsingular formal curve $\Gamma \subset M$ (the construction of Γ is similar to the argument used in the proof of Proposition 1). By Lemma 3, Γ is invariant by $\mathcal{F}$ and since the blow-ups π_i are nondicritical, it follows that the p_i are pre-simple singularities. We arrive to an absurd and we are done.

□

This concludes the first part of the proof of Theorem 2, which is to get pre-simple singularities. Now we move on to the passage from pre-simple to simple.

Proposition 6 *Let $p \in M$ be a pre-simple singularity of $\mathcal{F}$. There exits a morphism $\pi = M_N \to M = M_0$, composition of finitely many blow-ups centered at points, such that all the singularities of $\pi^*\mathcal{F} = \mathcal{F}_N$ are simple.*

Proof: Put

$$Res(\mathcal{F},p) = \begin{cases} 0 & \text{if } \alpha \notin \mathbb{Q}_{>0} \\ m+n & \text{if } \alpha = \frac{m}{n} \text{ is an irreducible fraction, } m,n \in \mathbb{Z}_{>0} \end{cases} ,$$

where α is the quotient of the eigenvalues of the matrix $J_p(\omega; x, y)$. So a singularity q is simple if and only if $Res(\mathcal{F}, q) = 0$. The argument is that, after blowing-up a pre-simple singularity, if singularities which are not yet simple (and they must be pre-simple, due to the stability property) appear, then the residue strictly decreases. Since it cannot decrease infinitely, after finitely many blow-ups the residue of the singularities in the last divisor will be zero.

If p is a pre-simple but not simple singularity, after a linear change of coordinates the matrix $J_p(\omega; x, y)$ has one of the following forms:

1. $JD(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
2. $JD(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;
3. $JD(0,0) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$.

The first case corresponds to π_1 being a dicritical blow-up; in the divisor $D_1^1 = \pi_1^{-1}(p)$ there are no singularities, and the result follows.

In the second case, we will find only one singular point of $\mathcal{F}_1$, $p' \in D_1^1$, which is the origin of the first local chart $x = x', y = x'y'$. We have that

$$J_{p'}(\omega'; x', y') = \begin{pmatrix} 1 & 0 \\ \star & 0 \end{pmatrix} ,$$

and p' is, therefore, a simple singularity. The origin of the second local chart $x = x''y'', y = y''$ is not a singularity of $\mathcal{F}_1$.

Finally, in the third case, we will find two singularities on D_1^1, the origins of the local charts; call them p_0 and p_∞. The eigenvalues of the jacobian matrix of p_0 are $m, n-m$, whereas the eigenvalues of the jacobian matrix of p_∞ are $n, m-n$. So we have that

$$Res(\mathcal{F},p) = m+n > n = m+n-m = Res(\mathcal{F}_1, p_0) ,$$
$$Res(\mathcal{F},p) = m+n > m = n+m-n = Res(\mathcal{F}_1, p_\infty) .$$

□

1.5 Camacho-Sad's theorem

In [3], C. Camacho and P. Sad proved that every holomorphic foliation $\mathcal{F}$ of $(\mathbb{C}^2, \underline{0})$ admits an invariant analytic curve. If, during the reduction of singularities of $\mathcal{F}$, one component of the exceptional divisor happens to be dicritical, then each leaf of the final transform of $\mathcal{F}$ which intersects this component is projected onto an analytic curve; hence, in this case, $\mathcal{F}$ indeed admits infinitely many invariant analytic curves. In this section we will exhibit a method, due to J. Cano (see [9]), for constructing invariant analytic curves in the case of nondicritical foliations.

Let $\mathcal{F}$ be a foliation of $(\mathbb{C}^2, \underline{0})$ and let Γ be an invariant curve which is not singular. Given local coordinates x, y at the origin, we may assume that $\Gamma = \{y = 0\}$ and that $\mathcal{F}$ is generated by the 1-form

$$\omega = y\tilde{a}(x,y)dx + b(x,y)dy,\ \tilde{a}(x,y), b(x,y) \in \mathbb{C}\{x,y\}\ .$$

The ***index*** $I(\mathcal{F}, \Gamma; \underline{0})$ of $\mathcal{F}$ relative to Γ at $\underline{0} \in \mathbb{C}^2$ is defined by

$$I(\mathcal{F}, \Gamma; \underline{0}) = \text{residue at } \underline{0} \in \mathbb{C}^2 \text{ of } -\frac{\tilde{a}(x,0)}{b(x,0)}\ .$$

That is to say, if

$$-\frac{\tilde{a}(x,0)}{b(x,0)} = \sum c_i x^i \in \mathbb{C}\{x\}[x^{-1}]\ ,$$

then $I(\mathcal{F}, \Gamma; \underline{0}) = c_{-1}$. The index does not depend on the choice of coordinates nor on the choice of the generator ω of $\mathcal{F}$.

We are interested in the behavior of the index under blow-ups, and also on calculating directly the index at simple singularities. The proof of the following result is based on the classical Residue Theorem of one complex variable. For further details see [6].

Proposition 7 *Let* $\pi : M' \to M_0 = (\mathbb{C}^2, \underline{0})$ *be a blow-up centered at the origin,* $E = \pi^{-1}(\underline{0}) \subset M'$ *be the exceptional divisor,* $\mathcal{F}' = \pi^*\mathcal{F}$ *be the strict transform of* $\mathcal{F}$ *and* Γ' *be the strict transform of* Γ. *Suppose* π *is not dicritical. Then*

- $\sum_{p' \in E} I(\mathcal{F}', E; p') = -1$
- $I(\mathcal{F}', \Gamma'; q') = I(\mathcal{F}, \Gamma; \underline{0}) - 1$ *where* $q' = \Gamma' \cap E$.

Now let's suppose $\underline{0} \in \mathbb{C}^2$ is a simple singularity of the foliation $\mathcal{F}$. We recall (Remark 1) that there are two formal invariant curves at the origin, Γ_x and Γ_y, which are tangent to $L_x = T_{\underline{0}}(x = 0)$ and $L_y = T_{\underline{0}}(y = 0)$ respectively. If $\mu \neq 0$ then L_x is a strong proper direction, and it is weak otherwise; the same for L_y. If L_x is strong, then Γ_x is convergent.

Lemma 8 *In the situation above, we have that*

- *if L_x is a weak direction, then $I(\mathcal{F}, \Gamma_y; \underline{0}) = 0$;*
- *if $\lambda\mu \neq 0$, then $I(\mathcal{F}, \Gamma_x; \underline{0}) \cdot I(\mathcal{F}, \Gamma_y; \underline{0}) = 1$.*

Proof: Firstly let's suppose L_x is a weak direction; thus $\mu = 0$ and the origin is a saddle-node singularity. Thus L_y is necessarily strong, and therefore Γ_y is convergent. Choosing local coordinates x, y, we may write $\Gamma_y = (y = 0)$ and $\mathcal{F}$ is generated by the 1-form

$$\omega = y\tilde{a}(x, y)dx + b(x, y)dy$$

where $\tilde{a}(0, 0) = 0$ and $b(x, 0) = xu(x), u(0) \neq 0$. Thus we can write

$$\tilde{a}(x, y) = \sum_{i+j\geq 1} a_{ij}x^i y^j = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots$$

$$b(x, y) = xu(x) + y(\cdots) \ .$$

So

$$\frac{-\tilde{a}(x, 0)}{b(x, 0)} = \frac{-(a_{10}x + a_{20}x^2 + \cdots + a_{k0}x^k + \cdots)}{xu(x)} = -\frac{a_{10}}{u(x)} - \frac{x}{u(x)}(\cdots) \in \mathbb{C}\{x\} \ ,$$

and therefore $I(\mathcal{F}, \Gamma_y; \underline{0}) = 0$. That is to say, the index of the "strong" curve of a saddle-node singularity is zero.

Now suppose $\lambda\mu \neq 0$: then the origin is a complex hyperbolic singularity and both directions L_x, L_y are strong, hence both curves Γ_x and Γ_y are convergent. Thus we may assume that $\Gamma_x = (x = 0)$, $\Gamma_y = (y = 0)$ and we can write ω as follows:

$$\omega = y(-\mu + a_1(x, y))dx + x(\lambda + b_1(x, y))dy$$

where $a_1(0, 0) = b_1(0, 0) = 0$. Writing

$$a_1(x, y) = \sum_{i+j\geq 1} a_{ij}x^i y^j = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots$$

$$b_1(x,y) = \sum_{i+j\geq 1} b_{ij}x^i y^j = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \cdots \ ,$$

we have that

$$\frac{-(-\mu + a_1(x,0))}{x(\lambda + b_1(x,0))} = \frac{\mu - (a_{10}x + a_{20}x^2 + \cdots + a_{k0}x^k + \cdots)}{\lambda x + x(b_{10}x + b_{20}x^2 + \cdots + b_{k0}x^k + \cdots)}$$

and therefore $I(\mathcal{F}, \Gamma_y; \underline{0}) = c_{-1} = \mu/\lambda$. On the other hand we have that

$$\frac{-(\lambda + b_1(0,y))}{y(-\mu + a_1(0,y))} = \frac{\lambda - (b_{01}y + b_{02}y^2 + \cdots + b_{0k}y^k + \cdots)}{\mu y - y(a_{01}y + a_{02}y^2 + \cdots + a_{0k}y^k + \cdots)}$$

and therefore $I(\mathcal{F}, \Gamma_x; \underline{0}) = c_{-1} = \lambda/\mu$, and the second assertion follows.

□

We are especially interested in calculating the index of a foliation at a nodal singularity. Suppose that $\underline{0} \in \mathbb{C}^2$ is a nodal point of $\mathcal{F}$. Thus choosing local coordinates x, y, the local generator ω can be written as follows (Definition 2):

$$\omega_p = \lambda_1 y dx + \lambda_2 x dy$$

where $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{R}_{<0} \setminus \mathbb{Q}_{<0}$. Thus $I(\mathcal{F}, \Gamma_x; \underline{0}) = -\lambda_2/\lambda_1 \in \mathbb{R}_{>0}$ and $I(\mathcal{F}, \Gamma_y; \underline{0}) = -\lambda_1/\lambda_2 \in \mathbb{R}_{>0}$.

Remark 5 Let $\mathcal{F}$ be a foliation on $M = (\mathbb{C}^2, \underline{0})$ and assume that the blow-up $\pi : M_1 \to M_0 = (\mathbb{C}^2, \underline{0})$ is not dicritical. If there is a nodal singularity $p \in \pi_1^{-1}(\underline{0}) = E^1$ of $\mathcal{F}_1 = \pi_1^*\mathcal{F}$, there exists another singular point $q \neq p$, $q \in E^1$, such that q is not a nodal singularity of $\mathcal{F}'$. Indeed, we have just seen that $I(\mathcal{F}_1, E^1; p) \in \mathbb{R}_{>0}$; since

$$\sum_{p' \in E^1} I(\mathcal{F}_1, E^1; p') = -1 \ ,$$

there must exist a point $q \in E^1$ such that $I(\mathcal{F}_1, E^1; q) \notin \mathbb{R}_{>0}$. Hence q is not a nodal singularity of $\mathcal{F}_1$.

Now we will exhibit the algorithm for constructing invariant curves of a foliation $\mathcal{F}$ of $(\mathbb{C}^2, \underline{0})$. As we have remarked before, we will suppose that $\mathcal{F}$ is not dicritical, otherwise we would find infinitely many invariant curves.

Definition 4 (J. Cano) *Let $\mathcal{F}$ be a foliation over an analytic manifold M of dimension two; consider a normal crossings divisor $E \subset M$ without dicritical components of $\mathcal{F}$. We will say the pair $(\mathcal{F}, E)$ satisfies property $(\star)$ at a point $p \in M$ if one of the following properties is satisfied:*

$(\star) - 1$ *The point p belongs to only one irreducible component of E (that is to say, $e_p(E) = 1$) and*

$$I(\mathcal{F}, E; p) \notin \mathbb{Q}_{\geq 0} \ .$$

$(\star) - 2$ *The point p belongs to two irreducible components D_1, D_2 of E (that is to say, $e_p(E) = 2$) and there exists a real number $a > 0$ such that*

$$I(\mathcal{F}, D_1; p) \in \mathbb{Q}_{\leq -a} \ ,$$

$$I(\mathcal{F}, D_2; p) \notin \mathbb{Q}_{\geq -1/a} \ .$$

Theorem 4 *Suppose $(\mathcal{F}, E)$ satisfies property $(\star)$ at the point $p \in M$. Let $\pi : M' \to M$ be the blow-up centered at p and suppose π is not dicritical. Let $\mathcal{F}' = \pi^*\mathcal{F}$, $D = \pi^{-1}(p)$, $E' = D \cup \pi^{-1}(E)$. Then there exists a point $p' \in D$ such that $(\mathcal{F}', E')$ satisfies property $(\star)$ at p'.*

Proof: (see [9]). Suppose, by absurd, that the assertion is false. Firstly let's consider the case $e_p(E) = 1$: let D_1 be the irreducible component of E which contains p and D_1' be its transform by π. Since the pair $(\mathcal{F}, E)$ satisfies property $(\star)$ at p, we have that $I(\mathcal{F}, E; p) \notin \mathbb{Q}_{\geq 0}$. Let $q = D \cap D_1'$. Then for every point $p' \in D$, $p' \neq q$, we have that $e_{p'}(E') = 1$ and, since $(\mathcal{F}', E')$ does not satisfy property $(\star)$ at p', we have that $I(\mathcal{F}', D; p') \in \mathbb{Q}_{\geq 0}$. Due to Proposition 7 it follows that

$$I(\mathcal{F}', D; q) = -1 - \sum_{p' \neq q} I(\mathcal{F}', D; p') \in \mathbb{Q}_{\leq -1} \ .$$

However, since $(\mathcal{F}', E')$ does not satisfy property $(\star)$ at q, it follows that

$$I(\mathcal{F}', D_1'; q) \in \mathbb{Q}_{\geq -1} \text{ (take } a = 1) \ .$$

But also due to Proposition 7 we have that

$$I(\mathcal{F}, E; p) = I(\mathcal{F}', D_1'; q) + 1 \in \mathbb{Q}_{\geq 0} \ ,$$

which is an absurd and we are done.

Now suppose $e_p(E) = 2$: $p \in D_1 \cap D_2$, $D_1, D_2 \subset E$. So there exists a real number $a > 0$ such that

$$I(\mathcal{F}, D_1; p) \in \mathbb{Q}_{\leq -a} \ ,$$

$$I(\mathcal{F}, D_2; p) \notin \mathbb{Q}_{\geq -1/a} \ .$$

Let D_1', D_2' be the transforms of D_1, D_2 respectively and let $q_1 = D_1' \cap D$, $q_2 = D_2' \cap D$. So, like in the other case, for every $p' \in D$, $p' \neq q_1, q_2$ we have by hypothesis that $I(\mathcal{F}', D; p') \in \mathbb{Q}_{\geq 0}$. Therefore

$$I(\mathcal{F}', D; q_1) + I(\mathcal{F}', D; q_2) = -1 - \sum_{p' \neq q_1, q_2} I(\mathcal{F}', D; p') \in \mathbb{Q}_{\leq -1} \ .$$

However, we have that

$$I(\mathcal{F}', D_1'; q_1) = I(\mathcal{F}, D_1; p) - 1 \in \mathbb{Q}_{-(a+1)} \ ;$$

since $(\mathcal{F}', E')$ does not satisfy property $(\star)$ at q_1, it follows that

$$I(\mathcal{F}', D; q_1) \in \mathbb{Q}_{\geq -1/(a+1)} \ .$$

This implies that

$$I(\mathcal{F}', D; q_2) = \Big(I(\mathcal{F}', D; q_1) + I(\mathcal{F}', D; q_2)\Big) - I(\mathcal{F}', D; q_1) \in \mathbb{Q}_{\leq -a/(a+1)} \ .$$

But since $(\mathcal{F}', E')$ does not satisfy property $(\star)$ at q_2, it follows that

$$I(\mathcal{F}', D_2'; q_2) \in \mathbb{Q}_{-(a+1)/a} \ .$$

However, this implies that

$$I(\mathcal{F}, D_2; p) = I(\mathcal{F}', D_2'; q_2) + 1 \in \mathbb{Q}_{\geq -1/a} \ ,$$

which is an absurd and we are done.

□

Note that if $p \in E$ is a simple singularity of $\mathcal{F}$ such that $e_p(E_{inv}) = 2$, then $(\mathcal{F}, E)$ does not satisfy property $(\star)$ at p. Indeed, suppose $p = D_1 \cap D_2$ where D_1, D_2 are irreducible components of E invariant by $\mathcal{F}$. Then if $I(\mathcal{F}, D_1; p) \cdot I(\mathcal{F}, D_2; p) = 0$ (that is to say, p is a saddle-node singularity), clearly $(\mathcal{F}, E)$ does not satisfy property $(\star) - 2$ at p. In the case $I(\mathcal{F}, D_1; p) \cdot I(\mathcal{F}, D_2; p) = 1$, then p is a complex hyperbolic singularity and $I(\mathcal{F}, D_1; p) = \mu/\lambda$, $I(\mathcal{F}, D_2; p) = \lambda/\mu$ (Lemma 8). So if $I(\mathcal{F}, D_1; p) \in \mathbb{Q}_{\leq -a}$ then $I(\mathcal{F}, D_2; p) \in \mathbb{Q}_{\geq -1/a}$ and once again $(\mathcal{F}, E)$ does not satisfy property $(\star) - 2$ at p. Hence if p is a simple singularity of $\mathcal{F}$ such that $(\mathcal{F}, E)$ satisfies property $(\star)$ at p then $e_p(E_{inv}) = 1$. In this case we have

$$I(\mathcal{F}, E; p) \neq 0 \ .$$

Assume we can choose local coordinates x, y at p such that $p = (x, y) = (0, 0)$ and $E = (y = 0)$. By Lemma 8, $I(\mathcal{F}, E; p) \neq 0$ implies that $L_x = (x = 0)$ is not a weak direction; that is to say, L_x is a strong direction and therefore the formal curve Γ_x at p (which exists because p is a simple singularity) is convergent. Thus we may write $\Gamma_x = (x = 0)$.

Now suppose we have a nondicritical foliation $\mathcal{F}$ of $(\mathbb{C}^2, \underline{0})$; we wish to construct an analytic curve Γ which is invariant for $\mathcal{F}$. Let $\pi_1 : M_1 \to M_0 = (\mathbb{C}^2, \underline{0})$ be the blow-up of the origin, $\mathcal{F}_1 = \pi^* \mathcal{F}$, $E = \pi_1^{-1}(\underline{0})$. Since

$$\sum_{p \in E} I(\mathcal{F}_1, E; p) = -1 \ ,$$

there exists a point $p_1 \in E$ such that $(\mathcal{F}, E)$ satisfies property $(\star)$ at p_1. Note that $(\mathcal{F}, E)$ in fact satisfies property $(\star) - 1$, since $e_{p_1}(E) = 1$. If p_1 is simple, we have just seen that there exists a convergent analytic curve Γ_1 transversal to E and invariant for $\mathcal{F}$; thus $\Gamma = \pi_1(\Gamma_1)$ and we are done. If p_1 is not simple, we blow-up p_1. By the theorem of reduction of singularities in dimension two (Theorem 2), after a finite number of blow-ups we will find a point p_k which is simple and such that $(\mathcal{F}_k, E_k)$ satisfies property $(\star)$ at p_k. Most importantly, as remarked above, we must have $e_{p_k}(E_k) = 1$; thus we find an analytic curve Γ_k which is invariant for $\mathcal{F}_k$ and is projected onto an analytic curve Γ invariant for $\mathcal{F}$.

In [21] the authors give a proof of a stronger version of Camacho-Sad's theorem.

1.6 Dimensional type

Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, \underline{0})$.

Definition 5 *We say that* $\mathcal{F}$ has dimensional type $\leq n - k$ at the origin *if there are* k *germs of nonsingular vector fields* $\xi_1, \xi_2, \ldots, \xi_k$ *tangent to* $\mathcal{F}$ *such that*

$$\xi_1(0), \xi_2(0), \ldots, \xi_k(0)$$

are $\mathbb{C}$*-linearly independent tangent vectors. In this case there is a submersion*

$$\phi : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}^{n-k}, \underline{0})$$

and a codimension one foliation $\mathcal{G}$ *on* $(\mathbb{C}^{n-k}, \underline{0})$ *such that* $\mathcal{F} = \phi^* \mathcal{G}$.

In other words, there are local coordinates $x_1, x_2, \ldots, x_n$ at the origin $\underline{0} \in \mathbb{C}^n$ such that $\mathcal{F}$ is generated by the integrable 1-form

$$\omega = \sum_{i=1}^{n-k} a_i(x_1, x_2, \ldots, x_k) dx_i \ .$$

We say that τ *is the dimensional type of* $\mathcal{F}$ if $\tau = n - k$ where k is the maximum possible with the above property.

Due to Frobenius Theorem, we have that $\tau = 1$ if and only if $\mathcal{F}$ is nonsingular. We remark that the dimensional type of $\mathcal{F}$ has been defined as the largest number of variables needed to generate $\mathcal{F}$ locally at the origin; we could also say, for short, that τ is the *dimensional type of the origin*. Moreover, for every point p near it, the dimensional type of p (or equivalently, of $\mathcal{F}$ at p) is smaller or equal to the dimensional type of the origin $\underline{0} \in \mathbb{C}^n$. Hence we may interchangeably say "dimensional type of a point" to mean "dimensional type of the foliation at the point". The dimensional type of the foliation is simply the largest number found when computing the dimensional type of each point individually.

For instance, if $n = 2$, then every singular foliation $\mathcal{F}$ has dimensional type two. If $n = 3$, a singular foliation $\mathcal{F}$ of $(\mathbb{C}^3, \underline{0})$ has dimensional type two if there are local coordinates x, y, z at the origin such that $\mathcal{F}$ is generated by a 1-form

$$\omega = a(x, y)dx + b(x, y)dy \ .$$

That is to say, $\mathcal{F}$ is a cylinder in $(\mathbb{C}^3, \underline{0})$ over the foliation $\mathcal{G}$ of $(\mathbb{C}^2, \underline{0})$ given by the 1-form

$$\omega_{\mathcal{G}} = a(x, y)dx + b(x, y)dy \ .$$

In this case, the z-axis $Z = \{x = y = 0\}$ is contained in the singular locus of $\mathcal{F}$. Furthermore, for every two-dimensional section Δ_p transversal to Z at a point $p = (0, 0, p)$ we have that p is a singularity in dimension two of the induced two-dimensional foliation $\mathcal{F}|_{\Delta_p}$. Indeed, we have that $\mathcal{F}|_{\Delta_p} = \mathcal{G}$ for every $p \in Z$. Note that Z may be contained in one or two irreducible components of a normal crossings divisor $E \subset (\mathbb{C}^3, \underline{0})$ which are invariant for the foliation $\mathcal{F}$. That is to say, we also include the possibility of existence of a normal crossings divisor E such that $Z \subset E$. For instance, in the local coordinates x, y, z we may have $E = \{x = 0\}$, $E = \{y = 0\}$ or $E = \{xy = 0\}$. So $\mathcal{F}$ has dimensional type two, and every point of Z also has dimensional type two.

1.7 Simple singularities in dimension $n \geq 2$

Let $\mathcal{F}$ be a codimension one foliation in $M = (\mathbb{C}^n, \underline{0})$, $n \geq 2$ and $E \subset M$ a normal crossings divisor. Let τ be the dimensional type of the origin. If $\tau = 1$, as a consequence of Frobenius Theorem, the origin is a regular point of $\mathcal{F}$. So suppose $\tau \geq 2$.

Definition 6 (CH-simple) *We will say that the origin is a* Complex Hyperbolic Simple Point of $\mathcal{F}$ *if there exist convergent local coordinates* $x_1, x_2, \ldots, x_n$ *such that* $\mathcal{F}$ *is given by* $\omega = 0$ *where*

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + a_i(x_1, x_2, \ldots, x_\tau)) \frac{dx_i}{x_i}$$

with $a_i(0) = 0$ *for all* $i = 1, 2, \ldots, \tau$ *and* $\sum \lambda_i m_i \neq 0$ *if* $\underline{m} \neq 0, m_i \in \mathbb{Z}_{>0}$.

We will decompose the divisor $E \subset M$ as follows:

$$E = E_{inv} \cup E_{dic}$$

where E_{inv} is the union of the irreducible components of E invariant by $\mathcal{F}$ and E_{dic} is the union of the components of E that are generically transversal to $\mathcal{F}$ (dicritical components). The origin is *CH-simple for the pair* $\mathcal{F}, E$ (or "adapted to E") if and only if, in addition, the coordinates $x_1, x_2, \ldots, x_n$ may be chosen in such a way that

$$E \subset \left(\prod_{i=1}^{n} x_i = 0 \right); \; E_{dic} \subset \left(\prod_{i=\tau+1}^{n} x_i = 0 \right); \qquad (2)$$

$$\left(\prod_{i=1}^{\tau-1} x_i = 0 \right) \subset E_{inv} \subset \left(\prod_{i=1}^{\tau} x_i = 0 \right) .$$

In the case that $E_{inv} = \left(\prod_{i=1}^{\tau-1} x_i = 0 \right)$ we say that we have a *trace CH-simple point*. In this case we find an invariant hypersurface $H = (x_\tau = 0)$ such that $E \cup H$ is a normal crossings divisor. If $E_{inv} = \left(\prod_{i=1}^{\tau} x_i = 0 \right)$ we say that we have a *CH-simple corner*.

Remark 6 The hypersurfaces $(x_i = 0)$, $i = 1, 2, \ldots, \tau$ are the only invariant hypersurfaces at a CH-simple singularity. This can be viewed by performing a single blow-up of the origin and considering a plane section. On the other hand, $(x_j = 0)$

are generically transversal to $\mathcal{F}$ for $j = \tau+1, \tau+2, \ldots, n$. Note also that the singular locus is given by

$$\text{Sing}\,\mathcal{F} = \bigcup_{1\leq i<j\leq\tau} (x_i = x_j = 0)\ .$$

All the singularities around a CH-simple point are also CH-simple. Note that there are no saddle-nodes with $\tau = 2$ around a CH-simple singularity. There is a more general definition of simple singularity (see [5], [4]) that allows the existence of two-dimensional saddle-nodes. Although we are only interested in CH-simple singularities, we include the general definition for the sake of completeness.

Definition 7 *We say that the origin is a* simple singularity *if it has one of the following types:*

A *There exist formal local coordinates $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\tau$ and a function*

$$\varphi : u = \hat{x}_1^{\lambda_1}\hat{x}_2^{\lambda_2}\cdots\hat{x}_\tau^{\lambda_\tau} \ \textit{with} \ \sum \lambda_i m_i \neq 0 \ \textit{if} \ \underline{m} \neq 0,\ m_i \in \mathbb{Z}_{>0}\ ,$$

such that $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega = \varphi^*\alpha\ , \quad \alpha = \frac{du}{u}\ .$$

That is to say,

$$\omega = \sum_{i=1}^{\tau} \lambda_i \frac{d\hat{x}_i}{\hat{x}_i}\ .$$

B *There exist formal local coordinates $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\tau$ and a function*

$$\varphi : \left\{ \begin{array}{l} u = \hat{x}_1^{p_1}\hat{x}_2^{p_2}\cdots\hat{x}_k^{p_k},\ k \leq \tau \\ v = \hat{x}_2^{\lambda_2}\hat{x}_3^{\lambda_3}\cdots\hat{x}_\tau^{\lambda_\tau} \end{array} \right. \quad \textit{with} \ \sum_{i\geq k+1}^{\tau} \lambda_i m_i \neq \ \ \textit{if} \ \underline{m} \neq 0,\ m_i \in \mathbb{Z}_{>0}\ ,$$

such that $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega = \varphi^*\alpha\ , \quad \alpha = \frac{du}{u} + \psi(u)\frac{dv}{v},\ \psi(0) = 0\ .$$

That is to say,

$$\omega = \sum_{i=1}^{k} p_i \frac{d\hat{x}_i}{\hat{x}_i} + \psi(\hat{x}_1^{p_1}\hat{x}_2^{p_2}\cdots\hat{x}_k^{p_k}) \cdot \sum_{i=2}^{\tau} \lambda_i \frac{d\hat{x}_i}{\hat{x}_i}\ .$$

Note also that $\prod_{i=1}^{\tau} x_i = 0$ are the only invariant formal hyperplanes at a simple singularity. As in the case CH-simple, we say that the singularity is *adapted to E* if the formal coordinates may be chosen in such a way that we have (2) and we take a similar definition for corner and trace points.

1.8 Reduction of singularities in dimension three

The reduction of singularities in dimension two has been generalized in dimension three by F. Cano and D. Cerveau in the nondicritical case in [5] and later in the general case by F. Cano in [4]. The main result of [4] is the following

Theorem 5 *Let X be a three-dimensional germ, around a compact analytic set, of nonsingular complex analytic space. Let $\mathcal{F}$ be a holomorphic singular foliation of codimension one and D be a normal crossings divisor on X. Then there is a morphism $\pi : X' \to X$ composition of a finite sequence of blow-ups with nonsingular centers such that:*

(1) *Each center is invariant by the strict transform of $\mathcal{F}$ and has normal crossings with the total transform of D.*

(2) *The strict transform $\mathcal{F}'$ of $\mathcal{F}$ in X' has normal crossings with the total transform D' of D and it has at most simple singularities adapted to D'.*

Essentially, Theorem 5 asserts that given a holomorphic codimension one foliation $\mathcal{F}$ of $(\mathbb{C}^3, \underline{0})$ it is possible to find a morphism $\pi : M' \to M = (\mathbb{C}^3, \underline{0})$, composition of a finite number of blow-ups with adequate centers, such that every point of $\mathcal{F}' = \pi^*\mathcal{F}$ is *simple* as described in Section 1.7.

We will introduce the notation that will be used throughout the text. Note that it is very similar to the notation given in Remark 3.

Remark 7 Let $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N : M_N \to M_0 = (\mathbb{C}^3, 0)$ be a morphism of reduction of singularities of $\mathcal{F}$,

$$(\mathbb{C}^3, 0) = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N \ . \tag{3}$$

For $1 \le s \le N$, we will denote:

- $\sigma_s = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_s$
- $\rho_s = \pi_{s+1} \circ \pi_{s+2} \circ \cdots \circ \pi_N$
- Y_{s-1} is the center of π_s
- $D_s^s = \pi_s^{-1}(Y_{s-1})$
- D_i^s is the strict transform by π_s of D_i^{s-1}, $i < s$
- $E^s = D_1^s \cup D_2^s \cup \cdots \cup D_s^s$ is the exceptional divisor in each step, $E^s \subset M_s$
- $\mathcal{F}_1 = \pi_1^*\mathcal{F} \quad , \ \ldots \ , \quad \mathcal{F}_s = \pi_s^*\mathcal{F}_{s-1} \quad , \ \ldots \ , \quad \mathcal{F}_N = \pi_N^*\mathcal{F}_{N-1} = \pi^*\mathcal{F}$.

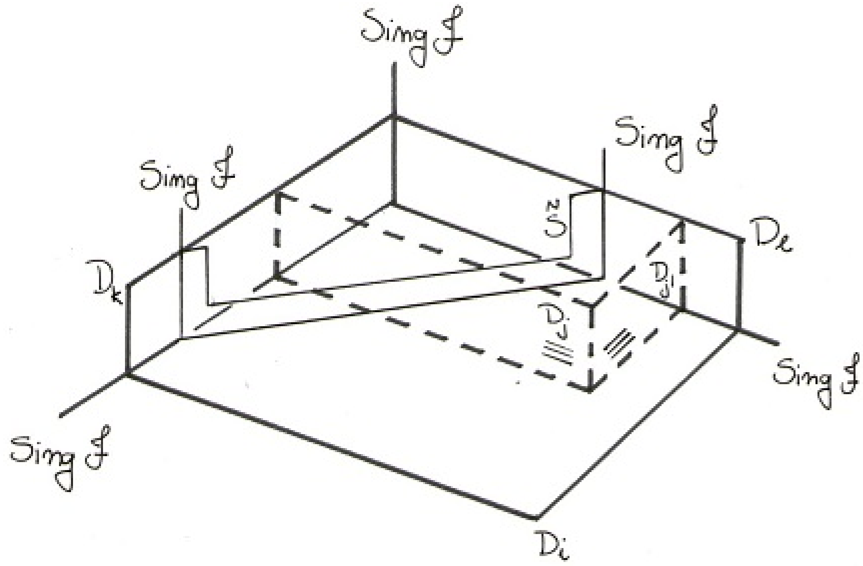


Figure 2: Example of arrangement of the irreducible components of the divisor. D_i, D_k, D_l are invariant components, D_j and $D_{j'}$ are dicritical components and $\tilde{S}$ is a convergent separatrix.

As in the two-dimensional case, we denote by $e_p(E)$ the number of irreducible components of the normal crossings divisor E through p. We remark that in each intermediate ambient space M_s, the irreducible components of the divisor E^s and Y_s (the center of π_{s+1}) have normal crossings. We may write $E^s = E^s_{inv} \cup E^s_{dic}$ as before. Theorem 5 assures that for every point $p \in M_N$, we have that $1 \leq \tau_p \leq 3$, $0 \leq e_p(E^N) \leq 3$ and the following inequality holds:

$$\tau_p - 1 \leq e_p(E^N_{inv}) \leq \tau_p \ .$$

Hence if $e_p(E^N_{inv}) = \tau_p - 1$, p is a *trace* point; and it is a *corner* in the case $e_p(E^N_{inv}) = \tau_p$.

The set $\sigma_s^{-1}(\underline{0}) \subset M_s$ is a compact analytic subset that can be described as follows. First, if $s = 0$, we have $\sigma_0^{-1}(\underline{0}) = \{\underline{0}\}$. If $s \geq 1$, we have that $\sigma_s^{-1}(\underline{0}) \subset E^s$

and

$$\sigma_s^{-1}(\underline{0}) = \pi_s^{-1}(\sigma_{s-1}^{-1}(\underline{0})) \ .$$

Moreover if $Y_{s-1} = \{p\}$ is a point, we have that $p \in \sigma_{s-1}^{-1}(\underline{0})$ and thus $D_s^s \subset \sigma_s^{-1}(\underline{0})$. If Y_{s-1} is a curve, we have two possibilities:

a) Y_{s-1} is a compact curve, $Y_{s-1} \subset \sigma_{s-1}^{-1}(\underline{0})$. In this case $D_s^s = \pi_s^{-1}(Y_{s-1})$ is a compact divisor contained in $\sigma_s^{-1}(\underline{0})$.

b) Y_{s-1} is a germ of curve at a single point $Y_{s-1} \cap \sigma_{s-1}^{-1}(\underline{0}) = q$. In this case $\pi_s^{-1}(q)$ is a compact curve contained in D_s^s and D_s^s is a germ of hypersurface around $\pi_s^{-1}(q)$. In particular D_s^s is a noncompact component of E^s.

As we will see further, in this work we will never encounter possibility a).

1.9 The argument of Cano-Cerveau

In [5], F. Cano and D. Cerveau exhibit a method for constructing an invariant germ of surface once you have a reduction of singularities as in Theorem 5 and Remark 7. It essentially says that, if there are no dicritical components in the exceptional divisor $E^N \subset M_N$, it is possible to continue each germ of invariant surface that rests on a curve of the singular locus whose points are trace singularities of dimensional type two and thus construct, by projection, an invariant germ of surface at the origin $\underline{0} \in \mathbb{C}^3$.

The main result of [5] is the following

Theorem 6 (Existence of separatrices in dimension three) *If $\mathcal{F}$ is a germ of holomorphic singular foliation of codimension one over $(\mathbb{C}^3, \underline{0})$ given by $\omega = 0$ then one of the following properties are satisfied:*

(i) *$\mathcal{F}$ has a germ of invariant surface.*

(ii) *There is an analytic mapping $\sigma : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^3, \underline{0})$ such that $\sigma^*\omega$ is not identically zero and the foliation given by $\sigma^*\omega = 0$ has infinitely many analytic solutions through the origin.*

We shall give here a brief description of Cano-Cerveau's argument. Suppose $\mathcal{F}$ does not fulfill (ii). Then after the reduction of singularities we do not encounter dicritical irreducible components in $E^N \subset M_N$ that project onto the origin (compact components). Consider the set

$$\text{Tr Sing } \mathcal{F}_N = \bigcup \{Y;\ Y \text{ is an irreducible component of Sing } \mathcal{F}_N \text{ which is generically contained in only one irreducible component of } E^N\}.$$

Thus Tr Sing $\mathcal{F}_N$ is a union of curves such that generically, every point of each curve is a trace singularity of dimensional type two. Furthermore, it is clear that each $Y \subset$ Tr Sing $\mathcal{F}_N$ has normal crossings with E^N and satisfies the following properties:

$-Y$ is not a singular curve;
$-$ Either $Y \subset \pi^{-1}(\underline{0})$ and in this case it is a global compact curve; or $Y \cap \pi^{-1}(\underline{0}) = p_Y$ is one point and Y is a germ at p_Y, in which case $\pi(Y)$ is a germ of curve at the origin $\underline{0} \in \mathbb{C}^3$;
$-$ If $Y \neq Y'$, then either $Y \cap Y' = \emptyset$ or $Y \cap Y' = p_{Y,Y'}$ is a single point.

Let Γ be a connected component of Tr Sing $\mathcal{F}_N$: $\Gamma = Y_1 \cup Y_2 \cup \cdots \cup Y_l$ where the generical point of each Y_i is a trace singularity of dimensional type two, and the intersection points $Y_i \cap Y_j$ are trace singularities of dimensional type three. Then for each point $p \in \Gamma$ there exists a formal separatrix S_p; let us assume S_p is convergent. By analytic triviality we may continue in an analytic way S_p to the points $q \in \Gamma$ such that $e_q(E^N_{inv}) \leq e_p(E^N_{inv})$. The difficulty lies in continuing S_p to the points q' where $e_{q'}(E^N_{inv}) = 2$, that is to say, to the points of Γ which are trace singularities with dimensional type three. Nevertheless, it can also be done. Therefore we can "glue" the local separatrices S_p in order to obtain a closed hypersurface S_Γ which gives, locally, a separatrix at each point of Γ. Due to the fact that the reduction of singularities (3) is a proper morphism, S_Γ is projected to a convergent germ of separatrix $S \subset (\mathbb{C}^3, \underline{0})$ of the foliation $\mathcal{F}$.

It remains to show that there is at least one connected component Γ of Tr Sing $\mathcal{F}_N$ that supports a convergent separatrix as above. Let Δ be a non-degenerate two-dimensional section of $\mathcal{F}$. Such a section exists due to the Transversality Theorem of [19]. So $\mathcal{F}|_\Delta$ is a codimension one foliation in a two dimension ambient space such that the origin $\underline{0} \in \Delta$ is an isolated singularity. Since it is the restriction of a nondicritical foliation, $\mathcal{F}|_\Delta$ is also nondicritical in dimension two. By [3], there exists at least one convergent separatrix γ of $\mathcal{F}|_\Delta$. So γ is a nonsingular invariant curve of $\mathcal{F}$ that is not contained in Sing $\mathcal{F}$. We have the following

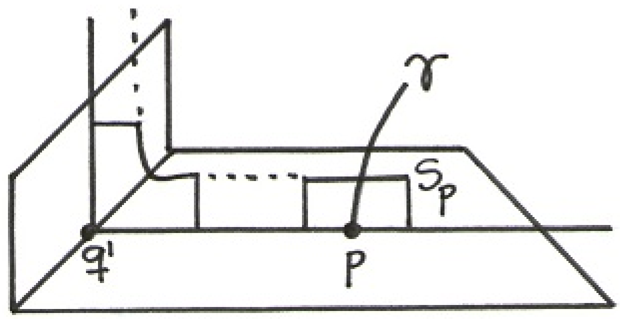


Figure 3: The argument of Cano-Cerveau

Lemma 9 [5] *The strict transform of* γ *under the reduction of singularities of* $\mathcal{F}$*,* $\gamma_N = \pi^*\gamma$*, is nonsingular, not contained in Sing* $\mathcal{F}_N$ *and transversal to* E^N *at a point* $p \in E^N$ *such that* $e_p(E^N_{inv}) = 1$*.*

With this lemma we conclude that the set Tr Sing $\mathcal{F}_N$ is not empty by showing that the final transform of γ intersects E^N at a point p which is a trace singularity of dimensional type two; thus we find a curve contained in Tr Sing $\mathcal{F}_N$ passing through it (see Figure 3).

The presence of a compact dicritical component can prevent the extension process of constructing the separatrix $S \subset (\mathbb{C}^3, \underline{0})$. For instance, in Jouanolou's example (see [12]), it is possible to construct a conic foliation $\mathcal{F}$ of $(\mathbb{C}^3, \underline{0})$ such that after a single blow-up centered at the origin we obtain only simple singularities, and the exceptional divisor - which only has one compact component - is generically transversal to the strict transform $\mathcal{F}'$. Let $\mathcal{G}$ be a codimension one foliation of $(\mathbb{C}^2, \underline{0})$ with only simple singularities and which has no invariant curves. We may build $\mathcal{F}$ so that the intersection of $\mathcal{F}'$ and the exceptional divisor is precisely the foliation $\mathcal{G}$. Since $\mathcal{G}$ has no invariant curves and $\mathcal{F}$ is a conic foliation of $(\mathbb{C}^3, \underline{0})$, we have that $\mathcal{F}$ has no invariant surfaces. The foliation $\mathcal{F}$ is given by the differential 1-form

$$\omega = (x^m y - z^{m+1})dx + (y^m z - x^{m+1})dy + (z^m x - y^{m+1})dz, \ m \geq 2 \ .$$

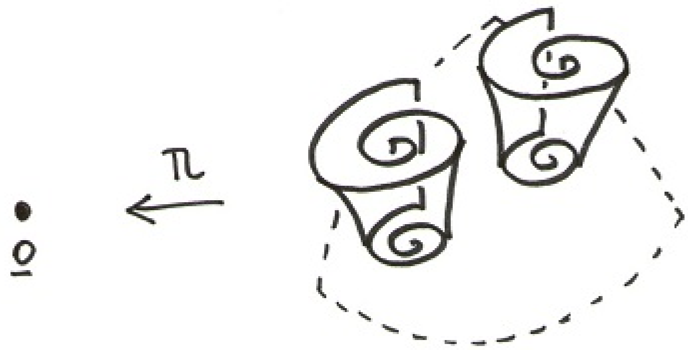


Figure 4: Jouanolou's example

This is the primary example of a holomorphic codimension one foliation in ambient space with dimension higher than two that has no invariant hypersurfaces.

However, there are other situations in which Cano-Cerveau's argument works even if there are dicritical irreducible components in the exceptional divisor. For instance, if Sing $\mathcal{F}$ has codimension two, in ambient space dimension three this means that Sing $\mathcal{F}$ is a union of germs of curves at $\underline{0} \in \mathbb{C}^3$. If all dicritical irreducible components generated during the reduction of singularities of $\mathcal{F}$ are projected onto these germs of curves, Cano-Cerveau's argument is still valid. That is to say, if the dicritical components of the divisor are not compact, there is no risk of losing the compactness of the prolongation of the local invariant surfaces S_p when intersecting with these components, since the intersection of S_p and a dicritical component results in a germ of curve.

The argument is still valid when we have meromorphic first integrals in the compact dicritical components (see [23]).

1.10 Generic equireduction

In this section we assume that the ambient space M has dimension three, although most of the properties are also true in higher dimension. Thus we consider a complex analytic manifold M of dimension three (we will assume that M is either compact or a germ over a compact set), a codimension one singular foliation $\mathcal{F}$ on M, and we also fix a normal crossings divisor $E \subset M$.

We start by defining the *adapted singular locus* $\mathrm{Sing}(\mathcal{F}, E)$ of $\mathcal{F}$ relatively to E. We recall that $\mathcal{F}$ and E have *normal crossings* at a point $p \in M$ if and only if $\mathcal{F}$ is nonsingular at p and

$$E \cup H$$

defines a normal crossings divisor locally at p, where H is the only invariant hypersurface of $\mathcal{F}$ through p. Then we define

$$\mathrm{Sing}(\mathcal{F}, E) = \left\{ p \in M;\ \mathcal{F} \text{ and } E \text{ do not have normal crossings at } p \right\} .$$

By definition, we have that $\mathrm{Sing}\,\mathcal{F} \subset \mathrm{Sing}\,(\mathcal{F}, E)$. Moreover $\mathrm{Sing}(\mathcal{F}, E)$ is a closed analytic subset of M of codimension at least two (to see this it is enough to remark that if $\mathcal{F}$ is tangent to a hypersurface D, then $\mathcal{F}$ and D have normal crossings at a generic point of D). Let us also remark that

$$\mathrm{Sing}\,\mathcal{F} = \mathrm{Sing}(\mathcal{F}, \emptyset) .$$

Before giving the precise definition of *point of equireduction*, let us introduce the *finite equireduction bamboos*. Given a point $p \in M$, a *finite equireduction bamboo* for $\mathcal{F}, E$ of length $N \geq 0$ over p is given by

$$\mathcal{B} : \quad \left\{ (M_k, \mathcal{F}_k, E^k, Y_k, p_k; U_k) \right\}_{k=0}^{N}$$

where we have the following properties:

1. $U_k \overset{i_k}{\hookrightarrow} M_k$ is an open subset of M_k, $k = 0, 1, \ldots, N$.

2. $Y_k \subset U_k$ is a closed connected nonsingular curve having normal crossings with $E^k \cap U_k$, $k = 0, 1, \ldots, N$.

3. $M_0 = M$ and $\pi_k : M_k \to U_{k-1}$ is the blow-up with center Y_{k-1} for $k = 1, 2, \ldots, N$.

4. $E^0 = E$ and $E^k = \pi_k^{-1}(Y_{k-1} \cup (E^{k-1} \cap U_{k-1}))$ for $k = 1, 2, \ldots, N$.

5. $p_0 = p$ and $\pi_k(p_k) = p_{k-1}$ for $k = 1, 2, \ldots, N$.

6. $p_k \in Y_k$, $k = 0, 1, \ldots, N$.

7. $i_k \circ \pi_k$ induces an étale morphism $Y_k \to Y_{k-1}$ at p_k.

Moreover, $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_k$ is the transform by π_k of $\mathcal{F}_{k-1}|_{U_{k-1}}$ and we add the conditions

8. $\mathrm{Sing}(\mathcal{F}_k|_{U_k}, E^k \cap U_k) = Y_k$.

9. If $D_k^k = \pi_k^{-1}(Y_{k-1})$ is dicritical for $\mathcal{F}_k$, we have one of the following properties:

   a) (complete transversality) For each $q \in Y_{k-1}$ the fiber $\pi_k^{-1}(q)$ is generically transversal to $\mathcal{F}_k$.

   b) (verticality) For each $q \in Y_{k-1}$ the fiber $\pi_k^{-1}(q)$ is invariant by $\mathcal{F}_k$.

Remark 8 The existence of a finite equireduction bamboo of length $N = 0$ simply means that Sing $(\mathcal{F}, E)$ is a nonsingular curve at p having normal crossings with E. Note that this property is satisfied at the generic points of the curves contained in $\mathrm{Sing}(\mathcal{F}, E)$.

We can represent such a bamboo in a displayed way by the diagram

$$\begin{array}{cccccccccccc} \mathcal{B}: & M & \overset{i_0}{\hookleftarrow} & U_0 & \overset{\pi_1}{\leftarrow} & M_1 & \overset{i_1}{\hookleftarrow} & U_1 & \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_N}{\leftarrow} & M_N & \overset{i_N}{\hookleftarrow} & U_N \\ & \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ & E, \mathcal{F} & & Y_0 \ni p_0 & & E^1, \mathcal{F}_1 & & Y_1 \ni p_1 & & E^N, \mathcal{F}_N & & Y_N \ni p_N \end{array}$$

Definition 8 *We say that a point $p \in M$ is a* point of equireduction *for $\mathcal{F}, E$ if Sing$(\mathcal{F}, E)$ is a nonsingular curve at p having normal crossings with E and for any finite equireduction bamboo*

$$\mathcal{B}: \quad \left\{(M_k, \mathcal{F}_k, E^k, Y_k, p_k; U_k)\right\}_{k=0}^{N}$$

there is an open set $W \subset U_N$, $p_N \in W$ such that the blow-up

$$\sigma : \tilde{M} \to W$$

with center $Y_N \cap W$ satisfies

1. *If $\tilde{E} = \sigma^{-1}(Y_N \cup (E^N \cap W))$ and $\tilde{\mathcal{F}}$ is the transform of $\mathcal{F}_N$ by σ, then Sing$(\tilde{\mathcal{F}}, \tilde{E})$ is a (possibly empty) union of nonsingular curves having normal crossings with $\tilde{E}$.*

2. *For any $q \in$ Sing$(\tilde{\mathcal{F}}, \tilde{E})$ the induced morphism Sing$(\tilde{\mathcal{F}}, \tilde{E}) \to Y_N \cap W$ is étale.*

3. If σ is a dicritical blow-up, the we have either a) *or* b) *where*

   a) *Each fiber $\sigma^{-1}(r)$ is generically transversal for $r \in Y_N \cap W$.*

   b) *Each fiber $\sigma^{-1}(r)$, $r \in Y_N \cap W$, is invariant by $\tilde{\mathcal{F}}$.*

Remark 9 The bamboo $\mathcal{B}$ may be extended to several branches of length $N+1$, except in the case that $\text{Sing}(\tilde{\mathcal{F}}, \tilde{E}) = \emptyset$. This case only occurs for a dicritical (completely transversal) morphism σ.

Let $p \in M$ be a point with dimensional type $\tau_p = 2$. So there exists a neighborhood $U \subset M$, $p \in U$, and a germ of nonsingular vector field ξ in U which is tangent to $\mathcal{F}$. Hence $\text{Sing}(\mathcal{F}, E) \cap U$ is a nonsingular curve; moreover, it is contained in each invariant component of $E \cap U$ passing through p. If $\pi : M' \to M$ is a blow-up centered at $Y = \text{Sing}(\mathcal{F}, E) \cap U$, we have that $\mathcal{F}_1$ is tangent to the vector field ξ_1, the transform of ξ by π. Therefore, in the case that $\text{Sing}(\mathcal{F}_1, E^1) \neq \emptyset$, for any $q \in \text{Sing}(\mathcal{F}_1, E^1)$ we can find a neighborhood $U_q \subset M_1$ such that $\text{Sing}(\mathcal{F}_1, E^1) \cap U_q$ is a nonsingular curve contained in each invariant component of E^1. Repeating the argument, we conclude that p is an equireduction point.

However, the properties "$p \in M$ is an equireduction point for $\mathcal{F}, E$" and "the dimensional type of p is two" are not equivalent. Take for instance $M = (\mathbb{C}^3, \underline{0})$, $E = \emptyset$ and $\mathcal{F}$ is the foliation given by $\omega = 0$ where

$$\omega = d[xy(x-y)(y+(z+1)x)] \ .$$

So $\text{Sing}(\mathcal{F}, E) = \text{Sing}\,\mathcal{F} = (x = y = 0)$. By performing just one nondicritical blow-up $\pi_1 : M_1 \to M_0 = M$ centered at $Y_0 = (x = y = 0)$, we obtain that every point of $\text{Sing}(\mathcal{F}_1, E^1)$ is simple. Note that $\text{Sing}(\mathcal{F}_1, E^1)$ is the union of four nonsingular curves which are locally isomorphic to Y_0. If we continue performing blow-ups we will only obtain simple singularities. Hence every point $p \in \text{Sing}(\mathcal{F}, E)$ is an equireduction point for $\mathcal{F}, E$. However, note that the dimensional type of every point $p \in \text{Sing}(\mathcal{F}, E)$ is not two. Indeed, suppose we have $\tau_p = 2$. Then locally at p the vector field $\xi = \partial/\partial z$ is tangent to $\mathcal{F}$. Hence $\mathcal{F}_1$ is (locally) tangent to the vector field $\xi_1 = \partial/\partial z$, the transform of ξ bi π_1. We have that $\pi_1^{-1}(p) \cap \text{Sing}(\mathcal{F}_1, E^1)$ gives four points; call them p'_1, p'_2, p'_3, p'_4. Since $\mathcal{F}_1$ is (locally) tangent to ξ_1, for $q \in \text{Sing}(\mathcal{F}, E)$ near p we have that $\pi_1^{-1}(q) \cap \text{Sing}(\mathcal{F}, E)$ gives the same four points p'_1, p'_2, p'_3, p'_4. This is an absurd due to the fact that one of these points corresponds to the intersection of $\pi_1^{-1}(q)$ with the transform S' of the invariant hypersurface $S = (y + (z+1)x = 0)$, which depends on the point p (see Figure 5).

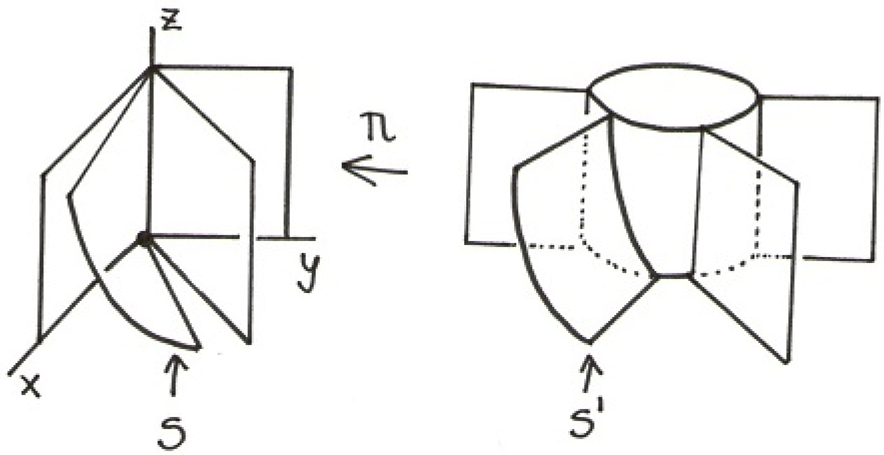


Figure 5: Equireduction is not equivalent to dimensional type two.

Let us state the main results on equireduction that we need in this work. The precise proofs may be found in [4], [8].

Proposition 10 *Let $p \in M$ be an equireduction point for $\mathcal{F}, E$. Then there is an open set $U \subset M$, $p \in U$, and a finite sequence of blow-ups*

$$U \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \longleftarrow \cdots \xleftarrow{\pi_N} M_N$$

that gives a reduction of singularities of $\mathcal{F}, E$ and has the following properties (with the notation as usual):

1. *The center of π_k is Sing$(\mathcal{F}_{k-1}, E^{k-1})$, which is a nonsingular curve having normal crossings with E^{k-1}, for $k = 1, 2, \ldots, N$.*
2. *The induced morphism Sing$(\mathcal{F}_k, E^k) \to$ Sing$(\mathcal{F}_{k-1}, E^{k-1})$ is étale.*
3. *Each dicritical component of the exceptional divisor of π_k is either vertical (condition* b) *of Definition 8) or has no invariant fibers.*
4. *All the points in Sing$(\mathcal{F}_N, E^N)$ are simple points of dimensional type $\tau = 2$.*

As a consequence of Proposition 10 we obtain the genericity of the equireduction property.

Proposition 11 *The set of points $p \in M$ that are not points of equireduction for $\mathcal{F}, E$ is a finite set of points.*

The properties above allow us to define the *generic character* of an irreducible curve $\Gamma \subset \mathrm{Sing}(\mathcal{F}, E)$. We can consider an open set $U \subset M$ such that $U \cap \Gamma$ is connected,

$$\mathrm{Sing}(\mathcal{F}, E) \cap U = U \cap \Gamma$$

and $U \cap \Gamma$ is precisely the set of equireduction points in Γ. Now, we can take U as in Proposition 10 and perform a canonical reduction of singularities

$$\mathcal{R}_\Gamma : \quad U \stackrel{\pi_1}{\longleftarrow} M_1 \longleftarrow \cdots \stackrel{\pi_N}{\longleftarrow} M_N \ .$$

The behavior of this reduction of singularities gives the generic character of Γ.

Definition 9 *We say that $\mathcal{F}$ is* generically dicritical along Γ *if and only if one of the blow-ups π_k of $\mathcal{R}_\Gamma$ is dicritical; otherwise we say that $\mathcal{F}$ is* generically nondicritical along Γ*. In the second case, we say that $\mathcal{F}$ is* generically nodal along Γ *if and only if there is an irreducible component of Sing$(\mathcal{F}_N, E^N)$ that corresponds to a two-dimensional nodal simple singularity (recall that the dimensional type of the singularities in M_N is two).*

It is interesting to indicate the relationship between a two-dimensional transversal section and the generic behavior of $\mathcal{F}, E$ along a curve.

Proposition 12 *Assume that $\mathcal{F}, E$ is generically nondicritical along Γ or that it is generically dicritical but without vertical components along Γ. Take an equireduction point $p \in \Gamma$ and a two-dimensional plane Δ, $p \in \Delta$, transversal to $\mathcal{F}$. Then the sequence $\mathcal{R}_\Gamma$ induces by section a reduction of singularities of $\mathcal{F}|_\Delta$ where the dicritical and nondicritical components coincide with the ones for $\mathcal{F}$.*

We finish this section with the following result:

Proposition 13 *Let $M = (\mathbb{C}^3, \underline{0})$, $E = \emptyset$ and $\mathcal{F}$ be a codimension one foliation on M. Suppose the origin $\underline{0} \in M$ is an equireduction point for $\mathcal{F}, E$. Then there exists a germ of analytic surface $S \subset M$ invariant by $\mathcal{F}$.*

Proof: Let $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N : M_N \to M_0 = M$ be the reduction of singularities given by Proposition 10. Hence each component of the final exceptional divisor $E^N \subset M_N$ is noncompact and we are done.

$\square$

2 Brunella's local alternative for RICH foliations

2.1 Complex Hyperbolic foliations

In this section we will only consider Complex Hyperbolic simple singularities (Definition 6). These singularities are the high-dimension version of the simple singularities in the sense of Seidenberg [24] given by vector fields with two nonzero eigenvalues.

Let us define Complex Hyperbolic Foliations. In dimension two, these foliations are exactly the "generalized curves" introduced by C. Camacho, A. Lins-Neto and P. Sad in [2]. We recall that a foliation $\mathcal{F}$ of $(\mathbb{C}^2, \underline{0})$ is called a generalized curve if, and only if, there is a reduction of singularities of $\mathcal{F}$

$$M_0 = (\mathbb{C}^2, \underline{0}) \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

such that all the singular points of the final transform $\mathcal{F}_N$ of $\mathcal{F}$ are of complex hyperbolic type; that is, we do not accept saddle-nodes in the reduction of singularities of $\mathcal{F}$. Let us remark that "dicritical generalized curves" are allowed. The definition of a generalized curve in dimension two does not depend on the particular reduction of singularities. Moreover, even in the dicritical case, it is known that the reduction of singularities of a generalized curve coincides with the reduction of singularities of its set of invariant curves. Also, all the formal invariant curves of a generalized curve are convergent, since the existence of non-convergent invariant curves implies the rising of saddle-nodes in the reduction of singularities.

In dimension three, D. Cerveau uses the terminology "quasi-regular" foliation to denote nondicritical germs of foliation having a reduction of singularities such that the generic points of the lines of singularities are not saddle-nodes for a transversal section (see [11]). Also in the nondicritical case J. Mozo and P. Fernndez use the terminology "generalized surface" [20]. Anyway, as we shall see, we are mainly interested in the study of dicritical situations.

For dimension $n \geq 2$, we give the following definition:

Definition 10 (CH foliation) *Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, \underline{0})$. We say that $\mathcal{F}$ is a* Complex Hyperbolic Foliation *(for short,* CH foliation*) at the origin if for any map*

$$\varphi : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^n, \underline{0})$$

generically transversal to $\mathcal{F}$ we have that $\varphi^*\mathcal{F}$ is a generalized curve in the sense of Camacho, Lins-Neto and Sad [2], that is, there are no saddle-nodes singularities in the reduction of $\mathcal{G}$.

Proposition 14 *Let $\mathcal{F}$ be a germ of CH foliation on $M = (\mathbb{C}^3, \underline{0})$ and assume that $\pi : M' \to M$ defines a reduction of singularities in the sense of Theorem 5. Then all the points in M are CH-simple for $\pi^*\mathcal{F}$.*

Proof: Fix a reduction π and suppose, by absurd, that there are points of Sing $\mathcal{F}'$ which are not CH simple for $\mathcal{F}'$, where $\mathcal{F}' = \pi^*\mathcal{F}$. Let $\Delta \subset M'$ be a plane section transversal to $\mathcal{F}'$ and $\Delta \overset{i}{\hookrightarrow} M'$ be the canonical immersion. Hence

$$\tilde{\varphi} = \pi \circ i : \Delta \to M'$$

is a map generically transversal to $\mathcal{F}$ such that the foliation $\tilde{\varphi}^*\mathcal{F} = \mathcal{F}'|_\Delta$ is not a generalized curve, which is an absurd.

□

Lemma 15 *Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, \underline{0})$ having a CH-simple point at the origin. Then $\mathcal{F}$ is a CH foliation.*

Proof: Let $\phi : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^n, \underline{0})$ be a map which is generically transversal to $\mathcal{F}$. Suppose, by absurd, that $\mathcal{G} = \phi^*\mathcal{F}$ is not a generalized curve. So, up to doing the reduction of singularities of $\mathcal{G}$, we can find another map $\tilde{\phi} : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^n, \underline{0})$ such that the origin is a saddle-node singularity of $\tilde{\mathcal{G}} = \tilde{\phi}^*\mathcal{F}$. By performing finitely many local blow-ups of $(\mathbb{C}^2, \underline{0})$, there is a map $\pi : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}^2, \underline{0})$ such that if $\psi = \tilde{\phi} \circ \pi$ we have that

1. The foliation $\psi^*\mathcal{F}$ has a saddle node at the origin.
2. If we write $\psi(z_1, z_2) = (\psi_1(z_1, z_2), \psi_2(z_1, z_2), \ldots, \psi_n(z_1, z_2))$, then

   $$\psi_i(z_1, z_2) = u_i(z_1, z_2) z_1^{a_i} z_2^{b_i};\ i = 1, 2, \ldots, n$$

   where $a_i, b_i \in \mathbb{Z}_{\geq 0}$ and $u_i(0, 0) = 0$.

Since $\mathcal{F}$ is given by a 1-form of the type

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + B_i(x_1, x_2, \ldots, x_n)) \frac{dx_i}{x_i},\ B_i \in \mathbb{C}\{x_1, \ldots, x_n\},\ B_i(0) = 0\ ,$$

we have that

$$\begin{aligned}
\psi^*\omega &= \sum_{i=1}^{\tau} (\lambda_i + B_i \circ \psi) \frac{d\psi_i}{\psi_i} = \sum_{i=1}^{\tau} (\lambda_i + B_i \circ \psi) \Big\{ a_i \frac{dz_1}{z_1} + b_i \frac{dz_2}{z_2} + \frac{du_i}{du_i} \Big\} \\
&= \sum_{i=1}^{\tau} a_i (\lambda_i + B_i \circ \psi) \frac{dz_1}{z_1} + \sum_{i=1}^{\tau} b_i (\lambda_i + B_i \circ \psi) \frac{dz_2}{z_2} + \sum_{i=1}^{\tau} (\lambda_i + B_i \circ \psi) \frac{du_i}{u_i}
\end{aligned}$$

Hence the foliation $\psi^*\mathcal{F}$ is given by a vector field whose eigenvalues are

$$\alpha = \sum_{i=1}^{\tau} b_i\lambda_i,\ \beta = -\sum_{i=1}^{\tau} a_i\lambda_i\ .$$

We are assuming that $\alpha\beta = 0$; suppose $\alpha = 0$. In view of the non-resonance of the residual vector, we have that $b_i = 0$ for all $i = 1, 2, \ldots, \tau$. This implies that

$$\psi_i(z_1, z_2) = u_i(z_1, z_2) z_2^{b_i} \text{ for } i = 1, 2, \ldots, \tau\ .$$

However, since $\psi(0) = 0$, we have that $b_i \neq 0$ for some $i \in \{1, 2, \ldots, \tau\}$. Hence $\psi^*\mathcal{F}$ is nonsingular at the origin, which is an absurd and we are done.

□

Proposition 16 *Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, \underline{0})$. Assume that there exists a sequence of blow-ups*

$$(\mathbb{C}^n, 0) = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

such that for any $1 \leq s \leq n$ we have

1. *The center $Y_{s-1} \subset M_{s-1}$ os the blow-up π_s is nonsingular, has normal crossings with the total exceptional divisor $E^{s-1} \subset M_{s-1}$ and is invariant by the transform $\mathcal{F}_{s-1}$ of $\mathcal{F}$.*
2. *Each $p \in M_N$ is a complex Hyperbolic Simple Point for the pair $(\mathcal{F}_N, E^N)$.*

Then $\mathcal{F}$ is a CH foliation.

Proof: Let $\pi : M_N \to M_0$ be the composition $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N$ of all the blow-ups. Consider a map $\phi : (\mathbb{C}^2, \underline{0}) \to M_0$ generically transversal to $\mathcal{F}$. By the universal property of the blow-up, there exists a map

$$\sigma : \tilde{\Delta} \to (\mathbb{C}^2, \underline{0})$$

that is the composition of a sequence of blow-ups and lifts ϕ. That is to say, there is a map

$$\tilde{\phi} : \tilde{\Delta} \to M_N$$

such that $\pi \circ \tilde{\phi} = \phi \circ \sigma$. Applying Lemma 15 to the foliation $\tilde{\mathcal{G}} = \tilde{\phi}^*\pi^*\mathcal{F} = \sigma^*\phi^*\mathcal{F}$, we have that $\tilde{\mathcal{G}}$ only has points of type generalized curve. Hence $\mathcal{G} = \sigma^*\mathcal{F}$ is a generalized curve and the result follows.

□

We do not exclude the possibility of a CH foliation being dicritical. For instance, the foliation in $(\mathbb{C}^3, \underline{0})$ given by the 1-form

$$\omega = (y^{m+1} - zx^m)dx + (z^{m+1} - xy^m)dy + (x^{m+1} - yz^m)dz$$

is Complex Hyperbolic and dicritical. Moreover it is known that this foliation has no invariant surface [12]. Another example of CH foliations are the logarithmic foliations given by a 1-form ω of the type

$$\omega = \sum_{i=1}^{k} \lambda_i \frac{df_i}{f_i}; \quad \lambda_i \in \mathbb{C},\ i = 1, 2, \ldots, k\ ,$$

which correspond to the levels of the multivaluated function $f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_k^{\lambda_k}$.

Remark 10 In ambient space dimension three, any germ of codimension one foliation admits a reduction of singularities [4]. A reduction of singularities is called *Complex Hyperbolic* if all the points of $\pi^*\mathcal{F}$ are CH-simple. Due to Proposition 16, we have that if $\mathcal{F}$ admits a complex hyperbolic reduction of singularities then $\mathcal{F}$ itself is a CH foliation and thus all the reduction of singularities of $\mathcal{F}$ are also complex hyperbolic. This condition has been considered as a definition in [20], where the authors regard the so called *generalized surfaces*, which are the nondicritical CH foliations in ambient space dimension three. In the nondicritical case, it is proved in [20] that the reduction of singularities of the invariant surfaces automatically gives the reduction of singularities of the considered CH foliation. Next we state a result of this nature in any ambient dimension which can be proved as in the three dimensional case.

Proposition 17 *Let $\mathcal{F}$ be a germ of nondicritical CH foliation on $(\mathbb{C}^n, \underline{0})$ of dimensional type n. Assume that the invariant hypersurfaces of $\mathcal{F}$ are exactly the coordinate hyperplanes $\prod_{i=1}^{n} x_i = 0$. Then the origin is a CH-simple point.*

Proof: See [20].

□

2.2 Relatively Isolated CH foliations

The RICH foliations that we introduce here define the main class of foliations of $(\mathbb{C}^3, \underline{0})$ we are going to consider. Its name comes from

Relatively Isolated Complex Hyperbolic.

In the previous section we have given the definition of CH foliations; now we will ask, in addition, that the foliation presents a few non-restrictive conditions during its reduction of singularities.

The expression "absolutely isolated singularities" usually refers to singularities that can be desingularized by using only punctual blow-ups. In the case of vector fields this kind of singularities were studied in any dimension in [1]. There is a work describing codimension one singularities of foliations desingularized by punctual blow-ups [7], where the authors ask for nondicritical conditions and specific additional properties for the line of singularities (Poincaré type).

Anyway, for the case of codimension one foliations it is very special to encounter isolated singularities in ambient space dimension $n \geq 3$. In fact, the Singular Frobenius Theorem of Malgrange [14] assures that is this is the case, the foliation has a first integral and then it coincides with the "levels" of a given function. Moreover, the definition of simple singularity for a triple $(\mathcal{F}, E, M)$ gives always a non-isolated singularity for $n \geq 3$, $n = \dim M$.

Thus, there is only a small interest in considering the case of isolated singularities and, anyway, they will never produce isolated singularities at the end of the reduction of singularities.

On the other hand, we have seen that in ambient space dimension three we have generic equireduction properties for the curves of singularities. So it is reasonable to ask isolated isolated properties only for the singular locus created over the origin by the reduction of singularities. These reasons justify the next definition.

Definition 11 (RICH) *Let $\mathcal{F}$ be a foliation of $(\mathbb{C}^3, \underline{0})$. We will say $\mathcal{F}$ is a* Relatively Isolated Complex Hyperbolic Foliation *if $\mathcal{F}$ is a CH foliation and there exists a reduction of singularities of $\mathcal{F}$*

$$\mathcal{S}: \quad (\mathbb{C}^n, 0) = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

such that for any $1 \leq s \leq N$

1− *The center $Y_{s-1} \subset M_{s-1}$ of the blow-up π_s is nonsingular, has normal crossings with the total exceptional divisor $E^{s-1} \subset M_{s-1}$ and is invariant by the transform $\mathcal{F}_{k-1}$ of $\mathcal{F}$.*

2− *The intersection $Y_{k-1} \cap (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{s-1})^{-1}(\underline{0})$ is a single point.*

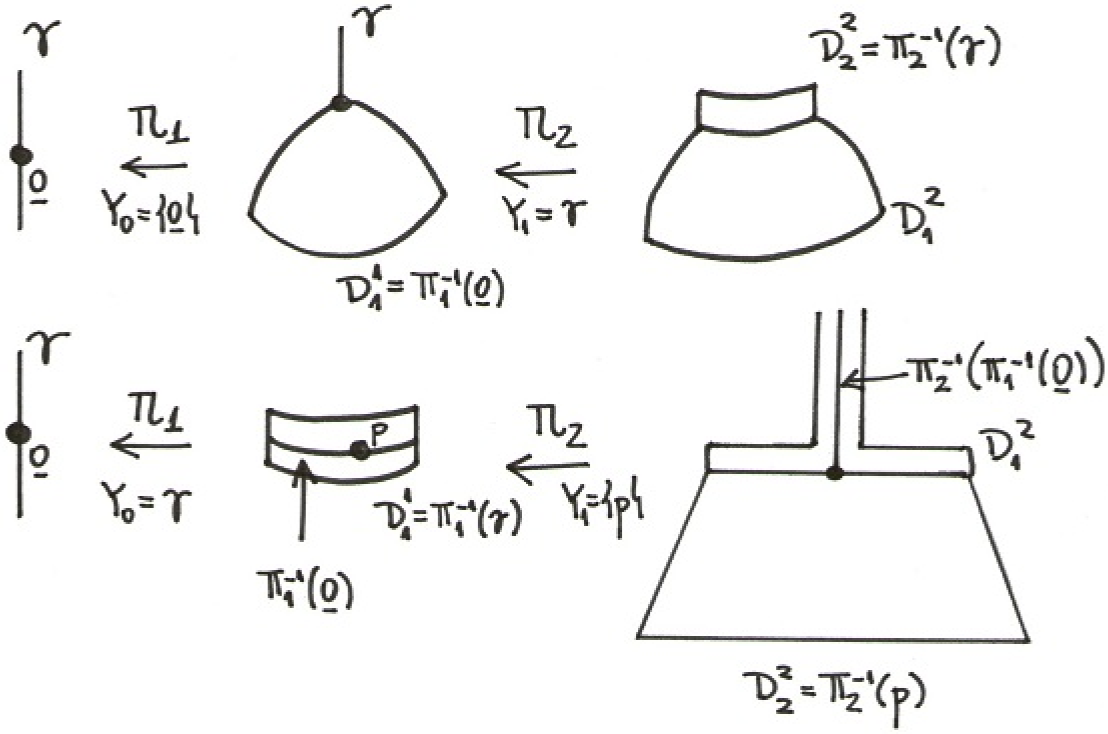


Figure 6: The type of blow-ups allowed for RICH foliations

Remark 11 As a consequence of item $2-$ of the above definition, the center of each π_s of the reduction $\mathcal{S}$ is never a compact curve; it is either a point or a germ of curve, for all $s = 1, 2, \ldots, N-1$.

Remark 12 Each time a compact curve appears in the singular locus Sing $\mathcal{F}_s$, for $s = 1, 2, \ldots, N-1$, this curve is generically simple.

Remark 13 As a consequence of Proposition 14 all the points in M_N are CH simple.

The condition "relatively isolated" is less restrictive than "absolutely isolated". One example of relatively isolated is the case of equireduction along a curve; another example are foliations of the type $df = 0$ where $f =$ is a germ at the origin of surface with isolated singularity.

2.3 Main result

The main result of this work may be stated as follows:

Theorem 7 *Let $\mathcal{F}$ be a RICH foliation in $(\mathbb{C}^3, \underline{0})$. Assume that there is no germ of invariant analytic surface for $\mathcal{F}$. Then one of the two properties holds:*

(1) *There is a neighborhood W of the origin $\underline{0} \in \mathbb{C}^3$ such that for each leaf $L \subset W$ of $\mathcal{F}$ in W there is an analytic curve $\gamma \subset L$ with $\underline{0} \in \gamma$.*

(2) *There is an analytic curve $\Gamma \subset (\mathbb{C}^3, \underline{0})$ contained in the singular locus Sing $\mathcal{F}$ such that $\mathcal{F}$ is generically dicritical or generically nodal along Γ.*

3 The case without nodal components

In this chapter we are going to prove Theorem 7 in the particular case that there are no nodal components, to be introduced in the text. In this case, Theorem 7 may be stated as follows:

Theorem 8 *Let $\mathcal{F}$ be a RICH foliation in $(\mathbb{C}^3, \underline{0})$ and assume that there is no germ of invariant analytic surface for $\mathcal{F}$. If there are no nodal components in Sing $\mathcal{F}_N$, then there is a neighborhood W of the origin $\underline{0} \in \mathbb{C}^3$ such that for each leaf $L \subset W$ of $\mathcal{F}$ in W there is an analytic curve $\gamma \subset L$ with $\underline{0} \in \gamma$.*

3.1 Nodal components

In this section we will define the concept of *nodal components* of a RICH foliation in dimension three and exhibit some of its properties. But first, let us generalize the definition of *nodal point* in dimension n. Let $\mathcal{F}$ be a codimension one foliation on $(\mathbb{C}^n, \underline{0})$. Recall (Definition 6) that the origin is a CH-simple point if we can find local coordinates $x_1, x_2, \ldots, x_n$ such that $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + a_i(x_1, x_2, \cdots, x_n)) \frac{dx_i}{x_i}$$

with $a_i(0) = 0$ for all $i = 1, 2, \ldots, \tau$ and $\sum \lambda_i m_i \neq 0$ if $\underline{m} \neq 0$, $m_1 \in \mathbb{Z}_{>0}$.

We remark that the residual vector is well defined as an element of the projective space $[\underline{\lambda}] \in \mathbb{P}_{\mathbb{C}}^{\tau-1}$ up to permutation of the coordinates.

Definition 12 *Assume $\mathcal{F}$ has a CH-simple point at the origin with $\tau \geq 2$. We say that the origin is a* nodal singularity *if and only if the residual vector can be chosen as*

$$\underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{\tau-1}, -1)$$

where $\lambda_i \in \mathbb{R}_{>0}$ for $i = 1, 2, \ldots, \tau - 1$.

Remark 14 Nodal singularities may be normalized in a convergent way.

Remark 15 In dimension three a nodal singularity of dimensional type three is contained in exactly two generically nodal curves of the singular locus of the foliation. Indeed, the foliation is given by

$$\omega = (\lambda + a(x, y, z)) \frac{dx}{x} + (\mu + b(x, y, z)) \frac{dy}{z} + (-1 + c(x, y, z)) \frac{dz}{z}$$

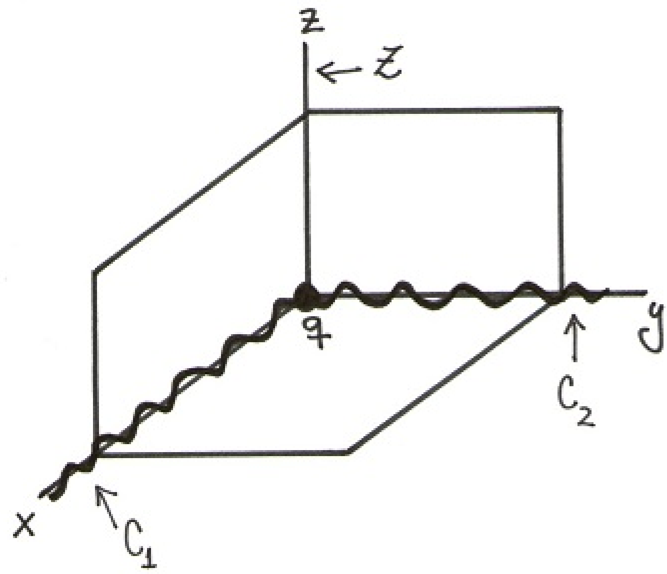


Figure 7: Nodal singularity of dimensional type three. We will always denote generically nodal curves as we do in this picture (curves C_1 and C_2). Curves which are not generically nodal will always be denote by "clean" lines (like the curve Z).

where $a(0) = b(0) = c(0) = 0$ and $\lambda, \mu \in \mathbb{R}_{>0}$. Call $C_1 = (y = z = 0)$, $C_2 = (x = z = 0)$, $Z = (x = y = 0)$. The curves C_1 and C_2 are generically nodal; on the other hand, the curve Z is not.

Proposition 18 *Let $\mathcal{F}$ be a foliation on $M = (\mathbb{C}^3, \underline{0})$ and assume the origin is a nodal singularity of $\mathcal{F}$ with dimensional type three. Call $\Gamma_1, \Gamma_2 \subset Sing\ \mathcal{F}$ the two curves passing through the origin which are generically nodal. Let q be a point in Γ_1, $q \neq \underline{0}$, and let q' be a point in Γ_2, $q' \neq \underline{0}$. There exists an open set $U \subset M$, $q \in U$ such that*

$$q' \in \mathrm{Sat}_{\mathcal{F}}(U) \ .$$

Proof: Due to Remark 14, the proof is done by direct integration.

□

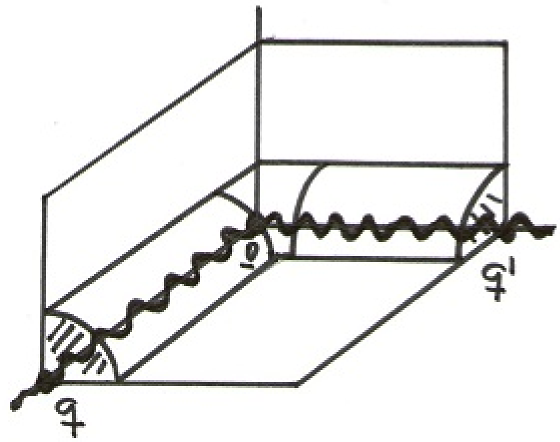


Figure 8: The neighborhood of the union of two generically nodal curves given by Proposition 18.

Now let $\mathcal{F}$ be a RICH foliation in $M = (\mathbb{C}^3, \underline{0})$ and let us fix a reduction of singularities

$$\mathcal{S} : M_0 = M \overset{\pi_1}{\longleftarrow} M_1 \overset{\pi_2}{\longleftarrow} \cdots \overset{\pi_N}{\longleftarrow} M_N$$

such that for any $1 \leq s \leq N$ we have

1. The center $Y_{s-1} \subset M_{s-1}$ of the blow-up π_s is nonsingular, has codimension at least two, has normal crossings with the exceptional divisor $E^{s-1} \subset M_{s-1}$ and is invariant by the transform $\mathcal{F}_{s-1}$ of $\mathcal{F}$.
2. The intersection $Y_{s-1} \cap \sigma_{s-1}^{-1}(\underline{0})$ is a single point, where $\sigma_{s-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{s-1}$.
3. All the points of M_N are CH-simple for the pair $\mathcal{F}_N, E^N$.

Let us give some notations associated to this particular sequence of blow-ups. Given $0 \leq s \leq s' \leq N$ we denote $\pi_{ss} = \mathrm{id}_{M_s}$ and

$$\pi_{ss'} = \pi_{s+1} \circ \pi_{s+2} \circ \cdots \circ \pi_{s'} : M_{s'} \to M_s$$

is $s < s'$. We take special notations for some particular cases:

$$\rho_s = \pi_{Ns} : M_N \to M_s; \;\; \sigma_s = \pi_{s0} : M_s \to M_0 = (\mathbb{C}^3, \underline{0}) \; .$$

Finally, we denote $\pi = \pi_{N0} = \rho_0 = \sigma_N : M_N \to M_0$ the morphism of reduction of singularities. We decompose the exceptional divisor E^s into irreducible components

$$E^s = D_1^s \cup D_2^s \cup \cdots \cup D_s^s$$

where D_i^s is the strict transform by π_s of D_i^{s-1} for $i < s$ and $D_s^s = \pi_s^{-1}(Y_{s-1})$. We write

$$E_{inv}^s \subset E, \text{ respectively } E_{dic}^s \subset E^s$$

the union of the irreducible components of E^s invariant by $\mathcal{F}_s$, respectively the generically transversal (dicritical) components of E^s. This notation will be used again in Chapter 4.

Remark 16 The morphisms $\pi_{ss'}$ are morphisms of germs

$$\pi_{ss'} : (M_{s'}, \sigma_{s'}^{-1}(\underline{0})) \to (M_s, \sigma_s^{-1}(\underline{0}))$$

around the compact sets $\sigma_s^{-1}(\underline{0}) \subset M_s$ and $\sigma_{s'}^{-1}(\underline{0}) \subset M_{s'}$. In particular π is a morphism of germs

$$\pi : (M_N, \pi^{-1}(\underline{0})) \to (\mathbb{C}^3, \underline{0}) \ .$$

In view of the properties of the sequence of reduction of singularities, an irreducible component D_i^s of the exceptional divisor E^s is compact if and only if the center Y_{i-1} of π_{i-1} is a single point; moreover, this is equivalent to saying that $D_i^s \subset \sigma_s^{-1}(\underline{0})$. Conversely, the irreducible component D_i^s is not compact if and only if the center Y_{i-1} is a germ of curve (that is necessarily not contained in $\sigma_{i-1}^{-1}(\underline{0})$).

Take a connected component $\mathcal{C}$ of the singular locus Sing $\mathcal{F}_N$. Note that $\mathcal{C}$ is a connected union of non-singular irreducible curves

$$\mathcal{C} = C_1 \cup C_2 \cup \cdots \cup C_m$$

and that $\mathcal{C}$ has normal crossings with the exceptional divisor E^N.

Definition 13 *We say that $\mathcal{C}$ is a* nodal component *for $\mathcal{F}, \mathcal{S}$ if and only if all the points of $\mathcal{C}$ are nodal singularities of $\mathcal{F}$. We say that $\mathcal{F}$* has no nodal components *if there exists a reduction of singularities $\mathcal{S}$ as before such that there are no nodal components for $\mathcal{F}, \mathcal{S}$.*

Let $C \subset$ Sing $\mathcal{F}_N$ be an irreducible curve which is generically nodal. It is possible that C is not contained in a nodal component. In this case there exists a point $q \in C$ such that q has dimensional type three but is not a nodal singularity of $\mathcal{F}_N$. Such

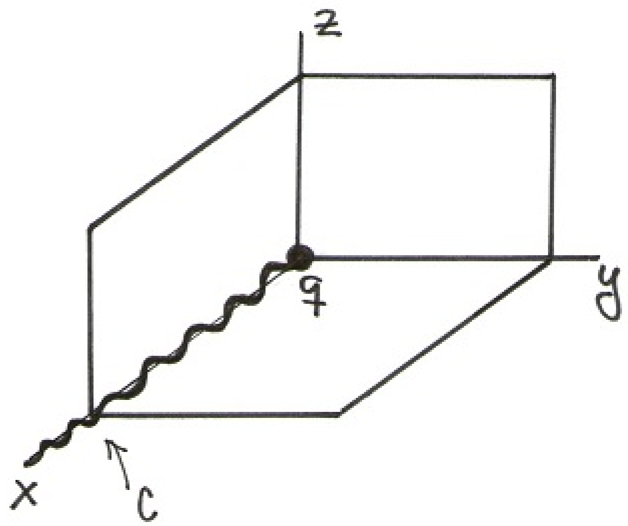


Figure 9: Example of a generically nodal curve C which is not contained in a nodal component.

a point q is called a *point of interruption* for the nodality. Thus a nodal component is a connected component of Sing $\mathcal{F}_N$ union of generically nodal curves and without interruption points. An example of this situation is given as follows.Choose local coordinates x, y, z at q such that $\mathcal{F}_N$ is given by

$$\omega = (\lambda + a(x,y,z))\frac{dx}{x} + (\mu + b(x,y,z))\frac{dy}{z} + (-1 + c(x,y,z))\frac{dz}{z}$$

where $a(0) = b(0) = c(0) = 0$ and $\lambda \notin \mathbb{R}_{>0}$, $\mu \in \mathbb{R}_{>0}$, $\mu/\lambda \notin \mathbb{R}_{>0}$. Hence C is the curve $(y = z = 0)$. Note that the other curves of the singular locus containing q, which are locally given by $(x = y = 0)$, $(x = z = 0)$, are not generically nodal.

Remark 17 A nodal component $\mathcal{C}$ is irreducible if and only if all of its points have dimensional type two.

3.2 Local study of complex hyperbolic simple singularities

In this section we continue the study of the behavior of leaves of a foliation near complex hyperbolic simple points in dimensions two and three. Our first result is given by Proposition 18. In this section we prove the following result for $n = 2, 3$:

Proposition 19 *Assume that $\mathcal{F}$ has a CH-simple point at the origin of $\mathbb{C}^n$ which is not a nodal singularity and suppose that $\mathcal{F}$ is defined in a neighborhood U of the origin by the integrable 1-form*

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + b_i(x_1, x_2, \cdots, x_n)) \frac{dx_i}{x_i}, \ b_i(0) = 0 \ .$$

Take an index $j \in \{1, 2, \ldots, \tau\}$, a point $R \in (x_j = 0) \setminus$ Sing $\mathcal{F}$ close enough to the origin and consider the transversal curve to $x_j = 0$ given by

$$\Delta = U \cap pr_j^{-1}(pr_j(R)) \ ,$$

where $pr_j : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is the linear projection that avoids the j-th coordinate. Then the saturation by the foliation of Δ jointly with the coordinate hyperplanes $\prod_{i=1}^{\tau} x_i = 0$ define a neighborhood of the origin.

Initially let us regard the two-dimensional case. Let $\mathcal{F}$ be a codimension one foliation on $M = (\mathbb{C}^2, \underline{0})$ such that the origin is a CH-simple point of $\mathcal{F}$ and let $E \subset M$ be a normal crossings divisor. Firstly, we will consider the cases where the 1-form ω above is linearizable. We will fix $U = \mathbb{D}_r^2 = \mathbb{D}_r \times \mathbb{D}_r$, where

$$\mathbb{D}_r = \{x \in \mathbb{C};\ |x| \leq r\},\ 0 < r < 1 \ .$$

The following result has been presented by J.F. Mattei and D. Marín in [15].

Lemma 20 *Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2, \underline{0})$ given by $\omega = 0$ where*

$$\omega = y dx + \lambda x dy \ ,\ \lambda \in \mathbb{C} \setminus \mathbb{R}_{<0} \ .$$

Let $\Delta = \{(r, 0)\} \times \mathbb{D}_\varepsilon$, $\mathbb{D}_\varepsilon = \{y \in \mathbb{C};\ |y| \leq \varepsilon\}$, $0 < \varepsilon \leq r$. Then there exists $0 < \delta \leq r$ such that

$$\mathbb{D}_\delta^2 = \mathbb{D}_\delta \times \mathbb{D}_\delta \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta) \ .$$

Proof: Let $q_0 = (x_0, y_0) \notin E = (xy = 0)$ be a point in $\mathbb{D}_r^2$. The leaf of $\mathcal{F}|_{\mathbb{D}_r^2}$ passing through q_0 is given by $L_{q_0} = (x = c_0 y^{-\lambda})$ where $c_0 = x_0 y_0^{\lambda} \in \mathbb{C}^*$. Hence

$$\varphi_{q_0}(t) = (x_0 e^{\lambda t}, y_0 e^{-t}), t \in \mathbb{C}$$

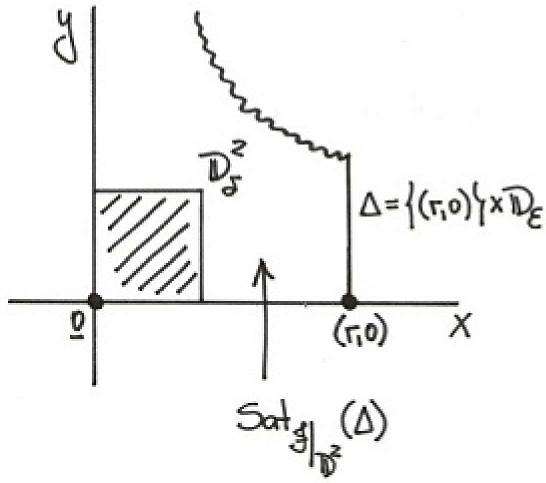


Figure 10: Scheme of Lemma 20.

is a parametrization of L_{q_0}. We are only interested in the values of t such that $\varphi_{q_0}(t) \in \mathbb{D}^2_r$. Consider the set

$$A_{q_0} = \{t \in \mathbb{C};\ \varphi_{q_0}(t) \in \mathbb{D}^2_r\} = \{t \in \mathbb{C};\ |x_0 e^{\lambda t}| \le r,\ |y_0 e^{-t}| \le r\}\ .$$

So A_{q_0} is a connected subset of $\mathbb{C}$ such that $0 \in A_{q_0}$; thus $L_{q_0} = \varphi_{q_0}(A_{q_0})$ is connected. By writing

$$\lambda = \lambda_1 + i\lambda_2\ ,\ \ \lambda_1, \lambda_2 \in \mathbb{R}$$

$$t = t_1 + it_2\ ,\ \ t_1, t_2 \in \mathbb{R}$$

we have that

$$|x_0 e^{\lambda t}| \le r \text{ is equivalent to } \lambda_1 t_1 - \lambda_2 t_2 \le \ln(r/|x_0|)\ ,$$

$$|y_0 e^{-t}| \le r \text{ is equivalent to } t_1 \ge -\ln(r/|y_0|)\ .$$

The argument of the proof is the following: we wish to find a polydisc $\mathbb{D}^2_\delta$ such that for each point $q_0 \in \mathbb{D}^2_\delta$, $q \notin E$, the leaf L_{q_0} of $\mathcal{F}|_{\mathbb{D}^2_r}$ passing through q_0 cuts the transversal section Δ. Hence $\mathbb{D}^2_\delta \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}^2_r}}(\Delta)$. That is to say, if

$$\varphi_{q_0}(t) = (x_0 e^{\lambda t}, y_0 e^{-t}) = (\varphi_{q_0,1}(t), \varphi_{q_0,2}(t))\ ,$$

we want to see that it is possible to find a suitable $\tilde{t}$ such that $\varphi_{q_0}(\tilde{t}) \in \Delta$, or, in other words,

$$\varphi_{q_0,1}(\tilde{t}) = r,\ |\varphi_{q_0,2}(\tilde{t})| < \varepsilon\ .$$

The points $t_k = t_{1k} + it_{2k}$ such that $\varphi_{q_0,1}(t_k) = r$ are the solutions of the system

$$\begin{cases} \lambda_1 t_{1k} - \lambda_2 t_{2k} = \ln\left(\frac{r}{|x_0|}\right) \\ \lambda_1 t_{2k} + \lambda_2 t_{1k} = \theta_{\frac{r}{x_0}} + 2\pi k,\ k \in \mathbb{Z}\ . \end{cases} \tag{4}$$

In the system above, $\theta_{\frac{r}{x_0}}$ denotes the argument of the complex number $\dfrac{r}{x_0}$. We have the following cases to consider:

Case 1: $\lambda_1 > 0,\ \lambda_2 > 0$.

Rewriting system (4), we obtain

$$\begin{cases} t_{2k} = \dfrac{\lambda_1}{\lambda_2} t_{1k} - \dfrac{\ln\left(r/|x_0|\right)}{\lambda_2} & \text{(a)} \\[2ex] t_{2k} = \dfrac{-\lambda_2}{\lambda_1} t_{1k} + \dfrac{1}{\lambda_1}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right),\ k \in \mathbb{Z}\ . & \text{(b)} \end{cases}$$

Equation (a) represents a real line R with slope $\dfrac{\lambda_1}{\lambda_2} > 0$ and passing through the point $\left(0, -\frac{\ln(r/|x_0|)}{\lambda_2}\right) \in \mathbb{R}^2$. For each $k \in \mathbb{Z}$, (b) represents a real line S_k which is orthogonal to R; we are looking for the intersection points $\{R \cap S_k,\ k \in \mathbb{Z}\}$.

Isolating t_{1k}, we obtain:

$$\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2}\ t_{1k} = \frac{\ln(r/|x_0|)}{\lambda_2} + \frac{1}{\lambda_1}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right),\ k \in \mathbb{Z}\ .$$

Since $\lambda_1, \lambda_2 > 0$, we have that $t_{1k} \to \infty$ when $k \to \infty$ and therefore

$$\lim_{k\to\infty} |\varphi_{q_0,2}(t_k)| = \lim_{k\to\infty} |y_0 e^{-t_k}| = \lim_{k\to\infty} \frac{|y_0|}{e^{\mathrm{Re}(t_k)}} = \lim_{k\to\infty} \frac{|y_0|}{e^{t_{1k}}} = 0\ .$$

Note that, with the sequence of values $t_k = t_{1k} + it_{2k}$ found above, we have that

$$\lim_{k\to\infty} |y_0 e^{-t_k}| = 0$$

for any starting point $q_0 = (x_0, y_0) \in \mathbb{D}_r^2 \setminus E = \{xy = 0\}$; in other words,

$$\mathbb{D}_r^2 \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta) \ .$$

In the case $\lambda_1 < 0, \lambda_2 < 0$, we have that $|\varphi_{q_0,2}(t_k)| = |y_0 e^{-t_k}|$ tends to zero when $k \to -\infty$; hence we also conclude that $\mathbb{D}_r^2$ is contained in $\mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta)$. By symmetry, in both cases we have that $\mathbb{D}_r^2$ is contained in $\mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta')$ where $\Delta' = \mathbb{D}_\varepsilon \times \{y = r\}$.

Case 2: $\lambda_1 > 0, \lambda_2 < 0$.

Rewriting system (4) we obtain

$$\begin{cases} t_{2k} = -\dfrac{\lambda_1}{|\lambda_2|} t_{1k} + \dfrac{\ln(r/|x_0|)}{|\lambda_2|} & \text{(a)} \\ \\ t_{2k} = \dfrac{|\lambda_2|}{\lambda_1} t_{1k} + \dfrac{1}{\lambda_1}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right), \ k \in \mathbb{Z} \ . & \text{(b)} \end{cases}$$

Equation (a) represents a real line R with slope $-\dfrac{\lambda_1}{|\lambda_2|} < 0$ and for each $k \in \mathbb{Z}$ (b) represents a real line S_k which is orthogonal to R.

Isolating t_{1k}, we obtain:

$$\frac{-(\lambda_1^2 + \lambda_2^2)}{\lambda_1 |\lambda_2|} \ t_{1k} = -\frac{\ln(r/|x_0|)}{|\lambda_2|} + \frac{1}{\lambda_1}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right), \ k \in \mathbb{Z} \ .$$

Hence when $k \to \infty$, we have that $t_{1k} \to -\infty, t_{2k} \to \infty$. So it follows that

$$\lim_{k \to \infty} |\varphi_{q_0,2}(t_k)| = \lim_{k \to -\infty} |y_0 e^{-t_k}| = \lim_{k \to -\infty} \frac{|y_0|}{e^{\mathrm{Re}(t_k)}} = \lim_{k \to -\infty} \frac{|y_0|}{e^{t_{1k}}} = 0 \ .$$

Once again, we remark that in this case, as in Case 1, we have that

$$\mathbb{D}_r^2 \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta) \ .$$

In the case $\lambda_1 < 0, \lambda_2 > 0$, we have that $t_{1k} \to \infty, t_{2k} \to -\infty$ when $k \to \infty$ and hence

$$\lim_{k \to \infty} |\varphi_{q_0,2}(t_k)| = \lim_{k \to \infty} |y_0 e^{-t_k}| = \lim_{k \to \infty} \frac{|y_0|}{e^{\mathrm{Re}(t_k)}} = \lim_{k \to \infty} \frac{|y_0|}{e^{t_{1k}}} = 0 \ .$$

And once again, we conclude that $\mathbb{D}_r^2 \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta)$. By symmetry, in both cases we have that $\mathbb{D}_r^2$ is contained in $\mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta')$ where $\Delta' = \mathbb{D}_\varepsilon \times \{y = r\}$.

Case 3: $\lambda_1 = 0$.

So system (4) can be rewritten as

$$\begin{cases} t_{2k} = -\dfrac{1}{\lambda_2}\ln\Big(r/|x_0|\Big) & \text{(a)} \\ \\ t_{1k} = \dfrac{1}{\lambda_2}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right),\ k \in \mathbb{Z}\ . & \text{(b)} \end{cases}$$

Therefore (a) is a horizontal line in $\mathbb{R}^2$ and for each $k \in \mathbb{Z}$, (b) is a real vertical line. Given that

$$\begin{array}{ll} \text{if } \lambda_2 > 0 \text{ then } t_{1k} \to \infty \text{ when } k \to \infty & \text{(i)} \\ \text{if } \lambda_2 < 0 \text{ then } t_{1k} \to \infty \text{ when } k \to -\infty\ , & \text{(ii)} \end{array}$$

it follows that

$$|\varphi_{q_0,2}(t_k)| = |y_0 e^{-t_k}| \to 0 \text{ when } k \to \infty \text{ in case (i)},$$
$$|\varphi_{q_0,2}(t_k)| = |y_0 e^{-t_k}| \to 0 \text{ when } k \to -\infty \text{ in case (ii)}.$$

Again we conclude that $\mathbb{D}^2_r$ is contained in $\mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}^2_r}}(\Delta)$. By symmetry, we also have $\mathbb{D}^2_r \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}^2_r}}\Delta'$, where Δ' is as before.

Case 4: $\lambda_2 = 0, \lambda_1 \in \mathbb{R}_{>0}$.

We remark this is the last case to consider. Since $\lambda = \lambda_1 \in \mathbb{R}_{>0}$, the set A_{q_0} relative to a point $q_0 = (x_0, y_0) \in \mathbb{D}^2_r \setminus E = \{xy = 0\}$ is

$$A_{q_0} = \left\{ t = t_1 + it_2 \in \mathbb{C}\ ;\ -\ln\Big(r/|y_0|\Big) \le t_1 \le \frac{\ln\Big(r/|x_0|\Big)}{\lambda} \right\}\ .$$

We are interested in the points $t_k \in A_{q_0}$ such that $\varphi_{q_0,1}(t_k) = r$. Then, rewriting system (4), they will be the points $t_k = t_{1k} + it_{2k} \in A_{q_0}$ which satisfy

$$\begin{cases} t_{1k} = \dfrac{1}{\lambda}\ln\Big(r/|x_0|\Big) \\ \\ t_{2k} = \dfrac{1}{\lambda}\left(\theta_{\frac{r}{|x_0|}} + 2\pi k\right)\ . \end{cases}$$

So in this case we have that

$$|\varphi_{q_0,2}(t_k)| = |y_0 e^{-t_k}| = \frac{|y_0|}{e^{\mathrm{Re}(t_k)}} = \frac{|y_0|}{e^{t_{1k}}} = \frac{|y_0|}{\Big(r/|x_0|\Big)^{1/\lambda}} = |y_0|\left(\frac{|x_0|}{r}\right)^{1/\lambda}\ ,$$

which does not depend on $k \in \mathbb{Z}$. So in order to have $|\varphi_{q_0,2}(t_k)| < \varepsilon$, we must have

$$|y_0|(|x_0|/r)^{1/\lambda} < \varepsilon \ .$$

Consider the mapping

$$\begin{array}{rccc} \rho : & M & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (|x|,|y|) \end{array} .$$

The image of the set

$$\{(x,y) \in M;\ |y|(|x|/r)^{1/\lambda} < \varepsilon\} \subset M$$

by ρ is the set

$$\{\tilde{y}(\tilde{x}/r)^{1/\lambda} < \varepsilon\} \subset \mathbb{R}^2_{\geq 0} \ ,$$

where $\tilde{y} = |y|$, $\tilde{x} = |x|$. Since $\lambda > 0$, $\{\tilde{y}\tilde{x}^{1/\lambda} = \varepsilon\}$ is a hyperbole and the polydisc

$$\tilde{\mathbb{D}}^2 = \left\{ (\tilde{x},\tilde{y}) \in \mathbb{R}^2_{\geq 0} \ ; \quad \begin{array}{c} \tilde{x} < r\varepsilon^{\lambda/2} \ , \\ \tilde{y} < \varepsilon^{1/2} \end{array} \right\}$$

is contained in $\{|y|(|x_0|/r)^{1/\lambda} < \varepsilon\}$. In M we put

$$\mathbb{D}^2_\delta = \left\{ (x,y) \in \mathbb{D}^2_r \ ; \ |x|,|y| < \delta \text{ where } \delta = \min\{\varepsilon^{\lambda/2}, \varepsilon^{1/2}\} \right\} \ .$$

Hence if $q_0 \in \mathbb{D}^2_\delta$ and t_k are the solutions of system (4), $|\varphi_{q_0,2}(t_k)| < \varepsilon$. Therefore $\mathbb{D}^2_\delta$ is contained in $\mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}^2_r}}(\Delta)$, and, by symmetry, $\mathbb{D}^2_\delta \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}^2_r}} \Delta'$, where Δ' is as before.

□

Now we will regard the cases when the 1-form ω is not linearizable. Due to the following result, we may assume that

$$\omega = y(\lambda + f(x,y))dx + xdy \text{ where } \lambda \in \mathbb{R}_{>0} \ .$$

Theorem 9 (Poincaré)[22] *Let*

$$\omega = xdy + \alpha ydx + \cdots \quad \textit{with } \alpha \notin \mathbb{R}_{>0} \cup \mathbb{Q}_{<0} \ .$$

Then the foliation $\mathcal{F}$ given by $(\omega = 0)$ is linearizable, i. e. there exist holomorphic coordinates X, Y such that $\mathcal{F}$ is given by

$$Xdy + \alpha Y dx \ .$$

Lemma 21 *Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2, \underline{0})$ given by*

$$\omega = y(\lambda + f(x,y))dx + xdy$$

where $\lambda \in \mathbb{R}_{>0}$, $|\lambda + f(x,y)| > \tilde{\lambda}$, $|\lambda + \mathrm{Re}f(x,y)| > \tilde{\lambda}$ and $|\lambda + \mathrm{Im}f(x,y)| > \tilde{\lambda}$ for some $\tilde{\lambda} > 0$. Let $\Delta = \{(r,0)\} \times \mathbb{D}_\varepsilon$. Then there exists $\delta = \delta(\tilde{\lambda}, \varepsilon)$ depending only on $\tilde{\lambda}, \varepsilon$, with $0 < \delta \leq r$ such that

$$\mathbb{D}_\delta \times \mathbb{D}_\delta \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^2}}(\Delta) \ .$$

Proof: We want to find a $\delta > 0$ which depends only on $\tilde{\lambda}, \varepsilon$, such that for each point $p_0 = (x_0, y_0) \in \mathbb{D}_\delta^2$ there exists a path

$$\begin{aligned} \gamma : [0,1] &\rightarrow \mathbb{D}_r^2 \\ t &\mapsto \gamma(t) = (\gamma_1(t), 0) \end{aligned}$$

whose lifting $\tilde{\gamma}$ of γ from p_0 (that is to say, the pre-image of γ in the leaf of $\mathcal{F}|_{\mathbb{D}_r^2}$ which contains the point p_0) is contained in $\mathbb{D}_r^2$ and satisfies

$$\tilde{\gamma}(1) = (r, \tilde{\gamma}_2(1)) \text{ with } |\tilde{\gamma}_2(1)| < \varepsilon \ .$$

Consider $\mathbb{D}_r \times \{0\}$ and the point $(x_0, 0) \in \mathbb{D}_r \times \{0\}$. Let

$$\alpha(t) = (x_0 e^{2\pi i t}, 0), \ t \in [0,1] \ .$$

There exists a t_0 such that $x_0 e^{1\pi i t_0} \in [0, r]$; put $s_0 = x_0 e^{2\pi i t_0}$ and let

$$\beta(t) = (s_0 + t(r - s_0), 0), \ t\imath[0,1] \ .$$

We will fix

$$\gamma = \beta \circ \alpha \ .$$

Let $\tilde{\alpha}$, $\tilde{\beta}$ be the liftings of α, β. Hence $\tilde{\gamma} = \tilde{\beta} \circ \tilde{\alpha}$. We can write

$$\tilde{\alpha}(t) = (x_0 e^{2\pi i t}, \tilde{\alpha}_2(t))$$

$$\tilde{\beta}(t) = (s_0 + t(r - s_0), \tilde{\beta}_2(t)).$$

If $\tilde{\beta}(0) = (s_0, \tilde{y}_0)$, we have that $|\tilde{\beta}_2(1)| < |\tilde{y}_0|$. So we need to find a δ with the following property: if $\tilde{\alpha}(0) = (x_0, y_0)$ with $|x_0| < \delta$, $|y_0| < \delta$, then for all $t \in [0,1]$ we have that

$$|\tilde{\alpha}_2(1)| < \varepsilon \ .$$

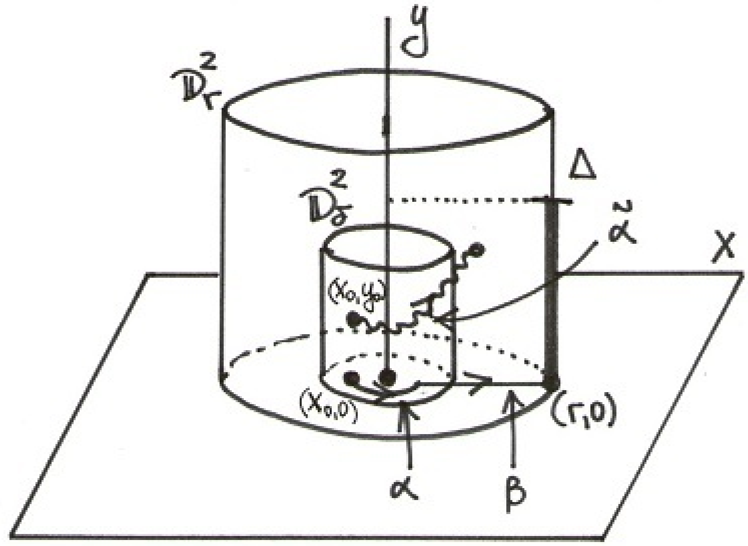
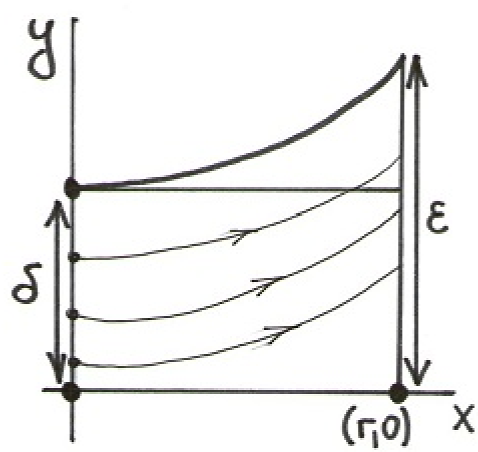


Figure 11: Scheme of Lemma 21.



We have $\tilde{\alpha}'(t) = (2\pi i x_0 e^{2\pi i t}, \tilde{\alpha}_2'(t))$. Since $\omega|_{\tilde{\alpha}(t)}(\tilde{\alpha}'(t)) \equiv 0$, it follows that

$$\omega|_{\tilde{\alpha}(t)}(\tilde{\alpha}'(t)) = \tilde{\alpha}_2(t) \cdot 2\pi i x_0 e^{2\pi i t} \cdot (\lambda + f) + x_0 e^{2\pi i t} \tilde{\alpha}_2'(t) \ = 0.$$

Hence

$$\tilde{\alpha}_2'(t) = -2\pi i \tilde{\alpha}_2(t)(\lambda + f) \ .$$

Let $\rho(t) = |\tilde{\alpha}_2(t)|^2 = \tilde{\alpha}_2(t) \cdot \overline{\tilde{\alpha}}_2(t)$. So

$$\rho'(t) = \tilde{\alpha}_2(t) \cdot \overline{\tilde{\alpha}}_2'(t) + \tilde{\alpha}_2'(t) \cdot \overline{\tilde{\alpha}}_2(t) \ .$$

We have that

$$\overline{\tilde{\alpha}}_2'(t) = 2\pi i \, \overline{\tilde{\alpha}}_2(t) \overline{(\lambda + f)} \ ;$$

hence

$$\begin{aligned} \rho'(t) &= 2\pi i \, \tilde{\alpha}_2(t) \overline{\tilde{\alpha}}_2(t) \overline{(\lambda + f)} - 2\pi i \, \tilde{\alpha}_2(t) \overline{\tilde{\alpha}}_2(t)(\lambda + f) \\ &= 2\pi i \, \rho(t) \left[\overline{(\lambda + f)} - (\lambda + f) \right] \\ &= 2\pi i \, \rho(t) \cdot (-2i \, \mathrm{Im}(f)) \\ &= 4\pi \, \rho(t) \mathrm{Im}(f) \in \mathbb{R} \ . \end{aligned}$$

However, since $|\lambda + \mathrm{Im} f| > \tilde{\lambda} > 0$, we have that $|4\pi \mathrm{Im}(f)| < K$, where $K \in \mathbb{R}_{>0}$ is a constant who depends on $\tilde{\lambda}$. Hence for $t \in [0, 1]$ we have that

$$|\tilde{\alpha}_2(t)|^2 = \rho(t) < \delta e^{Kt}, \delta \in \mathbb{R}_{>0} \ .$$

We take $\delta = \varepsilon e^K$ and the result follows.

□

Now let us go to the proof of Proposition 19 for $n = 3$.

Assume that $\mathcal{F}$ is defined in $\mathbb{D}_r^3$ where $\mathbb{D}_r = \{x \in \mathbb{C}; \ |x| \leq r\}$, $r > 0$. If the origin is a CH-simple point of $\mathcal{F}$ with dimensional type two, we can write the generator of $\mathcal{F}$ in $\mathbb{D}_r^2$ as follows:

$$\omega = \frac{dx}{x} + (\lambda + b(x, y)) \frac{dy}{y} \text{ where } a(0) = b(0) = 0 \text{ and } \lambda \notin \mathbb{R}_{<0} \ .$$

Let $R = (r, 0, 0) \in (y = 0)$, $\Delta = \{(r, y, 0); \ y \in \mathbb{D}_\varepsilon\}$. Consider the plane section $\Gamma = (z = 0) \cap \mathbb{D}_r^3$. We have that the origin is a CH-simple point of $\mathcal{F}|_\Gamma$. Due to the previous lemmas we have that $\mathrm{Sat}_{\mathcal{F}|_\Gamma}(\Delta) \cup ((xy = 0) \cap \Gamma)$ is a neighborhood of the origin in the section Γ. However, since the origin is a point with dimensional type two, there is a nonsingular vector field ξ tangent to $\mathcal{F}$ and so it follows that $\mathrm{Sat}_{\mathcal{F}}(\Delta) \cup (xy = 0)$ is a neighborhood of the origin.

Now suppose the origin has dimensional type three. Thus we can write the generator of $\mathcal{F}$ as

$$\omega = \frac{dx}{x} + (\lambda + b(x,y,z))\frac{dy}{y} + (\mu + c(x,y,z))\frac{dz}{z}$$

with $b(0) = c(0) = 0$ and $\lambda \notin \mathbb{R}_{<0}$, $\mu \notin \mathbb{R}_{<0}$. Let $R = (r,r,0) \in (z = 0)$, $\Delta = \{(r,r,z);\ z \in \mathbb{D}_\varepsilon\}$. If ω can be normalized, the result follows by applying Lemma 20 to $\mathcal{F}$ restricted to, for instance, the plane sections $\Gamma = \{x = r\}$, $\Gamma' = \{y = r\}$. So suppose ω cannot be normalized; thus we have $\lambda, \mu \in \mathbb{R}_{>0}$. There exists a neighborhood $U \subset \mathbb{D}_r^3$ of the origin where b and c are sufficiently small. Take a point $(0, y_0, 0) \in U$ and consider the section $\Gamma = \{y = y_0\}$. So $\mathcal{F}|_\Gamma$ is given by

$$\omega|_\Gamma = \frac{dx}{x} + (\mu + c(x, y_0, z))\frac{dz}{z}\ .$$

Due to Lemma 21, we find $\delta > 0$ such that the polydisc $\mathbb{D}_\delta \times \{y_0\} \times \mathbb{D}_\delta$ is contained in $\mathrm{Sat}_{\mathcal{F}|_\Gamma}(\Delta_\Gamma)$ where $\Delta_\Gamma \subset \Gamma$ is a one-dimensional section transversal to $\{z = 0\}$ such that $\Delta_\Gamma \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^3}}$. Since there is uniformity in the existence of δ, we have that

$$\mathbb{D}_\delta \times \mathbb{D}_\delta \times \mathbb{D}_\delta \subset \mathrm{Sat}_{\mathcal{F}|_{\mathbb{D}_r^3}}(\Delta)$$

and the result follows.

□

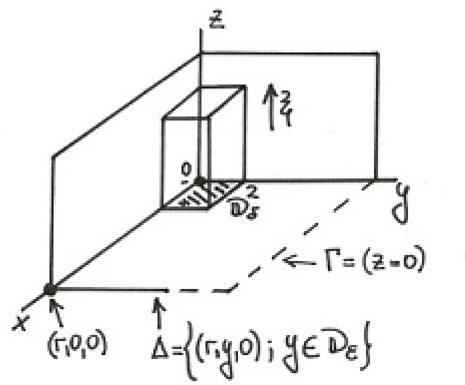


Figure 12: Proof of Proposition 19 for dimensional type two.

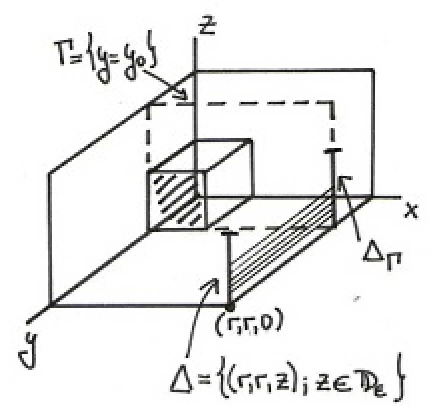


Figure 13: Proof of Proposition 19 for dimensional type three.

3.3 Brunella's alternative without nodal components

In this section we consider a RICH foliation $\mathcal{F}$ in $M = (\mathbb{C}^3, \underline{0})$ which does not admit a germ of invariant surface. Let

$$\mathcal{S}: \quad \pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N : M_N \to M_0 = M$$

be a reduction of singularities of $\mathcal{F}$ as in Section 3.1. We will assume that $\mathcal{F}$ has no nodal components.

We want to find a neighborhood $W \subset M$ of the origin such that each leaf of $\mathcal{F}$ in W has a germ of analytic invariant curve. We are supposing $\mathcal{F}$ does not admit a germ of invariant surface; as remarked in Sections 1.9 and 1.10, this implies that there exists a compact component D_j^N of the exceptional divisor $E^N \subset M_N$ which is dicritical. Write

$$E^N_{c,dic} \subset E^N_{dic}$$

the union of the compact components of E^N_{dic} and consider the set

$$H = \text{ union of leaves of } \mathcal{F}_N \text{ which intersect } E^N_{c,dic} \; .$$

Note that H is a saturated set. For each leaf $L \in H$ we fix a regular point

$$q \in L \cap D_j^N \text{ where } D_j^N \subset E^N_{c,dic} \; .$$

Hence we can find a germ of analytic curve $\gamma \subset L$. Indeed, if x, y, z are local coordinates at q we have that $\mathcal{F}$ is given by $dz = 0$ (see Figure 14),

$$D_j^N = (y = 0), \; L = \{z = 0\}, \; \gamma = (x = z = 0) \; .$$

Since $\pi(E^N_{c,dic}) = \underline{0}$ we have that $\pi(\gamma) = \tilde{\gamma}$ is a germ of analytic curve such that

$$\tilde{\gamma} \subset \pi(L) \cup \{\underline{0}\} \; .$$

We are going to show that $H \cup E^N$ is a neighborhood of $\pi^{-1}(\underline{0})$.

Remark 18 Assume $H \cup E^N$ is a neighborhood of $\pi^{-1}(\underline{0})$. Then $W = \pi(H \cup E^N)$ is a neighborhood of the origin $\underline{0} \in \mathbb{C}^3$ such that for each leaf $L \subset W$ of $\mathcal{F}$ in W there is an analytic curve $\gamma \subset L$ with $\underline{0} \in \gamma$ and therefore we have Theorem 8.

Proposition 22 *$H \cup E^N$ is a neighborhood of $\pi^{-1}(\underline{0})$.*

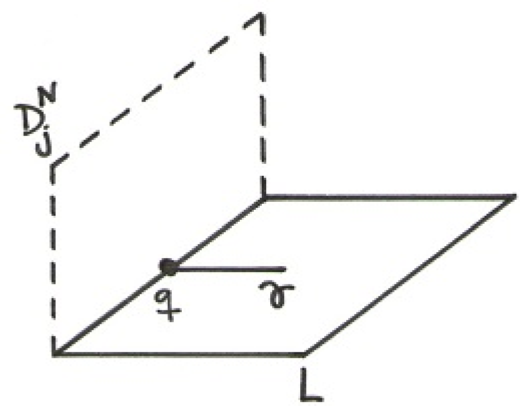


Figure 14: A leaf L intersecting a dicritical component D_j^N.

In order to simplify the exhibition of the next arguments, we give the following definition:

Definition 14 *Let $D_i^N \subset E^N$ be an invariant irreducible component. We will say D_i^N is* partially covered *if $H \cup E^N$ is a neighborhood of $D_i^N \setminus (Sing\ \mathcal{F}_N \cap D_i^N)$. A subset $A \subset M_N$ is said to be* well covered *if $H \cup E^N$ is a neighborhood of A.*

Lemma 23 *Let D_i^N be an invariant irreducible component of $\pi^{-1}(\underline{0})$. If there exists a one-dimensional section $\Delta \subset M_N$ transversal to D_i^N at a regular point q such that $\Delta \setminus \{q\} \subset H$, then D_i^N is partially covered.*

Proof: Let $\Delta_q \subset M_N$ be a one-dimensional section transversal to D_i^N at a regular point $q \in D_i^N$ and such that $\Delta_q \setminus \{q\} \subset H$. Since H is a saturated set we have that

$$\mathrm{Sat}_{\mathcal{F}_N}(\Delta_q) \subset H \cup E^N \ .$$

Let $q' \in D_i^N$, $q' \neq q$ be a regular point. There exists a compact path

$$\alpha : I = [0,1] \to D_i^N$$

such that $\alpha(0) = q$, $\alpha(1) = q'$ and $\alpha(I) \cap (\mathrm{Sing}\ \mathcal{F}_N \cap D_i^N) = \emptyset$.

Since q is a regular point, there exists a neighborhood of q, $V_q \subset M_N$, such that $\mathcal{F}_N$ is given by $dz = 0$ in V_q. That is to say, V_q is an open foliated set. Taking a smaller Δ_q if necessary, we may assume that $\Delta_q \subset V_q$. We cover the path α with a finite number of open foliated sets such that the last one of these, $V_{q'}$, contains the point q'. Let $\Delta_{q'} \subset M_N$ be a one-dimensional section transversal to D_i^N at the point q' such that $\Delta_{q'} \subset V_{q'}$. So

$$\Delta_{q'} \subset \mathrm{Sat}_{\mathcal{F}_N|_{V_q}}(\Delta), \text{ which implies that } \Delta_{q'} \setminus \{q'\} \subset H \ .$$

Thus $H \cup E^N$ is a neighborhood of every regular point of D_i^N and the result follows.

□

Lemma 23 asserts that the behavior of the section Δ can be "pushed" from one regular point to another along a compact path which avoids the singularities of the foliation and in that way it is possible to cover all the regular points of the component.

Now let us go to the proof of Proposition 22.

First reduction: it is enough to show that all the invariant components of E^N are partially covered. If we have this result, we can cover any curve of the singular set which is not generically nodal. Moreover, if we have a curve $\Gamma \subset \mathrm{Sing}\, \mathcal{F}_N$ which is generically nodal, we can find a finite chain (see Figure 15)

$$\Gamma, p_1, \Gamma_1, p_2, \Gamma_2, \ldots, p_k, \Gamma_k, q$$

where $p_1 \in \Gamma \cap \Gamma_1$, $p_2 \in \Gamma_1 \cap \Gamma_2$, $\ldots$, $p_k \in \Gamma_{k-1} \cap \Gamma_k$ are nodal points with dimensional type three, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are generically nodal and finally $q \in \Gamma_k$ is not nodal (in particular $\tau_q = 3$). Taking an invariant component $D_j^N \ni q$ we see by Proposition 19 that q is well covered. Now we follow the sequence $\Gamma_k, \Gamma_{k-1}, \ldots, \Gamma_1, \Gamma$ through the points $p_k, p_{k-1}, \ldots, p_1$ to deduce that Γ is also well covered in view of Proposition 18.

Now, let us show that all the invariant components of E^N are partially covered. In view of the connectedness of the dual graph of E^N (note that any component intersecting a dicritical component is partially covered), it is enough to show that if D_i^N and D_j^N are two invariant components of E^N such that $D_i^N \cap D_j^N \neq \emptyset$, then we have:

$$D_i^N \text{ partially covered } \Rightarrow \ D_j^N \text{ partially covered.}$$

Let us show this.

In order to be complete in our proof, we extend the divisor E^N to a bigger one $\widetilde{E}^N$ locally around the invariant components by adding an additional irreducible

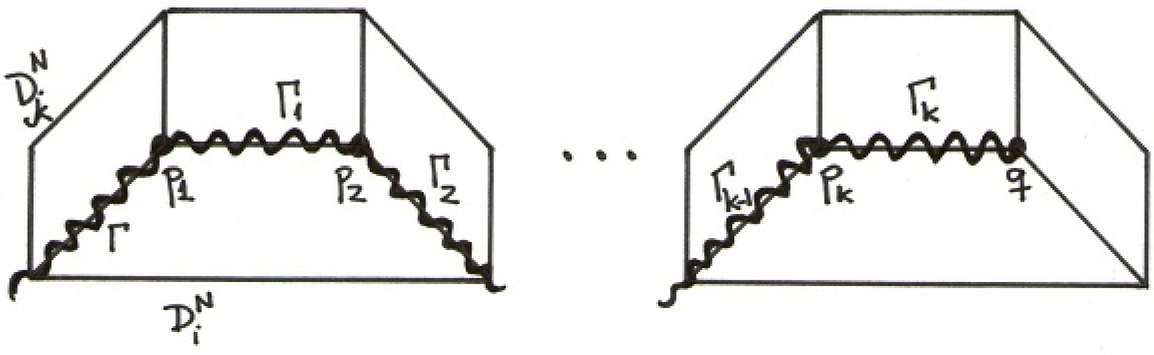


Figure 15: Scheme of the sequence $\Gamma, p_1, \Gamma_1, p_2, \Gamma_2, \ldots, p_k, \Gamma_k, q$.

invariant component associated to each connected component of the union of trace curves (see the description of Cano-Cerveau's argument, Section 1.9). This technical trick allows us to do our argument assuming that all the points of our interest are in fact simple corners, we do not insist more on this.

So we have $D_i^N \cap D_j^N \neq \emptyset$ and D_i^N is partially covered. We have to prove that D_j^N is also partially covered. If $D_i^N \cap D_j^N$ is not generically nodal, we are done by Proposition 19. So assume that $D_i^N \cap D_j^N$ is generically nodal; call $\Gamma = D_i^N \cap D_j^N$. Since Γ is not contained in a nodal component, we find a point of interruption q in the connected component of the union of generically nodal curves that contains Γ. This implies the existence of a sequence

$$\Gamma, p_1, \Gamma_1, p_2, \Gamma_2, \ldots, p_k, \Gamma_k, q$$

as in the argument of the first reduction. Now, let us show that for each $s = 0, 1, 2, \ldots, k$ there is a partially covered component $\widetilde{D}_{i_s}$ of $\widetilde{E}^N$ such that $\Gamma_s \subset \widetilde{D}_{i_s}$. Once we prove this, we see that $q \in \Gamma_k \subset \widetilde{D}_{i_k}$ is well covered in view of Proposition 19. Now we finish by applying Proposition 18 along $\Gamma_k, p_k, \Gamma_{k-1}, \ldots, \Gamma_1, p_1, \Gamma$ to see that Γ is well covered and hence $D_j^N \supset \Gamma$ is necessarily partially covered.

It remains to prove the existence of $\widetilde{D}_{i_s}$. We take $\widetilde{D}_{i_0} = D_i^N$ and we proceed by induction assuming that $\widetilde{D}_{i_{s-1}}$ exists. Consider the point $p_s \in \Gamma_{s-1} \subset \widetilde{D}_{i_{s-1}}$. Now, if $\Gamma_s \subset \widetilde{D}_{i_{s-1}}$, we take $\widetilde{D}_{i_s} = \widetilde{D}_{i_{s-1}}$; otherwise, the point p_s is a corner and we take $\widetilde{D}_{i_s}$ such that $\widetilde{D}_{i_{s-1}} \cap \widetilde{D}_{i_s}$ is not generically nodal and $\Gamma_s \subset \widetilde{D}_{i_s}$. Of course, by Proposition 19, $\widetilde{D}_{i_s}$ is partially covered.

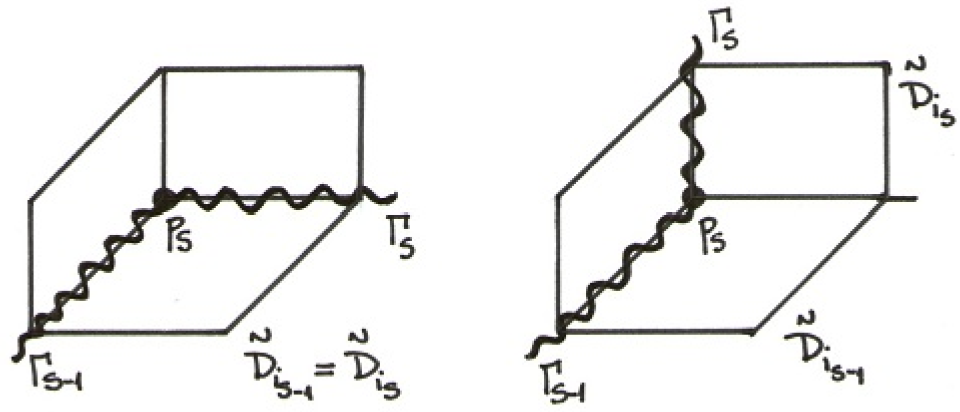


Figure 16: Existence of $\widetilde{D}_{i_s}$.

$\square$

4 Infinitesimal singular locus of RICH foliations

In this chapter we will study the behavior of a nodal component for a RICH foliation in intermediate steps of the reduction of singularities and prove Brunella's local alternative as given by Theorem 7.

4.1 CH pre-simple corners

Let $\mathcal{F}$ be a germ of codimension one foliation on $M = (\mathbb{C}^n, \underline{0})$ of dimensional type τ and $E \subset M$ be a normal crossings divisor. In this section, we will give the definition of *Complex Hyperbolic pre-simple corner* for the pair $\mathcal{F}, E$ and describe the behavior of these points under blow-up in the case $n = 3$.

Definition 15 *Let $\mathcal{F}$ be a germ of codimension one foliation on $M = (\mathbb{C}^n, \underline{0})$ of dimensional type τ and let $E \subset M$ be a normal crossings divisor. We say that the pair $\mathcal{F}, E$ has a* Complex Hyperbolic pre-simple corner at the origin *if and only if there are local coordinates $x_1, x_2, \ldots, x_n$ such that $\mathcal{F}$ is given by $\omega = 0$ where*

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + b_i(x_1, x_2, \ldots, x_\tau)) \frac{dx_i}{x_i}, \; b_i \in \mathbb{C}\{x_2, x_2, \ldots, x_\tau\}, b_i(0) = 0$$

with $\prod_{i=1}^{\tau} \lambda_i \neq 0$ and

$$E_{inv} = \left(\prod_{i=1}^{\tau} x_i = 0 \right) \;, \quad E_{dic} \subset \left(\prod_{i=\tau+1}^{n} x_i = 0 \right) \;.$$

Note that if we add the non-resonance condition, we recover a simple CH corner (Definition 6).

Remark 19 Suppose $n = 3$ and consider a point $p \in M$. As before, let $e_p(E_{inv})$ denote the number of invariant components of E passing through p. If $e_p(E_{inv}) = 1$ then p is a CH pre-simple corner if and only if p is a regular point of $\mathcal{F}$. Or, equivalently, we have that $p \in \mathrm{Sing}(\mathcal{F}, E)$ and p is a CH pre-simple corner then $e_p(E_{inv}) \geq 2$.

For the rest of this section, we will assume that $\mathcal{F}$ is a CH foliation of $M = (\mathbb{C}^3, \underline{0})$ and $E \subset M$ is a normal crossings divisor. Let $Y \subset M$ be a nonsingular variety with codimension at least two having normal crossings with E and invariant by $\mathcal{F}$. Let

$$\pi : M' \to M$$

be the blow-up centered at Y and consider the normal crossings divisor $E' = \pi^{-1}(E \cup Y) \subset M'$. Denote $\mathcal{F}' = \pi^*\mathcal{F}$ the transformed foliation of $\mathcal{F}$ by π, $D = \pi^{-1}(Y)$. We will denote $e_0(E_{inv})$ the number of invariant components of E through the origin.

We will prove the following results:

Proposition 24 *In the conditions above, if the origin is a CH pre-simple corner for $\mathcal{F}, E$ than any point $p \in \pi^{-1}(\underline{0})$ is a CH pre-simple corner for $\mathcal{F}', E'$.*

Proposition 25 *In the conditions above, if the origin is not a CH pre-simple corner for $\mathcal{F}, E$, then there exists a point $q \in \pi^{-1}(\underline{0})$ which is not a CH pre-simple corner for the pair $\mathcal{F}', E'$.*

Let us begin with the proof of Proposition 24.

Proof of Proposition 24: Suppose $\underline{0} \in \mathbb{C}^3$ is a CH pre-simple corner for $\mathcal{F}, E$. The case that the origin has dimensional type one is immediate. So first let's suppose that the origin has dimensional type two. Then $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega = (\lambda + a(x,y))\frac{dx}{x} + (\mu + b(x,y))\frac{dy}{y}$$

with $a(0) = b(0) = 0$ and $\lambda\mu \neq 0$. So $E = (xy = 0)$. There are several cases to consider.

First case: $Y = \{\underline{0}\}$ and π is not dicritical. Hence $\lambda + \mu \neq 0$. The first local chart is $x' = x, y' = y/x, z' = z/x$ and we have

$$\pi^*\omega = (\lambda + \mu + x(\cdots))\frac{dx}{x} + (\mu + x(\cdots))\frac{dy'}{y'} \ .$$

So all the points in the first local chart are CH pre-simple corners for $\mathcal{F}', E'$. The second local chart is $x' = x/y, y' = y, z' = z/y$ and we have

$$\pi^*\omega = (\lambda + y(\cdots))\frac{dx'}{x'} + (\lambda + \mu + y(\cdots))\frac{dy}{y}$$

and once again we see that all the points of the second local chart are CH pre-simple corners for $\mathcal{F}', E'$. Finally, the third local chart is $x' = x/z, y' = y/z, z' = z$ and we have

$$\pi^*\omega = (\lambda + z(\cdots))\frac{dx'}{x'} + (\mu + z(\cdots))\frac{dy'}{y'} + (\lambda + \mu + z(\cdots))\frac{dz}{z} \ .$$

Hence all the points in the third local are also CH pre-simple corners for $\mathcal{F}', E'$.

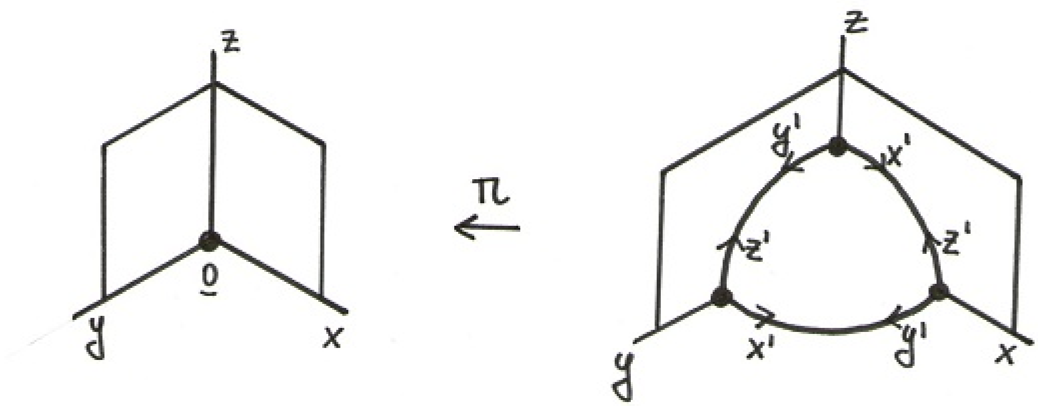


Figure 17: First case for dimensional type two: $Y = \{\underline{0}\}$ and π is not dicritical.

Second case: $Y = \underline{0}$ and π is dicritical. Hence $\lambda + \mu = 0$. In the first local chart we have

$$\pi^*\omega = (\cdots)dx + (\mu + x(\cdots))\frac{dy'}{y'} \ .$$

So there are no singular points in the first local chart; by symmetry, there are no singular points in the second local chart as well. In the third local chart we have

$$\pi^*\omega = (\lambda + z(\cdots))\frac{dx'}{x'} + (\mu + z(\cdots))\frac{dy'}{y'} + (\cdots)dz \ .$$

The only singular point in $\pi^{-1}(\underline{0})$ is the origin of the third local chart. However we see that the vector field

$$\xi = (\lambda + z(\cdots))\frac{\partial}{\partial x'} - zc'\frac{\partial}{\partial z}$$

is nonsingular and tangent to $\mathcal{F}'$ (c' is the coefficient of dz in $\pi^*\omega$). Hence the origin of the third chart is a CH pre-simple corner for $\mathcal{F}', E'$ with dimensional type two.

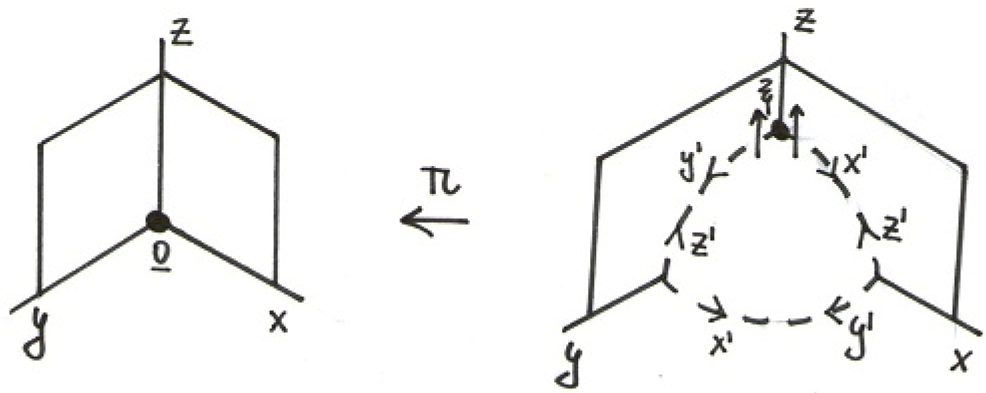


Figure 18: Second case for dimensional type two: $Y = \{\underline{0}\}$ and π is dicritical.

Third case: $Y = (x = y = 0)$ and π is not dicritical. Hence $\lambda + \mu \neq 0$. The first local chart is $x' = x, y' = y/x, z' = z$ and we have

$$\pi^*\omega = (\lambda + \mu + x(\cdots))\frac{dx}{x} + (\mu + x(\cdots))\frac{dy'}{y'} \ .$$

The second local chart is $x' = x/y, y' = y', z' = z$ and

$$\pi^*\omega = (\lambda + y(\cdots))\frac{dx'}{x'} + (\lambda + \mu + y(\cdots))\frac{dy}{y} \ .$$

In both cases, we only find CH pre-simple corners in $\pi^{-1}(\underline{0})$.

Fourth case: $Y = (x = y = 0)$ and π is dicritical. Hence $\lambda + \mu = 0$. In the first local chart we have

$$\pi^*\omega = (\cdots)dx + (\mu + x(\cdots))\frac{dy'}{y'} \ .$$

Hence there are no singular points in $\pi^{-1}(\underline{0})$ in the first local chart; by symmetry, there are no singular points in the second local chart as well. So all the points in $\pi^{-1}(\underline{0})$ are CH pre-simple corners with dimensional type one.

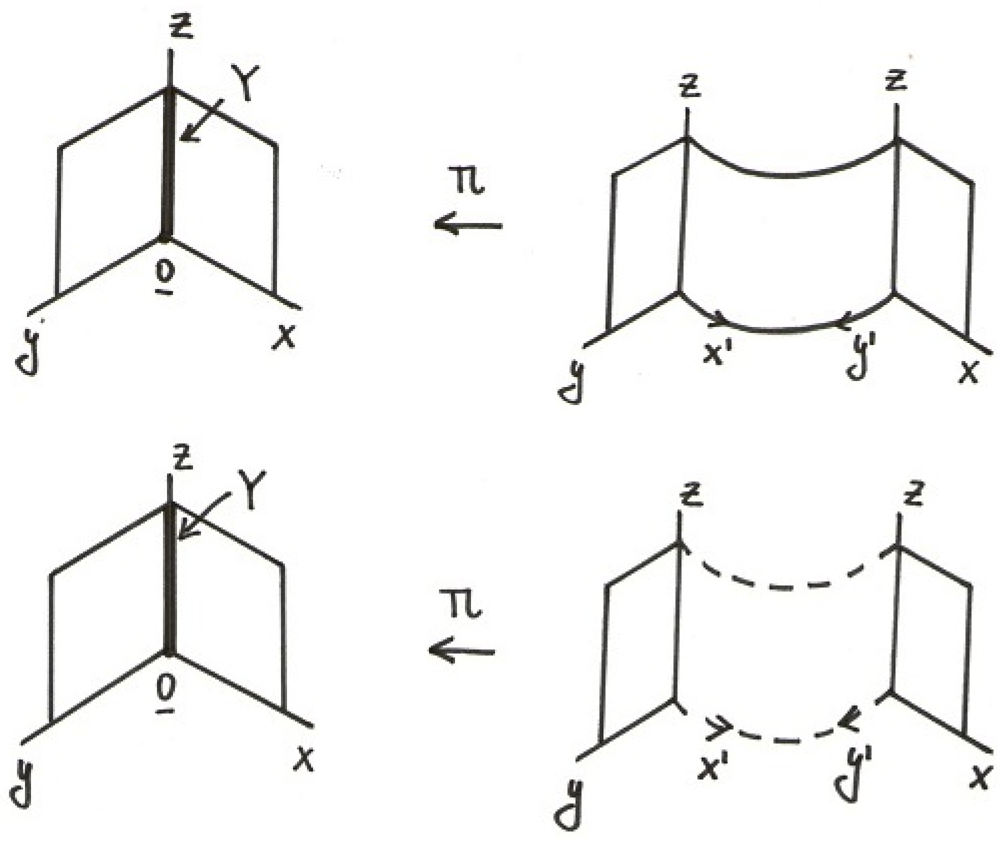


Figure 19: The image on the top shows the third case for dimensional type two: $Y = (x = y = 0)$ and π is not dicritical. The image at the bottom shows the fourth case for dimensional type two: $Y = (x = y = 0)$ and π is dicritical.

Now suppose the origin has dimensional type three. Then $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega = (\lambda + a(x,y,z))\frac{dx}{x} + (\mu + b(x,y,z))\frac{dz}{z} + (\delta + c(x,y,z))\frac{dz}{z}$$

with $a(0) = b(0) = c(0) = 0$ and $\lambda\mu\delta \neq 0$. Once again, there are several cases to consider.

First case: $Y = \{\underline{0}\}$ and π is not dicritical. Hence $\lambda + \mu + \delta \neq 0$. In the first local chart we have

$$\pi^*\omega = (\lambda + \mu + \delta + x(\cdots))\frac{dx}{x} + (\mu + x(\cdots))\frac{dy'}{y'} + (\delta + x(\cdots))\frac{dz'}{z'} \ .$$

So we see that all the points of $\pi^{-1}(\underline{0})$ are CH pre-simple corners for $\mathcal{F}', E'$ in the first local chart. By symmetry, all the points of $\pi^{-1}(\underline{0})$ are CH pre-simple corners for $\mathcal{F}', E'$ in the second and third local charts as well.

Second case: $Y = \{\underline{0}\}$ and π is dicritical. Hence $\lambda + \mu + \delta = 0$. In the first local chart we have

$$\pi^*\omega = (\cdots)dx + (\mu + x(\cdots))\frac{dy'}{y'} + (\delta + x(\cdots))\frac{dz'}{z'} \ .$$

The origin of the first local chart is the only singular point. However, we see that the vector field

$$\xi = (\mu + x(\cdots))\frac{\partial}{\partial x} - y'a'\frac{\partial}{\partial y'}$$

is nonsingular and tangent to $\mathcal{F}'$ (a' is the coefficient of dx in $\pi^*\omega$). Hence the origin of the first chart is a CH pre-simple corner for $\mathcal{F}', E'$. By symmetry, we have the same situation in the second and third local charts.

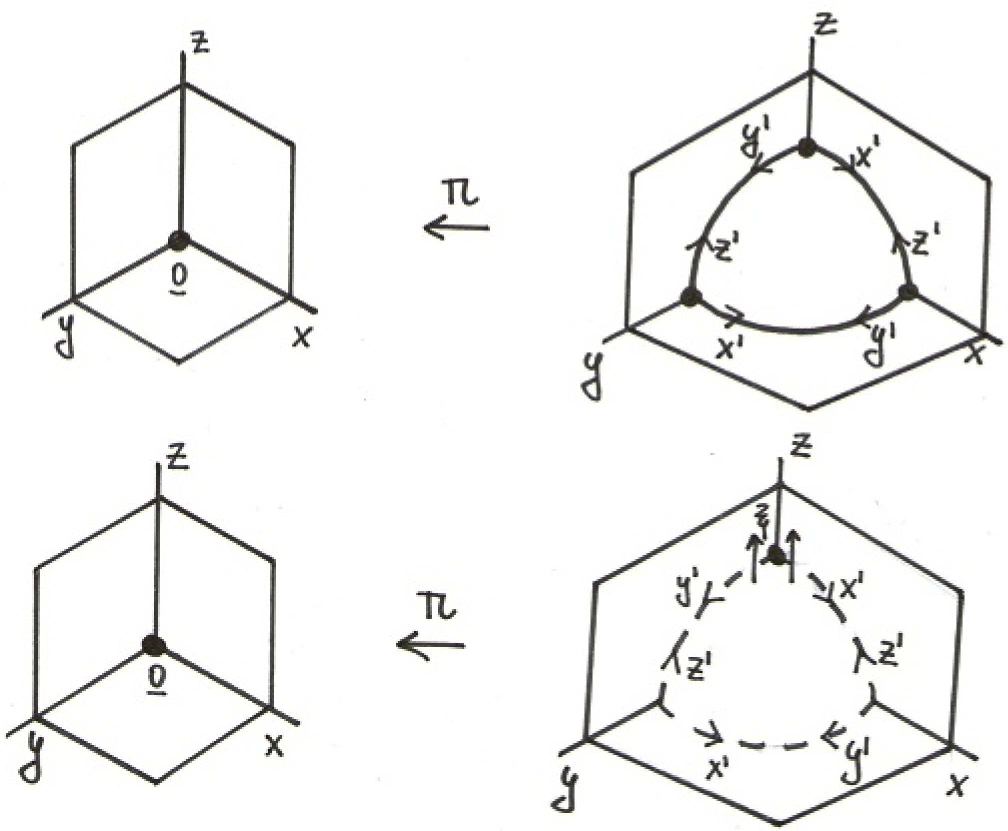


Figure 20: The image on the top shows the first case for dimensional type three: $Y = \{\underline{0}\}$ and π is not dicritical. The image at the bottom shows the second case for dimensional type three: $Y = \{\underline{0}\}$ and π is dicritical.

Third case: $Y = (x = y = 0)$ and $\mu + \lambda \neq 0$. Hence π is not dicritical. In the first local chart we have

$$\pi^*\omega = (\lambda + \mu + \cdots)\frac{dx}{x} + (\mu + \cdots)\frac{dy'}{y'} + (\delta + \cdots)\frac{dz'}{z'} \ .$$

So we see that all the points of $\pi^{-1}(\underline{0})$ are CH pre-simple corners for $\mathcal{F}', E'$ in the first local chart. By symmetry, all the points of $\pi^{-1}(\underline{0})$ are CH pre-simple corners for $\mathcal{F}', E'$ in the second local chart as well.

Fourth case: $Y = (x = y = 0)$, $\mu + \lambda = 0$ and π is dicritical. In the first local chart we have

$$\pi^*\omega = (\cdots)dx + (\mu + \cdots)\frac{dy'}{y'} + (\delta + \cdots)\frac{dz'}{z'} \ .$$

So the origin of the first local chart is the only singular point. However, the vector field

$$\xi = (\mu + \cdots)\frac{\partial}{\partial x} - y'a'\frac{\partial}{\partial y'}$$

is nonsingular and tangent to $\mathcal{F}'$ (a' is the coefficient of dx in $\pi^*\omega$). Hence the origin is a CH pre-simple corner for $\mathcal{F}', E'$ with dimensional type two. By symmetry, we have the same situation in the second local chart.

Fifth case: $Y = (x = y = 0)$, $\mu + \lambda = 0$ and π is not dicritical. In the first local chart we have

$$\pi^*\omega = f(x, y', z')\frac{dx}{x} + (\mu + g(x, y', z'))\frac{dy'}{y'} + (\delta + h(x, y', z'))\frac{dz'}{z'}$$

with $f(0) = g(0) = h(0) = 0$ and $x \not| f$. We can write

$$f(x, y', z') = xa'(x, y', z') + z'\phi(z') \ ,$$
$$g(x, y', z') = xb'(x, y', z') + z'\psi(z') \ ,$$
$$h(x, y', z') = xc'(x, y', z') + z'\eta(z') \ .$$

All points of dimensional type two are CH pre-simple corners. Suppose, by absurd, that the origin of the first chart q is not a CH pre-simple corner for $\mathcal{F}', E'$. Consider a point $R = (0, y_0, 0)$ and the plane section $\Delta = \{y = y_0\}$. We have that $\mathcal{F}'|_\Delta$ is given by the 1-form

$$(xa'(x, y_0, z') + z'\phi(z'))\frac{dx}{x} + (\delta + h(x, y', z'))\frac{dz'}{z'} \ .$$

So the point $(0, y_0, 0)$ is a saddle-node singularity of $\mathcal{F}'|_\Delta$. However, since $\mathcal{F}$ is a CH foliation, there cannot exist saddle-node singularities in the restriction to plane sections after blow-ups. Hence q is a CH pre-simple corner for $\mathcal{F}', E'$. By symmetry we have the same situation in the second local chart.

□

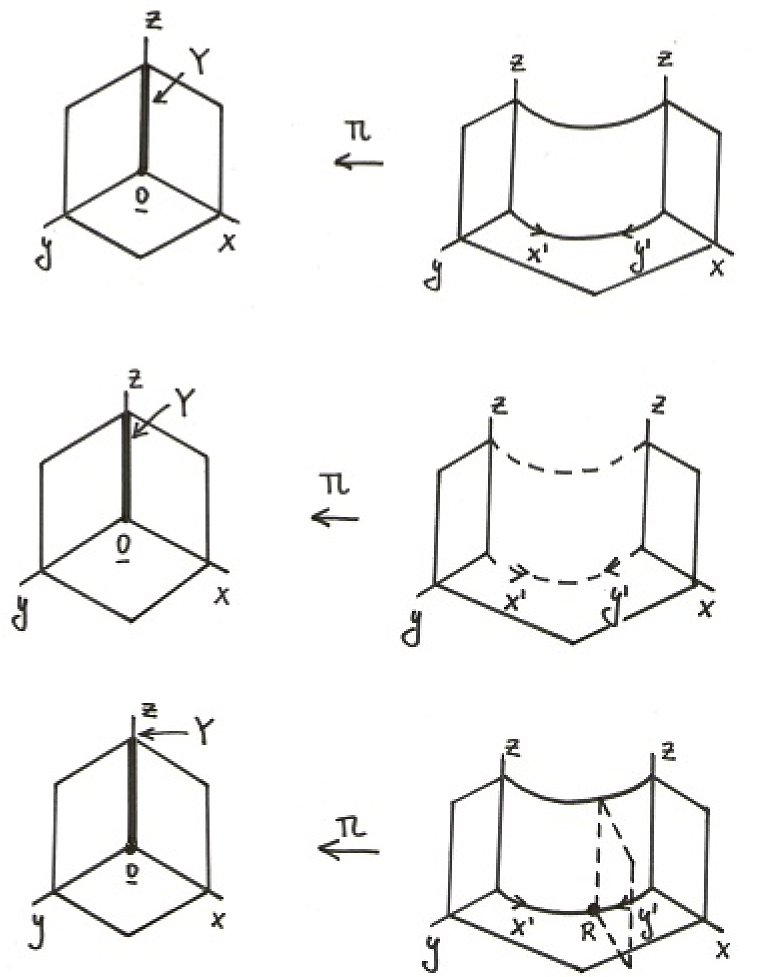


Figure 21: The image on the top shows the third case for dimensional type three: $Y = (x = y = 0)$ and $\mu + \lambda \neq 0$. The image in the middle shows the fourth case: $Y = (x = y = 0)$, $\mu + \lambda = 0$ and π is dicritical. The image at the bottom shows the fifth case: $Y = (x = y = 0)$, $\mu + \lambda = 0$ and π is not dicritical.

Now let us prove Proposition 25. We consider several situations that will be treated next. The first one is when $Y = \{\underline{0}\}$ and π is dicritical. Since $D = \pi^{-1}(\underline{0})$ is generically transversal, $\mathcal{F}'|_D$ defines a foliation in $\mathbb{P}^2 \simeq D$. Note that if $e_0(E_{inv}) = 0$ Proposition 25 is immediate: there exists a point $q \in D$ which is singular for $\mathcal{F}'|_D$; this point is not a CH pre-simple corner for $\mathcal{F}', E'$.

Lemma 26 *Suppose $Y = \{\underline{0}\}$ and that π is dicritical. If the origin is not a CH pre-simple corner for $\mathcal{F}, E$ then for each invariant component $E_i \subset E_{inv}$ there exists a point $q \in E'_i = \pi^{-1}(E_i) \cap D$ which is not a simple corner for the pair $\mathcal{F}', E'$.*

Proof: Let us recall the relationship between the degree of a foliation $\mathcal{G}$ in $\mathbb{P}^2$ and the singular points along invariant lines (see [6]). Suppose L is a projective line invariant by $\mathcal{G}$. Given local coordinates x, y, we may put $L = (y = 0)$. Let

$$\xi = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$$

be a vector field which is tangent to $\mathcal{G}$. Hence $\xi|_L = a(x, 0)\partial/\partial x$. For every point $p \in L$, let $\alpha_p = \nu_p(a(x, 0))$. Thus p is a singular point of $\mathcal{G}$ if and only if $\alpha_p \geq 1$. We have that

$$\sum_{p \in L} \alpha_p = \partial^0 \mathcal{G} + 1 \ .$$

There are several cases to consider.

First case: $e_0(E_{inv}) = 1$. Let E_1 be the invariant component of E through the origin. Due to Remark 19, we want to see that the set Sing $\mathcal{F}' \cap (E'_1 \cap D)$ is not empty. Suppose, by absurd, that every point of $E'_1 \cap D$ is a CH pre-simple corner for $\mathcal{F}', E'$. Hence all the points of the projective line $E'_1 \cap D$ are regular points of $\mathcal{F}'|_D$; this implies that $\partial^0 \mathcal{F}'|_D = -1$ and we arrive to an absurd.

Second case: $e_0(E_{inv}) = 2$. Let E_1, E_2 be the invariant components of E through the origin. Suppose, by absurd, that every point of $(E'_1 \cap D) \cup (E'_2 \cap D)$ are CH pre-simple corners for $\mathcal{F}', E'$. So the point $p = (E'_1 \cap E'_2) \cap D$ is the only singular CH pre-simple corner; in particular, we have that $\partial^0 \mathcal{F}'|_D = 0$. By a result of [18], we obtain that the dimensional type of the origin $\underline{0} \in \mathbb{C}^3$ is two, which is an absurd. Hence $\partial^0 \mathcal{F}'|_D \geq 1$ and we have the following options:

1. $\alpha_q \geq 2$, in which case q is not a CH pre-simple corner for $\mathcal{F}', E'$ and we are done.

2. $\alpha_q = 1$; in this case we find a pair of points $p_1 \in E_1' \cap D$, $p_2 \in E_2' \cap D$, $p_1, p_2 \neq q$, such that $\alpha_{p_1}, \alpha_{p_2} \geq 1$. Hence p_1, p_2 are not CH pre-simple corners for $\mathcal{F}', E'$ are we are done.

Third case: $e_0(E_{inv}) = 3$. Let E_1, E_2, E_3 be the components of E through the origin. Suppose, by absurd, that every point of

$$(E_1' \cap D) \cup (E_2' \cap D) \cup (E_3' \cap D)$$

is a CH pre-simple corner for $\mathcal{F}', E'$. This implies that $\partial^0 \mathcal{F}'|_D = 1$. Let us show that this cannot be. Note, due to Remark 19, that there is transversality between $\mathcal{F}'$ and D along each invariant projective line $E_i' \cap D$.

We have that $\mathcal{F}$ is locally given by $\omega = 0$ where

$$\omega = a(x,y,z)\frac{dx}{x} + b(x,y,z)\frac{dy}{z} + c(x,y,z)\frac{dz}{z}$$

where a, b, c have no common factor and $x \nmid a$, $y \nmid b$, $z \nmid c$. Put $r = \mathrm{order}_0(a,b,c)$. So we may write

$$\begin{aligned} a(x,y,z) &= A_r(x,y,z) + \cdots \\ b(x,y,z) &= B_r(x,y,z) + \cdots \\ c(x,y,z) &= C_r(x,y,z) + \cdots \end{aligned}$$

where A_r, B_r, C_r are homogeneous polynomials of degree r. Let

$$\Omega = XYZ\omega(X,Y,Z) = \Omega_{r+2} + \Omega_{r+3} + \cdots \ .$$

Since π is a dicritical blow-up, we have that

$$\Omega_{r+2}(R) \equiv 0$$

where R is the radial vector field. Hence it follows that

$$A_r + B_r + C_r \equiv 0 \ .$$

Now we will see that $\mathcal{F}'|_D$ as a foliation in $\mathbb{P}^2$ is the same foliation given by $\Omega_{r+2} = 0$. It is only necessary to check if both foliations coincide in the first chart of the blow-up π. Let V_0 be the open set related to the first local chart. We have the change of coordinate maps

$$V_0 \stackrel{y=Y/X}{\longrightarrow} \mathbb{C} \ , \ V_0 \stackrel{z=Z/X}{\longrightarrow} \mathbb{C} \ .$$

So

$$\Omega_{r+2} = YZA_r(X,Y,Z)dX + XZB_r(X,Y,Z) + XYC_r(X,Y,Z)dZ \ .$$

In the first chart, the foliation $\Omega_{r+2} = 0$ is given by the restriction of Ω_{r+2} to the projective space $X = 1$; hence

$$\Omega_{r+2}\Big|_{X=1} = zB_r(1,y,z)dy + yC_r(1,y,z)dz \ .$$

On the other hand, in the first local chart the change of coordinates is $x' = x$, $y' = y/x$, $z' = z/x$ and hence

$$\pi^*\omega = (\tilde{a} + \tilde{b} + \tilde{c})\frac{dx}{x} + \tilde{b}\frac{dy'}{y'} + \tilde{c}\frac{dz'}{z'}$$

where

$$\begin{aligned}
\tilde{a} &= a(x, xy', xz') = x^r(A_r(1,y',z') + x(\cdots)) \ , \\
\tilde{b} &= b(x, xy', xz') = x^r(B_r(1,y',z') + x(\cdots)) \ , \\
\tilde{c} &= c(x, xy', xz') = x^r(C_r(1,y',z') + x(\cdots)) \ .
\end{aligned}$$

Call $\omega' = \dfrac{1}{x^r}\pi^*\omega$. Since $A_r + b_r + C_r \equiv 0$, it follows that

$$\omega' = \frac{1}{x^r}\left[x(\cdots)\frac{dx}{x} + \Big(B_r(1,y',z') + x(\cdots)\Big)\frac{dy'}{y'} + \Big(C_r(1,y',z') + x(\cdots)\frac{dz'}{z'}\Big)\right] \ .$$

We have that $\mathcal{F}'|_D$ is given by $\omega'|_{x=0}$ where

$$\omega'\Big|_{x=0} = B_r(1,y',z')\frac{dy'}{y'} + C_r(1,y',z')\frac{dz'}{z'} \ .$$

So we conclude that $\mathcal{F}'|_D$ and the foliation given by $\Omega_{r+2} = 0$ are the same. Note that since $A_r + B_r + C_r \equiv 0$ we have that $B_r(1,y',z')$ and $C_r(1,y',z')$ have no common factor.

As a consequence of this it follows that the degree of the $\mathcal{F}'|_D$ is $r+1$. Hence, in the case $\partial^0\mathcal{F}'|_D = 1$, we have that $r = 0$. So we can write the coefficients of ω as follows:

$$\begin{aligned}
a(x,y,z) &= \lambda + \cdots \\
b(x,y,z) &= \mu + \cdots \\
c(x,y,z) &= \delta + \cdots
\end{aligned}$$

with $\lambda+\mu+\delta=0$. Suppose we have $\mu=0$. So $\mathcal{F}'$ is given by $\omega'=0$ where

$$\omega' = y'z'(\cdots)dx + xz'(\cdots)dy' + y'(\delta + x(\cdots))dz' \ .$$

Let us see that there is tangency between $\mathcal{F}'$ and D in this case. In the first chart, the tangency points of $\mathcal{F}'$ and $D=(x=0)$ are given by the set $(\omega\wedge dx=0)\cap(x=0)$, which gives us the set $(x=y'=0)$. This is an absurd, as we have remark earlier. Therefore we have that $\lambda\mu\delta\neq 0$, which implies that the origin is a CH pre-simple point. This is an absurd.

So we conclude that $\partial^0\mathcal{F}'|_D \geq 2$. By a direct computation of the degree along each invariant line $E_i'\cap D$ we see that there exist three points, at least two of them distinct, $p_1\in(E_1'\cap E_2')\cap D$, $p_2\in(E_2'\cap E_3')\cap D$, $p_3\in(E_1'\cap E_3')\cap D$ which are not CH pre-simple corners for $\mathcal{F}'|_D$.

□

Lemma 27 *Suppose $Y=\{\underline{0}\}$ and that π is not dicritical. If the origin is not a CH pre-simple corner for $\mathcal{F},E$ then there exists a point $q\in D$ which is not a CH pre-simple corner for the pair $\mathcal{F}',E'$.*

Proof: There are several cases to consider.

First case: $e_0(E_{inv})=0$. Let $\Delta\subset M$ be a plane section generically transversal ([19]) to $\mathcal{F}$ and Δ' its transform by π. There exists a point $q\in D\cap\Delta'$ which is singular for $\mathcal{F}'|_\Delta'$; hence q is not a CH pre-simple corner for $\mathcal{F}',E'$ and we are done.

Second case: $e_0(E_{inv})=1$. Let E_1 be the component of E_{inv} through the origin, $\Delta\subset M$ a plane section transversal to E_1 at the origin and Δ' its transform by π. Suppose, by absurd, that every point of D is a CH pre-simple corner for $\mathcal{F}',E'$. Hence $\mathcal{F}'|_{\Delta'}$ has only one singularity, which moreover is a CH pre-simple corner for $\mathcal{F}'|_{\Delta'},E'\cap\Delta'$. This implies that $\underline{0}\in\Delta$ is a CH pre-simple corner for $\mathcal{F}|_\Delta,E\cap\Delta$. However, since $e_0(E_{inv})=1$, it follows that $\underline{0}\in\Delta$ is a regular point of $\mathcal{F}|_\Delta$; this implies that $\underline{0}\in\mathbb{C}^3$ is a regular point of $\mathcal{F}$, which is an absurd.

Third case: $e_0(E_{inv})=2$. Let E_1,E_2 be the components of E_{inv} through the origin, $\Delta\subset M$ a plane section transversal to $E_1\cap E_2$ at the origin and Δ' its transform by π. We may assume that the point $E_1'\cap E_2'\cap E_3'$ does not belong to Δ'. Suppose, by absurd, that all the points in D are CH pre-simple corners for $\mathcal{F}',E'$. Hence $\mathcal{F}'|_{\Delta'}$ has two singular points which are both CH pre-simple corners for $\mathcal{F}'|_\Delta,E'\cap\Delta'$. Then $\underline{0}\in\Delta$ is a CH pre-simple corner for $\mathcal{F}|_\Delta,E\cap\Delta$. By a result in [18], this

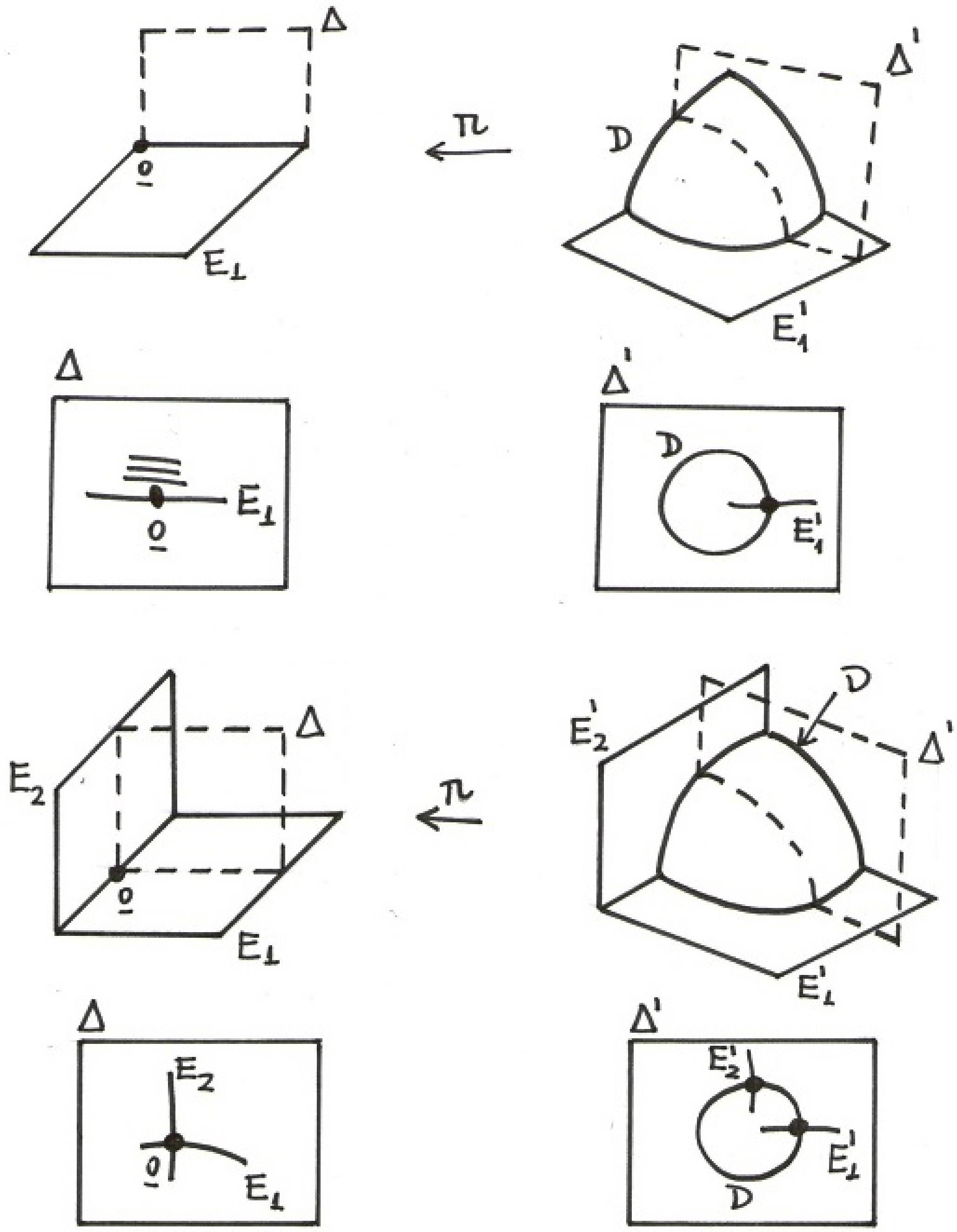


Figure 22: Lemma 27. The image at the top shows the first case, and the image of the bottom shows the second case.

implies that the dimensional type of the origin is two and thus a CH pre-simple corner given by the saturation of $\mathcal{F}|_\Delta$ by a nonsingular vector field transversal to Δ. This is an absurd since $\underline{0} \in \mathbb{C}^3$ is not a CH pre-simple point for $\mathcal{F}, E$.

Fourth case: $e_0(E_{inv}) = 3$. Suppose, by absurd, that every point of D is a CH pre-simple corner for $\mathcal{F}', E'$. We have that $\mathcal{F}$ is locally given by $\omega = 0$ where

$$\omega = a(x,y,z)\frac{dx}{x} + b(x,y,z)\frac{dy}{z} + c(x,y,z)\frac{dz}{z}$$

where a, b, c have no common factor and $x \not| a$, $y \not| b$, $z \not| c$. Let $r = \text{order}_0(a,b,c)$. Then

$$a(x,y,z) = A_r(x,y,z) + \cdots$$
$$b(x,y,z) = B_r(x,y,z) + \cdots$$
$$c(x,y,z) = C_r(x,y,z) + \cdots .$$

Let $P_r = A_r + B_r + C_r$. Since π is not dicritical, we have that $P_r \not\equiv 0$. In the firs local chart $x' = x, y' = y/x, z' = z/x$ we have that

$$\begin{aligned} \pi^*\omega &= x^r\Big[(P_r(1,y',z') + x(\cdots))\frac{dx}{x} + (B_r(1,y',z') + x(\cdots))\frac{dy'}{z'} \\ &\quad +(C_r(1,y',z') + x(\cdots))\frac{dz'}{z'}\Big] . \end{aligned}$$

Since $\underline{0} \in M$ is not a CH pre-simple corner for $\mathcal{F}, E$ we have that either P_r is not a constant polynomial or $r = 0$ and $A_0B_0C_0 = 0$. Suppose we have the first case. We may assume that, in the first chart, $P_r(1,y',z')$ is a polynomial with degree d, $1 \leq d \leq r$. The set

$$\{P_r(1,y',z') = 0\} \cap (x = 0)$$

gives a curve of the singular locus Sing $\mathcal{F}'$ such that each of its points is not a CH pre-simple corner for $\mathcal{F}', E'$. If $r = 0$ and $A_0B_0C_0 = 0$ we find two-dimensional saddle-nodes for $\mathcal{F}'$ and $\mathcal{F}$ cannot be a CH foliation, which is an absurd.

□

For the following two lemmas we will assume $Y \subset \text{Sing}\,\mathcal{F}$ is a germ of curve. Let $e_Y(E_{inv})$ denote the number of invariant components of E which generically contain Y; hence we have that $0 \leq e_Y(E_{inv}) \leq 2$.

Lemma 28 *Suppose that $Y \subset Sing\ \mathcal{F}$ is a germ of curve such that $e_Y(E_{inv}) = 0$ and that π is centered at Y. If the origin is not a CH pre-simple corner for $\mathcal{F}, E$ and $e_0(E_{inv}) = 1$, then there exists a point $q \in \pi^{-1}(\underline{0})$ which is not a CH pre-simple corner for $\mathcal{F}', E'$. In particular, if π is not dicritical, there exists an invariant germ of curve $\gamma \subset Sing\ \mathcal{F}'$ such that $\gamma \subset D = \pi^{-1}(Y)$ and γ has normal crossings with E'.*

Proof: We take local coordinates x, y, z at the origin such that

$$E_1 = (z = 0)\ ,$$

$$Y = (x = y = 0)\ .$$

Hence $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega(x, y, z) = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)\frac{dz}{z}\ .$$

Put $\nu = \mathrm{order}_{(x,y)}(a, b)$ and $\mu = \mathrm{order}_{(x,y)}(c)$. Note that $\mu \geq 1$ since Y is invariant by $\mathcal{F}$. We can write

$$a(x, y, z) = A_\nu(x, y; z) + A_{\nu+1}(x, y; z) + \cdots\ ,$$

$$b(x, y, z) = B_\nu(x, y; z) + B_{\nu+1}(x, y; z) + \cdots\ ,$$

$$c(x, y, z) = C_\mu(x, y; z) + C_{\mu+1}(x, y; z) + \cdots\ .$$

In the first local chart we have $x' = x,\ y' = y/x,\ z' = z$. Let

$$P_{\nu+1}(x, y; z) = xA_\nu(x, y; z) + yB_\nu(x, y; z)\ .$$

Then $P_{\nu+1}(x, xy'; z) = x^{\nu+1}P_{\nu+1}(1, y'; z)$. Hence

$$\begin{aligned}\pi^*\omega &= x^{\nu+1}\Big(P_{\nu+1}(1, y', z) + x(\cdots)\Big)\frac{dx}{x} + x^{\nu+1}\Big(B_\nu(1, y', z) + x(\cdots)\Big)dy' \\ &+ x^{\mu}\Big(C_\mu(1, y', z) + x(\cdots)\Big)\frac{dz}{z}\ .\end{aligned}$$

Firstly, suppose that $\mu \geq \nu + 1$. In this case, we can divide $\pi^*\omega$ by $x^{\nu+1}$:

$$\begin{aligned}\omega' = \frac{1}{x^{\nu+1}}\pi^*\omega &= \Big(P_{\nu+1}(1, y'; z) + x(\cdots)\Big)\frac{dx}{x} + \Big(B_\nu(1, y'; z) + x(\cdots)\Big)dy' \\ &+ x^{\mu-\nu-1}\Big(C_\mu(1, y'; z) + x(\cdots)\Big)\frac{dz}{z}\ .\end{aligned}$$

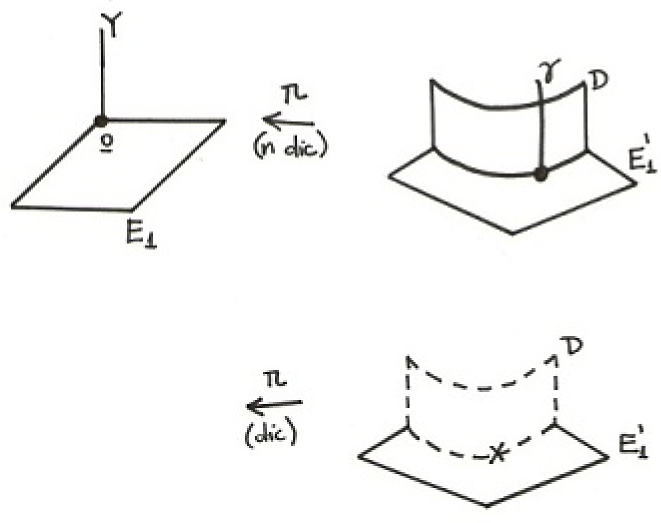


Figure 23: Lemma 28.

If $P_{\nu+1}(1,y';z) \not\equiv 0$ then $(x=0) = D$ is invariant by $\mathcal{F}'$ and π is nondicritical. If, on the other hand, $P_{\nu+1}(1,y';z) \equiv 0$, then

$$\omega' = (\cdots)\, dx + \Big(B_\nu(1,y';z) + x(\cdots)\Big)dy' + x^{\mu-\nu-1}\Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z}\ .$$

Note that if $P_{\nu+1}(1,y';z) \equiv 0$ then $B_\nu(1,y';z) \neq 0$ (otherwise we would have $A_\nu(1,y';z) = 0 = B_\nu(1,y';z)$, which is an absurd). Therefore, in this case, D is not invariant by $\mathcal{F}'$ and the blow-up π is dicritical.

Now suppose that $1 \le \mu \le \nu$. In this case, we can divide $\omega(x,xy',z)$ by x^μ:

$$\begin{aligned} \omega' = \frac{1}{x^\mu}\pi^*\omega &= x^{\nu-\mu}\Big(P_{\nu+1}(1,y';z) + x(\cdots)\Big)dx + x^{\nu-\mu+1}\Big(B_\nu(1,y';z) + x(\cdots)\Big)dy' \\ &+ \Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z}\ . \end{aligned}$$

Therefore we also have that D is a dicritical component of E' and the blow-up π is also dicritical.

Nondicritical case: The set

$$\{P_{\nu+1}(1,y',z) = 0\} \cap (x=0)$$

gives a curve of the singular locus distinct from $(x = z = 0) = \pi^{-1}(\underline{0})$. Moreover, there exists a point of intersection of $\{P_{\nu+1}(1, y', z) = 0\}$ and $(x = z = 0) = \pi^{-1}(\underline{0})$; this point is not a CH pre-simple corner for $\mathcal{F}', E'$ and we are done.

Dicritical cases: In both cases, we are looking for the singular points of $\mathcal{F}'$ in $\pi^{-1}(\underline{0})$. In the first chart, $\pi^{-1}(\underline{0})$ is the y'-axis $(x = z = 0)$. Firstly suppose $\mu \geq \nu + 1$, $P_{\nu+1}(1, y'; z) \equiv 0$. Then Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{x^{\nu-\mu-1} C_\mu(1, y'; z) = 0\} \ .$$

If $\mu > \nu + 1$ then $\mathcal{F}'$ is tangent to D along the curve $(x = z = 0)$and hence every point $p \in (x = z = 0)$ is not a CH pre-simple corner for $\mathcal{F}', E'$. If $\mu = \nu + 1$ we have

$$\mathrm{Sing}(z\omega') \cap (x = z = 0) = \{C_\mu(1, y'; z) = 0\} \ .$$

We have that $C_\mu(1, y', z) = P(\tilde{y}, \tilde{z})$ has a zero of the form $(\tilde{y}_0, 0)$; hence the point $(0, \tilde{y}_0, 0) \in \pi^{-1}(\underline{0}) \cap D$ is a point which is not a CH pre-simple corner for $\mathcal{F}', E'$. In the case $1 \leq \nu \leq \mu$ we have that Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{C_\mu(1, y'; z) = 0\}$$

and we repeat the previous argument.

□

Lemma 29 *Suppose Y is a germ of curve in the singular locus Sing $\mathcal{F}$ with $e_Y(E_{inv}) = 1, 2$. Assume that there exists $E_i \subset E_{inv}$ such that $\underline{0} \in E_i$, $Y \not\subset E_i$. If the origin is not a CH pre-simple corner for $\mathcal{F}, E$, then there exists a point $q \in \pi^{-1}(\underline{0})$ which is not a CH pre-simple corner for $\mathcal{F}', E'$. In particular, if π is not dicritical, there exists an invariant germ of curve $\gamma \subset$ Sing $\mathcal{F}'$ such that $\gamma \subset D = \pi^{-1}(Y)$ and γ has normal crossings with E'.*

Proof: There are two cases to consider: $e_0(E_{inv}) = 2$ and $e_0(E_{inv}) = 3$.

First case: $e_0(E_{inv}) = 2$. We take local coordinates x, y, z at the origin such that

$$E_1 = (z = 0) \subset E_{inv} \ ,$$

$$E_2 = (y = 0) \subset E_{inv} \ ,$$

$$Y = (x = y = 0) \subset E_2.$$

Hence $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega(x, y, z) = a(x, y, z)dx + b(x, y, z)\frac{dy}{y} + c(x, y, z)\frac{dz}{z} \ .$$

If $\nu = \text{order}_{(x,y)}(a) = \text{order}_{(x,y)}(b) - 1$, $\mu = \text{order}_{(x,y)}(c)$, we can write

$$a(x,y,z) = A_\nu(x,y;z) + A_{\nu+1}(x,y;z) + \cdots$$
$$b(x,y,z) = B_{\nu+1}(x,y;z) + B_{\nu+2}(x,y;z) + \cdots$$
$$c(x,y,z) = C_\mu(x,y;z) + C_{\mu+1}(x,y;z) + \cdots$$

where A_i, B_i, C_i, as in Lemma 28, are homogeneous polynomials in the variables x, y of degree i. Thus we have $(A_\nu, B_{\nu+1}) \neq (0,0)$, $C_\mu \neq 0$.

In the first local chart we have $x = x, y = xy', z = z$. Let

$$P_{\nu+1}(x,y;z) = xA_\nu(x,y,;z) + B_{\nu+1}(x,y;z) \ .$$

Then $P_{\nu+1}(x,xy';z) = x^{\nu+1}P_{\nu+1}(1,y';z)$. Hence

$$\begin{aligned} \pi^*\omega &= x^{\nu+1}\Big(P_{\nu+1}(1,y',z) + x(\cdots)\Big)\frac{dx}{x} + x^{\nu+1}\Big(B_{\nu+1}(1,y',z) + x(\cdots)\Big)\frac{dy'}{y'} \\ &+ x^{\mu}\Big(C_\mu(1,y',z) + x(\cdots)\Big)\frac{dz}{z} \ . \end{aligned}$$

Firstly suppose that $\mu \geq \nu + 1$. Then we may divide $\omega(x,xy',z)$ by $x^{\nu+1}$:

$$\begin{aligned} \omega' = \frac{1}{x^{\nu+1}}\ \pi^*\omega &= \Big(P_{\nu+1}(1,y';z) + x(\cdots)\Big)\frac{dx}{x} + \Big(B_{\nu+1}(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'} \\ &+ x^{\mu-\nu-1}\Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z} \ . \end{aligned}$$

If $P_{\nu+1}(1,y';z) \not\equiv 0$ then $(x=0) = D$ is invariant by $\mathcal{F}'$ and π is nondicritical. If, on the other hand, we have $P_{\nu+1}(1,y';z) \equiv 0$, then

$$\omega' = (\cdots)dx + \Big(B_{\nu+1}(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'} + x^{\mu-\nu-1}\Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z} \ .$$

Note that $P_{\nu+1}(1,y';z) \equiv 0$ implies that $B_{\nu+1}(1,y';z) \neq 0$; otherwise, we would have $B_{\nu+1}(1,y';z) = A_\nu(1,y';z) = 0$, which is an absurd. Thus $(x=0) = D$ is not invariant by $\mathcal{F}'$ and π is dicritical.

Now suppose $1 \leq \mu \leq \nu$. Then we may divide $\omega(x,xy',z)$ by at most x^μ:

$$\begin{aligned} \omega' = \frac{1}{x^\mu}\pi^*\omega &= x^{\nu-\mu}\Big(P_{\nu+1}(1,y';z) + x(\cdots)\Big)dx + x^{\nu-\mu+1}\Big(B_{\nu+1}(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'} \\ &+ \Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z} \ . \end{aligned}$$

Since $C_\mu(1, y'; z) \neq 0$, we have that $(x = 0) = D$ is not invariant for $\mathcal{F}'$ and π is dicritical.

Nondicritical case: The set

$$\{P_{\nu+1}(1, y', z) = 0\} \cap (x = 0)$$

gives a curve of the singular locus distinct from $(x = z = 0) = \pi^{-1}(\underline{0})$. Hence there exists a point of intersection of $\{P_{\nu+1}(1, y', z) = 0\}$ and $(x = z = 0) = \pi^{-1}(\underline{0})$; this point is not a CH pre-simple corner for $\mathcal{F}', E'$ and we are done.

Dicritical cases: In both cases, we are looking for the singular points of $\mathcal{F}'$ in $\pi^{-1}(\underline{0})$. In the first chart, $\pi^{-1}(\underline{0})$ is the y'-axis $(x = z = 0)$. Firstly suppose $\mu \geq \nu + 1$, $P_{\nu+1}(1, y'; z) \equiv 0$. Then Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{x^{\nu-\mu-1} C_\mu(1, y'; z) = 0\} \ .$$

If $\mu > \nu + 1$ then $\mathcal{F}'$ is tangent to D along the curve $(x = z = 0)$ and all the points in this curve are not CH pre-simple corners for $\mathcal{F}', E'$. If $\mu = \nu + 1$ then

$$\mathrm{Sing}(z\omega') \cap (x = z = 0) = \{C_\mu(1, y'; z) = 0\} \ .$$

We have that $C_\mu(1, y', z) = P(\tilde{y}, \tilde{z})$ has a zero of the form $(\tilde{y}_0, 0)$; hence the point $(0, \tilde{y}_0, 0) \in \pi^{-1}(\underline{0}) \cap D$ is a point which is not a CH pre-simple corner for $\mathcal{F}', E'$. In the case $1 \leq \nu \leq \mu$ we have that Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{C_\mu(1, y'; z) = 0\}$$

and we repeat the previous argument.

Second case: $e_0(E_{inv}) = 3$. We take local coordinates x, y, z at the origin such that

$$E_1 = (z = 0) \subset E_{inv} \ ,$$

$$E_2 = (y = 0) \subset E_{inv} \ ,$$

$$E_3 = (x = 0) \subset E_{inv} \ ,$$

$$Y = (x = y = 0) = E_2 \cap E_3 \ .$$

Hence $\mathcal{F}$ is given by $\omega = 0$ where

$$\omega(x, y, z) = a(x, y, z)\frac{dx}{x} + b(x, y, z)\frac{dy}{y} + c(x, y, z)\frac{dz}{z} \ .$$

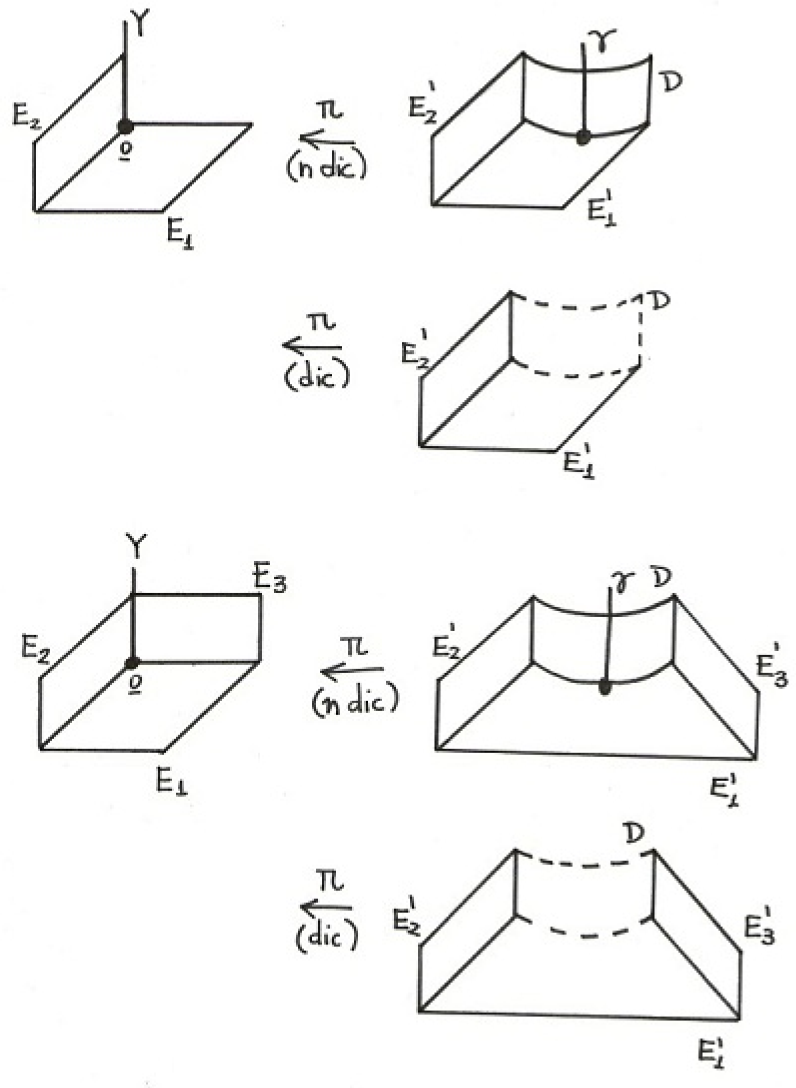


Figure 24: Lemma 29. The image at the top shows the case $e_Y(E_{inv}) = 1$, and the image at the bottom show the case $e_Y(E_{inv}) = 2$.

If $\nu = \text{order}_{(x,y)}(a,b)$, $\mu = \text{order }_{(x,y)}(c)$, we write

$$a(x,y,z) = A_\nu(x,y;z) + A_{\nu+1}(x,y;z) + \cdots$$
$$b(x,y,z) = B_\nu(x,y;z) + B_{\nu+1}(x,y;z) + \cdots$$
$$c(x,y,z) = C_\mu(x,y;z) + C_{\mu+1}(x,y;z) + \cdots$$

where A_i, B_i, C_i, once again, are homogeneous polynomials in the variables x, y of degree i. Thus $(A_\nu, B_\nu) \neq (0,0)$, $C_\mu \neq 0$.

In the first local chart we have $x = x', y = x'y', z = z'$. Let

$$P_\nu(x,y;z) = A_\nu(x,y,;z) + B_\nu(x,y;z)\ .$$

Then $P_\nu(x, xy'; z) = x^\nu P_\nu(1, y'; z)$. Hence

$$\begin{aligned}\pi^*\omega &= x^\nu\Big(P_\nu(1,y',z) + x(\cdots)\Big)\frac{dx}{x} + x^\nu\Big(B_\nu(1,y',z) + x(\cdots)\Big)\frac{dy'}{y'}\\ &+ x^\nu\Big(C_\mu(1,y',z) + x(\cdots)\Big)\frac{dz}{z}\ .\end{aligned}$$

Suppose $\mu \geq \nu$. Then we may divide $\omega(x, xy', z)$ by x^ν:

$$\begin{aligned}\omega' = \frac{1}{x^\nu}\pi^*\omega &= \Big(P_\nu(1,y';z) + x(\cdots)\Big)\frac{dx}{x} + \Big(B_\nu(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'}\\ &+ x^{\mu-\nu}\Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z}\ .\end{aligned}$$

If $P_\nu(1,y';z) \not\equiv 0$ then $(x=0) = D$ is invariant by $\mathcal{F}'$ and π is nondicritical. If, on the other hand, we have $P_\nu(1,y';z) \equiv 0$, then

$$\omega' = (\cdots)dx + \Big(B_\nu(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'} + x^{\mu-\nu}\Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z}\ .$$

Note that $P_\nu(1,y';z) \equiv 0$ implies that $B_\nu(1,y';z) \neq 0$; otherwise, we would have $B_\nu(1,y';z) = A_\nu(1,y';z) = 0$, which is an absurd. Thus $(x=0) = D$ is not invariant for $\mathcal{F}'$, and π is dicritical.

Now suppose $1 \leq \mu < \nu$. Then we may divide $\omega(x, xy', z)$ by x^μ:

$$\begin{aligned}\omega' = \frac{1}{x^\mu}\pi^*\omega &= x^{\nu-\mu-1}\Big(P_\nu(1,y';z) + x(\cdots)\Big)dx + x^{\nu-\mu}\Big(B_\nu(1,y';z) + x(\cdots)\Big)\frac{dy'}{y'}\\ &+ \Big(C_\mu(1,y';z) + x(\cdots)\Big)\frac{dz}{z}\ .\end{aligned}$$

Since $C_\mu(1, y'; z) \neq 0$, we have that $(x = 0) = D$ is not invariant by $\mathcal{F}'$ and π is dicritical.

Nondicritical case: The set

$$\{P_\nu(1, y', z) = 0\} \cap (x = 0)$$

gives a curve of the singular locus distinct from $(x = z = 0) = \pi^{-1}(\underline{0})$. Hence there exists a point of intersection of $\{P_\nu(1, y', z) = 0\}$ and $(x = z = 0) = \pi^{-1}(\underline{0})$; this point is not a CH pre-simple corner for $\mathcal{F}', E'$ and we are done.

Dicritical cases: In both cases, we are looking for the singular points of $\mathcal{F}'$ in $\pi^{-1}(\underline{0})$. In the first chart, $\pi^{-1}(\underline{0})$ is the y'-axis $(x = z = 0)$. Firstly suppose $\mu \geq \nu + 1$, $P_\nu(1, y'; z) \equiv 0$. Then Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{x^{\nu-\mu-1} C_\mu(1, y'; z) = 0\} \ .$$

If $\mu > \nu + 1$ then $\mathcal{F}'$ is tangent to D along the curve $(x = z = 0)$ and all the points in this curve are not CH pre-simple corners for $\mathcal{F}', E'$. If $\mu = \nu + 1$ then

$$\mathrm{Sing}(z\omega') \cap (x = z = 0) = \{C_\mu(1, y'; z) = 0\} \ .$$

We have that $C_\mu(1, y', z) = P(\tilde{y}, \tilde{z})$ has a zero of the form $(\tilde{y}_0, 0)$; hence the point $(0, \tilde{y}_0, 0) \in \pi^{-1}(\underline{0}) \cap D$ is a point which is not a CH pre-simple corner for $\mathcal{F}', E'$. In the case $1 \leq \nu \leq \mu$ we have that Sing $(z\omega') \cap (x = z = 0)$ is the set

$$(z = 0) \cap \{C_\mu(1, y'; z) = 0\}$$

and we repeat the previous argument.

$\square$

Remark 20 Lemmas 28 and 29 are still valid in the case that there does not exist an invariant component of E which is transversal to Y.

4.2 Singular locus of a RICH foliation

Let $\mathcal{F}$ be a RICH foliation in $M = (\mathbb{C}^3, \underline{0})$ and let's fix a reduction of singularities

$$\mathcal{S} : M_0 = M \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

as in Section 3.1. Recall that for $0 \leq s \leq s' \leq N$ we denote $\pi_{ss} = \mathrm{id}_{M_s}$ and

$$\pi_{ss'} = \pi_{s+1} \circ \pi_{s+2} \circ \cdots \circ \pi_{s'} : M_{s'} \to M_s$$

is $s < s'$. In particular,

$$\rho_s = \pi_{Ns} : M_N \to M_s; \ \sigma_s = \pi_{s0} : M_s \to M_0 = (\mathbb{C}^3, \underline{0})$$

and $\pi = \pi_{N0} = \rho_0 = \sigma_N : M_N \to M_0$ is the morphism of reduction of singularities. We decompose the exceptional divisor E^s into irreducible components

$$E^s = D_1^s \cup D_2^s \cup \cdots \cup D_s^s$$

where D_i^s is the strict transform by π_s of D_i^{s-1} for $i < s$ and $D_s^s = \pi_s^{-1}(Y_{s-1})$ and we write $E^s_{inv} \subset E, \ E^s_{dic} \subset E^s$.

We will frequently do arguments by induction on the *height* $q \in \sigma_s^{-1}(\underline{0})$. This number is defined by

$$\mathrm{ht}(q) = \sharp\{s' \geq s; \ q \in \pi_{ss'}(Y_{s'})\} \ .$$

Note that $\mathrm{ht}(q) = 0$ if and only if q is a CH-simple point of $\mathcal{F}_s$. Moreover, if $q \in Y_s$ then $\mathrm{ht}(q') < \mathrm{ht}(q)$ for all $q' \in \pi_{s+1}^{-1}(q)$.

This section is devoted to proving the following result:

Proposition 30 *Let $q \in \sigma_s^{-1}(\underline{0})$ be a point which is not a CH pre-simple corner for $\mathcal{F}_s, E^s$. Take a nondicritical irreducible component D_i^s of E^s (that is, $D_i^s \subset E^s_{inv}$) with $q \in D_i^s$. Assume that there is no germ of curve $Y_{s'}$, with $s \leq s'$, such that*

$$q \in \pi_{ss'}(Y_{s'}) \subset D_i^s \ .$$

Then there exists a curve Γ contained in the singular locus Sing $\mathcal{F}_s$ such that

$$q \in \Gamma \subset D_i^s$$

and $\Gamma \not\subset D_j^s$ for any $j \neq i$.

Proof: We do induction on the height $\mathrm{ht}(q)$. If $\mathrm{ht}(q) = 0$, q is a CH trace point and we are done. Assume that $\mathrm{ht}(q) \geq 1$. Let $b > s$ be the first index such that $q \in \pi_{s(b-1)}(Y_{b-1})$. There are several cases to consider.

First case: the center Y_{b-1} is a point Q and π_b is a nondicritical blow-up. We perform the blow-up π_b; by Proposition 25 there exists a point $q' \in \pi_b^{-1}(Q) = D_b^b$ that is not a CH pre-simple corner for $\mathcal{F}_b, E_b$. We apply the induction hypothesis to q'; since D_b^b is compact (it is isomorphic to $\mathbb{P}^2$) and invariant we find a compact curve $\tilde{\Gamma} \subset D_b^b$ in the singular locus of $\mathcal{F}_b$ which is not contained in another invariant component of the exceptional divisor E^b. The curve $\tilde{\Gamma}$ must intersect the projective line $D_i^b \cap D_b^b$ at a point q''. It follows that the point q'' is not a CH pre-simple corner; we apply the induction hypothesis to D_i^b at q'' to find Γ' and we take $\Gamma = \pi_{kb}(\Gamma')$.

Second case: $Y_{b-1} = \{Q\}$ and π_b is a dicritical blow-up. By Proposition 25, there exists a point $q' \in D_i^b \cap D_b^b$ which is not a CH pre-simple corner for $\mathcal{F}_b, E^b$. We apply the induction hypothesis to the point q' as before.

Third case: Y_{b-1} is a curve. In this case D_i^{b-1} is transversal to Y_{b-1} in view of the hypothesis on D_i^s. By Proposition 25 there exists a non CH pre-simple corner $q' \in \pi_b^{-1}(Q) = D_i^b \cap D_b^b$ where Q is the (only) point over q such that $\pi_{s(b-1)}(Q) = q$. We apply induction hypothesis to D_i^b at q' to obtain a curve $\Gamma' \subset D_i^b$ and we put $\Gamma = \pi_{kb}(\Gamma')$.

□

Remark 21 Note that it is possible to have $\Gamma = \pi_{ss'}(Y_{s'})$ with the properties that $p \in \Gamma \subset \text{Sing } \mathcal{F}$, $\Gamma \subset E_i^s$ and there is no $j \neq i$ with $\Gamma \subset E_j^s$.

4.3 Structural results: continuation of nodal curves

In this section we consider a RICH foliation $\mathcal{F}$ in $M = (\mathbb{C}^3, \underline{0})$ without invariant analytic surface and we fix a reduction of singularities

$$\mathcal{S} : M_0 = M \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

as in Section 3.1. We recall that $\pi : M_N \to M_0$ denotes the composition of all the blow-ups in the sequence $\mathcal{S}$.

Definition 16 *Let $\mathcal{C} \subset$ Sing $\mathcal{F}_N$ be an nodal component of $\mathcal{F}, \mathcal{S}$. We say that $\mathcal{C}$ is of* good type *if and only if we have one of the following possibilities:*

1. *$\mathcal{C} \cap E_{dic}^N \neq \emptyset$.*

2. *$\pi(\mathcal{C}) \neq \{\underline{0}\}$, that is to say, there exists a germ of curve in $\pi(\mathcal{C})$.*

The following statement is the technical result that we need to complete the proof of Theorem 7.

Proposition 31 *All the nodal components of $\mathcal{F},\mathcal{S}$ are of good type.*

Let us start the proof of Proposition 31. We do an argument by contradiction, by considering a fixed nodal component $\mathcal{C}$ that is not of good type. That is, we assume that $\mathcal{C}\cap E^N_{dic}=\emptyset$ and $\pi(\mathcal{C})=\{\underline{0}\}$. For any $0\leq s\leq N$, we denote $\mathcal{C}^s=\rho_s(\mathcal{C})$. We have that $\mathcal{C}^s\subset\sigma_s^{-1}(\underline{0})$ and hence $\mathcal{C}^s$ is a connected union of compact analytic subsets of E^s. We have two possibilities: either $\mathcal{C}^s$ is a single point or it is a finite union of compact irreducible analytic curves

$$\mathcal{C}^s=\Gamma_1^s\cup\Gamma_2^s\cup\cdots\cup\Gamma_{k_s}^s\ .$$

Let us remark that the curves $\Gamma_i^s\subset\sigma_s^{-1}(\underline{0})$ will never be used a center of blow-up in the sequence $\mathcal{S}$. In particular, the generic points of Γ_i^s are CH simple for $\mathcal{F}_s,E^s$ and only finitely many points in Γ_i^s will be modified by subsequent blow-ups. Also $\mathcal{C}^{s+1}$ has the form

$$\mathcal{C}^{s+1}=\Gamma_1^{s+1}\cup\Gamma_2^{s+1}\cup\cdots\cup\Gamma_{k_{s+1}}^{s+1}$$

where $k_{s+1}\geq k_s$ and for each $1\leq i\leq k_s$ the curve Γ_i^{s+1} is the strict transform of Γ_i^s by π_{s+1}.

Lemma 32 *Take a point $q\in\sigma_s^{-1}(\underline{0})$. Then the following properties are satisfied:*

A [Continuation at corners] *Suppose $e_q(E^s_{inv})=3$ and let D_i^s,D_k^s,D_l^s be the three irreducible components of E^s containing q. Suppose $D_i^s\cap D_k^s\subset\mathcal{C}^s$. Then $D_k^s\cap D_l^s\subset\mathcal{C}^s$ or $D_i^s\cap D_l^s\subset\mathcal{C}^s$, and moreover, in the first case we have $D_i^s\cap D_l^s\not\subset\mathcal{C}^s$, and in the second case we have $D_k^s\cap D_l^s\not\subset\mathcal{C}^s$.*

C' [Nondicriticalness of corners] *Suppose $e_q(E^s_{inv})=2$ and let D_i^s,D_k^s be the two irreducible components of E^s containing q. If $D_i^s\cap D_k^s\subset\mathcal{C}^s$ then $q\notin E^s_{dic}$.*

Proof: As in Proposition 30, the proof will be done by induction on the height of q. Suppose initially that $\mathrm{ht}(q)=0$. Then q is already a simple point and will not be a center of explosion. Furthermore, if q belongs to a germ of curve γ, then γ will also not be a center of explosion. So there exists only one point $q'\in M_N$ such that $\rho_s(q')=q$.

C' Since $q'\in\mathcal{C}\subset M_N$ and $\mathcal{C}$ is not of good type, it follows directly that $q'\notin E^N_{dic}$. Hence q does not belong to E^s_{dic}.

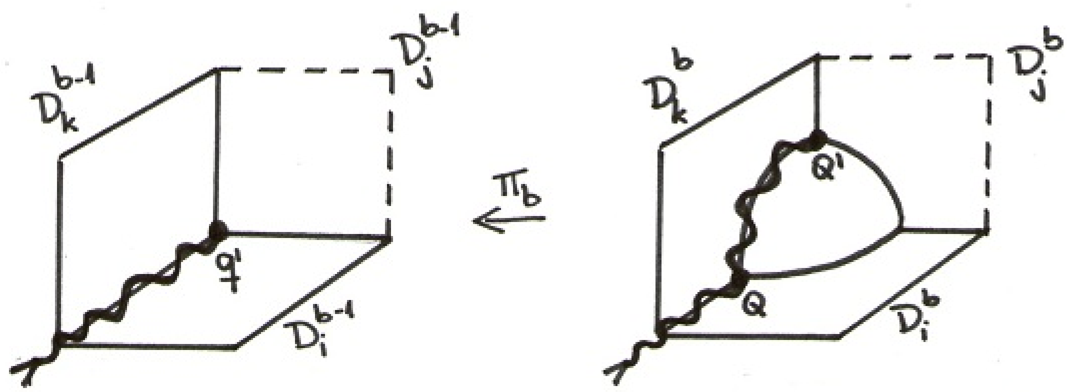


Figure 25: Case **C'**. We denote the hypothesis data with an arrow pointing towards inside the curve.

A We have that $e_{q'}(E^N_{inv}) = 3$, $D^N_i \cap D^N_k \subset \mathcal{C}$ and $\mathcal{C}$ is a nodal component; therefore exactly one of the curves $D^N_i \cap D^N_l$ or $D^N_k \cap D^N_l$ is an irreducible component of $\mathcal{C}$ whereas the other is not, and the result follows.

Now suppose that **A** and **C'** are true for every point with height $\leq h-1$ and assume ht(q) =h. Let $b > s$ be the first index such that $q \in \pi_{s(b-1)}(Y_{b-1})$. Let $q' \in Y_{b-1}$ be the point such that $\pi_{s(b-1)}(q') = q$. Note that since **C'** is valid for every point with height $\leq h-1$ it follows that π_b is nondicritical. We recall we denote $\pi_b^{-1}(Y_{b-1}) = D^b_b$.

C' (see Figure 25) Suppose q belongs to a dicritical component $D^s_j \subset E^s_{dic}$: so $q' \in D^{b-1}_j \subset E^{b-1}_{dic}$. Since π_b is nondicritical, by the induction hypothesis we may apply **A** to the point $Q = D^b_b \cap D^b_i \cap D^b_k$. Thus we find a point $Q' \in D^b_i \cap D^b_b$ (or $Q' \in D^b_k \cap D^b_b$) such that $Q' \in D^b_j \subset E^b_{dic}$, which is an absurd.

A (see Figure 26) The result follows by applying the induction hypothesis to the points $Q_1 = D^b_b \cap D^b_i \cap D^b_k$, $Q_2 = D^b_b \cap D^b_k \cap D^b_l$ and $Q_3 = D^b_b \cap D^b_i \cap D^b_l$ in the case $Y_{b-1} = \{q'\}$; and by applying the induction hypothesis to the points $Q_1 = D^b_b \cap D^b_i \cap D^b_k$ and $Q_2 = D^b_b \cap D^b_i \cap D^b_l$ in the case that Y_{b-1} is a germ of curve of the singular locus Sing $\mathcal{F}_{b-1}$.

□

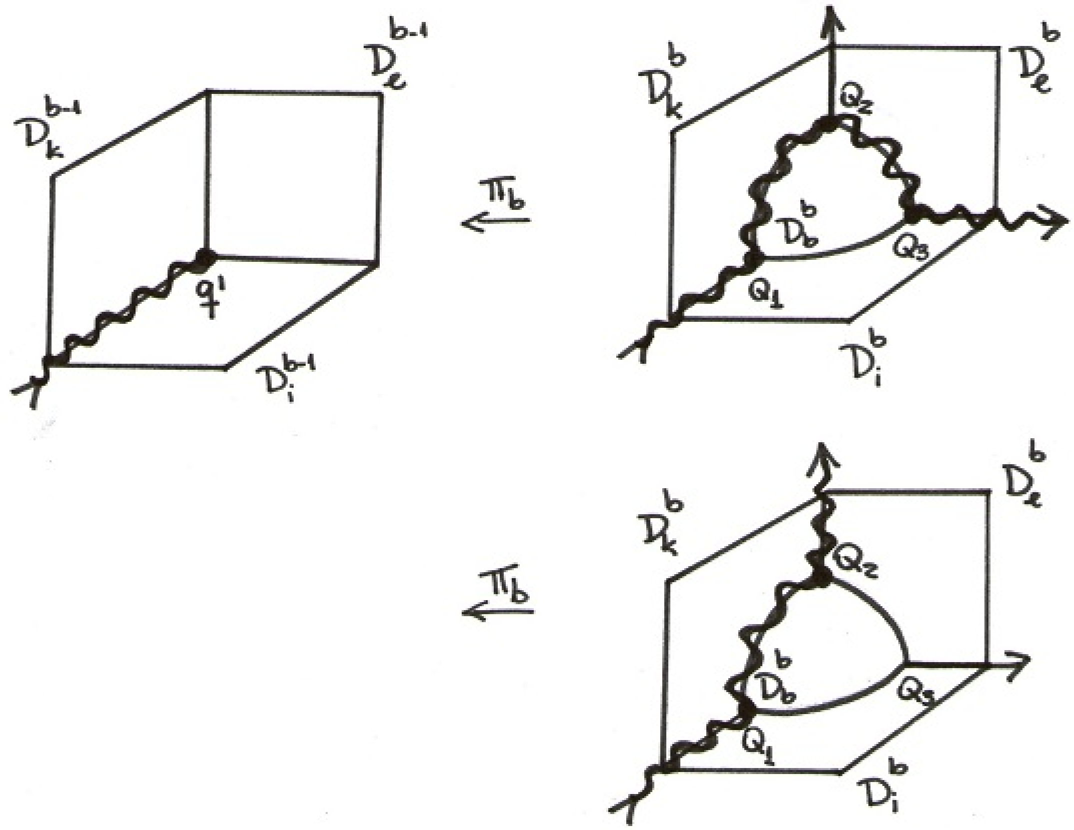


Figure 26: Case **A**. We denote the hypothesis data with an arrow pointing towards inside the curve, and the conclusion data with an arrow point towards outside the curve.

With Lemma 32 we can see that the properties that $\mathcal{C}$ fulfills at the end of the reduction of singularities can already be seen at intermediate stages in the case of curves which are generically corner. The next result assures the same is true for irreducible components of $\mathcal{C}^s$ which are trace curves (by trace curve we mean to say a curve of the singular locus which is generically contained in only one invariant component of the exceptional divisor).

Lemma 33 *Take a point $q \in \sigma_s^{-1}(\underline{0})$ with $e_q(E^s_{inv}) \geq 2$. If $e_q(E^s_{inv}) = 2$, D^s_i, D^s_k will denote the two invariant components of E^s containing q. If $e_q(E^s_{inv}) = 3$, D^s_i, D^s_k, D^s_l will denote the three invariant components of E^s containing q. Assume that there is no germ of curve $Y_{s'}$, with $s \leq s'$, such that*

$$q \in \pi_{ss'}(Y_{s'}).$$

Then the following properties are satisfied:

B [Transition of trace curves]

- **B.1** *Suppose $e_q(E^s_{inv}) = 2$ and that $D^s_i \cap D^s_k \subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \not\subset \mathcal{C}^s$, then there exists a trace curve $\Gamma_k \subset D^s_k$ such that $\Gamma_k \subset \mathcal{C}^s$ and $q \in \Gamma_k$.*
- **B.1'** *Suppose $e_q(E^s_{inv}) = 2$ and that $D^s_i \cap D^s_k \subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \subset \mathcal{C}^s$, then there exists a trace curve $\Gamma_k \subset D^s_k$ such that $\Gamma_k \not\subset \mathcal{C}^s$ and $q \in \Gamma_k$.*
- **B.2** *Suppose $e_q(E^s_{inv}) = 2$ and that $D^s_i \cap D^s_k \not\subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \subset \mathcal{C}^s$, then there exists a trace curve $\Gamma_k \subset D^s_k$ such that $\Gamma_k \subset \mathcal{C}^s$ and $q \in \Gamma_k$.*
- **B.2'** *Suppose $e_q(E^s_{inv}) = 2$ and that $D^s_i \cap D^s_k \not\subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \not\subset \mathcal{C}^s$, then there exists a trace curve $\Gamma_k \subset D^s_k$ such that $\Gamma_k \not\subset \mathcal{C}^s$ and $q \in \Gamma_k$.*
- **B.3** *Suppose $e_q(E^s_{inv}) = 3$ and that $D^s_i \cap D^s_k, D^s_i \cap D^s_l, D^s_k \cap D^s_l \not\subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \subset \mathcal{C}^s$, then there exist trace curves $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ such that $\Gamma_k, \Gamma_l \subset \mathcal{C}^s$ and $q \in \Gamma_k, \Gamma_l$.*
- **B.3'** *Suppose $e_q(E^s_{inv}) = 3$ and that $D^s_i \cap D^s_k, D^s_i \cap D^s_l, D^s_k \cap D^s_l \not\subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \not\subset \mathcal{C}^s$, then there exist trace curves $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ such that $\Gamma_k, \Gamma_l \not\subset \mathcal{C}^s$ and $q \in \Gamma_k, \Gamma_l$.*
- **B.4** *Suppose $e_q(E^s_{inv}) = 3$ and that $D^s_i \cap D^s_k, D^s_k \cap D^s_l \subset \mathcal{C}^s$ but $D^s_i \cap D^s_l \not\subset \mathcal{C}^s$. If there exists a trace curve $\Gamma_i \subset D^s_i$ such that $q \in \Gamma_i$ and $\Gamma_i \subset \mathcal{C}^s$, then there exist trace curves $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ such that $\Gamma_k \not\subset \mathcal{C}^s$, $\Gamma_l \subset \mathcal{C}^s$ and $q \in \Gamma_k, \Gamma_l$.*

B.4' *Suppose* $e_q(E^s_{inv}) = 3$ *and that* $D^s_i \cap D^s_k, D^s_k \cap D^s_l \subset \mathcal{C}^s$ *but* $D^s_i \cap D^s_l \not\subset \mathcal{C}^s$. *If there exists a trace curve* $\Gamma_i \subset D^s_i$ *such that* $q \in \Gamma_i$ *and* $\Gamma_i \not\subset \mathcal{C}^s$, *then there exist trace curves* $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ *such that* $\Gamma_k \subset \mathcal{C}^s$, $\Gamma_l \not\subset \mathcal{C}^s$ *and* $q \in \Gamma_k, \Gamma_l$.

B.5 *Suppose* $e_q(E^s_{inv}) = 3$ *and that* $D^s_i \cap D^s_k, D^s_i \cap D^s_l \subset \mathcal{C}^s$ *but* $D^s_k \cap D^s_l \not\subset \mathcal{C}^s$. *If there exists a trace curve* $\Gamma_i \subset D^s_i$ *such that* $q \in \Gamma_i$ *and* $\Gamma_i \subset \mathcal{C}^s$, *then there exist trace curves* $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ *such that* $\Gamma_k, \Gamma_l \not\subset \mathcal{C}^s$ *and* $q \in \Gamma_k, \Gamma_l$.

B.5' *Suppose* $e_q(E^s_{inv}) = 3$ *and that* $D^s_i \cap D^s_k, D^s_i \cap D^s_l \subset \mathcal{C}^s$ *but* $D^s_k \cap D^s_l \not\subset \mathcal{C}^s$. *If there exists a trace curve* $\Gamma_i \subset D^s_i$ *such that* $q \in \Gamma_i$ *and* $\Gamma_i \not\subset \mathcal{C}^s$, *then there exist trace curves* $\Gamma_k \subset D^s_k$, $\Gamma_l \subset D^s_l$ *such that* $\Gamma_k, \Gamma_l \subset \mathcal{C}^s$ *and* $q \in \Gamma_k, \Gamma_l$.

C [Nondicriticalness] *If* $q \in \mathcal{C}^s$, *then* $q \notin E^s_{dic}$.

Proof: As before, the proof is done by induction on the height of q. Suppose $\mathrm{ht}(q) = 0$. Then q is already a simple point and will not be a center of explosion. Furthermore, if q belongs to a germ of curve γ, then γ will also not be a center of explosion. So there exists only one point $q' \in M_N$ such that $\rho_s(q') = q$.

C Since $q' \in \mathcal{C} \subset M_N$ and $\mathcal{C}$ is not of good type, it follows directly that $q' \notin E^N_{dic}$. Hence $q \notin E^s_{dic}$.

B We have that q' is a simple point with dimensional type three. Thus **B.3 - B.5'** cannot occur in the case $\mathrm{ht}(q) = 0$. Cases **B.1 - B.2'** follow directly from the facts that since q' is a simple point with dimensional type three there are three curves of the singular locus containing q' and that $\mathcal{C}$ is a nodal component. Note that in the case **B.2'** we obtain that $q' \notin \mathcal{C}$. Since $\rho_s(q') = q$, we obtain the same results for the point q in the stage M_s.

Assume $\mathrm{ht}(q) \geq 1$. Let $b > s$ be the first index such that $q \in \pi_{s(b-1)}(Y_{b-1})$. Let $q' \in Y_{b-1}$ be the point such that $\pi_{s(b-1)}(q') = q$. Hence we may "push" all the hypothesis on q to the point q'. Note that since **C** is valid for every point with height $\leq h-1$ it follows that π_b is nondicritical. Moreover, in view of the hypothesis, $Y_{b-1} = \{q'\}$. We recall we denote $\pi_b^{-1}(Y_{b-1}) = D^b_b$.

C (see Figure 27) Suppose q belongs to a dicritical component $D^s_j \subset E^s_{dic}$; hence $q' \in D^{b-1}_j \subset E^{b-1}_{dic}$. Call $\Gamma \subset \mathcal{C}^{b-1}$ the irreducible component of $\mathcal{C}^{b-1}$ which contains q' (that is to say, Γ is the transform by $\pi_{s(b-1)}$ of the irreducible component of $\mathcal{C}^s$

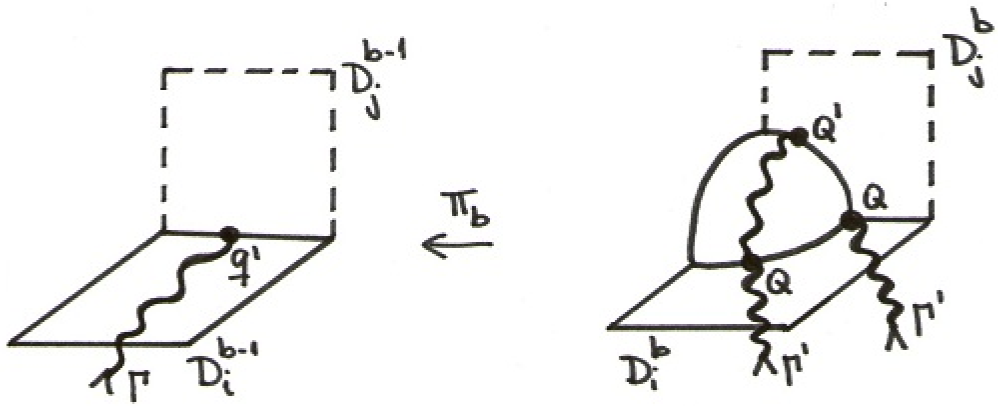


Figure 27: Case **C**.

which contains q). We perform the blow-up π_b. Let Γ' denote the transform by π_b of Γ and let $Q = \Gamma' \cap (D_b^b \cap D_i^b)$. If $Q \in D_j^b$, by applying the induction hypothesis to Q we come to an absurd. If $Q \notin D_j^b$, we apply **B** or **A** of Proposition 32 to q' (depending on whether Γ' is a trace curve or corner curve). Either way, we will find a point $Q' \in \mathcal{C}^b$ such that $Q \in D_j^b$, which is an absurd and the result follows.

B We will still denote by $\Gamma_i \subset \text{Sing}\ \mathcal{F}_{b-1}$ the transform by $\pi_{s(b-1)}$ of $\Gamma_i \subset \text{Sing}\ \mathcal{F}_s$. We perform the nondicritical blow-up π_b centered at q'. First let's consider cases **B.1 - B.2'** (see Figures 28 - 31). We will use the following notation:

$$Q_1 = D_i^b \cap D_k^b \cap D_b^b \ ,$$
$$Q_2 = \Gamma_i' \cap (D_i^b \cap D_b^b) \ ,$$

where $\Gamma_i' \subset \text{Sing}\ \mathcal{F}_b$ is the transform by π_b of $\Gamma_i \subset \text{Sing}\ \mathcal{F}_{b-1}$. We apply Lemma 32 to the point Q_1 in order to see which of the curves (if any) $D_i^b \cap D_b^b$, $D_k^b \cap D_b^b$ are irreducible components of $\mathcal{C}^s$. Then, we apply the induction hypothesis to the point Q_2: if $Q_2 = Q_1$, we apply one of the cases **B.3 - B.5'** and the result will follow immediately; if $Q_2 \neq Q_1$, we apply one of the cases **B.1 - B.2'** to the point Q_2. In this case, we will find a trace curve $\Gamma_b \subset D_b^b$ (which may be an irreducible component of $\mathcal{C}^s$ or not, depending on the case) such that $Q_2 \in \Gamma_b$. Finally, we apply the induction hypothesis to the point $Q_3 = \Gamma_b \cap (D_k^b \cap D_b^b)$ and we find a trace curve $\tilde{\Gamma}_k \subset D_k^b$, $Q_3 \in \tilde{\Gamma}_k$ (again, we will apply cases **B.3 - B.5'** if $Q_3 = Q_1$, and cases **B.1 - B.2'** otherwise). We take $\Gamma_k = \pi_{sb}(\tilde{\Gamma}_k)$ and the result follows .

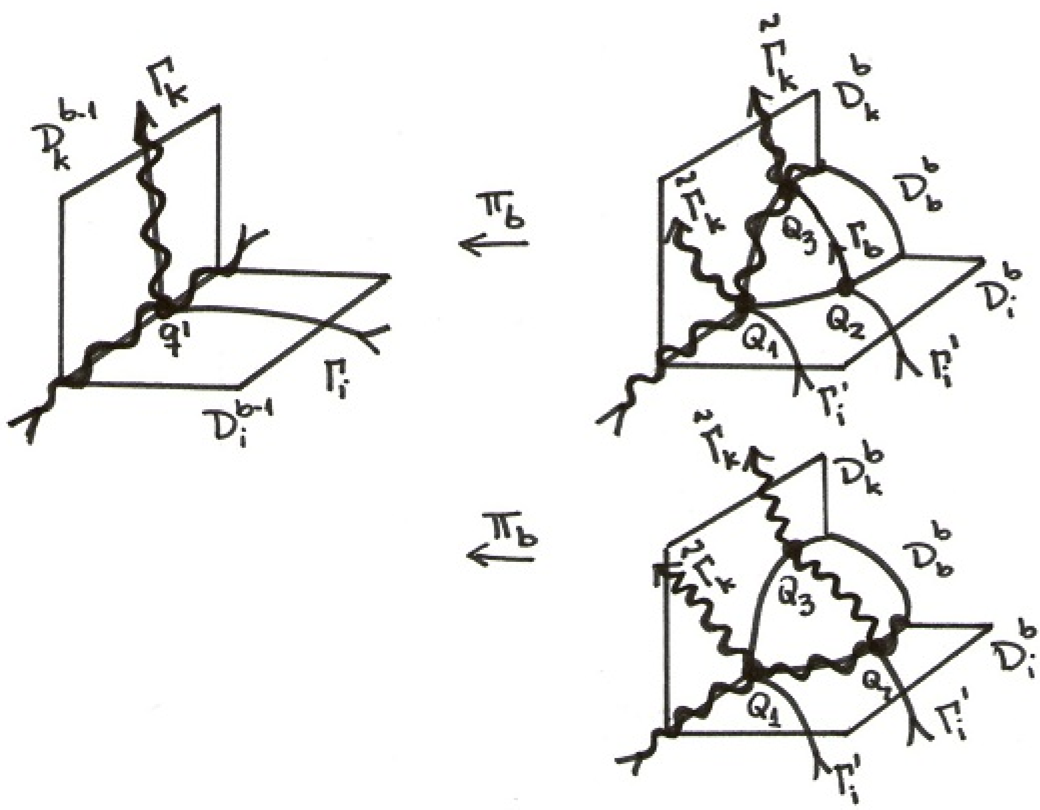


Figure 28: Case **B.1**. We recall we denote the hypothesis data with an arrow pointing towards inside the curve, and the conclusion data with an arrow point towards outside the curve.

For example, let's see in detail case **B.1** (see Figure 28). We apply **A** of Lemma 32 to the point Q_1: since $D_i^b \cap D_k^b \subset \mathcal{C}^b$, we have that either $D_k^b \cap D_b^b \subset \mathcal{C}^b$ (and in this case $D_i^b \cap D_b^b \not\subset \mathcal{C}^b$) or $D_i^b \cap D_b^b \not\subset \mathcal{C}^b$ (in which case $D_k^b \cap D_b^b \not\subset \mathcal{C}^b$). *Suppose we have the first option.* Now consider the point Q_2. If $Q_2 = Q_1$, we apply **B.4'** to Q_2 and the result follows directly: we find a curve $\tilde{\Gamma}_k \subset D_k^b$, $\tilde{\Gamma}_k \subset \mathcal{C}^b$; now put $\Gamma_k = \pi_{sb}(\tilde{\Gamma}_k)$. If $Q_2 \neq Q_1$, we apply **B.2'** to Q_2 and obtain a curve $\Gamma_b \subset D_b^b$, $\Gamma_b \not\subset \mathcal{C}^b$. Now we look at the point Q_3. If $Q_3 = Q_1$, we apply **B.4'** to Q_3 and as before, the result follows. If $Q_3 \neq Q_1$, we apply **B.1** to Q_3 and find the curve $\tilde{\Gamma}_k \subset D_k^s$ and the result follows. *Now suppose we have* $D_i^b \cap D_b^b \subset \mathcal{C}^b$ (hence $D_k^b \cap D_b^b \not\subset \mathcal{C}^b$). If $Q_2 = Q_1$, we apply **B.5'** to Q_2 and find the curve $\tilde{\Gamma}_k \subset D_k^b$, $\tilde{\Gamma}_k \subset \mathcal{C}^b$ directly. If $Q_2 \neq Q_1$, we apply **B.1** to Q_2 and obtain a curve $\Gamma_b \subset D_b^b$ such that $\Gamma_b \subset \mathcal{C}^b$. Finally, we look to the point Q_3: if $Q_3 = Q_1$ we apply **B.4** to Q_3; and if $Q_3 \neq Q_1$, we apply **B.2** to Q_3. Either way, we find the curve $\tilde{\Gamma}_k \subset D_k^b$, $\tilde{\Gamma}_k \subset \mathcal{C}^b$ and we are done.

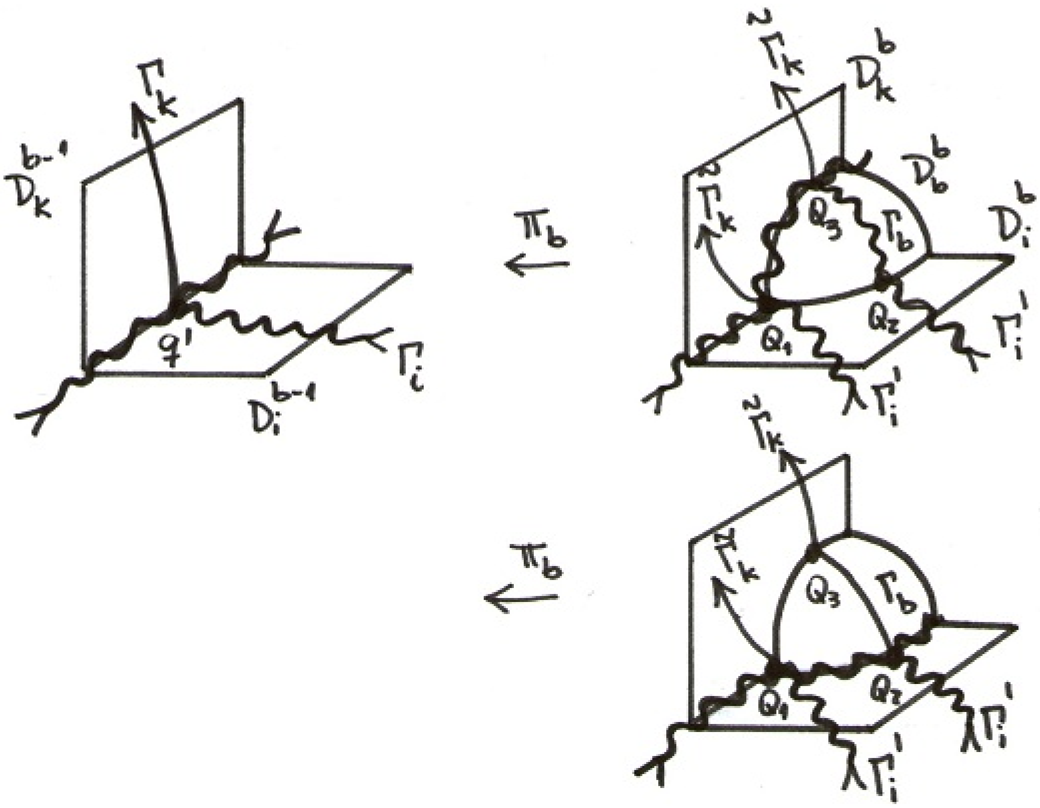


Figure 29: Case **B.1'**

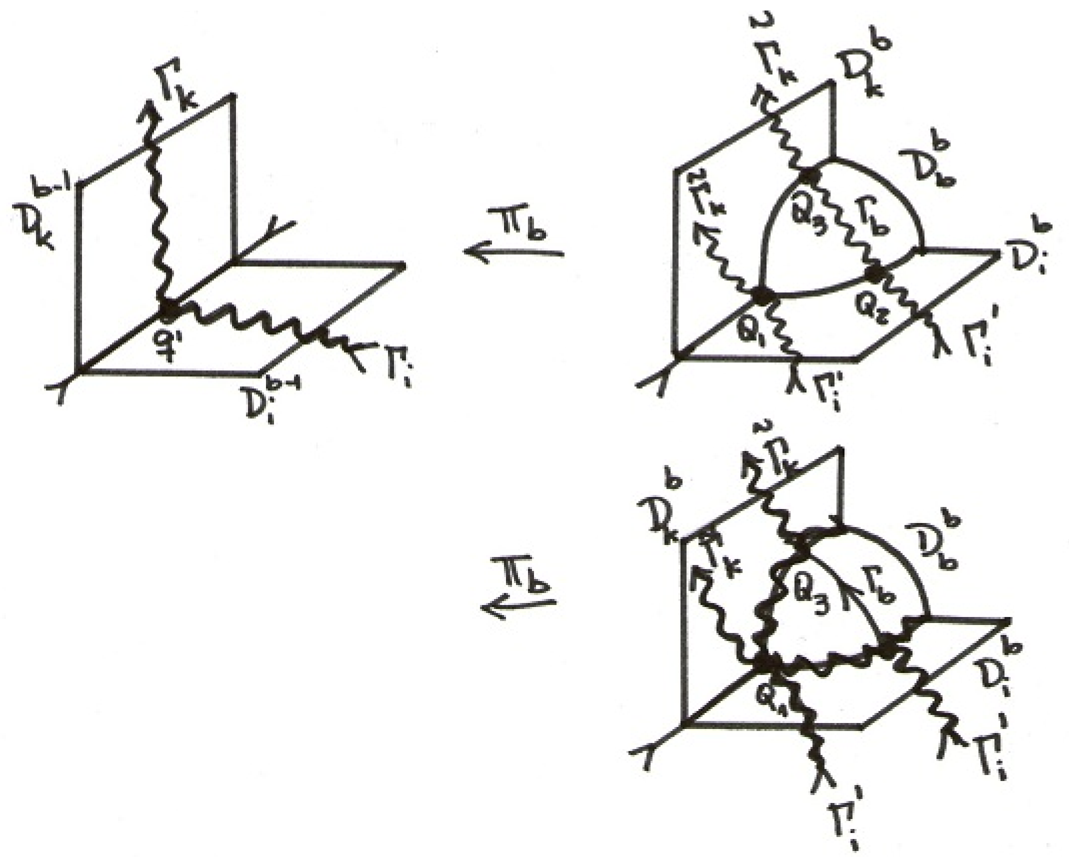


Figure 30: Case **B.2**

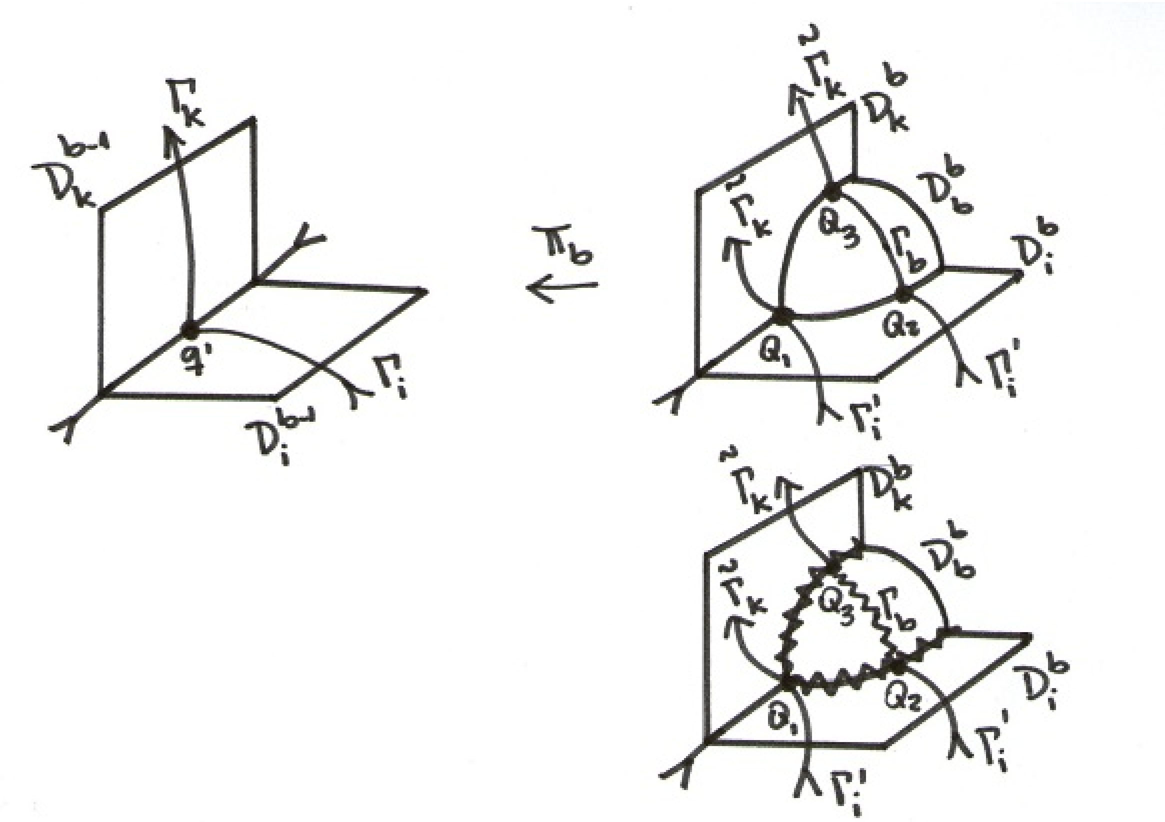


Figure 31: Case **B.2’**

Now we move on to cases **B.3 - B.5'** (see Figures 32 - 37). We will use the following notation:

$$Q_1 = D_i^b \cap D_k^b \cap D_b^b \ ,$$

$$Q_2 = D_k^b \cap D_l^b \cap D_b^b \ ,$$

$$Q_3 = D_i^b \cap D_l^b \cap D_b^b \ ,$$

$$Q_4 = \Gamma_i \cap (D_i^b \cap D_b^b) \ ,$$

where $\Gamma_i' \subset \text{Sing } \mathcal{F}_b$ once again denotes the transform by π_b of $\Gamma_i \subset \text{Sing } \mathcal{F}_{b-1}$. We apply Lemma 32 to the points Q_1, Q_2, Q_3 in order to see which of the curves $D_i^b \cap D_b^b$, $D_k^b \cap D_b^b$, $D_l^b \cap D_b^b$ are irreducible components of $\mathcal{C}^b$. We apply the induction hypothesis to the point Q_4 in order to find a trace curve $\Gamma_b \subset D_b^b$, $Q_4 \in \Gamma_b$. Depending on the case, Γ_b belongs to $\mathcal{C}^s$ or not. Note that we may have $Q_4 = Q_1$ or $Q_4 = Q_3$; in these cases we will be applying cases **B.3 - B.5'** to the point Q_4. Otherwise, we will be applying cases **B.1 - B.2'** to Q_4. Now let

$$Q_k = \Gamma_b \cap (D_k^b \cap D_b^b) \ ,$$

$$Q_l = \Gamma_b \cap (D_l^b \cap D_b^b) \ .$$

These points exist because $D_b^b \simeq \mathbb{P}^2$ and Γ_b, $D_k^b \cap D_b^b$, $D_l^b \cap D_b^b$ are projective lines in D_b^b. We apply the induction hypothesis to the points Q_k, Q_l and find trace curves $\tilde{\Gamma}_k \subset D_k^b$, $\tilde{\Gamma}_l \subset D_l^b$, $Q_k \in \tilde{\Gamma}_k$, $Q_l \in \tilde{\Gamma}_l$. We take $\Gamma_k = \pi_{s,b}(\tilde{\Gamma}_k)$, $\Gamma_l = \pi_{sb}(\tilde{\Gamma}_l)$ and the result follows. Notice that we may have $Q_k = Q_1$ or Q_2, and $Q_l = Q_2$ or Q_3; in these cases, we will apply cases **B.3 - B.5'** to the points Q_k, Q_l. Otherwise, we apply cases **B.1 - B.2'**.

□

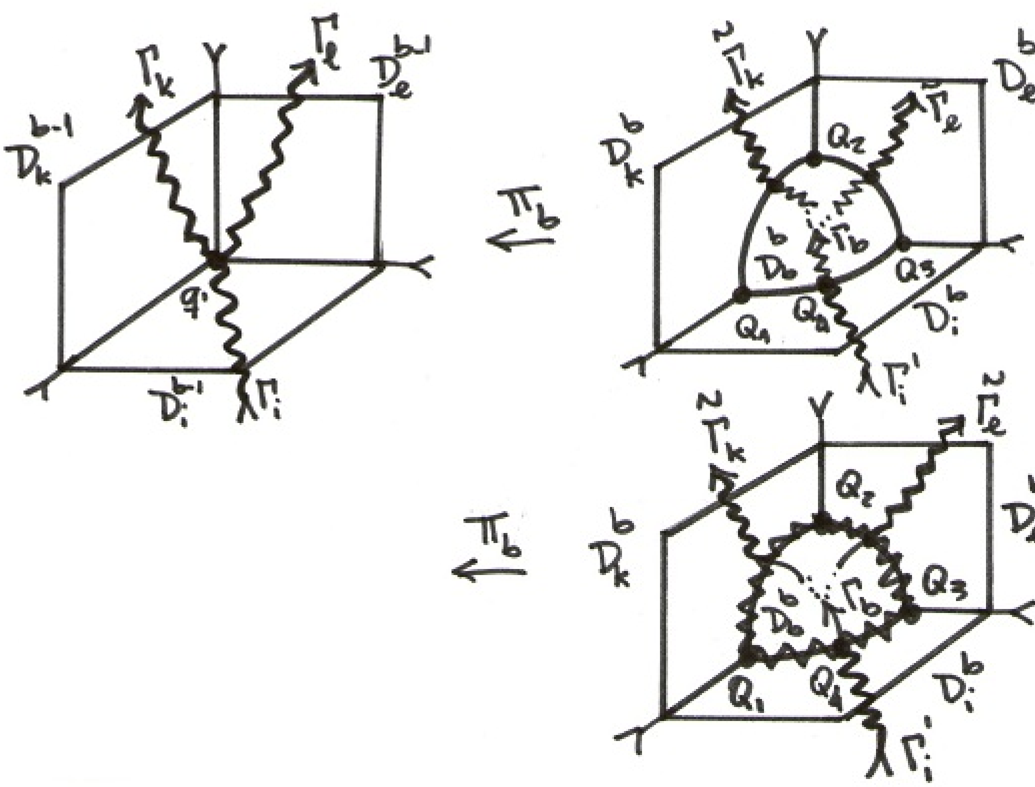


Figure 32: Case **B.3**. We recall we denote the hypothesis data with an arrow pointing towards inside the curve, and the conclusion data with an arrow point towards outside the curve.

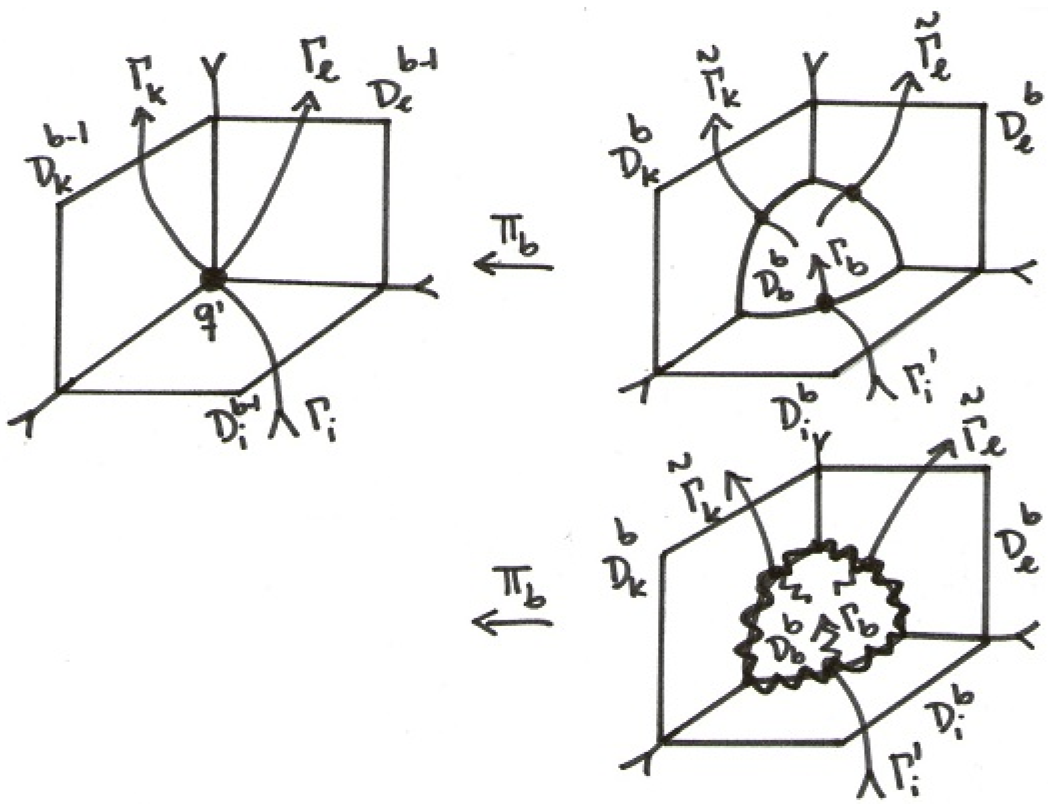


Figure 33: Case **B.3'**.

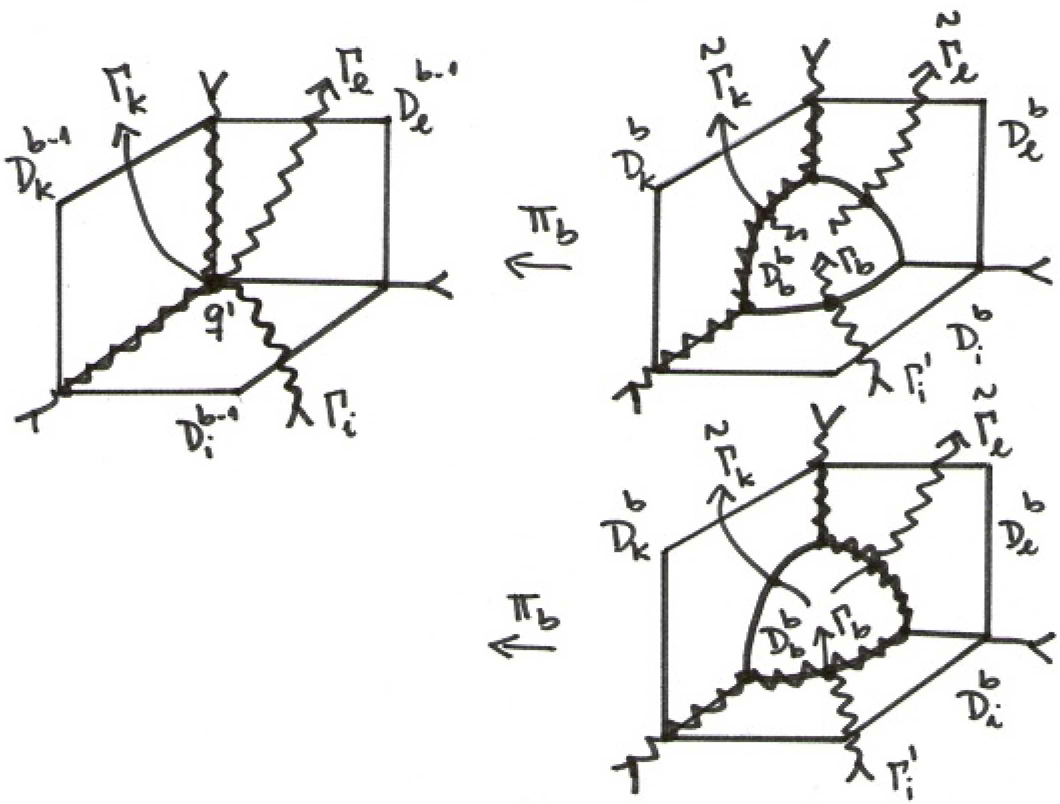


Figure 34: Case **B.4**.

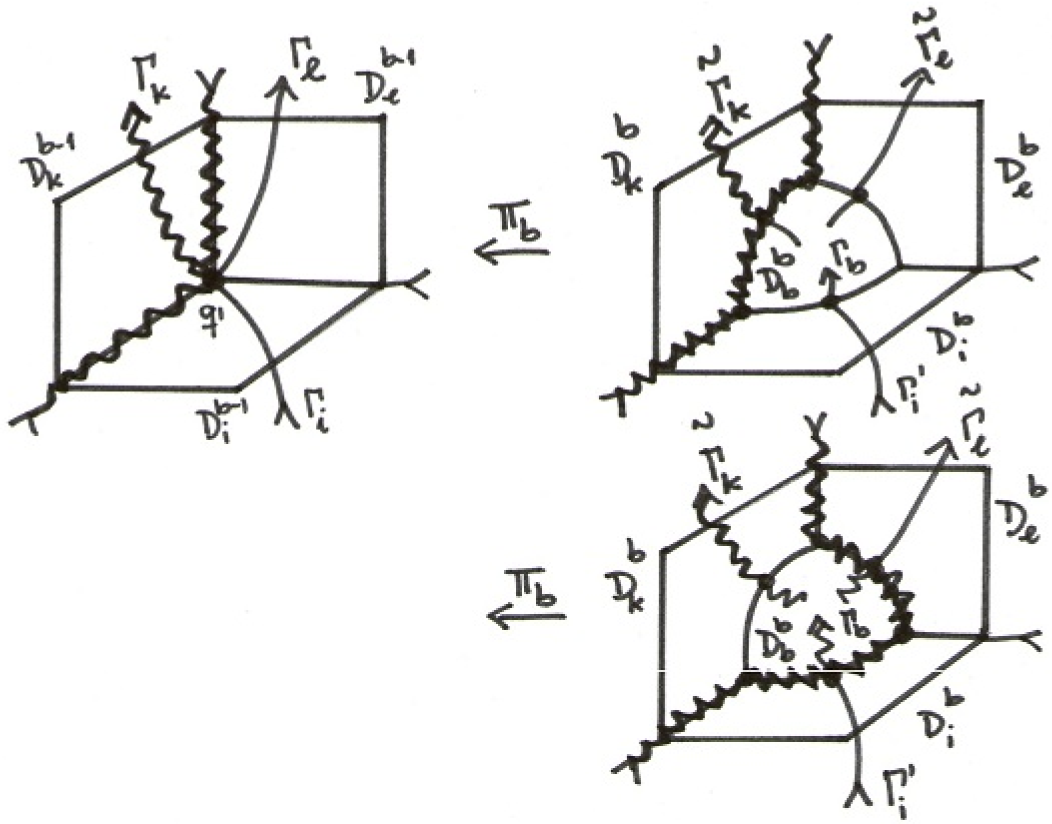


Figure 35: Case **B.4'**

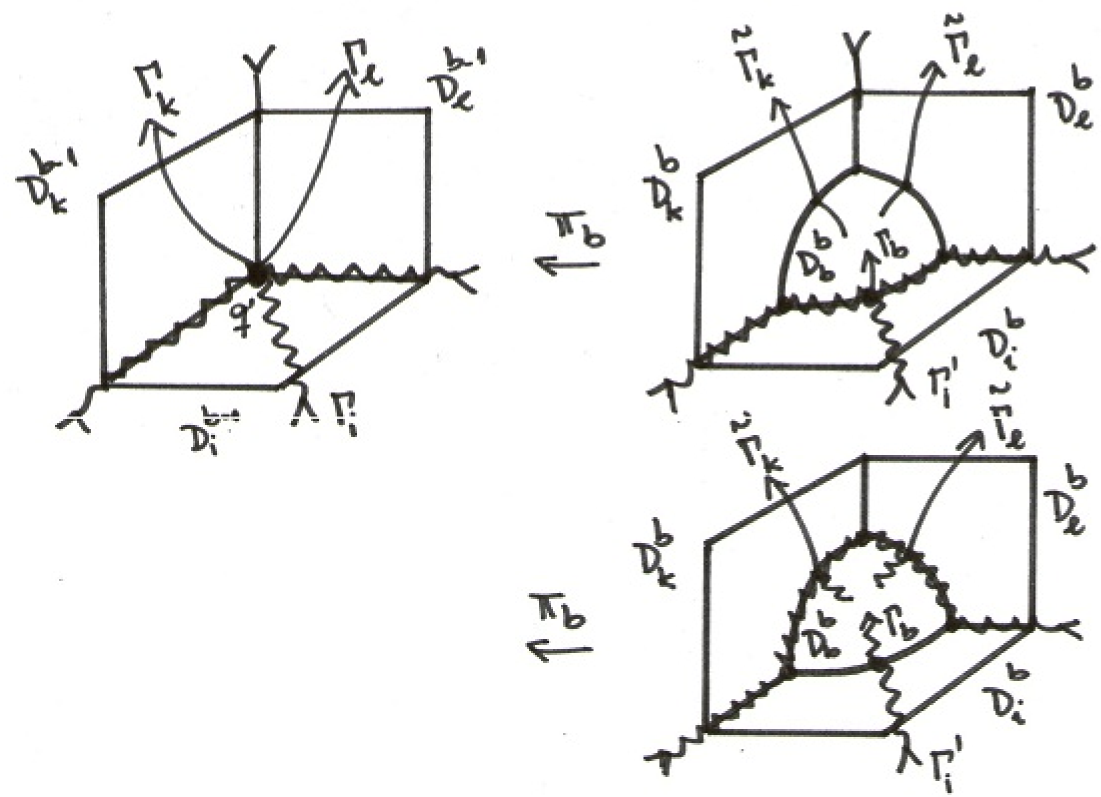


Figure 36: Case **B.5**.

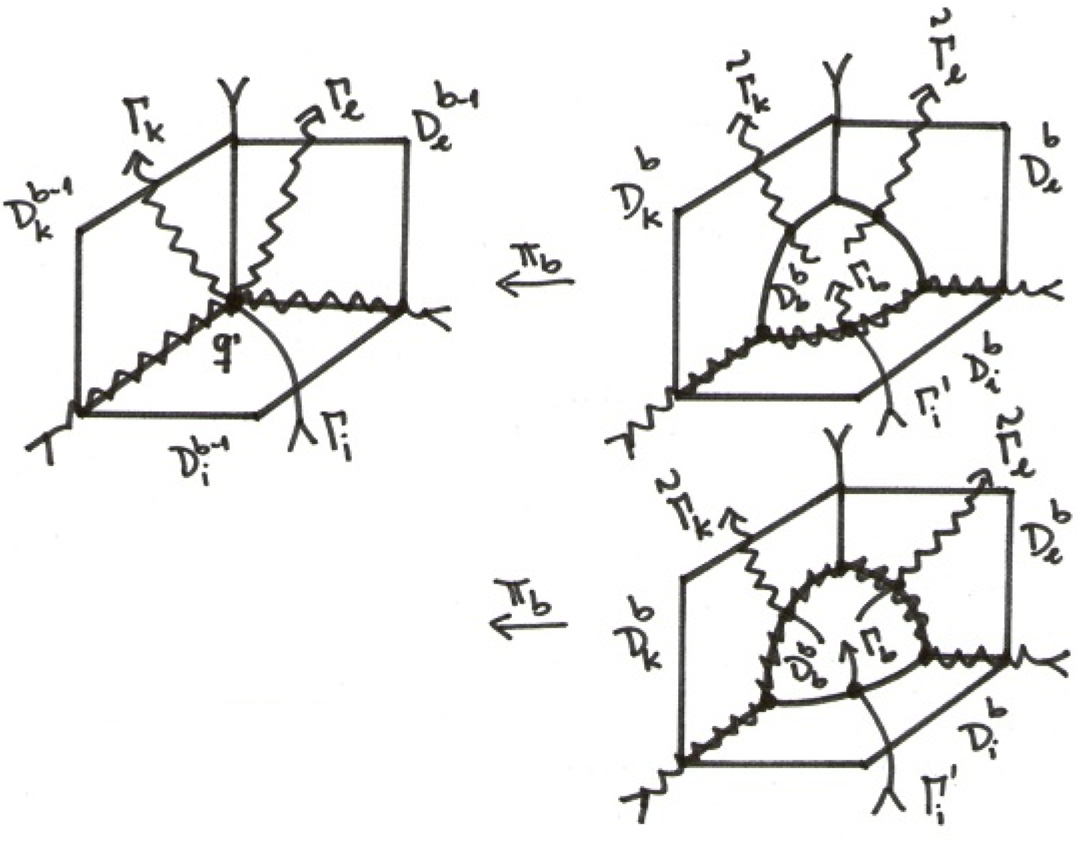


Figure 37: Case **B.5'**.

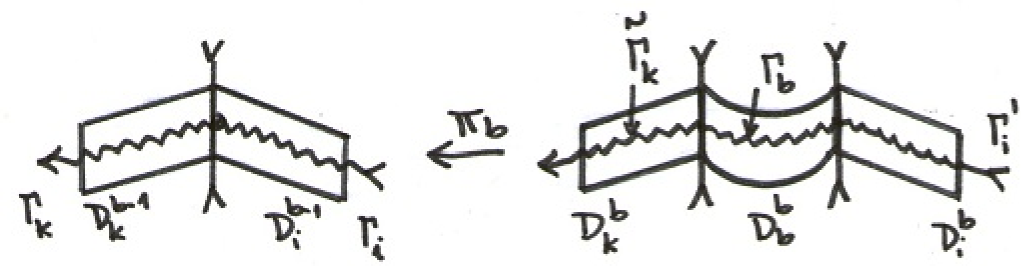


Figure 38: Remark 22.

Remark 22 Case **B.2** is true in the case that D_i^{b-1} and D_k^{b-1} are noncompact invariant components of E^{b-1} and $Y_{b-1} = D_i^{b-1} \cap D_k^{b-1}$. We have that $\Gamma_i = \sigma_{b-1}^{-1}(\underline{0}) \cap D_i^{b-1}$ and we obtain that $\Gamma_k = \sigma_{b-1}^{-1}(\underline{0}) \cap D_k^{b-1}$ (see Figure 38). Let us prove this. Firstly suppose $\mathrm{ht}(q) = 0$. Then q is a simple point and there exists only one point $q' \in M_N$ such that $\rho_s(q') = q$. Hence q' is a simple point with dimensional type three and since $\mathcal{C} \subset \mathrm{Sing}\, \mathcal{F}_N$ is not of good type, it follows directly that

$$\tilde{\Gamma}_k = \pi^{-1}(\underline{0}) \cap D_k^N$$

is an irreducible component of $\mathcal{C}$. Hence $\Gamma_k = \rho_s(\tilde{\Gamma}_k) = \sigma_s^{-1}(\underline{0}) \cap D_k^s$ is an irreducible component of $\mathcal{C}^s$ and the result follows. Now suppose that $\mathrm{ht}(q) \geq 1$. Let $b > s$ be the first index such that $q \in \pi_{s(b-1)}(Y_{b-1})$. Let $q' \in Y_{b-1}$ be the point such that $\pi_{s(b-1)}(q') = q$. Note that due to **C** we have that π_b is a nondicritical blow-up. We will use the same notation as before.

We apply the induction hypothesis to the point $Q_1 = \Gamma_i' \cap D_b^b$. Since $\mathcal{C}$ is not of good type, we have that $D_i^b \cap D_b^b \not\subset \mathcal{C}^b$. Hence

$$\Gamma_b = \sigma_b^{-1}(\underline{0}) \cap D_b^b \subset \mathcal{C}^b \ .$$

Applying the induction hypothesis to the point $Q_2 = \Gamma_b \cap D_k^b$ we obtain that

$$\tilde{\Gamma}_k = \sigma_b^{-1}(\underline{0}) \cap D_k^b \subset \mathcal{C}^b \ .$$

Hence $\Gamma_k = \pi_{sb}(\tilde{\Gamma}_k) \subset \mathcal{C}^s$ and we are done.

4.4 Structural results: incompatibility of trace curves

In this section we continue with the proof of Proposition 31. The following result is another property that the nodal component $\mathcal{C} \subset \text{Sing } \mathcal{F}_N$ we have fixed satisfies at intermediate steps of the reduction of singularities $\mathcal{S}$.

Lemma 34 [Incompatibility] *Let $q \in \sigma_s^{-1}(\underline{0})$ be a point of $\mathcal{C}^s$ such that there is one irreducible component Γ^s of $\mathcal{C}^s$ containing q which is generically contained in only one component $D_i^s \subset E_{inv}^s$. Then there does not exist a curve $\Upsilon^s \subset Sing\ \mathcal{F}_s$ such that*

1. *D_i^s is the only invariant component of E_{inv}^s which contains Υ^s.*
2. *$\Upsilon^s \not\subset \mathcal{C}^s$.*
3. *$q \in \Upsilon^s$.*

Proof: Suppose, by absurd, that there exists a curve $\Upsilon^s \subset Sing\mathcal{F}_s$ which satisfies 1-3. As before, we will use induction on the height of q. If $\text{ht}(q) = 0$, we have that q is a simple point. Hence there cannot exist two trace curves contained in the same invariant component intersecting at q, and the result follows.

Assume that $\text{ht}(q) \geq 1$. Let $b > s$ be the first index such that $q \in \pi_{s(b-1)}(Y_{b-1})$. Let $q' \in Y_{b-1}$ be the point such that $\pi_{s(b-1)}(q') = q$. From Lemma 33 it follows that π_b is nondicritical. We recall we denote $\pi_b^{-1}(Y_{b-1}) = D_b^b$. We will call Γ^{b-1}, Υ^{b-1} the transforms by $\pi_{s,(b-1)}$ of Γ^s, Υ^s. The transforms of Γ^{b-1}, Υ^{b-1}, by π_b will be denoted Γ^b, Υ^b. Call

$$Q_1 = \Gamma^b \cap (D_i^b \cap D_b^b) \ ,$$
$$Q_2 = \Upsilon^b \cap (D_i^b \cap D_b^b) \ .$$

There are several cases to consider.

First case: $Y_{b-1} = \{q'\}$ (see Figure 39). We perform the blow-up π_b. If $Q_1 = Q_2$, by the induction hypothesis we are done. If $Q_1 \neq Q_2$, by Lemma 33 there exist two trace curves $\tilde{\Gamma}$, $\tilde{\Upsilon} \subset D_b^b$ such that one of them is an irreducible component of $\mathcal{C}^b$ and the other is not. Since $D_b^b \simeq \mathbb{P}^2$, the projective lines $\tilde{\Gamma}$, $\tilde{\Upsilon}$ must intersect. We apply the induction hypothesis to the point $Q_3 = \tilde{\Gamma} \cap \tilde{\Upsilon}$.

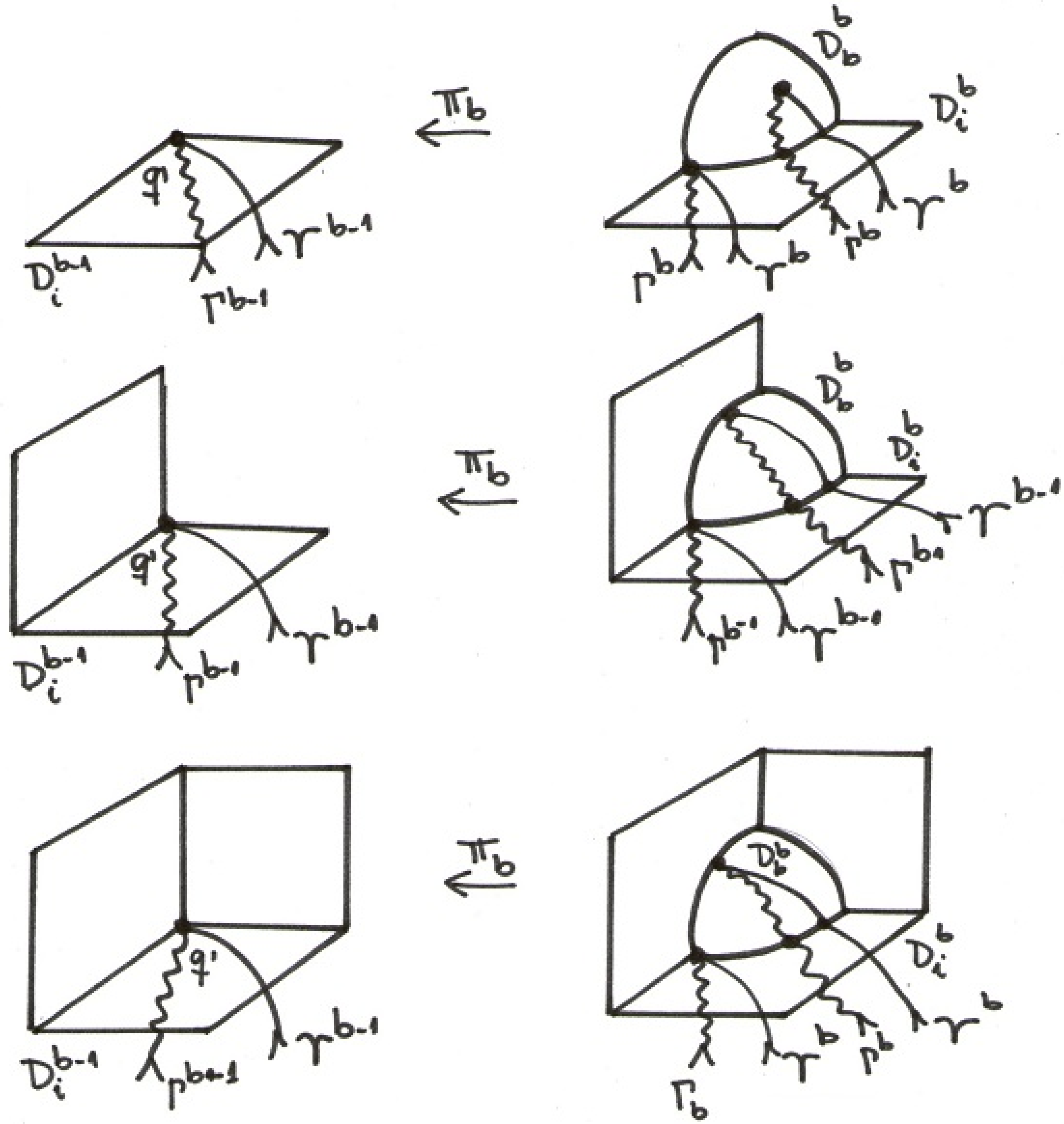


Figure 39: These are the cases $Y_{b-1} = \{q'\}$. The image at the top shows the case $e_{q'}(E_{inv}^{b-1}) = 1$, the image in the middle shows the case $e_{q'}(E_{inv}^{b-1}) = 2$ and the image at the bottom shows the case $e_{q'}(E_{inv}^{b-1}) = 3$.

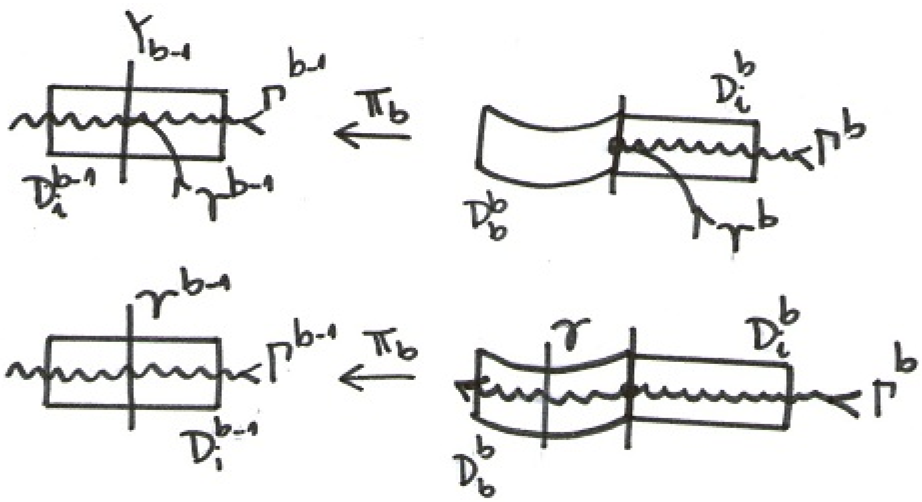


Figure 40: This figure exhibits the cases when Y_{b-1} is a germ of curve. The image at the top shows the second case and the image at the bottom shows the third case.

Second case: Y_{b-1} *is a germ of curve transversal to* D_i^{b-1} (see Figure 40). Let Q_1, Q_2 be as above. If $Q_1 = Q_2$, by the induction hypothesis the result follows. If $Q_1 \neq Q_2$, by Lemma 33 we find an irreducible component of $\mathcal{C}^b$ which is contained in the noncompact component D_b^b. This is an absurd since $\mathcal{C}$ is not of good type.

Third case: Y_{b-1} *is the curve* Υ^{b-1} (see Figure 40). In this case we have that D_i^s is not compact and therefore $\Gamma^{b-1} = \sigma_{b-1}^{-1}(\underline{0}) \cap D_i^{b-1}$. Let Q_1 be as before. Since $\mathcal{C}$ is not of good type, we have that $D_i^b \cap D_b^b$ is not an irreducible component of $\mathcal{C}^b$. By Lemma 33, the curve $\tilde{\Gamma} = \sigma_b^{-1}(\underline{0}) \cap D_b^b$ is an irreducible component of $\mathcal{C}^s$; moreover, D_b^b is the only invariant component of E^b which generically contains $\tilde{\Gamma}$. By Lemma **??** there exists a trace curve $\tilde{\Upsilon} \subset \text{Sing } \mathcal{F}_b$, $\tilde{\Upsilon} \subset D_b^b$. We apply the induction hypothesis to the point $Q_2 = \tilde{\Upsilon} \cap \tilde{\Gamma}$ and the result follows.

□

4.5 The goodness of nodal components

In this section we finish the proof of Proposition 31. The nodal component $\mathcal{C} \subset \text{Sing } \mathcal{F}_N$ we have fixed satisfies Lemmas 30, 32, 33 and 34. We want to show that such a $\mathcal{C}$ cannot exist.

We will descend to the stage M_s of the reduction of singularities $\mathcal{S}$ where it appears, for the first time, an irreducible component of $\mathcal{C}$. In other words: let $\pi_s : M_s \to M_{s-1}$ be the blow-up centered at $Y_{s-1} \subset M_{s-1}$. So in M_{s-1} there are no irreducible components of $\mathcal{C}$, that is to say, $\mathcal{C}^{s-1} = \emptyset$; and in M_s we see, for the first time, an irreducible component of $\mathcal{C}$: $\mathcal{C}^s \neq \emptyset$. Note that, due to Lemmas 32 and 33, π_s is not dicritical.

First case: $s = 1$ (see Figure 41). The nondicritical blow-up $\pi_1 : M_1 \to M_0 = (\mathbb{C}^3, \underline{0})$ is centered at $Y_0 \subset M_0$ and $E^1 = D_1^1 = \pi_1^{-1}(Y_0) \subset M_1$. Suppose $Y_0 = \{\underline{0}\}$: so $D_1^1 \simeq \mathbb{P}^2$ is compact and invariant (since π_1 is not dicritical). So $\mathcal{C}^1 \subset D_1^1$ is a trace curve. Let $\Delta \subset M_0$ be a plane section which is generically transversal to $\mathcal{F}$ and let Δ' be its transform by π_1. We may assume that the section Δ satisfies the following properties:

1. $\underline{0} \in \Delta$ is a singularity in dimension two of the induced foliation $\mathcal{F}|_{\Delta}$.
2. Every point in Δ' is a singularity in dimension two.

In Δ' we have the following situation: $D_1^1 \cap \Delta' \simeq \mathbb{P}^1$ is an invariant divisor and the point $p = \mathcal{C}^1 \cap \Delta'$ is a nodal singularity in dimension two; due to Remark 5 there exists a point $q \in D_1^1 \cap \Delta'$ such that *q is a trace singularity which is not nodal*. Since q is a simple singularity of dimensional type two of $\mathcal{F}_1$, there exists a trace curve $\Gamma \subset \operatorname{Sing} \mathcal{F}_1$, $\Gamma \subset D_1^1$, such that $q \in \Gamma$ and $\Gamma \not\subset \mathcal{C}^1$. However, Γ and the irreducible component of $\mathcal{C}^1$ through p are two projective lines in D_1^1; therefore they must intersect. This is an absurd due to Lemma 34.

Now suppose Y_0 is a germ of curve contained in $\operatorname{Sing} \mathcal{F}$. Thus $E^1 = D_1^1 = \pi_1^{-1}(Y_0)$ is not compact and since $\mathcal{C}$ is not of good type it follows that $\mathcal{C}^1 = \pi_1^{-1}(\underline{0})$. Due to Remark 20 there exists a trace curve $\gamma \subset \operatorname{Sing} \mathcal{F}_1$ such that $\pi_1(\gamma) = Y_0$. Thus $\gamma \cap \mathcal{C}^1 \neq \emptyset$, which is an absurd due to Lemma 34.

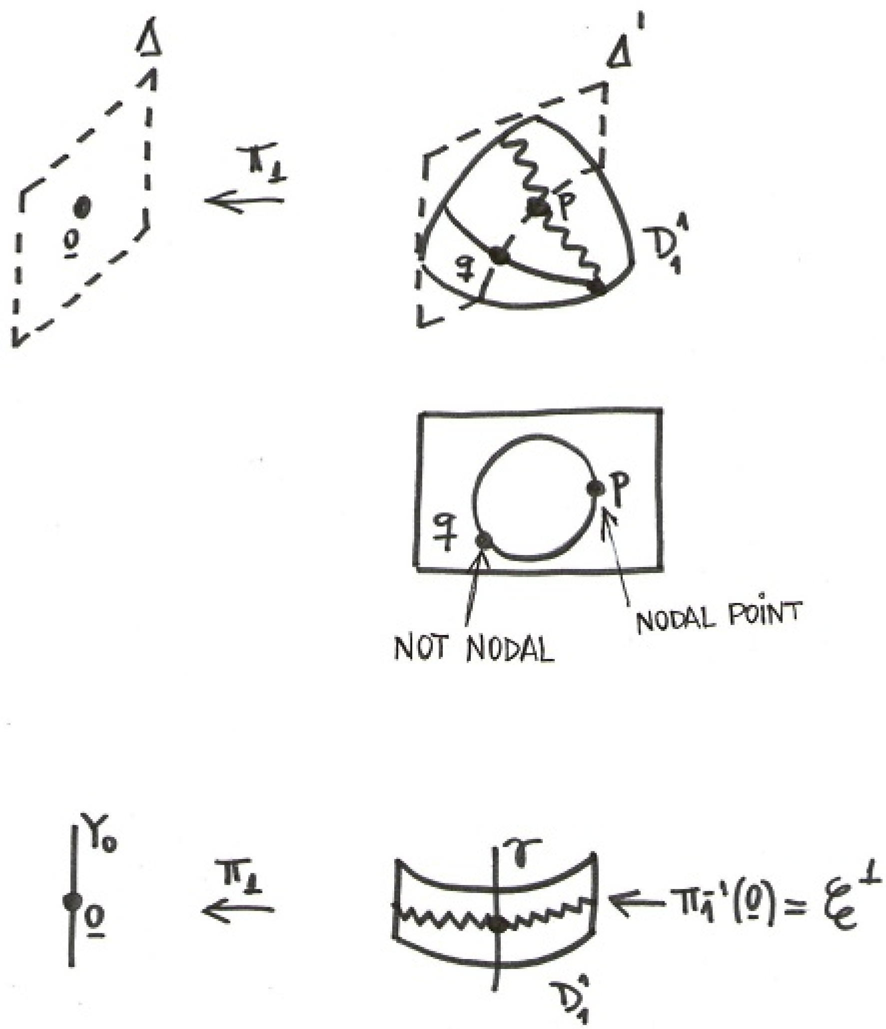


Figure 41: Case $s = 1$.

Second case: $s \geq 2$. The nondicritical blow-up $\pi_s : M_s \to M_{s-1}$ is centered at $Y_{s-1} \subset M_{s-1}$. So $\pi_s^{-1}(Y_{s-1}) = D_s^s$ and $\mathcal{F}_s$ is the transform of $\mathcal{F}$. First suppose Y_{s-1} is a point $\{q\}$. Since $\mathcal{C}$ is not of good type, $q \notin E_{dic}^{s-1}$ and $e_q(E_{inv}^{s-1}) \geq 1$. Assume first that $e_q(E_{inv}^{s-1}) = 1$. So $q \in D_i^{s-1} \subset E_{inv}^{s-1}$. The component $D_s^s = \pi_s^{-1}(q)$ is compact, invariant and $\mathcal{C}^s \subset D_s^s$. Let Γ_s be an irreducible component of $\mathcal{C}^s$. Since Γ^s, $D_i^s \cap D_s^s$ are projective lines in D_s^s, they must intersect. At the point of intersection of Γ^s and $D_i^s \cap D_s^s$ we apply Lemma 33: so either $D_i^s \cap D_s^s \subset \mathcal{C}^s$ or there exists a trace curve $\Gamma_i \subset \text{Sing } \mathcal{F}_s$, $\Gamma_i \subset D_i^s$ such that $\Gamma_i \subset \mathcal{C}^s$. However, the second case would imply that $\pi_s(\Gamma_i)$ is a curve of the singular locus Sing $\mathcal{F}_{s-1}$ such that $\pi_s(\Gamma_i) \subset D_i^{s-1}$, $\pi_s(\Gamma_i) \subset \mathcal{C}^{s-1}$ and hence $\mathcal{C}^{s-1} \neq \emptyset$, which is an absurd.

Therefore we have $D_i^s \cap D_s^s \subset \mathcal{C}^s$ (see Figure 42). We take a plane section $\Delta \subset M_{s-1}$ generically transversal to $\mathcal{F}_{s-1}$ and which satisfies conditions 1. and 2. above. In Δ' there are two invariant divisors, $D_s^s \cap \Delta' \simeq \mathbb{P}^1$ and $D_i^s \cap \Delta'$. The corner point $p = \Delta' \cap (D_i^s \cap D_s^s)$ is a nodal singularity in dimension two. Due to Remark 5, there exists a point $q' \in D_s^s \cap \Delta'$, $q' \neq p$, that is a trace singularity which is not nodal. Since q' is a singularity of dimensional type two of $\mathcal{F}_s$, there exists a curve $\Gamma \subset$ Sing $\mathcal{F}_s$, $\Gamma \subset D_s^s$, such that $\Gamma \not\subset \mathcal{C}^s$ and $q' \in \Gamma$. At the point $Q = \Gamma \cap (D_i^s \cap D_s^s)$ we apply Lemma 33 and we find a trace curve $\Gamma_i \subset \text{Sing } \mathcal{F}_s$, $\Gamma_i \subset D_i^s$ such that $\Gamma_i \subset \mathcal{C}^s$. However as we have seen this implies that $\pi_s(\Gamma_i) \subset \mathcal{C}^{s-1}$ and hence $\mathcal{C}^{s-1} \neq \emptyset$, which is an absurd and the result follows.

For the case $e_q(E_{inv}^{s-1}) = 2$, we have that $q \in D_i^{s-1} \cap D_k^{s-1}$. We conclude that both $D_i^s \cap D_s^s$ and $D_k^s \cap D_s^s$ must be irreducible components of $\mathcal{C}^s$ and repeat the argument above. And for $e_q(E_{inv}^{s-1}) = 3$, we have that $q \in D_i^{s-1} \cap D_k^{s-1} \cap D_l^{s-1}$, and all curves $D_i^s \cap D_s^s$, $D_k^s \cap D_s^s$ and $D_l^s \cap D_s^s$ are irreducible components of $\mathcal{C}^s$. Once again, we repeat the argument above (see Figures 43 and 44).

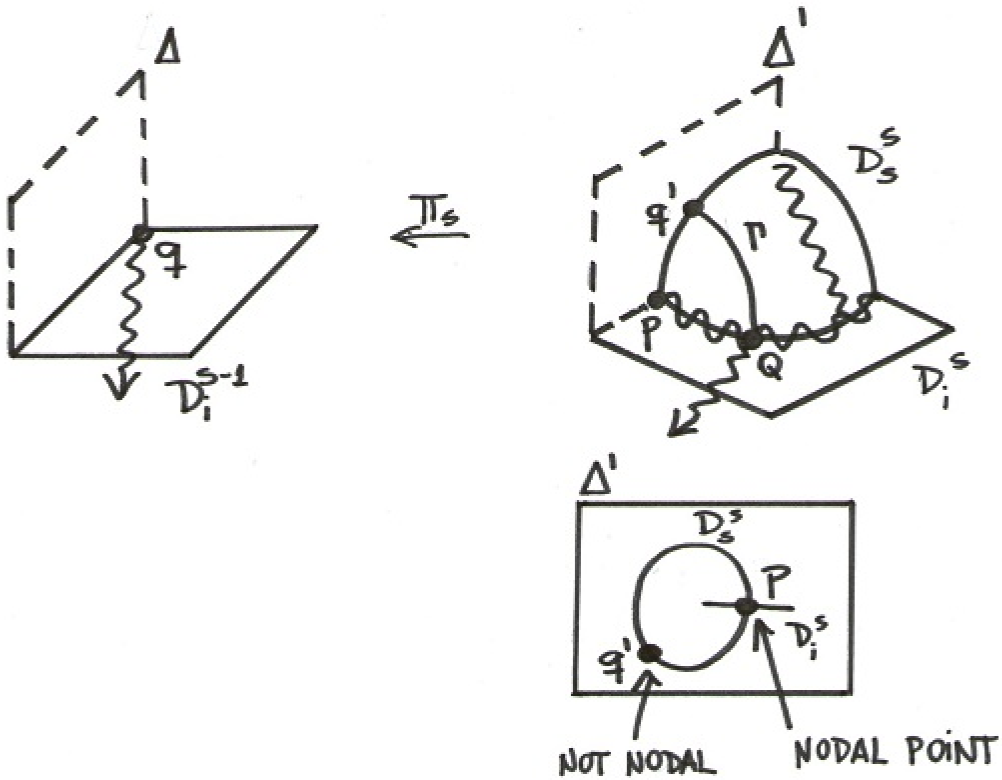


Figure 42: Case $s \geq 2$, π_s is a nondicritical blow-up centered at q and $e_q(E_{inv}^{s-1}) = 1$.

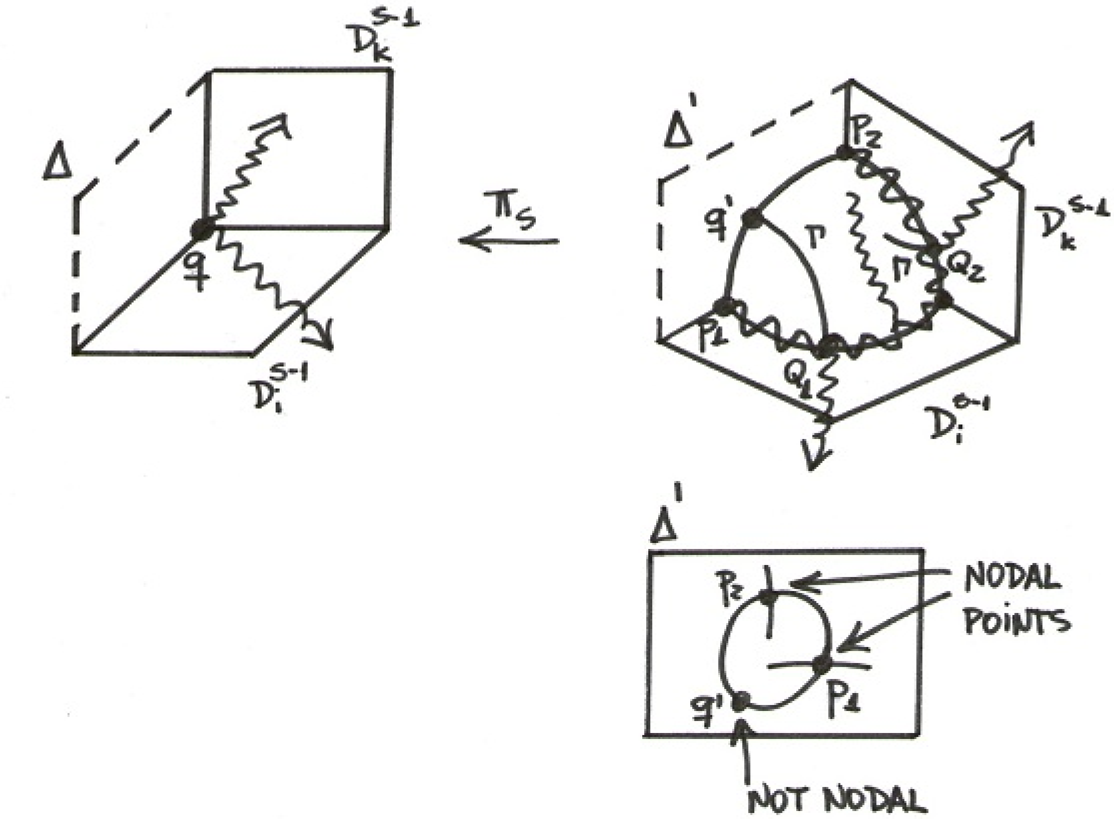


Figure 43: Case $s \geq 2$, π_s is a nondicritical blow-up centered at q and $e_q(E_{inv}^{s-1}) = 2$.

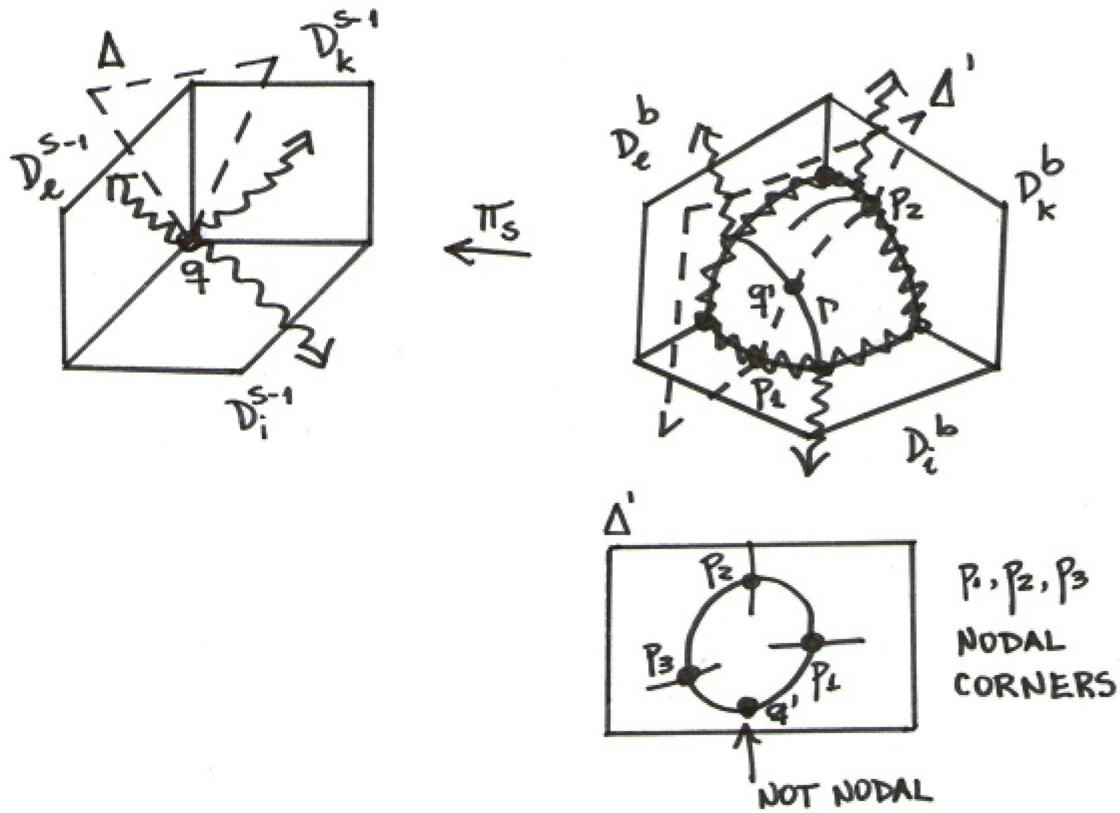


Figure 44: Case $s \geq 2$, π_s is a nondicritical blow-up centered at q and $e_q(E_{inv}^{s-1}) = 3$.

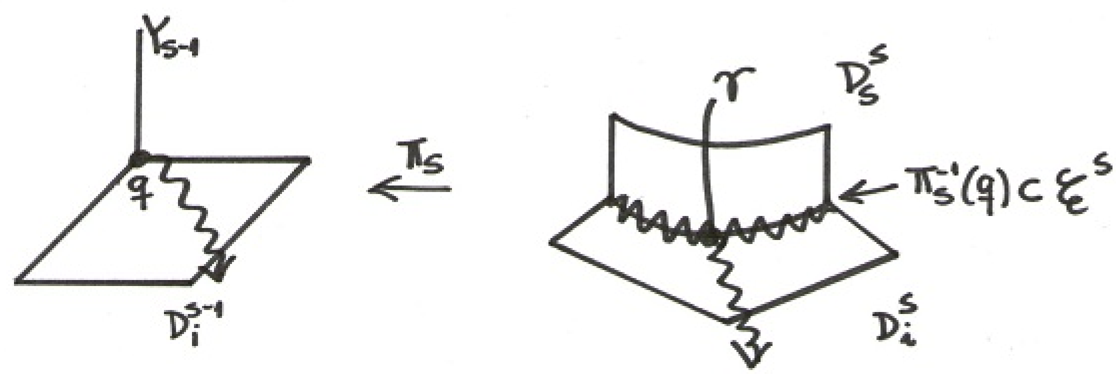


Figure 45: Case $s \geq 2$, π_s is a nondicritical blow-up centered at Y and $e_Y(E_{inv}^{s-1}) = 0$.

Finally, suppose Y_{s-1} is a germ of curve contained in Sing $\mathcal{F}_{s-1}$. Firstly assume that $e_Y(E_{inv}^{s-1}) = 0$ (that is to say, Y is not contained in any invariant divisor). There exists a point $q \in Y$ such that $q \in D_i^{s-1}$ where D_i^{s-1} is a compact component of E_{inv}^{s-1} (see Figure 45). The component $D_s^s = \pi_s^{-1}(Y_{s-1})$ is not compact and $\mathcal{C}^s = \pi_s^{-1}(q) \subset D_s^s$. Due to Lemma 28 there exists an invariant curve $\gamma \subset$ Sing $\mathcal{F}_s$, $\gamma \subset D_s^s$ such that $\pi_s(\gamma) = Y$ (in particular, γ is a trace curve). Since $\mathcal{C}$ is of not of good type, $\gamma \not\subset \mathcal{C}^s$. We apply Lemma 33 to the point $q' = \gamma \cap \mathcal{C}^s$ and obtain a trace curve $\Gamma_i \subset$ Sing $\mathcal{F}_s$, $\Gamma_i \subset D_i^s$ such that $q' \in \Gamma_i \subset \mathcal{C}^s$. Hence, as in the previous cases, we obtain that $\mathcal{C}^{s-1} \neq \emptyset$, which is an absurd.

Now assume that $e_Y(E_{inv}^{s-1}) \geq 1$: Y is contained in an invariant component $D_i^{s-1} \subset E_{inv}^{s-1}$ (we remark that the following argument is the same in the case $\gamma = D_i^{s-1} \cap D_k^{s-1}$ with D_i^{s-1}, $D_k^{s-1} \subset E_{inv}^{s-1}$). So $\mathcal{C}^s$ is the compact curve $\sigma_s^{-1}(\underline{0}) \cap D_s^s$ (see Figures 46 and 47). Since $D_i^s \cap D_s^s$ is a noncompact curve, it is not an irreducible component of $\mathcal{C}^s$. Due to Remark 22 we obtain that the curve $\sigma_s^{-1}(\underline{0}) \cap D_i^s$ is an irreducible component of $\mathcal{C}^s$; however, this implies that $\mathcal{C}^{s-1} \neq \emptyset$, which is an absurd.

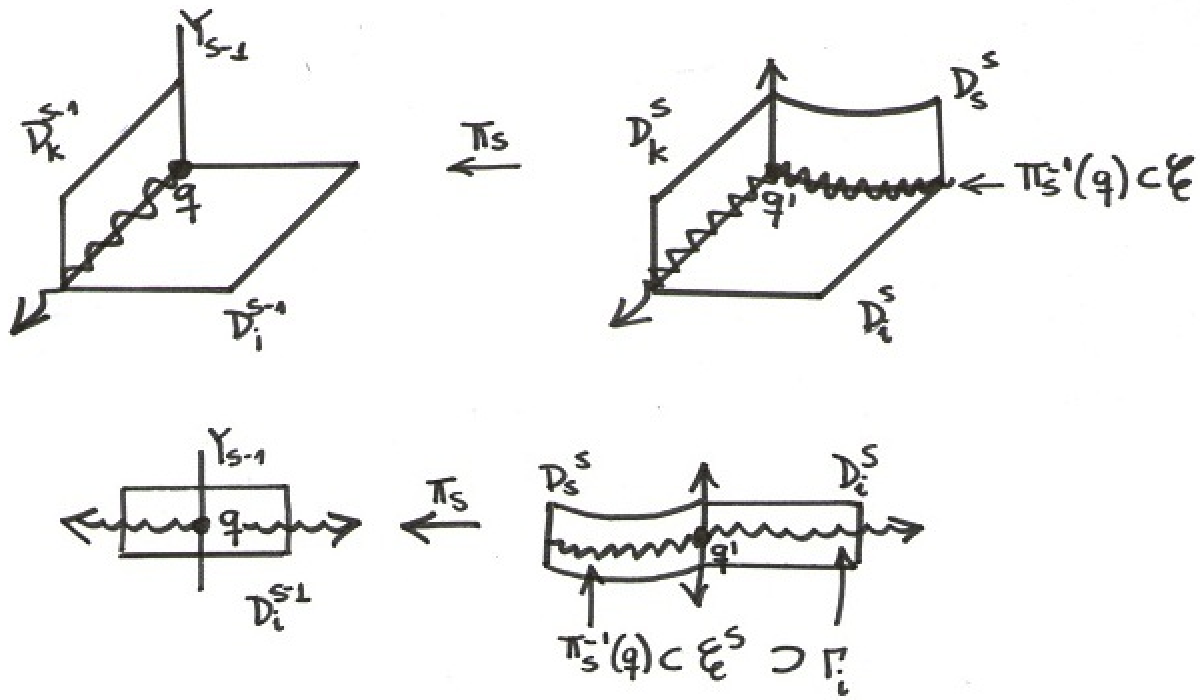


Figure 46: Case $s \geq 2$, π_s is a nondicritical blow-up centered at Y and $e_Y(E_{inv}^{s-1}) = 1$.

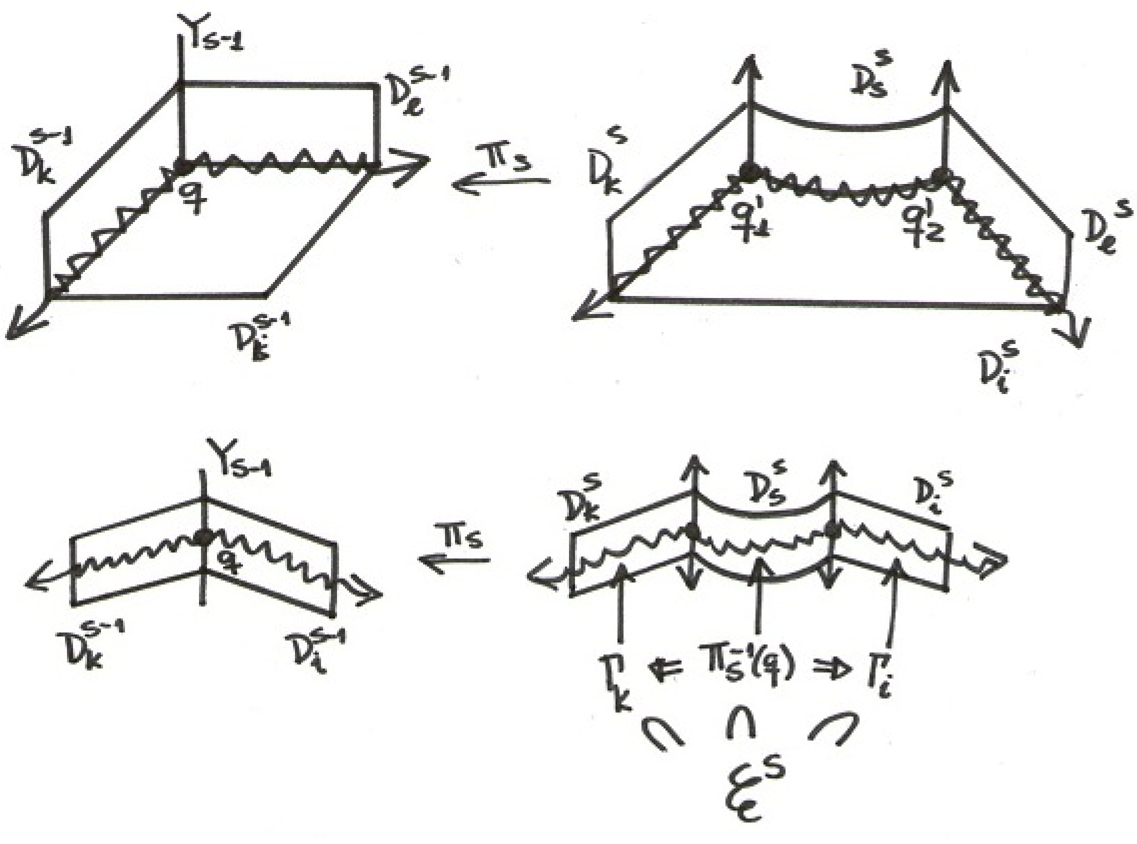


Figure 47: Case $s \geq 2$, π_s is a nondicritical blow-up centered at Y and $e_Y(E_{inv}^{s-1}) = 2$.

4.6 Brunella's local alternative with nodal components

Now we prove Theorem 7 in the general case. If there are no nodal components in Sing $\mathcal{F}_N$, we are done (Theorem 8). If there exists a nodal component $\mathcal{C} \subset \text{Sing}\,\mathcal{F}_N$, by Proposition 31 we have that $\mathcal{C}$ is of good type. We have two possibilities:

a) All the nodal components intersect at least one compact dicritical irreducible component of the exceptional divisor. In this case, we extend the arguments of the case without nodal components and we find a neighborhood W of the origin such that for each leaf $L \subset W$ of $\mathcal{F}$ there is an analytic curve $\gamma \subset L$ with $\underline{0} \in \gamma$.

b) There is a nodal component $\mathcal{C} \subset \text{Sing}\,\mathcal{F}_N$ which does not intersect any compact dicritical irreducible component of the divisor. Since $\mathcal{C}$ is of good type, either it intersects a noncompact dicritical component of the divisor or it contains a noncompact irreducible curve. Thus we find an analytic curve $\Gamma \subset (\mathbb{C}^3, \underline{0})$ in the singular locus Sing $\mathcal{F}$ such that $\mathcal{F}$ is generically dicritical or generically nodal along Γ.

5 Epilogue: Semitranscendental leaves

In some sense the global alternative of Brunella may be interpreted as a property concerning the "concentration-diffusion"of the non-transcendency of the leaves of a foliation: either we concentrate the non-transcendency in an algebraic leaf or all the leaves are not completely transcendental in the sense that they are foliated by algebraic curves. In our local situation we have an analogous of this phenomenon based on the concept of an *end of leaf.* We state this result as follows:

Theorem 10 *Let $\mathcal{F}$ be a RICH foliation in $(\mathbb{C}^3, \underline{0})$ that has no germ of invariant analytic surface. Then there is a neighborhood U of the origin $\underline{0} \in \mathbb{C}^3$ such that each leaf $L \subset U$ of $\mathcal{F}$ is U has at least one end b_L which is semi-transcendental.*

The definition of the set $\mathbb{B}_0(\mathcal{F})$ of *ends of leaves* is given by an inductive limit by means of *any* fundamental system of neighborhoods $\{U_i\}_{i \in I}$ of the origin $\underline{0} \in \mathbb{C}^3$, where we assume that $\mathcal{F}$ is defined in an open set U with $U_i \subset U$ for all $i \in I$. Denote by $Q_i(\mathcal{F})$ the space of leaves of the restriction of $\mathcal{F}$ to U_i. An *end of leaf* $b \in \mathbb{B}_0(\mathcal{F})$ is an element

$$b = (L_i)_{i \in I} \in \prod_{i \in I} Q_i(\mathcal{F})$$

such that $L_i \subset L_j$ if $U_i \subset U_j$. Given a leaf $L \subset U$, we say that *b is an end of L* if $L_i \subset L$ for all $i \in I$. We say that *b contains a germ of analytic curve γ* if $\gamma \setminus \{\underline{0}\}$ is contained in the germ at the origin of L_i for all $i \in I$.

An end of leaf $b \in \mathbb{B}_0(\mathcal{F})$ is called *semitranscendental* if it contains a germ of analytic curve or there exists a reduction of singularities

$$(\mathbb{C}^3, \underline{0}) = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_N} M_N$$

for $\mathcal{F}$ such that the lifted end $\tilde{b}$ accumulates only at the singular locus of $\mathcal{F}_N$.

Let us give an idea of the proof of Theorem 10. Take a nodal component $\mathcal{C}$ corresponding to the fixed reduction of singularities of the RICH foliation $\mathcal{F}$. We have that

$$\mathcal{C} \subset \mathrm{Sing}\ \mathcal{F}$$

is a union of generically nodal curves such that all the points are in fact nodal points. We have two possibilities:

a) There are "secondary holonomies" inside $\mathcal{C}$ that modify the transversal set of leaves in a component of $\mathcal{C}$ that approach the leaves to the divisors.

b) Such secondary holonomies do not exist and the leaves $\mathcal{C}$ are at a "fixed" distance of the divisor.

In the first case, the nodal component is not a "barrier" from the propagation of the leaves and we can proceed as in the case without nodal components. In the second case, the leaves around $\mathcal{C}$ correspond to semitranscendental ends of leaves accumulating at $\mathcal{C}$. Since the only "barrier" we found are of this type, Theorem 10 follows.

Note of course that the fact that the nodal components are always of good type, the semitranscendental ends of leaves that we found either contain a germ of analytic curve or they accumulate (in a very precise way) at a curve Γ in the initial singular locus Sing $\mathcal{F}$, where $\mathcal{F}$ is either generically dicritical or generically nodal along Γ.

In the future we will undertake a more accurate study of semitranscendental ends in order to precise the ideas above. We acknowledge J. F. Mattei for the helpful and stimulating discussions about the behavior of semitranscendental ends.

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