On universal realizability of spectra.*

Ana I. Julio[a], Carlos Marijuán[b], Miriam Pisonero[b], Ricardo L. Soto[a†]

[a]Dpto. Matemáticas, Universidad Católica del Norte, Casilla 1280
Antofagasta, Chile.

[b]Dpto. Matemática Aplicada, Universidad de Valladolid/IMUVA, Spain.

Abstract

A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is said to be *realizable* if it is the spectrum of an entrywise nonnegative matrix. The list Λ is said to be *universally realizable* ($\mathcal{UR}$) if it is the spectrum of a nonnegative matrix for each possible Jordan canonical form allowed by Λ. It is well known that an $n \times n$ nonnegative matrix A is co-spectral to a nonnegative matrix B with constant row sums. In this paper, we extend the co-spectrality between A and B to a similarity between A and B, when the Perron eigenvalue is simple. We also show that if $\epsilon \geq 0$ and $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$, then $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $\mathcal{UR}$. We give counter-examples for the cases: $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$ implies $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$ is $\mathcal{UR}$, and Λ_1, Λ_2 are $\mathcal{UR}$ implies $\Lambda_1 \cup \Lambda_2$ is $\mathcal{UR}$.

Key words: nonnegative matrix, inverse eigenvalue problem, universal realizability.

*Supported by Fondecyt 1170313, Chile; conicyt-pai 79160002, 2016, Chile; MTM2015-365764-C-1 (MINECO/FEDER); MTM2017-85996-R (MINECO/FEDER); Consejería de Educación de la Junta de Castilla y León (Spain) VA128G18.

†*Corresponding author: rsoto@ucn.cl (R.L. Soto), ajulio@ucn.cl (A.I. Julio), marijuan@mat.uva.es (C. Marijuán), mpisoner@maf.uva.es (M. Pisonero).

1 Introduction

Let M_n denote the set of $n \times n$ real matrices and $M_{k,l}$ the set of $k \times l$ real matrices. Let $A \in M_n$ and let

$$J(A) = S^{-1}AS = diag\,(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k))$$

be the *Jordan canonical form* of A (hereafter JCF of A), where the $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \; i = 1, \ldots, k,$$

are called the *Jordan blocks* of $J(A)$. The *elementary divisors* of A are the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \ldots, k$. The *nonnegative inverse elementary divisors problem* (hereafter NIEDP) is the problem of determining necessary and sufficient conditions for the existence of an $n \times n$ entrywise nonnegative matrix with prescribed elementary divisors [3, 5, 10, 11, 13, 14, 15, 16]. If there exists a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for each possible Jordan canonical form allowed by Λ, we say that Λ is *universally realizable* ($\mathcal{UR}$). If Λ is the spectrum of a nonnegative diagonalizable matrix, then Λ is said to be *diagonalizably realizable* ($\mathcal{DR}$).

The NIEDP is closely related to the *nonnegative inverse eigenvalue problem* (hereafter NIEP), which is the problem of characterizing all possible spectra of entrywise nonnegative matrices. If there is a nonnegative matrix A with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we say that Λ is *realizable* and that A is a *realizing matrix*. Both problems, the NIEDP and the NIEP, remain unsolved. A complete solution for the NIEP is known only for $n \leq 4$.

Throughout this paper, the first written element of a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *i.e.* λ_1, is the Perron eigenvalue of Λ, $\lambda_1 = \max\{|\lambda_i|\,,\ \lambda_i \in \Lambda\}$. If Λ is the spectrum of a nonnegative matrix A, we write $\rho(A) = \lambda_1$ for the spectral radius of A.

In this paper, we ask whether certain properties of the NIEP, such as the three rules that characterize the C-realizability of lists (see [2]), extend or not to the NIEDP. In particular, we ask:

1) If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$, is $\{\lambda_1+\epsilon, \lambda_2, \ldots, \lambda_n\}$ also $\mathcal{UR}$ for any $\epsilon > 0$?
2) If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$ and λ_2 is real, is $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$

also $\mathcal{UR}$ for any $\epsilon > 0$?
3) If the lists Λ_1 and Λ_2 are $\mathcal{UR}$, is $\Lambda_1 \cup \Lambda_2$ also $\mathcal{UR}$?

In [4], Cronin and Laffey examine the subtle difference between the *symmetric nonnegative inverse eigenvalue problem* (SNIEP), in which the realizing matrix is required to be symmetric, and the *real diagonalizable nonnegative inverse eigenvalue problem* (DRNIEP), in which the realizing matrix is diagonalizable. The authors in [4] give examples of lists of real numbers, which can be the spectrum of a nonnegative matrix, but not the spectrum of a diagonalizable nonnegative matrix.

The set of all $n \times n$ real matrices with constant row sums equal to $\alpha \in \mathbb{R}$ will be denoted by CS_α. It is clear that $\mathbf{e} = [1, 1, \ldots, 1]^T$ is an eigenvector of any matrix $A \in CS_\alpha$, corresponding to the eigenvalue α. Denote by $\mathbf{e}_k$ the vector with 1 in the k^{th} position and zeros elsewhere. The importance of matrices with constant row sums is due to the well known fact that an $n \times n$ nonnegative matrix A with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, λ_1 being the Perron eigenvalue, is co-spectral to a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ [7, 6]. In this paper, we extend the co-spectrality between A and B to similarity between A and B, when λ_1 is simple, and therefore $J(A) = J(B)$. In what follows, we use the following notations and results: we write $A \geq 0$ if A is a nonnegative matrix, and $A > 0$ if A is a positive matrix, that is, if all its entries are positive. We shall use the same notation for vectors.

Theorem 1.1 [1, (2.7) Theorem p. 141] *Let $A \in \{M = (m_{ij}) \in M_n : m_{ij} \leq 0, i \neq j\}$ be an irreducible matrix. Then each one of the following conditions is equivalent to the statement: "A is a nonsingular M-matrix".*
i) A^{-1} is positive.
ii) $A\mathbf{x} \geq 0$ and $A\mathbf{x} \neq 0$ for some $\mathbf{x}$ positive.

Theorem 1.2 [13] *Let $\mathbf{q} = [q_1, \ldots, q_n]^T$ be an arbitrary n-dimensional vector and $E_{11} \in M_n$ with 1 in the $(1, 1)$ position and zeros elsewhere. Let $A \in \mathcal{CS}_{\lambda_1}$ with JCF*

$$J(A) = S^{-1}AS = diag\left(J_1(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)\right).$$

If $\lambda_1 + \sum_{i=1}^{n} q_i \neq \lambda_i$, $i = 2, \ldots, n$, then the matrix $A + \mathbf{e}\mathbf{q}^T$ has Jordan canonical form $J(A) + (\sum_{i=1}^{n} q_i)E_{11}$. In particular, if $\sum_{i=1}^{n} q_i = 0$, then A and $A + \mathbf{e}\mathbf{q}^T$ are similar.

This paper is organized as follows: In Section 2, we extend the co-spectrality between a nonnegative matrix A and a nonnegative matrix B

with constant row sums to a similarity between A and B, when the Perron eigenvalue is simple. In Section 3, we show that if a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $\mathcal{UR}$, then $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $\mathcal{UR}$ for any $\epsilon > 0$. We also consider the universal realizability of the Guo perturbation $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$, and of the union of two universally realizable lists Λ_1 and Λ_2. In Section 4, we study the nonsymmetric realizablity of lists of size 5 with trace zero and three negative elements.

2 Nonnegative matrices similar to nonnegative matrices with constant row sums

It is well known that if A is an irreducible nonnegative matrix, then A has a positive eigenvector associated to its Perron eigenvalue. In this section, we extend this result to reducible matrices under certain conditions. As a consequence, in both cases, A is similar to a nonnegative matrix B with constant row sums when the Perron eigenvalue is simple. In this way, we extend a result attributed to Johnson [7], about the co-spectrality between a nonnegative matrix A and a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$.

Lemma 2.1 *Let $A \in M_n$ be a nonnegative matrix of the form*

$$A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},$$

with $A_1 \in \mathcal{CS}_{\lambda_1}$, $A_3 \neq 0$, A_2 irreducible and $\lambda_1 = \rho(A) = \rho(A_1) > \rho(A_2)$. Then A has a positive eigenvector associated to λ_1. Moreover, there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to A.

Proof. Let $A_1 \in M_k$ and $A_2 \in M_{n-k}$. Let $\mathbf{x} = \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \end{bmatrix}$ with $\mathbf{e} \in M_{k,1}$, $\mathbf{y} \in M_{n-k,1}$. Then, for

$$\begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A_1\mathbf{e} \\ A_3\mathbf{e} + A_2\mathbf{y} \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{e} \\ \lambda_1\mathbf{y} \end{bmatrix},$$

we have $A_3\mathbf{e} = (\lambda_1 I - A_2)\mathbf{y}$, where $\lambda_1 I - A_2$ is an irreducible nonsingular M-matrix. Then, from Theorem 1.1, $(\lambda_1 I - A_2)^{-1} > 0$. Therefore,

$$\mathbf{y} = (\lambda_1 I - A_2)^{-1}(A_3\mathbf{e}) > 0, \tag{1}$$

and so $\mathbf{x}^T = \left[\mathbf{e}^T, \mathbf{y}^T\right] = [x_1, \ldots, x_n]$ is positive. Then, for $D = diag\,(x_1, \ldots, x_n)\,,$ $B = D^{-1}AD$ is similar to A. Since

$$B\mathbf{e} = D^{-1}AD\mathbf{e} = \lambda_1 \mathbf{e},$$

then $B \in \mathcal{CS}_{\lambda_1}$. ■

Remark 2.1 *Note that the eigenvector* $\mathbf{x}$ *obtained in the proof of Lemma 2.1 is* $\mathbf{x}^T = [\mathbf{e}^T, \mathbf{y}^T]$, *where* $\mathbf{e}$ *has the number of rows* A_1 *and*

$$\mathbf{y} = (\lambda_1 I - A_2)^{-1}(A_3\mathbf{e}) = [y_1, \ldots, y_{n-k}]^T > 0.$$

Let $Y = diag(y_1, \ldots, y_{n-k})$, *then a matrix* $B \in \mathcal{CS}_{\lambda_1}$ *similar to* A *is of the form*

$$B = \begin{bmatrix} A_1 & 0 \\ Y^{-1}A_3 & Y^{-1}A_2Y \end{bmatrix}.$$

Note that in Lemma 2.1 it is not necessary that the spectral radius of A be simple, as shown in matrix

$$A = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 2 & 0 \\ \hline 2 & 0 & 1 \end{array}\right],$$

which has a positive eigenvector $[1, 1, 2]$ associated to the double eigenvalue $\lambda_1 = 2$.

Now, suppose that A is a block diagonal matrix. Then, for this case, we have the following result:

Lemma 2.2 *Let* $A \in M_n$ *be a nonnegative matrix of the form*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

with $A_1 \in \mathcal{CS}_{\lambda_1}$, A_2 *irreducible and* $\lambda_1 = \rho(A) = \rho(A_1) > \rho(A_2)$. *Then* A *is similar to a nonnegative matrix* $\widetilde{A} = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}$, *with* $A_3 \neq 0$. *Moreover, there exists a nonnegative matrix* $B \in \mathcal{CS}_{\lambda_1}$ *similar to* A.

Proof. Let $A_1 \in M_k$ and $A_2 \in M_{n-k}$. We suppose, without loss of generality, that $A_2 \in \mathcal{CS}_{\rho(A_2)}$. Define the nonsingular matrix

$$S = \begin{bmatrix} I_k & 0 \\ -Z & I_{n-k} \end{bmatrix}, \quad \text{with} \quad S^{-1} = \begin{bmatrix} I_k & 0 \\ Z & I_{n-k} \end{bmatrix},$$

where $Z = \mathbf{e}\mathbf{z}^T \in M_{n-k,k}$, with $\mathbf{z}$ being an eigenvector of A_1^T associated to λ_1. Then

$$\widetilde{A} = S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ ZA_1 - A_2Z & A_2 \end{bmatrix}.$$

We show that $A_3 = ZA_1 - A_2Z$ is a nonzero nonnegative matrix. The entry in position (r, j) of the matrix A_3 is,

$$\begin{aligned} \mathbf{e}_r^T(ZA_1 - A_2Z)\mathbf{e}_j &= \mathbf{z}^T col_j(A_1) - z_j row_r(A_2)\mathbf{e} \\ &= \sum_{i=1}^{k} a_{ij}z_i - z_j\rho(A_2), \end{aligned}$$

for all $r = 1, \ldots, n-k$, $j = 1, \ldots, k$. Therefore, $ZA_1 - A_2Z$ has all its rows equal, which can be expressed as

$$(A_1^T - \rho(A_2)I_k)\mathbf{z}. \tag{2}$$

Since $A_1^T - \rho(A_2)I_k$ and A_1^T have the same eigenvectors, then from (2)

$$\begin{aligned} (A_1^T - \rho(A_2)I_k)\mathbf{z} &= A_1^T\mathbf{z} - \rho(A_2)\mathbf{z} \\ &= \lambda_1\mathbf{z} - \rho(A_2)\mathbf{z} \\ &= (\lambda_1 - \rho(A_2))\mathbf{z} \geq 0. \end{aligned}$$

Therefore $A_3 = ZA_1 - A_2Z$ is a nonzero nonnegative matrix. Since A and $\widetilde{A}$ are similar with A_3 nonzero nonnegative, then from Lemma 2.1 there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to A. ■

Remark 2.2 *Note that the matrix A_3 in the proof of Lemma 2.2 is*

$$A_3 = \mathbf{e}\mathbf{z}^T A_1 - A_2\mathbf{e}\mathbf{z}^T, \tag{3}$$

with $\mathbf{z}$ being an eigenvector of A_1^T associated to λ_1. Then, from Lemma 2.1, $\widetilde{A}$ has a positive eigenvector $\mathbf{x} = [\mathbf{e}^T, \mathbf{y}^T]$ associated to λ_1, where

$$\mathbf{y} = (\lambda_1 I - A_2)^{-1}(A_3\mathbf{e}) = [y_1, \ldots, y_{n-k}]^T, \text{ with } A_3 \text{ as in } (3).$$

Let $Y = diag\{y_1, \ldots, y_{n-k}\}$, then a matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to $\widetilde{A}$ is of the form

$$B = \begin{bmatrix} A_1 & 0 \\ Y^{-1}A_3 & Y^{-1}A_2Y \end{bmatrix}.$$

Next we prove the main result in this section. This result extends the co-spectrality between a nonnegative matrix A and a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$, to a similarity between A and B.

Theorem 2.1 *Let $A \in M_n$ be a nonnegative matrix with $\lambda_1 = \rho(A)$ simple. Then there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to A.*

Proof. If A is irreducible, then A has a positive eigenvector $\mathbf{x} = [x_1, \ldots, x_n]^T$ associated to λ_1. Let $D = diag\,(x_1, \ldots, x_n)$. Then $B = D^{-1}AD \in \mathcal{CS}_{\lambda_1}$ is nonnegative and similar to A.

If A is reducible, then A is permutationally similar to

$$\widetilde{A} = \begin{bmatrix} A_{11} & & & & & & \\ A_{21} & A_{22} & & & & & \\ \vdots & \ddots & \ddots & & & & \\ A_{k1} & \cdots & A_{k,k-1} & A_{kk} & & & \\ 0 & \cdots & 0 & 0 & A_{k+1,k+1} & & \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & A_{k+r,k+r} \end{bmatrix}$$

with blocks A_{ii} irreducible of order n_i, or zero of size 1×1, such that $\sum_{i=1}^{k+r} n_i = n$, and $\begin{bmatrix} A_{i1} & A_{i2} & \cdots A_{i,i-1} \end{bmatrix}$ nonzero, $i = 2, \ldots, k$. We may assume, without loss of generality, that λ_1 is an eigenvalue of $A_{11} \in \mathcal{CS}_{\lambda_1}$, and $A_{ii} \in \mathcal{CS}_{\rho(A_{ii})}$, $i = 2, 3, \ldots, k + r$.

From Lemma 2.1, the submatrix

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

in the left upper corner of $\widetilde{A}$, is similar to a nonnegative matrix $B_1 \in \mathcal{CS}_{\lambda_1}$, with $B_1 = {D_1}^{-1}A_1D_1$.

We define $\widetilde{D}_1 = \begin{bmatrix} D_1 & \\ & I_{n-(n_1+n_2)} \end{bmatrix}$. Then

$$\widetilde{D}_1^{-1}\widetilde{A}\widetilde{D}_1 = \begin{bmatrix} B_1 & & & & & & \\ * & A_{33} & & & & & \\ \vdots & \ddots & \ddots & & & & \\ * & \cdots & * & A_{kk} & & & \\ 0 & \cdots & 0 & 0 & A_{k+1,k+1} & & \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & A_{k+r,k+r} \end{bmatrix}.$$

Again, from Lemma 2.1, the left upper corner submatrix of $\widetilde{D}_1^{-1}\widetilde{A}\widetilde{D}_1$,

$$A_2 = \begin{bmatrix} B_1 & 0 \\ * & A_{33} \end{bmatrix},$$

is similar to a nonnegative matrix $B_2 \in \mathcal{CS}_{\lambda_1}$, with $B_2 = D_2{}^{-1}A_2D_2$. Then we define $\widetilde{D}_2 = \begin{bmatrix} D_2 & \\ & I_{n-(n_1+n_2+n_3)} \end{bmatrix}$ and we obtain

$$\widetilde{D}_2^{-1}\widetilde{D}_1^{-1}\widetilde{A}\widetilde{D}_1\widetilde{D}_2 = \begin{bmatrix} B_2 & & & & & & \\ * & A_{44} & & & & & \\ \vdots & \ddots & \ddots & & & & \\ * & \cdots & * & A_{kk} & & & \\ 0 & \cdots & 0 & 0 & A_{k+1,k+1} & & \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & A_{k+r,k+r} \end{bmatrix}.$$

Proceeding in a similar way, after $k-1$ steps, we obtain

$$\widetilde{D}_{k-1}^{-1}\cdots\widetilde{D}_1^{-1}\widetilde{A}\widetilde{D}_1\cdots\widetilde{D}_{k-1} = \begin{bmatrix} B_{k-1} & & & \\ & A_{k+1,k+1} & & \\ & & \ddots & \\ & & & A_{k+r,k+r} \end{bmatrix},$$

which is a block diagonal matrix, with $B_{k-1} \in \mathcal{CS}_{\lambda_1}$. Now, from Lemma 2.2, the submatrix

$$A'_k = \begin{bmatrix} B_{k-1} & \\ & A_{k+1,k+1} \end{bmatrix},$$

is similar to a nonnegative matrix $B'_k \in \mathcal{CS}_{\lambda_1}$, $B'_k = {D_k}^{-1}{S_k}^{-1}A'_k S_k D_k$, where $S_k = \begin{bmatrix} I_{n_1+\cdots+n_k} & \\ -\mathbf{e}\mathbf{z}_k^T & I_{k+1,k+1} \end{bmatrix}$, with $\mathbf{z}_k$ being an eigenvector of B_{k-1}^T associated to λ_1.
We define $\widetilde{D}_k = \begin{bmatrix} S_k D_k & \\ & I_{n-(n_1+\cdots+n_{k+1})} \end{bmatrix}$. Then,

$$\widetilde{D}_k^{-1} \cdots \widetilde{D}_1^{-1} \widetilde{A} \widetilde{D}_1 \cdots \widetilde{D}_k = \begin{bmatrix} B'_k & & & \\ & A_{k+2,k+2} & & \\ & & \ddots & \\ & & & A_{k+r,k+r} \end{bmatrix}.$$

Proceeding in a similar way, after $r-1$ steps, we obtain a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to A. ■

Remark 2.3 *Note that the condition of simple Perron eigenvalue cannot be deleted from Theorem 2.1, as shown in matrix*

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Observe also that this means that it is not always possible to work with matrices with constant row sums in the NIEDP, this fact does not apply to the NIEP.

3 Perturbation of universally realizable lists

Guo in 1997 [6] proved that increasing the Perron eigenvalue of a realizable list preserves the realizability. We extend this result to $\mathcal{UR}$ lists.

Theorem 3.1 *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with λ_1 simple. If Λ is $\mathcal{UR}$, then $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $\mathcal{UR}$ for any $\epsilon > 0$.*

Proof. Let $\epsilon > 0$ and

$$J_\epsilon = J_1(\lambda_1 + \epsilon) \bigoplus_{i=2}^{k} J_{n_i}(\lambda_i)$$

be a JCF allowed by Λ_ϵ. The matrix

$$J = J_1(\lambda_1) \bigoplus_{i=2}^{k} J_{n_i}(\lambda_i)$$

is an allowed JCF by Λ. Because Λ is $\mathcal{UR}$, there exists a nonnegative matrix A with spectrum Λ and Jordan canonical form J. Besides, from Theorem 2.1, there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with $J(B) = J$. Then, from Theorem 1.2, for B and $\mathbf{q}^T = [\frac{\epsilon}{n}, \ldots, \frac{\epsilon}{n}]$, we have that the matrix $A_\epsilon = B + \mathbf{e}\mathbf{q}^T$ is nonnegative with spectrum Λ_ϵ and JCF

$$J(A_\epsilon) = J(B) + \epsilon E_{11} = J + \epsilon E_{11} = J_1(\lambda_1 + \epsilon) \bigoplus_{i=2}^{k} J_{n_i}(\lambda_i).$$

Thus, Λ_ϵ is $\mathcal{UR}$. ■

Guo in 1997 [6] also proved that increasing by ϵ a Perron eigenvalue and decreasing by ϵ another real eigenvalue of a realizable list preserves the realizability. Soto and Ccapa in 2008 [13] proved that a list of real numbers of Suleĭmanova type, that is, a list $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_i \leq 0$ for $i = 2, \ldots, n$, and $\sum_{i=1}^{n} \lambda_1 \geq 0$, is $\mathcal{UR}$. As a consequence, the perturbed list $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$ with $\epsilon > 0$ is $\mathcal{UR}$ for nonnegative lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and also for Suleĭmanova type lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. As we show below, this is not true for general lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The construction of a counter-example is based on the study of $\mathcal{UR}$ lists of size 5 with trace zero and three negative elements. This construction has been motivated by the work of Cronin and Laffey [4]. They show that a realizable list is not necessarily diagonalizably realizable. In particular, they observe that the lists $\{3+t, 3-t, -2+\epsilon, -2, -2-\epsilon\}$ are realizable for small positive values of ϵ and values of t close to 0.44, but they are symmetrically realizable only for $t \geq 1-\epsilon$ [17, Theorem 3]. Note that these lists are diagonalizably realizable, since the eigenvalues are distinct. However, this is not a continuous property in ϵ as Cronin and Laffey show via the following result.

Proposition 3.2 [4] *Suppose* $\{3+t, 3-t, -2, -2, -2\}$ *is diagonalizably realizable, then* $t \geq 1$.

Note that the list $\{3+t, 3-t, -2, -2, -2\}$ represents any list of size 5 with trace zero, simple Perron eigenvalue and three negative elements all equal, *i.e.*, lists of the form $\{\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_3\}$ with $\lambda_1 > \lambda_2 \geq 0 > \lambda_3$ and $\lambda_1 + \lambda_2 + 3\lambda_3 = 0$. This list can be scaled by $-2/\lambda_3$ to

$$\left\{\frac{-2\lambda_1}{\lambda_3}, \frac{-2\lambda_2}{\lambda_3}, -2, -2, -2\right\}$$

and taking $t = \frac{-2\lambda_1}{\lambda_3} - 3 = 3 + \frac{2\lambda_2}{\lambda_3}$ we have

$$\Lambda_{\pm t} = \{3+t, 3-t, -2, -2, -2\}, \quad 0 < t \leq 3.$$

Analogously:

- The list $\Lambda_t^{t_0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$, with $0 < t_0 < \min\{1+t, 2t\} < 2$ and $0 < t \leq 3$, represents the lists $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4\}$ with $\lambda_1 > \lambda_2 \geq 0 > \lambda_3 > \lambda_4$ and $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 = 0$ (scaling by $-2/\lambda_4$ and taking $t_0 = 2 - \frac{2\lambda_3}{\lambda_4}$ and $t = \frac{-2\lambda_1}{\lambda_4} - 3 + t_0 = 3 + \frac{2\lambda_2}{\lambda_4}$).

- The list $\Lambda_t^{'t_0} = \{3+t+t_0, 3-t, -2, -2, -2-t_0\}$, with $t_0 > \max\{0, -2t\}$ and $-1 < t \leq 3$, represents the lists $\{\lambda_1, \lambda_2, \lambda_3, \lambda_3, \lambda_4\}$ with $\lambda_1 > \lambda_2 \geq 0 > \lambda_3 > \lambda_4 > -\lambda_1$ and $\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 = 0$ (scaling by $-2/\lambda_3$ and taking $t_0 = -2 + \frac{2\lambda_4}{\lambda_3}$ and $t = \frac{-2\lambda_1}{\lambda_3} - 3 - t_0 = 3 + \frac{2\lambda_2}{\lambda_3}$).

We need the following result due to Šmigoc:

Lemma 3.1 [12, Lemma 5] *Suppose B is an $m \times m$ matrix with Jordan canonical form $J(B)$ that contains at least one 1×1 Jordan block corresponding to the eigenvalue c:*

$$J(B) = \begin{bmatrix} c & 0 \\ 0 & I(B) \end{bmatrix}.$$

Let $\mathbf{u}$ and $\mathbf{v}$, respectively, be left and right eigenvectors of B associated with the 1×1 Jordan block in the above canonical form. Furthermore, we normalize vectors $\mathbf{u}$ and $\mathbf{v}$ so that $\mathbf{u}^T\mathbf{v} = 1$. Let $J(A)$ be a Jordan canonical form for an $n \times n$ matrix

$$A = \begin{bmatrix} A_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix},$$

where A_1 is an $(n-1) \times (n-1)$ matrix and $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{C}^{n-1}$. Then the matrix

$$C = \begin{bmatrix} A_1 & \mathbf{a}\mathbf{u}^T \\ \mathbf{v}\mathbf{b}^T & B \end{bmatrix}$$

has Jordan canonical form

$$J(C) = \begin{bmatrix} J(A) & 0 \\ 0 & I(B) \end{bmatrix}.$$

We consider the lists $\Lambda_t^{t_0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$ and $\Lambda_t^{\prime t_0} = \{3 + t + t_0, 3 - t, -2, -2, -2 - t_0\}$ that have a better behavior than $\Lambda_{\pm t}$ with respect to the Guo result applied to $\mathcal{UR}$.

Theorem 3.3 *i) Let* $\Lambda_t^{t_0} = \{3+t-t_0, 3-t, -2+t_0, -2, -2\}$ *with* $0 < t_0 < 2$ *and* $\frac{t_0}{2} < t \leq 3$. *If* $\Lambda_t^{t_0}$ *is realizable, then it is* $\mathcal{UR}$.
ii) Let $\Lambda_t^{\prime t_0} = \{3 + t + t_0, 3 - t, -2, -2, -2 - t_0\}$ *with* $t_0 > \max\{0, -2t\}$ *and* $t \leq 3$. *If* $\Lambda_t^{\prime t_0}$ *is realizable, then it is* $\mathcal{UR}$.

Proof. *i)* Observe that the list $\Lambda_t^{t_0}$ has two possible JCF, since the only repeated eigenvalue is -2 with double multiplicity.

Under the realizability conditions in [9, 18], the realizing matrices for lists $\Lambda_t^{t_0}$ have the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & * & 0 & 1 \\ * & * & * & * & 0 \end{bmatrix},$$

then $\operatorname{rank}(A + 2I) = 4$ and A has a JCF with a Jordan block of size two $J_2(-2)$.

If $\Lambda_t^{t_0}$ is symmetrically realizable (see Spector conditions in [17, Theorem 3]), then $\Lambda_t^{t_0}$ is $\mathcal{DR}$.

If $\Lambda_t^{t_0}$ is realizable but not symmetrically realizable, which means that $t < 1$ (see next section), we show that $\Lambda_t^{t_0}$ is $\mathcal{DR}$ via the Šmigoc method given in Lemma 3.1. Let

$$\Gamma_1 = \{3 + t - t_0, 3 - t, -2 + t_0, -2\} \text{ and } \Gamma_2 = \{\operatorname{tr}(\Gamma_1), -2\} = \{2, -2\}.$$

Note that these spectra are realizable because they satisfy the Perron and trace conditions. The matrix

$$B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \approx J(B) = \begin{bmatrix} c = 2 & 0 \\ 0 & -2 \end{bmatrix}$$

realizes Γ_2. Let $\mathbf{u}^T = [1/2, 1/2]$ and $\mathbf{v}^T = [1, 1]$ be, respectively, left and right normalized eigenvectors of B.

We need to find a realization of Γ_1 with diagonal $(0, 0, 0, c = \operatorname{tr}(\Gamma_1) = 2)$ and the only realization that we know with this diagonal is the one given in

[16, Theorem 14] which is of the form

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ d_1 & 0 & 1 & 0 \\ b & 0 & 0 & 1 \\ \hline a & 0 & d_3 & 2 \end{array}\right].$$

The characteristic polynomial of A is

$$\begin{array}{rcl} P_A(x) & = & x^4 - 2x^3 - (d_1 + d_3)x^2 + (2d_1 - b)x + 2b + d_1 d_3 - a \\ & = & (x - (3 + t - t_0))(x - (3 - t))(x - (-2 + t_0))(x + 2) \\ & = & x^4 + k_1 x^3 + k_2 x^2 + k_3 x + k_4 \end{array}$$

with

$$\begin{array}{rcl} k_2 & = & -(t^2 - t_0 t + t_0^2 - 5t_0 + 11), \\ k_3 & = & (t_0 - 4)t^2 + t_0(4 - t_0)t + t_0^2 - 5t_0 + 12, \\ k_4 & = & 2(t_0 - 2)(t - t_0 + 3)(t - 3). \end{array}$$

Identifying coefficients we have the system:

$$d_1 + d_3 = -k_2, \quad 2d_1 - b = k_3, \quad 2b + d_1 d_3 - a = k_4 \tag{4}$$

which allows us to obtain realizations of Γ_1, in function of d_1, of the form

$$A(d_1) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ d_1 & 0 & 1 & 0 \\ 2d_1 - k_3 & 0 & 0 & 1 \\ \hline -d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2 \end{array}\right]$$

that has JCF

$$J(A(d_1)) = \left[\begin{array}{cccc} 3 + t - t_0 & 0 & 0 & 0 \\ 0 & 3 - t & 0 & 0 \\ 0 & 0 & -2 + t_0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

Now, by Lemma 3.1, the bonding of matrices $A(d_1)$ and B leads to the matrix

$$C(d_1) = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ d_1 & 0 & 1 & 0 & 0 \\ 2d_1 - k_3 & 0 & 0 & 1/2 & 1/2 \\ \hline -d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 0 & 2 \\ -d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2 & 0 \end{array}\right]$$

which realizes diagonally the list $\Lambda_t^{t_0}$.

Finally, $\Lambda_t^{t_0}$ is $\mathcal{UR}$.

ii) Analogously, under the realizability conditions in [9, 18], the realizing matrices for lists $\Lambda_t^{\prime t_0}$ have a JCF with a Jordan block of size two $J_2(-2)$.

If $\Lambda_t^{\prime t_0}$ is symmetrically realizable, then $\Lambda_t^{\prime t_0}$ is $\mathcal{DR}$.

If $\Lambda_t^{\prime t_0}$ is realizable but not symmetrically realizable (for $t < 1$), we apply the Šmigoc method to the spectra

$$\Gamma_1^{'} = \{3 + t + t_0, 3 - t, -2, -2 - t_0\} \text{ and } \Gamma_2 = \{2, -2\}$$

and, in the same way, we obtain the following $\mathcal{DR}$ realization of $\Lambda_t^{\prime t_0}$

$$C(d_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ d_1 & 0 & 1 & 0 & 0 \\ 2d_1 - k_3 & 0 & 0 & 1/2 & 1/2 \\ -d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 0 & 2 \\ -d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2 & 0 \end{bmatrix}$$

for the system (4), with

$$\begin{aligned} k_2 &= -(t^2 - t_0 t + t_0^2 + 5t_0 + 11), \\ k_3 &= (t_0 + 4)t^2 + t_0(4 + t_0)t - t_0^2 - 5t_0 - 12, \\ k_4 &= 2(t_0 + 2)(t + t_0 + 3)(3 - t). \end{aligned}$$

Hence, $\Lambda_t^{\prime t_0}$ is $\mathcal{UR}$. ■

Corollary 3.1 *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a $\mathcal{UR}$ list with λ_2 real. The list $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$, for $\epsilon > 0$, is not necessarily $\mathcal{UR}$.*

Proof. Let

$$\Lambda = \Lambda_t^{t_0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$$

be a $\mathcal{UR}$ list as in Theorem 3.3 with $t < 1$ (see Lemma 4.1 for its existence). Now, applying Wuwen perturbation with $\epsilon = t_0$, we obtain the list

$$\{3 + t, 3 - t, -2, -2, -2\}$$

which is not diagonalizably realizable by Proposition 3.2 and therefore it is not $\mathcal{UR}$. ■

It is easy to see that if Λ and Γ are lists of nonnegative real numbers, then $\Lambda \cup \Gamma$ is $\mathcal{UR}$. Let

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \ \text{ and } \ \Gamma = \{\mu_1, \mu_2, \ldots, \mu_m\}$$

be lists of real numbers of Suleĭmanova type with trace zero and $\lambda_1 > \mu_1$, the Perron eigenvalues of Λ and Γ respectively. Then, from [3], $\Lambda \cup \Gamma$ is $\mathcal{UR}$. Now we show that this is not true for general lists.

Lemma 3.2 *Let $\Lambda = \{\lambda_1, \lambda_1, \lambda_2, \lambda_2\}$ be a list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq -\lambda_1$ and $\lambda_1 + 2\lambda_2 < 0$. Then Λ has no nonnegative realization with Jordan canonical form*

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}.$$

Proof. Suppose there exists a nonnegative realization A of Λ with Jordan canonical form $J(A) = J$. As $\lambda_1 + 2\lambda_2 < 0$, then Λ only admits reducible realizations and must be partitioned as $\{\lambda_1, \lambda_2\} \cup \{\lambda_1, \lambda_2\}$. So we assume, without loss of generality, that A is of the form

$$A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are irreducible matrices with spectrum $\{\lambda_1, \lambda_2\}$. Therefore, from the minimal polynomial of B and D, we have

$$B^2 = (\lambda_1 + \lambda_2)B - \lambda_1\lambda_2 I \ \text{ and } \ D^2 = (\lambda_1 + \lambda_2)D - \lambda_1\lambda_2 I.$$

Since the minimal polynomial of A is

$$x^3 + (-\lambda_1 - 2\lambda_2)x^2 + (2\lambda_1\lambda_2 + \lambda_2^2)x - \lambda_1\lambda_2^2,$$

then

$$A^3 + (-\lambda_1 - 2\lambda_2)A^2 + (2\lambda_1\lambda_2 + \lambda_2^2)A - \lambda_1\lambda_2^2 I = 0,$$

with

$$A^2 = \begin{bmatrix} B^2 & 0 \\ CB + DC & D^2 \end{bmatrix} = \begin{bmatrix} (\lambda_1 + \lambda_2)B - \lambda_1\lambda_2 I & 0 \\ CB + DC & (\lambda_1 + \lambda_2)D - \lambda_1\lambda_2 I \end{bmatrix},$$

and

$$A^3 = AA^2 = \begin{bmatrix} (\lambda_1+\lambda_2)B^2 - \lambda_1\lambda_2 B & 0 \\ (\lambda_1+\lambda_2)CB - \lambda_1\lambda_2 C + DCB + D^2C & (\lambda_1+\lambda_2)D^2 - \lambda_1\lambda_2 D \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)B - (\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)I & 0 \\ (\lambda_1+\lambda_2)(CB+DC) + DCB - 2\lambda_1\lambda_2 C & (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)D - (\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)I \end{bmatrix}.$$

Therefore,
$A^3 - (\lambda_1+2\lambda_2)A^2 + (2\lambda_1\lambda_2+\lambda_2^2)A - \lambda_1\lambda_2^2 I$

$$= \begin{bmatrix} (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)B - (\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)I & 0 \\ (\lambda_1+\lambda_2)(CB+DC) + DCB - 2\lambda_1\lambda_2 C & (\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)D - (\lambda_1^2\lambda_2+\lambda_1\lambda_2^2)I \end{bmatrix}$$

$$-(\lambda_1+2\lambda_2)\begin{bmatrix} (\lambda_1+\lambda_2)B - \lambda_1\lambda_2 I & 0 \\ CB+DC & (\lambda_1+\lambda_2)D - \lambda_1\lambda_2 I \end{bmatrix} + (2\lambda_1\lambda_2+\lambda_2^2)\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

$-\lambda_1\lambda_2^2 I = 0.$
Now, by equalizing the block in position $(2,1)$ to zero, we have:

$$(\lambda_1+\lambda_2)(CB+DC) + DCB - 2\lambda_1\lambda_2 C - (\lambda_1+2\lambda_2)(CB+DC) + (2\lambda_1\lambda_2+\lambda_2^2)C$$

$$= -\lambda_2(CB+DC) + DCB + \lambda_2^2 C = 0.$$

Since the matrices involved in the last equality are nonnegative and $\lambda_2 < 0$, this is only possible if each addend is zero. In particular, $C = 0$. Then

$$\begin{aligned} \dim(\ker(A-\lambda_2 I)) &= 4 - \operatorname{rank}(A-\lambda_2 I) \\ &= 4 - \operatorname{rank}\begin{bmatrix} B-\lambda_2 I & 0 \\ 0 & D-\lambda_2 I \end{bmatrix} \\ &= 4 - (\operatorname{rank}(B-\lambda_2 I) + \operatorname{rank}(D-\lambda_2 I)) \\ &= 4 - (1+1) = 2. \end{aligned}$$

However, from $J(A) = J$ we have

$$\dim(\ker(A-\lambda_2 I)) = 4 - \operatorname{rank}\begin{bmatrix} \lambda_1-\lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1-\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1,$$

which contradicts the existence of a nonnegative realization A with Jordan canonical form J. ■

As an example, consider $\Lambda = \{1, -1\}$. It is clear that Λ is $\mathcal{UR}$. However, from Lemma 3.2, the list $\Lambda \cup \Lambda = \{1, 1, -1, -1\}$ has no nonnegative realization with JCF

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Therefore, $\Lambda \cup \Lambda$ is not $\mathcal{UR}$.

4 Lists of size 5 with trace zero and three negative elements

We are interested in the realizability of the lists with size 5 and trace zero

$$\Lambda_{\pm t} = \{3+t, 3-t, -2, -2, -2\},$$

$$\Lambda_t^{t_0} = \{3+t-t_0, 3-t, -2+t_0, -2, -2\},$$
$$\Lambda_t^{\prime t_0} = \{3+t+t_0, 3-t, -2, -2, -2-t_0\}$$

introduced in Section 3. It is well known that the list $\Lambda_{\pm t}$ is realizable if and only if $t \geq \sqrt{16\sqrt{6}-39} = 0.43799\cdots$ (see [8]), and symmetrically realizable if and only if $t \geq 1$ (see [17]). Now, we study when the lists $\Lambda_t^{t_0}$ and $\Lambda_t^{\prime t_0}$ are realizable but not symmetrically realizable. We need the following result:

Theorem 4.1 [18, Theorem 39 for $n = 5$ and $p = 2$] *Let $P(x) = x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$. Then the following statements are equivalent:*

i) $P(x)$ is the characteristic polynomial of a nonnegative matrix;

ii) the coefficients of $P(x)$ satisfy:

a) $k_2, k_3 \leq 0$;

b) $k_4 \leq \frac{k_2^2}{4}$;

c) $k_5 \leq \begin{cases} k_2k_3 & \text{if } k_4 \leq 0, \\ k_3\left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right) & \text{if } k_4 > 0. \end{cases}$

Lemma 4.1 *1.* $\Lambda_t^{t_0} = \{3+t-t_0, 3-t, -2+t_0, -2, -2\}$ *with* $0 < t_0 < 2t < 2$ *is realizable, but not symmetrically realizable, in the region*

$$t \geq \frac{t_0 + \sqrt{16\sqrt{6-t_0}(4-t_0) - 3t_0^2 + 52t_0 - 156}}{2}. \tag{5}$$

2. $\Lambda_t'^{t_0} = \{3+t+t_0, 3-t, -2, -2, -2-t_0\}$ *with* $0 < t_0, t < 1$ *and* $t+t_0 < 1$ *is realizable, but not symmetrically realizable, in the region*

$$t \geq \frac{-t_0 + \sqrt{16\sqrt{6+t_0}(4+t_0) - 3t_0^2 - 52t_0 - 156}}{2}. \tag{6}$$

Proof. *1.* Note that (t_0, t) varies in the interior of the triangle T with vertices $(0,0), (0,1)$ and $(2,1)$. The hypothesis $t < 1$ guarantees that $\Lambda_t^{t_0}$ is not symmetrically realizable (see [17, Theorem 3]). Let us see that $\Lambda_t^{t_0}$ is realizable using Theorem 4.1.

The characteristic polynomial $x^5 + k_2x^3 + k_3x^2 + k_4x + k_5$ of $\Lambda_t^{t_0}$ is

$$(x-(3+t-t_0))(x-(3-t))(x-(-2+t_0))(x+2)^2$$

where

$$\begin{aligned}
k_2 &= -t^2 + t_0t - t_0^2 + 5t_0 - 15\\
k_3 &= -(6-t_0)t^2 + t_0(6-t_0)t - t_0^2 + 5t_0 - 10\\
k_4 &= 4((t_0-3)t^2 + t_0(3-t_0)t + 2t_0^2 - 10t_0 + 15)\\
k_5 &= 4(t-3)(t-t_0+3)(t_0-2).
\end{aligned}$$

Clearly k_2 is negative in the triangle T because $k_2 < t_0t + 5t_0 - 15 < -3$. The derivative of k_3 with respect to t is $k_3' = -2(6-t_0)t + t_0(6-t_0)$, which is 0 in $t = t_0/2$ and then the maximum value of k_3 is $k_3(t_0/2) = (2-t_0)(t_0^2-20)/4$ which is negative for $0 < t_0 < 2$ and so k_3 is also negative in T.

The inequality $k_4 \leq \frac{k_2^2}{4}$ holds if and only if $k_2^2 - 4k_4$ is nonnegative. We have

$$k_2^2 - 4k_4 = (t^2 - t_0t + 4(4-t_0)\sqrt{6-t_0} + t_0^2 - 13t_0 + 39)(t^2 - t_0t - 4(4-t_0)\sqrt{6-t_0} + t_0^2 - 13t_0 + 39)$$

where the first factor is positive and the second is nonnegative in the triangle T if

$$t \geq \frac{t_0 + \sqrt{16\sqrt{6-t_0}(4-t_0) - 3t_0^2 + 52t_0 - 156}}{2}.$$

The coefficient k_4 is positive in T because

$$k_4 > 4(t_0^3/4 - 3 + (3 - t_0)t_0^2/2 + 2t_0^2 - 10t_0 + 15) = -t_0^3 + 14t_0^2 - 40t_0 + 48 > 0,$$

and $k_5 \le k_3 \left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4} \right)$ in T if the inequality (5) holds.

Therefore, by Theorem 4.1, we conclude that $\Lambda_t^{t_0}$ is realizable in the region (5).

2. Now (t_0, t) varies in the interior of the triangle R with vertices $(0, 0), (0, 1)$ and $(1, 0)$. Again, the hypothesis $t < 1$ implies no symmetric realization of $\Lambda_t^{'t_0}$ (see [17, Theorem 3]).

The characteristic polynomial $x^5 + k_1x^4 + k_2x^3 + k_3x^2 + k_4x + k_5$ of $\Lambda_t^{'t_0}$ is

$$(x - (3 + t + t_0))(x - (3 - t))(x + 2)^2(x - (-2 - t_0))$$

where

$$\begin{aligned}
k_2 &= -(t^2 + t_0t + t_0^2 + 5t_0 + 15) \\
k_3 &= -((t_0 + 6)t^2 + t_0(t_0 + 6)t + t_0^2 + 5t_0 + 10) \\
k_4 &= -4((t_0 + 3)t^2 + t_0(t_0 + 3)t - 2t_0^2 - 10t_0 - 15) \\
k_5 &= 4(3 - t)(t + t_0 + 3)(t_0 + 2).
\end{aligned}$$

Clearly k_2 and k_3 are negative in the triangle R. For $k_4 \le \frac{k_2^2}{4}$ we have

$$k_2^2 - 4k_4 = (t^2 + t_0t + 4(4 + t_0)\sqrt{6 + t_0} + t_0^2 + 13t_0 + 39)(t^2 + t_0t - 4(4 + t_0)\sqrt{6 + t_0} + t_0^2 + 13t_0 + 39)$$

where the first factor is positive and the second is nonnegative in the triangle R if

$$t \ge \frac{-t_0 + \sqrt{16\sqrt{6 + t_0}(4 + t_0) - 3t_0^2 - 52t_0 - 156}}{2}.$$

The coefficient k_4 is positive in T because

$$k_4 > -4((t_0+3)(1-t_0)^2 + t_0(t_0+3)(1-t_0) - 2t_0^2 - 10t_0 - 15) = 12(t_0^2 + 4t_0 + 4) > 0,$$

and $k_5 \le k_3 \left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4} \right)$ in R if the inequality (6) holds.

Therefore, by Theorem 4.1, we conclude that $\Lambda_t^{'t_0}$ is realizable in the region (6). ■

Figure 1 and Figure 2 show graphically the regions of realizability (the grey regions) of $\Lambda_t^{t_0}$ and $\Lambda_t^{'t_0}$ respectively, described in the previous lemma.

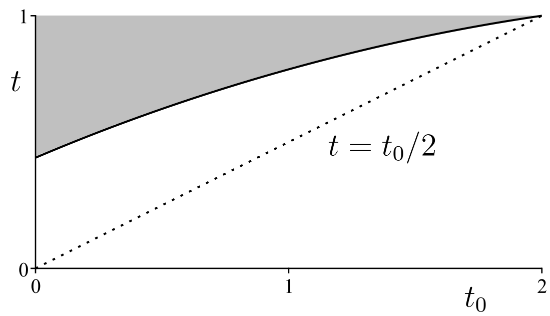

Figure 1: List $\Lambda_t^{t_0}$.

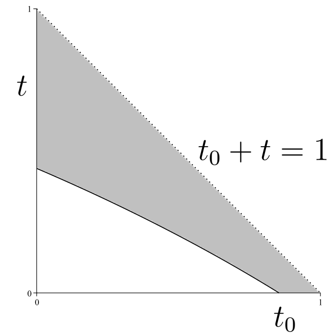

Figure 2: List $\Lambda_t^{\prime t_0}$.

In the following example we give a diagonalizable nonsymmetric realization of the lists $\Lambda_t^{t_0}$ and $\Lambda_t^{\prime t_0}$ for particular values of (t_0, t) in the corresponding regions.

Example 4.1 *Let us consider the list $\Lambda_t^{t_0}$ for $t_0 = 1$. By Lemma 4.1, the list $\Lambda_t^1 = \{2+t, 3-t, -1, -2, -2\}$ is realizable for $t \geq \frac{1}{2}(1+\sqrt{48\sqrt{5}-107}) = 0.7877\cdots$. Let us consider $t = 0.8$ and realize diagonalizably the list $\Lambda_{0.8}^1 = \{2.8, 2.2, -1, -2, -2\}$. The characteristic polynomial of the list $\Gamma_1 = \{2.8, 2.2, -1, -2\}$ is $x^4 - 2x^3 - \frac{171}{25}x^2 + \frac{212}{25}x + \frac{308}{25}$. Following the proof of Theorem 3.3 we obtain*

$$d_3 = \frac{171}{25} - d_1, \quad b = 2d_1 - \frac{212}{25}, \quad a = -d_1^2 + \frac{271}{25}d_1 - \frac{732}{25}.$$

The entries d_3 and b are nonnegative for $\frac{106}{25} \leq d_1 \leq \frac{171}{25}$. The entry a is nonnegative for $d_1 \in [\frac{271-\sqrt{241}}{50}, \frac{271+\sqrt{241}}{50}] = [5.10951\cdots, 5.73048\cdots]$. Then the rank of a is between 0 and its maximum value attained in $d_1 = \frac{271}{50}$, i.e., $a \in [0, 0.094]$. If we take $d_1 = 5.5$ we obtain the matrices

$$A(5.5) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.5 & 0 & 1 & 0 \\ 2.52 & 0 & 0 & 1 \\ 0.09 & 0 & 2.58 & 2 \end{bmatrix} \quad \text{and} \quad C(5.5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 5.5 & 0 & 1 & 0 & 0 \\ 2.52 & 0 & 0 & 0.5 & 0.5 \\ 0.09 & 0 & 2.58 & 0 & 2 \\ 0.09 & 0 & 2.58 & 2 & 0 \end{bmatrix}$$

that realize Γ_1 and $\Lambda_{0.8}^1$ respectively.

Finally, we consider the list $\Lambda_t^{\prime 0.5} = \{3.5+t, 3-t, -2, -2, -2.5\}$ that, by Lemma 4.1, is realizable for $t \geq \frac{-1+\sqrt{144\sqrt{26}-731}}{4} = 0.2013\cdots$. Let us consider

$t = 0.3$ *and realize diagonalizably the list* $\Lambda_{0.3}^{\prime 0.5} = \{3.8, 2.7, -2, -2, -2.5\}$. *The characteristic polynomial of the list* $\Gamma_1' = \{3.8, 2.7, -2, -2.5\}$ *is* $x^4 - 2x^3 - \frac{1399}{100}x^2 + \frac{1367}{100}x + \frac{513}{10}$. *From the proof of Theorem 3.3 we obtain*

$$d_3 = \frac{1399}{100} - d_1, \quad b = 2d_1 - \frac{1367}{100}, \quad a = -d_1^2 + \frac{1799}{100}d_1 - \frac{1966}{25}.$$

The entries d_3 *and* b *are nonnegative for* $\frac{1367}{200} \leq d_1 \leq \frac{1399}{100}$. *The entry* a *is nonnegative for* $d_1 \in [\frac{1799-9\sqrt{1121}}{200}, \frac{1799+9\sqrt{1121}}{200}] = [7.483\cdots, 10.501\cdots]$. *Then the rank of* a *is between* 0 *and its maximum value attained in* $d_1 = \frac{1799}{200}$, *i.e.,* $a \in [0, 2.270025]$. *If we take* $d_1 = 9$ *we obtain the matrices*

$$A(9) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 1 & 0 \\ 4.33 & 0 & 0 & 1 \\ 2.27 & 0 & 4.99 & 2 \end{bmatrix} \quad \text{and} \quad C(9) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 1 & 0 & 0 \\ 4.33 & 0 & 0 & 0.5 & 0.5 \\ 2.27 & 0 & 4.99 & 0 & 2 \\ 2.27 & 0 & 4.99 & 2 & 0 \end{bmatrix}$$

that realize Γ_1' *and* $\Lambda_{0.3}^{\prime 0.5}$ *respectively.*

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