SUOWA operators: A review of the state of the art

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Abstract

SUOWA operators are a particular case of Choquet integral that simultaneously generalize weighted means and OWA operators. Since they are constructed by using normalized capacities, they possess properties such as continuity, monotonicity, idempotency, compensativeness and homogeneity of degree 1. Besides these ones, some articles published in recent years have shown that SUOWA operators also exhibit other interesting properties. So, we think that the time has come to summarize existing knowledge on these operators. The aim of this paper is to collect the main results obtained so far on SUOWA operators. Moreover, we also introduce some new results and illustrate the usefulness of SUOWA operators by using an example given by Beliakov (2018).

1. INTRODUCTION

Aggregation operators have received much attention in recent years due to their wide range of applications in a variety of areas. Two of the best-known aggregation operators are the weighted means and the ordered weighted averaging (OWA) operators.[1] Although both families of functions are defined through weighting vectors, their behavior is completely different: in the case of weighted means, the values are weighted according to the reliability of the information sources, while in the case of OWA operators, the values are weighted in accordance with their relative size. The need of both weightings in several fields such as robotics, vision, fuzzy logic controllers, constraint satisfaction problems, scheduling, multicriteria aggregation problems and decision making has been reported by some authors (see, for instance, Refs. 2–7, and the references therein).

It is important to highlight the main features of weighted means and OWA operators. In the case of weighted means, the components of the weighting vector tell us the importance (or "weight") of the information sources. For their part, OWA operators are in fact convex combinations of order statistics. Hence, a main feature of these operators is that extreme values (also known as outliers) can be discarded by using appropriate weighting vectors. For instance, trimmed and Winsorized means are well-known examples of OWA operators that possess this property.

In view of the previous comments, an interesting topic is the construction of functions that allow us to combine weighted means and OWA operators in a single function. The usual approach followed in the literature is to consider functions parametrized by two weighting vectors, $\boldsymbol{p}$ for the weighted mean and $\boldsymbol{w}$ for the OWA operator, so that we can recover the weighted mean when $\boldsymbol{w} = (1/n, \dots, 1/n)$ and the OWA operator when $\boldsymbol{p} = (1/n, \dots, 1/n)$. Among the solutions proposed in the literature,[8,9] the weighted ordered weighted averaging (WOWA) operators[10] and the semiuninorm-based ordered weighted averaging (SUOWA) operators[6] are very interesting because they can be expressed through Choquet integrals with respect to known normalized capacities. In this way, both families of functions are monotonic, compensative, idempotent, continuous, and homogeneous of degree 1.

In addition to satisfying the above properties, it would be very interesting to keep the aforementioned characteristics of weighted means and OWA operators, so that we are able to discard

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extreme values while each information source has the desired weight. In this regard, it is worthy of note that in the framework of games and capacities, the "weight" of each information source is determined through an importance index (usually the Shapley value).

The study of SUOWA operators has been carried out in several papers, where it has been shown that these functions exhibit some interesting properties. In particular, some specific classes of SUOWA operators are located between two order statistics (thus discarding the extreme values) while each information source is weighted by the desired weight. Hence, it seems appropriate to present in a single paper the existing knowledge so far on these operators, which is the main aim of this work. Moreover, we also introduce some new results and illustrate the usefulness of SUOWA operators by using an example given by Beliakov.[9]

The remainder of the paper is organized as follows. In Section 2 we recall the notion of Choquet integral and some indices used in the study of these functions: orness degree, Shapley value, veto and favor indices, and k-conjunctiveness and k-disjunctiveness indices. Section 3 is devoted to present semiuninorms, uninorms and SUOWA operators. Section 4 summarizes the main results obtained so far for SUOWA operators. In Section 5 we show some new results and, in Section 6, we illustrate the usefulness of SUOWA operators in an example given by Beliakov.[9] Finally, some concluding remarks are provided in Section 7.

2. PRELIMINARIES

Throughout the paper we will use the following notation: $N = \{1, \dots, n\}$; given $A \subseteq N$, $|A|$ denotes the cardinality of A; vectors are denoted in bold, $\boldsymbol{\eta}$ denotes the tuple $(1/n, \dots, 1/n) \in \mathbb{R}^n$ and, for each $k \in N$, $\boldsymbol{e}_k$ denotes the vector with 1 in the kth coordinate and 0 elsewhere. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in N$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$. Moreover, given $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the floor of a; i.e., the largest integer smaller than or equal to a.

Given $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, some well-known properties of F are the following:

1. Symmetry: $F(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = F(x_1, \dots, x_n)$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all permutation σ of N.

2. Monotonicity: $\boldsymbol{x} \geq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \geq F(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

3. Idempotency: $F(x, \dots, x) = x$ for all $x \in \mathbb{R}$.

4. Compensativeness (or internality): $\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

5. Homogeneity of degree 1 (or ratio scale invariance): $F(r\boldsymbol{x}) = rF(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$.

2.1. Choquet integral

The notion of Choquet integral was introduced by Choquet in 1953.[11] Since then, it has been used by many authors in several fields due mainly to its simplicity, versatility and good properties. Choquet integral is based on the concept of capacity[11], which is also known in the literature as fuzzy measure[12]. The notion of capacity is similar to that of probability measure, where the additivity property is changed by monotonicity. And a game is a generalization of a capacity without the monotonicity assumption.

Definition 1.

1. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing) = 0$.

2. A capacity (or fuzzy measure) μ on N is a game on N satisfying $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. In particular, it follows that $\mu : 2^N \longrightarrow [0, \infty)$. A capacity μ is said to be normalized if $\mu(N) = 1$.

A straightforward way to get a capacity from a game is to consider the monotonic cover of the game.[13,14]

Definition 2. Let υ be a game on N. The monotonic cover of υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A) = \max_{B \subseteq A} \upsilon(B).$$

By construction $\hat{\upsilon}$ is a capacity on N and, when υ is a capacity, $\hat{\upsilon} = \upsilon$. Moreover, the monotonic cover of a game satisfies the following property.

Remark 1. Let υ be a game on N. If $\upsilon(A) \leq 1$ for all $A \subseteq N$ and $\upsilon(N) = 1$, then $\hat{\upsilon}$ is a normalized capacity.

The dual of a normalized capacity is defined as follows.

Definition 3. Let μ be a normalized capacity on N.

1. The dual capacity of μ, denoted as $\overline{\mu}$, is the normalized capacity defined by

$$\overline{\mu}(A) = 1 - \mu(A^{\mathsf{c}}) \qquad (A \subseteq N).$$

2. μ is self-dual if $\overline{\mu} = \mu$.

Although the Choquet integral is usually defined as a functional,[11,15,16] it can also be seen, in the discrete case, as an aggregation function over $\mathbb{R}^n$ (see, for instance, Ref. 17). Moreover, by similarity with the original definition of OWA operators, we represent it by using nonincreasing sequences of values.[6,18]

Definition 4. Let μ be a capacity on N. The Choquet integral with respect to μ is the function $\mathcal{C}_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big), \tag{1}$$

where $A_{[i]} = \{[1], \ldots, [i]\}$, and we adopt the convention that $x_{[n+1]} = 0$.

From the previous expression, it is straightforward to show explicitly the weights of the values $x_{[i]}$ by representing the Choquet integral as follows:

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \big(\mu(A_{[i]}) - \mu(A_{[i-1]})\big)x_{[i]},$$

where we use the convention $A_{[0]} = \varnothing$. It is worth noting that the Choquet integral fulfills some properties which are useful in certain information aggregation contexts.[17]

Remark 2. If μ is a normalized capacity on N, then $\mathcal{C}_\mu$ is continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

Two of the most popular particular cases of Choquet integral are the weighted means and the OWA operators.[1] Both are defined by using weight distributions that add up to 1.[a]

Definition 5. A vector $\boldsymbol{q} \in [0, 1]^n$ is a weighting vector if $\sum_{i=1}^{n} q_i = 1$.

Definition 6. Let $\boldsymbol{p}$ be a weighting vector. The weighted mean associated with $\boldsymbol{p}$ is the function $M_{\boldsymbol{p}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

[a] It is worth noting that the choice of the weight distribution has generated a large literature (in the case of OWA operators, see, for instance, Refs. 19–22).

Two relevant special cases of weighted means are the arithmetic mean (when $\boldsymbol{p} = \boldsymbol{\eta}$) and the kth projection ($\mathsf{P}_k(\boldsymbol{x}) = x_k$, obtained when $\boldsymbol{p} = \boldsymbol{e}_k$).

Definition 7. Let $\boldsymbol{w}$ be a weighting vector. The OWA operator associated with $\boldsymbol{w}$ is the function $O_{\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

As in the case of weighted means, the arithmetic mean is a particular case of OWA operators (when $\boldsymbol{w} = \boldsymbol{\eta}$). Likewise, the kth order statistic $\big(\mathrm{OS}_k(\boldsymbol{x}) = x_{(k)}\big)$ is also a special case of OWA operators when $\boldsymbol{w} = \boldsymbol{e}_{n-k+1}$.

Besides the previous ones, centered[23] and unimodal[24] weighting vectors give rise to two important families of OWA operators.

Definition 8. A weighting vector $\boldsymbol{w}$ is said to be centered if it satisfies the following conditions:

1. $w_i = w_{n+1-i}, \quad i = 1, \dots, \lfloor n/2 \rfloor$.
2. $w_i < w_j$ whenever $i < j \leq \lfloor (n+1)/2 \rfloor$.
3. $w_1 > 0$.

Definition 9. A weighting vector $\boldsymbol{w}$ is unimodal if there exists an index k such that

$$w_1 \leq \cdots \leq w_{k-1} \leq w_k \geq w_{k+1} \geq \cdots \geq w_n.$$

Notice that unimodal weighting vectors embrace, among others, nondecreasing ($w_1 \leq \cdots \leq w_n$), nonincreasing ($w_1 \geq \cdots \geq w_n$), and centered weighting vectors. We finish this subsection by noting that both weighted means and OWA operators are a special type of Choquet integral.[6,25–27]

Remark 3.

1. If $\boldsymbol{p}$ is a weighting vector, then the weighted mean $M_{\boldsymbol{p}}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i \in A} p_i$.
2. If $\boldsymbol{w}$ is a weighting vector, then the OWA operator $O_{\boldsymbol{w}}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$.

Notice that $\mu_{\boldsymbol{p}}$ is self-dual; that is, $\overline{\mu}_{\boldsymbol{p}} = \mu_{\boldsymbol{p}}$, and that $\overline{\mu}_{|\boldsymbol{w}|} = \mu_{|\overline{\boldsymbol{w}}|}$, where $\overline{\boldsymbol{w}}$ is the dual of $\boldsymbol{w}$; that is, the weighting vector defined by $\overline{w}_i = w_{n+1-i}$, $i = 1, \dots, n$. Besides, according to Remark 2, weighted means and OWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. And, since the values of the variables are previously ordered in a nonincreasing way, OWA operators are also symmetric.

2.2. Indices for Choquet integrals

Several indices have been proposed in the literature to provide information on the behavior of operators used in the aggregation processes: orness and andness degrees, importance and interaction indices, tolerance indices, dispersion indices, etc. In this subsection we recall (in the particular case of the Choquet integral) the definitions of the following indices: orness degree, Shapley value, veto and favor indices, and k-conjunctiveness and k-disjunctiveness indices.

The notion of orness allows us to measure the degree to which the aggregation is disjunctive (i.e., it is like an *or* operation). This concept was proposed by Yager in the analysis of OWA operators,[1] and later, it was generalized by Marichal to the case of Choquet integrals by using the concept of average value.[28] The same author[29] gave an expression to represent the orness degree in terms of the normalized capacity.

Remark 4. Let μ be a normalized capacity on N. Then

$$\text{orness}(\mathcal{C}_\mu) = \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\mu(T).$$

Notice that the degree of orness preserves the usual order between Choquet integrals; that is, if μ_1 and μ_2 are two normalized capacities on N such that $\mu_1 \le \mu_2$ (which is equivalent to $\mathcal{C}_{\mu_1} \le \mathcal{C}_{\mu_2}$), then $\text{orness}(\mathcal{C}_{\mu_1}) \le \text{orness}(\mathcal{C}_{\mu_2})$.[17]

In the theory of cooperative games, Shapley introduced a solution to the problem of distributing the amount $\mu(N)$ among the players.[30] This solution is known as the Shapley value and it can be interpreted as a kind of average value of the contribution of element j alone in all coalitions. In the MCDM field, the Shapley value can also be seen as the importance of each criterion.[31]

Definition 10. Let $j \in N$ and let μ be a normalized capacity on N. The Shapley value of criterion j with respect to μ is defined by

$$\phi(\mathcal{C}_\mu, j) = \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\Big(\mu(T\cup\{j\}) - \mu(T)\Big).$$

The concepts of veto and favor[32] were introduced in the context of social choice functions (where "favor" was called "dictator") and, afterwards, in the field of multicriteria decision making.[33]

Definition 11. Let $j \in N$ and let μ be a normalized capacity on N.

1. j is a veto for $\mathcal{C}_\mu$ if $\mathcal{C}_\mu(\boldsymbol{x}) \le x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. j is a favor for $\mathcal{C}_\mu$ if $\mathcal{C}_\mu(\boldsymbol{x}) \ge x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

Since veto and favor criteria are infrequent in practice, Marichal proposed two indices for measuring the degree with which a criterion behaves like a veto or a favor.[29,34]

Definition 12. Let $j \in N$ and let μ be a normalized capacity on N. The veto and favor indices of criterion j with respect to μ are defined by

$$\text{veto}(\mathcal{C}_\mu, j) = 1 - \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\mu(T),$$

$$\text{favor}(\mathcal{C}_\mu, j) = \frac{1}{n-1}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\mu(T\cup\{j\}) - \frac{1}{n-1}.$$

The veto index can be interpreted as the degree to which the decision maker demands that criterion j is satisfied. Analogously, the favor index is the degree to which the decision maker considers that a good score along criterion j is sufficient to be satisfied. It is worth noting that it is possible to establish a relationship among veto, favor and Shapley value of a criterion.[29]

Remark 5. Let $j \in N$ and let μ be a normalized capacity on N. Then,

$$\text{veto}(\mathcal{C}_\mu, j) + \text{favor}(\mathcal{C}_\mu, j) = 1 + \frac{n\phi(\mathcal{C}_\mu, j) - 1}{n-1}.$$

The concepts of k-conjunctive and k-disjunctive functions (originally called *at most k-intolerant* and *at most k-tolerant* functions) where introduced for determining the conjunctive/disjunctive character of aggregation.[34]

Definition 13. Let $k \in N$ and let μ be a normalized capacity on N.

1. $\mathcal{C}_\mu$ is k-conjunctive if $\mathcal{C}_\mu \leq \mathrm{OS}_k$; i.e., $\mathcal{C}_\mu(\boldsymbol{x}) \leq x_{(k)}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. $\mathcal{C}_\mu$ is k-disjunctive if $\mathcal{C}_\mu \geq \mathrm{OS}_{n-k+1}$; i.e., $\mathcal{C}_\mu(\boldsymbol{x}) \geq x_{(n-k+1)} = x_{[k]}$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

As is the case of veto and favor, k-conjunctive and k-disjunctive Choquet integrals are infrequent in practice. So, Marichal suggested two indices for measuring the degree to which a Choquet integral is k-conjunctive or k-disjunctive.[34]

Definition 14. Let $k \in N \setminus \{n\}$ and let μ be a normalized capacity on N. The k-conjunctiveness and k-disjunctiveness indices for $\mathcal{C}_\mu$ are defined by

$$\mathrm{conj}_k(\mathcal{C}_\mu) = 1 - \frac{1}{n-k} \sum_{t=1}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T),$$

$$\mathrm{disj}_k(\mathcal{C}_\mu) = \frac{1}{n-k} \sum_{t=k}^{n} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T) - \frac{1}{n-k} = \frac{1}{n-k} \sum_{t=k}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \mu(T).$$

In Table 1 we collect the orness, the Shapley values, and the veto, favor, k-conjunctiveness and k-disjunctiveness indices for weighted means and OWA operators (see, for instance, Refs. 29 and 17).[b]

Table 1: Some indices for weighted means and OWA operators.

Indice	$M_{\boldsymbol{p}}$	$O_{\boldsymbol{w}}$
$\mathrm{orness}(\mathcal{C}_\mu)$	$\frac{1}{2}$	$\frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i$
$\phi(\mathcal{C}_\mu, j)$	p_j	$\frac{1}{n}$
$\mathrm{veto}(\mathcal{C}_\mu, j)$	$\frac{1}{2} + \frac{np_j - 1}{2(n-1)}$	$\frac{1}{n-1} \sum_{i=1}^{n} (i-1) w_i$
$\mathrm{favor}(\mathcal{C}_\mu, j)$	$\frac{1}{2} + \frac{np_j - 1}{2(n-1)}$	$\frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i$
$\mathrm{conj}_k(\mathcal{C}_\mu)$	$\frac{n+k-1}{2n}$	$1 - \frac{1}{n-k} \sum_{i=1}^{n-k} (n+1-k-i) w_i$
$\mathrm{disj}_k(\mathcal{C}_\mu)$	$\frac{n+k-1}{2n}$	$1 - \frac{1}{n-k} \sum_{i=k+1}^{n} (i-k) w_i$

3. SUOWA OPERATORS

SUOWA operators[6] were introduced for dealing with situations where both the importance of information sources and the importance of the relative size of the values have to be taken into account. In their definition, semiuninorms and uninorms play a fundamental role.

3.1. Semiuninorms and uninorms

Semiuninorms[35] are monotonic functions having a neutral element in the interval $[0, 1]$. They were introduced as a generalization of uninorms,[c] which, in turn, were proposed as a generalization of t-norms and t-conorms.[37]

[b]Notice that we are considering the original definition of OWA operators given by Yager, where the components of $\boldsymbol{x}$ are ordered in a nonincreasing way. For this reason, the orness, and the veto, favor, k-conjunctiveness and k-disjunctiveness indices of OWA operators do not match with those shown in Refs. 29 and 17, where the components are ordered in a nondecreasing way.

[c]An interesting survey on uninorms is given in Ref. 36.

Definition 15. Let $U : [0,1]^2 \longrightarrow [0,1]$.

1. U is a semiuninorm if it is monotonic and possesses a neutral element $e \in [0,1]$ ($U(e,x) = U(x,e) = x$ for all $x \in [0,1]$).

2. U is a uninorm if it is a symmetric and associative ($U(x,U(y,z)) = U(U(x,y),z)$ for all $x,y,z \in [0,1]$) semiuninorm.

We denote by $\mathcal{U}^e$ (respectively, $\mathcal{U}_{\mathrm{i}}^e$) the set of semiuninorms (respectively, idempotent semiuninorms) with neutral element $e \in [0,1]$. The semiuninorms employed in the definition of SUOWA operators have $1/n$ as neutral element. Moreover, they have to belong to the following subset[6]:

$$\widetilde{\mathcal{U}}^{1/n} = \left\{ U \in \mathcal{U}^{1/n} \mid U(1/k, 1/k) \leq 1/k \text{ for all } k \in N \right\}.$$

Obviously $\mathcal{U}_{\mathrm{i}}^{1/n} \subseteq \widetilde{\mathcal{U}}^{1/n}$. Note that the smallest and the largest elements of $\widetilde{\mathcal{U}}^{1/n}$ are, respectively, the following semiuninorms:

$$U_{\perp}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ 0 & \text{if } (x,y) \in [0, 1/n)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$U_{\top}(x,y) = \begin{cases} 1/k & \text{if } (x,y) \in I_k \setminus I_{k+1}, \text{where } I_k = \big(1/n, 1/k\big]^2 \ \big(k \in N \setminus \{n\}\big), \\ \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Notice also that the smallest and the largest elements of $\mathcal{U}_{\mathrm{i}}^{1/n}$ are, respectively, the following uninorms[37]:

$$U_{\min}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$U_{\max}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

The following idempotent semiuninorms were introduced in Ref. 38 in order to show some interesting properties of SUOWA operators:

$$U_{\min}^{\max}(x,y) = \begin{cases} \min(x,y) & \text{if } y < 1/n, \\ x & \text{if } y = 1/n, \\ \max(x,y) & \text{if } y > 1/n. \end{cases}$$

$$U_{\mathrm{minmax}}(x,y) = \begin{cases} \min(x,y) & \text{if } x < 1/n, \\ y & \text{if } x = 1/n, \\ \max(x,y) & \text{if } x > 1/n. \end{cases}$$

In addition to the previous ones, continuous semiuninorms can be obtained through a procedure introduced in Ref. 39. Some of the most relevant are the following:

$$U_{T_{\mathbf{L}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \max(x+y-1/n,0) & \text{otherwise.} \end{cases}$$

$$U_{\widetilde{P}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ nxy & \text{otherwise.} \end{cases}$$

$$U_{T_{\mathbf{M}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,1/n)^2, \\ x+y-1/n & \text{otherwise.} \end{cases}$$

$$U_{P}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n,1]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,1/n)^2, \\ nxy & \text{otherwise.} \end{cases}$$

Notice that the last two semiuninorms are also idempotent. In Table 2 we show which of the previous semiuninorms satisfy the properties of idempotency, continuity and symmetry.

Table 2: Properties satisfied by some semi-uninorms.

	Idempotency	Continuity	Symmetry
$U_\top$			✓
$U_{\max}$	✓		✓
$U_{\min}^{\max}$	✓		
U_{minmax}	✓		
$U_{T_{\mathbf{M}}}$	✓	✓	✓
U_P	✓	✓	✓
$U_{\min}$	✓		✓
$U_{T_{\mathbf{L}}}$		✓	✓
$U_{\widetilde{P}}$		✓	✓
$U_\bot$			✓

It is also worth noting that we can define a partial order in the set $\widetilde{\mathcal{U}}^{1/n}$ by considering the usual partial order between functions; that is, given $U_1, U_2 \in \widetilde{\mathcal{U}}^{1/n}$, $U_1 \le U_2$ if $U_1(x,y) \le U_2(x,y)$ for all $(x,y) \in [0,1]^2$. The Hasse diagram of this partial order for the semiuninorms considered above is shown in Figure 1.

3.2. SUOWA operators

As we have seen in the previous section, capacities play a fundamental role in the definition of Choquet integrals. In the case of SUOWA operators, the capacities are the monotonic cover of certain games, which are defined by using semiuninorms with neutral element $1/n$ and the values of the capacities associated with the weighted means and the OWA operators. To be specific, the games from which SUOWA operators are built are defined as follows.

Definition 16. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$.

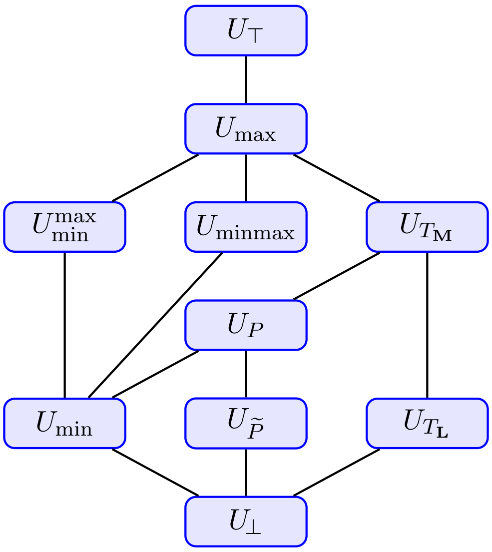

Figure 1: Hasse diagram showing the usual partial order between some semiuninorms.

1. The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}} : 2^N \longrightarrow \mathbb{R}$ defined by

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right) = |A|\, U\left(\frac{\sum_{i\in A} p_i}{|A|}, \frac{\sum_{i=1}^{|A|} w_i}{|A|}\right),$$

   if $A \neq \varnothing$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$.

2. $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$, the monotonic cover of the game $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.

Notice that $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$. Moreover, since $U \in \widetilde{\mathcal{U}}^{1/n}$ we have $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq 1$ for any $A \subseteq N$.[6] Therefore, according to Remark 1, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is always a normalized capacity.

Definition 17. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$. The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the function $S^U_{\boldsymbol{p},\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$

where $s_i = \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}) - \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i-1]})$ for all $i \in N$, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is the capacity associated with $\boldsymbol{p}, \boldsymbol{w}$ and U, and $A_{[i]} = \big\{[1], \dots, [i]\big\}$ (with the convention that $A_{[0]} = \varnothing$).

According to expression (1), the SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U can also be written as

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]})\big(x_{[i]} - x_{[i+1]}\big).$$

By the choice of $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ we have $S^U_{\boldsymbol{p},\boldsymbol{\eta}} = M_{\boldsymbol{p}}$ and $S^U_{\boldsymbol{\eta},\boldsymbol{w}} = O_{\boldsymbol{w}}$ for any $U \in \widetilde{\mathcal{U}}^{1/n}$. Moreover, by Remark 2 and given that $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity, SUOWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

4. A SUMMARY OF RESULTS ON SUOWA OPERATORS

The aim of this section is to gather the main results obtained so far on SUOWA operators. Firstly, we are going to show some general results. Next, we are going to focus on convex combination of semiuninorms, and the families of symmetric and idempotent semiuninorms. Lastly, we will give specific results for some of the semiuninorms presented in the paper: $U_{\max}$, $U_{\min}^{\max}$, U_{minmax}, $U_{\min}$, $U_{T_{\mathbf{L}}}$, and $U_{\widetilde{P}}$.

4.1. General results

We start showing the behavior of SUOWA operators when we consider simple cases of weighting vectors; namely, when $\boldsymbol{p} = \boldsymbol{e}_k$ and $\boldsymbol{w} = \boldsymbol{e}_l$, where $k, l \in N$.

Proposition 1. (Ref. 38). *Let $k, l \in N$ and $U \in \widetilde{\mathcal{U}}^{1/n}$. Then the following holds:*

1. *Given $\boldsymbol{x} \in \mathbb{R}^n$, $\min(x_k, x_{[l]}) \leq S^U_{\boldsymbol{e}_k,\boldsymbol{e}_l}(\boldsymbol{x}) \leq \max(x_k, x_{[l]})$.*
2. *If $U_\perp \leq U \leq U_{\min}$, then $S^U_{\boldsymbol{e}_k,\boldsymbol{e}_l}(\boldsymbol{x}) = \min(x_k, x_{[l]})$ for any $\boldsymbol{x} \in \mathbb{R}^n$.*
3. *If $U_{\max} \leq U \leq U_\top$, then $S^U_{\boldsymbol{e}_k,\boldsymbol{e}_l}(\boldsymbol{x}) = \max(x_k, x_{[l]})$ for any $\boldsymbol{x} \in \mathbb{R}^n$.*

It is worth mentioning that the partial order of the semiuninorms is preserved by the games, the capacities, the SUOWA operators, and the orness degree.

Proposition 2. (Refs. 6 and 40). *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors. Then the following holds:*

1. *If $U_1, U_2 \in \widetilde{\mathcal{U}}^{1/n}$, and $U_1 \leq U_2$, then*

$$\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}} \leq \upsilon^{U_2}_{\boldsymbol{p},\boldsymbol{w}}, \qquad \hat{\upsilon}^{U_1}_{\boldsymbol{p},\boldsymbol{w}} \leq \hat{\upsilon}^{U_2}_{\boldsymbol{p},\boldsymbol{w}},$$
$$S^{U_1}_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U_2}_{\boldsymbol{p},\boldsymbol{w}}, \qquad \text{orness}\left(S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \text{orness}\left(S^{U_2}_{\boldsymbol{p},\boldsymbol{w}}\right).$$

2. *If $U \in \widetilde{\mathcal{U}}^{1/n}$, then*

$$S^{U_\perp}_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U}_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U_\top}_{\boldsymbol{p},\boldsymbol{w}},$$
$$\text{orness}\left(S^{U_\perp}_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \text{orness}\left(S^{U}_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \text{orness}\left(S^{U_\top}_{\boldsymbol{p},\boldsymbol{w}}\right).$$

4.2. Convex combination of semiuninorms

The capacities associated with SUOWA operators have an interesting property. Suppose we consider a convex combination of semiuninorms such that the games associated with these semiuninorms are normalized capacities. Then the game associated with the new semiuninorm is also a normalized capacity, and it can be straightforward obtained by using the same convex combination of the capacities associated with the former semiuninorms.

From this property on the capacities, it is possible to obtain similar properties about the SUOWA operators, the orness degree, the Shapley value, etc.[d] They are gathered in the following proposition.

Proposition 3. (Refs. 39, 40 and 42). *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, \upsilon^{U_m}_{\boldsymbol{p},\boldsymbol{w}}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{j=1}^m \lambda_j U_j$. Then,*

1. *$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity on N and, for any subset A of N,*

$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = \sum_{j=1}^m \lambda_j \upsilon^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(A).$$

2. *For any $\boldsymbol{x} \in \mathbb{R}^n$,*

$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{j=1}^m \lambda_j S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}).$$

[d] Note that these properties are very interesting from a practical point of view; see, for instance, Refs. 38–41.

3. *Given $\boldsymbol{x} \in \mathbb{R}^n$ and $i \in N$,*

$$s_i^U = \sum_{j=1}^{m} \lambda_j s_i^{U_j},$$

where s_i^U (respectively, $s_i^{U_j}$) is the weight associated with $x_{[i]}$ in the SUOWA operator $S^U_{\boldsymbol{p},\boldsymbol{w}}$ (respectively, $S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}$).

4. $\operatorname{orness}\left(S^U_{\boldsymbol{p},\boldsymbol{w}}\right) = \sum_{j=1}^{m} \lambda_j \operatorname{orness}\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}\right).$

5. *For each $i \in N$,*

$$\begin{aligned}
\phi\left(S^U_{\boldsymbol{p},\boldsymbol{w}}, i\right) &= \sum_{j=1}^{m} \lambda_j \phi\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}, i\right),\\
\operatorname{veto}\left(S^U_{\boldsymbol{p},\boldsymbol{w}}, i\right) &= \sum_{j=1}^{m} \lambda_j \operatorname{veto}\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}, i\right),\\
\operatorname{favor}\left(S^U_{\boldsymbol{p},\boldsymbol{w}}, i\right) &= \sum_{j=1}^{m} \lambda_j \operatorname{favor}\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}, i\right).
\end{aligned}$$

6. *For each $k \in N$,*

$$\begin{aligned}
S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \le \mathrm{OS}_k \;&\Rightarrow\; S^U_{\boldsymbol{p},\boldsymbol{w}} \le \mathrm{OS}_k,\\
S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \ge \mathrm{OS}_k \;&\Rightarrow\; S^U_{\boldsymbol{p},\boldsymbol{w}} \ge \mathrm{OS}_k\,.
\end{aligned}$$

7. *For each $k \in N \setminus \{1\}$,*

$$\left.\begin{matrix} S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \le \mathrm{OS}_k \\ S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \not\le \mathrm{OS}_{k-1} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} S^U_{\boldsymbol{p},\boldsymbol{w}} \le \mathrm{OS}_k \\ S^U_{\boldsymbol{p},\boldsymbol{w}} \not\le \mathrm{OS}_{k-1}\,. \end{matrix}\right.$$

8. *For each $k \in N \setminus \{n\}$,*

$$\left.\begin{matrix} S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \ge \mathrm{OS}_k \\ S^{U_1}_{\boldsymbol{p},\boldsymbol{w}}, \dots, S^{U_m}_{\boldsymbol{p},\boldsymbol{w}} \not\ge \mathrm{OS}_{k+1} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} S^U_{\boldsymbol{p},\boldsymbol{w}} \ge \mathrm{OS}_k \\ S^U_{\boldsymbol{p},\boldsymbol{w}} \not\ge \mathrm{OS}_{k+1}\,. \end{matrix}\right.$$

9. *For each $k \in N \setminus \{n\}$*

$$\begin{aligned}
\operatorname{conj}_k\left(S^U_{\boldsymbol{p},\boldsymbol{w}}\right) &= \sum_{j=1}^{m} \lambda_j \operatorname{conj}_k\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}\right),\\
\operatorname{disj}_k\left(S^U_{\boldsymbol{p},\boldsymbol{w}}\right) &= \sum_{j=1}^{m} \lambda_j \operatorname{disj}_k\left(S^{U_j}_{\boldsymbol{p},\boldsymbol{w}}\right).
\end{aligned}$$

4.3. Symmetric semiuninorms

The use of these semiuninorms allows SUOWA operators to have a symmetrical behavior between the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$.

Proposition 4. (Ref. 39). *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, U a symmetrical semiuninorm belonging to $\widetilde{\mathcal{U}}^{1/n}$, and $\boldsymbol{x} \in \mathbb{R}^n$. If $\boldsymbol{p}'$ and $\boldsymbol{w}'$ are two weighting vectors such that $p'_{[i]} = w_i$ and $w'_i = p_{[i]}$ for any $i \in N$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$ and $\upsilon^U_{\boldsymbol{p}',\boldsymbol{w}'}$ are normalized capacities, then $S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = S^U_{\boldsymbol{p}',\boldsymbol{w}'}(\boldsymbol{x})$.*

4.4. Idempotent semiuninorms

These semiuninorms are located between $U_{\min}$ and $U_{\max}$ and they fulfill several properties, but one of them stands out particularly: the capacity (and also the game) associated with an idempotent semiuninorm ranges between the capacities of the corresponding weighted mean and OWA operator.

Proposition 5. (Ref. 6). *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, and $U \in \mathcal{U}_{\mathrm{i}}^{1/n}$. Then the following holds:*

1. *For any nonempty subset A of N, we have*
$$\min\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big) \leq \upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \max\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big).$$
Moreover, if $\mu_{\boldsymbol{p}}(A) = \mu_{|\boldsymbol{w}|}(A)$, then $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A) = \mu_{\boldsymbol{p}}(A)$.
2. *Let $\boldsymbol{x} \in \mathbb{R}^n$ such that $p_{[i]} = w_i$ for any $i \in N$. Then $s_i^U = p_{[i]} = w_i$ for any $i \in N$, where s_i^U is the weight associated with $x_{[i]}$ in the SUOWA operator $S^U_{\boldsymbol{p},\boldsymbol{w}}$, and, consequently,*
$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = M_{\boldsymbol{p}}(\boldsymbol{x}) = O_{\boldsymbol{w}}(\boldsymbol{x}).$$
3. $S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}} \leq S^U_{\boldsymbol{p},\boldsymbol{w}} \leq S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$.
4. $\operatorname{orness}\left(S^{U_{\min}}_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \operatorname{orness}\left(S^U_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \operatorname{orness}\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right)$.
5. *If $\mu_{\boldsymbol{p}} \leq \mu_{|\boldsymbol{w}|}$, then $M_{\boldsymbol{p}} \leq S^U_{\boldsymbol{p},\boldsymbol{w}} \leq O_{\boldsymbol{w}}$.*
6. *If $\mu_{|\boldsymbol{w}|} \leq \mu_{\boldsymbol{p}}$, then $O_{\boldsymbol{w}} \leq S^U_{\boldsymbol{p},\boldsymbol{w}} \leq M_{\boldsymbol{p}}$.*

4.5. The uninorm $U_{\max}$

This uninorm is the largest idempotent semiuninorm; so, it satisfies the properties given in Proposition 5. Besides, when the weighting vector $\boldsymbol{w}$ satisfies that $\sum_{i=1}^{j} w_i > j/n$ for any $j \in N \setminus \{n\}$,[e] the SUOWA operator associated with this uninorm has interesting properties, which are collected in the following proposition.

Proposition 6. (Refs. 39, 40 and 42). *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i > j/n$ for any $j \in N \setminus \{n\}$. Then, for any weighting vector $\boldsymbol{p}$, we have*

1. *$\upsilon^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity on N given by*
$$\upsilon^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(A) = \max\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big).$$
2. *$S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) \geq \max\big(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\big)$ for any $\boldsymbol{x} \in \mathbb{R}^n$.*
3. *$\min(p_{[i]}, w_i) \leq s_i^{U_{\max}} \leq \max(p_{[i]}, w_i)$ for any $\boldsymbol{x} \in \mathbb{R}^n$ and for any $i \in N$.*
4. $\operatorname{orness}\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) \geq \operatorname{orness}(O_{\boldsymbol{w}}) > 0.5$.
5. *$\operatorname{disj}_k\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) \geq \operatorname{disj}_k(O_{\boldsymbol{w}})$ for any $k \in N \setminus \{n\}$.*
6. *$\operatorname{conj}_k\left(S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}}\right) \leq \operatorname{conj}_k(O_{\boldsymbol{w}})$ for any $k \in N \setminus \{n\}$.*

It is also worth mentioning that the SUOWA operator obtained with this uninorm retains the disjunctive character of the OWA operator associated with it.

Proposition 7. (Ref. 42). *Let $\boldsymbol{w}$ be a weighting vector. Then:*

1. *If there exists $k \in N$ such that $O_{\boldsymbol{w}} \geq \mathrm{OS}_k$, then $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}} \geq \mathrm{OS}_k$ for any weighting vector $\boldsymbol{p}$.*
2. *If there exists $k \in N \setminus \{n\}$ such that $O_{\boldsymbol{w}} \geq \mathrm{OS}_k$ and $O_{\boldsymbol{w}} \ngeq \mathrm{OS}_{k+1}$, then $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}} \geq \mathrm{OS}_k$ and $S^{U_{\max}}_{\boldsymbol{p},\boldsymbol{w}} \ngeq \mathrm{OS}_{k+1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq n+1-k$.*

[e] Notice that, for instance, nonincreasing weighting vectors fulfill this requirement.

4.6. The semiuninorm $U_{\min}^{\max}$

This semiuninorm[38] was introduced to show that, when we consider $\boldsymbol{w} = \boldsymbol{e}_{n-k+1}$, the order statistic OS_k can be recovered from SUOWA operators (it is worth noting that this fact is independent of the weighting vector $\boldsymbol{p}$ considered).

Proposition 8. (Ref. 38). *If $k \in N$, then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{e}_{n-k+1}}^{U_{\min}^{\max}}$ is a normalized capacity on N and $S_{\boldsymbol{p},\boldsymbol{e}_{n-k+1}}^{U_{\min}^{\max}} = O_{\boldsymbol{e}_{n-k+1}} = \mathrm{OS}_k$.*

Notice that this semiuninorm is idempotent; therefore, it satisfies the properties given in Proposition 5. Besides, the game associated with this semiuninorm has interesting properties when the weighting vector $\boldsymbol{w}$ is unimodal; namely, it is a normalized capacity on N, its expression is very simple, the weight $s_i^{U_{\min}^{\max}}$ which affecting the component $x_{[i]}$ of $\boldsymbol{x}$ is located between $p_{[i]}$ and w_i $(i = 1, \ldots, n)$, it is subadditive (respectively, superadditive) when $\boldsymbol{w}$ is nonincreasing (respectively, nondecreasing), and, as in OWA operators, its dual capacity can be easily obtained by using the dual of the weighting vector $\boldsymbol{w}$.

Proposition 9. (Ref. 24). *Let $\boldsymbol{w}$ be a unimodal weighting vector. Then, for any weighting vector $\boldsymbol{p}$, we have*

1. *$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$ is a normalized capacity on N given by*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}(A) = \begin{cases} \min\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big), & \text{if } \mu_{|\boldsymbol{w}|}(A) < |A|/n, \\ \mu_{\boldsymbol{p}}(A), & \text{if } \mu_{|\boldsymbol{w}|}(A) = |A|/n, \\ \max\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big), & \text{if } \mu_{|\boldsymbol{w}|}(A) > |A|/n. \end{cases}$$

2. $\min(p_{[i]}, w_i) \leq s_i^{U_{\min}^{\max}} \leq \max(p_{[i]}, w_i)$ *for any $\boldsymbol{x} \in \mathbb{R}^n$ and for any $i \in N$.*

3. *If $\boldsymbol{w}$ is nonincreasing, then $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$ is subadditive; that is, $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any $A, B \subseteq N$ such that $A \cap B = \varnothing$.*

4. *If $\boldsymbol{w}$ is nondecreasing, then $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$ is superadditive; that is, $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for any $A, B \subseteq N$ such that $A \cap B = \varnothing$.*

5. $\overline{\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}} = \upsilon_{\boldsymbol{p},\overline{\boldsymbol{w}}}^{U_{\min}^{\max}}$.

It is also worth noting that the SUOWA operators constructed by using the semiuninorm $U_{\min}^{\max}$ preserve the conjunctive/disjuntive character of the OWA operator associated with them. In this way, it is possible to get operators located between two order statistics that take into account the weights of the information sources.

Proposition 10. (Ref. 42). *Let $\boldsymbol{w}$ be a weighting vector. Then:*

1. *If there exist $k, k' \in N$ such that $\mathrm{OS}_k \leq O_{\boldsymbol{w}} \leq \mathrm{OS}_{k'}$, then $\mathrm{OS}_k \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}} \leq \mathrm{OS}_{k'}$ for any weighting vector $\boldsymbol{p}$.*

2. *If there exist $k \in N \setminus \{n\}$ and $k' \in N \setminus \{1\}$ such that $\mathrm{OS}_k \leq O_{\boldsymbol{w}} \leq \mathrm{OS}_{k'}$ and $\mathrm{OS}_{k+1} \nleq O_{\boldsymbol{w}} \nleq \mathrm{OS}_{k'-1}$, then $\mathrm{OS}_k \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}} \leq \mathrm{OS}_{k'}$ and $\mathrm{OS}_{k+1} \nleq S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}} \nleq \mathrm{OS}_{k'-1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq \max(n+1-k, k')$.*

An interesting family of functions, which generalizes the well-known family of Winsorized means, is the class of the (r, s)-fold Winsorized weighted means. In these functions, given a vector of values $\boldsymbol{x}$ to be aggregated, the r lowest values and the s highest values of $\boldsymbol{x}$ are replaced by $x_{(r+1)}$ and $x_{(n-s)}$, respectively, and after that, the weighted mean associated with a weighting vector $\boldsymbol{p}$ is considered.

Definition 18. (Ref. 43). Let $\mathcal{R} = \{(r,s) \in \{0,1,\dots,n-1\}^2 \mid r+s \le n-1\}$, let $(r,s) \in \mathcal{R}$, and let $\boldsymbol{p}$ be a weighting vector. The (r,s)-fold Winsorized weighted mean is defined by

$$M_{\boldsymbol{p}}^{(r,s)}(\boldsymbol{x}) = \left(\sum_{i=1}^{r} p_{(i)}\right) x_{(r+1)} + \sum_{i=r+1}^{n-s} p_{(i)} x_{(i)} + \left(\sum_{i=n-s+1}^{n} p_{(i)}\right) x_{(n-s)}.$$

The (r,s)-fold Winsorized weighted means can be obtained through SUOWA operators by using the semiuninorm $U_{\min}^{\max}$. Moreover, closed-form expressions for the orness degree, the Shapley values, and the veto, favor, k-conjunctiveness and k-disjunctiveness indices can be given.

Proposition 11. (Refs. 24 and 43). *Let $(r,s) \in \mathcal{R}$ and let $\boldsymbol{w}$ be the weighting vector defined by*

$$w_i = \begin{cases} 0 & \text{if } i = 1,\dots,s, \\ \frac{s+1}{n} & \text{if } i = s+1, \\ \frac{1}{n} & \text{if } i = s+2,\dots,n-r-1, \\ \frac{r+1}{n} & \text{if } i = n-r, \\ 0 & \text{otherwise,} \end{cases}$$

when $r+s \le n-2$, and $\boldsymbol{w} = \boldsymbol{e}_{s+1}$ when $r+s = n-1$. Then, for any weighting vector $\boldsymbol{p}$, we have

1. $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}$ *is a normalized capacity on N given by*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}(A) = \begin{cases} 0, & \text{if } |A| \le s, \\ \sum_{i\in A} p_i, & \text{if } s < |A| < n-r, \\ 1, & \text{if } |A| \ge n-r. \end{cases}$$

2. *For any $\boldsymbol{x} \in \mathbb{R}^n$,*

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}(\boldsymbol{x}) = \left(\sum_{i=1}^{s} p_{[i]}\right) x_{[s+1]} + \sum_{i=s+1}^{n-r} p_{[i]} x_{[i]} + \left(\sum_{i=n-r+1}^{n} p_{[i]}\right) x_{[n-r]} = M_{\boldsymbol{p}}^{(r,s)}(\boldsymbol{x}).$$

3. $\text{orness}\left(M_{\boldsymbol{p}}^{(r,s)}\right) = \frac{1}{2} + \frac{r(r+1)-s(s+1)}{2n(n-1)}$.

4. *For each $j \in N$,*

$$\phi\left(M_{\boldsymbol{p}}^{(r,s)}, j\right) = \frac{r+s}{n-1}\,\frac{1}{n} + \left(1 - \frac{r+s}{n-1}\right) p_j, \tag{2}$$

$$\text{veto}\left(M_{\boldsymbol{p}}^{(r,s)}, j\right) = 1 - \frac{r}{n-1} - \frac{(1-p_j)\big((n-r)(n-r-1) - s(s+1)\big)}{2(n-1)^2},$$

$$\text{favor}\left(M_{\boldsymbol{p}}^{(r,s)}, j\right) = (1-p_j)\frac{(n-1)(n-2) + r(r+1) - s(s-1)}{2(n-1)^2} + p_j\left(1 - \frac{s}{n-1}\right).$$

5. *For each $k \in N \setminus \{n\}$,*

$$\text{conj}_k\left(M_{\boldsymbol{p}}^{(r,s)}\right) = \begin{cases} 1 & \text{if } k \ge n-s, \\ \frac{n(n-1)+s(s+1)-k(k-1)}{2n(n-k)} & \text{if } r < k < n-s, \\ \frac{n(n-1)+s(s+1)-r(r+1)}{2n(n-k)} & \text{if } k \le r. \end{cases}$$

$$\text{disj}_k\left(M_{\boldsymbol{p}}^{(r,s)}\right) = \begin{cases} 1 & \text{if } k \ge n-r, \\ \frac{n(n-1)+r(r+1)-k(k-1)}{2n(n-k)} & \text{if } s < k < n-r, \\ \frac{n(n-1)+r(r+1)-s(s+1)}{2n(n-k)} & \text{if } k \le s. \end{cases}$$

Since the Shapley values reflect the overall importance of criteria, it is very important to be able to determine weighting vectors that allow to obtain Shapley values previously fixed. The following proposition explicitly shows these weighting vectors.

Proposition 12. (Ref. 43). *Let $(\phi_1, \dots, \phi_n)$ be a weighting vector. Given $(r, s) \in \mathcal{R}$ such that $r + s < n - 1$, the following conditions are equivalent:*

1. $\min_{j \in N} \phi_j \geq \dfrac{r+s}{n(n-1)}$.

2. *The vector $\boldsymbol{p}$ defined by*

$$p_j = \frac{1}{n} + \frac{n-1}{n-1-(r+s)} \left(\phi_j - \frac{1}{n} \right), \quad j = 1, \dots, n,$$

is a weighting vector such that $\phi\Big(M_{\boldsymbol{p}}^{(r,s)}, j\Big) = \phi_j$ for any $j \in N$.

4.7. The semiuninorm U_{minmax}

This semiuninorm[38] was introduced to show that, when we consider $\boldsymbol{p} = \boldsymbol{e}_k$, the projection P_k can be recovered from SUOWA operators (it is worth noting that this fact is independent of the weighting vector $\boldsymbol{w}$ considered).

Proposition 13. (Ref. 38). *If $k \in N$, then, for any weighting vector $\boldsymbol{w}$, $\upsilon_{\boldsymbol{e}_k, \boldsymbol{w}}^{U_{\text{minmax}}}$ is a normalized capacity on N and $S_{\boldsymbol{e}_k, \boldsymbol{w}}^{U_{\text{minmax}}} = M_{\boldsymbol{e}_k} = \mathsf{P}_k$.*

4.8. The uninorm $U_{\min}$

This uninorm is the smallest idempotent semiuninorm; so, it satisfies the properties given in Proposition 5. Moreover, when the weighting vector $\boldsymbol{w}$ satisfies that $\sum_{i=1}^{j} w_i < j/n$ for any $j \in N \setminus \{n\}$,[f] the SUOWA operator associated with this uninorm possesses the properties listed in the following proposition.

Proposition 14. (Refs. 39, 40 and 42). *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i < j/n$ for any $j \in N \setminus \{n\}$. Then, for any weighting vector $\boldsymbol{p}$, we have*

1. $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ *is a normalized capacity on N given by*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(A) = \min\big(\mu_{\boldsymbol{p}}(A), \mu_{|\boldsymbol{w}|}(A)\big).$$

2. $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(\boldsymbol{x}) \leq \min\big(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\big)$ *for any $\boldsymbol{x} \in \mathbb{R}^n$.*

3. $\min(p_{[i]}, w_i) \leq s_i^{U_{\min}} \leq \max(p_{[i]}, w_i)$ *for any $\boldsymbol{x} \in \mathbb{R}^n$ and for any $i \in N$.*

4. $\operatorname{orness}\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\big) \leq \operatorname{orness}(O_{\boldsymbol{w}}) < 0.5$.

5. $\operatorname{disj}_k\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\big) \leq \operatorname{disj}_k(O_{\boldsymbol{w}})$ *for any $k \in N \setminus \{n\}$.*

6. $\operatorname{conj}_k\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\big) \geq \operatorname{conj}_k(O_{\boldsymbol{w}})$ *for any $k \in N \setminus \{n\}$.*

It is also worthy of note that the SUOWA operator obtained with this uninorm maintains the conjunctive character of the OWA operator associated with it.

Proposition 15. (Ref. 42). *Let $\boldsymbol{w}$ be a weighting vector. Then:*

1. *If there exists $k \in N$ such that $O_{\boldsymbol{w}} \leq \mathrm{OS}_k$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \leq \mathrm{OS}_k$ for any weighting vector $\boldsymbol{p}$.*

2. *If there exists $k \in N \setminus \{1\}$ such that $O_{\boldsymbol{w}} \leq \mathrm{OS}_k$ and $O_{\boldsymbol{w}} \not\leq \mathrm{OS}_{k-1}$, then $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \leq \mathrm{OS}_k$ and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \not\leq \mathrm{OS}_{k-1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq k$.*

[f]Notice that, for instance, nondecreasing weighting vectors fulfill this requirement.

4.9. The semiuninorm $U_{T_{\mathbf{L}}}$

When the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ satisfy some conditions, the game associated with this semiuninorm is a normalized capacity, and we can give closed-form expressions for the orness degree, the Shapley values, and the veto, favor, k-conjunctiveness and k-disjunctiveness indices.

Proposition 16. (Refs. 38, 40 and 42). *Let $\mathbf{p}$ and $\boldsymbol{w}$ be two weighting vectors such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$ and $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$. Then:*

1. *$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$ is a normalized capacity on N and, for any nonempty $A \subseteq N$,*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(A) = \sum_{i\in A} p_i + \sum_{i=1}^{|A|} w_i - \frac{|A|}{n}.$$

2. *For any $\boldsymbol{x} \in \mathbb{R}^n$,*

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(\boldsymbol{x}) = M_{\boldsymbol{p}}(\boldsymbol{x}) + O_{\boldsymbol{w}}(\boldsymbol{x}) - \overline{\boldsymbol{x}},$$

   where $\overline{\boldsymbol{x}}$ is the average of $\boldsymbol{x}$.

3. $\operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) = \operatorname{orness}(O_{\boldsymbol{w}}) \leq 0.5$.

4. *For each $j \in N$,*

$$\begin{aligned}
\phi\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) &= \phi(M_{\boldsymbol{p}}, j) = p_j,\\
\operatorname{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) &= \frac{np_j - 1}{2(n-1)} + 1 - \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right),\\
\operatorname{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) &= \frac{np_j - 1}{2(n-1)} + \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right).
\end{aligned}$$

5. *For each $k \in N \setminus \{n\}$,*

$$\begin{aligned}
\operatorname{conj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) &= \operatorname{conj}_k(O_{\boldsymbol{w}}),\\
\operatorname{disj}_k\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) &= \operatorname{disj}_k(O_{\boldsymbol{w}}).
\end{aligned}$$

4.10. The semiuninorm $U_{\widetilde{P}}$

When the weighting vector $\boldsymbol{w}$ fulfills a requirement, we can known the game associated with this semiuninorm. Although, in general, the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is not a capacity, we can guarantee this property when $\boldsymbol{w}$ is nondecreasing.

Proposition 17. (Ref. 40). *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. Then, for any weighting vector $\mathbf{p}$ and any nonempty subset A of N, we have*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(A) = \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right).$$

Moreover, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N when $\boldsymbol{w}$ is nondecreasing.

If $\boldsymbol{w}$ satisfies the condition given in the previous proposition and $\boldsymbol{p}$ is such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N, then we can give closed-form expressions for the orness degree, the Shapley values, and the veto, favor, k-conjunctiveness and k-disjunctiveness indices.

Proposition 18. (Refs. 40 and 42). *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $\mathbf{p}$ is a weighting vector such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N, then:*

1. orness $\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \text{orness}(O_{\boldsymbol{w}}) \leq 0.5.$

2. For each $j \in N$,

$$\phi\Big(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\Big) = \frac{1}{n-1}\left(1 - p_j + (np_j - 1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right) w_i\right), \tag{3}$$

$$\text{veto}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) = 1 - \frac{n}{n-1}(1-p_j)\,\text{orness}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}\right),$$

$$\text{favor}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) = 1 - \text{veto}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) + \frac{1}{n-1}\Big(n\phi\Big(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\Big) - 1\Big).$$

3. For each $k \in N \setminus \{n\}$,

$$\text{conj}_k\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \text{conj}_k(O_{\boldsymbol{w}}),$$

$$\text{disj}_k\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}\right) = \text{disj}_k(O_{\boldsymbol{w}}).$$

It is also worth mentioning that the SUOWA operator obtained with this semiuninorm retains the conjunctive character of the OWA operator associated with it.

Proposition 19. (Ref. 42). *Let $\boldsymbol{w}$ be a weighting vector. Then:*

1. *If there exists $k \in N$ such that $O_{\boldsymbol{w}} \leq \text{OS}_k$, then $S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}} \leq \text{OS}_k$ for any weighting vector $\boldsymbol{p}$.*
2. *If there exists $k \in N \setminus \{1\}$ such that $O_{\boldsymbol{w}} \leq \text{OS}_k$ and $O_{\boldsymbol{w}} \not\leq \text{OS}_{k-1}$, then $S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}} \leq \text{OS}_k$ and $S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}} \not\leq \text{OS}_{k-1}$ for any weighting vector $\boldsymbol{p}$ such that $|\{i \in N \mid p_i > 0\}| \geq k$.*

5. SOME NEW RESULTS

In this section we show some additional results on the semiuninorms U_P and $U_{\widetilde{P}}$. In both cases we establish conditions that allow us to obtain normalized capacities.

5.1. The semiuninorm U_P

In the following proposition we show that we can get normalized capacities for any weighting vector $\boldsymbol{p}$ when the weighting vector $\boldsymbol{w}$ satisfies certain conditions.

Proposition 20. *Let $\boldsymbol{w}$ be a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1, \ldots, n-1\}$. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$ is a normalized capacity on N.*

Proof. If $\boldsymbol{w}$ is a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1, \ldots, n-1\}$, then, taking into account the definition of U_P and its continuity, we have

$$\upsilon^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) = \begin{cases} \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) & \text{if } \sum_{i\in A} p_i < |A|/n, \\[2ex] \dfrac{n}{|A|}\left(\displaystyle\sum_{i\in A} p_i\right)\left(\displaystyle\sum_{i=1}^{|A|} w_i\right) & \text{if } \sum_{i\in A} p_i \geq |A|/n, \end{cases}$$

whenever $A \neq \varnothing$.

To prove the monotonicity of $\upsilon^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$, it is sufficient to show that $\upsilon^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) \leq \upsilon^{U_P}_{\boldsymbol{p},\boldsymbol{w}}\big(A \cup \{j\}\big)$ for any $A \subsetneq N$ such that $|A| \geq 1$, and $j \in N \setminus A$. We distinguish four cases:

1. If $\sum_{i\in A} p_i < |A|/n$ and $\sum_{i\in A} p_i + p_j < (|A|+1)/n$, then

$$v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) = \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) \leq \min\left(\sum_{i\in A} p_i + p_j, \sum_{i=1}^{|A|+1} w_i\right) = v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}\big(A \cup \{j\}\big).$$

2. If $\sum_{i\in A} p_i < |A|/n$ and $\sum_{i\in A} p_i + p_j \geq (|A|+1)/n$, then

$$\begin{aligned} v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) = \min\left(\sum_{i\in A} p_i, \sum_{i=1}^{|A|} w_i\right) &\leq \frac{n}{|A|+1}\frac{|A|+1}{n}\sum_{i=1}^{|A|+1} w_i \\ &\leq \frac{n}{|A|+1}\left(\sum_{i\in A} p_i + p_j\right)\left(\sum_{i=1}^{|A|+1} w_i\right) = v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}\big(A \cup \{j\}\big). \end{aligned}$$

3. If $\sum_{i\in A} p_i \geq |A|/n$ and $\sum_{i\in A} p_i + p_j < (|A|+1)/n$, then notice that

$$\frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right) \leq \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\frac{|A|}{n} \leq \sum_{i\in A} p_i + p_j$$

and

$$\begin{aligned} \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right) \leq \frac{n}{|A|}\left(\sum_{i\in A} p_i + p_j\right)\left(\sum_{i=1}^{|A|} w_i\right) &\leq \frac{n}{|A|}\frac{|A|+1}{n}\sum_{i=1}^{|A|} w_i \\ &= \sum_{i=1}^{|A|} w_i + \frac{1}{|A|}\sum_{i=1}^{|A|} w_i \leq \sum_{i=1}^{|A|+1} w_i, \end{aligned}$$

where the last inequality is satisfied since, by hypothesis, $\frac{1}{j}\sum_{i=1}^{j} w_i \leq w_{j+1}$ for all $j \in \{1,\ldots,n-1\}$. Therefore

$$v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) = \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right) \leq \min\left(\sum_{i\in A} p_i + p_j, \sum_{i=1}^{|A|+1} w_i\right) = v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}\big(A \cup \{j\}\big).$$

4. If $\sum_{i\in A} p_i \geq |A|/n$ and $\sum_{i\in A} p_i + p_j \geq (|A|+1)/n$, then notice that, as in the previous case,

$$\frac{|A|+1}{|A|}\sum_{i=1}^{|A|} w_i = \sum_{i=1}^{|A|} w_i + \frac{1}{|A|}\sum_{i=1}^{|A|} w_i \leq \sum_{i=1}^{|A|+1} w_i.$$

Therefore,

$$\begin{aligned} v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}(A) = \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right) &\leq \frac{n}{|A|+1}\frac{|A|+1}{|A|}\left(\sum_{i\in A} p_i + p_j\right)\left(\sum_{i=1}^{|A|} w_i\right) \\ &\leq \frac{n}{|A|+1}\left(\sum_{i\in A} p_i + p_j\right)\left(\sum_{i=1}^{|A|+1} w_i\right) = v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}\big(A \cup \{j\}\big). \end{aligned}$$ □

It is easy to check that nondecreasing weighting vectors satisfy the hypotheses of the above proposition.

Corollary 1. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $w_1 \leq w_2 \leq \cdots \leq w_n$. *Then, for any weighting vector* $\boldsymbol{p}$, $v^{U_P}_{\boldsymbol{p},\boldsymbol{w}}$ *is a normalized capacity on* N.

Proof. The proof is immediate taking into account that if $w_1 \leq w_2 \leq \cdots \leq w_n$, then $\boldsymbol{w} = \boldsymbol{\eta}$ or $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{1,\ldots,n-1\}$ (see Lemma 1 in Ref. 39). So, in both cases, $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\ldots,n-1\}$. □

5.2. The semiuninorm $U_{\widetilde{P}}$

As in the case of the semiuninorm U_P, we can impose conditions on the weighting vector $\boldsymbol{w}$ that allow us to obtain normalized capacities for any weighting vector $\boldsymbol{p}$.

Proposition 21. *Let $\boldsymbol{w}$ be a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\dots,n-1\}$. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N.*

Proof. If $\boldsymbol{w}$ is a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\dots,n-1\}$, then, taking into account the definition of $U_{\widetilde{P}}$ and its continuity, we have

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(A) = \frac{n}{|A|}\left(\sum_{i\in A} p_i\right)\left(\sum_{i=1}^{|A|} w_i\right)$$

whenever $A \neq \varnothing$. So, the proof is similar to the one given in the fourth case of Proposition 20. □

Since the Shapley value reflects the global importance of each information source, it is very interesting to be able to determine the weights that allow us to obtain Shapley values previously fixed. If $\boldsymbol{w}$ is a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\dots,n-1\}$, then by expression (3) we can express the weight p_j in terms of $\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big)$. Notice that $\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right) w_i = \sum_{t=1}^{n}\frac{1}{t}\sum_{i=1}^{t} w_i$. If we use the notation $W = \sum_{t=1}^{n}\frac{1}{t}\sum_{i=1}^{t} w_i$, from

$$\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big) = \frac{1}{n-1}\big(1 - p_j + (np_j - 1)W\big)$$

we get

$$p_j = \frac{(n-1)\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big) + W - 1}{nW - 1}. \tag{4}$$

Notice that the above expression lacks of sense when $nW - 1 = 0$. Since

$$nW - 1 = n\sum_{t=1}^{n-1}\frac{1}{t}\sum_{i=1}^{t} w_i,$$

we have $nW - 1 \geq 0$ for any weighting vector $\boldsymbol{w}$ such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\dots,n-1\}$, and $nW - 1 = 0$ if and only if $\boldsymbol{w} = (0,\dots,0,1)$.[g]

From expression (4) it is easy to check that $p_j \geq 0$ if and only if $\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big) \geq \frac{1-W}{n-1}$,[h] and $\sum_{j=1}^{n} p_j = 1$. Therefore, we have the following proposition.

Proposition 22. *Let $\boldsymbol{w}$ be a weighting vector such that $\frac{1}{j}\sum_{i=1}^{j} w_i \leq \min(w_{j+1}, 1/n)$ for all $j \in \{1,\dots,n-1\}$, with $\boldsymbol{w} \neq (0,\dots,0,1)$, and let $W = \sum_{t=1}^{n}\frac{1}{t}\sum_{i=1}^{t} w_i$. Given a weighting vector $(\phi_1,\dots,\phi_n)$, the following conditions are equivalent:*

1. $\displaystyle\min_{j\in N}\phi_j \geq \frac{1-W}{n-1}$.

2. *The vector $\boldsymbol{p}$ defined by*

$$p_j = \frac{(n-1)\phi_j + W - 1}{nW - 1}, \quad j = 1,\dots,n,$$

is a weighting vector such that $\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big) = \phi_j$ for any $j \in N$.

---

[g] Note that when $\boldsymbol{w} = (0,\dots,0,1)$ we get $\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\Big) = \frac{1}{n}$ for any $j \in N$.

[h] Notice that $W = \sum_{t=1}^{n}\frac{1}{t}\sum_{i=1}^{t} w_i \leq \sum_{t=1}^{n}\frac{1}{n} = 1$, that is, $1 - W \geq 0$.

6. DISCUSSION

In this section we are going to show the usefulness of SUOWA operators in an example borrowed from Ref. 9. Consider a selection committee composed of three professors from distinct research fields, two other academics, and the head of department as the chair. The importance of each member is expressed through the weighting vector $\boldsymbol{p} = (2/11, 2/11, 2/11, 1/11, 1/11, 3/11)$. The members of the committee evaluate each applicant and return scores which are aggregated to obtain a global mark. Suppose the professors want to strengthen their research team. Then they would give high scores to applicants whose research falls in their area. To avoid this bias, an OWA type aggregation should be used so that extreme scores are discarded. In Ref. 9, the OWA operator associated with the weighting vector $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0)$ is considered (notice that this OWA operator is also known as a trimmed mean).

It should be noted again that in this example the use of the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ has a twofold purpose. On the one hand, we want to discard extreme scores (for which the weighting vector $\boldsymbol{w}$ is used). On the other hand, we want each member of the committee to have a certain weight (given through the weighting vector $\boldsymbol{p}$).

Consider now Table 3, taken from Ref. 9, which shows the scores given by the members of the committee to several applicants, together with the global score obtained by means of three different family of functions (see Ref. 9 for more details).[i]

Table 3: Individual evaluations and global score obtained by the applicants.

Applicant	Evaluations	PnTA	Torra	Implicit
A	$(1, \mathbf{0}, 1, 1, 1, 1,)$	1	0.98	0.85
B	$(\mathbf{1}, 0.5, 0.5, 0.5, 0.5, 0.5)$	0.5	0.5	0.57
C	$(0.8, 0.8, \mathbf{0}, 0.8, 0.8, 0.8)$	0.8	0.78	0.68
D	$(0.8, 0.8, 0.8, \mathbf{0}, 0.8, 0.8)$	0.8	0.8	0.8
E	$(0.8, 0.8, 0.8, 0.8, 0.8, \mathbf{0})$	0.65	0.67	0.68

Notice that, as it has been pointed out in Ref. 9, none of the three methods used rule out extreme scores. In fact, in these prototypical applicants, where all the scores except one (the outlier) are the same, it would be expected that the overall score coincided with them. Hence, none of the three methods used achieve the first goal. In relation to the second objective, it is worth noting that in the framework of games and capacities, the "weight" of each individual is determined through an importance index (usually the Shapley value). Since the implicit method is not based on Choquet integrals and in the PnTA method the capacities are unknown, we focus on Torra's functions, also known as WOWA operators.[10]

It is well known[38,44] that WOWA operators are Choquet integrals with respect to the normalized capacities $\mu^{Q}_{\boldsymbol{p},\boldsymbol{w}}(A) = Q\big(\mu_{\boldsymbol{p}}(A)\big) = Q\left(\sum_{i\in A} p_i\right)$, where Q is a quantifier generating the weighting vector $\boldsymbol{w}$.[j] In our case, since $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0)$, we have to choose a quantifier interpolating the points $(0, 0)$, $(1/6, 0)$, $(2/6, 0.25)$, $(3/6, 0.5)$, $(4/6, 0.75)$, $(5/6, 1)$, and $(1, 1)$. If we consider a linear interpolation, which is the most usual choice, we have the quantifier given by

$$Q(x) = \begin{cases} 0 & \text{if } x \le 1/6, \\ 1.5x - 0.25 & \text{if } 1/6 < x < 5/6, \\ 1 & \text{if } x \ge 5/6, \end{cases}$$

which is depicted in Figure 2.

[i] The values in boldface indicate outliers.

[j] The WOWA operator associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and Q will be denoted as $W^{Q}_{\boldsymbol{p},\boldsymbol{w}}$.

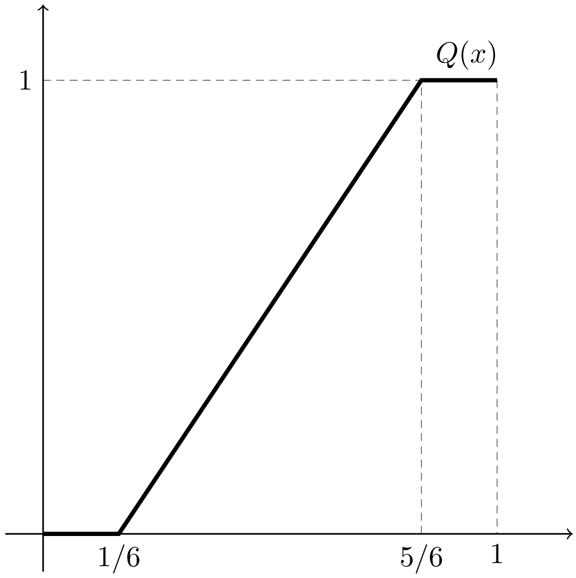

Figure 2: Quantifier associated to the weighting vector $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0)$.

The Shapley values can be obtained by using the *Kappalab R package*[45]:

$$\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 1\big) = \phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 2\big) = \phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 3\big) = 0.17\overline{42},$$
$$\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 4\big) = \phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 5\big) = 0.08\overline{3},$$
$$\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 6\big) = 0.31\overline{06}.$$

Notice that these values are relatively close to $2/11 = 0.\overline{18}$, $1/11 = 0.\overline{09}$, and $3/11 = 0.\overline{27}$. Hence, it could be considered that, in this example, the WOWA operator achieves the second objective. In any case, note that this WOWA operator overestimates the larger weight, which seems to be a common behavior in WOWA operators when the weighting vector $\boldsymbol{w}$ corresponds to an olympic OWA operator (see, for instance, Example 1 in Ref. 8, where $\boldsymbol{p} = (0.5, 0.2, 0.2, 0.1)$, $\boldsymbol{w} = (0, 0.5, 0.5, 0)$, and the Shapley values are $\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 1\big) = 0.\overline{6}$, $\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 2\big) = \phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 3\big) = 0.1\overline{3}$, and $\phi\big(W^Q_{\boldsymbol{p},\boldsymbol{w}}, 4\big) = 0.0\overline{6}$).

Let us see now the behavior of some SUOWA operators. As we have seen in Subsection 4.6, the semiuninorm $U^{\max}_{\min}$ allow us to obtain operators located between two order statistics; that is, since $\boldsymbol{w} = (0, 0.25, 0.25, 0.25, 0.25, 0)$ we have $\mathrm{OS}_2 \le O_{\boldsymbol{w}} \le \mathrm{OS}_5$ and, in accordance with the first item of Proposition 10, $\mathrm{OS}_2 \le S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}} \le \mathrm{OS}_5$ for any weighting vector $\boldsymbol{p}$. Therefore, if $\boldsymbol{x}$ is a vector of individual evaluations where all the scores except one are the same (see Table 3), then $\mathrm{OS}_2(\boldsymbol{x}) = \mathrm{OS}_5(\boldsymbol{x})$ and, consequently, $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \mathrm{OS}_2(\boldsymbol{x}) = \mathrm{OS}_5(\boldsymbol{x})$. Hence, the operator $S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}$ allows us to achieve the first goal.

The Shapley values have also been obtained by using the *Kappalab R package*:

$$\phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 1\Big) = \phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 2\Big) = \phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 3\Big) = 0.171\overline{96},$$
$$\phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 4\Big) = \phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 5\Big) = 0.14\overline{69},$$
$$\phi\Big(S^{U^{\max}_{\min}}_{\boldsymbol{p},\boldsymbol{w}}, 6\Big) = 0.190\overline{15}.$$

Notice that the values obtained for the last three members of the committee are not what would be desired. It is possible to get values closer to 1/11 and 3/11 by changing the weighting vector

$\boldsymbol{p}$. For instance, if we consider $\boldsymbol{p} = (0.15, 0.15, 0.15, 0, 0, 0.55)$, the Shapley values are

$$\begin{aligned}
&\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 1\Big) = \phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 2\Big) = \phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 3\Big) = 0.171\overline{6},\\
&\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 4\Big) = \phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 5\Big) = 0.10\overline{6},\\
&\phi\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}^{\max}}, 6\Big) = 0.271\overline{6},
\end{aligned}$$

which are relatively close to $2/11 = 0.\overline{18}$, $1/11 = 0.\overline{09}$, and $3/11 = 0.\overline{27}$. However, notice that we can not get the exact values because we do not know expressions that relate the weighting vector $\boldsymbol{p}$ to the Shapley values. This shortcoming can be solved by using a Winsorized weighted mean $M_{\boldsymbol{p}}^{(1,1)}$, which, as we have seen, is a specific case of SUOWA operators. Notice that in this case the weighting vector considered is $\boldsymbol{w} = (0, 2/6, 1/6, 1/6, 2/6, 0)$; that is, the OWA operator $O_{\boldsymbol{w}}$ is a Winsorized mean instead of a trimmed mean. Nevertheless, it also fulfills the first purpose which is to discard extreme values. Moreover, as in the previous case, we also obtain that $M_{\boldsymbol{p}}^{(1,1)}(\boldsymbol{x}) = \mathrm{OS}_2(\boldsymbol{x}) = \mathrm{OS}_5(\boldsymbol{x})$ for any vector $\boldsymbol{x}$ of individual evaluations where all the scores except one are the same. Hence, $M_{\boldsymbol{p}}^{(1,1)}$ allows us to get the first objective.

The Shapley values of $M_{\boldsymbol{p}}^{(1,1)}$ can be determined by using expression (2). These values are

$$\begin{aligned}
&\phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 1\Big) = \phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 2\Big) = \phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 3\Big) = \frac{29}{165} = 0.1\overline{75},\\
&\phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 4\Big) = \phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 5\Big) = \frac{4}{33} = 0.\overline{12},\\
&\phi\Big(M_{\boldsymbol{p}}^{(1,1)}, 6\Big) = \frac{38}{165} = 0.2\overline{30}.
\end{aligned}$$

But, in the case of the Winsorized weighted means, Proposition 12 allows us to get a weighting vector $\boldsymbol{p}$ so that the Shapley values of $M_{\boldsymbol{p}}^{(1,1)}$ are $(2/11, 2/11, 2/11, 1/11, 1/11, 3/11)$. This weighting vector is $\boldsymbol{p} = (19/99, 19/99, 19/99, 4/99, 4/99, 34/99)$. Hence, this Winsorized weighted mean allows us to achieve the objectives we pursued in this example.

7. CONCLUSION

SUOWA operators were introduced in the literature for dealing with situations where it is necessary to take into account both the importance of the information sources and the relative size of the values provided by the information sources. Given that they are Choquet integrals with respect to normalized capacities, they have some natural properties such as continuity, monotonicity, idempotency, compensativeness and homogeneity of degree 1. Besides these properties, several recently published papers have shown that SUOWA operators also exhibit other appealing properties. So, in this paper we have presented the main results obtained to date on SUOWA operators and we have introduced some new results. Moreover, we have illustrated the application of SUOWA operators in an example taken from Ref. 9. Of special interest are the Winsorized weighted means, since they allow us to discard extreme values at the time that each information source has the desired weight.

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