Closed-form expressions for some indices of SUOWA operators

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Abstract

Aggregation operators are an important tool in various scientific fields to determine the overall evaluation of an alternative from individual values. So, it seems relevant to know as many features as possible of the operators used in the aggregation processes. For this purpose, several indices have appeared in the literature, among which worth mentioning are the orness degree, the Shapley value, and the veto and favor indices. However, closed-form expressions of these indices are only known for few operators. The aim of this paper is to provide closed-form expressions of the previously said indices for some specific cases of SUOWA operators (which are a special case of Choquet integral), and to show the usefulness of these operators in a classical example given by Grabisch [M. Grabisch, Fuzzy integral in multicriteria decision making, Fuzzy Sets and Systems 69 (1995) 279–298].

Keywords: Choquet integral, SUOWA operators, orness degree, Shapley value, veto and favor indices.

1. Introduction

Aggregation operators are a powerful tool to aggregate values in several scientific fields. Among the large number of existing operators, Choquet integral plays an important role due to its versatility (see, for instance, Grabisch [9, 11], Grabisch and Roubens [17], Grabisch and Labreuche [14] and Yager [40]). In this regard, it is worth mentioning that Choquet integral generalizes two well-known families of operators, the weighted means and the ordered weighted averaging (OWA) operators (Yager [39]), which have been frequently used in the literature.

Weighted means and OWA operators are both defined through weighting vectors, but their role in the definition of both families of functions is very different. Weighted means allow weighting each element (for instance, criteria in a multicriteria decision making problem) in relation to their importance while OWA operators allow weighting the values in accordance with their relative position. The need of both weightings in several fields has been reported by some authors (see, for instance, Torra and Godo [37, pp. 160–161], Torra and Narukawa [38, pp. 150–151], Roy [32], Yager and Alajlan [41] and Section 6 of this paper). This fact has prompted the emergence of specific functions to deal with this class of problems.

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The approach generally followed in the literature is to consider functions parametrized by two weighting vectors, one for the weighted mean and the other one for the OWA type aggregation, so that the weighted mean (or the OWA operator) is recovered when the other weighting vector is $(1/n, \dots, 1/n)$ (see Llamazares [20] for a study of some of these families of functions). Weighted OWA (WOWA) operators, proposed by Torra [35], and the semi-uninorm based ordered weighted averaging (SUOWA) operators, introduced by Llamazares [21], are two of the most interesting solutions. This is due to the fact that WOWA and SUOWA operators can be represented by using the Choquet integral with respect to normalized capacities. So, they are continuous, monotonic, idempotent, compensative and homogeneous of degree 1 functions.

In addition to knowing that WOWA and SUOWA operators satisfy the above properties, it is also interesting to know the behavior of these functions with respect to other characteristics. A first study has been carried out by Llamazares [24] regarding some simple cases of weighting vectors, the capacities from which they are built, the weights affecting the components of each vector, and the values they return. Another approach reported in the literature is to provide additional information about operators by means of several indices: orness and andness degrees, importance and interaction indices, tolerance indices, dispersion indices, etc. Some relevant indices are the following:

1. The orness, which measures the degree to which the aggregation is disjunctive.
2. The Shapley value, which expresses the global importance of each criterion.
3. The veto and favor indices, which measure the degree to which a criterion behaves like a veto (a criterion that bounds the overall score from above) or a favor (a criterion that bounds the overall score from below).

These indices provide useful information about the behavior of aggregation functions but closed-form expressions of them are only known for few operators. The aim of this paper is to provide closed-form expressions of the previously said indices for some specific cases of SUOWA operators. In this regard, some results given in this paper are only valid when the weighting vector of the OWA type aggregation satisfies a certain condition. However, it is worth noting that some useful weighting vectors satisfy that condition. For instance, those ones with nondecreasing weights, which allow to characterize the Schur-concavity of OWA operators (see Bortot and Marques Pereira [3][1]). Likewise, nondecreasing weights also appear when the weights form a (nondecreasing) arithmetic progression, as in 2-additive symmetric normalized capacities (see, for instance, Beliakov et al. [1, p. 86], and Bortot and Marques Pereira [2]) and in some models proposed in the literature to determine the OWA weighting vector (see, for instance, Liu [19]).

Another interesting issue addressed in this paper concerns a classical example given by Grabisch [9], where the usefulness of Choquet integral to model interaction among criteria is shown. We consider his example under the perspective of SUOWA operators and illustrate how we can apply the results obtained in this paper to achieve appropriate operators.

[1]Notice that in Proposition 2 in Bortot and Marques Pereira [3] nondecreasing weights allow to characterize Schur-convexity instead of Schur-concavity because in the definition of OWA operators these authors consider that the components of $\boldsymbol{x}$ are ordered in a nondecreasing way.

The remainder of the paper is organized as follows. In Section 2 we collect some well-known properties of aggregation functions. Likewise, we recall the notions of semi-uninorms and uninorms and give some examples of such functions. In Section 3 we do a brief survey of Choquet integral, embracing the weighted means, OWA operators, and SUOWA operators. In Section 4 we present some usual indices in the context of Choquet integral. Section 5 collects the main results of the paper. In Section 6 we show the usefulness of SUOWA operators in a classical example given by Grabisch [9]. Finally, some concluding remarks are provided in Section 7.

2. Preliminaries

Let $\mathcal{A}$ be a finite set of alternatives and let $N = \{1, \dots, n\}$ be a finite set of criteria in a multicriteria decision making problem. Each alternative $a \in \mathcal{A}$ is associated with a vector $\boldsymbol{x}^a = (x_1^a, \dots, x_n^a) \in \mathbb{R}^n$, where x_i^a represents the score of a with respect to the criterion i. In many cases, the global score of each alternative is obtained through aggregation operators which take into account the importance of the criteria. In this context, the following notation will be used throughout the paper: given $A \subseteq N$, $|A|$ denotes the cardinality of A; vectors are denoted in bold; $\boldsymbol{\eta}$ denotes the tuple $(1/n, \dots, 1/n) \in \mathbb{R}^n$. We write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $i \in N$. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $[\cdot]$ and $(\cdot)$ denote permutations such that $x_{[1]} \geq \cdots \geq x_{[n]}$ and $x_{(1)} \leq \cdots \leq x_{(n)}$.

Some interesting properties of functions ($F : \mathbb{R}^n \longrightarrow \mathbb{R}$) are given next.

1. Symmetry: $F(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = F(x_1, \dots, x_n)$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all permutation σ of N.
2. Monotonicity: $\boldsymbol{x} \geq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \geq F(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.
3. Idempotency: $F(x, \dots, x) = x$ for all $x \in \mathbb{R}$.
4. Compensativeness (or internality): $\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
5. Homogeneity of degree 1 (or ratio scale invariance): $F(r\boldsymbol{x}) = rF(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and for all $r > 0$.

SUOWA operators are defined by means of semi-uninorms (see Liu [18]), which are monotonic functions with a neutral element in the interval $[0, 1]$. They were suggested as a generalization of uninorms by dispensing with the symmetry and associativity properties. In turn, uninorms were introduced by Yager and Rybalov [42] as a generalization of t-norms and t-conorms.

Definition 1. Let $U : [0, 1]^2 \longrightarrow [0, 1]$.

1. U is a semi-uninorm if it is monotonic and possesses a neutral element $e \in [0, 1]$ ($U(e, x) = U(x, e) = x$ for all $x \in [0, 1]$).
2. U is a uninorm if it is a symmetric and associative ($U(x, U(y, z)) = U(U(x, y), z)$ for all $x, y, z \in [0, 1]$) semi-uninorm.

We denote by $\mathcal{U}^e$ (respectively, $\mathcal{U}_{\mathrm{i}}^e$) the set of semi-uninorms (respectively, idempotent semi-uninorms) with neutral element $e \in [0, 1]$. The structure of semi-uninorms and idempotent semi-uninorms has been studied by Liu [18] (see a graphic representation in Llamazares [21]).

The semi-uninorms employed in the definition of SUOWA operators have to meet two requirements: the neutral element has to be $1/n$ and they have to belong to the following subset (see Llamazares [21]):

$$\widetilde{\mathcal{U}}^{1/n} = \left\{ U \in \mathcal{U}^{1/n} \mid U(1/k, 1/k) \leq 1/k \text{ for all } k \in N \right\}.$$

Obviously $\mathcal{U}_{\mathrm{i}}^{1/n} \subseteq \widetilde{\mathcal{U}}^{1/n}$. Moreover, it is easy to check that the smallest and the largest elements of $\widetilde{\mathcal{U}}^{1/n}$ are, respectively, the following semi-uninorms:

$$U_{\perp}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ 0 & \text{if } (x,y) \in [0, 1/n)^2, \\ \min(x,y) & \text{otherwise}, \end{cases}$$

and

$$U_{\top}(x,y) = \begin{cases} 1/k & \text{if } (x,y) \in I_k \backslash I_{k+1}, \text{where} \\ & I_k = (1/n, 1/k]^2 \;\; (k \in N \backslash \{n\}), \\ \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise}. \end{cases}$$

In the case of idempotent semi-uninorms, the smallest and the largest elements of $\mathcal{U}_{\mathrm{i}}^{1/n}$ are, respectively, the following uninorms (which were given by Yager and Rybalov [42]):

$$U_{\min}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ \min(x,y) & \text{otherwise}, \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0, 1/n]^2, \\ \max(x,y) & \text{otherwise}. \end{cases}$$

In addition to the previous ones, several procedures to construct semi-uninorms have been introduced by Llamazares [25]. One of them, which is based on ordinal sums of aggregation operators, allows us to get continuous semi-uninorms. Two of the most relevant continuous semi-uninorms obtained are the following:

$$U_{T_{\mathrm{L}}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ \max(x+y-1/n, 0) & \text{otherwise}, \end{cases}$$

and

$$U_{\widetilde{P}}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [1/n, 1]^2, \\ nxy & \text{otherwise}. \end{cases}$$

Notice that by the continuity of $U_{T_{\mathrm{L}}}$ and $U_{\widetilde{P}}$ these semi-uninorms can be also written as[2]

$$U_{T_{\mathrm{L}}}(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in(1/n,1]^2,\\ \max(x+y-1/n,0) & \text{otherwise,}\end{cases}$$

and

$$U_{\widetilde{P}}(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in(1/n,1]^2,\\ nxy & \text{otherwise.}\end{cases} \tag{1}$$

The plots of all these semi-uninorms for the case $n=4$ can be found in Llamazares [23].

3. Choquet integral

The Choquet integral was introduced in 1953 by Choquet [4], and due to its simplicity and versatility, it has had since then a wide variety of applications (see, for instance, Grabisch et al. [16] and Grabisch and Labreuche [14]). Choquet integral is based on the concept of capacity (see Choquet [4]), which was also introduced as fuzzy measure by Sugeno [34]. The notion of capacity resembles that of probability measure but in the definition of the former additivity is replaced by monotonicity. And a game is a generalization of a capacity where the monotonicity is ruled out.

Definition 2.

1. A game υ on N is a set function, $\upsilon : 2^N \longrightarrow \mathbb{R}$ satisfying $\upsilon(\varnothing)=0$.
2. A capacity (or fuzzy measure) μ on N is a game on N satisfying $\mu(A)\le\mu(B)$ whenever $A\subseteq B$. In particular, it follows that $\mu : 2^N \longrightarrow [0,\infty)$. A capacity μ is said to be normalized if $\mu(N)=1$.

Given a game, the monotonic cover is the smallest capacity that contains it (see Maschler and Peleg [29] and Maschler et al. [30]).

Definition 3. Let υ be a game on N. The monotonic cover of υ is the set function $\hat{\upsilon}$ given by

$$\hat{\upsilon}(A)=\max_{B\subseteq A}\upsilon(B).$$

The monotonic cover of a game satisfies the properties given next.

Remark 1. Let υ be a game on N. Then:

1. $\hat{\upsilon}$ is a capacity.
2. If υ is a capacity, then $\hat{\upsilon}=\upsilon$.

[2] These expressions of $U_{T_{\mathrm{L}}}$ and $U_{\widetilde{P}}$ will be used for obtaining the results shown in Subsections 5.2 and 5.3.

3. If $\upsilon(A) \leq 1$ for all $A \subseteq N$ and $\upsilon(N) = 1$, then $\hat{\upsilon}$ is a normalized capacity.

Although the Choquet integral is usually defined as a functional (see, for example, Choquet [4], Murofushi and Sugeno [31] and Denneberg [5]), in the discrete case, and once the capacity has been chosen, it can be seen as an aggregation function over $\mathbb{R}^n$ (see, for instance, Grabisch et al. [15, p. 181]). This is the approach taken in this paper. Moreover, by analogy with the original definition of OWA operators, we represent it by using nonincreasing sequences of values (see Torra [36] and Llamazares [21]).

Definition 4. Let μ be a capacity on N. The Choquet integral with respect to μ is the function $C_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \mu(A_{[i]})(x_{[i]} - x_{[i+1]}), \tag{2}$$

where $A_{[i]} = \{[1], \dots, [i]\}$, and we use the convention $x_{[n+1]} = 0$.

It is immediate to express the Choquet integral as follows:

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\mu(A_{[i]}) - \mu(A_{[i-1]})\right) x_{[i]}, \tag{3}$$

where the weights of the components $x_{[i]}$ are explicitly shown (we use the convention $A_{[0]} = \varnothing$). It is also worth noting that the Choquet integral has very interesting properties (see, for instance, Grabisch et al. [15, pp. 192–196]).

Remark 2. Let μ be a capacity on N. Then C_μ is continuous, monotonic and homogeneous of degree 1. Moreover, it is idempotent and compensative when μ is a normalized capacity.

Remark 3. Let μ_1 and μ_2 be two capacities on N. Then $\mu_1 \leq \mu_2$ if and only if $C_{\mu_1} \leq C_{\mu_2}$.

In the following subsections we recollect the most important particular cases of Choquet integral: weighted means, OWA operators and SUOWA operators.

3.1. Weighted means and OWA operators

Weighted means and OWA operators (introduced by Yager [39]) are both defined in terms of weight distributions that add up to 1.

Definition 5. A vector $\boldsymbol{q} \in [0, 1]^n$ is a weighting vector if $\sum_{i=1}^{n} q_i = 1$.

Definition 6. Let $\boldsymbol{p}$ be a weighting vector. The weighted mean associated with $\boldsymbol{p}$ is the function $M_{\boldsymbol{p}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$M_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

Definition 7. Let $\boldsymbol{w}$ be a weighting vector. The OWA operator associated with $\boldsymbol{w}$ is the function $O_{\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$O_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

It is well known that weighted means and OWA operators are Choquet integrals with respect to normalized capacities (see, for instance, Fodor et al. [8], Grabisch [9, 10] or Llamazares [21]).

Remark 4.

1. If $\boldsymbol{p}$ is a weighting vector, then the weighted mean $M_{\boldsymbol{p}}$ is the Choquet integral with respect to the normalized capacity $\mu_{\boldsymbol{p}}(A) = \sum_{i\in A} p_i$.
2. If $\boldsymbol{w}$ is a weighting vector, then the OWA operator $O_{\boldsymbol{w}}$ is the Choquet integral with respect to the normalized capacity $\mu_{|\boldsymbol{w}|}(A) = \sum_{i=1}^{|A|} w_i$.

So, according to Remark 2, weighted means and OWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1. Moreover, OWA operators are also symmetric given that the values of the variables are previously ordered in a nonincreasing way.

3.2. SUOWA operators

SUOWA operators were introduced by Llamazares [21] as a useful tool for problems where both the importance of values and the importance of criteria have to be taken into account. They are a particular case of the Choquet integral where their capacities are the monotonic cover of certain games, which are constructed through semi-uninorms with neutral element $1/n$ and the values of the capacities associated with the weighted means and the OWA operators. Specifically, the definition of these games is given next.

Definition 8. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$.

1. The game associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U is the set function $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}} : 2^N \longrightarrow \mathbb{R}$ defined by
$$\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) = |A|\, U\left(\frac{\mu_{\boldsymbol{p}}(A)}{|A|}, \frac{\mu_{|\boldsymbol{w}|}(A)}{|A|}\right)$$
if $A \neq \varnothing$, and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(\varnothing) = 0$.
2. $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$, the monotonic cover of the game $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}$, will be called the capacity associated with $\boldsymbol{p}$, $\boldsymbol{w}$ and U.

Notice that $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(A) \leq 1$ for all $A \subseteq N$ and $\upsilon^U_{\boldsymbol{p},\boldsymbol{w}}(N) = 1$. Therefore, according to the third item of Remark 1, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is always a normalized capacity.

Definition 9. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors and let $U \in \widetilde{\mathcal{U}}^{1/n}$. The SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U is the function $S^U_{\boldsymbol{p},\boldsymbol{w}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by
$$S^U_{\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_{[i]},$$
where $s_i = \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i]}) - \hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}(A_{[i-1]})$ for all $i \in N$, $\hat{\upsilon}^U_{\boldsymbol{p},\boldsymbol{w}}$ is the capacity associated with $\boldsymbol{p}, \boldsymbol{w}$ and U, and $A_{[i]} = \{[1], \ldots, [i]\}$ (with the convention that $A_{[0]} = \varnothing$).

According to expression (2), the SUOWA operator associated with $\boldsymbol{p}, \boldsymbol{w}$ and U can also be written as

$$S_{\boldsymbol{p},\boldsymbol{w}}^{U}(\boldsymbol{x}) = \sum_{i=1}^{n} \hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}(A_{[i]})(x_{[i]} - x_{[i+1]}). \tag{4}$$

By the choice of $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ we have $S_{\boldsymbol{p},\boldsymbol{\eta}}^{U} = M_{\boldsymbol{p}}$ and $S_{\boldsymbol{\eta},\boldsymbol{w}}^{U} = O_{\boldsymbol{w}}$ for any $U \in \widetilde{\mathcal{U}}^{1/n}$. Moreover, by Remark 2 and given that $\hat{\upsilon}_{\boldsymbol{p},\boldsymbol{w}}^{U}$ is a normalized capacity, SUOWA operators are continuous, monotonic, idempotent, compensative and homogeneous of degree 1.

It is worthy of note that SUOWA operators preserve the order of the corresponding semi-uninorms. As an immediate consequence of this fact, we know the bounds of SUOWA operators when we consider generic semi-uninorms or idempotent semi-uninorms.

Proposition 1 (Llamazares [21])**.** *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors. Then the following holds:*

1. *If* $U_1, U_2 \in \widetilde{\mathcal{U}}^{1/n}$ *and* $U_1 \leq U_2$*, then* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_1} \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U_2}$*.*
2. *If* $U \in \widetilde{\mathcal{U}}^{1/n}$*, then* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\perp}} \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U} \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\top}}$*.*
3. *If* $U \in \mathcal{U}_{\mathrm{i}}^{1/n}$*, then* $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U} \leq S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$*.*

4. Indices for Choquet integrals

Aggregation operators are often used to calculate the overall evaluation of an alternative with respect to several criteria. So, it seems very interesting to know the behavior of the operator employed for that task. For this purpose, several indices have been suggested in the literature to provide information on various characteristics of operators: orness and andness degrees, importance and interaction indices, tolerance indices, dispersion indices, etc. In this section we focus on the following indices: orness degree, Shapley value, and veto and favor indices; making special emphasis on their representation in the case of Choquet integrals.

The notion of orness allows us to measure the degree to which the aggregation is like an *or* operation (i.e. disjunctive), and it can be seen as a measure of global tolerance of criteria. This concept was proposed by Dujmović [7] in the analysis of the root-mean-powers and, in an independent way, by Yager [39] in the study of OWA operators. Afterwards, and by employing the concept of average value, Marichal [26] suggested an orness measure for Choquet integrals.

Definition 10. Let μ be a normalized capacity on N.

1. The average value of C_μ is defined by
$$E(C_\mu) = \int_{[0,1]^n} C_\mu(\boldsymbol{x})\, d\boldsymbol{x}.$$
2. The orness degree of C_μ is defined by
$$\operatorname{orness}(C_\mu) = \frac{E(C_\mu) - E(\min)}{E(\max) - E(\min)}.$$

The orness of C_μ can be expressed in terms of μ.

Remark 5 (Marichal [27])**.** Let μ be a normalized capacity on N. Then

$$\mathrm{orness}(C_\mu) = \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\mu(T).$$

It is worthy of note that the degree of orness preserves the usual order between Choquet integrals (see Grabisch et al. [15, p. 354]).

Remark 6. Let μ_1 and μ_2 be two normalized capacities on N. If $C_{\mu_1} \leq C_{\mu_2}$ (which, by Remark 3, is equivalent to $\mu_1 \leq \mu_2$), then $\mathrm{orness}(C_{\mu_1}) \leq \mathrm{orness}(C_{\mu_2})$.

The Shapley value was introduced by Shapley [33] in the context of cooperative games as a solution to the problem of distributing the amount $\mu(N)$ among the players. It can be interpreted as a kind of average value of the contribution of element j alone in all coalitions.

Definition 11. Let $j \in N$ and let μ be a normalized capacity on N. The Shapley value of criterion j with respect to μ is defined by

$$\phi(\mu, j) = \sum_{T\subseteq N\setminus\{j\}}\frac{(n-t-1)!t!}{n!}\Big(\mu(T\cup\{j\}) - \mu(T)\Big).$$

Remark 7 (Marichal [28])**.** It is easy to check that the Shapley value of criterion j with respect to a normalized capacity μ can be written as

$$\phi(\mu, j) = \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\Big(\mu(T\cup\{j\}) - \mu(T)\Big).$$

The concepts of veto and favor were introduced in the context of social choice functions by Dubois and Koning [6] (where "favor" was called "dictator") and, afterwards, by Grabisch [12] in the field of multicriteria decision making.

Definition 12. Let $j \in N$ and let μ be a normalized capacity on N.

1. j is a veto for C_μ if $C_\mu(\boldsymbol{x}) \leq x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
2. j is a favor for C_μ if $C_\mu(\boldsymbol{x}) \geq x_j$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

So, when criterion j is a veto and the score of j is small, then the global score will be small too. Analogously, if criterion j is a favor and the score of j is large, then the global score will be large too.

Since veto and favor criteria are infrequent in practice, it seems relevant to define indices measuring the degree with which a criterion behaves like a veto or a favor.

Definition 13 (Marichal [27])**.** Let $j \in N$ and let μ be a normalized capacity on N. The veto and favor indices of criterion j with respect to μ are defined by

$$\mathrm{veto}(C_\mu, j) = 1 - \frac{1}{n-1} \sum_{T \subseteq N \setminus \{j\}} \frac{1}{\binom{n-1}{t}} \mu(T),$$
$$\mathrm{favor}(C_\mu, j) = \frac{1}{n-1} \sum_{T \subseteq N \setminus \{j\}} \frac{1}{\binom{n-1}{t}} \mu(T \cup \{j\}) - \frac{1}{n-1}.$$

The veto index, $\mathrm{veto}(C_\mu, j)$, is more or less the degree to which the decision maker demands that criterion j is satisfied. Analogously, the favor index, $\mathrm{favor}(C_\mu, j)$, is the degree to which the decision maker considers that a good score along criterion j is sufficient to be satisfied.

Remark 8 (Marichal [28])**.** It is easy to check that the veto and favor indices of criterion j with respect to a normalized capacity μ can be written as

$$\mathrm{veto}(C_\mu, j) = 1 - \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \mu(T),^{3}$$
$$\mathrm{favor}(C_\mu, j) = \frac{1}{n-1} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \mu(T \cup \{j\}) - \frac{1}{n-1}.$$

Likewise, it is also possible to establish a relationship among veto, favor and Shapley value of a criterion.

Remark 9 (Marichal [27])**.** Let $j \in N$ and let μ be a normalized capacity on N. Then,

$$\mathrm{veto}(C_\mu, j) + \mathrm{favor}(C_\mu, j) = 1 + \frac{n\phi(\mu, j) - 1}{n-1}.$$

5. Indices for SUOWA operators

The indices that we have seen in the previous section provide a useful information about Choquet integrals. However, closed-form expressions of these indices are only known for few operators (see, for instance, Marichal [27] and Grabisch et al. [15, pp. 353, 364, 375]). In Table 1 we gather the orness, the Shapley values, and the veto and favor indices for weighted means and OWA operators.[4]

The aim of this section is to show some interesting results about these indices in the case of SUOWA operators. For instance, as an immediate consequence of Proposition 1 and Remark 6 we get the following outcomes about the orness.

[3]Notice that the summation begins in $t = 1$ because when $t = 0$ we have $T = \varnothing$ and, therefore, $\mu(T) = 0$.

[4]Note that in this paper we consider the original definition of OWA operators given by Yager [39], where the components of $\boldsymbol{x}$ are ordered in a nonincreasing way. For this reason, the orness, and the veto and favor indices of OWA operators do not match with those shown by Marichal [27] and Grabisch et al. [15], where the components are ordered in a nondecreasing way.

Table 1: Some indices for weighted means and OWA operators.

Indice	$M_{\boldsymbol{p}}$	$O_{\boldsymbol{w}}$
$\text{orness}(C_\mu)$	$\frac{1}{2}$	$\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$
$\phi(\mu, j)$	p_j	$\frac{1}{n}$
$\text{veto}(C_\mu, j)$	$\frac{1}{2}+\frac{np_j-1}{2(n-1)}$	$\frac{1}{n-1}\sum_{i=1}^{n}(i-1)w_i$
$\text{favor}(C_\mu, j)$	$\frac{1}{2}+\frac{np_j-1}{2(n-1)}$	$\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$

Remark 10. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors. Then the following holds:

1. If $U_1, U_2 \in \widetilde{\mathcal{U}}^{1/n}$ and $U_1 \le U_2$, then
$$\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_1}) \le \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_2}).$$
2. If $U \in \widetilde{\mathcal{U}}^{1/n}$, then
$$\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_\perp}) \le \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U}) \le \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_\top}).$$
3. If $U \in \mathcal{U}_{\text{i}}^{1/n}$, then
$$\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}) \le \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U}) \le \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}).$$

In Propositions 3 and 4 we show some useful properties of the indices when we consider convex combination of semi-uninorms and the games associated with the semi-uninorms are normalized capacities. Then, the orness, the Shapley values, and the veto and favor indices of the SUOWA operator associated with the constructed semi-uninorm can be straightforward obtained by using the same convex combination of the indices of the SUOWA operators associated with the initial semi-uninorms. These outcomes are obtained from the following result.[5]

Proposition 2 (Llamazares [25])**.** *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_m}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{i=1}^{m} \lambda_i U_i$. Then, for any subset T of N,*
$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}(T) = \sum_{i=1}^{m} \lambda_i \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}(T),$$
and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}$ is a normalized capacity.

Proposition 3. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_m}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{i=1}^{m} \lambda_i U_i$. Then,*
$$\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U}\right) = \sum_{i=1}^{m} \lambda_i \,\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}\right).$$

[5]Notice that a similar result to that of Proposition 3 was given without proof by Llamazares [22] for the case of two semi-uninorms.

Proposition 4. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors, let $U_1, \dots, U_m \in \widetilde{\mathcal{U}}^{1/n}$ such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_1}, \dots, \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_m}$ be normalized capacities, let $\boldsymbol{\lambda}$ be a weighting vector, and let $U = \sum_{i=1}^{m} \lambda_i U_i$. Then, for each $j \in N$,*

$$\phi\big(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, j\big) = \sum_{i=1}^{m} \lambda_i \phi\big(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_i}, j\big),$$

$$\operatorname{veto}\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U}, j\big) = \sum_{i=1}^{m} \lambda_i \operatorname{veto}\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}, j\big),$$

$$\operatorname{favor}\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U}, j\big) = \sum_{i=1}^{m} \lambda_i \operatorname{favor}\big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_i}, j\big).$$

In the following subsections we give explicitly the orness, the Shapley values, and the veto and favor indices for some particular cases of SUOWA operators. More specifically, we analyze the operators $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$, and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ for some specific cases of $\boldsymbol{p}$ and $\boldsymbol{w}$. In the case of the weighting vector $\boldsymbol{w}$, the usual condition that we demand is $\sum_{i=1}^{j} w_i \leq j/n$ (or $\sum_{i=1}^{j} w_i < j/n$) for all $j \in N$. Notice that some useful weighting vectors satisfy the previous conditions. For instance, those ones with nondecreasing weights satisfy the condition $\sum_{i=1}^{j} w_i \leq j/n$.

Lemma 1 (Llamazares [25])**.** *Let $\boldsymbol{w}$ be a weighting vector such that $w_1 \leq w_2 \leq \cdots \leq w_n$. Then $\boldsymbol{w} = \boldsymbol{\eta}$ or $\sum_{i=1}^{j} w_i < j/n$ for all $j \in \{1, \dots, n-1\}$.*

The following statements on the orness of OWA operators will be used later.

Proposition 5. *Let $\boldsymbol{w}$ be a weighting vector.*

1. *If $\sum_{i=1}^{j} w_i < j/n$ for all $j \in N$, then* $\operatorname{orness}(O_{\boldsymbol{w}}) < 0.5$.
2. *If $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$, then* $\operatorname{orness}(O_{\boldsymbol{w}}) \leq 0.5$.
3. *If $\sum_{i=1}^{j} w_i > j/n$ for all $j \in N$, then* $\operatorname{orness}(O_{\boldsymbol{w}}) > 0.5$.
4. *If $\sum_{i=1}^{j} w_i \geq j/n$ for all $j \in N$, then* $\operatorname{orness}(O_{\boldsymbol{w}}) \geq 0.5$.

5.1. The uninorms $U_{\min}$ and $U_{\max}$

In the sequel we establish bounds for the orness of $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$. In the case of $U_{\min}$, the following proposition, where a bound for the SUOWA operator $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is given, will be useful.

Proposition 6 (Llamazares [25])**.** *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i < j/n$ for all $j \in N$. Then, for all weighting vector $\boldsymbol{p}$, we have*

1. *$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$ is a normalized capacity on N, and for any $T \subseteq N$ such that $|T| = t \geq 1$,*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(T) = \min\left(\sum_{i \in T} p_i, \sum_{i=1}^{t} w_i \right).$$

2. *For all $\boldsymbol{x} \in \mathbb{R}^n$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}(\boldsymbol{x}) \leq \min\left(M_{\boldsymbol{p}}(\boldsymbol{x}), O_{\boldsymbol{w}}(\boldsymbol{x})\right)$, and, consequently, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}} \leq O_{\boldsymbol{w}}$.*

As an immediate consequence of the second item of Proposition 6, Remark 6 and the first item of Proposition 5, we get the following result.

Corollary 1. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $\sum_{i=1}^{j} w_i < j/n$ *for all* $j \in N$*. Then, for all weighting vector* $\boldsymbol{p}$*, we have*

$$\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}) \leq \text{orness}(O_{\boldsymbol{w}}) < 0.5.$$

A similar result can be found for the uninorm $U_{\max}$ and weighting vectors $\boldsymbol{w}$ such that $\sum_{i=1}^{j} w_i > j/n$ for all $j \in N$.

Corollary 2. *Let* $\boldsymbol{w}$ *be a weighting vector such that* $\sum_{i=1}^{j} w_i > j/n$ *for all* $j \in N$*. Then, for all weighting vector* $\boldsymbol{p}$*, we have*

$$\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}) \geq \text{orness}(O_{\boldsymbol{w}}) > 0.5.$$

5.2. The semi-uninorm $U_{T_{\mathrm{L}}}$

We begin by showing the capacity associated with the semi-uninorm $U_{T_{\mathrm{L}}}$ when the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ fulfill some additional properties.

Proposition 7 (Llamazares [24])**.** *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors such that* $\sum_{i=1}^{j} w_i \leq j/n$ *for all* $j \in N$ *and* $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$*. Then, for any* $T \subseteq N$ *such that* $|T| = t \geq 1$*,*

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \sum_{i\in T} p_i + \sum_{i=1}^{t} w_i - \frac{t}{n},$$

and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$ *is a normalized capacity on* N*.*

The following propositions allow us to know the orness, the Shapley values and the veto and favor indices when we consider the semi-uninorm $U_{T_{\mathrm{L}}}$ and the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ satisfy the required properties.

Proposition 8. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors such that* $\sum_{i=1}^{j} w_i \leq j/n$ *for all* $j \in N$ *and* $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$*. Then,*

$$\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right) = \text{orness}(O_{\boldsymbol{w}}) \leq 0.5.$$

Proposition 9. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors such that* $\sum_{i=1}^{j} w_i \leq j/n$ *for all* $j \in N$ *and* $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$*. Then, for each* $j \in N$*,*

$$\begin{aligned}
\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}, j\right) &= p_j,\\
\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}, j\right) &= \frac{np_j - 1}{2(n-1)} + 1 - \text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right),\\
\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}, j\right) &= \frac{np_j - 1}{2(n-1)} + \text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right).
\end{aligned}$$

Notice that:

1. The orness of $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$ coincides with that of $O_{\boldsymbol{w}}$.

2. The Shapley values of criterion j with respect to the capacities $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$ and $\mu_{\boldsymbol{p}}$ are equal (see Table 1).
3. The expressions given for the veto and favor indices of criterion j with respect to the capacity $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$ are also valid when we consider the capacity $\mu_{\boldsymbol{p}}$ (see Table 1).

5.3. The semi-uninorm $U_{\widetilde{P}}$

We first show the capacity associated with the semi-uninorm $U_{\widetilde{P}}$ when the weighting vectors $\boldsymbol{p}$ and $\boldsymbol{w}$ fulfill some additional properties.

Remark 11. Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. Since $\left(\sum_{i=1}^{j} w_i\right)/j \leq 1/n$ for all $j \in N$, then, by expression (1), for any weighting vector $\boldsymbol{p}$ and any $T \subseteq N$ such that $|T| = t \geq 1$, we have

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = tU_{\widetilde{P}}\left(\frac{\sum\limits_{i\in T} p_i}{t}, \frac{\sum\limits_{i=1}^{t} w_i}{t}\right) = \frac{n}{t}\left(\sum_{i\in T} p_i\right)\left(\sum_{i=1}^{t} w_i\right).$$

Notice that, in general, the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is not a capacity. For instance, consider the weighting vectors $\boldsymbol{p} = (0.4, 0.2, 0.4)$ and $\boldsymbol{w} = (1/3, 0, 2/3)$. Then,

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(\{1\}) = 0.4 > 0.3 = \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(\{1, 2\}).$$

However, we can guarantee that the game $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a capacity when the weighting vector $\boldsymbol{w}$ is a nondecreasing sequence of weights.

Proposition 10. *Let $\boldsymbol{w}$ be a weighting vector such that $w_1 \leq w_2 \leq \cdots \leq w_n$. Then, for any weighting vector $\boldsymbol{p}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N.*

In the following propositions we show that for the studied weighting vectors, the orness of $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ coincides with that of $O_{\boldsymbol{w}}$, and we also give closed-form expressions for the Shapley values, and the veto and favor indices.

Proposition 11. *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $\boldsymbol{p}$ is a weighting vector such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N, then:*

$$\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right) = \text{orness}(O_{\boldsymbol{w}}) \leq 0.5.$$

Proposition 12. *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $\boldsymbol{p}$ is a weighting vector such that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ is a normalized capacity on N, then, for each $j \in N$,*

$$\begin{aligned}
\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right) &= \frac{1}{n-1}\left(1 - p_j + (np_j - 1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right),\\
\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right) &= 1 - \frac{n}{n-1}(1 - p_j)\,\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right),\\
\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right) &= 1 - \text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right) + \frac{n}{(n-1)^2}\\
&\quad\cdot\left(1 - p_j + (np_j - 1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right) - \frac{1}{n-1}.
\end{aligned}$$

We finish this section by collecting in Table 2 the orness, the Shapley values, and the veto and favor indices for the SUOWA operators $S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}$ and $S^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}$. Notice that these values are valid when the hypothesis of Propositions 8, 9, 11, and 12 are fulfilled.

Table 2: Some indices for $S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}$ and $S^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}$.

Indice	$S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}$	$S^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}$
orness(C_μ)	$\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$	$\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i$
$\phi(\mu, j)$	p_j	$\frac{1}{n-1}\left(1-p_j+(np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right)$
veto(C_μ, j)	$\frac{np_j-1}{2(n-1)}+1-\text{orness}\left(S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}\right)$	$1-\frac{n}{n-1}(1-p_j)\,\text{orness}\left(S^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}\right)$
favor(C_μ, j)	$\frac{np_j-1}{2(n-1)}+\text{orness}\left(S^{U_{T_{\mathbf{L}}}}_{\boldsymbol{p},\boldsymbol{w}}\right)$	$1-\text{veto}\left(S^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right)+\frac{1}{n-1}\left(n\phi\left(\upsilon^{U_{\overline{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right)-1\right)$

6. Discussion

In this section we are going to show the usefulness of SUOWA operators in a classical example given by Grabisch [9] (see also Grabisch [11] and Marichal [27]). Consider the problem of evaluating students in a high school with respect to three subjects: mathematics (M), physics (P), and literature (L). Usually, this is done by a simple weighted mean, whose weights are the coefficients of importance of the different subjects. Suppose that the school is more scientifically than literary oriented, so that weights could be, for example, proportional to 3, 3, and 2, respectively (that is, $\boldsymbol{p} = (3/8, 3/8, 2/8) = (0.375, 0.375, 0.25)$). Then the global evaluation given by the weighted mean $M_{\boldsymbol{p}}$ to three students A, B, and C are collected in the fifth column of Table 3 (marks are given on a scale from 0 to 20).

Table 3: Global evaluation by using $M_{\boldsymbol{p}}$ and C_μ.

Student	M	P	L	$M_{\boldsymbol{p}}$	C_μ
A	18	16	10	15.25	13.9
B	10	12	18	12.75	13.6
C	14	15	15	14.625	14.9

If the school wants to favor well equilibrated students without weak points then student C should be considered better than student A, who has a severe weakness in literature. However, as it has been pointed out by Marichal [27], no weight vector $\boldsymbol{p} = (p_1, p_2, p_3)$ satisfying $p_1 = p_2 > p_3$ is able to favor student C.

To solve this problem, Grabisch [9] introduces the Choquet integral for aggregating the marks. A simple calculation allows us to see that the order C $\succ$ A is obtained when the normalized capacity μ satisfies

$$\mu(\{P, L\}) > 2\,\mu(\{M\}) + 6\,\mu(\{M, P\}) - 4.$$

The normalized capacity considered by Grabisch [9] is defined by:

$$\mu(\{M\}) = 0.45, \qquad \mu(\{P\}) = 0.45, \qquad \mu(\{L\}) = 0.3,$$
$$\mu(\{M, P\}) = 0.5, \qquad \mu(\{M, L\}) = 0.9, \qquad \mu(\{P, L\}) = 0.9.$$

According to Grabisch [11], this capacity allows to keep unchanged the initial ratio of weights $(3, 3, 2)$ (given that $0.45/0.3 = 3/2$), and avoids some overlap effect between mathematics and physics (since, usually, students good at mathematics are also good at physics). By using this capacity, the score obtained by student C is larger than that obtained by student A (the global evaluation given by C_μ to the three students A, B, and C is collected in the sixth column of Table 3).

Although this capacity allows us to get the desired student, it does not satisfy some of the initial purposes:

1. The orness of C_μ is $3.5/6 = 0.58\overline{3}$ (see Marichal [27]). However, if the school wants to favor well equilibrated students, the students' score should be calculated through an operator closer to the minimum than to the maximum; that is, its orness should be smaller than 0.5.
2. The Shapley value of L is larger than the Shapley value of M and P: $\phi(\mu, L) = 0.41\overline{6}$, and $\phi(\mu, M) = \phi(\mu, P) = 0.291\overline{6}$ (see Grabisch [11] and Marichal [27]). Given that the Shapley value reflects the overall importance of each subject, the Shapley value of M and P should be larger than that of L. In fact, it seems that the appropriate Shapley values of M, P and L should be 0.375, 0.375, and 0.25, respectively (notice that these ones are the Shapley values of the capacity associated with the weighted mean $M_{\boldsymbol{p}}$ when $\boldsymbol{p} = (0.375, 0.375, 0.25)$).

In order to favor well equilibrated students, one possibility would be to use the minimum or an OWA operator close to it; that is, whose orness be smaller than 0.5. But, in this case, the Shapley value of M, P, and L is equal to $1/3$ (see Table 1) and the initial purpose of giving more important to M and P than L is not satisfied.

An alternative way of tackling this problem is by using SUOWA operators. In this way we can combine the fact that the school is more scientifically than literary oriented (through the weighting vector $\boldsymbol{p}$) with the idea of favoring well equilibrated students (by means of the weighting vector $\boldsymbol{w}$). So, given that the initial ratio of weights is $(3, 3, 2)$, we can consider $\boldsymbol{p} = (0.375, 0.375, 0.25)$. On the other hand, given that the school wants to favor well equilibrated students, we can take, for instance, $\boldsymbol{w} = (0.2, 0.3, 0.5)$.

In Table 4 we show the capacities associated with the analyzed semi-uninorms: $U_{\min}$, $U_{\max}$, $U_{T_{\mathbf{L}}}$ and $U_{\widetilde{P}}$. It is worth noting that the games considered in this example, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$, and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$, are in fact capacities.

Notice also that the capacity $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ keeps the initial ratio $(3, 3, 2)$ among the values $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(\{M\})$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(\{P\})$ and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(\{L\})$ (given that $0.225/0.15 = 3/2$).

Table 4: Capacities associated to $U_{\min}$, $U_{\max}$, $U_{T_{\mathrm{L}}}$ and $U_{\widetilde{P}}$.

Set	$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$	$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$
{M}	0.2	0.375	$0.241\overline{6}$	0.225
{P}	0.2	0.375	$0.241\overline{6}$	0.225
{L}	0.2	0.2	$0.11\overline{6}$	0.15
{M, P}	0.5	0.75	$0.58\overline{3}$	0.5625
{M, L}	0.5	0.5	$0.458\overline{3}$	0.46875
{P, L}	0.5	0.5	$0.458\overline{3}$	0.46875
{M, P, L}	1	1	1	1

Table 5: Global evaluation by using $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$, and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$.

Student	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$	$S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$
A	13.4	15.25	$13.98\overline{3}$	13.825
B	12.2	12.2	$11.61\overline{6}$	11.8375
C	14.5	14.5	$14.458\overline{3}$	14.46875

The global evaluation given by the SUOWA operators $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$, $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$, and $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ to the students A, B and C is collected in Table 5. Note that we get the desired order C $\succ$ A in all cases except when the operator is $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$.

In relation to the orness of the analyzed operators, in the four cases is less than 0.5. In fact, we get

$$\begin{aligned}
&\operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}\right) = \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right) = \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right) = 0.35,\\
&\operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}\right) = 0.45.
\end{aligned}$$

Note that the orness of operators whose value is 0.35 can be easily obtained taking into account that $\operatorname{orness}(O_{\boldsymbol{w}}) = 0.35$, the fifth item of Proposition 8 in Llamazares [25], and Propositions 8 and 11 in this paper.

With respect to the Shapley values, the condition

$$\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \mathrm{M}\right) = \phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \mathrm{P}\right) \geq \phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \mathrm{L}\right)$$

is obtained in the four cases analyzed in this paper (the Shapley values, and the veto and favor indices associated with $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$, $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}$, and $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$ are collected in Tables 6–9).

Table 6: Shapley values, and veto and favor indices associated to $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}$.

	M	P	L
$\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}, j\right)$	$0.\overline{3}$	$0.\overline{3}$	$0.\overline{3}$
$\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}, j\right)$	0.65	0.65	0.65
$\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}, j\right)$	0.35	0.35	0.35

Table 7: Shapley values, and veto and favor indices associated to $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}$.

	M	P	L
$\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}, j\right)$	$0.4041\overline{6}$	$0.4041\overline{6}$	$0.191\overline{6}$
$\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}, j\right)$	0.60625	0.60625	0.4375
$\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}, j\right)$	0.5	0.5	0.35

Table 8: Shapley values, and veto and favor indices associated to $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$.

	M	P	L
$\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right)$	0.375	0.375	0.25
$\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right)$	0.68125	0.68125	0.5875
$\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right)$	0.38125	0.38125	0.2875

Notice that in the case $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$ the Shapley values are $\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \text{M}\right) = \phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \text{P}\right) = 0.375$ and $\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \text{L}\right) = 0.25$, which seem the most suitable values as we have said. Moreover, given that $\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) < 0.5$ and this operator provides the order C $\succ$ A, it seems that $S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}$ is a good choice for aggregating the marks of the students.

To finish this section, we are going to show how to obtain a SUOWA operator with a specific degree of orness (or a specific Shapley value of a criterion). Notice that, by the third item of Remark 10, for any idempotent semi-uninorm U we get

$$0.35 \leq \text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U}) \leq 0.45.$$

Suppose now that we seek an idempotent semi-uninorm U with $\text{orness}\,(S_{\boldsymbol{p},\boldsymbol{w}}^{U}) = 0.4$. By Proposition 3, and given

Table 9: Shapley values, and veto and favor indices associated to $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$.

	M	P	L
$\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right)$	$0.361458\overline{3}$	$0.361458\overline{3}$	$0.27708\overline{3}$
$\text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right)$	0.671875	0.671875	0.60625
$\text{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right)$	0.3703125	0.3703125	0.309375

that $0.4 = 0.5 \cdot 0.35 + 0.5 \cdot 0.45$, it is sufficient to consider $U_{\text{am}} = 0.5\, U_{\min} + 0.5\, U_{\max}$, that is:

$$U_{\text{am}}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in \left[0, 0.\overline{3}\right]^2, \\ \max(x,y) & \text{if } (x,y) \in \left[0.\overline{3}, 1\right]^2 \setminus \left\{\left(0.\overline{3}, 0.\overline{3}\right)\right\}, \\ (x+y)/2 & \text{otherwise.} \end{cases}$$

Analogously, suppose we seek an idempotent semi-uninorm U with $\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U}, \text{L}\right) = 0.25$. Given that $\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\min}}, \text{L}\right) = 0.\overline{3}$ and $\phi\left(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\max}}, \text{L}\right) = 0.191\overline{6}$, and $0.25 = 7/17 \cdot 0.\overline{3} + 10/17 \cdot 0.191\overline{6}$, then, by Proposition 4, the idempotent semi-uninorm $U = 7/17\, U_{\min} + 10/17\, U_{\max}$ fulfills the requirement.

7. Concluding remarks

The application of the discrete Choquet integral in multicriteria decision making has been proposed by several authors throughout the last years (see, for instance, Grabisch [9, 11], Grabisch and Roubens [17], Grabisch and Labreuche [14], and the references therein). This is due mainly to the discrete Choquet integral allows to take into account the interaction that often exists among the criteria (the classical example proposed by Grabisch [9] and reproduced in Section 6 highlights this fact). However, the use of the discrete Choquet integral requires the initial choice of a capacity, and this choice is not always obvious (see Grabisch et al. [13]). In some contexts (for instance, in the example of Section 6) it is possible to model the problem using a "mixture" of weighted means and OWA operators: the weighted mean type aggregation allows to take into account the importance of each criterion whereas the OWA type aggregation allows to reflect the attitudinal character of the decision maker in the decision process.

Knowing the behavior of functions is essential for choosing an appropriate operator, and several indices have been suggested in the literature for this purpose. Among them, the orness degree and the Shapley values are crucial to know the degree to which the aggregation is disjunctive and the global importance of each criterion. Nevertheless, closed-form expressions of these indices are only know for few operators (basically, weighted means and OWA operators).

In this paper we have provided closed-form expressions of the orness degree, the Shapley values, and the veto and favor indices for some specific cases of SUOWA operators, which allow us to deal with decision problems where both weighted mean and OWA type aggregations are necessary. A natural line of research is to extend the results obtained in Section 5 to other SUOWA operators and/or indices proposed in the literature.

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Appendix A. Proofs

Proof of Proposition 3. According to Remark 5 and Proposition 2, we have

$$
\begin{aligned}
\operatorname{orness}\left(S^{U}_{\boldsymbol{p},\boldsymbol{w}}\right) &= \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\left(\sum_{i=1}^{m}\lambda_i \upsilon^{U_i}_{\boldsymbol{p},\boldsymbol{w}}(T)\right)\\
&= \sum_{i=1}^{m}\lambda_i\left(\frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\|T|=t}}\upsilon^{U_i}_{\boldsymbol{p},\boldsymbol{w}}(T)\right)\\
&= \sum_{i=1}^{m}\lambda_i \operatorname{orness}\left(S^{U_i}_{\boldsymbol{p},\boldsymbol{w}}\right). \qquad \square
\end{aligned}
$$

Proof of Proposition 4. The proofs are similar to that of Proposition 3 (taking into account that $\sum_{i=1}^{m}\lambda_i = 1$) and, therefore, they are omitted here. □

Proof of Proposition 5. We only prove the first statement, since the proofs of the remaining statements are similar. Given a weighting vector $\boldsymbol{w}$ such that $\sum_{i=1}^{j} w_i < j/n$ for all $j \in N$, we have

$$
\begin{aligned}
\operatorname{orness}(O_{\boldsymbol{w}}) &= \frac{1}{n-1}\sum_{i=1}^{n-1}(n-i)w_i = \frac{1}{n-1}\sum_{j=1}^{n-1}\sum_{i=1}^{j} w_i\\
&< \frac{1}{n-1}\sum_{j=1}^{n-1}\frac{j}{n} = 0.5. \qquad \square
\end{aligned}
$$

In some of the remaining proofs we will use the following remarks.

Remark 12. Let $\boldsymbol{p}$ be a weighting vector. If $t \geq 1$, then

$$
\sum_{\substack{T\subseteq N\\|T|=t}}\sum_{i\in T} p_i = \binom{n-1}{t-1}\sum_{i=1}^{n} p_i = \binom{n-1}{t-1} = \binom{n}{t}\frac{t}{n},
$$

and, for any $j \in N$,

$$\sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \sum_{i\in T} p_i = \binom{n-2}{t-1} \sum_{\substack{i=1\\ i\neq j}}^{n} p_i = \binom{n-2}{t-1}(1-p_j)$$

$$= \binom{n-1}{t} \frac{t(1-p_j)}{n-1}.$$

Remark 13. Let $\boldsymbol{w}$ be a weighting vector. Then

$$\frac{1}{n-1} \sum_{t=1}^{n-1} \sum_{i=1}^{t} w_i = \frac{1}{n-1} \sum_{i=1}^{n-1} (n-i) w_i = \text{orness}(O_{\boldsymbol{w}}).$$

Before we give the proof of Proposition 8, we previously establish the following lemma.

Lemma 2. *Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$ and $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$. If $t \geq 1$, then*

$$\sum_{\substack{T\subseteq N\\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \binom{n}{t} \sum_{i=1}^{t} w_i,$$

and, for any $j \in N$,

$$\sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \binom{n-1}{t} \left(\left(\frac{1-p_j}{n-1} - \frac{1}{n} \right) t + \sum_{i=1}^{t} w_i \right).$$

Proof. By Proposition 7 and Remark 12, we have

$$\sum_{\substack{T\subseteq N\\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \sum_{\substack{T\subseteq N\\ |T|=t}} \sum_{i\in T} p_i + \sum_{\substack{T\subseteq N\\ |T|=t}} \sum_{i=1}^{t} w_i - \sum_{\substack{T\subseteq N\\ |T|=t}} \frac{t}{n}$$

$$= \binom{n}{t} \frac{t}{n} + \binom{n}{t} \sum_{i=1}^{t} w_i - \binom{n}{t} \frac{t}{n} = \binom{n}{t} \sum_{i=1}^{t} w_i,$$

and

$$\sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \sum_{i\in T} p_i + \sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \sum_{i=1}^{t} w_i - \sum_{\substack{T\subseteq N\setminus\{j\}\\ |T|=t}} \frac{t}{n}$$

$$= \binom{n-1}{t} \left(\left(\frac{1-p_j}{n-1} - \frac{1}{n} \right) t + \sum_{i=1}^{t} w_i \right).$$

□

Proof of Proposition 8. By Remarks 5 and 13, and Lemma 2 we have

$$\text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}\right) = \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T\subseteq N\\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathrm{L}}}}(T) = \frac{1}{n-1} \sum_{t=1}^{n-1} \sum_{i=1}^{t} w_i$$

$$= \text{orness}(O_{\boldsymbol{w}}) \leq 0.5,$$

where the last inequality is obtained by the second item of Proposition 5. □

In some of the remaining proofs we will make use of the following remark.

Remark 14. Let $\boldsymbol{p}$ and $\boldsymbol{w}$ be two weighting vectors such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$ and $\min_{i\in N} p_i + \min_{i\in N} w_i \geq 1/n$. By Proposition 7, for any $j \in N$ and $T \subseteq N \setminus \{j\}$ with $|T| = t$ we get

$$\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(T \cup \{j\}) - \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(T) = p_j + w_{t+1} - \frac{1}{n}.$$

Proof of Proposition 9. Let $j \in N$. By Remarks 7 and 14 we have

$$\begin{aligned}\phi(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j) &= \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\left(p_j + w_{t+1} - \frac{1}{n}\right)\\ &= \frac{1}{n}\sum_{t=0}^{n-1}\left(p_j + w_{t+1} - \frac{1}{n}\right) = p_j.\end{aligned}$$

By Remarks 8 and 13, Lemma 2, and Proposition 8 we have

$$\begin{aligned}\operatorname{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) &= 1 - \frac{1}{n-1}\sum_{t=1}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}}\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}(T)\\ &= 1 - \frac{1}{n-1}\left(\left(\frac{1-p_j}{n-1} - \frac{1}{n}\right)\sum_{t=1}^{n-1} t + \sum_{t=1}^{n-1}\sum_{i=1}^{t} w_i\right)\\ &= 1 - \frac{1-np_j}{2(n-1)} - \operatorname{orness}(O_{\boldsymbol{w}})\\ &= \frac{np_j - 1}{2(n-1)} + 1 - \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right).\end{aligned}$$

Lastly, by Remark 9 we get

$$\begin{aligned}\operatorname{favor}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) &= 1 - \operatorname{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j\right) + \frac{n\phi(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}, j) - 1}{n-1}\\ &= -\frac{np_j - 1}{2(n-1)} + \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right) + \frac{np_j - 1}{n-1}\\ &= \frac{np_j - 1}{2(n-1)} + \operatorname{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{T_{\mathbf{L}}}}\right).\end{aligned}$$

□

Proof of Proposition 10. Let us see the monotonicity of $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}$. For this, it is sufficient to show that $\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) \leq \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T \cup \{j\})$ for any $T \subsetneq N$ such that $|T| = t \geq 1$, and $j \in N \setminus T$. By Lemma 1 and Remark 11 we have

$$\begin{aligned}&\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) \leq \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T \cup \{j\})\\ &\Leftrightarrow \frac{n}{t}\left(\sum_{i\in T} p_i\right)\left(\sum_{i=1}^{t} w_i\right) \leq \frac{n}{t+1}\left(\sum_{i\in T} p_i + p_j\right)\left(\sum_{i=1}^{t} w_i + w_{t+1}\right)\\ &\Leftrightarrow (t+1)\left(\sum_{i\in T} p_i\right)\left(\sum_{i=1}^{t} w_i\right) \leq t\left(\sum_{i\in T} p_i + p_j\right)\left(\sum_{i=1}^{t} w_i + w_{t+1}\right)\\ &\Leftrightarrow \left(\sum_{i\in T} p_i\right)\left(\sum_{i=1}^{t} w_i\right) \leq t w_{t+1}\sum_{i\in T} p_i + t p_j \sum_{i=1}^{t} w_i + t p_j w_{t+1},\end{aligned}$$

which is true given that, by hypothesis, $w_1 \leq w_2 \leq \cdots \leq w_n$ and, consequently, $\sum_{i=1}^{t} w_i \leq t w_{t+1}$. □

The following lemma will be used in some of the remaining proofs.

Lemma 3. *Let $\boldsymbol{w}$ be a weighting vector such that $\sum_{i=1}^{j} w_i \leq j/n$ for all $j \in N$. If $t \geq 1$, then,*

$$\sum_{\substack{T \subseteq N \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \binom{n}{t} \sum_{i=1}^{t} w_i,$$

and, for any $j \in N$,

$$\sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \frac{n}{n-1}(1-p_j)\binom{n-1}{t} \sum_{i=1}^{t} w_i.$$

Proof. By Remarks 11 and 12, we have

$$\sum_{\substack{T \subseteq N \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \frac{n}{t}\left(\sum_{i=1}^{t} w_i\right) \sum_{\substack{T \subseteq N \\ |T|=t}} \sum_{i \in T} p_i = \binom{n}{t} \sum_{i=1}^{t} w_i.$$

and

$$\begin{aligned} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) &= \frac{n}{t}\left(\sum_{i=1}^{t} w_i\right) \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \sum_{i \in T} p_i \\ &= \frac{n}{n-1}(1-p_j)\binom{n-1}{t} \sum_{i=1}^{t} w_i. \end{aligned}$$

□

Proof of Proposition 11. By Remarks 5 and 13, and Lemma 3 we have

$$\begin{aligned} \text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right) &= \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) = \frac{1}{n-1} \sum_{t=1}^{n-1} \sum_{i=1}^{t} w_i \\ &= \text{orness}(O_{\boldsymbol{w}}) \leq 0.5, \end{aligned}$$

where the last inequality is obtained by the second item of Proposition 5. □

Proof of Proposition 12. Given $j \in N$, by Remarks 8 and 13, Lemma 3, and Proposition 11 we have

$$\begin{aligned} \text{veto}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}, j\right) &= 1 - \frac{1}{n-1} \sum_{t=1}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T|=t}} \upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}(T) \\ &= 1 - \frac{1}{n-1} \frac{n}{n-1}(1-p_j) \sum_{t=1}^{n-1} \sum_{i=1}^{t} w_i \\ &= 1 - (1-p_j)\frac{n}{n-1} \text{orness}(O_{\boldsymbol{w}}) \\ &= 1 - (1-p_j)\frac{n}{n-1} \text{orness}\left(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\widetilde{P}}}\right). \end{aligned}$$

Likewise, taking into account Remarks 11 and 12, for any $t \geq 1$ we get

$$\begin{aligned}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}} \upsilon^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}(T\cup\{j\}) &= \sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}} \frac{n}{t+1}\left(\sum_{i\in T} p_i + p_j\right)\left(\sum_{i=1}^{t+1} w_i\right)\\ &= \frac{n}{t+1}\left(\sum_{i=1}^{t+1} w_i\right)\binom{n-1}{t}\left(\frac{t(1-p_j)}{n-1}+p_j\right).\end{aligned}$$

Notice that this expression is also valid when $t = 0$, given that by Remark 11,

$$\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=0}} \upsilon^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}(T\cup\{j\}) = \upsilon^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}(\{j\}) = np_j w_1.$$

Therefore, by Remark 8 we have

$$\begin{aligned}\operatorname{favor}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) &= \frac{1}{n-1}\sum_{t=0}^{n-1}\frac{n}{t+1}\left(\sum_{i=1}^{t+1} w_i\right)\left(\frac{t(1-p_j)}{n-1}+p_j\right) - \frac{1}{n-1}\\ &= \frac{n}{(n-1)^2}\sum_{t=0}^{n-1}\frac{(n-1-t)p_j + t}{t+1}\left(\sum_{i=1}^{t+1} w_i\right) - \frac{1}{n-1}\\ &= \frac{n}{(n-1)^2}\sum_{t=1}^{n}\frac{(n-t)p_j + t - 1}{t}\left(\sum_{i=1}^{t} w_i\right) - \frac{1}{n-1}\\ &= \frac{n}{(n-1)^2}\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{(n-t)p_j + t - 1}{t}\right) w_i - \frac{1}{n-1}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{(n-t)p_j+t-1}{t}\right)w_i = \sum_{i=1}^{n}\left(\sum_{t=i}^{n}\left((1-p_j)+\frac{np_j-1}{t}\right)\right)w_i\\ &= \sum_{i=1}^{n}\left((1-p_j)(n-i+1) + (np_j-1)\sum_{t=i}^{n}\frac{1}{t}\right)w_i\\ &= (1-p_j)\left(\sum_{i=1}^{n}(n-i)w_i + 1\right) + (np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i,\end{aligned}$$

and taking into account the value of $\operatorname{orness}(O_{\boldsymbol{w}})$ and Proposition 11, we get

$$\begin{aligned}\operatorname{favor}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) &= \frac{n}{n-1}(1-p_j)\operatorname{orness}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) + \frac{n}{(n-1)^2}\\ &\quad\cdot\left(1-p_j+(np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right) - \frac{1}{n-1}\\ &= 1-\operatorname{veto}\left(S^{U_{\widetilde{P}}}_{\boldsymbol{p},\boldsymbol{w}}, j\right) + \frac{n}{(n-1)^2}\\ &\quad\cdot\left(1-p_j+(np_j-1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right) - \frac{1}{n-1}.\end{aligned}$$

Lastly, by Remark 9 we have

$$\phi(\upsilon_{\boldsymbol{p},\boldsymbol{w}}^{U_{\overline{P}}}, j) = \frac{1}{n}\Big((n-1)\Big(\operatorname{veto}\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\overline{P}}}, j\Big) + \operatorname{favor}\Big(S_{\boldsymbol{p},\boldsymbol{w}}^{U_{\overline{P}}}, j\Big) - 1\Big) + 1\Big)$$
$$= \frac{1}{n-1}\left(1 - p_j + (np_j - 1)\sum_{i=1}^{n}\left(\sum_{t=i}^{n}\frac{1}{t}\right)w_i\right). \qquad \square$$

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