Simple and absolute special majorities generated by OWA operators

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Abstract

Simple and absolute special majorities are decision procedures used very often in real life. However, these rules do not allow individuals to express the intensity with which they prefer some alternatives to others. In order to consider this situation, individual preferences can be represented by fuzzy preferences through values located between 0 and 1. Then the collective preference is obtained by means of aggregation functions. In this paper we use OWA operators in order to aggregate individual preferences and we generalize simple and absolute special majorities by means of OWA operators.

Key words: Fuzzy sets, Simple majority, Absolute special majorities, Aggregation functions, OWA operators.

1 Introduction

Numerous procedures determine a collective preference over a set of alternatives, taking into account the individual preferences of m agents over the alternatives. The simplest situation consists of choosing an alternative between x and y when individuals do not grade their preferences. In this case, some inconsistency issue such as Arrow impossibility theorems or voting paradoxes are avoided. Moreover, in this framework, one of the most common systems is simple majority (May (1952) characterizes simple majority by means of three properties: anonymity, neutrality and positive responsiveness).

On the other hand, Fishburn (1973) presents an exhaustive study of several classes of majorities, including absolute special majorities. In these procedures,

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the alternative x is chosen when the number of individuals who prefer x to y, the *status quo*, is greater than a fixed percentage of the total number of voters; otherwise, y is chosen. Consequently, indifference between x and y is not possible. Moreover, Fishburn (1973, p. 67) characterizes these rules by means of several properties.

In this paper only two alternatives are considered under duality assumption (neutrality in May (1952)). This property guarantees an egalitarian treatment between the alternatives, which is very usual in the framework of decision making. Thus, the collective decision does not depends on the label of the options. In this case, for absolute special majorities an alternative is chosen if it has greater support than the fixed percentage of the total number of voters; otherwise, the two alternatives are collectively indifferent. Under duality assumption, these majorities have also been studied when there are more than two alternatives. Craven (1971) and Ferejohn and Grether (1974) determine the values of the percentage mentioned earlier for ensuring acyclicity in the collective preference.

However, individuals generally prefer one alternative to another with different levels of intensity. For instance, Fishburn (1973, p. 10) describes the following example:

> *the husband would rather stay home than go to a movie, but he really doesn't feel strongly about this; on the other hand, his wife is "dying to get out of the house" and has a very "strong" preference for "movie" over "stay home".*

Intensities of preference have been studied, under different points of view, by Harsanyi (1955), Sen (1970), Jech (1989) and Harvey (1999), among others. In the paper we have considered intensity of preferences by means of fuzzy preferences, evaluating the levels of preference intensity between 0 and 1. In this respect, see Nurmi (1981), Tanino (1984) and García-Lapresta and Llamazares (2000), among others.

However, it is important to emphasize that the use of an absolute scale, the same for all individuals, has been the target of several criticisms (see, for instance, French (1984)). On the other hand, the choice of and available scale in the ordinal case has been studied by Yager (2002).

In order to aggregate individual preferences, we consider aggregation operators which assign a collective intensity of preference to each profile of individual intensities of preference. There exist numerous aggregation functions utilized in multicriteria decision making (see, for instance, Grabisch et al. (1998), Marichal (1998) and Calvo et al. (2002), among others). In this work we use OWA operators, a class of aggregation functions introduced by Yager (1988). According to the collective intensity of preference obtained by means of an aggregation function, we can decide if an alternative is chosen or if the two alternatives are collectively indifferent. For this, we use a kind of strong α-cuts, where $\alpha \in [\frac{1}{2}, 1)$. The α-cuts allow to obtain asymmetric ordinary binary relations from reciprocal fuzzy binary relations. On this, see García-Lapresta and Llamazares (2000). Moreover, α-cuts have been utilized by some authors for defining fuzzy majorities. A survey of these developments can be found in Kacprzyk and Nurmi (1998).

When individuals do not grade their preferences, which is represented through the values 0, 1/2 and 1, the above procedure allows us to generalize some well-known decision rules. So, García-Lapresta and Llamazares (2001) generalize two classes of majorities based on difference of votes, using quasiarithmetic means and window OWA ("Ordered Weighted Averaging") operators as aggregation functions. Here, we obtain generalizations of simple and absolute special majorities by means of OWA operators. Therefore, the previous procedure allows us to extend simple and absolute special majorities to fuzzy framework through OWA operators.

OWA operators have been used in the aggregation of intensity preferences by Kacprzyk et al. (1997), Montero and Cutello (1997) and Chiclana et al. (2003), among others. On the other hand, arithmetic and weighted means have been characterized and used by Intriligator (1973, 1982) in order to determine social probabilities (from individual probabilities) and individual choices (from conflicting criteria), respectively.

The organization of the paper is as follows. In Section 2 we introduce aggregation functions and discrete aggregation functions. Moreover, some properties of aggregation functions (such as anonymity, duality, monotonicity and strict monotonicity) are introduced. We also characterize simple and absolute special majorities. In Section 3 we introduce OWA operators and we determine those ones that satisfy duality and strict monotonicity properties. Finally, in Section 4 we give the main results of the paper, the characterization of the OWA operators which generalize simple and absolute special majorities.

2 Aggregation functions

We consider m voters, with $m \geq 3$, and two alternatives x and y. Voters represent their preferences between x and y through variables r_i. If the individuals grade their preferences, then $r_i \in [0, 1]$ and it denotes the *intensity* with which voter i prefers x to y. We also suppose that $1 - r_i$ is the intensity with which voter i prefers y to x. If the individuals do not grade their preferences, then $r_i \in \{0, \frac{1}{2}, 1\}$ and it represents that voter i prefers x to y, prefers y to x or is indifferent between the two alternatives, if r_i is 1, 0 or $\frac{1}{2}$, respectively.

A *profile of preferences* is a vector $(r_1, \ldots, r_m)$ which describes the voters' preferences between the alternative x and the alternative y. Obviously, $(1 - r_1, \ldots, 1 - r_m)$ shows the voters' preferences between y and x. The collective preference will be obtained by means of an aggregation function, for each profile of preferences.

Definition 1. *An aggregation function is a mapping* $F : [0,1]^m \longrightarrow [0,1]$. *A discrete aggregation function (DAF) is a mapping* $F : \{0, \frac{1}{2}, 1\}^m \longrightarrow \{0, \frac{1}{2}, 1\}$.

The interpretation of collective preference is consistent with the foregoing interpretation for individual preferences. So, if F is an aggregation function, then $F(r_1, \ldots, r_m)$ is the intensity with which x is collectively preferred to y. When F is a DAF, then $F(r_1, \ldots, r_m)$ show us if an alternative is collectively preferred to another or the alternatives are collectively indifferent, according to whether $F(r_1, \ldots, r_m)$ is 1, 0 or $\frac{1}{2}$, respectively.

Next we present some properties of aggregation functions very used in the literature: anonymity, duality, monotonicity and strict monotonicity. Anonymity, also referred to as equality and symmetry, means that collective intensity of preference depends on only the set of individual intensity of preferences, but not on which individuals have these preferences. Duality, also referred to as neutrality (May (1952)), means that if everyone reverses his or her preferences between x and y, then the collective preference is also reversed. Monotonicity means that collective intensity of preference does not decrease if no individual intensity decreases. And strict monotonicity means that collective intensity of preference increases if some individual intensity increases. A characterization of the DAF's which simultaneously satisfy the three first properties can be found in Fishburn (1973, p. 56).

Definition 2. *Let* F *be an aggregation function or a DAF.*

(1) F *is anonymous if and only if for all profile* $(r_1, \ldots, r_m)$ *and all permutation* σ *of* $\{1, \ldots, m\}$ *the following condition is satisfied*

$$F(r_1, \ldots, r_m) = F(r_{\sigma(1)}, \ldots, r_{\sigma(m)}).$$

(2) F *is dual if and only if for all profile* $(r_1, \ldots, r_m)$ *the following condition is satisfied*

$$F(1 - r_1, \ldots, 1 - r_m) = 1 - F(r_1, \ldots, r_m).$$

(3) F *is monotonic if and only if for all pair of profiles* $(r_1, \ldots, r_m)$ *and* $(s_1, \ldots, s_m)$ *the following condition is satisfied*

$$\forall i \in \{1, \ldots, m\} \;\; r_i \geq s_i \;\; \Rightarrow \;\; F(r_1, \ldots, r_m) \geq F(s_1, \ldots, s_m).$$

(4) F *is strictly monotonic if and only if for all pair of profiles* $(r_1, \ldots, r_m)$

and $(s_1, \ldots, s_m)$ *such that* $(r_1, \ldots, r_m) \neq (s_1, \ldots, s_m)$ *the following condition is satisfied*

$$\forall i \in \{1, \ldots, m\} \;\; r_i \geq s_i \;\;\Rightarrow\;\; F(r_1, \ldots, r_m) > F(s_1, \ldots, s_m).$$

Now some consequences of the previous properties are obtained. The cardinal of a set will be denoted by $\#$.

Remark 3. *If* F *is an anonymous DAF, then* $F(r_1, \ldots, r_m)$ *depends on only the number of 1,* $\frac{1}{2}$ *and 0. Given a profile* $(r_1, \ldots, r_m)$*, if we consider:*

(1) $m_1 = \#\{i \mid r_i = 1\}$*, the number of voters who prefer* x *to* y*,*
(2) $m_2 = \#\{i \mid r_i = \frac{1}{2}\}$*, the number of voters who are indifferent between* x *and* y*,*
(3) $m_3 = \#\{i \mid r_i = 0\}$*, the number of voters who prefer* y *to* x*,*

then $m_1 + m_2 + m_3 = m$*.*

By Remark 3, every anonymous DAF can be represented by a mapping over the triples (m_1, m_2, m_3).

Definition 4. *Let* F *be an anonymous DAF and*

$$\mathcal{M} = \{(m_1, m_2, m_3) \in \{0, 1, \ldots, m\}^3 \mid m_1 + m_2 + m_3 = m\}.$$

We say that F *is represented by the mapping* $f : \mathcal{M} \longrightarrow \{0, \frac{1}{2}, 1\}$*, defined by*

$$f(m_1, m_2, m_3) = F(1, \overset{(m_1)}{\cdots}, 1, \tfrac{1}{2}, \overset{(m_2)}{\cdots}, \tfrac{1}{2}, 0, \overset{(m_3)}{\cdots}, 0).$$

Remark 5. *If* F *is an anonymous DAF represented by* f*, then it is dual if and only if*

$$f(m_3, m_2, m_1) = 1 - f(m_1, m_2, m_3)$$

for all $(m_1, m_2, m_3) \in \mathcal{M}$*. In this case,* F *is characterized by the set* $f^{-1}(\{1\})$*, since*

$$f^{-1}(\{0\}) = \{(m_1, m_2, m_3) \in \mathcal{M} \mid (m_3, m_2, m_1) \in f^{-1}(\{1\})\},$$
$$f^{-1}(\{\tfrac{1}{2}\}) = \mathcal{M} - (f^{-1}(\{1\}) \cup f^{-1}(\{0\})).$$

When a DAF is dual, the two alternatives have an egalitarian treatment. Therefore, if the DAF is also anonymous and the number of voters who prefer x to y coincides with the number of voters who prefer y to x, then x and y are collectively indifferent.

Remark 6. *If* F *is an anonymous and dual DAF represented by* f*, then* $f(m_1, m_2, m_3) = \frac{1}{2}$ *for all* $(m_1, m_2, m_3) \in \mathcal{M}$ *such that* $m_1 = m_3$.

Definition 7. *The binary relation* $\succeq$ *on* $\mathcal{M}$ *is defined by*

$$(m_1, m_2, m_3) \succeq (n_1, n_2, n_3) \quad \Leftrightarrow \quad m_1 \geq n_1 \;\; \textit{and} \;\; m_1 + m_2 \geq n_1 + n_2.$$

We note that $\succeq$ *is a partial order on* $\mathcal{M}$ *(reflexive, antisymmetric and transitive binary relation).*

Monotonicity of an anonymous DAF only depends on the number of 1, $\frac{1}{2}$ and 0, such as we show in the following remark.

Remark 8. *If* F *is an anonymous DAF represented by* f*, then it is monotonic if and only if*

$$(m_1, m_2, m_3) \succeq (n_1, n_2, n_3) \;\Rightarrow\; f(m_1, m_2, m_3) \geq f(n_1, n_2, n_3),$$

for all $(m_1, m_2, m_3), (n_1, n_2, n_3) \in \mathcal{M}$.

By Remark 5 it is possible to define an anonymous and dual DAF, F, by means of the elements $(m_1, m_2, m_3) \in \mathcal{M}$ where the mapping which represents F takes the value 1. Based on this, we now show some DAF's widely used in real decisions.

Definition 9.

(1) The simple majority, F_S*, is the anonymous and dual DAF defined by*

$$f(m_1, m_2, m_3) = 1 \;\Leftrightarrow\; m_1 > m_3, \;\; \textit{for all} \;\; (m_1, m_2, m_3) \in \mathcal{M}.$$

(2) The absolute majority, F_A*, is the anonymous and dual DAF defined by*

$$f(m_1, m_2, m_3) = 1 \;\Leftrightarrow\; m_1 > \frac{m}{2}, \;\; \textit{for all} \;\; (m_1, m_2, m_3) \in \mathcal{M}.$$

(3) The unanimous majority, F_U*, is the anonymous and dual DAF defined by*

$$f(m_1, m_2, m_3) = 1 \;\Leftrightarrow\; m_1 = m, \;\; \textit{for all} \;\; (m_1, m_2, m_3) \in \mathcal{M}.$$

Next we present absolute special majorities under the assumption of duality. These majorities have been characterized by Fishburn (1973, p. 67) without the hypothesis of duality.

Definition 10. *Given* $\beta \in [\frac{1}{2}, 1)$*, the absolute special majority* Q_β *is the anonymous and dual DAF represented by the mapping* $f : \mathcal{M} \longrightarrow \{0, \frac{1}{2}, 1\}$, *which is defined by*

$$f(m_1, m_2, m_3) = 1 \;\Leftrightarrow\; m_1 > \beta m, \quad \textit{for all} \;\; (m_1, m_2, m_3) \in \mathcal{M}.$$

Obviously, absolute special majorities are monotonic. Moreover, it is possible to obtain absolute and unanimous majorities as particular cases of absolute special majorities. We denote by $[a]$ the *integer part of* a, i.e., the largest integer smaller than or equal to a.

Remark 11. *Given $\beta \in [\frac{1}{2}, 1)$, then*

(1) $Q_\beta = F_A \;\Leftrightarrow\; \begin{cases} \frac{1}{2} - \frac{1}{2m} \leq \beta < \frac{1}{2} + \frac{1}{2m}, & \textit{if } m \textit{ is odd,} \\ \frac{1}{2} \leq \beta < \frac{1}{2} + \frac{1}{m}, & \textit{if } m \textit{ is even.} \end{cases}$

(2) $Q_\beta = F_U \;\Leftrightarrow\; 1 - \frac{1}{m} \leq \beta < 1.$

In order to generalize simple and absolute special majorities by means of OWA operators, we are going to characterize these majorities through some elements $(m_1, m_2, m_3) \in \mathcal{M}$. The monotonicity of simple and absolute special majorities play an essential role in the following characterizations. Next, we characterize simple majority through the elements $(m_1, m_2, m_3) \in \mathcal{M}$ such that $m_1 = m_3 + 1$.

Proposition 12. *Let F be an anonymous, dual and monotonic DAF represented by f. Then the following statements are equivalent:*

(1) $F = F_S$.

(2) $f(m_3 + 1, m - (2m_3 + 1), m_3) = 1$ *for all* $m_3 \in \{0, \ldots, [\frac{m-1}{2}]\}$.

PROOF.

$1 \Rightarrow 2$: Obvious.

$2 \Rightarrow 1$: Given $(m_1, m_2, m_3) \in \mathcal{M}$, we distinguish three cases:

(a) If $m_1 > m_3$ then $(m_1, m_2, m_3) \succeq (m_3 + 1, m - (2m_3 + 1), m_3)$ and, by monotonicity of F, we have

$$f(m_1, m_2, m_3) \geq f(m_3 + 1, m - (2m_3 + 1), m_3) = 1.$$

(b) If $m_1 = m_3$ then by Remark 6, we have $f(m_1, m_2, m_3) = \frac{1}{2}$.

(c) If $m_1 < m_3$ then $f(m_1, m_2, m_3) = 1 - f(m_3, m_2, m_1) = 1 - 1 = 0$.

Therefore, $f(m_1, m_2, m_3) = 1 \;\Leftrightarrow\; m_1 > m_3$. $\square$

In the following proposition absolute special majorities are characterized by means of two elements of $\mathcal{M}$. The first one corresponds to the minimum support that the alternative x needs to be selected. The second one corresponds to the maximum support that the alternative x can have without being selected.

Proposition 13. *Let F be an anonymous, dual and monotonic DAF represented by f and $\beta \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

(1) $F = Q_\beta$.
(2) $f([\beta m] + 1, 0, m - [\beta m] - 1) = 1$ *and* $f([\beta m], m - [\beta m], 0) < 1$.

PROOF.

$1 \Rightarrow 2$: Obvious.

$2 \Rightarrow 1$: Given $(m_1, m_2, m_3) \in \mathcal{M}$, we distinguish two cases:

(a) If $m_1 \geq [\beta m] + 1$ then $(m_1, m_2, m_3) \succeq ([\beta m] + 1, 0, m - [\beta m] - 1)$ and, by monotonicity of F, we have

$$f(m_1, m_2, m_3) \geq f([\beta m] + 1, 0, m - [\beta m] - 1) = 1.$$

(b) If $m_1 \leq [\beta m]$ then $([\beta m], m - [\beta m], 0) \succeq (m_1, m_2, m_3)$ and, by monotonicity of F, we have

$$f(m_1, m_2, m_3) \leq f([\beta m], m - [\beta m], 0) < 1.$$

Therefore, $f(m_1, m_2, m_3) = 1 \Leftrightarrow m_1 \geq [\beta m] + 1 \Leftrightarrow m_1 > \beta m$. □

Given an aggregation function, we can generate different DAF's by means of a parameter $\alpha \in [\frac{1}{2}, 1)$. Moreover, it is easy to check that these DAF's are anonymous, dual and monotonic when the original aggregation function satisfies these properties.

Definition 14. *Let F be an aggregation function and $\alpha \in [\frac{1}{2}, 1)$. Then the α–DAF associated with F is the DAF F_α defined by*

$$F_\alpha(r_1, \ldots, r_m) = \begin{cases} 1, \text{ if } F(r_1, \ldots, r_m) > \alpha, \\ \frac{1}{2}, \text{ if } 1 - \alpha \leq F(r_1, \ldots, r_m) \leq \alpha, \\ 0, \text{ if } F(r_1, \ldots, r_m) < 1 - \alpha. \end{cases}$$

Remark 15. *Given an aggregation function F, for all $\alpha \in [\frac{1}{2}, 1)$ the following statements hold:*

(1) If F is anonymous, then F_α is also anonymous.
(2) If F is dual, then F_α is also dual.
(3) If F is monotonic, then F_α is also monotonic.

In a similar way to anonymous DAF's, when F is an anonymous aggregation function, the restriction $F|_{\{0,\frac{1}{2},1\}^m}$ can be represented by a mapping $f : \mathcal{M} \longrightarrow [0,1]$. Now we show the relationship between f and the family of mappings f_α which represent the α–DAF's associated with F.

Remark 16. *Let F be an anonymous aggregation function and $\alpha \in [\frac{1}{2}, 1)$. Then F_α and $F|_{\{0,\frac{1}{2},1\}^m}$ can be represented by the mappings f_α and f respectively. There exists the following relationship between these mappings:*

$$f_\alpha(m_1, m_2, m_3) = \begin{cases} 1, \text{ if and only if } f(m_1, m_2, m_3) > \alpha, \\ \\ \frac{1}{2}, \text{ if and only if } 1 - \alpha \leq f(m_1, m_2, m_3) \leq \alpha, \\ \\ 0, \text{ if and only if } f(m_1, m_2, m_3) < 1 - \alpha. \end{cases}$$

3 OWA operators

Yager (1988) introduced the ordered weighted averaging (OWA) operators as a tool for aggregation procedures in multicriteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered from more to less. Thus, contrary to the weighted means, the weights are not associated with concrete variables. Consequently, OWA operators satisfy anonymity. Moreover, OWA operators generalize arithmetic mean and they verify other interesting properties, such as monotonicity and that the value of an OWA operator is located between the minimum and the maximum values of the variables. Because of these properties, OWA operators have been widely used in the literature (for instance, see Yager and Kacprzyk (1997)).

Usually, OWA operators are defined as functions whose domain is $\mathbb{R}^m$. Since in this paper individual intensities of preference vary between 0 and 1, we have restricted their domain to $[0,1]^m$.

Definition 17. *Let $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0,1]^m$ satisfying $\sum_{i=1}^{m} w_i = 1$. The OWA operator associated with $\boldsymbol{w}$ is the aggregation function $F^{\boldsymbol{w}}$ defined by*

$$F^{\boldsymbol{w}}(r_1, \ldots, r_m) = \sum_{i=1}^{m} w_i \, r_{\sigma(i)},$$

where σ is a permutation of $\{1, \ldots, m\}$ such that $r_{\sigma(1)} \geq \cdots \geq r_{\sigma(m)}$.

Remark 18. *By definition, OWA operators are anonymous and monotonic aggregation functions. Moreover, the restriction $F^{\boldsymbol{w}}|_{\{0,\frac{1}{2},1\}^m}$ is represented by the mapping $f^{\boldsymbol{w}}$ defined by*

$$f^{\boldsymbol{w}}(m_1, m_2, m_3) = \sum_{i=1}^{m_1} w_i + \frac{1}{2} \sum_{i=1}^{m_2} w_{m_1+i}.$$

In the following proposition we characterize strictly monotonic OWA operators by means of weighting vectors where no weight is null.

Proposition 19. *Given $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0,1]^m$ satisfying $\sum_{i=1}^m w_i = 1$, the following statements are equivalent:*

(1) $F^{\boldsymbol{w}}$ is strictly monotonic.
(2) $w_i > 0$ for all $i \in \{1, \ldots, m\}$.

PROOF.

$1 \Rightarrow 2$: Suppose that there exists $j \in \{1, \ldots, m\}$ such that $w_j = 0$. Let $(r_1, \ldots, r_m)$, $(s_1, \ldots, s_m) \in [0,1]^m$ such that $r_i = 1$ for all $i \in \{1, \ldots, m\}$ and

$$s_i = \begin{cases} 1, \text{ if } i \neq j, \\ 0, \text{ if } i = j. \end{cases}$$

Then we have $(r_1, \ldots, r_m) \neq (s_1, \ldots, s_m)$, $r_i \geq s_i$ for all $i \in \{1, \ldots, m\}$ and

$$F^{\boldsymbol{w}}(r_1, \ldots, r_m) = F^{\boldsymbol{w}}(s_1, \ldots, s_m) = 1,$$

which contradicts the hypothesis.

$2 \Rightarrow 1$: Suppose that $F^{\boldsymbol{w}}$ is not strictly monotonic. Then, by definition 2, there exist $(r_1, \ldots, r_m)$, $(s_1, \ldots, s_m) \in [0,1]^m$ such that $r_i \geq s_i$ for all $i \in \{1, \ldots, m\}$, $(r_1, \ldots, r_m) \neq (s_1, \ldots, s_m)$ and

$$F^{\boldsymbol{w}}(r_1, \ldots, r_m) = F^{\boldsymbol{w}}(s_1, \ldots, s_m).$$

Let σ_1 and σ_2 be permutations of $\{1, \ldots, m\}$ such that $r_{\sigma_1(1)} \geq \cdots \geq r_{\sigma_1(m)}$ and $s_{\sigma_2(1)} \geq \cdots \geq s_{\sigma_2(m)}$. Then we have $r_{\sigma_1(i)} \geq s_{\sigma_2(i)}$ for all $i \in \{1, \ldots, m\}$,

$(r_{\sigma_1(1)}, \ldots, r_{\sigma_1(m)}) \neq (s_{\sigma_2(1)}, \ldots, s_{\sigma_2(m)})$ and

$$\sum_{i=1}^{m} w_i \, r_{\sigma_1(i)} = \sum_{i=1}^{m} w_i \, s_{\sigma_2(i)}$$

or, equivalently

$$\sum_{i=1}^{m} w_i (r_{\sigma_1(i)} - s_{\sigma_2(i)}) = 0.$$

Since $r_{\sigma_1(i)} \geq s_{\sigma_2(i)}$ for all $i \in \{1, \ldots, m\}$, then $w_i(r_{\sigma_1(i)} - s_{\sigma_2(i)}) = 0$ for all $i \in \{1, \ldots, m\}$. Moreover, since $(r_{\sigma_1(1)}, \ldots, r_{\sigma_1(m)}) \neq (s_{\sigma_2(1)}, \ldots, s_{\sigma_2(m)})$, there exists $j \in \{1, \ldots, m\}$ such that $r_{\sigma_1(j)} > s_{\sigma_2(j)}$. Therefore, we have $w_j = 0$, which contradicts the hypothesis. □

Dual OWA operators have been characterized by García-Lapresta and Llamazares (2001). An OWA operator is dual if and only if the weights that are equidistant from the extremes are equal.

Proposition 20. *Given* $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0,1]^m$ *satisfying* $\sum_{i=1}^{m} w_i = 1$, *the following statements are equivalent:*

(1) $F^{\boldsymbol{w}}$ *is dual.*
(2) $w_{m+1-i} = w_i$ *for all* $i \in \{1, \ldots, [\frac{m}{2}]\}$.

If $F^{\boldsymbol{w}}$ is dual, then we have the following relationship among their weights.

Remark 21. *Given the weighting vector* $\boldsymbol{w} \in [0,1]^m$ *of a dual OWA operator, we have*

(1) If m *is odd then:* $2 \sum_{i=1}^{\frac{m-1}{2}} w_i + w_{\frac{m+1}{2}} = 1$.

(2) If m *is even then:* $2 \sum_{i=1}^{\frac{m}{2}} w_i = 1$.

In order to determine the α–DAF associated with a dual OWA operator, $F^{\boldsymbol{w}}$, which coincides with simple majority or an absolute special majority, we need to know the values that $f^{\boldsymbol{w}}$ takes on the elements which characterize those majorities. In the following remarks, we give these values.

Remark 22. *Let* $F^{\boldsymbol{w}}$ *be a dual OWA operator represented by* $f^{\boldsymbol{w}}$. *By Proposition 20 and Remark 21, in the elements* $(m_1, m_2, m_3) \in \mathcal{M}$ *which charac-*

terize simple majority (Proposition 12), the mapping $f^{\boldsymbol{w}}$ takes the following values:

(1) m odd:

(a) If $m_3 < \dfrac{m-1}{2}$:

$$\begin{aligned} f^{\boldsymbol{w}}(m_3+1, m-(2m_3+1), m_3) = \\ &= \sum_{i=1}^{m_3} w_i + \frac{3}{2} w_{m_3+1} + \sum_{i=m_3+2}^{\frac{m-1}{2}} w_i + \frac{1}{2} w_{\frac{m+1}{2}} \\ &= \frac{1}{2} + \frac{1}{2} w_{m_3+1}. \end{aligned}$$

(b) If $m_3 = \dfrac{m-1}{2}$:

$$f^{\boldsymbol{w}}(\tfrac{m+1}{2}, 0, \tfrac{m-1}{2}) = \sum_{i=1}^{\frac{m+1}{2}} w_i = \frac{1}{2} + \frac{1}{2} w_{\frac{m+1}{2}}.$$

(2) m even:

(a) If $m_3 < \dfrac{m}{2} - 1$:

$$\begin{aligned} f^{\boldsymbol{w}}(m_3+1, m-(2m_3+1), m_3) &= \sum_{i=1}^{m_3} w_i + \frac{3}{2} w_{m_3+1} + \sum_{i=m_3+2}^{\frac{m}{2}} w_i \\ &= \frac{1}{2} + \frac{1}{2} w_{m_3+1}. \end{aligned}$$

(b) If $m_3 = \dfrac{m}{2} - 1$:

$$f^{\boldsymbol{w}}(\tfrac{m}{2}, 1, \tfrac{m}{2} - 1) = \sum_{i=1}^{\frac{m}{2}-1} w_i + \frac{3}{2} w_{\frac{m}{2}} = \frac{1}{2} + \frac{1}{2} w_{\frac{m}{2}}.$$

Remark 23. *Let $F^{\boldsymbol{w}}$ be a dual OWA operator represented by $f^{\boldsymbol{w}}$. By Proposition 20 and Remark 21, in the elements $(m_1, m_2, m_3) \in \mathcal{M}$ which characterize absolute special majorities (Proposition 13), the mapping $f^{\boldsymbol{w}}$ takes the following values:*

(1) m odd:

(a) If $[\beta m] = \dfrac{m-1}{2}$:

$$f^{\boldsymbol{w}}(\tfrac{m+1}{2}, 0, \tfrac{m-1}{2}) = \sum_{i=1}^{\frac{m+1}{2}} w_i = \frac{1}{2} + \frac{1}{2} w_{\frac{m+1}{2}}.$$

$$f^{\boldsymbol{w}}(\tfrac{m-1}{2}, \tfrac{m+1}{2}, 0) = \frac{3}{2}\sum_{i=1}^{\frac{m-1}{2}} w_i + \frac{1}{2}w_{\frac{m+1}{2}} = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{\frac{m-1}{2}} w_i.$$

(b) If $[\beta m] > \dfrac{m-1}{2}$*:*

$$\begin{aligned} f^{\boldsymbol{w}}([\beta m]+1, 0, m-[\beta m]-1) &= \\ &= \sum_{i=1}^{m-[\beta m]-1} w_i + 2\sum_{i=m-[\beta m]}^{\frac{m-1}{2}} w_i + w_{\frac{m+1}{2}} \\ &= 1 - \sum_{i=1}^{m-[\beta m]-1} w_i. \end{aligned}$$

$$\begin{aligned} f^{\boldsymbol{w}}([\beta m], m-[\beta m], 0) &= \frac{3}{2}\sum_{i=1}^{m-[\beta m]} w_i + 2\sum_{i=m-[\beta m]+1}^{\frac{m-1}{2}} w_i + w_{\frac{m+1}{2}} \\ &= 1 - \frac{1}{2}\sum_{i=1}^{m-[\beta m]} w_i. \end{aligned}$$

(2) m even:

(a) If $[\beta m] = \dfrac{m}{2}$*:*

$$f^{\boldsymbol{w}}(\tfrac{m}{2}+1, 0, \tfrac{m}{2}-1) = \sum_{i=1}^{\frac{m}{2}-1} w_i + 2w_{\frac{m}{2}} = \frac{1}{2} + w_{\frac{m}{2}}.$$

$$f^{\boldsymbol{w}}(\tfrac{m}{2}, \tfrac{m}{2}, 0) = \frac{3}{2}\sum_{i=1}^{\frac{m}{2}} w_i = \frac{3}{4}.$$

(b) If $[\beta m] > \dfrac{m}{2}$*:*

$$\begin{aligned} f^{\boldsymbol{w}}([\beta m]+1, 0, m-[\beta m]-1) &= \sum_{i=1}^{m-[\beta m]-1} w_i + 2\sum_{i=m-[\beta m]}^{\frac{m}{2}} w_i \\ &= 1 - \sum_{i=1}^{m-[\beta m]-1} w_i. \end{aligned}$$

$$f^{\boldsymbol{w}}([\beta m], m-[\beta m], 0) = \frac{3}{2}\sum_{i=1}^{m-[\beta m]} w_i + 2\sum_{i=m-[\beta m]+1}^{\frac{m}{2}} w_i$$

$$= 1 - \frac{1}{2} \sum_{i=1}^{m-[\beta m]} w_i.$$

4 Generalization of majorities through OWA operators

In this section we establish the main results of the paper. Simple and absolute special majorities are generated by means of α–DAF's associated with dual OWA operators. Thus, the outcomes of this section allow to extend these majorities to fuzzy framework by means of OWA operators.

The results obtained for simple and absolute special majorities are similar. Firstly, we characterize the OWA operators for which we can generate simple and absolute special majorities. Next, we give the values of $\alpha \in [\frac{1}{2}, 1)$ for which these majorities can be obtained by means of a dual OWA operator.

We begin justifying that simple majority coincides with a class of α–DAF's associated with dual and strictly monotonic OWA operators.

Theorem 24. *Let $F^{\boldsymbol{w}}$ be a dual OWA operator and $\alpha \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

(1) $F^{\boldsymbol{w}}_\alpha = F_S$.
(2) $F^{\boldsymbol{w}}$ is strictly monotonic and $\alpha < \dfrac{1}{2}\Big(1 + \min\limits_i w_i\Big)$.

PROOF. Let $f^{\boldsymbol{w}}$ be the mapping which represents $F^{\boldsymbol{w}}$. By Proposition 12 and Remark 16, we have that the condition $F^{\boldsymbol{w}}_\alpha = F_S$ is equivalent to $f^{\boldsymbol{w}}(m_3 + 1, m - (2m_3 + 1), m_3) > \alpha$ for all $m_3 \in \{0, \ldots, [\frac{m-1}{2}]\}$. By Remark 22 we distinguish the following cases:

(1) m odd:

(a) If $m_3 < \dfrac{m-1}{2}$:

$$f^{\boldsymbol{w}}(m_3 + 1, m - (2m_3 + 1), m_3) > \alpha \quad \Leftrightarrow \quad 1 + w_{m_3+1} > 2\alpha.$$

(b) If $m_3 = \dfrac{m-1}{2}$:

$$f^{\boldsymbol{w}}(\tfrac{m+1}{2}, 0, \tfrac{m-1}{2}) > \alpha \quad \Leftrightarrow \quad 1 + w_{\frac{m+1}{2}} > 2\alpha.$$

Therefore, since $\alpha \in [\frac{1}{2}, 1)$, we have

$$F_\alpha^{\boldsymbol{w}} = F_S \quad \Leftrightarrow \quad \begin{array}{l} w_i > 0 \ \text{ for all } \ i \in \{1, \ldots, \frac{m+1}{2}\} \ \text{ and} \\ \alpha < \frac{1}{2}\Big(1 + \min\limits_i w_i\Big). \end{array}$$

(2) m even:

(a) If $m_3 < \dfrac{m}{2} - 1$:

$f^{\boldsymbol{w}}(m_3 + 1, m - (2m_3 + 1), m_3) > \alpha \quad \Leftrightarrow \quad 1 + w_{m_3+1} > 2\alpha.$

(b) If $m_3 = \dfrac{m}{2} - 1$:

$f^{\boldsymbol{w}}(\frac{m}{2}, 1, \frac{m}{2} - 1) > \alpha \quad \Leftrightarrow \quad 1 + w_{\frac{m}{2}} > 2\alpha.$

Therefore, since $\alpha \in [\frac{1}{2}, 1)$, we have

$$F_\alpha^{\boldsymbol{w}} = F_S \quad \Leftrightarrow \quad \begin{array}{l} w_i > 0 \ \text{ for all } \ i \in \{1, \ldots, \frac{m+1}{2}\} \ \text{ and} \\ \alpha < \frac{1}{2}\Big(1 + \min\limits_i w_i\Big). \end{array} \qquad \square$$

Next we show the values of α for which simple majority can be generated by means of the α–DAF associated with a dual OWA operator.

Corollary 25. *Given $\alpha \in [\frac{1}{2}, 1)$, the following statements are equivalent:*

(1) There exists a dual OWA operator $F^{\boldsymbol{w}}$ such that $F_\alpha^{\boldsymbol{w}} = F_S$.
(2) $\alpha < \dfrac{m+1}{2m}$.

PROOF.

1 ⇒ 2: By Theorem 24, we have $\alpha < \frac{1}{2}(1 + \min\limits_i w_i)$. Suppose that $\alpha \geq \frac{m+1}{2m}$. Then it is satisfied

$$\frac{1}{2}\left(1 + \frac{1}{m}\right) = \frac{m+1}{2m} \leq \alpha < \frac{1}{2}\Big(1 + \min_i \ w_i\Big).$$

Therefore, $\min\limits_i w_i > \dfrac{1}{m}$ and, consequently, it is verified

$$\sum_{i=1}^{m} w_i \geq m \min_i \ w_i > m\frac{1}{m} = 1,$$

which contradicts $\sum\limits_{i=1}^{m} w_i = 1$.

$2 \Rightarrow 1$: By Theorem 24, it is sufficient to take the dual OWA operator $F^{\boldsymbol{w}}$ associated with $\boldsymbol{w} = (\frac{1}{m}, \ldots, \frac{1}{m})$. □

In the following Theorem we give a necessary and sufficient condition in order to obtain absolute special majorities by means of α–DAF's associated with dual OWA operators.

Theorem 26. *Let $F^{\boldsymbol{w}}$ be a dual OWA operator and $\beta \in [\frac{1}{2}, 1)$. Then the following statements are equivalent:*

(1) There exists $\alpha \in [\frac{1}{2}, 1)$ such that $F^{\boldsymbol{w}}_\alpha = Q_\beta$.

(2) $w_{m-[\beta m]} > \sum_{i=1}^{m-[\beta m]-1} w_i$.

PROOF. Let $f^{\boldsymbol{w}}$ be the mapping which represents $F^{\boldsymbol{w}}$. By Proposition 13 and Remark 16, we have that the condition $F^{\boldsymbol{w}}_\alpha = Q_\beta$ is equivalent to $f^{\boldsymbol{w}}([\beta m]+1, 0, m-[\beta m]-1) > \alpha$ and $f^{\boldsymbol{w}}([\beta m], m-[\beta m], 0) \leq \alpha$. By Remark 23, we distinguish the following cases:

(1) m odd:

(a) If $[\beta m] = \dfrac{m-1}{2}$:

$$F^{\boldsymbol{w}}_\alpha = Q_\beta \quad \Leftrightarrow \quad \frac{1}{2} + \frac{1}{2} w_{\frac{m+1}{2}} > \alpha \quad \text{and} \quad \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{\frac{m-1}{2}} w_i \leq \alpha.$$

Hence, there exists $\alpha \in [\frac{1}{2}, 1)$ such that $F^{\boldsymbol{w}}_\alpha = Q_\beta$ if and only if

$$\frac{1}{2} + \frac{1}{2} w_{\frac{m+1}{2}} > \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{\frac{m-1}{2}} w_i \quad \Leftrightarrow \quad w_{\frac{m+1}{2}} > \sum_{i=1}^{\frac{m-1}{2}} w_i.$$

(b) If $[\beta m] > \dfrac{m-1}{2}$:

$$F^{\boldsymbol{w}}_\alpha = Q_\beta \quad \Leftrightarrow \quad 1 - \sum_{i=1}^{m-[\beta m]-1} w_i > \alpha \quad \text{and} \quad 1 - \frac{1}{2} \sum_{i=1}^{m-[\beta m]} w_i \leq \alpha.$$

Therefore, there exists $\alpha \in [\frac{1}{2}, 1)$ such that $F^{\boldsymbol{w}}_\alpha = Q_\beta$ if and only if

$$1 - \sum_{i=1}^{m-[\beta m]-1} w_i > 1 - \frac{1}{2} \sum_{i=1}^{m-[\beta m]} w_i \quad \Leftrightarrow \quad w_{m-[\beta m]} > \sum_{i=1}^{m-[\beta m]-1} w_i.$$

(2) m even:

(a) If $[\beta m] = \dfrac{m}{2}$:

$$F_\alpha^{\boldsymbol{w}} = Q_\beta \;\Leftrightarrow\; \frac{1}{2} + w_{\frac{m}{2}} > \alpha \quad \text{and} \quad \frac{3}{4} \le \alpha.$$

Hence, there exists $\alpha \in [\frac{1}{2}, 1)$ such that $F_\alpha^{\boldsymbol{w}} = Q_\beta$ if and only if

$$\frac{1}{2} + w_{\frac{m}{2}} > \frac{3}{4} \;\Leftrightarrow\; w_{\frac{m}{2}} > \frac{1}{2}\sum_{i=1}^{\frac{m}{2}} w_i \;\Leftrightarrow\; w_{\frac{m}{2}} > \sum_{i=1}^{\frac{m}{2}-1} w_i.$$

(b) If $[\beta m] > \dfrac{m}{2}$:

$$F_\alpha^{\boldsymbol{w}} = Q_\beta \;\Leftrightarrow\; 1 - \sum_{i=1}^{m-[\beta m]-1} w_i > \alpha \quad \text{and} \quad 1 - \frac{1}{2}\sum_{i=1}^{m-[\beta m]} w_i \le \alpha.$$

Therefore, there exists $\alpha \in [\frac{1}{2}, 1)$ such that $F_\alpha^{\boldsymbol{w}} = Q_\beta$ if and only if

$$1 - \sum_{i=1}^{m-[\beta m]-1} w_i > 1 - \frac{1}{2}\sum_{i=1}^{m-[\beta m]} w_i \;\Leftrightarrow\; w_{m-[\beta m]} > \sum_{i=1}^{m-[\beta m]-1} w_i.$$

□

Now, we show the values of α for which absolute special majorities can be generated by means of the α–DAF associated with a dual OWA operator.

Proposition 27.

(1) If $[\beta m] = \frac{m-1}{2}$, then for all $\alpha \in [\frac{1}{2}, 1)$ there exists a dual OWA operator $F^{\boldsymbol{w}}$ such that $F_\alpha^{\boldsymbol{w}} = Q_\beta$.

(2) If $[\beta m] \neq \frac{m-1}{2}$, then there exists a dual OWA operator $F^{\boldsymbol{w}}$ such that $F_\alpha^{\boldsymbol{w}} = Q_\beta$ if and only if $\alpha \in [\frac{3}{4}, 1)$.

PROOF.

(1) It is sufficient to consider the dual OWA operator defined by

$$w_i = \begin{cases} 1, \text{ if } i = \frac{m+1}{2}, \\ 0, \text{ otherwise.} \end{cases}$$

(2) $\Rightarrow$) By Proposition 13 and Remarks 16 and 23 we have:

(a) If m is odd and $[\beta m] > \frac{m-1}{2}$, then

$$F_\alpha^{\boldsymbol{w}} = Q_\beta \;\; \Rightarrow \;\; \alpha \,\geq\, 1 - \frac{1}{2} \sum_{i=1}^{m-[\beta m]} w_i \,\geq\, 1 - \frac{1}{4} \,=\, \frac{3}{4}.$$

(b) If m is even and $[\beta m] = \frac{m}{2}$, then

$$F_\alpha^{\boldsymbol{w}} = Q_\beta \;\; \Rightarrow \;\; \alpha \,\geq\, \frac{3}{4}.$$

(c) If m is even and $[\beta m] > \frac{m}{2}$, then

$$F_\alpha^{\boldsymbol{w}} = Q_\beta \;\; \Rightarrow \;\; \alpha \,\geq\, 1 - \frac{1}{2} \sum_{i=1}^{m-[\beta m]} w_i \,\geq\, 1 - \frac{1}{4} \,=\, \frac{3}{4}.$$

$\Leftarrow$) We only need to consider the dual OWA operator defined by

$$w_i = \begin{cases} \frac{1}{2}, \text{ if } \; i = m - [\beta m], [\beta m]+1, \\ 0, \text{ otherwise.} \end{cases}$$ □

Lastly, we give the necessary and sufficient conditions in order to obtain absolute and unanimous majorities by means of α–DAF's associated with dual OWA operators.

Corollary 28. *If $F^{\boldsymbol{w}}$ is a dual OWA operator then the following statements hold:*

(1) (a) If m is odd:

$$F_\alpha^{\boldsymbol{w}} = F_A \;\; \Leftrightarrow \;\; \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{\frac{m-1}{2}} w_i \,\leq\, \alpha \,<\, \frac{1}{2} + \frac{1}{2} w_{\frac{m+1}{2}}.$$

(b) If m is even:

$$F_\alpha^{\boldsymbol{w}} = F_A \;\; \Leftrightarrow \;\; \frac{3}{4} \,\leq\, \alpha \,<\, \frac{1}{2} + w_{\frac{m}{2}}.$$

(2) $F_\alpha^{\boldsymbol{w}} = F_U \;\; \Leftrightarrow \;\; w_1 > 0$ *and* $\alpha \geq 1 - \frac{1}{2} w_1.$

PROOF.

(1) (a) If m is odd, by Remark 11, we have $Q_\beta = F_A \Leftrightarrow [\beta m] = \frac{m-1}{2}$. The result is obtained by 1 (a) in the proof of Theorem 26.

(b) If m is even, by Remark 11, we have $Q_\beta = F_A \Leftrightarrow [\beta m] = \frac{m}{2}$. The result is obtained by 2 (a) in the proof of Theorem 26.

(2) By Remark 11, $Q_\beta = F_U \Leftrightarrow [\beta m] = m - 1$. The result is obtained, for m odd and for m even, by proof of Theorem 26. □

We can conclude that simple majority can be extended for considering intensity of preferences by means of α-cuts of dual strictly monotonic OWA operators. Analogously, absolute special majorities can be also extended through α-cuts of dual OWA operators whose weights satisfy an additional condition.

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