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Preference intensities and majority decisions based on difference of support between alternatives

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Abstract Simple majority is one of the most used decision rules in practice. However, under this decision rule, an alternative can defeat another one with very poor support. For this reason, other decision rules have been considered in the literature, such as qualified and special majorities as well as other majorities based on difference of votes. In this paper we generalize the latter mentioned voting systems by considering individual intensities of preference, and we provide some axiomatic characterizations.

Key words voting systems; simple majority; majorities based on difference of votes; intensities of preference; fuzzy decision rules

1 Introduction

May [29] provided his well-known axiomatic characterization of *simple majority*[1] –one of the most widely used voting systems in real life– that introduced a fruitful field of research within the Social Choice Theory. So much so that most of voting systems have axiomatic characterizations that allow knowing in-depth how they work. After May [29], other characterizations of simple majority can be found in Fishburn [11, p. 58] and [12], Campbell [3,4], Maskin [28], Campbell and Kelly [5,6], Aşan and Sanver [1], Woeginger [38,39], Miroiu [31], Yi [40], and Llamazares [24].

Since simple majority requires very poor support for declaring an alternative as a winner, other majorities have been introduced and studied in the literature (see

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[1] According to simple majority, x defeats y when the number of individuals who prefer x to y is greater than the number of individuals who prefer y to x.

Fishburn [11, chapter 6], Ferejohn and Grether [10], Saari [36, pp. 122-123], and García-Lapresta and Llamazares [15], among others).

In order to avoid some drawbacks of simple and absolute majorities, and other voting systems, in García-Lapresta and Llamazares [15] we introduced and analyzed M_k *majorities*, a class of voting systems based on difference of votes. Given two alternatives, x and y, for M_k, x is collectively preferred to y, when the number of individuals who prefer x to y exceeds the number of individuals who prefer y to x by at least a fixed integer k from 0 to $m-1$, where m is the number of voters. We note that M_k majorities are located between simple majority and unanimity, in the extreme cases of $k = 0$ and $k = m-1$, respectively. Subsequently, M_k majorities have been characterized axiomatically by Llamazares [24] and Houy [21].

A feature of simple majority, and other classic voting systems, is that they require individuals to declare dichotomous preferences: they can only declare if an alternative is preferred to another, or if they are indifferent. All kinds of preference modalities are identified and voters' opinions are misrepresented. According to Sen [37, p. 162], "... the method of majority decision takes no account of intensities of preference, and it is certainly arguable that what matters is not merely the *number* who prefer x to y and the *number* who prefer y to x, but also by how much each prefers one alternative to the other". This idea had already been considered in the 18th Century by the Spanish mathematician J.I. Morales, who in [32] states that "opinion is not something that can be quantified but rather something which has to be weighed" (see English translation in McLean and Urken [30, p. 204]), or "... majority opinion ... is something which is independent of any fixed number of votes or, which is the same, it has a varying relationship with this figure" (see English translation in McLean and Urken [30, p. 214]).

Fuzzy and linguistic preferences have been introduced for dealing with preference intensities in different fields of decision theory. Fuzzy preferences generalize ordinary (or crisp) preferences by allowing individuals to show preference degrees among alternatives, by means of numerical values within the unit interval (see for instance Nurmi [33,35]). On the other hand, individuals could declare preference degrees among alternatives through linguistic terms with appropriate semantics (see for instance Zadeh [41], Herrera, Herrera-Viedma and Verdegay [19,20], and Herrera and Herrera-Viedma [18]), or through terms of a finite scale (see for instance Grabisch [17]).

The importance of considering intensities of preference in the design of appropriate voting systems has been advocated by Nurmi [35]. In this way, García-Lapresta and Llamazares [14] provide some axiomatic characterizations of several decision rules that aggregate fuzzy preferences through different kinds of means. Additionally, in [14, Prop. 2], simple majority has been obtained as a specific case of the mentioned decision rules. Likewise, another kind of majorities can be obtained through operators that aggregate fuzzy preferences (on this, see Llamazares and García-Lapresta [26,27] and Llamazares [23,25]). In addition, García-Lapresta [13] generalizes simple majority by allowing individuals to show preference degrees in a linguistic manner, and García-Lapresta, Martínez-Panero and Meneses [16] provide some generalizations of the Borda rule to the field of linguistic preferences.

In this paper we introduce two generalizations of M_k majorities by considering individual intensities of preference. Thus, individuals might show "by how much each prefers one alternative to the other" (see again Sen [37, p. 162]) by a number between 0 and 1. Given that threshold k can now be a non integer number, infinite fuzzy majorities are obtained. Based on the axiomatic characterization of M_k majorities recently provided by Llamazares [24], in this paper we will be characterizing M_k fuzzy majorities.

The paper is organized as follows. In Section 2 we introduce ordinary and fuzzy decision rules, including ordinary and fuzzy majorities based on difference of votes. Section 3 is devoted to defining some properties for ordinary and fuzzy decision rules. Section 4 contains axiomatic characterizations of several fuzzy majorities based on difference of support between alternatives. Finally, some conclusions are included in Section 5.

2 Decision rules

Suppose m individuals, with $m \geq 3$, who show their preferences between two alternatives x and y. We distinguish between ordinary and fuzzy decision rules, depending on whether we consider ordinary or fuzzy individual preferences. In both cases we denote by N the classical negation operator on $\{0, 0.5, 1\}$ or $[0,1]$, defined by $N(d) = 1 - d$.

2.1 Ordinary decision rules

May [29] and Fishburn [11] use an index D for distinguishing among the three possible cases of ordinary preference and indifference between x and y:

$$D = \begin{cases} 1, & \text{if } x \text{ is preferred to } y, \\ 0, & \text{if } x \text{ is indifferent to } y, \\ -1, & \text{if } y \text{ is preferred to } x. \end{cases}$$

If we define a new index $d = \dfrac{D+1}{2}$, then we have:

$$d = \begin{cases} 1, & \text{if } x \text{ is preferred to } y, \\ 0.5, & \text{if } x \text{ is indifferent to } y, \\ 0, & \text{if } y \text{ is preferred to } x. \end{cases}$$

Following this notation, we use d_i to denote the opinion of individual i between x and y. A *profile of crisp preferences* is a vector $(d_1, \ldots, d_m)$ of $\{0, 0.5, 1\}^m$ that contains the opinions of the m individuals between x and y. Under this construction, $N(d_i)$, the reverse of d_i, shows the opinion of individual i between y and x:

$$N(d_i) = \begin{cases} 1, & \text{if } i \text{ prefers } y \text{ to } x, \\ 0.5, & \text{if } i \text{ is indifferent between } y \text{ and } x, \\ 0, & \text{if } i \text{ prefers } x \text{ to } y. \end{cases}$$

Definition 1 An *ordinary decision rule* is a mapping

$$F : \{0, 0.5, 1\}^m \longrightarrow \{0, 0.5, 1\}$$

that assigns 0, 0.5 or 1 to each profile $(d_1,\dots,d_m) \in \{0,0.5,1\}^m$, depending on whether y defeats x, x and y tie, or x defeats y, respectively.

Definition 2 Given an integer number $k \in \{0,1,\dots,m-1\}$, the M_k *majority* is the ordinary decision rule defined by:

$$M_k(d_1,\dots,d_m) = \begin{cases} 1, \text{ if } \displaystyle\sum_{i=1}^m d_i > \sum_{i=1}^m N(d_i) + k, \\ 0.5, \text{ if } \left| \displaystyle\sum_{i=1}^m d_i - \sum_{i=1}^m N(d_i) \right| \le k, \\ 0, \text{ if } \displaystyle\sum_{i=1}^m d_i < \sum_{i=1}^m N(d_i) - k, \end{cases}$$

for every profile $(d_1,\dots,d_m) \in \{0,0.5,1\}^m$.

Similarly, given an integer number $k \in \{1,\dots,m-1,m\}$, the M'_k *majority* is the ordinary decision rule defined by:

$$M'_k(d_1,\dots,d_m) = \begin{cases} 1, \text{ if } \displaystyle\sum_{i=1}^m d_i \ge \sum_{i=1}^m N(d_i) + k, \\ 0.5, \text{ if } \left| \displaystyle\sum_{i=1}^m d_i - \sum_{i=1}^m N(d_i) \right| < k, \\ 0, \text{ if } \displaystyle\sum_{i=1}^m d_i \le \sum_{i=1}^m N(d_i) - k, \end{cases}$$

for every profile $(d_1,\dots,d_m) \in \{0,0.5,1\}^m$.

We note that $\sum_{i=1}^m d_i$ is the number of individuals who prefer x to y, plus half the number of individuals who are indifferent between these alternatives; and $\sum_{i=1}^m N(d_i) = m - \sum_{i=1}^m d_i$ is the number of individuals who prefer y to x, plus half the number of individuals who are indifferent between these alternatives. Since indifferences can be simplified in all the cases, we obtain that the winning alternative is the one with a number of votes exceeding those obtained by the other in the previously fixed quantity k.[2]

It is clear that $M_k = M'_{k+1}$ for every $k \in \{0,1,\dots,m-1\}$. Thus, M_k and M'_k define the same class of decision rules. However, their extensions to the fuzzy framework do not define exactly the same class of fuzzy decision rules.

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[2] When we consider the values -1, 0 and 1 for representing individual or collective preferences, the definition of M_k majorities is more straightforward (see, for instance, Llamazares [24]). However, in order to extend M_k majorities to the fuzzy framework, we need to consider the values 0, 0.5 and 1.

Remark 1 The class of the M_k majorities is ordered with respect to decisiveness according to k, being simple majority, M_0, the most decisive and unanimity, M_{m-1}, the less decisive. More concretely, for all $k, k' \in \{0, 1, \ldots, m-1\}$ and every profile $(d_1, \ldots, d_m) \in \{0, 0.5, 1\}^m$, if $k' \geq k$, then it holds:

1. $M_{k'}(d_1, \ldots, d_m) = 1 \Rightarrow M_k(d_1, \ldots, d_m) = 1$.
2. $M_{k'}(d_1, \ldots, d_m) = 0 \Rightarrow M_k(d_1, \ldots, d_m) = 0$.
3. $M_k(d_1, \ldots, d_m) = 0.5 \Rightarrow M_{k'}(d_1, \ldots, d_m) = 0.5$.

Simple, absolute and special majorities, as well as majorities based on difference of votes, M_k, have been considered mainly for only two alternatives (see Fishburn [11, Chapters 5 and 6]). The reason for this restriction is that when more than two alternatives are taken into account, the above mentioned majorities could produce inconsistencies and paradoxes[3]. In this sense, the best known drawback is the *voting paradox* that Condorcet [7] found for simple majority[4]. Nevertheless, when there are more than two alternatives, simple majority may be used in a sequential way. This is the case analyzed by Lehtinen [22] in two parliamentary agendas (amendment and elimination).

Next example shows some features of M_k majorities with respect to the voting paradox and the intransitivity of the collective preference and indifference relations associated with M_k.

Example 1 Suppose 20 individuals whose preferences over three alternatives x, y and z are the following: 5 individuals rank $x \succ y \succ z$, 6 individuals rank $y \succ z \succ x$ and the remaining 9 individuals rank $z \succ x \succ y$.

1. According to M_0 and M_1, x defeats y, y defeats z, and z defeats x, i.e., these majorities produces a cycle.
2. For M_k, $k = 2, \ldots, 7$, z defeats x, x defeats y, but z and y are collectively indifferent, i.e., the collective preference is not transitive.
3. According to M_8 and M_9, x and y as well as y and z are collectively indifferent, but z defeats x, i.e., the collective indifference is not transitive.
4. For M_k, $k \geq 10$, the three alternatives are collectively indifferent.

2.2 Fuzzy decision rules

Now suppose that individuals can show intensities of preference by means of numbers between 0 and 1 in a fuzzy manner: 0, when they prefer absolutely y to x; 0.5, when they are indifferent between x and y; 1, when they prefer absolutely x to y; and, whatever number different to 0, 0.5 and 1, for not extreme preferences, nor for indifference, in the sense that the closer the number is to 1, the more x is preferred to y (see García-Lapresta and Llamazares [14]).

[3] A systematic study of voting paradoxes can be found in Nurmi [34].

[4] Given three individuals who rank alternatives x, y and z in the following manner $x \succ y \succ z$, $y \succ z \succ x$ and $z \succ x \succ y$, simple majority produces a cycle among these alternatives: x defeats y, y defeats z, and z defeats x.

Definition 3 A *fuzzy decision rule* is a mapping $F : [0,1]^m \longrightarrow \{0, 0.5, 1\}$ that assigns 0, 0.5 or 1 to each profile $(d_1, \ldots, d_m) \in [0,1]^m$, depending on whether y defeats x, x and y tie, or x defeats y, respectively.

It is worth noting that fuzzy decision rules allow individuals to show preference intensities among alternatives, but that they also provide unequivocal outcomes – one alternative wins or both tie– as in ordinary decision rules. In this way, Barrett, Pattanaik and Salles [2] pointed out: "In real life, people often have vague preferences ... However, when confronted with an actual choice situation, where an alternative has to be chosen from a given feasible set of alternatives, the decision maker must make an unambiguous choice, even when his preferences are fuzzy". See also Dutta, Panda and Pattanaik [8] and Dutta [9].

We now extend M_k majorities to the context of fuzzy preferences.

Definition 4 Given a real number $k \in [0, m)$, the *fuzzy* $\widetilde{M}_k$ *majority* is the fuzzy decision rule defined by:

$$\widetilde{M}_k(d_1, \ldots, d_m) = \begin{cases} 1, \text{ if } \displaystyle\sum_{i=1}^m d_i > \sum_{i=1}^m N(d_i) + k, \\ 0.5, \text{ if } \left| \displaystyle\sum_{i=1}^m d_i - \sum_{i=1}^m N(d_i) \right| \leq k, \\ 0, \text{ if } \displaystyle\sum_{i=1}^m d_i < \sum_{i=1}^m N(d_i) - k, \end{cases}$$

for every profile $(d_1, \ldots, d_m) \in [0,1]^m$.

Similarly, given a real number $k \in (0, m]$, the *fuzzy* $\widetilde{M}'_k$ *majority* is the fuzzy decision rule defined by:

$$\widetilde{M}'_k(d_1, \ldots, d_m) = \begin{cases} 1, \text{ if } \displaystyle\sum_{i=1}^m d_i \geq \sum_{i=1}^m N(d_i) + k, \\ 0.5, \text{ if } \left| \displaystyle\sum_{i=1}^m d_i - \sum_{i=1}^m N(d_i) \right| < k, \\ 0, \text{ if } \displaystyle\sum_{i=1}^m d_i \leq \sum_{i=1}^m N(d_i) - k, \end{cases}$$

for every profile $(d_1, \ldots, d_m) \in [0,1]^m$.

Now, $\displaystyle\sum_{i=1}^m d_i$ is the *amount of opinion* obtained by x, taking into account all individual intensities of preference of x over y, and $\displaystyle\sum_{i=1}^m N(d_i) = m - \sum_{i=1}^m d_i$ is the *amount of opinion* obtained by y, taking into account all individual intensities of preference of y over x.

Notice that in the fuzzy framework, $\widetilde{M}_k$ and $\widetilde{M}'_k$ do not define exactly the same class of decision rules. For instance, the fuzzy extensions of simple majority and unanimity are only obtained by $\widetilde{M}_0$ and $\widetilde{M}'_m$, respectively.

Remark 2 $\widetilde{M}_k$ and $\widetilde{M}'_k$ majorities can be defined by means of the arithmetic mean of the individual intensities of preference.

Notice that

1. $\sum_{i=1}^{m} d_i > \sum_{i=1}^{m} N(d_i) + k \;\Leftrightarrow\; 2\sum_{i=1}^{m} d_i > m + k \;\Leftrightarrow\; \frac{1}{m}\sum_{i=1}^{m} d_i > 0.5 + \frac{k}{2m}$.

2. $\sum_{i=1}^{m} d_i < \sum_{i=1}^{m} N(d_i) - k \;\Leftrightarrow\; \frac{1}{m}\sum_{i=1}^{m} d_i < 0.5 - \frac{k}{2m}$.

3. $\left|\sum_{i=1}^{m} d_i - \sum_{i=1}^{m} N(d_i)\right| \leq k \;\Leftrightarrow\; \left|\frac{1}{m}\sum_{i=1}^{m} d_i - 0.5\right| \leq \frac{k}{2m}$.

Therefore, the fuzzy $\widetilde{M}_k$ majority can be also defined as:

$$\widetilde{M}_k(d_1,\ldots,d_m) = \begin{cases} 1, \text{ if } \dfrac{1}{m}\sum_{i=1}^{m} d_i > 0.5 + \dfrac{k}{2m}, \\ 0.5, \text{ if } \left|\dfrac{1}{m}\sum_{i=1}^{m} d_i - 0.5\right| \leq \dfrac{k}{2m}, \\ 0, \text{ if } \dfrac{1}{m}\sum_{i=1}^{m} d_i < 0.5 - \dfrac{k}{2m}, \end{cases}$$

for every profile $(d_1,\ldots,d_m) \in [0,1]^m$ and $k \in [0,m)$.

In a similar way, it is possible to define $\widetilde{M}'_k$ majorities by means of the arithmetic mean:

$$\widetilde{M}'_k(d_1,\ldots,d_m) = \begin{cases} 1, \text{ if } \dfrac{1}{m}\sum_{i=1}^{m} d_i \geq 0.5 + \dfrac{k}{2m}, \\ 0.5, \text{ if } \left|\dfrac{1}{m}\sum_{i=1}^{m} d_i - 0.5\right| < \dfrac{k}{2m}, \\ 0, \text{ if } \dfrac{1}{m}\sum_{i=1}^{m} d_i \leq 0.5 - \dfrac{k}{2m}, \end{cases}$$

for every profile $(d_1,\ldots,d_m) \in [0,1]^m$ and $k \in (0,m]$.

We note that arithmetic mean is the only neutral fuzzy aggregation rule satisfying decomposability, unanimity and anonymity. This result, other properties of, and references about the arithmetic mean as a fuzzy aggregation rule can be found in García-Lapresta and Llamazares [14].

It is worth emphasizing the importance of individuals being able to declare their intensities of preference among alternatives. In conventional voting systems, individuals can not show preference degrees, and they must declare crisp preferences (extreme preference or indifference) among alternatives.

Next example shows three different profiles of fuzzy preferences, all of them with the same crisp structure. Since all these crisp profiles are identical, they provide the same result for simple majority. However, we note that original fuzzy profiles generate different outcomes for fuzzy simple majority. This apparent paradox is due to the information we process. Taking into account only a part of the information that individuals provide, can result in the outcome being different to that obtained when all information contained in preference degrees among alternatives is considered. In fact, preference intensities not only produce more faithful outcomes, but also less paradoxes than ordinary preferences (see Nurmi [35]).

Example 2 Suppose five individuals who have to compare two alternatives x and y. Three of them prefer x to y and the remaining two prefer y to x, but with different intensities of preference. If we do not take into account these intensities, then x defeats y by simple majority. However, if we allow individuals to show their real preferences between these alternatives, we may obtain different outcomes by using the fuzzy simple majority $\widetilde{M}_0$.

Suppose the following three fuzzy profiles:

1. $(d_1,d_2,d_3,d_4,d_5) = (0.8,0.6,0.7,0.2,0.3)$: $\sum_{i=1}^{5} d_i = 2.6 > 2.4 = \sum_{i=1}^{5} N(d_i)$. Then, x defeats y for $\widetilde{M}_0$ ($\widetilde{M}_k$, for every $k \in [0,0.2)$).
2. $(d_1,d_2,d_3,d_4,d_5) = (0.8,0.6,0.7,0.1,0.3)$: $\sum_{i=1}^{5} d_i = 2.5 = \sum_{i=1}^{5} N(d_i)$. Then, x and y tie for $\widetilde{M}_0$ ($\widetilde{M}_k$, for every $k \in [0,5)$).
3. $(d_1,d_2,d_3,d_4,d_5) = (0.8,0.6,0.7,0.1,0)$: $\sum_{i=1}^{5} d_i = 2.2 < 2.8 = \sum_{i=1}^{5} N(d_i)$. Then, y defeats x for $\widetilde{M}_0$ ($\widetilde{M}_k$, for every $k \in [0,0.6)$).

Manipulability is a well-known drawback of a wide class of voting systems. In some cases, individuals can misrepresent their preferences in order to obtain an outcome closer to their desires. In the previous example, if individuals exaggerate their preferences, then they will show extreme degrees of preference, i.e. 1 or 0. Then, the result will coincide with that provided by simple majority. But some individuals with a slight inclination for an alternative could declare indifference instead of extreme preference. For instance, in the previous example, the second voter –who has only a 0.6 degree of preference of x over y– could declare indifference because that intensity is closer to indifference than the extreme preference. Anyway, it seems more appropriate to allow agents to show freely their preferences than to force them to declare extreme preferences.

3 Properties of decision rules

In this section we introduce some properties that will be used in the characterization of $\widetilde{M}_k$ and $\widetilde{M}'_k$ majorities.

Anonymity, neutrality, monotonicity and unanimity (weak and strong Pareto) are well-known in the literature and they can be defined simultaneously for ordinary and fuzzy decision rules. Anonymity means that the collective decision depends on the set of individual intensities of preference only, but not on which individuals have these preferences. Neutrality means that if all individuals reverse their preferences, then the collective decision is also reversed. Monotonicity means that if any of the individuals increase their preferences for an alternative over a second one, and the rest of the other individuals' preferences do not change, then the social preference of the first alternative over the second one should not decrease. Weak Pareto means that an alternative defeats another one whenever all individuals absolutely prefer the first alternative to the second one. And strong Pareto means that if some individuals prefer an alternative to a second one, and no individual prefers the second alternative to the first one, then the first alternative defeats the second one.

Given two profiles $(d_1,\dots,d_m),(d'_1,\dots,d'_m) \in [0,1]^m$, we use the following notation:

1. $(d'_1,\dots,d'_m) \geq (d_1,\dots,d_m)$ if $d'_i \geq d_i$ for every $i \in \{1,\dots,m\}$.
2. $(d'_1,\dots,d'_m) > (d_1,\dots,d_m)$ if $(d'_1,\dots,d'_m) \geq (d_1,\dots,d_m)$ and $d'_i > d_i$ for some $i \in \{1,\dots,m\}$.

Definition 5 Let F be an ordinary or fuzzy decision rule.

1. F is *anonymous* if for all bijection $\sigma : \{1,\dots,m\} \longrightarrow \{1,\dots,m\}$ and all profile $(d_1,\dots,d_m)$ it holds:
$$F(d_{\sigma(1)},\dots,d_{\sigma(m)}) = F(d_1,\dots,d_m).$$
2. F is *neutral* if for every profile $(d_1,\dots,d_m)$ it holds:
$$F(N(d_1),\dots,N(d_m)) = N(F(d_1,\dots,d_m)).$$
3. F is *monotonic* if for all pair of profiles $(d_1,\dots,d_m),(d'_1,\dots,d'_m)$ it holds:
$$(d'_1,\dots,d'_m) \geq (d_1,\dots,d_m) \ \Rightarrow\ F(d'_1,\dots,d'_m) \geq F(d_1,\dots,d_m).$$
4. F is *weak Pareto* if $F(1,\dots,1) = 1$ and $F(0,\dots,0) = 0$.
5. F is *strong Pareto* if for every profile $(d_1,\dots,d_m)$ it holds:
$$(d_1,\dots,d_m) > (0.5,\dots,0.5) \ \Rightarrow\ F(d_1,\dots,d_m) = 1$$
and
$$(0.5,\dots,0.5) > (d_1,\dots,d_m) \ \Rightarrow\ F(d_1,\dots,d_m) = 0.$$

Remark 3 If F is a neutral fuzzy decision rule, then:

1. $F(0.5,\dots,0.5) = 0.5$.
2. F is characterized by the set $F^{-1}(\{1\})$, since
$$F^{-1}(\{0\}) = \{(d_1,\dots,d_m) \in [0,1]^m \mid (N(d_1),\dots,N(d_m)) \in F^{-1}(\{1\})\}$$
and
$$F^{-1}(\{0.5\}) = [0,1]^m \setminus \left(F^{-1}(\{1\}) \cup F^{-1}(\{0\})\right).$$

Then, $\widetilde{M}_k$ and $\widetilde{M}'_k$ majorities are the neutral fuzzy decision rules given by

$$\widetilde{M}_k(d_1,\ldots,d_m) = 1 \;\Leftrightarrow\; \frac{1}{m}\sum_{i=1}^{m} d_i > 0.5 + \frac{k}{2m}, \text{ with } k \in [0,m),$$

$$\widetilde{M}'_k(d_1,\ldots,d_m) = 1 \;\Leftrightarrow\; \frac{1}{m}\sum_{i=1}^{m} d_i \geq 0.5 + \frac{k}{2m}, \text{ with } k \in (0,m].$$

Cancellativeness has been introduced by Llamazares [24] in his axiomatic characterization of the ordinary decision rules M_k. It establishes that, given two individuals with opposing preferences, i.e., one prefers x and the other prefers y, then the collective preference is the same as if both individuals were indifferent between x and y. We now introduce an extension of this property to fuzzy decision rules.

Definition 6 Let F be a fuzzy decision rule. F is *cancellative* if for all pair of profiles $(d_1,\ldots,d_m),(d'_1,\ldots,d'_m) \in [0,1]^m$ such that $d'_i = d_i + \varepsilon$ and $d'_j = d_j - \varepsilon$, for some $i,j \in \{1,\ldots,m\}$ and $\varepsilon > 0$, and $d'_l = d_l$ for all $l \neq i,j$, it holds $F(d'_1,\ldots,d'_m) = F(d_1,\ldots,d_m)$.

We note that if in the previous definition we consider ordinary preferences, $d_i = 0$, $d_j = 1$ and $\varepsilon = 0.5$, then we obtain the notion of cancellativeness introduced by Llamazares [24] for ordinary decision rules.

Remark 4 If F is a fuzzy decision rule, it is easy to see that F is cancellative if and only if for all pair of profiles $(d_1,\ldots,d_m),(d'_1,\ldots,d'_m) \in [0,1]^m$ such that $\sum_{i=1}^{m} d_i = \sum_{i=1}^{m} d'_i$ it holds $F(d'_1,\ldots,d'_m) = F(d_1,\ldots,d_m)$. Therefore, any cancellative fuzzy decision rule is also anonymous[5].

4 The results

We now present the results of the paper. Next theorem generalizes Llamazares [24, Theorem 8] from ordinary to fuzzy decision rules. We note that the original result requires anonymity. As mentioned above, every cancellative fuzzy decision rule is always anonymous, thus, assuming cancellativeness let us avoid anonymity in the following result.

Theorem 1 *A fuzzy decision rule F is a $\widetilde{M}_k$ or a $\widetilde{M}'_k$ majority if and only if it satisfies cancellativeness, monotonicity, weak Pareto and neutrality.*

PROOF:
$\Rightarrow$) Obvious.
$\Leftarrow$) By (2) of Remark 3, it is sufficient to prove that

[5] According to Llamazares [24, Prop. 5], this fact is not satisfied by ordinary decision rules.

$$F^{-1}(\{1\}) = \left\{ (d_1, \dots, d_m) \in [0,1]^m \;\middle|\; \frac{1}{m}\sum_{i=1}^{m} d_i > 0.5 + \frac{k}{2m} \right\},$$

with $k \in [0,m)$, or

$$F^{-1}(\{1\}) = \left\{ (d_1, \dots, d_m) \in [0,1]^m \;\middle|\; \frac{1}{m}\sum_{i=1}^{m} d_i \geq 0.5 + \frac{k}{2m} \right\},$$

with $k \in (0,m]$.

Given $(d_1,\dots,d_m),(d'_1,\dots,d'_m) \in [0,1]^m$ such as $\sum_{i=1}^{m} d'_i \geq \sum_{i=1}^{m} d_i$, we are going to show that $F(d'_1,\dots,d'_m) \geq F(d_1,\dots,d_m)$. For this, it is sufficient to take into account the following profile, defined recursively:

$$d''_i = \begin{cases} d'_i, & \text{if } \sum_{j=1}^{i-1} d''_j + d'_i < \sum_{j=1}^{m} d_j, \\ \sum_{j=1}^{m} d_j - \sum_{j=1}^{i-1} d''_j, & \text{if } \sum_{j=1}^{i-1} d''_j < \sum_{j=1}^{m} d_j \leq \sum_{j=1}^{i-1} d''_j + d'_i, \\ 0, & \text{if } \sum_{j=1}^{i-1} d''_j = \sum_{j=1}^{m} d_j. \end{cases}$$

It is easy to see that $d'_i \geq d''_i$ for every $i \in \{1,\dots,m\}$ and $\sum_{i=1}^{m} d''_i = \sum_{i=1}^{m} d_i$. By monotonicity and Remark 4, we have

$$F(d'_1,\dots,d'_m) \geq F(d''_1,\dots,d''_m) = F(d_1,\dots,d_m).$$

Therefore, if $(d_1,\dots,d_m) \in F^{-1}(\{1\})$ and $(d'_1,\dots,d'_m) \in [0,1]^m$ satisfy $\sum_{i=1}^{m} d'_i \geq \sum_{i=1}^{m} d_i$, then $(d'_1,\dots,d'_m) \in F^{-1}(\{1\})$.

Since $A = \left\{ \frac{1}{m}\sum_{i=1}^{m} d_i \;\middle|\; (d_1,\dots,d_m) \in F^{-1}(\{1\}) \right\}$ is bounded, let $\alpha = \inf A$. Two cases are possible:

1. If $\alpha \notin A$, then

$$F^{-1}(\{1\}) = \left\{ (d_1, \dots, d_m) \in [0,1]^m \;\middle|\; \frac{1}{m}\sum_{i=1}^{m} d_i > \alpha \right\}.$$

2. If $\alpha \in A$, then

$$F^{-1}(\{1\}) = \left\{ (d_1, \dots, d_m) \in [0,1]^m \;\middle|\; \frac{1}{m}\sum_{i=1}^{m} d_i \geq \alpha \right\}.$$

By weak Pareto, we have $\alpha < 1$ in the first case and $\alpha \leq 1$ in the second one. On the other hand, given a profile $(d_1,\dots,d_m)$ such that $\frac{1}{m}\sum_{i=1}^{m} d_i = 0.5$, by Remark 4 and (1) of Remark 3 we have $F(d_1,\dots,d_m) = F(0.5,\dots,0.5) = 0.5$. Therefore, $(d_1,\dots,d_m) \notin F^{-1}(\{1\})$. Then, $\alpha \in [0.5,1)$ in the first case, and $\alpha \in (0.5,1]$ in the second case. Since $\alpha = 0.5 + \frac{k}{2m} \Leftrightarrow k = 2m(\alpha - 0.5)$, then we have $F = \widetilde{M}_{2m(\alpha-0.5)}$ in the first case, and $F = \widetilde{M}'_{2m(\alpha-0.5)}$ in the second one. □

Remark 5 Axioms involved in the previous theorem are independent:

1. The fuzzy decision rule F defined by

$$F(d_1,\dots,d_m) = \begin{cases} 1, & \text{if } (d_1,\dots,d_m) > (0.5,\dots,0.5), \\ 0, & \text{if } (0.5,\dots,0.5) > (d_1,\dots,d_m), \\ d_1, & \text{otherwise}, \end{cases}$$

   satisfies monotonicity, weak Pareto and neutrality, but not anonymity, hence neither cancellativeness.
2. The fuzzy decision rule F defined by

$$F(d_1,\dots,d_m) = \begin{cases} 1, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \in (0,0.5) \cup \{1\}, \\ 0.5, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i = 0.5, \\ 0, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \in \{0\} \cup (0.5,1), \end{cases}$$

   satisfies cancellativeness, weak Pareto and neutrality, but not monotonicity.
3. The constant fuzzy decision rule F defined by $F(d_1,\dots,d_m) = 0.5$ for every profile $(d_1,\dots,d_m)$ satisfies cancellativeness, monotonicity and neutrality, but not weak Pareto.
4. The fuzzy decision rule F defined by

$$F(d_1,\dots,d_m) = \begin{cases} 1, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \geq 0.5, \\ 0, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i < 0.5, \end{cases}$$

   satisfies cancellativeness, monotonicity and weak Pareto, but not neutrality.

We now provide a characterization of the fuzzy version of simple majority by means of only three independent axioms.

Theorem 2 *A fuzzy decision rule F is the $\widetilde{M}_0$ majority if and only if it satisfies cancellativeness, strong Pareto and $F(0.5,\dots,0.5) = 0.5$.*

PROOF:
$\Rightarrow$) Obvious.
$\Leftarrow$) Let F be a cancellative and strong Pareto fuzzy decision rule that satisfies $F(0.5,\dots,0.5)=0.5$. Given a profile $(d_1,\dots,d_m)\in[0,1]^m$ such that $\frac{1}{m}\sum_{i=1}^{m} d_i >$ 0.5, consider the profile $(d'_1,\dots,d'_m)\in[0,1]^m$, where $d'_j=\frac{1}{m}\sum_{i=1}^{m} d_i$ for every $j\in\{1,\dots,m\}$. By Remark 4 and strong Pareto, we have

$$F(d_1,\dots,d_m)=F(d'_1,\dots,d'_m)=1.$$

In a similar way, it is possible to prove that if $(d_1,\dots,d_m)\in[0,1]^m$ is a profile that satisfies $\frac{1}{m}\sum_{i=1}^{m} d_i<0.5$, then $F(d_1,\dots,d_m)=0$. Finally, since $F(0.5,\dots,0.5)=$ 0.5, F is the $\widetilde{M}_0$ majority. $\square$

Remark 6 Axioms involved in the previous theorem are independent:

1. The fuzzy decision rule F defined in (1) of Remark 5 satisfies strong Pareto and $F(0.5,\dots,0.5)=0.5$, but not cancellativeness.
2. The fuzzy decision rule F defined in (3) of Remark 5 satisfies cancellativeness and $F(0.5,\dots,0.5)=0.5$, but not strong Pareto.
3. The fuzzy decision rule F defined in (4) of Remark 5 satisfies cancellativeness and strong Pareto, but $F(0.5,\dots,0.5)\neq 0.5$.

By (1) of Remark 3, any neutral fuzzy decision rule satisfies $F(0.5,\dots,0.5)=$ 0.5. Thus, we can also obtain the following characterization of the $\widetilde{M}_0$ majority.

Corollary 1 *A fuzzy decision rule F is the $\widetilde{M}_0$ majority if and only if it satisfies cancellativeness, strong Pareto and neutrality.*

Examples given in Remark 6 show that cancellativeness, strong Pareto and neutrality are also independent properties.

5 Concluding remarks

In this paper we have extended majorities based on difference of votes to the context of fuzzy preferences, by allowing agents to declare intensities of preference through numbers in the unit interval. With these new voting systems, an alternative defeats another one whenever the amount of opinion obtained by the first alternative exceeds the amount of opinion obtained by the second one in a previously fixed threshold.

The new voting systems, based on difference of support between alternatives, are quite flexible and are located between the fuzzy version of simple majority and the unanimous majority (and alternative defeats another one only if all the voters definitely prefer the first alternative to the second one).

A relevant issue in the field of the Social Choice Theory is to provide axiomatic characterizations of voting systems. This task allows to know deeply how voting systems work. Following this desideratum, we have obtained an axiomatic characterization of the introduced voting systems through four independent axioms. Three of them are classic in the Social Choice Theory: monotonicity, weak Pareto (unanimity with respect to the absolute preference), and neutrality. The fourth axiom, cancellativeness, is a stronger condition than that of anonymity and it requires that profiles with the same amount the opinion over the alternatives should produce the same outcome.

In addition, we have provided two characterizations of the fuzzy version of simple majority through three independent axioms: cancellativeness, strong Pareto and either unanimity with respect to the indifference or neutrality.

Finally, we note that in our proposal we have only considered two alternatives. This is a classic approach for dealing with simple majority and related voting systems. The reason is to avoid inconsistencies, paradoxes and strategic behaviour.

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