Annexes

In this document, we present in detail the calculations of our work, which lead to the results presented in the main text. The first annex corresponds to the computation of the $N = 1$ case. The last one includes the development of the $N = 2$ case. We have organised them in the same way as we present the results in the principal work.

A Annex I: N=1 Case

Here we include the expanded calculations of the first case considered in Section 3.

We start with the next Lagrangian:

$$B_1 = -\frac{1}{4\kappa^2} R^{ab} \wedge \Sigma_{ab} + \frac{i}{2}\bar{\psi} \wedge \gamma^5 \gamma_{(1)} \wedge D\psi \tag{1}$$

Where we use the conventions defined in Section 1 of the present work.

We have the following relations:

$$\begin{cases} \gamma_{(1)} := \gamma_a e^a \\ \gamma_{a_1 \ldots a_r} := \gamma_{[a_1 \ldots} \gamma_{a_r]} \to \gamma_{ab} = \gamma_{[a}\gamma_{b]} = \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a) \\ \rho \equiv D\psi := d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi \\ R^{ab} := d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \\ \Sigma_{a_1 \ldots a_r} := \frac{1}{(D-r)!}\epsilon_{a_1 \ldots a_r a_{r+1} \ldots a_D} e^{a_{r+1}} \wedge \ldots \wedge e^{a_D} \to \Sigma_{ab} := \frac{1}{2}\epsilon_{abcd} e^c \wedge e^d \end{cases} . \tag{2}$$

Then, we can write the Lagrangian like follows

$$B_1 = -\frac{1}{8\kappa^2}\epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + \frac{i}{2}\bar{\psi} \wedge \gamma^5 \gamma_a e^a \wedge \rho \tag{3}$$

A.1 H_1 calculation

We are going to calculate $H_1 = dB_1$. But, as we already know, we only have to calculate the covariant exterior differential of the Lagrangian, *i.e.*, $H_1 = DB_1$.

$$\begin{aligned} H_1 = & -\frac{1}{8\kappa^2}\epsilon_{abcd} DR^{ab} \wedge e^c \wedge e^d - \frac{1}{8\kappa^2}\epsilon_{abcd} R^{ab} \wedge De^c \wedge e^d + \frac{1}{8\kappa^2}\epsilon_{abcd} R^{ab} \wedge e^c \wedge De^d \\ & + \frac{i}{2}\bar{\rho} \wedge \gamma^5 \gamma_a e^a \wedge \rho - \frac{i}{2}\bar{\psi} \wedge \gamma^5 \gamma_a De^a \wedge \rho + \frac{i}{2}\bar{\psi} \wedge \gamma^5 \gamma_a e^a \wedge D\rho \end{aligned} \tag{4}$$

Firstly, it is easy to prove that DR^{ab} vanishes by the Bianchi identities:

$$
\begin{aligned}
DR^{ab} &= (dR + [\omega, R])^{ab} = (dR + \omega \wedge R - R \wedge \omega)^{ab} \\
&= (d^2\omega + d\omega \wedge \omega - \omega \wedge d\omega + \omega \wedge R - R \wedge \omega)^{ab} \\
&= (R \wedge \omega - \omega \wedge R + \omega \wedge R - R \wedge \omega)^{ab} = 0
\end{aligned} \tag{5}
$$

In the fourth step we have used that $d^2 = 0$ and $R^{ab} = d\omega^{ab}$, and we have proved what we wanted.

The second step consists in the fact that we can simplify the H_1 expression if we note that,

$$
-\frac{1}{8\kappa^2}\epsilon_{abcd}R^{ab} \wedge De^c \wedge e^d = \frac{1}{8\kappa^2}\epsilon_{abdc}R^{ab} \wedge e^c \wedge De^d \tag{6}
$$

Then, if we substitute this two results in (4) we have

$$
\begin{aligned}
H_1 =& \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge De^d + \frac{i}{2}\bar{\rho} \wedge \gamma^5\gamma_a e^a \wedge \rho \\
& - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a De^a \wedge \rho + \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a e^a \wedge D\rho
\end{aligned} \tag{7}
$$

At this point, we define (as we can see from the algebra)

$$
\begin{cases}
\Theta^a \equiv De^a := de^a + \omega^a{}_b \wedge e^b \\
T^a := De^a - \dfrac{i\kappa^2}{2}\bar{\psi}\gamma^a \wedge \psi
\end{cases} \tag{8}
$$

Expression which corresponds to the torsion. We take the expression of De^a:

$$
De^a = T^a + \frac{i\kappa^2}{2}\bar{\psi}\gamma^a \wedge \psi \tag{9}
$$

And we can write (7) as,

$$
\begin{aligned}
H_1 =& \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge (T^d + \frac{i\kappa^2}{2}\bar{\psi}\gamma^d \wedge \psi) + \frac{i}{2}\bar{\rho} \wedge \gamma^5\gamma_a e^a \wedge \rho \\
& - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a (T^a + \frac{i\kappa^2}{2}\bar{\psi}\gamma^a \wedge \psi) \wedge \rho + \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a e^a \wedge D\rho
\end{aligned} \tag{10}
$$

We follow calculating $D\rho$. We know that the covariant exterior differential for spinorial forms is given by:

$$
D\lambda := d\lambda + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \lambda \tag{11}
$$

That implies, as ρ is directly related with spinors by definiton, $\rho := D\psi \to D\rho = D(D\psi)$,

$$
\begin{aligned}
D\rho &= d\rho + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \rho = d(d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi) + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \rho \\
&= d^2\psi + \frac{1}{4}\gamma_{ab}d\omega^{ab} \wedge \psi - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \rho \\
&= \frac{1}{4}\gamma_{ab}R^{ab} \wedge \psi - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \rho + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \rho = \frac{1}{4}\gamma_{ab}R^{ab} \wedge \psi
\end{aligned} \tag{12}
$$

Substituting in (10),

$$
\begin{aligned}
H_1 =& \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c \wedge T^d + \frac{i}{8}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\bar{\psi}\gamma^d\wedge\psi + \frac{i}{2}\bar{\rho}\wedge\gamma^5\gamma_a e^a\wedge\rho \\
&- \frac{i}{2}\bar{\psi}\wedge\gamma^5\gamma_a T^a\wedge\rho - \frac{i}{2}\bar{\psi}\wedge\gamma^5\gamma_a(\frac{i\kappa^2}{2}\bar{\psi}\gamma^a\wedge\psi)\wedge\rho + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_a e^a\wedge\gamma_{bc}R^{bc}\wedge\psi
\end{aligned} \tag{13}
$$

Where, the term $-\frac{i}{2}\bar{\psi}\wedge\gamma^5\gamma_a(\frac{i\kappa^2}{2}\bar{\psi}\gamma^a\wedge\psi)\wedge\rho$ vanishes by the Fierz identity: $\bar{\psi}\gamma_a\wedge\bar{\psi}\gamma^a\wedge\psi = 0$.

We continue proving that,

$$
\frac{i}{8}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\bar{\psi}\gamma^d\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_a e^a\wedge\gamma_{bc}R^{bc}\wedge\psi = 0 \tag{14}
$$

To do that, we use which follows

$$
\gamma_a\gamma_{bc} = \gamma_{abc} + \eta_{ab}\gamma_c - \eta_{ac}\gamma_b \tag{15}
$$

Then,

$$
\begin{aligned}
&\frac{i}{8}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\bar{\psi}\gamma^d\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_a e^a\wedge\gamma_{bc}R^{bc}\wedge\psi \\
&= \frac{i}{8}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\bar{\psi}\gamma^d\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_{abc}e^a\wedge R^{bc}\wedge\psi \\
&+ \frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ab}\gamma_c e^a\wedge R^{bc}\wedge\psi - \frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ac}\gamma_b e^a\wedge R^{bc}\wedge\psi
\end{aligned} \tag{16}
$$

where the two last terms give us the next,

$$
\begin{aligned}
&\frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ab}\gamma_c e^a\wedge R^{bc}\wedge\psi - \frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ac}\gamma_b e^a\wedge R^{bc}\wedge\psi \\
&= \frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ab}\gamma_c e^a\wedge R^{bc}\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\eta_{ac}\gamma_b e^a\wedge R^{cb}\wedge\psi \\
&= \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_c e^a\wedge R^c{}_a\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_b e^a\wedge R^b{}_a\wedge\psi = \frac{i}{4}\bar{\psi}\wedge\gamma^5\gamma_b e^a\wedge R^b{}_a\wedge\psi
\end{aligned} \tag{17}
$$

Which vanishes by $\bar{\psi}\wedge\gamma^5\gamma_a\psi = 0$, because γ_b are symmetric matrices.

Finally, we have the terms

$$
\frac{i}{8}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\bar{\psi}\gamma^d\wedge\psi + \frac{i}{8}\bar{\psi}\wedge\gamma^5\gamma_{abc}e^a\wedge R^{bc}\wedge\psi, \tag{18}
$$

which we are going to prove that cancel each other. We know that,

$$
\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \to \gamma^5 \propto \epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d \to \gamma^5 \propto \gamma^{abcd} \tag{19}
$$

Then,

$$
\gamma^5\gamma_{abc} = \lambda\epsilon_{abcd}\gamma^d \to \gamma^5\gamma_{012} = \lambda\epsilon_{0123}\gamma^3 \tag{20}
$$

Where $\epsilon_{0123} = 1$ and $\gamma_{012} = \gamma_{[0}\gamma_1\gamma_{2]} = \gamma_0\gamma_1\gamma_2$, because the γ_a matrices anticommute each other (they have different indices).

We have

$$\begin{aligned}
&\gamma^0\gamma^1\gamma^2\gamma^3\gamma_0\gamma_1\gamma_2 = -\gamma^1\gamma^2\gamma^3(\gamma^0\gamma_0)\gamma_1\gamma_2 = -\gamma^1\gamma^2\gamma^3\gamma_1\gamma_2 \\
&= -\gamma^2\gamma^3(\gamma^1\gamma_1)\gamma_2 = -\gamma^2\gamma^3\gamma_2 = \gamma^3\gamma^2\gamma_2 = \gamma^3
\end{aligned} \tag{21}$$

Where we have used that: $\gamma^0\gamma_0 = I$, $\gamma^1\gamma_1 = I$ and $\gamma^2\gamma_2 = I$. That implies:

$$\gamma^3 = \lambda\gamma^3 \to \lambda = 1 \tag{22}$$

Then,

$$\gamma^5\gamma_{abc} = \epsilon_{abcd}\gamma^d \tag{23}$$

Substituting the last result in our expression, we have

$$\begin{aligned}
&\frac{i}{8}\epsilon_{abcd}R^{ab} \wedge e^c \wedge \bar{\psi}\gamma^d \wedge \psi + \frac{i}{8}\bar{\psi} \wedge \epsilon_{abcd}\gamma^d e^a \wedge R^{bc} \wedge \psi = \\
&\frac{i}{8}\epsilon_{abcd}R^{ab} \wedge e^c \wedge \bar{\psi}\gamma^d \wedge \psi - \frac{i}{8}\epsilon_{abcd}R^{bc} \wedge e^a \wedge \bar{\psi}\gamma^d \wedge \psi = 0
\end{aligned} \tag{24}$$

Finally, we arrive to the wanted expression of H_1,

$$H_1 = \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d + \frac{i}{2}\bar{\rho} \wedge \gamma^5\gamma_a e^a \wedge \rho - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a T^a \wedge \rho \tag{25}$$

We can write the last term as follows,

$$\frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a T^a \wedge \rho \to \frac{i}{2}\bar{\rho} \wedge \gamma^5\gamma_a T^a \wedge \psi \tag{26}$$

A.2 Motion equations

From (25) we are going to obtain the equations of motion. To do that, we need to calculate $I_{\alpha^{a_1\cdots a_n}}H_1$ ($\alpha^{a_1\cdots a_n}$ is a gauge field or curvature) and impose that it vanishes. We have three equations:

1. Eq. of $\omega^{ab} \to I_{R^{ab}}H_1 = 0$.
2. Eq. of $e^a \to I_{T^a}H_1 = 0$.
3. Eq. of $\bar{\psi} \to I_{\bar{\rho}}H_1 = 0$ (it is equivalent for ψ).

For this, we only have to derive with respect to the correspondent form in (25).

1. We must derive with respect to R^{ab} and R^{ba} (the last contribution has a plus sign):

$$I_{R^{ab}}H_1 = \frac{1}{2\kappa^2}\epsilon_{abcd}e^c \wedge T^d = 0 \tag{27}$$

2. The second one is,

$$I_{T^d}H_1 = \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_d\rho = 0 \tag{28}$$

3. As we said before, it is equivalent to derive with respect to ρ or $\bar{\rho}$, because we have real spinors:

$$I_{\bar{\rho}}H_1 = i\gamma^5\gamma_a e^a \wedge \rho - \frac{i}{2}\gamma^5\gamma_a T^a \wedge \psi = 0 \tag{29}$$

A.3 Vanishing Torsion

We can continue with the calculation of $I_{\bar{\psi}}H_1$, which takes the zero value when we substitute the equations of motion.

$$I_{\bar{\psi}}H_1 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho \tag{30}$$

From the equation of ω^{ab} we are going to obtain that $T^a = 0$ (vanishing-torsion). We note that if we put this result in the last relation, we will have what we want. Then, we are going to prove it.

As we said, from the first equation of motion we have (without a constant)

$$\epsilon_{abcd} e^c \wedge T^d = 0 \tag{31}$$

The forms e^a constitute a base of $T^*(M)$ (the cotangent space of the manifold M), which takes the expression $e^a = e^a{}_\mu dx^\mu$ with $e^a{}_\mu$ invertible. So, the torsion reads in this basis

$$\begin{aligned} T^d = T^d{}_{uv} \wedge e^u \wedge e^v \to \epsilon_{abcd} e^c \wedge T^d{}_{uv} \wedge e^u \wedge e^v = 0 \\ \to \epsilon_{abcd} e^c \wedge T^d{}_{uv} \wedge e^u \wedge e^v \propto \epsilon_{abcd} T^d{}_{uv} \wedge \epsilon^{cuvg} E_g = 0 \end{aligned} \tag{32}$$

Where we have,

$$E_g \propto \epsilon_{guvw} e^u \wedge e^v \wedge e^w \ (E_g \neq 0) \tag{33}$$

Because we know that the e^as form a basis, so they are linearly independent. That implies,

$$\epsilon_{abcd}\epsilon^{cuvg} T^d{}_{uv} = 0 \propto \delta^{uvg}_{abd} T^d{}_{uv} = 0 \tag{34}$$

In the first relation we have contracted the c index. Now, we know that

$$\begin{aligned} \delta^{uvg}_{abd} &= \delta^g{}_d \delta^{uv}{}_{ab} - \delta^g{}_b \delta^{uv}{}_{ad} + \delta^g{}_a \delta^{uv}{}_{bd} \\ &= \delta^g{}_d \delta^u{}_a \delta^v{}_b - \delta^g{}_d \delta^u{}_b \delta^v{}_a - \delta^g{}_b \delta^u{}_a \delta^v{}_d + \delta^g{}_b \delta^u{}_d \delta^v{}_a + \delta^g{}_a \delta^u{}_b \delta^v{}_d - \delta^g{}_a \delta^u{}_d \delta^v{}_b \end{aligned} \tag{35}$$

Substituting,

$$T^g{}_{ab} - T^g{}_{ba} - \delta^g{}_b T^d{}_{ad} + \delta^g{}_b T^d{}_{da} + \delta^g{}_a T^d{}_{bd} - \delta^g{}_a T^d{}_{db} = 0 \tag{36}$$

Interchanging the indices and reorganizing,

$$2T^g{}_{ab} - 2\delta^g{}_b T^d{}_{ad} + 2\delta^g{}_a T^d{}_{bd} = 0 \tag{37}$$

Contracting b with g:

$$2T^b{}_{ab} - 2T^d{}_{ad} - 2T^d{}_{ad} = 2T^b{}_{ab} - 4T^d{}_{ad} = 0 \to T^b{}_{ab} = 0 \tag{38}$$

Finally, we have

$$T^g{}_{ab} = 0 \to T^g = 0 \tag{39}$$

And it is what we wanted to prove.

So, we can do

$$I_{\bar{\psi}}H_1 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho \cong 0 \tag{40}$$

Like we mentioned.

A.4 Einstein and Rarita-Schwinger equations

We have used the first equation of motion to derive that the torsion vanishes. Now we use the other two to derive the Einstein and Rarita-Schwinger (R-S) equations, respectively.

a) Einstein equations:
From the second eq. of motion we have,

$$\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c - \frac{i}{2}\bar{\psi}\wedge\gamma^5\gamma_d\rho = 0 \tag{41}$$

We take the first term (without constants). Manipulating it,

$$\epsilon_{abcd}R^{ab}\wedge e^c = \epsilon_{abcd}R^{ab}{}_{uv}\wedge e^c\wedge e^d\wedge e^u\wedge e^v = \epsilon_{abcd}\epsilon^{cuvs}R^{ab}{}_{uv}\wedge E_s \tag{42}$$

where,

$$E_s \propto \epsilon_{cuvs}e^c\wedge e^u\wedge e^v \tag{43}$$

Then, we have ($\epsilon_{abcd}\epsilon^{cuvs} = -\delta^{uvs}_{abd}$):

$$\begin{aligned}
&-\delta^{uvs}_{abd}R^{ab}{}_{uv}\wedge E_s = (\delta^s{}_d\delta^{uv}{}_{ab} - \delta^s{}_b\delta^{uv}{}_{ad} + \delta^s{}_a\delta^{uv}{}_{bd})R^{ab}{}_{uv}\wedge(-E_s)\\
&= (\delta^s{}_d\delta^u{}_a\delta^v{}_b - \delta^s{}_d\delta^u{}_b\delta^v{}_a - \delta^s{}_b\delta^u{}_a\delta^v{}_d + \delta^s{}_b\delta^u{}_d\delta^v{}_a + \delta^s{}_a\delta^u{}_b\delta^v{}_d - \delta^s{}_a\delta^u{}_d\delta^v{}_b)R^{ab}{}_{uv}\wedge(-E_s)\\
&= [\delta^s{}_d(R^{uv}{}_{uv} - R^{uv}{}_{vu}) - \delta^s{}_b(R^{ab}{}_{ad} - R^{ab}{}_{da}) + \delta^s{}_a(R^{ab}{}_{bd} - R^{ab}{}_{db})]\wedge(-E_s)\\
&= [2\delta^s{}_dR^{uv}{}_{uv} - 2\delta^s{}_bR^{ab}{}_{ad} + 2\delta^s{}_aR^{ab}{}_{bd}]\wedge(-E_s) = [2\delta^s{}_dR^{uv}{}_{uv} - 4R^{sb}{}_{db}]\wedge(-E_s)
\end{aligned} \tag{44}$$

We take (-4) as common factor and the fact that $\delta^s{}_d \equiv \eta^s{}_d$:

$$[R^{sb}{}_{db} - \frac{1}{2}\eta^s{}_dR^{uv}{}_{uv}]\wedge(4E_s) \tag{45}$$

So,

$$\frac{1}{\kappa^2}[R^{sb}{}_{db} - \frac{1}{2}\eta^s{}_dR^{uv}{}_{uv}] = 0 \tag{46}$$

as we know for Einstein vacuum field equations.

b) R-S equations:
From the last eq.

$$i\gamma^5\gamma_ae^a\wedge\rho - \frac{i}{2}\gamma^5\gamma_aT^a\wedge\psi = 0 \tag{47}$$

The second term vanish if we impose that $T^a = 0$.
So, we have

$$\gamma^5\gamma_ae^a\wedge\rho = 0 \tag{48}$$

We write,

$$\gamma^5\gamma_ae^a_\mu\wedge\rho_{\nu\alpha}\wedge dx^\mu\wedge dx^\nu\wedge dx^\alpha = \gamma^5\gamma_\mu\epsilon^{\mu\nu\alpha\sigma}\wedge\rho_{\nu\alpha} = 0 \tag{49}$$

and,

$$\gamma^{\nu\alpha\sigma}\rho_{\nu\alpha} = 0 \tag{50}$$

the R-S eq.

A.5 Symmetries of the action

Finally, we would have to study the invariance. We know that $I_{\bar{\psi}}H_1 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho$. We would need to check that T^a is related to $I_{R^{ab}}H$, which implies that $I_{\bar{\psi}}H_1 = \frac{1}{2}X^{ab} \wedge I_{R^{ab}}H$. Here, X^{ab} is the term that we should calculate to assure the invariance. After an easy calculation, we obtain,

$$X^{ab} = -i\kappa^2\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge \rho_{ud} - \frac{i\kappa^2}{2}\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge \rho_{cd} \tag{51}$$

We already have all the ingredients to work out the field variations. According to the lemma, they are:

$$\begin{cases} \delta\omega^{ab} = -\bar{\epsilon}X^{ab} \\ \delta e^a = i\kappa^2\bar{\epsilon}\gamma^a\psi \\ \delta\psi = d\epsilon + \frac{1}{4}\gamma_{ab}\omega^{ab}\epsilon, \end{cases} \tag{52}$$

With ϵ the parameter of the gauge transformations. The difference respect to the original variations is that $\delta\omega^{ab} = 0$ in the gauge case. With these results we conclude our work in the $N = 1$ case.

B Annex II: N=2 Case

We present here the calculations of Section 4.

As in the $N = 1$ case, we are going to start considering the following Lagrangian,

$$B_2 := -\frac{1}{4\kappa^2}R^{ab} \wedge \Sigma_{ab} + \frac{i}{2}\bar{\psi}_A \wedge \gamma^5\gamma_{(1)} \wedge D\psi^A - \frac{1}{2}F \wedge_* F + ({}_*F - b) \wedge (dA - a) - \frac{1}{2}a \wedge b \tag{53}$$

Where we have that

$$\begin{cases} \gamma_{(1)} := \gamma_a e^a \\ \gamma_{ab} := \gamma_{[a}\gamma_{b]} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a) \\ R^{ab} := d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \\ \Sigma_{ab} := \frac{1}{2}\epsilon_{abcd}e^c \wedge e^d \\ \rho^A \equiv D\psi^A := d\psi^A + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi^A \\ a := \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B \\ b := \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge \psi^B \end{cases} \tag{54}$$

Where, if we remember, $A = 1, 2$, $\varepsilon^1{}_2 = -\varepsilon^2{}_1 = 1$, $\varepsilon^A{}_C\varepsilon^C{}_B = -\delta^A{}_B$ and the $*$ operator is the standard Hodge-duality.

We note that the first two terms are very similar to the last case, with the difference that now the spinors have the A index. It corresponds to the internal global $SO(2)$ R-symmetries. The new terms involve the two-forms a and b, which we have defined before, and the one-form A and the two-form F, as we stated in Section 4.

B.1 Action differential calculation

We are going to calculate $H_2 = DB_2$. But, we can rewrite the Lagrangian as follows,

$$B_2 = -\frac{1}{8\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d + \frac{i}{2}\bar{\psi}_A\wedge\gamma^5\gamma_a e^a\wedge\rho^A - \frac{1}{2}F\wedge_* F + {}_*F\wedge dA - {}_*F\wedge a - b\wedge dA + \frac{1}{2}a\wedge b \quad (55)$$

Now, we have to calculate the exterior covariant exterior differential. We do that term by term:

1. The first term is,

$$-\frac{1}{8\kappa^2}\epsilon_{abcd}D(R^{ab}\wedge e^c\wedge e^d) = \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c\wedge De^d \quad (56)$$

   We know that $DR^{ab} = 0$, and we can join the terms with De^a.

2. The second gives,

$$\frac{i}{2}D(\bar{\psi}_A\wedge\gamma^5\gamma_a e^a\wedge\rho^A) = \frac{i}{2}\bar{\rho}_A\wedge\gamma^5\gamma_a e^a\wedge\rho^A - \frac{i}{2}\bar{\psi}_A\wedge\gamma^5\gamma_a De^a\wedge\rho^A + \frac{i}{8}\bar{\psi}_A\wedge\gamma^5\gamma_a e^a\wedge\gamma_{bc}R^{bc}\wedge\psi^A \quad (57)$$

   Where we have, $D\rho^A = \frac{1}{4}\gamma_{ab}R^{ab}\wedge\psi^A$.

3. To calculate the next term, we write

$$\begin{cases} F = F_{ab}\wedge e^a\wedge e^b \\ {}_*F = \lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d \end{cases} \quad (58)$$

   Where we have defined the Hodge-duality with λ a constant that we do not need to find, at the moment (but we know that its value is 1/2, as we will see later).

   Then, we have

$$-\frac{1}{2}{}_*F\wedge F = -\frac{\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v \quad (59)$$

   So,

$$\begin{aligned} -\frac{\lambda}{2}\epsilon_{abcd}D(F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v) &= -\frac{\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v \\ &+ \lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge De^d\wedge F_{uv}\wedge e^u\wedge e^v - \frac{\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge DF_{uv}\wedge e^u\wedge e^v \\ &+ \lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge De^v \end{aligned} \quad (60)$$

4. The next reads,

$$\begin{aligned} D({}_*F\wedge dA) &= D({}_*F)\wedge dA +_* F\wedge D(dA) = D({}_*F)\wedge dA \\ &= \lambda\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge dA - 2\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge De^d\wedge dA \end{aligned} \quad (61)$$

5. Now follows,

$$
\begin{aligned}
&-D({}_*F\wedge a) = -D({}_*F)\wedge a - {}_*F\wedge Da = -\frac{i\kappa\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge \varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B \\
&+ i\kappa\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge De^d\wedge\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B + i\kappa\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\bar{\psi}_A\wedge\rho^B
\end{aligned}
\tag{62}
$$

In the last term we have used the properties of the spinors.

6. We continue with,

$$
-D(b\wedge dA) = -Db\wedge dA - b\wedge D(dA) = -Db\wedge dA = i\kappa\varepsilon^A{}_B\bar{\psi}_A\gamma^5\wedge\rho^B\wedge dA \tag{63}
$$

In a similar form as before.

7. The last is,

$$
\begin{aligned}
&\frac{1}{2}D(a\wedge b) = \frac{1}{2}(Da\wedge b + a\wedge Db) \\
&= \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A\wedge\rho^B\wedge\varepsilon^C{}_D\bar{\psi}_C\gamma^5\wedge\psi^D + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B\wedge\varepsilon^C{}_D\bar{\psi}_C\gamma^5\wedge\rho^D
\end{aligned}
\tag{64}
$$

It is the simplified result.

Useful expressions:
We have all the terms of H_2. But, before writing it, we are going to do two things. First, we are going to relate De^a with the torsion curvature, as in the $N=1$ case. The other thing is expressing the terms with dA in terms of the equation that involves A and F.

We can write (as we did before)

$$
\begin{cases}
\Theta^a \equiv De^a := de^a + \omega^a{}_b\wedge e^b \\
T^a := De^a - \dfrac{i\kappa^2}{2}\bar{\psi}_A\gamma^a\wedge\psi^A
\end{cases}
\tag{65}
$$

The last implies,

$$
De^a = T^a + \frac{i\kappa^2}{2}\bar{\psi}_A\gamma^a\wedge\psi^A \tag{66}
$$

Now, we can derive the equation that establishes the relation between F and A, considering the next expressions from B_2

$$
\begin{cases}
{}_*F\wedge F = \lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v \\
{}_*F\wedge(dA-a) = \lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge(dA-a)
\end{cases}
\tag{67}
$$

So,

$$
E\wedge F^{ab}\wedge F_{ab} = \epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge(dA-a) \tag{68}
$$

Which implies,

$$E \wedge F_{ab} = \epsilon_{abcd} e^c \wedge e^d \wedge (dA - a) \tag{69}$$

And $(dA - a) \equiv (dA - a)_{uv}$. Then,

$$E \wedge F_{ab} = \epsilon_{abcd}(dA - a)_{uv} \wedge \epsilon^{cduv} E = (dA - a)_{ab} \wedge E \tag{70}$$

Finally,

$$F = dA - a \tag{71}$$

With this, we can write the terms with dA in H_2 as follows,

$$(...)dA = (...)(dA - F - a) + (...)(F + a) \tag{72}$$

We have all the ingredients to give the expression of H_2. It reads,

$$\begin{aligned}
H_2 = &\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d + \frac{i}{8}\epsilon_{abcd}R^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A + \frac{i}{2}\bar{\rho_A} \wedge \gamma^5\gamma_a e^a \wedge \rho^A \\
&- \frac{i}{2}\bar{\psi_A} \wedge \gamma^5\gamma_a T^a \wedge \rho^A + \frac{\kappa^2}{4}\bar{\psi_A}\gamma^5\gamma_a \wedge \bar{\psi_B}\gamma^a \wedge \psi^B \wedge \rho^A \\
&+ \frac{i}{8}\bar{\psi_A} \wedge \gamma^5\gamma_a e^a \wedge \gamma_{bc}R^{bc} \wedge \psi^A - \frac{\lambda}{2}\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge e^v \\
&+ \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge F_{uv} \wedge e^u \wedge e^v + \frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A \wedge F_{uv} \wedge e^u \wedge e^v \\
&- \frac{\lambda}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge DF_{uv} \wedge e^u \wedge e^v + \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge T^v \\
&+ \frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge \bar{\psi_A}\gamma^v \wedge \psi^A + \lambda\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge (dA - F - a) \\
&+ \lambda\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge (F + a) - 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (dA - F - a) \\
&- 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (F + a) - i\lambda\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A \wedge (dA - F - a) \\
&- i\lambda\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A \wedge (F + a) - \frac{i\kappa\lambda}{2}\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge \varepsilon^A{}_B\bar{\psi_A} \wedge \psi^B \\
&+ i\kappa\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge \varepsilon^A{}_B\bar{\psi_A} \wedge \psi^B - \frac{\kappa^3\lambda}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A \wedge \varepsilon^B{}_C\bar{\psi_B} \wedge \psi^C \\
&+ i\kappa\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge \varepsilon^A{}_B\bar{\psi_A} \wedge \rho^B + i\kappa\varepsilon^A{}_B\bar{\psi_A}\gamma^5 \wedge \rho^B \wedge (dA - F - a) \\
&+ i\kappa\varepsilon^A{}_B\bar{\psi_A}\gamma^5 \wedge \rho^B \wedge (F + a) + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi_A} \wedge \rho^B \wedge \varepsilon^C{}_D\bar{\psi_C}\gamma^5 \wedge \psi^D \\
&+ \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi_A} \wedge \psi^B \wedge \varepsilon^C{}_D\bar{\psi_C}\gamma^5 \wedge \rho^D
\end{aligned} \tag{73}$$

B.2 Final H_2 expression

We can simplify a lot of terms of this equation and give the final expression of H_2.

a) We are going to prove that

$$\frac{i}{8}\epsilon_{abcd}R^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A + \frac{i}{8}\bar{\psi_A} \wedge \gamma^5\gamma_a e^a \wedge \gamma_{bc}R^{bc} \wedge \psi^A = 0 \tag{74}$$

We know,

$$\begin{cases}\gamma_a\gamma_{bc} = \gamma_{abc} + \eta_{ab}\gamma_c - \eta_{ac}\gamma_b \\ \gamma^5\gamma_{abc} = \epsilon_{abcd}\gamma^d\end{cases} \tag{75}$$

The second term gives us,

$$\frac{i}{8}\bar{\psi}_A \wedge \gamma^5\gamma_a e^a \wedge \gamma_{bc}R^{bc} \wedge \psi^A = \frac{i}{8}\bar{\psi}_A \wedge \epsilon_{abcd}\gamma^d e^a \wedge R^{bc} \wedge \psi^A + \frac{i}{4}\bar{\psi}_A \wedge \gamma^5\gamma_b e^a \wedge R^b{}_a \wedge \psi^A \tag{76}$$

And,

$$\frac{i}{4}\bar{\psi}_A \wedge \gamma^5\gamma_b e^a \wedge R^b{}_a \wedge \psi^A = 0 \tag{77}$$

It is because $\bar{\psi}_A \wedge \gamma^5\gamma_b \wedge \psi^A = 0$. So,

$$\frac{i}{8}\epsilon_{abcd}R^{ab} \wedge e^c \wedge \bar{\psi}_A\gamma^d \wedge \psi^A - \frac{i}{8}\epsilon_{abcd}R^{bc} \wedge e^a \wedge \bar{\psi}_A\gamma^d \wedge \psi^A = 0 \tag{78}$$

b) We take the terms that depend on torsion.

$$\begin{aligned}&\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d - \frac{i}{2}\bar{\psi}_A \wedge \gamma^5\gamma_a T^a \wedge \rho^A + \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge F_{uv} \wedge e^u \wedge e^v \\ &+ \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge T^v - 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (dA - F - a) \\ &- 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (F + a) + i\kappa\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge \varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B\end{aligned} \tag{79}$$

This can be simplified, if we note

$$\begin{aligned}-2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (F + a) =& - 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge F_{uv} \wedge e^u \wedge e^v \\ &- i\kappa\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge \varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B\end{aligned} \tag{80}$$

which gives,

$$\begin{aligned}&\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d - \frac{i}{2}\bar{\psi}_A \wedge \gamma^5\gamma_a T^a \wedge \rho^A - \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge F_{uv} \wedge e^u \wedge e^v \\ &+ \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge T^v - 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (dA - F - a) \\ &= \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d - \frac{i}{2}\bar{\psi}_A \wedge \gamma^5\gamma_a T^a \wedge \rho^A + \lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge T^v \\ &- 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (dA - a - \frac{F}{2})\end{aligned} \tag{81}$$

These terms do not give problems because, as we have proved (the result in $N = 1$ can be extended to any dimension in the same way), torsion vanishes, $T^a = 0$.

c) We can do the same with the terms that are proportional to $(dA - F - a)$.

$$\lambda\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge(dA-F-a) - i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge(dA-F-a) \\ + i\kappa\varepsilon^A{}_B\bar{\psi}_A\gamma^5\wedge\rho^B\wedge(dA-F-a) \tag{82}$$

They are going to vanish because one of the equations of motion implies that $(dA-F-a) = 0$, as we will see.

d) We continue with those that depend on F.

$$-\frac{\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v - \frac{\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge DF_{uv}\wedge e^u\wedge e^v \\ + \lambda\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v \\ = \frac{\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v - \frac{\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge DF_{uv}\wedge e^u\wedge e^v \tag{83}$$

e) Now, we consider those of second order in F and ψ.

$$\frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge F_{uv}\wedge e^u\wedge e^v \\ + \frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge\bar{\psi}_A\gamma^v\wedge\psi^A \\ - i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge F_{uv}\wedge e^u\wedge e^v \\ = -\frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge F_{uv}\wedge e^u\wedge e^v \\ + \frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge\bar{\psi}_A\gamma^v\wedge\psi^A \tag{84}$$

f) These two of second order in ψ and that they depend on DF^{ab} cancel each other.

$$\lambda\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge a - \frac{i\kappa\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B = \\ \frac{i\kappa\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B - \frac{i\kappa\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B = 0 \tag{85}$$

g) These two of order κ^3 vanish too.

$$-i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge a - \frac{\kappa^3\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge\varepsilon^B{}_C\bar{\psi}_B\wedge\psi^C \\ = \frac{\kappa^3\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge\varepsilon^B{}_C\bar{\psi}_B\wedge\psi^C \\ - \frac{\kappa^3\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi}_A\gamma^d\wedge\psi^A\wedge\varepsilon^B{}_C\bar{\psi}_B\wedge\psi^C = 0 \tag{86}$$

h) Finally, we are going to derive the Fierz identity in this case, which is more difficult than in $N = 1$.

$$
\begin{aligned}
&\frac{\kappa^2}{4}\bar{\psi}_A\gamma^5\gamma_a \wedge \bar{\psi}_B\gamma^a \wedge \psi^B \wedge \rho^A + i\kappa\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge \rho^B \wedge a + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \rho^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D \\
&+ \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \rho^D \\
&= \frac{\kappa^2}{4}\bar{\psi}_A\gamma^5\gamma_a \wedge \bar{\psi}_B\gamma^a \wedge \psi^B \wedge \rho^A - \frac{\kappa^2}{2}\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge \rho^B \wedge \varepsilon^C{}_D\bar{\psi}_C \wedge \psi^D \\
&+ \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \rho^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \rho^D \\
&= \frac{\kappa^2}{4}\bar{\psi}_A\gamma^5\gamma_a \wedge \bar{\psi}_B\gamma^a \wedge \psi^B \wedge \rho^A + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \rho^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D \\
&- \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \rho^D \\
&= \frac{\kappa^2}{4}\bar{\psi}_A\gamma^5\gamma_a \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) \wedge \rho^A + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge \rho^B \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) \\
&- \frac{\kappa^2}{4}(\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B) \wedge \varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \rho^D \\
&= \frac{\kappa^2}{4}\bar{\psi}_A\gamma^5\gamma_a \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) \wedge \rho^A + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) \wedge \rho^B \\
&- \frac{\kappa^2}{4}\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge (\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B) \wedge \rho^D
\end{aligned}
\tag{87}
$$

If we multiply all by γ^5 gives,

$$
\begin{aligned}
&- \frac{\kappa^2}{4}\bar{\psi}_A\gamma_a \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) \wedge \rho^A + \frac{\kappa^2}{4}\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) \wedge \rho^B \\
&+ \frac{\kappa^2}{4}\varepsilon^C{}_D\bar{\psi}_C \wedge (\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B) \wedge \rho^D = \\
&- \frac{\kappa^2}{4}\bar{\psi}_A\gamma_a \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) \wedge \rho^A - \frac{\kappa^2}{4}\varepsilon^B{}_A\bar{\psi}_B\gamma^5 \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) \wedge \rho^A \\
&- \frac{\kappa^2}{4}\varepsilon^D{}_C\bar{\psi}_D \wedge (\varepsilon^A{}_B\bar{\psi}_A \wedge \psi^B) \wedge \rho^C = 0
\end{aligned}
\tag{88}
$$

The expression vanishes by the Fierz rearrangement, $\gamma_a\psi^A \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) + \varepsilon^A{}_B\psi^B \wedge (\varepsilon^C{}_D\bar{\psi}_C \wedge \psi^D) + \varepsilon^A{}_B\gamma^5\psi^B \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) = 0$, as we said.

The rest can not be simplified, and we consider them individually in H_2.

So the final expression of H_2 is,

$$
\begin{aligned}
H_2 =&\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c\wedge T^d+\frac{i}{2}\bar{\rho_A}\wedge\gamma^5\gamma_a e^a\wedge\rho^A-\frac{i}{2}\bar{\psi_A}\wedge\gamma^5\gamma_a T^a\wedge\rho^A\\
&+\frac{\lambda}{2}\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge e^v-\frac{\lambda}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge DF_{uv}\wedge e^u\wedge e^v\\
&+\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge T^v-\frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi_A}\gamma^d\wedge\psi^A\wedge F_{uv}\wedge e^u\wedge e^v\\
&+\frac{i\lambda\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\wedge\bar{\psi_A}\gamma^v\wedge\psi^A-2\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge T^d\wedge(dA-a-\frac{F}{2})\\
&+i\kappa\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\bar{\psi_A}\wedge\rho^B+i\kappa\varepsilon^A{}_B\bar{\psi_A}\gamma^5\wedge\rho^B\wedge F_{uv}\wedge e^u\wedge e^v\\
&+\lambda\epsilon_{abcd}DF^{ab}\wedge e^c\wedge e^d\wedge(dA-F-a)-i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\wedge\bar{\psi_A}\gamma^d\wedge\psi^A\wedge(dA-F-a)\\
&+i\kappa\varepsilon^A{}_B\bar{\psi_A}\gamma^5\wedge\rho^B\wedge(dA-F-a)
\end{aligned}
\tag{89}
$$

As we can see, the first three terms are the same as the $N=1$ case. The rest are new.

B.3 Motion equations

We are going to derive the equations of motion. Proceeding like in the other case, we have:

1. The ω^{ab} equation:

$$
I_{R^{ab}}H_2=\frac{1}{2\kappa^2}\epsilon_{abcd}e^c\wedge T^d=0 \tag{90}
$$

2. Equation of e^d:

$$
\begin{aligned}
I_{T^d}H_2 =&\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c-\frac{i}{2}\bar{\psi_A}\gamma^5\gamma_d\wedge\rho^A+\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\\
&-2\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge(dA-a-\frac{1}{2}F)=0
\end{aligned}
\tag{91}
$$

Where we have,

$$
\begin{aligned}
\tau_d &= \frac{\partial}{\partial e_d}({}_*F\wedge(dA-a-\frac{1}{2}F)) = {}_*F\wedge\frac{\partial}{\partial e_d}(\frac{1}{2}F)-(\frac{\partial}{\partial e_d}{}_*F)\wedge(dA-a-\frac{1}{2}F)\\
&=\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u-2\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge(dA-a-\frac{1}{2}F)
\end{aligned}
\tag{92}
$$

So we can write,

$$
I_{T^d}H_2=\frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c-\frac{i}{2}\bar{\psi_A}\gamma^5\gamma_d\wedge\rho^A+\tau_d=0 \tag{93}
$$

3. $\bar{\psi_A}$ equation, which is the same that for ψ^A:

$$
\begin{aligned}
I_{\bar{\rho_A}}H_2 =&i\gamma^5\gamma_a e^a\wedge\rho^A-\frac{i}{2}\gamma^5\gamma_a T^a\wedge\psi^A-i\kappa\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\psi^B\\
&-i\kappa\varepsilon^A{}_B\psi^B\gamma^5\wedge F_{uv}\wedge e^u\wedge e^v-i\kappa\varepsilon^A{}_B\psi^B\gamma^5\wedge(dA-F-a)=0
\end{aligned}
\tag{94}
$$

And,

$$i\gamma^5\gamma_a e^a \wedge \hat{D}\psi^A = i\gamma^5\gamma_a e^a \wedge \rho^A - i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B \tag{95}$$

Then,

$$I_{\bar{\rho_A}} H_2 = i\gamma^5\gamma_a e^a \wedge \hat{D}\psi^A - \frac{i}{2}\gamma^5\gamma_a T^a \wedge \psi^A + i\kappa\varepsilon^A{}_B\psi^B\gamma^5 \wedge (F + a - dA) = 0 \tag{96}$$

4. We calculate the F^{ab} equation ($\lambda = 1/2$):

$$\begin{aligned} I_{DF^{ab}} H =& \frac{1}{4}\epsilon_{abcd} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge e^v - \frac{1}{4}\epsilon_{uvcd}F^{uv} \wedge e^c \wedge e^d \wedge e_a \wedge e_b \\ &+ \frac{1}{2}\epsilon_{abcd} \wedge e^c \wedge e^d \wedge (dA - F - a) = 0 \end{aligned} \tag{97}$$

We note that, $e^a \wedge e^b \wedge e^c \wedge e^d = \epsilon^{abcd}E$

$$\begin{aligned} &\frac{1}{4}\epsilon_{abcd} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge e^v - \frac{1}{4}\epsilon_{uvcd}F^{uv} \wedge e^c \wedge e^d \wedge e_a \wedge e_b \\ &= \frac{1}{4}\epsilon_{abcd}\epsilon^{cduv}E \wedge F_{uv} - \frac{1}{4}\epsilon_{uvcd}\epsilon^{cdab}E \wedge F^{uv} \\ &= \frac{1}{4}\epsilon_{abcd}\epsilon^{uvcd}E \wedge F_{uv} - \frac{1}{4}\epsilon_{abcd}\epsilon^{uvcd}E \wedge F_{uv} = 0 \end{aligned} \tag{98}$$

So,

$$I_{DF^{ab}} H = \frac{1}{2}\epsilon_{abcd} \wedge e^c \wedge e^d \wedge (dA - F - a) = 0 \tag{99}$$

5. Finally, for A, deriving respect to $\mathcal{F} = (dA - a)$

$$\begin{aligned} I_{\mathcal{F}} H_2 &= -2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d + \lambda\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \\ &- i\lambda\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A + i\kappa\varepsilon^A{}_B\bar{\psi_A}\gamma^5 \wedge \rho^B \\ &= -2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d + \lambda\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \\ &- i\lambda\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi_A}\gamma^d \wedge \psi^A - Db \\ &= \lambda\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d - 2\lambda\epsilon_{abcd}F^{ab} \wedge e^c \wedge (T^d + \frac{i\kappa^2}{2}\bar{\psi_A}\gamma^d \wedge \psi^A) - Db \\ &= D_*F - Db = 0 \end{aligned} \tag{100}$$

So,

$$I_{\mathcal{F}} H_2 = D({}_*F - b) = 0 \tag{101}$$

B.4 Vanishing $I_{\bar{\psi}_A}H_2$ by motion equations

We follow with the expression of $I_{\bar{\psi}_A}H_2$, which is going to be null when we impose the equations of motion. Then,

$$\begin{aligned} I_{\bar{\psi}_A}H_2 = &-\frac{i}{2}\gamma^5\gamma_a T^a\wedge\rho^A - i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F_{uv}\wedge e^u\wedge e^v \\ &+ i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A + i\kappa\lambda\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d\wedge\varepsilon^A{}_B\rho^B \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge F_{uv}\wedge e^u\wedge e^v - 2i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a) \end{aligned} \tag{102}$$

Simplifying,

$$\begin{aligned} I_{\bar{\psi}_A}H_2 = &-\frac{i}{2}\gamma^5\gamma_a T^a\wedge\rho^A - i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\ &+ i\kappa_*F\wedge\varepsilon^A{}_B\rho^B + i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge F - 2i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a) \end{aligned} \tag{103}$$

Finally,

$$\begin{aligned} I_{\bar{\psi}_A}H_2 = &-\frac{i}{2}\gamma^5\gamma_a T^a\wedge\rho^A - i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\ &+ i\kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\rho^B - 2i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a) \end{aligned} \tag{104}$$

Now, we are going to prove that it vanish imposing the field equations. We begin making use of the fact that, $T^a = 0$ and $(dA - F - a) = 0$. So,

$$I_{\bar{\psi}_A}H_2 = -i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A + i\kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\rho^B \tag{105}$$

We must show that these three terms are going to be cancelled. To prove it, we take (94),

$$\begin{aligned} I_{\bar{\rho}_A}H_2 &= i\gamma^5\gamma_a e^a\wedge\rho^A - \frac{i}{2}\gamma^5\gamma_a T^a\wedge\psi^A - i\kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\psi^B - i\kappa\varepsilon^A{}_B\psi^B\gamma^5\wedge(dA-F-a) \\ &= i\gamma^5\gamma_a e^a\wedge\rho^A - i\kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\psi^B = 0 \end{aligned} \tag{106}$$

where we have,

$$\gamma^5\gamma_a e^a\wedge\rho^A = \kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\psi^B \tag{107}$$

From this relation we obtain,

$$M\gamma^5\gamma_a e^a = ({}_*F+\gamma^5F) \tag{108}$$

If we find the value of the M matrix, we will prove what we are searching. At first, we take the slash notation: $\not{F} = \gamma_{ab}F^{ab}$ and $\not{e} = \gamma_a e^a$. Then, M is going to be of the form, $M \propto \not{e}\wedge\not{F}$. It implies that,

$$ {}_{*}F + \gamma^5 F \propto \not{e} \wedge \not{F} \wedge \gamma^5 \gamma_a e^a = \not{e} \wedge \not{F} \wedge \not{e}\gamma^5 \tag{109} $$

We expand the right-hand term,

$$ \not{e} \wedge \not{F} \wedge \not{e}\gamma^5 = \gamma_a e^a \wedge \gamma_{bc} F^{bc} \wedge \gamma_d e^d \gamma^5 \tag{110} $$

We can contract the indices in different ways:

$$ \begin{cases} \gamma_a e^a \wedge \gamma_{bc} F^{bc} \wedge \gamma_d e^d \gamma^5 = \\ e^a \wedge F^{bc} \wedge e^d \gamma_{abcd} \gamma^5 \text{ (0 contractions)} \\ +2e_a \wedge F^{ac} \wedge \gamma_{cd} e^d \gamma^5 \text{ (1 contraction)} \\ +2e^a \wedge \gamma_{ab} F^{bc} \wedge e_c \gamma^5 \text{ (1 contraction)} \\ +2e^a \wedge F_{ab} \wedge e^b \gamma^5 \text{ (2 contractions)} \end{cases} \tag{111} $$

The terms with one contraction cancel each other. That means,

$$ \begin{aligned} &\gamma_a e^a \wedge \gamma_{bc} F^{bc} \wedge \gamma_d e^d \gamma^5 = e^a \wedge F^{bc} \wedge e^d \gamma_{abcd} \gamma^5 + 2e^a \wedge F_{ab} \wedge e^b \gamma^5 \\ &= -e^a \wedge F^{bc} \wedge e^d \epsilon_{abcd} \gamma^5 \gamma^5 + 2e^a \wedge F_{ab} \wedge e^b \gamma^5 \end{aligned} \tag{112} $$

We know that, $(\gamma^5\gamma^5) = -I$ and $e^a \wedge F_{ab} \wedge e^b = F_{ab} \wedge e^a \wedge e^b = F$. Which that implies,

$$ \begin{aligned} &- e^a \wedge F^{bc} \wedge e^d \epsilon_{abcd} \gamma^5 \gamma^5 + 2e^a \wedge F_{ab} \wedge e^b \gamma^5 = \epsilon_{abcd} F^{bc} \wedge e^a \wedge e^d + 2\gamma^5 F \\ &= \epsilon_{bcad} F^{bc} \wedge e^a \wedge e^d + 2\gamma^5 F \end{aligned} \tag{113} $$

As we remember,

$$ \begin{aligned} &{}_{*}F = \lambda \epsilon_{abcd} F^{ab} \wedge e^c \wedge e^d \to \lambda = 1/2 \\ &\to {}_{*}F = \frac{1}{2} \epsilon_{abcd} F^{ab} \wedge e^c \wedge e^d \to 2{}_{*}F = \epsilon_{abcd} F^{ab} \wedge e^c \wedge e^d \end{aligned} \tag{114} $$

So,

$$ \epsilon_{bcad} F^{bc} \wedge e^a \wedge e^d + 2\gamma^5 F = 2{}_{*}F + 2\gamma^5 F = 2({}_{*}F + \gamma^5 F) \tag{115} $$

That is,

$$ \begin{aligned} &({}_{*}F + \gamma^5 F) = M\gamma^5 \gamma_a e^a \propto \not{e} \wedge \not{F} \wedge \gamma^5 \gamma_a e^a = 2({}_{*}F + \gamma^5 F) \\ &\to ({}_{*}F + \gamma^5 F) = \frac{1}{2} \not{e} \wedge \not{F} \wedge \gamma^5 \gamma_a e^a \end{aligned} \tag{116} $$

And as a conclusion,

$$ M = \frac{1}{2} \not{e} \wedge \not{F} \tag{117} $$

Now, as we said, we can complete our proof. We write,

$$
\begin{aligned}
I_{\bar{\psi}_A}H_2 &= -i\lambda\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\
&+ i\kappa({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\rho^B \\
&= -\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\
&+ \frac{i\kappa}{2}\not{e}\wedge\not{F}\wedge\gamma^5\gamma_a e^a\wedge\varepsilon^A{}_B\rho^B \\
&= -\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\
&+ \frac{i\kappa^2}{2}\not{e}\wedge\not{F}\wedge({}_*F+\gamma^5F)\wedge\varepsilon^A{}_B\varepsilon^B{}_C\psi^C \\
&= -\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\
&- \frac{i\kappa^2}{2}\not{e}\wedge\not{F}\wedge({}_*F+\gamma^5F)\wedge\psi^A
\end{aligned}
\tag{118}
$$

In the last step we have taken that, $\varepsilon^A{}_B\varepsilon^B{}_C = -\delta^A{}_C \to -\delta^A{}_C = -1$ if $A = C$. Afterwards, we consider the last term of the last line,

$$
\begin{aligned}
&-\frac{i\kappa^2}{2}\not{e}\wedge\not{F}\wedge({}_*F+\gamma^5F)\wedge\psi^A = -\frac{i\kappa^2}{2}\gamma_a e^a\wedge\gamma_{bc}F^{bc}\wedge({}_*F+\gamma^5F)\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}(e^a\wedge F^{bc}\gamma_{abc} + 2e^a\wedge F_{ab}\gamma^b)\wedge({}_*F+\gamma^5F)\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}e^a\wedge F^{bc}\gamma_{abc}\wedge{}_*F\wedge\psi^A - \frac{i\kappa^2}{2}e^a\wedge F^{bc}\wedge\gamma_{abc}\gamma^5F\wedge\psi^A - i\kappa^2e^a\wedge F_{ab}\gamma^b\wedge{}_*F\wedge\psi^A \\
&- i\kappa^2e^a\wedge F_{ab}\wedge\gamma^b\gamma^5F\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}e^a\wedge F^{bc}\gamma_{abc}\wedge{}_*F\wedge\psi^A + \frac{i\kappa^2}{2}e^a\wedge F^{bc}\wedge\epsilon_{abcd}\gamma^dF\wedge\psi^A - i\kappa^2e^a\wedge F_{ab}\gamma^b\wedge{}_*F\wedge\psi^A \\
&- i\kappa^2e^a\wedge F_{ab}\wedge\gamma^b\gamma^5F\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}e^a\wedge F^{bc}\gamma_{abc}\wedge{}_*F\wedge\psi^A + \frac{i\kappa^2}{2}\epsilon_{bcad}F^{bc}\wedge e^a\gamma^d\wedge\psi^A\wedge F - i\kappa^2{}_*F\wedge F_{ab}\wedge e^a\gamma^b\wedge\psi^A \\
&- i\kappa^2e^a\wedge F_{ab}\wedge\gamma^b\gamma^5F\wedge\psi^A
\end{aligned}
\tag{119}
$$

Coming back to $I_{\bar{\psi}_A}H_2$,

$$
\begin{aligned}
I_{\bar{\psi}_A}H_2 &= -\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A \\
&- \frac{i\kappa^2}{2}\not{e}\wedge\not{F}\wedge({}_*F+\gamma^5F)\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge F + i\kappa^2{}_*F\wedge F_{uv}\wedge e^u\gamma^v\wedge\psi^A - \frac{i\kappa^2}{2}e^a\wedge F^{bc}\gamma_{abc}\wedge{}_*F\wedge\psi^A \\
&+ \frac{i\kappa^2}{2}\epsilon_{bcad}F^{bc}\wedge e^a\gamma^d\wedge\psi^A\wedge F - i\kappa^2{}_*F\wedge F_{ab}\wedge e^a\gamma^b\wedge\psi^A - i\kappa^2e^a\wedge F_{ab}\wedge\gamma^b\gamma^5F\wedge\psi^A \\
&= -\frac{i\kappa^2}{2}e^a\wedge F^{bc}\gamma_{abc}\wedge{}_*F\wedge\psi^A - i\kappa^2e^a\wedge F_{ab}\wedge\gamma^b\gamma^5F\wedge\psi^A
\end{aligned}
\tag{120}
$$

We have,

$$
\begin{aligned}
I_{\bar{\psi}_A} H_2 =& -\frac{i\kappa^2}{2} e^a \wedge F^{bc}\gamma_{abc} \wedge {}_*F \wedge \psi^A - i\kappa^2 e^a \wedge F_{ab} \wedge \gamma^b\gamma^5 F \wedge \psi^A \\
=& -\frac{i\kappa^2}{4} e^a \wedge F^{bc}\gamma_{abc}\epsilon_{uvwt} \wedge F^{uv} \wedge e^w \wedge e^t \wedge \psi^A \\
& - i\kappa^2 e^a \wedge F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge e^u \wedge e^v \wedge \psi^A \\
=& -\frac{i\kappa^2}{4} e^a \wedge F^{bc}\epsilon_{abcd}\gamma^d\gamma^5\epsilon_{uvwt} \wedge F^{uv} \wedge e^w \wedge e^t \wedge \psi^A \\
& - i\kappa^2 e^a \wedge F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge e^u \wedge e^v \wedge \psi^A \\
=& -\frac{i\kappa^2}{4} F^{bc}\epsilon_{abcd}\gamma^d\gamma^5\epsilon_{uvwt} \wedge F^{uv} \wedge e^a \wedge e^w \wedge e^t \wedge \psi^A \\
& - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge e^a \wedge e^u \wedge e^v \wedge \psi^A
\end{aligned} \tag{121}
$$

We rewrite the expression using the next fact, $e^a \wedge e^b \wedge e^c = \epsilon^{abcd} E_d$.

$$
I_{\bar{\psi}_A} H_2 = -\frac{i\kappa^2}{4} F^{bc}\epsilon_{abcd}\gamma^d\gamma^5\epsilon_{uvwt} \wedge F^{uv} \wedge \epsilon^{awtg} E_g \wedge \psi^A - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A \tag{122}
$$

We can note that,

$$
\epsilon_{uvwt}\epsilon^{awtg} = \epsilon_{uvwt}\epsilon^{agwt} = -2(\delta^a{}_u\delta^g{}_v - \delta^a{}_v\delta^g{}_u) \tag{123}
$$

and the equation reads,

$$
\begin{aligned}
I_{\bar{\psi}_A} H_2 =& \frac{i\kappa^2}{2}(\delta^a{}_u\delta^g{}_v - \delta^a{}_v\delta^g{}_u) F^{bc}\epsilon_{abcd}\gamma^d\gamma^5 \wedge F^{uv} \wedge E_g \wedge \psi^A - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A \\
=& \frac{i\kappa^2}{2}(\delta^a{}_u - \delta^a{}_g\delta^g{}_u) F^{bc}\epsilon_{abcd}\gamma^d\gamma^5 \wedge F^{ug} \wedge E_g \wedge \psi^A - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A \\
=& i\kappa^2 F^{bc}\epsilon_{abcd}\gamma^d\gamma^5 \wedge F^{ag} \wedge E_g \wedge \psi^A - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A
\end{aligned} \tag{124}
$$

To do the third step we have taken: $\delta^a{}_u - \delta^a{}_g\delta^g{}_u = \delta^a{}_u + \delta^a{}_u = 2\delta^a{}_u$. At next, we interchange the g index in the first term,

$$
I_{\bar{\psi}A} H_2 = i\kappa^2 F^{bc}\epsilon_{abcd}\gamma^d\gamma^5 \wedge F^a{}_g \wedge E^g \wedge \psi^A - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A \tag{125}
$$

Now, we make contractions of the low indices a, b, c, d and d

$$
\begin{aligned}
I_{\bar{\psi}A} H_2 =& 2i\kappa^2 F^{bc}\epsilon_{agcd}\gamma^d\gamma^5 \wedge F^a{}_b \wedge E^g \wedge \psi^A + i\kappa^2 F^{bc}\epsilon_{abcg}\gamma^d\gamma^5 \wedge F^a{}_d \wedge E^g \wedge \psi^A \\
& - i\kappa^2 F_{ab} \wedge \gamma^b\gamma^5 F_{uv} \wedge \epsilon^{auvg} E_g \wedge \psi^A
\end{aligned} \tag{126}
$$

where it gives,

$$
2i\kappa^2 F^{bc}\epsilon_{agcd}\gamma^d\gamma^5 \wedge F^a{}_b \wedge E^g \wedge \psi^A = 0 \tag{127}
$$

Due to the contraction of the b index in $F^{bc} \wedge F^a{}_b$. So that, after reorganize:

$$
I_{\bar{\psi}A} H_2 = i\kappa^2 \epsilon^{abcg} F_{ad}\gamma^d\gamma^5 \wedge F_{bc} \wedge E_g \wedge \psi^A - i\kappa^2 \epsilon^{auvg} F_{ab}\gamma^b\gamma^5 \wedge F_{uv} \wedge E_g \wedge \psi^A = 0 \tag{128}
$$

And the proof is completed.

B.5 $I_{\bar{\psi}_A} H_2$ in terms of fermionic, vierbein and F^{ab} equations

Knowing that this equation vanishes when we substitute motion equations, we are going to work with it. We are going to manipulate it, writing this relation in a more comfortable way. What we want to do is writing it in terms of motion equations. If we remember, it was (the parameter $\lambda = 1/2$)

$$\begin{aligned} I_{\bar{\psi}_A} H_2 = &-\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A - \frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge F + i\kappa^2{}_*F \wedge F_{uv} \wedge e^u\gamma^v \wedge \psi^A \\ &+ i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\rho^B - i\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge (dA - F - a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B \wedge (dA - F - a) \end{aligned} \tag{129}$$

As we said before,

$$({}_*F + \gamma^5 F) = M\gamma^5\gamma_a e^a \tag{130}$$

Where we have, $M = \frac{1}{2}\not{e} \wedge \not{F}$. We use this fact to express our relation in terms of $I_{\bar{\rho}_A} H_2 \equiv E(\bar{\psi}_A)$. We have,

$$E(\bar{\psi}_A) = i\gamma^5\gamma_a e^a \wedge \rho^A - \frac{i}{2}\gamma^5\gamma_a T^a \wedge \psi^A - i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B - i\kappa\varepsilon^A{}_B\psi^B\gamma^5 \wedge (dA - F - a) = 0 \tag{131}$$

We multiply all by κM:

$$\begin{aligned} \kappa M E(\bar{\psi}_A) = &\, i\kappa M\gamma^5\gamma_a e^a \wedge \rho^A - \frac{i\kappa}{2} M\gamma^5\gamma_a T^a \wedge \psi^A \\ &- i\kappa^2 M({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B - i\kappa^2 M\varepsilon^A{}_B\psi^B\gamma^5 \wedge (dA - F - a) \end{aligned} \tag{132}$$

We add and substract this last expression in (129) obtaining

$$\begin{aligned} I_{\bar{\psi}_A} H_2 = &-\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A - \frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge F + i\kappa^2{}_*F \wedge F_{uv} \wedge e^u\gamma^v \wedge \psi^A \\ &+ i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\rho^B - i\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge (dA - F - a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B \wedge (dA - F - a) + \kappa M E(\bar{\psi}_A) - i\kappa M\gamma^5\gamma_a e^a \wedge \rho^A \\ &+ \frac{i\kappa}{2} M\gamma^5\gamma_a T^a \wedge \psi^A + i\kappa^2 M({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B + i\kappa^2 M\varepsilon^A{}_B\psi^B\gamma^5 \wedge (dA - F - a) \end{aligned} \tag{133}$$

Where it gives,

$$i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\rho^B - i\kappa M\gamma^5\gamma_a e^a \wedge \rho^A = 0 \tag{134}$$

and,

$$-\frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge F + i\kappa^2{}_*F \wedge F_{uv} \wedge e^u\gamma^v \wedge \psi^A + i\kappa^2 M({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B = 0 \tag{135}$$

Like we have proved before. This implies that:

$$I_{\bar{\psi}_A} H_2 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A - i\kappa^2 \epsilon_{abcd} F^{ab} \wedge e^c \gamma^d \wedge \psi^A \wedge (dA - F - a) + i\kappa \varepsilon^A{}_B \gamma^5 \rho^B \wedge (dA - F - a)$$
$$+ \kappa M E(\bar{\psi}_A) + \frac{i\kappa}{2} M \gamma^5 \gamma_a T^a \wedge \psi^A + i\kappa^2 M \varepsilon^A{}_B \gamma^5 \psi^B \wedge (dA - F - a) \tag{136}$$

The following step is changing T^a and $(dA-F-a)$ terms by others proportional to $I_{R^{ab}} H_2 \equiv E(\omega^{ab})$ and $I_{DF^{ab}} H_2 \equiv E(F^{ab})$, respectively. We are going to do it one by one:

- Firstly,
$$-\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A = \frac{1}{2} X_1^{ab,A} \wedge E(\omega^{ab}) \tag{137}$$
We have to calculate $X_1^{ab,A}$, which has the structure
$$X_1^{ab,A} = m\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge \rho^A_{ud} + n\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \tag{138}$$
We only need to find m and n. Then, we know that
$$E(\omega^{ab}) = \frac{1}{2\kappa^2}\epsilon_{abcd} e^c \wedge T^d \tag{139}$$
So,
$$-\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A = \frac{1}{4\kappa^2} m\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge \epsilon_{abrs} e^r \wedge T^s + \frac{1}{4\kappa^2} n\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge \epsilon_{abrs} e^r \wedge T^s \tag{140}$$
We deal with it separately. Starting with,
$$\begin{aligned}
&\frac{1}{4\kappa^2} m\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge \epsilon_{abrs} e^r \wedge T^s = -\frac{1}{2\kappa^2} m\delta^{cd}_{rs}\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge e^r \wedge T^s \\
&= -\frac{1}{2\kappa^2} m\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge e^c \wedge T^d + \frac{1}{2\kappa^2} m\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge e^d \wedge T^c \\
&= -\frac{1}{2\kappa^2} m\gamma^5\gamma_c e^u \wedge \rho^A_{ud} \wedge e^c \wedge T^d + \frac{1}{2\kappa^2} m\gamma^5\gamma_c T^c \wedge \rho^A
\end{aligned} \tag{141}$$
The other gives us,
$$\begin{aligned}
&\frac{1}{4\kappa^2} n\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge \epsilon_{abrs} e^r \wedge T^s = -\frac{1}{2\kappa^2} n\delta^{cd}_{rs}\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge e^r \wedge T^s \\
&= -\frac{1}{2\kappa^2} n\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge e^c \wedge T^d + \frac{1}{2\kappa^2} n\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge e^d \wedge T^c \\
&= -\frac{1}{2\kappa^2} n\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge e^c \wedge T^d - \frac{1}{2\kappa^2} n\gamma^5\gamma^u e_u \wedge \rho^A_{dc} \wedge e^d \wedge T^c \\
&= -\frac{1}{\kappa^2} n\gamma^5\gamma^u e_u \wedge \rho^A_{cd} \wedge e^c \wedge T^d
\end{aligned} \tag{142}$$
So we have,

$$\gamma^5\gamma_a T^a\wedge\rho^A = -\frac{i}{\kappa^2}m\gamma^5\gamma_c e^u\wedge\rho^A_{ud}\wedge e^c\wedge T^d + \frac{i}{\kappa^2}m\gamma^5\gamma_c T^c\wedge\rho^A - \frac{2i}{\kappa^2}n\gamma^5\gamma^u e_u\wedge\rho^A_{cd}\wedge e^c\wedge T^d \tag{143}$$

We note,

$$\begin{cases}\gamma^5\gamma_a T^a\wedge\rho^A = \frac{i}{\kappa^2}m\gamma^5\gamma_c T^c\wedge\rho^A \\ -\frac{i}{\kappa^2}m\gamma^5\gamma_c e^u\wedge\rho^A_{ud}\wedge e^c\wedge T^d - \frac{2i}{\kappa^2}n\gamma^5\gamma^u e_u\wedge\rho^A_{cd}\wedge e^c\wedge T^d = 0\end{cases} \tag{144}$$

Which gives, respectively,

$$\begin{cases}m = -i\kappa^2 \\ \frac{i}{\kappa^2}m\gamma^5\gamma_c e^c\wedge\rho^A_{ud}\wedge e^u\wedge T^d = \frac{2i}{\kappa^2}n\gamma^5\gamma^u e_u\wedge\rho^A_{cd}\wedge e^c\wedge T^d \to m = 2n \to n = \frac{-i\kappa^2}{2}\end{cases} \tag{145}$$

And,

$$X_1^{ab,A} = -i\kappa^2\epsilon^{abcd}\gamma^5\gamma_c e^u\wedge\rho^A_{ud} - \frac{i\kappa^2}{2}\epsilon^{abcd}\gamma^5\gamma^u e_u\wedge\rho^A_{cd} \tag{146}$$

- Secondly,

$$\frac{i\kappa}{2}M\gamma^5\gamma_a T^a\wedge\psi^A = \frac{1}{2}X_2^{ab,A}\wedge E(\omega^{ab}) \tag{147}$$

This case is very similar as before. The differences are: a minus sign, a κ factor and that we have to substitute ρ^A by $(M\psi)^A$. It means, if we introduce these changes in the first expression, we will directly obtain $X_2^{ab,A}$:

$$X_2^{ab,A} = i\kappa^3\epsilon^{abcd}\gamma^5\gamma_c e^u\wedge(M\psi)^A_{ud} + \frac{i\kappa^3}{2}\epsilon^{abcd}\gamma^5\gamma^u e_u\wedge(M\psi)^A_{cd} \tag{148}$$

- Thirdly,

$$i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a) = \frac{1}{2}X_3^{ab,A}E(F^{ab}) \tag{149}$$

Where the equation,

$$E(F^{ab}) = \frac{1}{2}\epsilon_{abcd}e^c\wedge e^d\wedge(dA-F-a) \tag{150}$$

That is,

$$i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a) = \frac{1}{4}X_3^{ab,A}\wedge\epsilon_{abcd}e^c\wedge e^d\wedge(dA-F-a) \tag{151}$$

Here, $X_3^{ab,A}$ is going to have the form,

$$X_3^{ab,A} = \lambda\epsilon^{abcd}\varepsilon^A{}_B\gamma^5\rho^B_{cd} \tag{152}$$

Which implies,

$$
\begin{aligned}
&\frac{\lambda}{4}\epsilon^{abcd}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge\epsilon_{abrs}e^r\wedge e^s\wedge(dA-F-a)=-\frac{\lambda}{2}\delta^{cd}_{rs}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge e^r\wedge e^s\wedge(dA-F-a)\\
&=-\frac{\lambda}{2}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge e^c\wedge e^d\wedge(dA-F-a)+\frac{\lambda}{2}\delta^{cd}_{rs}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge e^d\wedge e^c\wedge(dA-F-a)\\
&=-\frac{\lambda}{2}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge e^c\wedge e^d\wedge(dA-F-a)-\frac{\lambda}{2}\delta^{cd}_{rs}\varepsilon^A{}_B\gamma^5\rho^B_{cd}\wedge e^c\wedge e^d\wedge(dA-F-a)\\
&=-\frac{\lambda}{2}\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a)-\frac{\lambda}{2}\delta^{cd}_{rs}\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a)\\
&=-\lambda\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a)
\end{aligned}
\tag{153}
$$

Finally,

$$
-\lambda\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a)=i\kappa\varepsilon^A{}_B\gamma^5\rho^B\wedge(dA-F-a)\to\lambda=-i\kappa \tag{154}
$$

and

$$
X_3^{ab,A}=-i\kappa\epsilon^{abcd}\varepsilon^A{}_B\gamma^5\rho^B_{cd} \tag{155}
$$

- Fourthly,

$$
i\kappa^2M\varepsilon^A{}_B\gamma^5\psi^B\wedge(dA-F-a)=\frac{1}{2}X_4^{ab,A}E(^Fab) \tag{156}
$$

We can see that it is almost equal to the previous case. This one only varies on $\rho^B\to(M\psi)^B$ and a κ factor. Doing this it gives,

$$
X_4^{ab,A}=-i\kappa^2\epsilon^{abcd}\varepsilon^A{}_B\gamma^5(M\psi)^B_{cd} \tag{157}
$$

- Fifthly,

$$
-i\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a)=\frac{1}{2}X_5^{ab,A}E(F^{ab}) \tag{158}
$$

Substituting $E(F^{ab})$ we have

$$
-i\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a)=\frac{1}{4}X_5^{ab,A}\wedge\epsilon_{abcd}e^c\wedge e^d\wedge(dA-F-a) \tag{159}
$$

$X_5^{ab,A}$ takes the form,

$$
X_5^{ab,A}=\lambda F^{ab}\gamma_c\wedge\psi^{c,A}+\mu F^{[ac}\gamma^{b]}\wedge\psi^A_c \tag{160}
$$

Then,

$$\begin{aligned}
&\frac{\lambda}{4}F^{ab}\gamma_c \wedge \psi^{c,A} \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_c^A \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&= \frac{\lambda}{4}F^{ab}\gamma_c \wedge \psi^{c,A} \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_a^A \wedge \epsilon_{cbrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_b^A \wedge \epsilon_{acrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_r^A \wedge \epsilon_{abcs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_s^A \wedge \epsilon_{abrc}e^r \wedge e^s \wedge (dA - F - a)
\end{aligned} \tag{161}$$

We note that,

$$\frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_a^A \wedge \epsilon_{cbrs}e^r \wedge e^s \wedge (dA - F - a) = -\frac{\mu}{4}F^{ca}\gamma^b \wedge \psi_a^A \wedge \epsilon_{cbrs}e^r \wedge e^s \wedge (dA - F - a) \tag{162}$$

and, if we change it to the left-hand side:

$$\begin{aligned}
&\frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_c^A \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) + \frac{\mu}{4}F^{ca}\gamma^b \wedge \psi_a^A \wedge \epsilon_{cbrs}e^r \wedge e^s \wedge (dA - F - a) \\
&= \frac{\mu}{2}F^{ac}\gamma^b \wedge \psi_c^A \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a)
\end{aligned} \tag{163}$$

We can see too,

$$\begin{aligned}
&\frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_r^A \wedge \epsilon_{abcs}e^r \wedge e^s \wedge (dA - F - a) + \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_s^A \wedge \epsilon_{abrc}e^r \wedge e^s \wedge (dA - F - a) \\
&= \frac{\mu}{2}F^{ac}\gamma^b \wedge \psi_r^A \wedge \epsilon_{abcs}e^r \wedge e^s \wedge (dA - F - a)
\end{aligned} \tag{164}$$

So, operating

$$\begin{aligned}
&\frac{\lambda}{4}F^{ab}\gamma_c \wedge \psi^{c,A} \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_c^A \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&= \frac{\lambda}{4}F^{ab}\gamma_c \wedge \psi^{c,A} \wedge \epsilon_{abrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{8}F^{ac}\gamma^b \wedge \psi_b^A \wedge \epsilon_{acrs}e^r \wedge e^s \wedge (dA - F - a) \\
&+ \frac{\mu}{4}F^{ac}\gamma^b \wedge \psi_r^A \wedge \epsilon_{abcs}e^r \wedge e^s \wedge (dA - F - a)
\end{aligned} \tag{165}$$

Then, we can write

$$
\begin{aligned}
&\frac{\lambda}{4}F^{ab}\gamma_c\wedge\psi^{c,A}\wedge\epsilon_{abrs}e^r\wedge e^s\wedge(dA-F-a)+\frac{\mu}{8}F^{ac}\gamma^b\wedge\psi^A_b\wedge\epsilon_{acrs}e^r\wedge e^s\wedge(dA-F-a)\\
&+\frac{\mu}{4}F^{ac}\gamma^b\wedge\psi^A_r\wedge\epsilon_{abcs}e^r\wedge e^s\wedge(dA-F-a)=-i\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a)
\end{aligned} \tag{166}
$$

What gives us

$$
\begin{cases}
\frac{\mu}{4}F^{ac}\gamma^b\wedge\psi^A_r\wedge\epsilon_{abcs}e^r\wedge e^s\wedge(dA-F-a)\\
=-i\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a)\\
\\
\frac{\lambda}{4}F^{ab}\gamma_c\wedge\psi^{c,A}\wedge\epsilon_{abrs}e^r\wedge e^s\wedge(dA-F-a)\\
+\frac{\mu}{8}F^{ac}\gamma^b\wedge\psi^A_b\wedge\epsilon_{acrs}e^r\wedge e^s\wedge(dA-F-a)=0
\end{cases} \tag{167}
$$

Working with the first expression,

$$
-\frac{\mu}{4}\epsilon_{acsb}F^{ac}\wedge e^s\gamma^b\wedge\psi^A\wedge(dA-F-a)=-i\kappa^2\epsilon_{abcd}F^{ab}\wedge e^c\gamma^d\wedge\psi^A\wedge(dA-F-a)\rightarrow\mu=4i\kappa^2 \tag{168}
$$

and with the second one,

$$
\begin{aligned}
&\frac{\lambda}{4}F^{ab}\gamma_c\wedge\psi^{c,A}\wedge\epsilon_{abrs}e^r\wedge e^s\wedge(dA-F-a)\\
&=-\frac{\mu}{8}F^{ac}\gamma^b\wedge\psi^A_b\wedge\epsilon_{acrs}e^r\wedge e^s\wedge(dA-F-a)\\
&\rightarrow\lambda=-\frac{\mu}{2}\rightarrow\lambda=-2i\kappa^2
\end{aligned} \tag{169}
$$

Finally,

$$
X_5^{ab,A}=-2i\kappa^2F^{ab}\gamma_c\wedge\psi^{c,A}+4i\kappa^2F^{[ac}\gamma^{b]}\wedge\psi^A_c \tag{170}
$$

With all these results we can rewrite $I_{\bar{\psi}_A}H_2$ in terms of motion equations:

$$
\begin{aligned}
I_{\bar{\psi}_A}H_2=&\frac{1}{2}[X_1^{ab,A}\wedge E(\omega^{ab})+X_2^{ab,A}\wedge E(\omega^{ab})+X_3^{ab,A}E(F^{ab})\\
&+X_4^{ab,A}E(F^{ab})+X_5^{ab,A}E(F^{ab})]+\kappa ME(\bar{\psi}_A)
\end{aligned} \tag{171}
$$

If we call:

$$
\begin{cases}
X^{ab,A}=X_1^{ab,A}+X_2^{ab,A}\\
Y^{ab,A}=X_3^{ab,A}+X_4^{ab,A}+X_5^{ab,A}\\
Z^A{}_B=\kappa M
\end{cases} \tag{172}
$$

we can write,

$$I_{\bar{\psi}_A} H_2 = \frac{1}{2} X^{ab,A} \wedge E(\omega^{ab}) + \frac{1}{2} Y^{ab,A} E(F^{ab}) + Z^A{}_B \wedge E(\bar{\psi}_B) = 0 \tag{173}$$

Because, $E(\omega^{ab}) = 0$, $E(F^{ab}) = 0$ and $E(\bar{\psi}_A) = 0$. Substituting $X_i^{ab,A}$ $(i = 1, 2, 3, 4, 5)$ and simplifying:

$$\begin{aligned} I_{\bar{\psi}_A} H_2 = & -[\frac{i\kappa^2}{2} \epsilon^{abcd} \gamma^5 \gamma_c e^u \wedge (\rho^A_{ud} - \kappa (M\psi)^A_{ud}) + \frac{i\kappa^2}{4} \epsilon^{abcd} \gamma^5 \gamma^u e_u \wedge (\rho^A_{cd} - \kappa (M\psi)^A_{cd})] \wedge E(\omega^{ab}) \\ & - [i\kappa \epsilon^{abcd} \varepsilon^A{}_B \gamma^5 (\rho^B_{cd} + \kappa (M\psi)^B_{cd}) + 2i\kappa^2 F^{ab} \gamma_c \wedge \psi^{c,A} - 4i\kappa^2 F^{[ac} \gamma^{b]} \wedge \psi^A_c] \wedge E(F^{ab}) \\ & + Z^A{}_B \wedge E(\bar{\psi}_B) = 0 \end{aligned} \tag{174}$$

B.6 Supersymmetry transformations

We have just what we were searching for. To finish our work, we are going to study the supersymmetry transformations. Firstly, recalling the algebra,

$$\begin{cases} R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \\ T^a = de^a + \omega^a{}_b \wedge e^b - \dfrac{i\kappa^2}{2} \bar{\psi}_A \gamma^a \wedge \psi^A \\ \rho^A = d\psi^A + \dfrac{1}{4} \gamma_{ab} \omega^{ab} \wedge \psi^A \\ \mathcal{F} = dA - a \end{cases} \tag{175}$$

And the curvature associated to F^{ab} is DF^{ab}. If we set the curvatures equal to zero, we recover the Maurer-Cartan equations:

$$\begin{cases} d\omega^{ab} = -\omega^a{}_c \wedge \omega^{cb} \\ de^a = -\omega^a{}_b \wedge e^b + \dfrac{i\kappa^2}{2} \bar{\psi}_A \gamma^a \wedge \psi^A \\ d\psi^A = -\dfrac{1}{4} \gamma_{ab} \omega^{ab} \wedge \psi^A \\ dA = a \end{cases} \tag{176}$$

Working in the group space, we use the Lie derivative:

$$L_X = di_X + i_X d \tag{177}$$

Where we have,

$$i_X \psi^A = \epsilon^A \tag{178}$$

With ϵ^A the transformation parameter. Due to this fact, we can calculate easily the supersymmetry transformations of the gauge fields:

i) For ω^{ab},

$$L_X \omega^{ab} = \delta\omega^{ab} = di_X \omega^{ab} + i_X d\omega^{ab} = -i_X(\omega^a{}_c \wedge \omega^{cb}) = 0 \tag{179}$$

So,

$$\delta\omega^{ab} = 0 \tag{180}$$

ii) The e^a variation,

$$\begin{aligned} L_X e^a =& \delta e^a = d i_X e^a + i_X d e^a = -i_X(\omega^a{}_b \wedge e^b) + \frac{i\kappa^2}{2} i_X(\bar{\psi}_A \gamma^a \wedge \psi^A) \\ =& \frac{i\kappa^2}{2}(\bar{\epsilon}_A \gamma^a \wedge \psi^A - \bar{\psi}_A \gamma^a \wedge \epsilon^A) = i\kappa^2 \bar{\epsilon}_A \gamma^a \wedge \psi^A \end{aligned} \tag{181}$$

and the final result is,

$$\delta e^a = i\kappa^2 \bar{\epsilon}_A \gamma^a \wedge \psi^A \tag{182}$$

iii) The ψ^A one,

$$L_X \psi^A = \delta \psi^A = d i_X \psi^A + i_X d\psi^A = d\epsilon^A - \frac{1}{4}\gamma_{ab} i_X(\omega^{ab} \wedge \psi^A) = d\epsilon^A + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \epsilon^A \tag{183}$$

which gives,

$$\delta \psi^A = d\epsilon^A + \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \epsilon^A = D\epsilon^A \tag{184}$$

iv) For A,

$$\begin{aligned} L_X A =& \delta A = d i_X A + i_X dA = i_X a = \frac{i\kappa}{2}\varepsilon^A{}_B i_X(\bar{\psi}_A \wedge \psi^B) \\ =& \frac{i\kappa}{2}\varepsilon^A{}_B(\bar{\epsilon}_A \wedge \psi^B - \bar{\psi}_A \wedge \psi^B) = i\kappa \varepsilon^A{}_B \bar{\epsilon}_A \wedge \psi^B \end{aligned} \tag{185}$$

Then,

$$\delta A = i\kappa \varepsilon^A{}_B \bar{\epsilon}_A \wedge \psi^B \tag{186}$$

v) Finishing with the F^{ab} one,

$$L_X F^{ab} = \delta F^{ab} = d i_X F^{ab} + i_X dF^{ab} = 0 \tag{187}$$

So,

$$\delta F^{ab} = 0 \tag{188}$$

We have just obtained the original transformations, but we are looking for those which are modified by $X^{ab,A}$, $Y^{ab,A}$ and $Z^A{}_B$. These variations are calculated according to the Lemma defined in the second section of this work. It tells us that the variations are the same as before, but we have to add some new terms to them. As result of doing this,

$$\begin{cases} \delta\omega^{ab} = -\bar{\epsilon}_A X^{ab,A} \\ \delta e^a = i\kappa^2 \bar{\epsilon}_A \gamma^a \psi^A \\ \delta\psi^A = d\epsilon^A + \frac{1}{4}\gamma_{ab}\omega^{ab}\epsilon^A - Z^A{}_B \epsilon^B \\ \delta A = i\kappa \varepsilon^A{}_B \bar{\epsilon}_A \psi^B \\ \delta F^{ab} = -\bar{\epsilon}_A Y^{ab,A} \end{cases} \tag{189}$$

And we have what we wanted.